

Irreversible k -Threshold Conversion Processes on Graphs

by

Jane Wodlinger

B.Sc., University of Victoria, 2009

M.Sc., Simon Fraser University, 2012

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ABSTRACT

Given a graph G and an initial colouring of its vertices with two colours, say black and white, an irreversible k -threshold conversion process on G is an iterative process in which a white vertex becomes permanently coloured black at time t if at least k of its neighbours are coloured black at time $t - 1$. A set S of vertices is an irreversible k -threshold conversion set (k -conversion set) of G if the initial colouring in which the vertices of S are black and the others are white results in the whole vertex set becoming black eventually. In the case where G is $(k + 1)$ -regular, it can be shown that the k -conversion sets coincide with the so-called feedback vertex sets, or decycling sets.

In this dissertation we study the size, $c_k(G)$, and structure of minimum k -conversion sets in several classes of graphs, G . We examine conditions that lead to equality and inequality in existing bounds on $c_k(G)$ for k - and $(k + 1)$ -regular graphs. Furthermore, we derive new sharp lower bounds on $c_k(G)$ for regular graphs of degree ranging from $k + 1$ to $2k - 1$ and for graphs of maximum degree $k + 1$. We determine exact values of $c_k(G)$ for certain classes of trees.

We show that every $(k + 1)$ -regular graph has a minimum k -conversion set that

avoids certain structures in its induced subgraph. These results lead to new proofs of several known results on colourings and forest partitions of $(k + 1)$ -regular graphs and graphs of maximum degree $k + 1$.

Contents

Supervisory Committee	ii
Abstract	iii
Table of Contents	v
List of Tables	vii
List of Figures	viii
1 Introduction	1
1.1 Outline	4
2 Background and preliminaries	6
2.1 k -immune sets	9
2.2 The k -conversion number	10
2.2.1 Exact values (and near-exact values)	11
2.2.2 Bounds	18
2.3 Decycling sets, induced forests and k -conversion in $(k+1)$ -regular graphs	24
2.3.1 Lower bounds on the decycling number	26
2.3.2 Upper bounds on the decycling number	27
2.3.3 The decycling number of hypercubes	31
2.3.4 Cubic graphs that meet the lower bound	33
2.3.5 Interpolation	35
2.4 Complexity and algorithmic results	36
3 The k-conversion number of regular graphs	38
3.1 The k -conversion number of k -regular graphs	38
3.2 The k -conversion number of $(k+1)$ -regular graphs	40

3.2.1	k -conversion sets of size k in $(k + 1)$ -regular graphs	40
3.2.2	A lower bound on $c_k(G)$ for $(k + 1)$ -regular graphs	43
3.3	A lower bound on $c_k(G)$ for $(k + r)$ -regular graphs	45
4	The 2-conversion number of cubic graphs	48
4.1	Generalized fullerenes	48
4.2	Snarks and would-be snarks	55
4.3	3-edge connected cubic graphs	64
5	A lower bound on $c_k(G)$ for graphs of maximum degree $k + 1$	73
5.1	Definitions and preparation	73
5.2	The lower bound	83
5.2.1	Graphs for which the lower bound is sharp	85
5.2.2	Graphs of maximum degree $k + 1$ with large k -conversion number	86
6	k-conversion in trees	88
6.1	Caterpillars	89
6.2	Spiders and double spiders	91
7	Subgraph-avoiding minimum k-conversion sets in $(k + 1)$-regular graphs	94
7.1	Independent minimum 2-conversion sets in cubic graphs	96
7.2	Possible generalizations of Theorem 7.2	102
7.3	Avoiding K_k in minimum k -conversion sets of $(k + 1)$ -regular graphs .	104
7.4	Acyclic minimum 3-conversion sets in 4-regular graphs	109
7.5	Avoiding $(k - 1)$ -regular subgraphs in k -conversion sets of $(k + 1)$ -regular graphs	115
7.6	Consequences of Theorems 7.2, 7.13, 7.19 and 7.24	122
8	Conclusion and open problems	131
	Bibliography	135

List of Tables

4.1	Combinations of snark properties that permit equality/inequality in the lower bound on $c_2(G)$	60
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List of Figures

2.1	A 2-conversion process.	7
2.2	A minimum 4-conversion set (shown in black) in a 7×5 toroidal grid.	15
2.3	The Cartesian product of two graphs.	18
2.4	A 2-conversion set of the Cartesian product shown in Figure 2.3 as constructed in the proof of Theorem 2.18.	20
2.5	The tensor product of two graphs.	20
2.6	A 2-conversion set of the tensor product shown in Figure 2.5 as constructed in the proof of Theorem 2.20.	22
2.7	A graph achieving equality in the bounds of Proposition 2.27.	28
2.8	The graphs G_1 and G_2 of Theorem 2.29.	29
3.1	A 5-regular graphs with a 5-conversion set of size 5.	39
3.2	A graph that achieves equality in the bound of Proposition 3.6.	43
4.1	Fullerenes.	49
4.2	A 4 by 8 cylindrical grid.	50
4.3	An alternative picture of a generalized fullerene.	52
4.4	A minimum 2-conversion set in the Petersen graph.	56
4.5	The flower snark J_5	57
4.6	Building blocks for graphs that exceed the bound.	61
4.7	The triangle-replaced graph of the Petersen graph.	63
4.8	An example of the construction of a cubic graph $G \circ A^-$	66
5.1	A graph from which it is impossible to obtain a simple Δ -regular graph by adding a forest.	75
5.2	Illustration of the construction of a Δ -regular graph from a graph of maximum degree Δ	82
5.3	The construction of the Δ -regular graph G' from the proof of Theorem 5.10.	84

5.4	A subcubic graph for which $c_2(G)$ exceeds the lower bound.	86
6.1	Type 2 minimal 2-immune sets and $(k, k + 1)$ -paths in a caterpillar.	90
6.2	A spider and a double spider.	91
7.1	Illustration of seed shuffling for $k = 2$	95
7.2	The stopping condition for the algorithm from the proof of Theorem 7.2.	101
7.3	A graph in \mathcal{G}_3	103
7.4	The sequence of K_k 's in the proof of Theorem 7.13.	109
7.5	The sequence of cycles in the proof of Theorem 7.19 (Case 2).	114
7.6	The outcome of the algorithm in the proof of Theorem 7.24.	123

Chapter 1

Introduction

Network diffusion processes have been studied by social scientists since the mid-twentieth century to understand how ideas and information proliferate through communities, and how new technologies and behaviours become widely adopted. These studies set out to quantify, and often model, the extent to which people's behaviour and decisions are influenced by their peers [59, 87]. For example, an early network diffusion study aiming to explain the rapid acceptance of hybrid corn seed by Iowa farmers in the 1930s found that the most influential factor leading to farmers' acceptance of the new seed was their neighbours' attitudes towards it [96]. Other early studies analyzed the adoption of new drugs and medical technologies by physicians, examining the peer-to-peer transmission of attitudes towards the new treatments [31, 112].

The theory of network diffusion has obvious applications in marketing [24, 49, 59, 73], especially with the rise of viral marketing strategies, and in the context of public health campaigns. A company wishing to market its product efficiently may do so by identifying influential people in the network and, for example, giving them the product for free. A natural problem, then, is to find a small set of people whose adoption of the product will lead to the desired level of market saturation [26] (this is often called the *target set selection problem* [5]).

Other applications of network diffusion include epidemiology and disease control [4, 38], game theory [78, 115], and the design of computer networks and power grids [13, 38, 117]. Everett Rogers was the first to synthesize a large number of dif-

fusion studies from many disciplines into a common theory on the adoption of new innovations. His book *Diffusion of Innovations* [95], originally published in 1962 and now in its fifth edition, provides a comprehensive reference on the topic.

Unsurprisingly, network diffusion processes are highly dependent on the structure of the network in question, which is naturally modelled by a graph. To model the diffusion process, each vertex is in one of two states (say black or white) at time t and its state at time $t + 1$ is determined, according to some conversion rule, by the states of its neighbours at time t . Therefore the colours of the vertices are updated in parallel at each (discrete) time step; this is called a *graph conversion process*. The conversion rule, which is expressed as a threshold function, is chosen to reflect the extent to which agents (ie. the people or components represented by the vertices) are influenced by those around them.

The following example illustrates the use of a graph conversion process, with a specific conversion rule, to model the spread of faults through a computer network.

Example 1.1. Consider a distributed computing network in which all the processors store an identical bit of data. To maintain data consistency in the face of spontaneous faults (like rounding errors), the processors periodically poll their neighbours and adopt, for the bit in question, the value that is stored by the majority of the adjacent processors [38, 117]. This process is commonly modelled as described above, using a *majority conversion rule*: vertex v changes colour at time $t + 1$ if the majority of its neighbours have the opposite colour at time t . Therefore the threshold function is $f(v) = \frac{\deg(v)}{2}$. This model may be used to answer questions such as the following.

- What is the minimum number of corrupted processors that can lead all processors to eventually adopt the wrong value?
- What is the largest number m such that any set of m corrupted processors will still not lead to the corruption of the majority of the processors?
- Which network structures best impede the propagation of incorrect values?

In Example 1.1, as in general, the conversion rule involves a threshold function $f : V(G) \rightarrow \mathbb{Z}^+$, which dictates how many neighbours of v must be in the opposite state in order for v to change state. The conversion process in Example 1.1 is *reversible*,

in that vertices can change from white to black or from black to white. In other applications—for example, modelling the spread of information or the adoption of a new product—an *irreversible* conversion process provides a better model. In an irreversible conversion process, the conversion rule stipulates that vertices may change from white to black (say), but not from black to white. Graph conversion processes, with their various conversion rules, are related to several domination-type concepts, including alliances [63], coalitions [21], monopolies [60], local majorities [84] and k -dominating sets [42]. They can also be viewed as examples of symmetric neural networks, a special type of cellular automaton [37].

Two of the most commonly studied conversion processes are *majority conversion processes*, such as the one in Example 1.1, and *k -threshold conversion processes* (often abbreviated as *k -conversion processes*), in which the conversion threshold, $f(v)$, is k for all $v \in V(G)$. Both majority and k -threshold conversion processes can be reversible or irreversible; there are several versions of majority conversion processes, depending on whether open or closed neighbourhoods are considered and how ties are settled.

A central problem in the study of conversion processes is that of determining which “seed sets” S of initially black vertices result in the whole graph becoming black eventually for a given graph G and a given conversion rule. Such sets are called (ir)reversible *dynamic monopolies*¹ (or *dynamos*) in the case of an (ir)reversible majority conversion process² and (ir)reversible *k -threshold conversion sets* (or *k -conversion sets*) in the case of an (ir)reversible k -conversion process.

Of course, we are particularly interested in dynamos and k -conversion sets of minimum size. In this dissertation, we concentrate on minimum irreversible k -conversion sets. The size of such a set, for a graph G , is called the (*irreversible*) *k -conversion number* of G , and denoted by $c_k(G)$ (some authors use different notation). Our broad goals are

- to establish bounds, or improve existing bounds, on $c_k(G)$ for various classes of

¹A dynamic monopoly can be viewed as a generalization of a *monopoly*, which is a set S of vertices such that for each $v \in V(G)$, the majority of the neighbours of v are in S [60, 76, 77, 84]. In other words, a monopoly is a dynamic monopoly that converts the entire graph to black in one time step.

²As we have noted, there are many versions of majority conversion processes (both reversible and irreversible), depending on whether open or closed neighbourhoods are used and how ties are broken. A dynamic monopoly may arise from any of these specific conversion rules; therefore, when discussing dynamic monopolies, we must specify the details of the conversion rule.

graphs, and determine which graphs achieve equality in these bounds,

- to obtain formulas for the exact value of $c_k(G)$ where possible, and
- to determine structural properties of minimum k -conversion sets.

In Section 1.1 we provide an outline of the dissertation, with an overview of the purpose and the main results of each chapter. As noted in the outline, Chapter 2 contains a brief survey of previous results on irreversible k -conversion processes. Interested readers can find more information about reversible k -conversion processes in [26, 37, 36] and about majority conversion processes of all stripes in [4, 6, 19, 18, 26, 27, 28, 43, 46, 61, 62, 83]. General threshold conversion processes are studied in [104] and [117].

1.1 Outline

In Chapter 2 we present some preliminary results and give a brief survey of previous work on irreversible k -conversion processes and related concepts, with special attention paid to the ideas that we build upon in the dissertation.

Chapters 3 to 6 focus on the size of a minimum k -conversion set (that is, the k -conversion number) for various classes of graphs. Chapter 7 contains results on the structure of minimum k -conversion sets in $(k + 1)$ -regular graphs.

In Chapter 3 we study the k -conversion number of regular graphs of degree ranging from k to $2k - 1$. Beginning with a trivial bound on $c_k(G)$, we characterize the k -regular graphs that meet this bound and constrain the $(k + 1)$ -regular graphs that meet it. We then give a sharp lower bound on $c_k(G)$ for regular graphs of degree between $k + 1$ and $2k - 1$, improving an earlier bound.

The goal of Chapter 4 is to better understand which graphs meet the sharp lower bound on $c_k(G)$ for $(k + 1)$ -regular graphs (presented in Chapter 3) in the special case of 2-conversion in cubic graphs (that is, when $k = 2$). We present an infinite family of fullerenes and fullerene-like graphs that meet the bound. We then discuss the 2-conversion number of snarks and determine which combinations of the defining

properties of snarks admit graphs that meet the bound, and which combinations admit graphs that exceed the bound, and by how much.

In Chapter 5 we use the lower bound on $c_k(G)$ for $(k+1)$ -regular graphs to derive a lower bound on $c_k(G)$ for graphs of maximum degree $k+1$. For all $k \geq 2$ we define infinite families of graphs that meet and exceed the bound, respectively.

In Chapter 6 we determine exact values of $c_k(G)$ for certain classes of simple trees, namely caterpillars, spiders and double spiders.

Chapter 7 focuses on structures that can be avoided in minimum k -conversion sets of $(k+1)$ -regular graphs. In particular, we prove that any minimum k -conversion set of a $(k+1)$ -regular graph can be modified (over a series of steps) to produce a minimum k -conversion set that does not induce a certain subgraph H . Our results of this type range from a specific k and specific H (for example, for $k=2$ we can always avoid edges in a minimum 2-conversion set) to general k and a much more general definition of H (any $(k-1)$ -regular graph). At the end of the chapter we present consequences of our subgraph-avoidance results, including an upper bound on the k -conversion number of $(k+1)$ -regular graphs, and new proofs of several known results on *forest partitions*¹ of $(k+1)$ -regular graphs and graphs of maximum degree $k+1$.

Finally, in Chapter 8 we conclude the dissertation with a review of our results and collection of open problems arising from our work.

¹A *forest partition* of G is a partition of $V(G)$ such that each set of the partition induces a forest.

Chapter 2

Background and preliminaries

In this chapter we provide an overview of previous results on irreversible k -conversion processes and relevant related concepts that arise in the dissertation. We begin with a formal definition.

Definition 2.1. *An irreversible k -conversion process (sometimes called an irreversible k -threshold conversion process) on a graph G is a sequence of subsets S_0, S_1, \dots of $V(G)$ such that for $t = 1, 2, \dots$,*

$$S_t = S_{t-1} \cup \{v \in V - S_{t-1} : |N(v) \cap S_{t-1}| \geq k\}.$$

The set S_0 is called the seed set for the process, and if $S_t = V(G)$ for some finite t we call the seed set S_0 an irreversible k -conversion set of G .

It is common to think of the vertices in S_t as having the label 1 or being coloured black, and the vertices in $V - S_t$ as having the label 0 or being coloured white. Throughout this dissertation we refer to vertices in S_t as “converted” and vertices in $V - S_t$ as “unconverted”, and if a vertex v is in $V - S_{t-1}$ and in S_t , we say v “converts at time t ”.

Figure 2.1 illustrates an irreversible 2-conversion process on a graph G with a seed set of size 3. The conversion process stops at $t = 2$, after all but two vertices have converted. Clearly, all irreversible conversion processes on finite graphs eventually terminate; that is, there is a time after which no new vertices convert (this time is

called the *transient length* of the process). This contrasts sharply with reversible threshold processes, which may never reach such a fixed point. On a finite graph, these processes must clearly develop periodic behaviour. Remarkably, the period is either 1 or 2, even for a generalized threshold function (in which each vertex v has its own threshold $f(v)$ for changing colour) [50].

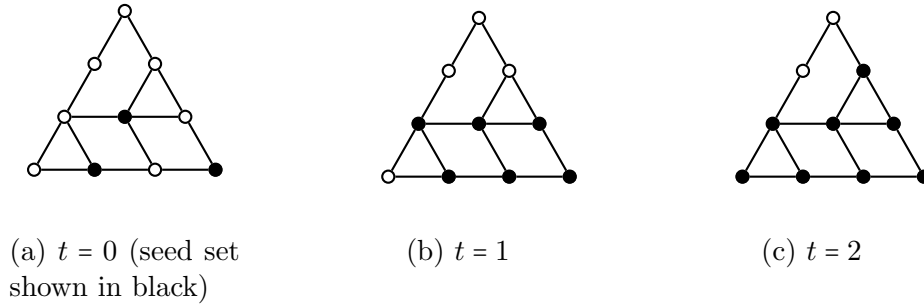


Figure 2.1: A 2-conversion process at $t = 0, 1$ and 2 , with converted vertices at each time step shown in black.

Irreversible and reversible k -conversion processes¹ were introduced in 2000 by Dreyer in his doctoral dissertation [37] (he called them *k-threshold networks*) as analogues of irreversible and reversible majority conversion processes. Majority conversion processes had previously been studied by several authors [17, 44, 45, 70, 79, 83]; they arose as a generalization of several domination-type concepts, notably monopolies [20].

In the study of majority conversion processes, the concept analogous to a (reversible/irreversible) k -conversion set is called a (reversible/irreversible) *dynamic monopoly* or *dynamo*; that is, a set of initially black vertices such that the whole graph eventually becomes black under a (reversible/irreversible) majority conversion process. Early research on majority conversion processes focused on determining or bounding the minimum size of a dynamo for certain classes of graphs; Dreyer answered analogous questions for reversible and irreversible k -conversion sets [37]. He also introduced the decision problems k -CONVERSION SET and IRR k -CONVERSION SET, which consist of determining whether a graph G has a reversible/irreversible (respectively) k -conversion set of a given size, and studied their complexity. The analogous problem for dynamic monopolies was studied later [75].

¹In a reversible k -conversion process, a vertex converts from white to black or from black to white at time t if at least k of its neighbours have the opposite colour at time $t - 1$.

Dreyer’s results on irreversible k -conversion processes, initially presented alongside his results on reversible k -conversion processes, were later published in [38]. Other authors have since resumed the study of reversible k -conversion processes; readers interested in this variant of the conversion process are referred to Dreyer’s dissertation [37] and to works by Centeno *et al.* [26] and Dourado *et al.* [36]. In the rest of this chapter, and in the rest of the dissertation, we limit our discussion to irreversible k -conversion processes and other concepts that we use in later chapters to obtain results about irreversible k -conversion. Since we will no longer need to distinguish between reversible and irreversible k -conversion, we will omit the word “irreversible” when discussing irreversible k -conversion processes and sets.

The goal for the rest of this chapter is to portray the current research landscape on k -conversion processes, which provides the setting for our new results. We present a selection of known results, chosen to convey the main ideas and central problems in the field, with a focus on results of a similar nature to our own. We include some proofs from other authors, particularly if they illustrate a common technique or general flavour of arguments about k -conversion processes.

We begin by formalizing, in Section 2.1, an idea used in many proofs about k -conversion sets, namely the idea of characterizing structures that cannot appear in the complement of a k -conversion set of a graph G . We introduce some definitions that allow us to present several proofs by different authors using common terminology. We also prove some preliminary results on the “forbidden” structures that we use in our own proofs throughout the rest of the dissertation. In Section 2.2 we present a brief survey of previous results on k -conversion sets, particularly work on determining their minimum size, from the results of Dreyer as presented in [38] to recent results by several other authors. Since our work in Chapters 3, 4 and 7 focuses on k -conversion in $(k + 1)$ -regular graphs, we discuss in Section 2.3 some well-studied structures that are equivalent and complementary, respectively, to k -conversion sets for these graphs, namely *decycling sets* and *induced forests*. We conclude our survey of previous work on k -conversion processes in Section 2.4 with a discussion of complexity results.

2.1 k -immune sets

It is clear that any k -conversion set of G must contain all vertices of degree less than k . This is our first and simplest example of a structure that cannot appear in the complement of a k -conversion set. We generalize this idea with the following definition.

Definition 2.2. *A nonempty set U of vertices of G is k -immune if, for all $v \in U$, $|N(v) - U| < k$. We say U is barely k -immune if for all $v \in U$, $|N(v) - U| = k - 1$. Finally, we say that a subgraph G' of G is (barely) k -immune if $V(G')$ is a (barely) k -immune subset of $V(G)$.*

Example 2.3. Consider the path P_n . We have already noted that the leaves u form 2-immune sets $U = \{u\}$, since u has only one neighbour, and therefore it has fewer than 2 neighbours outside $\{u\}$. In addition, any sub-path P' containing at least two vertices is also a 2-immune set of P_n , since each internal vertex of P' has no neighbours outside P' and each endpoint of P' has at most one neighbour outside P' .

The union of any two k -immune sets is k -immune. In particular, any set of vertices in a graph contains a largest k -immune set. This allows us to characterize *unconverted* vertices in terms of k -immune subgraphs (Proposition 2.4), and, in turn, to characterize k -conversion sets (Corollary 2.5).

Proposition 2.4. *Suppose G undergoes irreversible k -conversion with seed set S . If S is not a k -conversion set of G then the set of vertices which are unconverted at time $t = \infty$ is the largest k -immune subset of $V - S$.*

Proof. Let X be the set of vertices which are unconverted at time $t = \infty$ and let U be the largest k -immune subset of $V - S$. Clearly $U \subseteq X$ since no vertex in U ever converts. On the other hand, at time $t = \infty$, every vertex in $V - X$ is converted, so X is k -immune. \square

Corollary 2.5. *A set S of vertices of G is a k -conversion set of G if and only if $V - S$ does not contain a k -immune subset.*

In light of Corollary 2.5, we see that a set S of vertices of G is a k -conversion set if and only if it has a non-empty intersection with every k -immune set of G . In

particular, S must have a non-empty intersection with every minimal k -immune set of G (where minimality is with respect to containment). Therefore, identifying the minimal k -immune sets of G gives us information about the k -conversion sets, and is especially useful when we wish to argue that a particular k -conversion set has the smallest possible size.

Example 2.3, continued. The minimal 2-immune sets of P_n are the sets $\{u\}$ where u is a leaf of P_n , and $\{v, w\}$ where v and w are adjacent internal vertices. Therefore S is a 2-conversion set of P_n if and only if S contains both leaves and $V - S$ is independent.

The conclusion of Example 2.3 will allow us to determine the minimum number of vertices in a 2-immune set of P_n (see Proposition 2.7). Similarly, we will often argue about minimal k -immune sets of other graphs to make conclusions about the size and structure of their k -conversion sets. For the results we present in Section 2.2, where we summarize previous results on k -conversion sets, we may rephrase other authors' arguments to use the language of k -immune sets.

The following proposition makes a connection between barely k -immune sets and minimal k -immune sets.

Proposition 2.6. *If U is a barely k -immune set in a graph G and $G[U]$ is connected, then U is a minimal k -immune set of G .*

Proof. If $|U| = 1$ then the result is trivial, so assume $|U| \geq 2$. Since $G[U]$ is connected, removing a proper subset X of vertices from U leaves some vertex $y \in N(X) \cap U$ with k or more neighbours in $V(G) - U$, so $U - X$ is not k -immune. \square

2.2 The k -conversion number

As we have noted, much of the previous research on k -conversion processes has focused on determining or bounding the smallest possible size of a k -conversion set for a graph G . We call this number the *k -conversion number* of G , and denote it by $c_k(G)$. A k -conversion set of G of size $c_k(G)$ is called a *minimum k -conversion set* of G . In this section we present previous results on the k -conversion number for various infinite

families of graphs. In Section 2.2.1 we present the exact values that are known, and in Section 2.2.2 we present some bounds.

2.2.1 Exact values (and near-exact values)

We begin our survey with the k -conversion numbers of paths and cycles. As we have noted, any single vertex of degree less than k is k -immune, so it must be contained in every k -conversion set of G . In particular, for graphs G of maximum degree less than k , the only k -conversion set of G is $V(G)$. As well, the 1-conversion number of any graph is equal to the number of components (we will assume that all graphs are connected unless otherwise stated). Therefore the problem of determining $c_k(G)$ is only interesting when $2 \leq k \leq \Delta(G)$. When G is a path or a cycle, this means only $c_2(G)$ is interesting. Dreyer and Roberts [38] determine exact values of $c_2(G)$ for these graphs.

Proposition 2.7. [38] *For the path and cycle on n vertices, $c_2(P_n) = \lceil \frac{n+1}{2} \rceil$ and $c_2(C_n) = \lceil \frac{n}{2} \rceil$.*

Proof. We have shown in Example 2.3 that S is a 2-conversion set of P_n if and only if S contains the leaves of P_n and $V - S$ is independent. Therefore

$$c_2(P_n) = n - \alpha(P_{n-2}) = n - \left\lfloor \frac{n-2}{2} \right\rfloor = \left\lceil \frac{n+1}{2} \right\rceil,$$

as required. For C_n the 2-immune sets consist of paths of length at least 1, so the minimal 2-immune sets are the sets $\{u, v\}$ of adjacent vertices. Therefore S is a 2-conversion set of C_n if and only if $V - S$ is independent. Therefore

$$c_2(C_n) = n - \alpha(C_n) = n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil,$$

as required. □

Dreyer and Roberts also determine $c_k(G)$ for complete multipartite graphs. Adams *et al.* later give a simpler statement and a shorter proof of the same result; we state their version of the result below.

Theorem 2.8. [5, 38] *Let $G = K_{p_1, p_2, \dots, p_m}$ be a complete multipartite graph with $p_1 + p_2 + \dots + p_m = n$ and $p_1 \geq p_2 \geq \dots \geq p_m$. Let X be the set of all vertices with degree less than k . Then,*

$$c_k(G) = \begin{cases} \max\{|X|, k\}, & n > k, \\ n, & n \leq k. \end{cases}$$

Proof. If $n \geq k$ then $c_k(G) = n$, so assume $n > k$. For each $i = 1, 2, \dots, m$, let V_i be the partite set of G with p_i vertices. Note that a vertex belongs to X if and only if its entire partite set belongs to X . Therefore X is the union of some partite sets in G . If $k \leq |X|$, then X is a k -conversion set since X is a union of entire partite sets and therefore every vertex of $V - X$ is adjacent to every vertex in X . That is, with X as the seed set, $V - X$ converts at time $t = 1$. Therefore $c_k(G) \leq |X|$, but also every k -conversion set of G contains X so in fact $c_k(G) = |X|$.

Now suppose $n > k > |X|$. To show that $c_k(G) = k$ under these conditions, it suffices to exhibit a k -conversion set S of size k , since the inequality $c_k(G) \geq k$ is obvious. Let S consist of X together with $k - |X|$ additional vertices chosen such that $S - X$ contains a union of entire partite sets along with a (possibly empty) proper subset of some partite set V_j . Since X is also a union of entire partite sets, $S - V_j$ is the union of entire partite sets. In a k -conversion process with seed set S , the vertices in $V - S - V_j$ convert at time $t = 1$, since they are adjacent to the k vertices of S . To see that the remaining vertices in V_j (that is, the vertices of V_j that are not seed vertices) convert at $t = 2$, note that they have degree at least k , since they are not in X , and all of their neighbours (that is, all of the vertices outside V_j) have converted by $t = 1$. Therefore S is a k -conversion set of size k , as required. \square

Centeno *et al.* [26] determine the k -conversion number of $G \vee H$, the *join* of G and H , defined as the graph obtained from G and H by adding all edges uv where u is a vertex of G and v is a vertex of H . The analysis of k -conversion processes in $G \vee H$ is similar to that in a complete multipartite graph¹. The result of Centeno *et al.* is given in Proposition 2.9.

Proposition 2.9. [26] *Let G and H be graphs, and let K_1, K_2, \dots, K_m be the components of H .*

¹Perhaps this is not surprising since the complete multipartite graph K_{p_1, p_2, \dots, p_m} can be obtained by a sequence of joins. Specifically, let $G_1 = \overline{K_{p_1}}$, the empty graph on p_1 vertices, and for $i = 2, \dots, m$ let $G_i = G_{i-1} \vee \overline{K_i}$. Then $G_m = K_{p_1, p_2, \dots, p_m}$.

(a) If $|V(G)|, |V(H)| \geq k$ then $c_k(G \vee H) = k$, and

(b) if $|V(G)| = n < k$ and $|V(H)| \geq k$ then

$$c_k(G) = \max\{k, c_{k-n}(K_1) + c_{k-n}(K_2) + \cdots + c_{k-n}(K_m)\}.$$

Proof. For (a), it is clear that a set of k vertices from G is a k -conversion set of $G \vee H$. To prove (b), let S_i be a $(k-n)$ -conversion set of K_i for each $i = 1, 2, \dots, m$, and let $S = S_1 \cup S_2 \cup \cdots \cup S_m$. If $|S| \geq k$, then S converts all vertices of G and, subsequently, all vertices of H . Therefore S is a k -conversion set of $G \vee H$ of order $\max\{k, c_{k-n}(K_1) + c_{k-n}(K_2) + \cdots + c_{k-n}(K_m)\}$, so $c_k(G \vee H) \leq \max\{k, c_{k-n}(K_1) + c_{k-n}(K_2) + \cdots + c_{k-n}(K_m)\}$. Conversely, for any k -conversion set S of $G \vee H$, clearly $|S| \geq k$ and for $i = 1, 2, \dots, m$, $S \cap V(K_i)$ is a $(k_i - n)$ -conversion set of K_i . Therefore $c_k(G \vee H) \geq \max\{k, c_{k-n}(K_1) + c_{k-n}(K_2) + \cdots + c_{k-n}(K_m)\}$, and the result follows. \square

Several authors have studied conversion processes in $m \times n$ grids, cylindrical grids and toroidal grids. The $m \times n$ *standard grid*, which we denote by $G_{m,n}$, is the Cartesian product¹ of P_m and P_n . The $m \times n$ *toroidal grid*, which we denote by $T_{m,n}$, is the Cartesian product of C_m and C_n , and the $m \times n$ *cylindrical grid*, which we denote by $C_{m,n}$, is the Cartesian product of P_m and C_n . Exact values of $c_k(T_{m,n})$ are known for all k . For the standard grid, exact values of $c_2(G_{m,n})$ and $c_4(G_{m,n})$ are known for all m, n and exact values of $c_3(G_{3,n})$ are known. For $m \geq 4$, Dreyer [37] has derived upper and lower bounds on $c_3(G_{m,n})$. For any k , $c_k(C_{m,n})$ is bounded between the k -conversion numbers of the corresponding standard and toroidal grids. We present the known results on $c_k(G)$ for the various types of square grids below, beginning with toroidal grids.

Since toroidal grids are 4-regular, an irreversible 2-conversion process is equivalent to a *weak* irreversible majority process, where a vertex v changes from white to black at time t if *at least half* of its neighbours are black at time $t-1$, and an irreversible 3-conversion process is equivalent to a *strong* irreversible majority process, where v changes from white to black at time t if *more than* half of its neighbours are black at time $t-1$. An irreversible 3-conversion set in a 4-regular graph (more generally, a k -conversion set in a $(k+1)$ -regular graph) is also equivalent to a *decycling set*, which

¹We discuss Cartesian products and provide a definition on page 18.

is defined for any graph G as a set $S \subseteq V(G)$ such that $G[V - S]$ is acyclic. We will discuss decycling sets and demonstrate this equivalence in Section 2.3. Exact values of $c_2(T_{m,n})$ and $c_3(T_{m,n})$ have been derived from work on the minimum size of simple and strong dynamic monopolies and decycling sets in these graphs.

Flocchini *et al.* [46] determined the minimum size of a weak irreversible dynamic monopoly in $T_{m,n}$ by obtaining equal upper and lower bounds on this number. Their result translates to a formula for $c_2(T_{m,n})$, which we state in Proposition 2.10 (a).

For $c_3(T_{m,n})$, Luccio [71] determined a lower bound of $\lceil \frac{mn+2}{3} \rceil$ on the size of a decycling set in $T_{m,n}$. Pike and Zou [86] later showed that Luccio's bound is actually the exact decycling number (hence the exact 3-conversion number) of $T_{m,n}$ for $m, n \neq 4$, improving a previous upper bound by Flocchini *et al.* [44, 45]. (Kynčl *et al.* [65] later gave a simpler construction for a 3-conversion set of size $\lceil \frac{mn+2}{3} \rceil$.) Pike and Zou also gave exact values of c_3 for $T_{4,n}$ (and, by symmetry, $T_{m,4}$). We present all of the formulas for $c_3(T_{m,n})$ in Proposition 2.10 (b).

Finally, exact values of $c_4(T_{m,n})$, which we present in Proposition 2.10 (c), were obtained for all $m, n \geq 3$ by Dreyer and Roberts [38].

Proposition 2.10 (Exact k -conversion numbers for toroidal grids). [38, 46, 86]

$$(a) \quad c_2(T_{m,n}) = \lceil \frac{n+m}{2} \rceil - 1 .$$

$$(b) \quad c_3(T_{m,n}) = \begin{cases} \lceil \frac{mn+2}{3} \rceil & \text{if } m, n \neq 4 \\ \lceil \frac{3n}{2} \rceil & \text{if } m = 4 \\ \lceil \frac{3m}{2} \rceil & \text{if } n = 4. \end{cases}$$

$$(c) \quad c_4(T_{m,n}) = \begin{cases} \max\{n \lceil \frac{m}{2} \rceil, m \lceil \frac{n}{2} \rceil\} & \text{if } m \text{ or } n \text{ is odd} \\ \frac{mn}{2} & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

The proof of Proposition 2.10 (c) uses the characterization of the k -conversion sets of a k -regular graph G as complements of independent sets of G , which we present in Proposition 2.11. (In fact we have already used this result for the case $k = 2$ in the proof of Proposition 2.7 for cycles.) Dreyer and Roberts use this characterization to obtain a lower bound of $c_4(T_{m,n}) \geq \max\{n \lceil \frac{m}{2} \rceil, m \lceil \frac{n}{2} \rceil\}$, noting that, since $V - S$ is independent, the seed vertices in each row (column) must form a vertex cover for

that row (column)¹. They then construct a 4-conversion set (ie. a vertex cover) of that size. We give an example of such a 4-conversion set in a 5×7 toroidal grid in Figure 2.2.

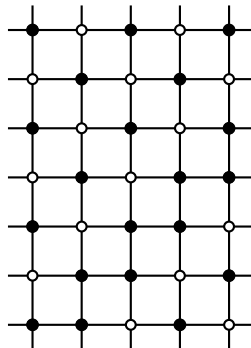


Figure 2.2: A minimum 4-conversion set (shown in black) in a 7×5 toroidal grid.

We now prove Dreyer and Robert’s characterization of k -conversion sets in k -regular graphs (Proposition 2.11), and subsequently generalize it to graphs of maximum degree k (Proposition 2.12). These characterizations lead to the determination of $c_k(G)$ for these graphs in terms of the independence number of a subgraph H of G (in the case where G is k -regular, $H = G$).

Proposition 2.11. [38] *Let G be a k -regular graph. Then S is a k -conversion set of G if and only if $V - S$ is independent. Consequently, $c_k(G) = n - \alpha(G)$.*

Proof. We claim that the minimal k -immune sets in G are the pairs of adjacent vertices. Let x and y be adjacent vertices in G and let $U = \{x, y\}$. Since x and y are adjacent to one vertex in U , each of them is adjacent to $k - 1$ vertices in $V - U$. Therefore U is barely k -immune, so by Proposition 2.6, it is a minimal k -immune set. Thus if S is a k -conversion set of G , $V - S$ is independent.

To prove the converse, note that if $V - S$ is independent and $v \in V - S$ then v has k neighbours in S , so v converts at $t = 1$.

Since every k -conversion set is the complement of an independent set of G , a minimum k -conversion set is the complement of a maximum independent set, which gives $c_k(G) = n - \alpha(G)$. \square

¹Recall that the complement of any independent set of a graph G is a *vertex cover*— that is, a set X of vertices such that each edge is incident with a vertex of X .

Proposition 2.12. [38] *Let G be a graph of order n and maximum degree k , and let H be the subgraph of G induced by its vertices of degree k . A set $S \subseteq V(G)$ is a k -conversion set of G if and only if $V(G) - S$ is an independent subset of $V(H)$. Consequently, $c_k(G) = n - \alpha(H)$.*

Proof. We claim that the minimal k -immune sets of G are

- (a) the sets $\{x\}$, where $\deg(x) < k$, and
- (b) the pairs $\{x, y\}$, where x and y are adjacent vertices of degree k .

For sets of Type (a), this claim is obvious. To prove the claim for Type (b), let $U = \{x, y\}$ be a set of this type. Then x and y each has exactly $k - 1$ neighbours outside U , so U is a barely k -immune set. Therefore, by Proposition 2.6, it is a minimal k -immune set. On the other hand, no other set of vertices is a minimal k -immune set since any such set contains a subset of Type (a) or a subset of Type (b).

It follows that $S \subseteq V(G)$ is a k -conversion set of G if and only if $V(G) - S$ is an independent set of H . Therefore $c_k(G) = n - \alpha(H)$. \square

We now turn our attention to standard grid graphs $G_{m,n}$. Dreyer [37] derives exact values of $c_2(G_{m,n})$ by modifying the result for toroidal grids (he finds that $c_2(G_{m,n}) = c_2(T_{m,n}) + 1$). He also determines exact values of $c_3(G_{3,n})$ and bounds for $c_3(G_{m,n})$ when $m \geq 4$. We summarize these results in Proposition 2.13. Dreyer also provides exact values of $c_4(G_{m,n})$. We present that result in Proposition 2.14; the proof illustrates an application of Proposition 2.12.

Proposition 2.13. [38, 44, 45, 71] *For the standard grid graph $G_{m,n}$*

- (a) $c_2(G_{m,n}) = \lceil \frac{m+n}{2} \rceil$,
- (b) $c_3(G_{3,n}) = \lfloor \frac{3n+2}{2} \rfloor$, and
- (c) $\frac{(m-1)(n-1)+1}{3} \leq c_3(G_{m,n}) \leq \frac{(m-1)(n-1)}{3} + \frac{3m+2n-3}{4} + 5$ for $4 \leq m \leq n$.

For the proof of Proposition 2.14, we label the vertices of $G_{m,n}$ as $v_{i,j}$ in the obvious way.

Proposition 2.14. [37] *For the standard grid graph $G_{m,n}$,*

$$c_4(G_{m,n}) = 2m + 2n - 4 + \left\lfloor \frac{(m-2)(n-2)}{2} \right\rfloor.$$

Proof. Let H be the graph induced by the interior vertices of $G = G_{m,n}$ and let S be a minimum 4-conversion set of G . By Proposition 2.12, S contains all border vertices of the grid, and $V - S$ is a maximum independent set of H . It can be shown that the larger of the two sets $\{v_{i,j} : i+j \text{ is odd}\} \cap H$ and $\{v_{i,j} : i+j \text{ is even}\} \cap H$ is a maximum independent set of H of order $\left\lfloor \frac{(m-2)(n-2)}{2} \right\rfloor$. Therefore S contains $\left\lfloor \frac{(m-2)(n-2)}{2} \right\rfloor$ interior vertices of $G_{m,n}$ together with the $2m + 2n - 4$ border vertices. \square

The three types of $m \times n$ grids we have mentioned—standard, cylindrical and toroidal—are closely related in that $E(G_{m,n}) \subset E(C_{m,n}) \subset E(T_{m,n})$, when m and n are large enough that all three grids are defined (ie. $m, n \geq 3$). This gives the following relationship between their k -conversion numbers.

Proposition 2.15. *For all k and all $m, n \geq 3$, $c_k(T_{m,n}) \leq c_k(C_{m,n}) \leq c_k(G_{m,n})$.*

Proof. Any k -conversion set of $G_{m,n}$ is a k -conversion set of $C_{m,n}$, and any k -conversion set of $C_{m,n}$ is a k -conversion set of $T_{m,n}$. \square

Proposition 2.15 allows us to bound $c_2(C_{m,n})$ to within 1 of the exact value, since $c_2(T_{m,n})$ and $c_2(G_{m,n})$ differ by one. It is easy to determine exact values of $c_2(C_{2,n})$ for any $n \geq 3$, since these graphs are cubic and thus their minimal 2-immune sets are cycles. Finally, Dreyer [37] determines exact values of $c_3(C_{3,n})$ for all $n \geq 3$. We summarize these results in Proposition 2.16.

Proposition 2.16. [37] *For the cylindrical grid $C_{m,n} = P_m \square C_n$,*

(a) $c_2(C_{2,n}) = \left\lceil \frac{n}{2} \right\rceil + 1$ for all $n \geq 3$,

(b) $\left\lceil \frac{m+n}{2} \right\rceil - 1 \leq c_2(C_{m,n}) \leq \left\lceil \frac{m+n}{2} \right\rceil$ for all $m, n \geq 3$, and

(c) $c_3(C_{3,n}) = \left\lceil \frac{3n}{2} \right\rceil$ for all $n \geq 3$.

2.2.2 Bounds

Exact values of $c_k(G)$ are not known for other well-known infinite families of graphs G^1 (although several authors have given examples of infinite families that achieve equality in various bounds on $c_k(G)$). We now shift our attention to the known bounds on $c_k(G)$. We begin with upper bounds on the k -conversion number of the Cartesian and tensor products, respectively, of two graphs (Theorems 2.18 and 2.20). We then present the known bounds for graphs with certain structural properties. Bounds on $c_k(G)$ are especially abundant for $(k+1)$ -regular graphs G , due to the connection between k -conversion sets and other structures (for example, decycling sets and maximum induced forests) in these graphs. We present these results in Section 2.3.

Definition 2.17. *The Cartesian product of two disjoint graphs G and H is the graph $G \square H$ such that $V(G \square H) = V(G) \times V(H)$ and two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and u' is adjacent to v' in H , or $u' = v'$ and u is adjacent to v in G .*

Figure 2.3 illustrates the Cartesian product of P_5 and the graph H obtained by joining a pendant vertex to one vertex of C_4 .

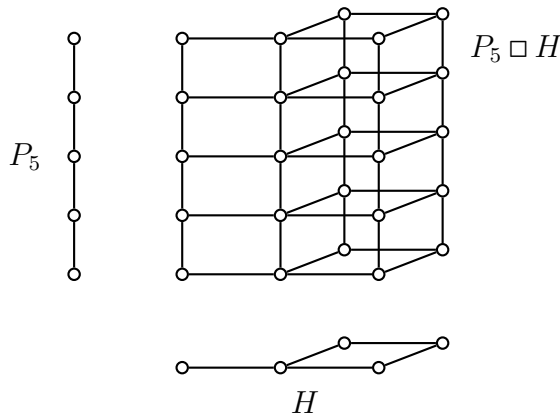


Figure 2.3: The Cartesian product of P_5 and H .

To prove their bound on $c_k(G \square H)$, the authors of [5] introduce the following notation. For a fixed vertex v in H , let G_v be the subgraph of $G \square H$ induced by the

¹While no formula exists to determine the k -conversion number of a tree, it can be computed in linear time. We discuss algorithmic results in Section 2.4.

vertices $\{(u, v) : u \in V(G)\}$; note that G_v is a copy of G . Similarly, for a fixed vertex u in G , let H_u be the subgraph of $G \square H$ induced by the vertices $\{(u, v) : v \in V(H)\}$; note that H_u is a copy of H . We now present the upper bound on $c_k(G \square H)$ due to Adams *et al.*, along with their proof.

Theorem 2.18. [5] *Let G and H be two graphs. Then*

$$c_k(G \square H) \leq c_k(G)c_k(H).$$

Proof. Let S_G and S_H be minimum k -conversion sets of G and H , respectively. Let S be the set of $c_k(G)c_k(H)$ vertices (u, v) in $G \square H$ such that $u \in S_G$ and $v \in S_H$. To verify the proposed upper bound, it suffices to show that S is a k -conversion set of $G \square H$.

Consider a k -conversion process on $G \square H$ with seed set S . For a fixed vertex u in S_G , H_u contains vertices (u, v) for all v in S_H . Since these vertices are in S and S_H is a k -conversion set of H , all vertices in H_u will eventually convert, as H_u is isomorphic to H and the conversion of H_u by S_H induces conversion of H_u by the set of vertices (u, v) for v in S_H .

To conclude the proof, it suffices to show that the union of all H_u for u in S_G is a k -conversion set of $G \square H$. Consider a k -conversion process on $G \square H$ with seed set $\cup_{u \in S_G} H_u$. For each v in H , G_v contains vertices (u, v) for all u in S_G . Since all the vertices in H_u for u in S_G are converted, and since S_G is a k -conversion set of G , all the vertices in G_v will eventually convert, as G_v is isomorphic to G and the conversion of G by S_G induces the conversion of G_v by the set of vertices (u, v) for u in S_G . Therefore S is a k -conversion set of $G \square H$, as claimed. \square

Figure 2.4 illustrates a 2-conversion set of the graph $P_5 \square H$ (defined in Figure 2.3) as constructed in the proof of Theorem 2.18.

Next we present an analogous bound for the tensor product of two graphs, which is also due to Adams *et al.* [5].

Definition 2.19. *The tensor product of two disjoint graphs G and H is the graph $G \times H$ such that $V(G \times H) = V(G) \times V(H)$ and two vertices (u, u') and (v, v') are adjacent if and only if u is adjacent to v in G and u' is adjacent to v' in H .*

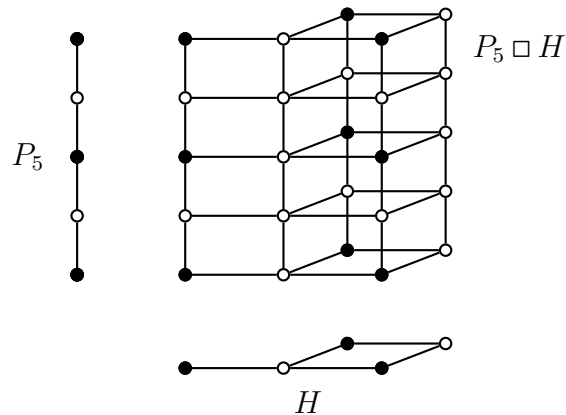


Figure 2.4: A 2-conversion set (shown in black) of $P_5 \square H$, constructed from 2-conversion sets of P_5 and H as in the proof of Theorem 2.18.

Figure 2.5 illustrates the tensor product of P_4 and the graph H defined in Figure 2.3.

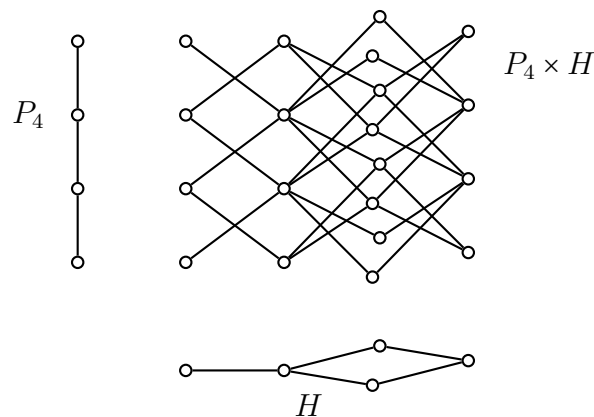


Figure 2.5: The tensor product of P_4 and H .

From the definition of tensor product it is clear that each isolate (that is, isolated vertex) of G generates $|V(H)|$ isolates in $G \times H$, and each isolate of H generates $|V(G)|$ isolates in $G \times H$. Moreover, there are no other isolates of $G \times H$. Therefore, if G and H contain i_G and i_H isolates, respectively, $G \times H$ contains $i_G|V(H)| + i_H|V(G)| - i_G i_H$ isolates, all of which must be contained in any k -conversion set of $G \times H$. To simplify the analysis of the bound, Adams *et al.* assume initially that G and H both have no isolates (or, equivalently, they ignore the isolates of G and H). Once they have bounded the k -conversion number of the isolate-free tensor product, they simply add

$i_G|V(H)| + i_H|V(G)| - i_G i_H$ to account for the isolates.

Theorem 2.20. [5] *Let G and H be two graphs without isolated vertices. Then*

$$c_k(G \times H) \leq \min\{c_k(G)|V(H)|, c_k(H)|V(G)|\}.$$

Proof. We may assume that $c_k(G)|V(H)| \leq c_k(H)|V(G)|$, since the tensor product is commutative. Let S_G be a minimum k -conversion set of G and let S be the set of $c_k(G)|V(H)|$ vertices (u, v) in $G \times H$ such that $u \in S_G$. To verify the proposed upper bound, it suffices to show that S is a k -conversion set of $G \times H$.

We first partition the vertices of $G \times H$. Suppose it takes T time steps to fully convert G from S_G ; for each $t = 0, 1, \dots, T$, let $H(t)$ be the set of vertices (u, v) such that u is a vertex that converts at time t in G (from S_G). The sets $H(0), H(1), \dots, H(T)$ partition the vertices of $G \times H$.

Consider a k -conversion process on $G \times H$ with seed set S . We will show that for each $t = 0, 1, \dots, T$, the vertices of $H(t)$, and hence all vertices of $G \times H$, will eventually convert. We proceed by induction on t .

As the base case, for every vertex in $(u, v) \in H(0)$, $u \in S_G$ so $(u, v) \in S$. Now suppose that all of the vertices in $H(0), H(1), \dots, H(t-1)$ have converted; we will show that the vertices of $H(t)$ now convert. Consider an arbitrary unconverted vertex (u, v) in $H(t)$. Note that $\deg_{G \times H}(u, v) = \deg_G(u) \deg_H(v)$. Since u converts at time t in G , at least k of its $\deg_G(u)$ neighbours in G had converted by time $t-1$. Each of these converted neighbours in G generates $\deg_H(v) > 0$ (since H has no isolated vertices) neighbours of (u, v) in $G \times H$. Specifically, these neighbours of (u, v) in $G \times H$ are in the union $H(0) \cup H(1) \cup \dots \cup H(t-1)$, which contains only converted vertices. Therefore, at least k of the $\deg_G(u) \deg_H(v)$ neighbours of (u, v) in $G \times H$ are converted, and therefore (u, v) converts at the next time step. It follows by induction that S is a k -conversion set of $G \times H$, as claimed. \square

Figure 2.6 illustrates a 2-conversion set of the graph $P_4 \square H$ (defined in Figure 2.3) as constructed in the proof of Theorem 2.20. Since $c_2(H)|V(P_4)| = (3)(4) < (3)(5) = c_2(P_4)|V(H)|$, we construct a 2-conversion set of $P_4 \times H$ by taking a 2-conversion set in every copy of H .

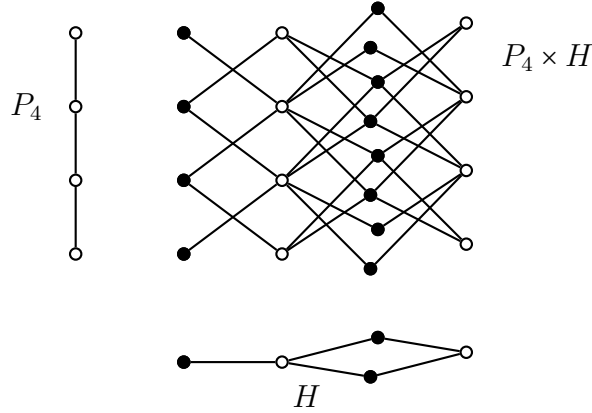


Figure 2.6: A 2-conversion set (shown in black) of $P_4 \times H$, constructed from a 2-conversion set of H as in the proof of Theorem 2.20.

The bound of Theorem 2.20 is sharp for all graphs $G \times K_2$ where G is any connected bipartite graph [5]. For any graph G , the theorem gives

$$c_k(G \times K_2) \leq \min\{2c_k(G), |V(G)|2\} = 2c_k(G),$$

since $c_k(G)$ is clearly at most $|V(G)|$. When G is connected and bipartite, Sampathkumar [98] has shown that $G \times K_2$ is isomorphic to $2G$, that is, two disjoint copies of G . Using this characterization, it is clear that $c_k(G \times K_2) = 2c_k(G)$.

The *strong product* of G and H is the graph $G \boxtimes H$ such that $V(G \boxtimes H) = V(G) \times V(H)$ and two vertices (u, u') and (v, v') are adjacent if and only if

- u is adjacent to v and $u' = v'$, or
- $u = v$ and u' is adjacent to v' , or
- u is adjacent to v and u' is adjacent to v' .

That is, the strong product of G and H has the same vertex set as the Cartesian product and the tensor product, and its edge set is the union of the edge sets of the Cartesian product and the tensor product. The k -conversion number of $G \boxtimes H$ does not appear to have been studied previously, but we make the straightforward observation that it is bounded above by both $c_k(G \square H)$ and $c_k(G \times H)$. We leave further study of $c_k(G \boxtimes H)$ as an open problem (see Chapter 8).

Many bounds on $c_k(G)$ exist for $(k+1)$ -regular graphs, where finding a k -conversion

set is equivalent to other problems that have been widely studied (specifically, finding decycling sets and maximum induced forests). We will discuss these bounds in Section 2.3. Our final bounds for this section apply to $(k+r)$ -regular graphs. The first, due to Dreyer and Roberts, holds for $0 \leq r < k$ (Proposition 2.21), and the second, due to Zaker, provides a tighter bound in the case $r = k - 1$ (Proposition 2.22).

To prove Proposition 2.21, Dreyer and Roberts introduce a slightly different conversion procedure. Instead of updating the graph synchronously at each time step (such that all unconverted vertices with at least k converted neighbours at time t convert simultaneously at time $t+1$), consider an *asynchronous* k -conversion process, in which only one vertex converts at each time step. That is, at each time step we choose one unconverted vertex that meets the conversion threshold, and we convert it. We repeat this process until no more vertices can be converted. Let S_t be the set of vertices that convert at time t under a synchronous conversion rule. Then, under an asynchronous conversion rule, we can convert all vertices of S_1 one at a time, followed by all vertices of S_2 , and so on. Therefore if S is a k -conversion set under a synchronous conversion rule, it is also a k -conversion set under an asynchronous conversion rule.

Proposition 2.21. [38] *Let G be a $(k+r)$ -regular graph, with $0 \leq r < k$. Then $c_k(G) \geq \frac{(k-r)n}{2k}$.*

Proof. Consider an asynchronous k -conversion process with seed set S , where S is a k -conversion set of G . For each time step t , let E_t be the number of edges between vertices of opposite states (converted/unconverted). Each time a vertex converts, this set loses at least k edges and gains at most r edges; that is, $E_{t+1} \leq E_t - (k-r)$. Therefore $E_{n-|S|} \leq E_0 - (n-|S|)(k-r)$. Since S is a k -conversion set, all vertices are converted at time $n-|S|$, so we have $E_{n-|S|} = 0$. As well, $E_0 \leq (k+r)|S|$. Combining these and rearranging gives $|S| \geq \frac{(k-r)n}{2k}$, as required. \square

In the case $r = k - 1$ (that is, when G is a $(2k-1)$ -regular graph), a k -conversion set is also a dynamic monopoly. Zaker [117] provides the following lower bound on the size of a dynamic monopoly in a $(2k-1)$ -regular graph, which improves the bound of Proposition 2.21 for these graphs.

Proposition 2.22. [117] *Let G be a $(2k-1)$ -regular graph of order n . Then $c_k(G) \geq \frac{n+2(k-1)}{2k}$.*

In the next section we shift our focus to k -conversion in $(k + 1)$ -regular graphs, in which k -conversion sets are equivalent to decycling sets and complementary to induced forests.

2.3 Decycling sets, induced forests and k -conversion in $(k + 1)$ -regular graphs

In Section 2.1 we introduced the idea of a k -immune set, that is, a set U of vertices such that each vertex in U has fewer than k neighbours outside U . It is clear that a k -conversion set of a graph G must intersect every k -immune set U of G , otherwise the vertices of U never convert. Therefore, we can characterize the k -conversion sets of G by characterizing the k -immune sets of G (in particular, the minimal k -immune sets of G), as we have in Propositions 2.7, 2.11 and 2.12, for example. In Proposition 2.23 we characterize the k -conversion sets of $(k + 1)$ -regular graphs by characterizing their minimal k -immune sets. This result was first proved by Dreyer and Roberts.

Proposition 2.23. [38] *Let G be a $(k + 1)$ -regular graph. A set S of vertices of G is a k -conversion set if and only if $G[V - S]$ is a forest.*

Proof. Let C be an induced cycle in a $(k + 1)$ -regular graph G . Then each vertex of C has exactly $k - 1$ neighbours in $V - C$, so $V(C)$ is a barely k -immune set. Therefore, by Proposition 2.6, $V(C)$ is a minimal k -immune set. That is, the minimal k -immune sets of G are the induced cycles of G . By Proposition 2.5, this implies that a set S of vertices is a k -conversion set of G if and only if $V - S$ does not induce any cycles. \square

In a graph G , a set S is a *decycling set*, or *feedback vertex set*, if and only if $G[V - S]$ is a forest. Therefore, as noted in [38], Proposition 2.23 states that, in a $(k + 1)$ -regular graph G , a set S is a k -conversion set if and only if it is a decycling set. Feedback vertex sets in undirected graphs were first studied by Harary in 1975 [52]. In directed graphs, feedback vertex sets and *feedback arc sets*, which are defined similarly, were motivated by applications in logic networks and circuit theory and have been studied since 1958 [33, 66, 100, 116]. In a logic network, one wishes to eliminate feedback paths. When the logic network is modelled by a digraph, feedback paths correspond

to directed cycles [99]. (More modern applications are given in [54]. Among them is deadlock prevention and recovery. In the wait-for (di)graph of an operating system, a directed cycle corresponds to a set of processes that are waiting for each other, causing a deadlock. At least one of the processes must be aborted in order to resolve the deadlock. A minimum feedback vertex set in this graph corresponds to a minimum set of processes that must be aborted to resolve all deadlocks.)

Since the complement of a decycling set induces a forest, the problem of finding a minimum decycling set is equivalent to the problem of finding a maximum induced forest, which has been studied since the 1980s (see, for example, [10, 22, 40, 94, 105, 119]). In $(k + 1)$ -regular graphs, both of these problems are equivalent to finding a minimum k -conversion set, and we will use these connections repeatedly in later sections to prove results about k -conversion in $(k + 1)$ -regular graphs. In particular, many existing bounds on the k -conversion number of a $(k + 1)$ -regular graph were originally proved as bounds on the size of a maximum induced forest or minimum size of a decycling set (feedback vertex set). Conversely, all of our results on k -conversion sets in $(k + 1)$ -regular graphs, which are found throughout Section 3.2 and in Chapters 4, 5 and 7, can be rephrased as results about decycling sets in these graphs.

In our brief survey of decycling sets and maximum induced forests, we will limit the scope of our discussion to results that apply to regular graphs, with a focus on bounds and structural results related to the problems we study later in the dissertation, namely bounds on their associated graph parameters. We discuss computational aspects of decycling sets and maximum induced forests in Section 2.4 as part of our summary of computational results on k -conversion sets.

As with research on k -conversion sets, research on decycling sets/feedback vertex sets has focused on determining the minimum size of such a set for a given graph G ; this number is the *decycling number* of G or the *feedback vertex number*. We will denote this parameter by $\phi(G)$, following Punnim [89, 90, 91, 92, 93], although some authors use $\nabla(G)$ [16, 85] or other notation. The order of a maximum induced forest of G is called the *forest number* of G and most commonly denoted by $a(G)$. As we have mentioned, S is a minimum decycling set of G if and only if $V - S$ induces a maximum forest of G . Therefore we have the trivial identity, $\phi(G) + a(G) = |V(G)|$. In our discussion of bounds on these parameters, we will state all results as bounds

on $\phi(G)$, since $\phi(G) = c_k(G)$ when G is $(k + 1)$ -regular.

In Section 2.3.1 we summarize previous work on lower bounds for $\phi(G)$ (corresponding to upper bounds on $a(G)$) that apply to regular graphs G . In Section 2.3.2 we present known upper bounds on $\phi(G)$, which focus on cubic graphs and planar graphs. More restrictive bounds have been proved for hypercubes and we present these in Section 2.3.3. We focus on the (sharp) lower bound for cubic graphs in Section 2.3.4 and present results pertaining to the graphs that meet this bound. (We study this question further in Chapter 4.) Finally, in Section 2.3.5 we shift our attention to the question of which values are realizable as $\phi(G)$ for some G in a given class \mathcal{C} (for example, planar graphs or regular graphs of order n). Specifically, are all integers between $\min\{\phi(G) : G \in \mathcal{C}\}$ and $\max\{\phi(G) : G \in \mathcal{C}\}$ realizable by a graph in \mathcal{C} ?

2.3.1 Lower bounds on the decycling number

In 1974, in one of the first publications on maximum induced forests, Jaeger [57] proved that for a cubic graph G of order n , $\phi(G) \geq \frac{n+2}{4}$. (This bound is sharp, and we discuss results on the graphs that meet it in Section 2.3.4.) Staton [105] extended Jaeger's result to r -regular graphs in 1984, with the bound of Proposition 2.24 (a). We give a proof of this result in Section 3.2.2, along with a condition for equality in the bound (Proposition 3.7). Punnim [90] later improved Staton's bound to $\phi(G) \geq r - 1$ for small values of n (Proposition 2.24 (b)). This result is trivial to prove using conversion sets.

Proposition 2.24. [90, 105] *Let G be an r -regular graph of order n . Then,*

$$(a) \phi(G) \geq \frac{(r-2)n+2}{2(r-1)}, \text{ and}$$

$$(b) \text{ if } r + 1 \leq n \leq 2r - 1 \text{ then } \phi(G) \geq r - 1 \geq \frac{(r-2)n+2}{2(r-1)}.$$

Proof. For the proof of (a), see Proposition 3.7. We prove (b) using conversion sets. A decycling set in an r -regular graph G is an $(r - 1)$ -conversion set of G , and therefore has size at least $r - 1$. The result follows since $r - 1 \geq \frac{(r-2)n+2}{2(r-1)}$ when $n < 2r$. \square

Beineke and Vandell [16] proved that for any connected graph G , with n vertices, m

edges and maximum degree Δ ,

$$\phi(G) \geq \frac{m - n + 1}{\Delta - 1}. \quad (2.1)$$

The bound of Proposition 2.24 (a) can be derived from (2.1) by setting $\Delta = r$ and $m = \frac{rn}{2}$.

We now turn our attention to upper bounds on the decycling number.

2.3.2 Upper bounds on the decycling number

Early upper bounds on the decycling number, and many since then, are restricted to specific classes of graphs, namely cubic graphs or planar graphs. The interest in planar graphs was largely motivated by a 1979 conjecture of Albertson and Berman [9] on the forest number of planar graphs. We state their conjecture below in terms of the decycling number.

Conjecture 2.25. [9] *Every planar graph of order n has a decycling set of size at most $\frac{n}{2}$.*

The conjecture remains open—the best known bound for general planar graphs is $\phi(G) \leq \frac{3n}{5}$ [23]—but it has been proved for some subclasses. Dross *et al.* [39] proved that every triangle-free planar graph of order n has a decycling set of size at most $\frac{5n-7}{11}$, improving an earlier bound by Salavatipour [97] confirming Conjecture 2.25 for triangle-free planar graphs. For outerplanar graphs G , $\phi(G) \leq \frac{n}{3}$ [55]. A history of the progress towards proving the conjecture can be found in [3, 39], and additional bounds for planar graphs of larger girth can be found in [40].

For bipartite planar graphs, Akiyama and Watanabe [7] and Albertson and Haas [8] independently propose the following conjecture, which also remains open.

Conjecture 2.26. [7, 8] *Every bipartite planar graph of order n has a decycling set of size at most $\frac{3n}{8}$.*

For non-planar graphs, the first upper bounds on the decycling number applied to cubic graphs. For cubic graphs without further restrictions, Speckenmeyer [102]

provides the following two bounds, both of which are obtained by constructing a decycling set from a maximal *non-separating independent set*, that is, an independent set X such that $G - X$ is connected.

Proposition 2.27. [102] *Let G be a cubic graph of order n and let $K(G)$ be the maximum number of disjoint cycles in G . Then,*

$$(a) \phi(G) \leq \left\lfloor \frac{n}{4} + \frac{K(G)+1}{2} \right\rfloor, \text{ and}$$

$$(b) \phi(G) \leq \left\lfloor \frac{3n}{8} + 1 \right\rfloor.$$

Moreover, both bounds are optimal.

To prove (a), they show that $F \cup X$ is a decycling set, where X is a non-separating independent set and F is a decycling set of $G - X$, and then prove a bound on $|X|$ in terms of $K(G)$. Then (b) is derived by modifying the procedure for finding a maximal non-separating independent set.

To prove the optimality of both bounds, Speckenmeyer constructs the graph H_ℓ , shown in Figure 2.7, from a caterpillar C whose ℓ internal vertices all have degree 3 by replacing each internal vertex of C with a triangle and each leaf with the graph H obtained by subdividing an edge of K_4 . This technique provides many important examples in later studies of decycling sets in cubic graphs. We also use it in Section 4.2 to construct examples of graphs with large k -conversion number, and generalize it to graphs of maximum degree $k + 1$ in Section 5.2.2.

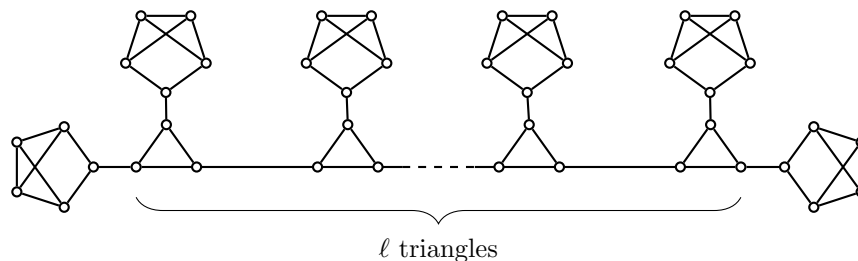


Figure 2.7: A graph H_ℓ achieving equality in both bounds of Proposition 2.27.

It is clear that $\phi(H_\ell) \geq 3\ell + 4$, since any decycling set requires two vertices from every copy of H and one from every triangle. On the other hand, $K(H_\ell) = 2\ell + 2$, since there are no two disjoint cycles in H , and H_ℓ has order $n_\ell = 8\ell + 10$. Therefore

the bound (a) gives

$$\phi(H_\ell) \leq \left\lfloor \frac{8\ell + 10}{4} + \frac{2\ell + 3}{2} \right\rfloor = 3\ell + 4.$$

The bound of (b) gives

$$\phi(H_\ell) \leq \left\lfloor \frac{3(8\ell + 10)}{8} \right\rfloor + 1 = 3\ell + \left\lfloor \frac{19}{4} \right\rfloor = 3\ell + 4.$$

Speckenmeyer notes that for the bound (a) there is also an infinite family of 2-connected graphs that meet the bound.

Writing $\phi(H_\ell)$ in terms of the order n_ℓ of H_ℓ gives $\phi(H_\ell) = \frac{3n_\ell}{8} + \frac{1}{4}$. Bondy *et al.* [22] later show that in fact all cubic graphs have decycling number at most $\frac{3n}{8} + \frac{1}{4}$. They also characterize the graphs that achieve equality in this bound by generalizing Speckenmeyer's construction of H_ℓ , and give an improved upper bound for cubic graphs of girth at least 4. These results follow from a more general result, due to Liu and Zhao [69], which bounds $\phi(G)$ in terms of the order and girth of G (improving an earlier bound of the same nature by Speckenmeyer [103]). We state this result in Theorem 2.28 and then we summarize the other results that can be obtained from it. First, we describe the generalization of Speckenmeyer's construction of the graphs H_ℓ .

A *cubic tree* is a tree whose internal vertices all have degree 3. We define \mathcal{G} to be the class of connected cubic graphs obtained from a cubic tree by replacing each degree 3 vertex with a K_3 and each leaf with the graph H obtained by subdividing an edge of K_4 . (In order to produce a cubic graph, the edge joining H to the rest of the graph must be incident with the degree 2 vertex of H .) We define G_1 and G_2 (not in \mathcal{G}) to be the two graphs shown in Figure 2.8.

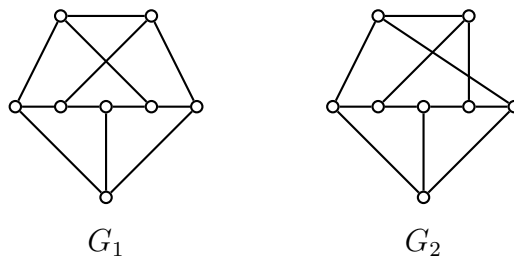


Figure 2.8: The graphs G_1 and G_2 of Theorem 2.29.

Theorem 2.28. [69] *Let G be a cubic graph of order n and girth g . Then*

$$\phi(G) \leq \frac{g}{4(g-1)}n + \frac{g-3}{2g-2},$$

except when $G \in \{K_4, G_1, G_2\} \cup \mathcal{G}$. If $g = 3$ and $G \in \mathcal{G}$ then $\phi(G) = \frac{3n}{8} + \frac{1}{4}$.

In [103], Speckenmeyer conjectured that all 2-connected triangle-free cubic graphs satisfy $\phi(G) \leq \frac{n}{3}$. Zheng and Lu [119] proved that this conjecture is true even without assuming 2-connectivity, with two exceptions, namely G_1 and G_2 . This result also follows from Theorem 2.28. We summarize the consequences of Theorem 2.28 in Theorem 2.29 (a) and (b). Dross *et al.* [40] improved the bound of Theorem 2.29 (a) in the case where G is 2-connected. We state their bound, which is sharp, in Theorem 2.29 (c).

Theorem 2.29. *Let G be a cubic graph of order $n > 4$.*

- (a) [22, 69] *If $G \in \mathcal{G}$ then $\phi(G) = \frac{3n+2}{8}$, and otherwise $\phi(G) \leq \frac{3n}{8}$.*
- (b) [119] *If G is triangle-free and $G \notin \{G_1, G_2\}$ then $\phi(G) \leq \frac{n}{3}$.*
- (c) [40] *If G is 2-connected then $\phi(G) \leq \frac{n+2}{3}$. Moreover, this bound is sharp.*

We define the *triangle-replaced* graph of a cubic graph G to be the graph obtained by replacing each vertex of G with a triangle. The bound of Theorem 2.29 (c) is attained by the triangle-replaced graph of any 2-connected cubic graph [40].

Few results on the maximum value of $\phi(G)$ exist for general regular graphs of degree greater than 3, but Punnim [91] gives the following bound for r -regular graphs, $r \geq 4$.

Theorem 2.30. [91] *Let G be a connected r -regular graph of order $n \geq 2r + 2$. Then $\phi(G) \leq \frac{r-2}{r}n$ for all $r \geq 4$.*

We prove this result for even r , and show that it is sharp in that case, in Section 7.6.

More restrictive bounds (both lower and upper) have been determined for the decycling number of hypercubes. We summarize these results in the next section.

2.3.3 The decycling number of hypercubes

For $n \geq 0$, the n -dimensional hypercube (also called the n -cube) is the graph Q_n whose vertices are the 2^n binary words of length n and two vertices are adjacent if and only if they differ in exactly one position. Therefore Q_n is n -regular.

Beineke and Vandell determine the exact values of $\phi(Q_n)$ for $n \leq 8$, and bounds for $n \geq 9$ [16]. The bounds were later improved by Bau *et al.* [15], Focardi *et al.*, who provide the best known upper bound [47], and Pike, who provides the best known lower bound [85]. We state these bounds in Theorem 2.31, but first we introduce some concepts from coding theory, which are used to prove the bounds of Theorem 2.31.

In coding theory, the number of positions in which two binary words u and v differ is called the *Hamming distance* $d_H(u, v)$ between u and v , and the number of 1 entries in u is called the *Hamming weight* (or simply the *weight*) of u and denoted by $w(u)$. Therefore $d(u, v) = w(u + v)$, where $u + v$ is the usual bitwise sum modulo 2, and the distance $d(u, v)$ between vertices u and v in Q_n is equal to the Hamming distance $d_H(u, v)$ between the two words. A *binary code of length n* is a subset A of $\{0, 1\}^n$ and the *minimum distance* of A is $\min\{d_H(u, v) : u, v \in A, u \neq v\}$. A binary code of length n and minimum distance d is often referred to as an (n, d) code.

The greater its minimum distance, the greater a code's ability to correct errors. At the same time, it is desirable to have as many code words as possible without increasing the length or decreasing the minimum distance. Therefore, a central problem in the study of error correcting codes is to determine the maximum cardinality $A(n, d)$ of an (n, d) code (see, for example, [68] and [72]).

Theorem 2.31. [47, 85] *For $n \geq 7$,*

$$2^{n-1} + \frac{n+1-2^{n-1}}{n-1} \leq \phi(Q_n) \leq 2^{n-1} - \frac{2^{n-1}}{2(n-1)}.$$

The bounds of Theorem 2.31 place the decycling number at slightly less than half of $|V(Q_n)| = 2^n$. They are both proved by observing that a decycling set of Q_n can be constructed using a binary code of minimum distance at least 4, and then applying bounds on the cardinality of such a code. We outline the proofs in the rest of this section.

The proof of Theorem 2.31 relies on the following lemma, guaranteeing that a maximum-cardinality $(n, 4)$ code can be constructed using only vertices (words) of even weight. We denote by V_0 and V_1 , respectively, the sets of even and odd weight vertices of Q_n .

Lemma 2.32. [72, 85] *There exists an $(n, 4)$ code A of cardinality $A(n, 4)$ such that $A \subseteq V_0$.*

By Lemma 2.32 we can partition $V(Q_n)$ into sets V_1 , A and $S = V_0 - A$, where A is an $(n, 4)$ -code of cardinality $A(n, 4)$. In the next lemma we show that S is a decycling set of Q_n .

Lemma 2.33. [47] *Let $A \subseteq V_0$ be an $(n, 4)$ code of cardinality $A(n, 4)$. Then $Q_n[V_1 \cup A]$ is acyclic.*

Proof. Let $G = Q_n[V_1 \cup A]$. We first argue that each component of G contains at most one vertex of A . To see this, consider a vertex $v \in A$ and let X denote the set of vertices at distance exactly 2 from v in Q_n . Then all of the vertices in X have even weight, and none of them is in A , by the minimum distance of A . Therefore none of them is in G . Thus X forms a vertex cut separating the component of G containing v from the rest of A .

It follows that G is acyclic, since any cycle in Q_n must contain at least two vertices of even weight. \square

Since the decycling set $S = V_1 - A$ of Lemma 2.33 has size $2^{n-1} - A(n, 4)$, we obtain the upper bound

$$\phi(Q_n) \leq 2^{n-1} - A(n, 4). \quad (2.2)$$

Focardi *et al.* [47] and Pike [85], respectively, prove the following bounds on $A(n, 4)$:

$$\frac{2^{n-2}}{n-1} \leq A(n, 4) \leq \frac{2^{n-1}}{n}. \quad (2.3)$$

The lower bound of (2.3) immediately gives the upper bound of Theorem 2.31 when applied to the the inequality (2.2).

To prove the lower bound, Pike first characterizes the cases where Q_n achieves equality in the bound (2.2).

Theorem 2.34. [85] *The hypercube Q_n satisfies $\phi(Q_n) = 2^{n-1} - A(n, 4)$ if and only if it has a minimum decycling set that is also an independent set.*

For any r -regular graph $G = (V, E)$ with decycling set S , Pike [85] proves the identity

$$|V| - |S| - \kappa(G - S) = |E| - r|S| + \epsilon(S), \quad (2.4)$$

where $\epsilon(S)$ is the number of edges in $G[S]$ and $\kappa(G - S)$ is the number of components of $G - S$, by counting in two ways the edges in $G - S$. When $G = Q_n$, (2.4) gives

$$|S| = 2^{n-1} + \frac{\kappa(G - S) + \epsilon(S) - 2^{n-1}}{n - 1},$$

after rearranging.

Pike then proves that $\kappa(G - S) + \epsilon(S) \geq n + 1$ to obtain the lower bound of Theorem 2.31. In the case where Q_n has an independent minimum decycling set, this result follows from Theorem 2.34 and the upper bound (2.3) on $A(n, 4)$. In the case where every minimum decycling set contains at least one edge, Pike proves that in fact it contains at least n edges.

2.3.4 Cubic graphs that meet the lower bound

In Proposition 2.24 (a), we saw that the decycling number of an r -regular graph G of order n is bounded below by $\frac{(r-2)n+2}{2(r-1)}$, or, more specifically, by the least integer that is not less than this number. When G is cubic, this gives

$$\phi(G) \geq \left\lceil \frac{n+2}{4} \right\rceil. \quad (2.5)$$

In [14], Bau and Beineke ask which cubic graphs achieve equality in the bound (2.5). Bondy *et al.* note in [22] that a result of Payan and Sakarovitch [82] provides a sufficient condition for equality in the bound; we state this result in Proposition 2.35.

A graph G is *cyclically k -edge connected* (*cyclically k -vertex connected*) if at least k

edges (vertices) must be removed in order to disconnect G into two subgraphs, each containing a cycle [109]. McCuaig [74] proves that a cubic graph $G \notin \{K_{3,3}, K_4\}$ is cyclically 4-edge connected if and only if it is cyclically 4-vertex connected, so we often simply say *cyclically 4-connected*.

Proposition 2.35. [82] *Let G be a cyclically 4-connected cubic graph of order n . Then $\phi(G) = \lceil \frac{n+2}{4} \rceil$.*

On the other hand, Punnim gives the following necessary condition on cubic graphs of order $n \equiv 2 \pmod{4}$ that meet the bound (2.5).

Proposition 2.36. [89, Lemma 2.1] *If G is a cubic graph of order $n \equiv 2 \pmod{4}$ and S is a decycling set of G of order $\lceil \frac{n+2}{4} \rceil$, then S is independent and $G[V - S]$ is a tree.*

We return to the problem of finding infinite families of graphs that meet the bound (2.5) in Chapter 4. In Section 7.1 (in particular, Theorem 7.2) we prove that every cubic graph with $n > 4$ has a minimum decycling set that is independent. Therefore the first conclusion of Proposition 2.36 is a special case of Theorem 7.2.

For the rest of this section we discuss results by Punnim asserting the class of cubic graphs of order n that attain equality in the bound $\phi(G) \geq \lceil \frac{n+2}{4} \rceil$ is closed under a graph operation called *switching*.

Let ab and cd be two edges in a graph G such that a, b, c and d are all distinct and ac and bd are not edges in G . We define the *switching* $\sigma(a, b; c, d)$ to be the graph operation consisting of deleting the edges ab and cd from G and adding the edges ac and bd . Following [89], we denote the resulting graph by $G^{\sigma(a, b; c, d)}$. It is clear that $G^{\sigma(a, b; c, d)}$ has the same degree sequence as G , so, for a given degree sequence $\mathbf{d} = (d_1, d_2, \dots, d_n)$, the class of graphs having degree sequence \mathbf{d} is closed under the switching operation. In addition, Havel [53] and Hakimi [51] have proved that there is only one orbit.

Theorem 2.37. [51, 53] *If G_1 and G_2 are two graphs with the same degree sequence, then G_2 can be obtained from G_1 by a finite sequence of switchings.*

For a given degree sequence \mathbf{d} , we define the graph $\mathcal{G}(\mathbf{d})$ whose vertices correspond to the graphs with degree sequence \mathbf{d} and where the vertices corresponding to G_1

and G_2 are adjacent if $G_1 = G_2^\sigma$ for some switching σ . It follows from Theorem 2.37 that $\mathcal{G}(\mathbf{d})$ is connected.

For cubic graphs of order n (that is, for the degree sequence $\mathbf{d} = 3^n$), Punnim [89] considers the subgraph of $\mathcal{G}(\mathbf{d})$ induced by the vertices corresponding to graphs G satisfying $\phi(G) = \lceil \frac{n+2}{4} \rceil$, which we denote by $\mathcal{G}(3^n, \lceil \frac{n+2}{4} \rceil)$. Punnim proves that for any two cubic graphs G_1 and G_2 that both achieve equality in the bound (2.5), G_2 can be obtained from G_1 by a finite sequence of switching such that every graph obtained in the sequence also achieves equality in the bound (2.5). We express this result with the following theorem.

Theorem 2.38. [89] *The subgraph $\mathcal{G}(3^n, \lceil \frac{n+2}{4} \rceil)$ of $\mathcal{G}(3^n)$ is connected.*

Punnim shows in [93] that the subgraph of $\mathcal{G}(3^n, \lceil \frac{n+2}{4} \rceil)$ induced by the vertices corresponding to planar graphs is also connected.

2.3.5 Interpolation

We have presented lower bounds (Proposition 2.24 (a)) and upper bounds (Proposition 2.29 (c) and Theorem 2.30) on $\phi(G)$ for r -regular graphs G , $r \geq 3$, as well as several narrower bounds for regular graphs with additional properties, such as girth restrictions. In [89], Punnim considers the problem of determining which integers are realizable as the decycling number of some graph G with a given degree sequence $\mathbf{d} = (d_1, \dots, d_n)$. Punnim considers this problem for all degree sequences; here, we are only interested in regular graphs, so we will state his results for $\mathbf{d} = r^n$.

Following [88], we call a graph parameter ρ an *interpolation parameter* with respect to a class of graphs \mathcal{J} if there exist integers $a \leq b$ such that $\{c \in \mathbb{Z} : a \leq c \leq b\} = \{\rho(G) : G \in \mathcal{J}\}$. That is, ρ is an interpolation parameter with respect to \mathcal{J} if every integer between $\min\{\rho(G) : G \in \mathcal{J}\}$ and $\max\{\rho(G) : G \in \mathcal{J}\}$ is realized as $\rho(G)$ for some $G \in \mathcal{J}$.

Punnim shows that ϕ is an interpolation parameter with respect to several classes of graphs. We provide a summary in Theorem 2.39.

Theorem 2.39. [92] *The decycling number ϕ is an interpolation parameter with respect to*

- (a) the class of all graphs with degree sequence \mathbf{d} , for any graphic sequence \mathbf{d} ,
- (b) the class of all connected graphs with degree sequence \mathbf{d} , for any graphic sequence \mathbf{d} , and
- (c) the class of planar cubic graphs.

Moreover, if \mathcal{J} is a class of graphs with degree sequence \mathbf{d} and the subgraph of $\mathcal{G}(\mathbf{d})$ induced by \mathcal{J} is connected, then ϕ is an interpolation parameter with respect to \mathcal{J} .

In the next section we return our attention to general graphs, where a k -conversion set is not necessarily a decycling set.

2.4 Complexity and algorithmic results

In this section we summarize known complexity results related to k -conversion processes. The IRREVERSIBLE k -CONVERSION SET problem is as follows:

IRREVERSIBLE k -CONVERSION SET (IRR k -CS)

Instance: A graph G and a positive integer s .

Question: Does G have an irreversible k -conversion set S where $|S| \leq s$? (Equivalently, is $c_k(G) \leq s$?)

This problem was first considered by Dreyer and Roberts [37, 38]. They proved that the problem is NP-complete for $k \geq 3$, by bounding the number of time steps required for a k -conversion process to terminate, called the *transient length* of the process, to show that the problem is in NP, and then proving a reduction from the INDEPENDENT SET problem.

The IRR k -CS problem is clearly polynomial for $k = 1$, since the conversion number is equal to the number of components, so Dreyer's and Roberts' result leaves the IRR k -CS problem open only for $k = 2$. Chen [30] published the first proof that it is NP-complete even for graphs of maximum degree 4, and this was independently proved by Centeno *et al.* [26] and Kyncl *et al.* [65]. On the other hand, the problem has long been recognized as polynomial for cubic graphs [111] (due to the equivalence between decycling sets and 2-conversion sets for these graphs) and for graphs of maximum

degree at most 2. Finally, Takoaka and Ueno [108] and Kyncl *et al.* [65] independently show that IRR 2-CS is also polynomial for non-regular graphs of maximum degree 3.

Centeno *et al.* [26] consider the optimization problem of finding a minimum k -conversion set of a graph, which is known as the MINIMUM IRREVERSIBLE k -CONVERSION SET problem and defined below.

MINIMUM IRREVERSIBLE k -CONVERSION SET (MIN IRR k -CS)

Instance: A connected graph G .¹

Task: Determine an irreversible k -conversion set S of G with $|S| = c_k(G)$.

Recall that a *block* of a graph G is a maximal subgraph of G without a cut vertex. Centeno *et al.* obtain a linear-time algorithm to compute $c_k(G)$ for trees and a quadratic-time algorithm to compute $c_2(G)$ for chordal graphs. Centeno *et al.* also present a counterexample demonstrating an error in an earlier algorithm by Dreyer and Roberts [38] designed to compute the $c_k(G)$ for trees. However, the algorithm of Centeno *et al.* uses ideas from that of Dreyer and Roberts, notably the technique of reducing the instance graph G into its blocks² (such that the cut vertex v of G appears in every block with which it is incident in G). The conversion number of the blocks is then determined and used to compute the conversion number of G , which Centeno *et al.* note can be done efficiently for graphs with simple blocks.

¹Centeno *et al.* consider the problem for an arbitrary threshold function $f : V(G) \rightarrow \mathbb{Z}$, so their formulation of the problem also includes this function as input. For our purposes every vertex has threshold k .

²Dreyer and Roberts reduce the instance graph slightly differently, since they are dealing only with trees (and therefore all blocks are isomorphic to K_2). They note that any vertex of degree at most $k - 1$ is necessarily in the conversion set, and therefore these vertices play the role of the cut vertices in the block-reduction algorithm of Centeno *et al.*

Chapter 3

The k -conversion number of regular graphs

In this chapter we study the k -conversion number of k -regular, $(k + 1)$ -regular and $(k + r)$ -regular graphs G . We present lower and upper bounds on $c_k(G)$ for these graphs, and study which graphs achieve equality in these bounds. (In Chapter 4 we further develop our study of k -conversion sets in $(k + 1)$ -regular graphs with a closer look at 2-conversion sets in cubic graphs, that is, the case $k = 2$.)

3.1 The k -conversion number of k -regular graphs

We begin the section with the straightforward observation that, in order for any conversion to occur in a k -conversion process, the seed set must contain at least k vertices. Therefore k is a trivial lower bound on $c_k(G)$ for any graph G with at least k vertices. More specifically, if G is a graph of order n and maximum degree Δ , then $c_k(G) = n$ if $\Delta < k$ and otherwise $c_k(G) \geq k$. Leaving aside the case where $c_k(G) = n$, we focus on graphs with maximum degree at least k and ask which graphs meet the bound $c_k(G) = k$.

Graphs that meet this bound are easy to find, and exist for any order $k + r$, where $r \geq 1$. (Take, for example, the complete bipartite graph $K_{k,r}$.) Imposing structural constraints on G naturally makes the bound harder to achieve. In this

section, we give a complete characterization of the k -regular graphs that meet this bound (Proposition 3.1). In Section 3.2 we will enlarge our investigation of the bound to include $(k + 1)$ -regular graphs. For graphs G and H , we define the *join* of G and H , denoted by $G \vee H$, to be the graph obtained by joining each vertex of G to each vertex of H .

Proposition 3.1. *A k -regular graph G has a k -conversion set of size k (that is, $c_k(G) = k$) if and only if $G = H \vee \overline{K_{k-t}}$, where H is a t -regular graph of order k , and $0 \leq t < k$.*

Proof. Let $G = H \vee \overline{K_{k-t}}$, where H and t are as above. Each vertex of $\overline{K_{k-t}}$ has k neighbours in H , so $V(H)$ is a k -conversion set of size k . Since vertices of $\overline{K_{k-t}}$ have no other neighbours, and each vertex of H has t neighbours in H and $k - t$ neighbours in $\overline{K_{k-t}}$, G is k -regular.

For the converse, let G be a k -regular graph with a k -conversion set S of order k . By Proposition 2.11, $V - S$ is independent. Since G is k -regular, $G[S]$ is t -regular for some $0 \leq t < k$ and $|V - S| = k - t$. The result follows with $H = G[S]$. \square

For $k = 5$ and $t = 2$, an example of a graph $G = H \vee \overline{K_{k-t}}$, as in Proposition 3.1, is shown in Figure 3.1. The black vertices induce the 2-regular graph H , and the white vertices form the independent set $\overline{K_3}$.

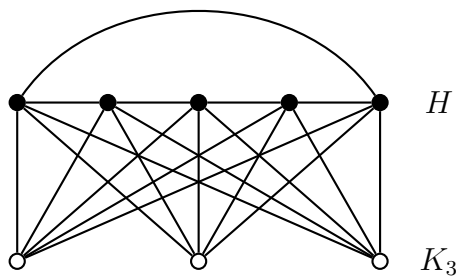


Figure 3.1: A 5-regular graph with a 5-conversion set of size 5, shown in black.

3.2 The k -conversion number of $(k + 1)$ -regular graphs

In this section we present upper and lower bounds on the k -conversion number of a $(k + 1)$ -regular graph, and we determine some properties of the graphs that meet these bounds. We begin with the trivial lower bound $c_k(G) \geq k$ (discussed previously for k -regular graphs), this time applied to $(k + 1)$ -regular graphs. Recall from Section 2.3 that for a $(k + 1)$ -regular graph G , a set S of vertices is a k -conversion set if and only if $G[V - S]$ is acyclic (Proposition 2.23). Such a set S is also known as a decycling set or a feedback vertex set. We rely heavily on this characterization of k -conversion sets in $(k + 1)$ -regular graphs throughout Section 3.2 and Chapters 4, 5 and 7.

3.2.1 k -conversion sets of size k in $(k + 1)$ -regular graphs

If $r \geq 1$ and G is a $(k + r)$ -regular graph with a k -conversion set S of size k , then every non-seed vertex has at least r neighbours outside of S . This introduces the possibility that complete conversion of the graph takes more than one time step. For $t \geq 0$, let S_t be the set of vertices that convert at time t , starting from a given seed set $S = S_0$. (It is worth noting that such a graph may still convert in one time step. For example, consider the 4-regular graph $G = \overline{K_3} \vee (K_2 + K_2)$,¹ with 3-conversion set $S = V(\overline{K_3})$.)

In Proposition 3.2 we derive a bound on the number of vertices in $\bigcup_{t \geq 2} S_t$ (that is, the number of vertices that convert at time $t = 2$ and later) for $(k + 1)$ -regular graphs with a k -conversion set of size k . This gives us a bound on the number of non-seed vertices (Corollary 3.3).

Proposition 3.2. *Let G be a $(k + 1)$ -regular graph with a k -conversion set of size k . Then*

$$\left| \bigcup_{t \geq 2} S_t \right| \leq \frac{k(k + 1) + |S_1|(1 - k) - 1}{k - 1}.$$

Proof. Let $Y = \bigcup_{t \geq 2} S_t$. We count the edges between Y and S_0 in two ways. First, since G is $(k + 1)$ -regular and each vertex of S_0 is adjacent to each vertex of S_1 , there

¹For graphs G and H , $G + H$ denotes the disjoint union of G and H , and $G \vee H$ denotes the *join* of G and H , obtained by adding all possible edges between G and H .

are at most $k(k+1-|S_1|)$ edges from S_0 to Y . On the other hand, each vertex in Y has at least k neighbours that convert before it. Therefore there are at least $|Y|k$ edges with at least one endpoint in Y . Since $G-S_0$ is a forest with $|Y|+|S_1|$ vertices, at most $|Y|+|S_1|-1$ have the other endpoint in $Y \cup S_1$. Therefore there are at least $|Y|k-|Y|-|S_1|+1$ edges from Y to S_0 . This gives $|Y|k-|Y|-|S_1|+1 \leq k(k+1-|S_1|)$, and the result follows. \square

Corollary 3.3. *Let G be a $(k+1)$ -regular graph and suppose that S_0 is a k -conversion set of size k . Then $|V(G) - S_0| < \frac{k(k+1)-1}{k-1}$.*

Proof. The left side of the bound in Proposition 3.2 equals $|V(G) - S_0| - |S_1|$. Rearranging gives the result. \square

In Proposition 3.5, we use Corollary 3.3 to get an upper bound on the order of a $(k+1)$ -regular graph having a k -conversion set of size k . We also prove by construction that the bound is sharp for each $k \geq 2$. The result of the construction for $k = 3$ is illustrated in Figure 3.2. We begin with a definition.

Definition 3.4. *Let v be a vertex such that $\deg(v) \leq \Delta$. We define the Δ -deficiency of v to be $\text{def}_\Delta(v) = \Delta - \deg(v)$.*

Proposition 3.5. *If G is a $(k+1)$ -regular graph having a k -conversion set of size k then the order of G is at most $2k+2$. Moreover, for every $k \geq 2$, there exists a $(k+1)$ -regular graph of order $2k+2$ which has a k -conversion set of size k .*

Proof. We obtain the bound for $k = 2$ by checking all examples (there are three cubic graphs having a 2-conversion set of size 2: K_4 and the two cubic graphs of order 6). For $k \geq 3$, $\frac{k(k+1)-1}{k-1} > k+3$, so the bound follows from Corollary 3.3. To prove that the bound is sharp, we construct a $(k+1)$ -regular graph of order $2k+2$ which has a k -conversion set of size k .

We begin with the graph $K_{2,k}$, where $S_0 = S$ is the set of size k (a k -conversion set) and $S_1 = \{u_1, v_1\}$ is the set of size 2 (the set of vertices that convert at time $t = 1$). For each $v \in S_0$ we now have $\text{def}_{k+1}(v) = k-1$ and for each $v \in S_1$ we have $\text{def}_{k+1}(v) = 1$. We will add vertex sets S_2, S_3, \dots such that the vertices of S_i convert at time $t = i$ from the k -conversion set S_0 . To achieve this, for each $i \geq 2$, we must

add at least k edges from S_i to $\cup_{j=0}^{i-1} S_j$. Some care is required in choosing the edges, in order to ensure that there will always be at least k distinct vertices available in $\cup_{j=0}^{i-1} S_j$.

For $i \geq 2$, if there are still at least $k - 1$ vertices in S_0 of deficiency at least 2, let $S_i = \{u_i, v_i\}$. Join u_i to u_{i-1} and to $k - 1$ vertices of S_0 , beginning with those of highest deficiency. Then join v_i to v_{i-1} and to $k - 1$ vertices of S_0 , once again beginning with those of highest deficiency. Joining u_i and v_i to u_{i-1} and v_{i-1} at each step means that the vertices of S_1, \dots, S_{i-1} have degree $k + 1$, so the only deficient vertices are the newly added ones and those in S_0 . Joining the new vertices first to the vertices of highest deficiency in S_0 guarantees that the deficiencies among the vertices of S_0 are always within 1 of each other. Therefore, the first time there fail to be at least $k - 1$ vertices in S_0 with deficiency at least 2, there are either no deficient vertices in S_0 (if k is even) or there are $k - 1$ deficient vertices in S_0 and their deficiency is 1 (if k is odd).

In the case where k is even, we add vertices u_i and v_i $\frac{k}{2}$ times before we run out of deficient vertices in S_0 . That is, the process stops when $i = \frac{k}{2} + 1$, and $|\cup_{i=2}^{\frac{k}{2}+1} S_i| = k$. Adding an edge between $u_{\frac{k}{2}+1}$ and $v_{\frac{k}{2}+1}$ yields a simple $(k + 1)$ -regular graph of order $2k + 2$ (including the k vertices of S_0 and the 2 vertices of S_1).

In the case where k is odd, we add $\frac{k-1}{2}$ pairs of vertices u_i and v_i before the deficiencies in S_0 become too small. That is, the process stops when $i = \frac{k+1}{2}$ and $|\cup_{i=2}^{\frac{k+1}{2}} S_i| = k - 1$. We complete the $(k + 1)$ -regular graph by adding one final vertex, w , and joining it to $u_{\frac{k+1}{2}}$, $v_{\frac{k+1}{2}}$ and to the $k - 1$ vertices of deficiency 1 in S_0 . The total number of vertices is now $2k + 2$, including the k vertices of S_0 and the 2 vertices of S_1 . \square

Proposition 3.6, below, provides another upper bound on the size of $\cup_{t \geq 2} S_t$ for $(k + 1)$ -regular graphs with a k -conversion set of size k . When $|S_1| \geq \frac{2k-1}{k-1}$, the bound provided by Proposition 3.2 is stronger than that of Proposition 3.6. However, the bound of Proposition 3.6 is sharp for small values of $|S_1|$, as shown by the graph in Figure 3.2.

Proposition 3.6. *Let G be a $(k + 1)$ -regular graph with a k -conversion set of size k . Then $|\cup_{t \geq 2} S_t| \leq k$.*

Proof. Let $Y = \cup_{t \geq 2} S_t$. By Corollary 2.23, $G - S_0$ is a forest F , and its leaves are the vertices in S_1 . Therefore, for every $v \in Y$, $\deg_F(v) \leq |S_1|$, and $\deg_G(v) = k + 1$, so v has at least $k + 1 - |S_1|$ neighbours in S_0 . Hence there are at least $|Y|(k + 1 - |S_1|)$ edges between Y and S_0 . On the other hand, there are at most $k(k + 1 - |S_1|)$ edges between S_0 and Y , by the argument given in the proof of Proposition 3.2. Therefore $|Y|(k + 1 - |S_1|) \leq k(k + 1 - |S_1|)$. \square

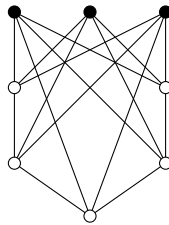


Figure 3.2: A 4-regular graph with $c_3(G) = 3 = |\cup_{t \geq 2} S_t|$, illustrating sharpness of the bound in Proposition 3.6. This graph also illustrates the construction in Proposition 3.5, with $k = 3$.

In this section and in Section 3.1, we began with a fixed seed set size (namely k , the minimum possible size for a nontrivial k -conversion set), and asked which graphs have a k -conversion set of this size. We obtained constraints on the structure and order (respectively) of k - and $(k + 1)$ -regular graphs with this property. For the rest of this chapter, beginning with Section 3.2.2, we will instead begin with a class of graphs, and ask how small a k -conversion set can be for a graph in this class.

3.2.2 A lower bound on $c_k(G)$ for $(k + 1)$ -regular graphs

As discussed in Section 3.1, k is a lower bound on the k -conversion number of any graph with order at least k . While it is possible to have arbitrarily large graphs that attain this bound, for many classes of graphs a k -conversion set of size k can only convert a limited number of vertices. Indeed, we showed in Proposition 3.5 that in the class of $(k + 1)$ -regular graphs, a k -conversion set of size k can convert at most $2k + 2$ vertices. For these graphs, as the order grows beyond the $2k + 2$ threshold, we require more than k seed vertices to convert the graph. In this case, k is no longer a good lower bound for the k -conversion number. In Proposition 2.24 we presented a lower bound, due to Staton [105], on the decycling number of an r -regular graph,

which corresponds to the $(r - 1)$ -conversion number. We now restate this result with $r = k + 1$ to obtain a bound on the k -conversion number of a $(k + 1)$ -regular graph. We present a proof which yields a condition for equality in the bound.

Proposition 3.7. *Let G be a $(k + 1)$ -regular graph of order n , $k \geq 2$. Then $c_k(G) \geq \left\lceil \frac{n(k-1)+2}{2k} \right\rceil$. Moreover, a minimum k -conversion set S of G has size $\frac{n(k-1)+2}{2k}$ if and only if S is independent and $G - S$ is a tree.*

Proof. Let S be a minimum k -conversion set of G , and let $\bar{S} = V \setminus S$. For $X \in \{S, \bar{S}\}$, let n_X and m_X denote the number of vertices and edges, respectively, in $G[X]$. Counting in two ways the number of edges between S and \bar{S} gives the identity

$$(k + 1)n_S - 2m_S = (k + 1)n_{\bar{S}} - 2m_{\bar{S}}.$$

By Corollary 2.23, $G[\bar{S}]$ is a forest; let y be its number of components. Then

$$(k + 1)n_S - 2m_S = (k + 1)n_{\bar{S}} - 2(n_{\bar{S}} - y).$$

Substituting $n_{\bar{S}} = n - n_S$, and rearranging, this gives

$$n_S = \frac{n(k - 1) + 2m_S + 2y}{2k}.$$

Therefore, $c_k(G) = n_S \geq \frac{n(k-1)+2}{2k}$, with equality if and only if S is independent and $G - S$ is a tree. In particular, $c_k(G) \geq \left\lceil \frac{n(k-1)+2}{2k} \right\rceil$. \square

The bound of Proposition 3.7 also follows from the following bound, due to Beineke and Vandell.

Proposition 3.8. [16, Corollary 1.2] *Let G be a graph with n vertices, m edges and maximum degree Δ . Then the decycling number of G is at least $\frac{m-n+1}{\Delta-1}$.*

In Chapter 4 we examine the bound of Proposition 3.7 in detail for the case $k = 2$; that is, the case of 2-conversion in cubic (i.e. 3-regular) graphs.

In the next section we prove a lower bound similar to that of Proposition 3.7 for $(k + r)$ -regular graphs.

3.3 A lower bound on $c_k(G)$ for $(k+r)$ -regular graphs

Dreyer and Roberts [38] give a lower bound of $\frac{(k-r)n}{2k}$ on $c_k(G)$ for $(k+r)$ -regular graphs of order n , for $0 \leq r < k$. In the case $r = k - 1$, where G is a $(2k - 1)$ -regular graph, Zaker [117] strengthens this bound to $c_k(G) \geq \frac{n+2(k-1)}{2k}$ (see Propositions 2.21 and 2.22). In this section we improve upon both of these previous bounds by providing, in Proposition 3.11, a new lower bound of $c_k(G) \geq \frac{(k-r)n+(r+1)r}{2k}$, which is sharp for $0 \leq r < k$.

Proposition 3.9 generalizes Proposition 2.23, characterizing the k -conversion sets S of $(k+r)$ -regular graphs in terms of a condition on $V - S$. A graph G is r -degenerate, for $r \geq 0$, if every induced subgraph of G has a vertex of degree at most r . We say that G is a *maximal r -degenerate* graph if G is r -degenerate but for every pair of non-adjacent vertices x, y in G , adding the edge xy to $E(G)$ produces a graph that is not r -degenerate. We note that a graph G is 0-degenerate if and only if it has no edges, and it is 1-degenerate if and only if it is acyclic.

Proposition 3.9. *Let G be a $(k+r)$ -regular graph, with $r \geq 0$. A set S of vertices of G is a k -conversion set if and only if $G[V - S]$ is r -degenerate.*

Proof. Suppose $V - S$ is r -degenerate, so every subgraph H of $V - S$ has a vertex of degree at most r . In other words, some vertex of H has at least k neighbours in $G - H$. Let $H_0 = V - S$ and let S_1 be the set of vertices of degree at most r in H_0 . These vertices have at least k neighbours in $G - H_0 = S$, so they convert at time $t = 1$. Let $H_1 = H_0 - S_1$ and let S_2 be the set of vertices of degree at most r in H_1 . These vertices have at least k neighbours in $V - H_1 = S \cup S_1$, so they convert at time $t = 2$. Continue this process until some $H_i = \emptyset$. At each step, the set $V - H_j$ is converted, so when $H_i = \emptyset$ the whole graph is converted. Therefore S is a k -conversion set.

On the other hand, if $V - S$ is not r -degenerate then there is some subgraph H of $V - S$ in which no vertex has k neighbours outside H . Therefore H is a k -immune set, so S is not a conversion set of G . \square

Proposition 3.11 generalizes Proposition 3.7, establishing a lower bound on $c_k(G)$ for $(k+r)$ -regular graphs G . The proof technique is the same as for Proposition 3.7,

but requires the following lemma, due to Lick and White, bounding the number of edges in an r -degenerate graph.

Lemma 3.10. [67, Proposition 3 and Corollary 1] *Let G be an r -degenerate graph with $n \geq r$ vertices and m edges. Then $m \leq rn - \binom{r+1}{2}$, with equality if and only if G is maximal r -degenerate.*

Proposition 3.11. *Let G be a $(k+r)$ -regular graph of order n , where $0 \leq r < k$. Then*

$$c_k(G) \geq \frac{(k-r)n + (r+1)r}{2k}.$$

Moreover, for $r \geq 1$, a minimum k -conversion set S of G has order $\frac{(k-r)n + (r+1)r}{2k}$ if and only if S is independent and $G[V - S]$ is a maximal r -degenerate graph.

Proof. First suppose $r = 0$. In this case, Proposition 2.11 gives $c_k(G) = n - \alpha(G)$. Since G is regular, $\alpha(G) \leq \frac{n}{2}$, and the result follows. Now let $r \geq 1$ and let G be a $(k+r)$ -regular graph with $n > k+r$ vertices. Let S be a k -conversion set of G and for $X \in \{S, \bar{S}\}$, let n_X and m_X denote the number of vertices in X and the number of edges in $G[X]$, respectively. Counting in two ways the edges between S and \bar{S} gives

$$(k+r)n_S - 2m_S = (k+r)n_{\bar{S}} - 2m_{\bar{S}}.$$

Applying the bound $m_{\bar{S}} \leq rn_{\bar{S}} - \binom{r+1}{2}$, as provided by Lemma 3.10, and simplifying gives

$$(k+r)n_S - 2m_S \geq (k-r)n_{\bar{S}} + (r+1)r,$$

with equality if and only if $G[\bar{S}]$ is maximal r -degenerate. By substituting $n_{\bar{S}} = n - n_S$ and rearranging, we obtain

$$n_S \geq \frac{(k-r)n + (r+1)r + 2m_S}{2k},$$

with equality if and only if $G[\bar{S}]$ is maximal r -degenerate. The result follows since $m_S \geq 0$ with equality if and only if S is an independent set. \square

We note that, by definition of maximal r -degeneracy, in order to determine whether a subgraph H of G (in particular, $H = G[\bar{S}]$) is maximal r -degenerate we must look

at all $x, y \in V(H)$ such that $xy \notin E(H)$ —regardless of whether $xy \in E(G)$ —and determine whether $H + xy$ is still r -degenerate. In other words the maximality of H with respect to r -degeneracy does not depend on whether we can add more vertices or edges of G into H without losing the r -degenerate property, but whether we can add an edge between two non-adjacent vertices of H . In particular, when $H = G[\overline{S}]$, H is an induced subgraph so any additional edge xy under consideration is necessarily absent from G .

Chapter 4

The 2-conversion number of cubic graphs

In Proposition 3.7 (Section 3.2.2) we presented a lower bound on the k -conversion number of $(k + 1)$ -regular graphs. For $k = 2$, Proposition 3.7 gives the lower bound

$$c_2(G) \geq \left\lceil \frac{n+2}{4} \right\rceil \quad (4.1)$$

for cubic graphs G of order n .

In this chapter we present classes of cubic graphs that attain this bound and others that exceed it. We prove in Section 4.1 that a class of fullerenes and “fullerene-like” graphs attain the bound (Theorem 4.1). We study the 2-conversion number of snarks, as well as graphs that have some of the defining properties of snarks, in Section 4.2. Our results in Section 4.2 lead us to study 3-connected cubic graphs in Section 4.3.

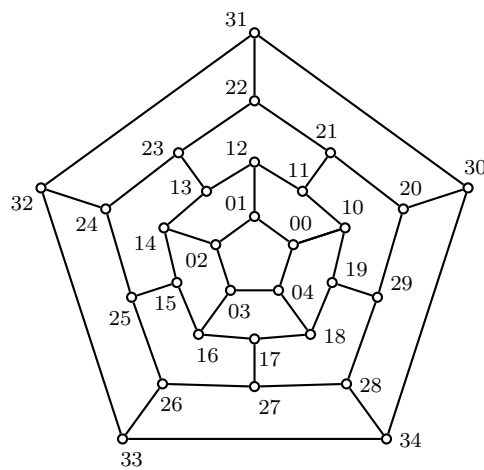
4.1 Generalized fullerenes

A *fullerene* is any planar cubic graph whose faces, including the outer face, all have size 5 or 6. Let $r = 5$ or 6 and $\ell \geq 0$. Consider the class of fullerenes whose vertices are

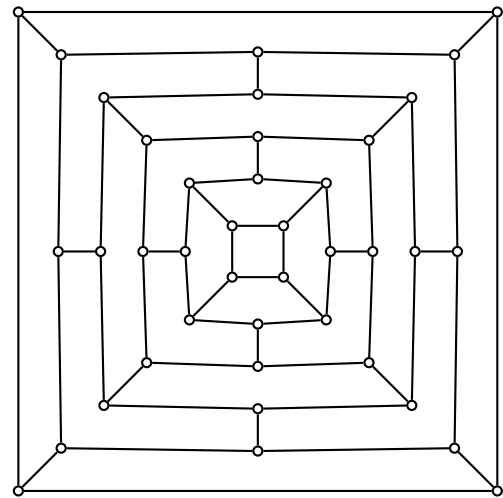
partitioned into two r -cycles¹, C^0 and $C^{\ell+1}$, and ℓ $2r$ -cycles, C^1, \dots, C^ℓ . Label the vertices in C^i $v_{i,j}$, for $0 \leq j \leq |C^i| - 1$ (in the obvious order). Define the adjacencies between the cycles as follows:

- (i) for $0 \leq j \leq r - 1$, $v_{0,j}$ is adjacent to $v_{1,2j}$;
- (ii) for $1 \leq i \leq \ell - 1$, $v_{i,j}$ is adjacent to $v_{i+1,j}$ if $j \equiv i \pmod{2}$;
- (iii) for $0 \leq j \leq r - 1$, $v_{\ell,2j+(\ell \bmod 2)}$ is adjacent to $v_{\ell+1,j}$.

A fullerene of this type with $r = 5$ and $\ell = 2$ is shown in Figure 4.1(a), with cycles C^0 to C^3 arranged from inside to outside. The vertices are labeled to illustrate the adjacencies of type (ii); for readability, vertex v_{ij} is labeled by its subscript ij only. We generalize this class of fullerenes to include any $r \geq 3$, and call such a graph constructed from two r -cycles and ℓ $2r$ -cycles an (r, ℓ) -generalized fullerene. If G is an (r, ℓ) -generalized fullerene then $|V(G)| = 2\ell r + 2r$. Figure 4.1(b) displays a $(4, 4)$ -generalized fullerene.



(a) A fullerene with $r = 5$ and $\ell = 2$



(b) A $(4, 4)$ -generalized fullerene

Figure 4.1: Fullerenes.

Došlić proved that all fullerenes are cyclically 4-edge connected [34, Theorem 8], and therefore by Proposition 2.35 they achieve equality in the lower bound (4.1). For

¹It follows from [29, Theorem 10.13] that every fullerene has exactly twelve faces of size 5. We note that these faces arise differently in the cases $r = 5$ and $r = 6$.

$r \geq 4$, (r, ℓ) -generalized fullerenes are also cyclically 4-edge connected², and therefore they achieve equality in the lower bound as well. We provide an alternate proof of this result (not using cyclic 4-connectivity) in Theorem 4.1 by constructing a 2-conversion set of size $\lceil \frac{n+2}{4} \rceil$ for any (r, ℓ) -generalized fullerene with $r \geq 3$ and $\ell \geq 0$.

Theorem 4.1. *If G is an (r, ℓ) -generalized fullerene with $|V(G)| = n$ then $c_2(G) = \lceil \frac{n+2}{4} \rceil$.*

We prove Theorem 4.1 later in this section. First, we must introduce some results on cylindrical hexagonal grids by Adams *et al.* [6], which are used in the proof of Theorem 4.1.

Definition 4.2. *Let m and n be two integers such that $m \geq 2$, $n \geq 4$ and n is even. An m by n cylindrical hexagonal grid is an array of n rows of m vertices (x, y) , with $0 \leq x \leq m-1$, $0 \leq y \leq n-1$, arranged in the first quadrant of a standard Cartesian plane such that each vertex (x, y) is adjacent to $(x, y+1)$ and, if y is even, also adjacent to $(x+1, y+1)$, provided that each vertex is in its allowed range (that is, the adjacency rule only applies to vertices with $0 \leq y \leq n-2$) and addition in the first coordinate is taken modulo m .*

An example of a 4 by 8 cylindrical hexagonal grid is displayed in Figure 4.2.

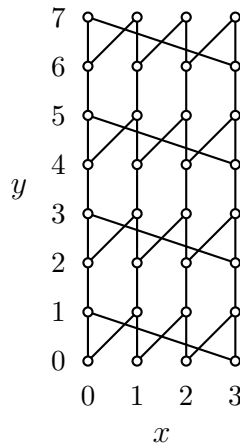


Figure 4.2: A 4 by 8 cylindrical grid.

The results in [6] on cylindrical hexagonal grids are for (weak) majority conversion processes, rather than k -conversion processes. Recall from Section 2.2 that in a weak

²This follows from the straightforward observation that there are at least four edge-disjoint paths between any pair of disjoint cycles in an (r, ℓ) -generalized fullerene, for $r \geq 4$.

majority conversion process, an unconverted vertex v converts at time t if at least half of its open neighbourhood is converted by time $t-1$. A set D of vertices of a graph G is a *dynamic monopoly (dynamo)* if a weak majority conversion process on G starting with seed set D eventually converts the whole graph¹. We denote the minimum size of a dynamic monopoly in G by $\min_D(G)$.

To prove Theorem 4.1 we construct a 2-conversion set of size at most $\lceil \frac{n+2}{4} \rceil$. The strategy for doing this is as follows. Adams *et al.* [6] provide an upper bound on the size of a minimum dynamic monopoly in a cylindrical hexagonal grid (Theorem 4.3). We show that every generalized fullerene F contains a cylindrical hexagonal grid as an induced subgraph G (Lemma 4.4). Then, to prove Theorem 4.1, we describe a construction for a 2-conversion set of F from a dynamic monopoly of G , and show that this 2-conversion set is not too much bigger than the dynamic monopoly of G .

Theorem 4.3 ([6], Theorem 4.2). *If G is an m by n cylindrical hexagonal grid, then*

$$\min_D(G) \leq \frac{n}{2} + \left(\left\lceil \frac{n}{4} \right\rceil - 1 \right) (m-2) + \frac{(n+2) \bmod 4}{2} \left\lfloor \frac{m-1}{2} \right\rfloor. \quad (4.2)$$

The proof of Theorem 4.3 (as presented in [6]) is constructive; for each m and n , it describes a dynamo of size less than or equal to the bound. We omit the proof of Theorem 4.3, but we will present the construction later (Construction 4.5) in order to use it to obtain a 2-conversion set in a generalized fullerene.

Lemma 4.4. *Every (r, ℓ) -generalized fullerene with $\ell \geq 2$ has an r by 2ℓ cylindrical hexagonal grid as an induced subgraph.*

Proof. Let F be an (r, ℓ) -generalized fullerene, with inner and outer r -cycles C^0 and $C^{\ell+1}$. The graph induced by $V(F) - V(C^0) - V(C^{\ell+1})$ is a cylindrical hexagonal grid. The vertices of C^i form rows $2(i-1)$ and $2(i-1)+1$ of the grid. \square

In light of Lemma 4.4, we present in Figure 4.3 an alternative picture of the $(4, 4)$ -generalized fullerene of Figure 4.1(b), which contains the 4 by 8 cylindrical hexagonal

¹This type of dynamic monopoly is more specifically called a *weak dynamic monopoly*. If a seed set D eventually converts the whole graph under a strong majority conversion process, where a vertex v only converts once more than half of its neighbours are converted, we call D a *strong dynamic monopoly*. In this discussion all dynamic monopolies will be weak, and we will simply call them dynamic monopolies.

grid of Figure 4.2. The alternative representation of the generalized fullerene makes it easy to see the cylindrical hexagonal grid subgraph.

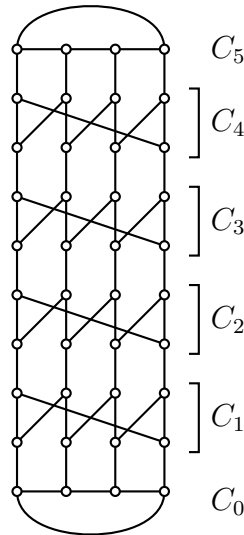


Figure 4.3: An alternative picture of the $(4, 4)$ -generalized fullerene of Figure 4.1(b).

We now present the construction, from the proof of Theorem 4.3, as given in [6], that yields a dynamo of size $\frac{n}{2} + \left(\left\lceil \frac{n}{4} \right\rceil - 1\right)(m-2) + \frac{(n+2) \bmod 4}{2} \left\lfloor \frac{m-1}{2} \right\rfloor$ (as in (4.2)) in a cylindrical hexagonal grid.

Construction 4.5 ([6], Theorem 4.2). Let G be an m by n cylindrical hexagonal grid. To construct a dynamo in G , we define the following sets of vertices:

1. Let $D_1 = \{(i, 4j + 2) : 0 \leq i \leq m - 3, 0 \leq j \leq \lceil \frac{n}{4} \rceil - 2\}$.

Call these *Type 1* vertices. They are the vertices in every 4th row, starting with row 2 (ie. the third row), and omitting the last two vertices in each row. If $n \equiv 0 \pmod{4}$, there are no Type 1 vertices in row $n - 2$.

2. If $n \equiv 0 \pmod{4}$ and m is odd, define $D_2 = \{(2i, n - 2) : 0 \leq i \leq \frac{m-3}{2}\}$, and otherwise define $D_2 = \emptyset$.

Call these *Type 2* vertices. They are every second vertex in row $n - 2$ (ie. the second-to-last row), starting with column 0 and ending with column $m - 3$.

3. If $n \equiv 0 \pmod{4}$ and m is even, define $D_3 = \{(2i + 1, n - 2) : 0 \leq i \leq \frac{m}{2} - 2\}$, and otherwise define $D_3 = \emptyset$.

Call these *Type 3* vertices. They are every second vertex in row $n - 2$ (ie. the second-to-last row), starting with column 1 and ending with column $m - 3$.

4. Let $D_4 = \{(m - 1, i) : 0 \leq i \leq n - 1, i \text{ even}\}$.

Call these *Type 4* vertices. They are every second vertex in the last column, starting with row 0.

Then $D = D_1 \cup D_2 \cup D_3 \cup D_4$ is a dynamo in G as shown in the proof of Theorem 4.3 [6]. Since the D_i 's are mutually disjoint,

$$|D| = \sum_{i=1}^4 |D_i| = \frac{n}{2} + \left(\left\lfloor \frac{n}{4} \right\rfloor - 1 \right) (m - 2) + \frac{(n + 2) \bmod 4}{2} \left\lfloor \frac{m - 1}{2} \right\rfloor. \quad (4.3)$$

We are now ready to prove Theorem 4.1, which is restated below. For $\ell \geq 2$ we use Lemma 4.4 and Construction 4.5 to construct a 2-conversion set in an (r, ℓ) -generalized fullerene. The cases $\ell = 0$ and $\ell = 1$ are considered separately.

Theorem 4.1 (again). If G is an (r, ℓ) -generalized fullerene with $|V(G)| = n$ then $c_2(G) = \left\lceil \frac{n+2}{4} \right\rceil$.

Proof. Let F be an (r, ℓ) -generalized fullerene, so $|V(F)| = 2\ell r + 2r$. First, suppose $\ell \geq 2$. By Lemma 4.4, F contains an r by 2ℓ cylindrical hexagonal grid G as an induced subgraph. Let D be the dynamo in G obtained from Construction 4.5. First, note that every degree 3 vertex of G eventually has two converted neighbours in G , since D is a dynamo of G . Further, each vertex of degree 2 in G eventually has at least one converted neighbour in G . Therefore, once all the vertices of C^0 and $C^{\ell+1}$ are converted, each vertex of degree 2 in G also has a converted neighbour outside of G , and therefore converts under a 2-conversion process.

Below, we define a subset D' of vertices of $C^0 \cup C^{\ell+1}$ such that $D \cup D'$ converts C^0 and $C^{\ell+1}$ by time $t = 2$. By the above reasoning, $D \cup D'$ is a 2-conversion set of F .

If r is even, define D' to be any minimum 2-conversion set of $C^0 \cup C^{\ell+1}$, so $|D'| = r$ (with $\frac{r}{2}$ seed vertices in each of the two cycles). By (4.3), with $n = 2\ell$ and $m = r$, $|D \cup D'| = \frac{\ell r}{2} + r + 1$. Since $|V(F)| = r(\ell + 2)$, we have $|D \cup D'| = \left\lceil \frac{|V(F)|+2}{4} \right\rceil$ for ℓ odd or even.

Now suppose r is odd. We consider the cases ℓ even and ℓ odd separately.

The case where ℓ is even corresponds to $n \equiv 0 \pmod{4}$ in Construction 4.5. In this case, one vertex of C^0 is adjacent to a seed vertex u of C^1 , so a set X of $\frac{r-1}{2}$ seed vertices in C^0 , together with u , converts C^0 . Similarly, one vertex, v , of $C^{\ell+1}$ is adjacent to a vertex of C^ℓ that has two neighbours in D (and therefore converts without help from seed vertices in $C^{\ell+1}$). Therefore, a set Y of $\frac{r-1}{2}$ seed vertices in $C^{\ell+1}$, together with v , converts $C^{\ell+1}$. Let $D' = X \cup Y$. Then $D \cup D'$ is a 2-conversion set of F and $|D \cup D'| = |D| + |D'|$. From above, we have $|D'| = r - 1$. We compute $|D|$ using (4.3) with $m = r$ and $n = 2\ell$, by Proposition 4.4, and simplify using the parities of ℓ (even) and r (odd). This gives

$$\begin{aligned} |D \cup D'| &= \ell + \left(\left\lfloor \frac{\ell}{2} \right\rfloor - 1 \right) (r - 2) + \frac{(2\ell + 2) \bmod 4}{2} \left\lfloor \frac{r-1}{2} \right\rfloor + r - 1 \\ &= \ell + \left(\frac{\ell}{2} - 1 \right) (r - 2) + \frac{r-1}{2} + r - 1 \\ &= \frac{\ell r + r + 1}{2}. \end{aligned} \tag{4.4}$$

On the other hand,

$$\left\lfloor \frac{|V(F)| + 2}{4} \right\rfloor = \left\lfloor \frac{2\ell r + 2r + 2}{4} \right\rfloor,$$

which simplifies to give the same expression as in (4.4), again using the parities of ℓ and r .

The case where ℓ is odd corresponds to $n \equiv 2 \pmod{4}$. Here, one vertex of C^0 is adjacent to a seed vertex u in C^1 , so a set X of $\frac{r-1}{2}$ seed vertices in C^0 , together with u , converts C^0 . However, no vertex in $C^{\ell+1}$ is adjacent to a vertex that converts without help from seed vertices in $C^{\ell+1}$. Therefore, a set Y of $\frac{r+1}{2}$ seed vertices is required to convert $C^{\ell+1}$. Let $D' = X \cup Y$. Then, as before, $D \cup D'$ is a 2-conversion set of F and $|D \cup D'| = |D| + |D'|$. From above we have $|D'| = r$, and once again we compute $|D|$ using (4.3) with $m = r$ and $n = 2\ell$. This time ℓ and r are both odd, so $(n+2) \bmod 4 = 0$ and we have

$$\begin{aligned} |D \cup D'| &= \ell + \left(\left\lceil \frac{\ell}{2} \right\rceil \right) (r - 2) + r \\ &= \ell + \left(\frac{\ell+2}{2} - 1 \right) (r - 2) + r \\ &= \frac{\ell r + r + 2}{2}. \end{aligned} \tag{4.5}$$

As in the previous case, $\left\lceil \frac{|V(F)|+2}{4} \right\rceil$ simplifies to give the same expression as (4.5).

It remains to prove the result when $\ell = 0$ and when $\ell = 1$. When $\ell = 0$, $|V(F)| = 2r$ and it is easy to verify that the set $\{v_{0,2j} : 0 \leq j < \lfloor \frac{r}{2} \rfloor\} \cup \{v_{1,r}\}$ is a 2-conversion set of order $\left\lceil \frac{2r+2}{4} \right\rceil$. Similarly, it is easy to show that F has a 2-conversion set consisting of $\lfloor \frac{r}{2} \rfloor$ vertices from each of C^0 and C^2 and one vertex from C^1 . Since $|V(F)| = 4r$ in this case, the result follows. \square

4.2 Snarks and would-be snarks

A *snark* is a connected, bridgeless cubic graph with chromatic index 4. (By Vizing's Theorem, every cubic graph has chromatic index 3 or 4; these cases define two classes of cubic graphs, called *class 1* and *class 2*, respectively. Snarks, therefore, are class 2 graphs.) They have been studied since the 1880's, when Tait proved that the Four Colour Theorem is equivalent to the statement that no snark is planar. To avoid degenerate cases, it has long been standard to require them to have girth at least five. We refer to such graphs (that is, bridgeless, class 2, triangle-free cubic graphs) as *Gardner snarks*, as this was the common definition of snarks when Martin Gardner gave them the name "snark" in 1975 [48]. The name, taken from the elusive creature in Lewis Carroll's poem *The Hunting of the Snark*, reflects the scarcity of examples in the years after Tait defined them. The smallest and earliest known example of a snark is the Petersen graph, discovered in 1898. Due to their connection with the Four Colour Theorem (Four Colour Conjecture, at the time), much attention was given to the pursuit of new examples of snarks (with the hope of finding a planar one, perhaps), but a second example was not discovered until 1946. Since then, more examples have been discovered, including infinite families.

Interest in snarks has remained steady, due in part to their connection to other important conjectures in graph theory, notably the Cycle Double Cover Conjecture [1, 106]. In 1985, Jaeger [58] proved that a smallest counterexample to the conjecture must be a snark; therefore, if the conjecture is true for snarks, it is true for all graphs.

More recently, more restrictive definitions of snarks have become the standard. It is now common to require snarks to have higher connectivity and larger girth. Some

authors use even more restrictive definitions in order to exclude snarks that can be obtained from other snarks. Some require them to be cyclically 4-edge connected, rather than simply triangle-free [2]. As mentioned in Section 2.3.4, a graph G is *cyclically k -edge connected* if at least k edges must be removed to disconnect G into two subgraphs that each contain a cycle. We call cyclically 4-edge connected snarks *strong snarks*. (As we noted in Section 2.3.4, a cubic graph is cyclically 4-edge connected if and only if it is cyclically 4-vertex connected [74], so we often simply say *cyclically 4-connected*.) A convenient overview of approximately the first century of snark research, including a discussion of modern definitions, can be found in [113].

We begin with the Petersen graph. By (4.1), the Petersen graph has 2-conversion number at least 3. The 2-conversion set given in Figure 4.4 shows that the Petersen graph achieves equality in the lower bound.

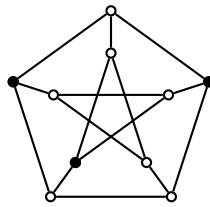


Figure 4.4: A minimum 2-conversion set in the Petersen graph.

In addition, Zhang *et al.* have proved that all *flower snarks* (defined below) meet the bound. (In fact, their result applies to a larger family of graphs that contains the flower snarks.) We state this result in Theorem 4.6, and in Proposition 4.7 we prove constructively that all permutation snarks meet the bound as well.

The flower snarks, discovered by Isaacs in 1975 [56], are the earliest-known infinite family of snarks. Before their discovery, only 4 examples of snarks were known: the Petersen graph, and snarks discovered by Blanuša (1946), Descartes¹ (1948) and Szekeres (1973). In the same paper, Isaacs presented a second infinite family, which he called the *BDS class* of graphs after the aforementioned authors, whose snarks belong to the class.

The flower snarks are constructed as follows. Start with $n \geq 3$ copies of $K_{1,3}$, with the central vertex of each copy denoted by a_i and the leaves denoted by b_i , c_i and d_i for

¹Blanche Descartes was the collective pseudonym of the English mathematicians R. Leonard Brooks, Arthur Harold Stone, Cedric Smith and William Tutte [101].

$i = 0, \dots, n - 1$. Add the n -cycle $(b_0 b_1 \dots b_{n-1})$ and the $2n$ -cycle $(c_0 \dots c_{n-1} d_0 \dots d_{n-1})$. The resulting cubic graph has $4n$ vertices and $6n$ edges. When n is odd and at least 5, the construction produces the flower snark J_n . The flower snark J_5 is shown in Figure 4.5. (When $n = 3$ the construction produces a class 2 graph with a triangle, known as the Tietze graph, and when n is even it produces a class 1 graph.) The result of Zhang *et al.* applies to graphs resulting from this construction for all $n \geq 3$. We call this class of graphs *generalized flower snarks*. Zhang *et al.* prove that generalized flower snarks meet the lower bound by constructing a 2-conversion set of the given size.

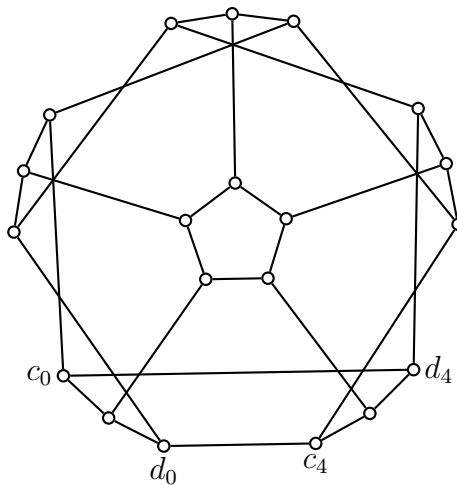


Figure 4.5: The flower snark J_5 .

Theorem 4.6 ([118], Theorem 1). *Let H_n be the generalized flower snark on $4n$ vertices, for $n \geq 3$. Then $c_2(H_n) = n + 1 = \left\lceil \frac{4n+2}{4} \right\rceil$.*

A *permutation snark* is a snark obtained by joining two odd cycles of the same order with a perfect matching. The Petersen graph is an example of a permutation snark. Proposition 4.7 implies that all permutation snarks achieve equality in the lower bound (4.1).

Proposition 4.7. *Let G be a graph obtained by joining two cycles of the same order with a perfect matching. Then $c_2(G) = \left\lceil \frac{|V(G)|+2}{4} \right\rceil$.*

Proof. Let $\frac{|V(G)|}{2} = r$ and let C_1 and C_2 be the two cycles of order r in G with a perfect matching between them. If r is even, then the set S containing every second

vertex on C_1 and any vertex on C_2 is a 2-conversion set of G . If r is odd, let v be a vertex on C_1 . Then the set S containing every second vertex of $C_1 - \{v\}$ and the neighbour of v on C_2 is a 2-conversion set of G . It is easy to verify that in both cases, $|S| = \left\lceil \frac{2r+2}{4} \right\rceil = \left\lceil \frac{|V(G)|+2}{4} \right\rceil$. \square

Both the flower snarks and the permutation snarks are strong snarks; that is, they are cyclically 4-connected. However, the proofs of Theorem 4.6 and Proposition 4.7 do not rely on the cyclic connectivity of the graphs. These results can also be obtained as corollaries of Proposition 2.35, which states that for any cyclically 4-connected cubic graph G , $c_2(G) = \left\lceil \frac{|V(G)|+2}{4} \right\rceil$.

Since all strong snarks achieve equality in the lower bound (4.1), it is natural to ask whether all snarks do. However, we will show in Section 4.3 that there exist infinitely many Gardner snarks that fail to meet the bound. In Table 4.1 we show that, in fact, all possible combinations of the three defining properties of Gardner snarks (bridgeless, class 2, triangle-free) admit graphs that meet the lower bound (4.1) and graphs that fail to meet it.

Theorem 4.8 and Proposition 4.9 give well known sufficient conditions for cubic graphs to be class 1 (chromatic index 3) and class 2 (chromatic index 4), respectively, which aids our search for examples in each category. Theorem 4.8 was shown by Tait in 1880 to be equivalent to the Four Colour Theorem.

Theorem 4.8. [11, 12, 107] *Every bridgeless planar cubic graph has chromatic index 3.*

Theorem 4.9. *Every bridged cubic graph has chromatic index 4.*

Proof. Let G be a cubic graph with a bridge e , and let H be a component of $G - e$. Then H can be obtained by subdividing an edge of a cubic graph (which may have parallel edges). Let $n = |V(H)|$. Then n is odd, and H has $\frac{3(n-1)+2}{2}$ edges. Consider a proper edge colouring of H . Since the edges in a given colour class are pairwise disjoint, there are at most $\frac{n-1}{2}$ edges in each colour class. This means that any three colour classes contain at most $\frac{3(n-1)+2}{2} - 1$ edges, so there are at least four colour classes. By Vizing's Theorem, $\chi'(H) = 4$, so $\chi'(G) = 4$ also. \square

Theorem 4.9 allows us to limit our investigation to graphs that are bridgeless or class 2, since there are no bridged, class 1 cubic graphs. All other combinations—that is, all allowable combinations—of the three defining characteristics of snarks admit graphs that meet the lower bound and graphs that do not meet the lower bound. Table 4.1 gives an example of a graph for each type for each of the combinations.

Table 4.1 indicates that all feasible combinations of the bridgeless, class 2 and triangle-free properties admit graphs that meet the lower bound (4.1) and graphs that do not meet this bound. For each combination of properties except bridgeless, class 2, triangle-free cubic graphs (i.e. Gardner snarks), we now show that the difference between the bound and the 2-conversion number can be arbitrarily large (Propositions 4.12 to 4.17). We address the remaining category in Section 4.3, where we consider 3-connected cubic graphs with arbitrary girth.

To prove that the difference between the bound and the 2-conversion number can be arbitrarily large for graphs with bridges, we use the following lemma.

Lemma 4.10. *Let G be a cubic graph with a bridge e , and let H_1 and H_2 be the components of $G - e$. Then $c_2(G) = c_2(H_1) + c_2(H_2)$.*

Proof. Clearly, $c_2(G) \leq c_2(H_1) + c_2(H_2)$. To show equality we show that the minimal 2-immune sets of H_1 and H_2 are the induced cycles. Let U be a minimal 2-immune set of H_i and let a be the vertex of degree 2 in H_i . First consider the case where $a \notin U$. Then every vertex in U has degree 3 in H_i . Since each of them also has at most one neighbour outside U , $H_i[U]$ has no leaves. By minimality of U , this implies that $H_i[U]$ is a chordless cycle. Now consider the case where $a \in U$. By minimality, if $H_i[U]$ contains a cycle then that cycle contains a (otherwise the cycle is a smaller 2-immune set). In this case, $H_i[U]$ is a chordless cycle containing a . On the other hand, if $H_i[U]$ does not contain a cycle then it has at least two leaves; one of these leaves is a vertex of degree 3 in H_i . This is a contradiction, since such a vertex has two neighbours outside U . Therefore the minimal 2-immune sets of H_i are the induced cycles of H_i .

Thus U is a minimal 2-immune set of G if and only if it is a minimal 2-immune set of H_1 or H_2 . Since H_1 and H_2 are disjoint, the result follows. \square

We construct several classes of graphs that exceed the bound from the four graphs

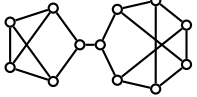
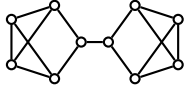
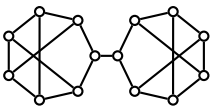
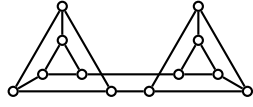
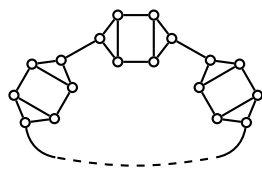
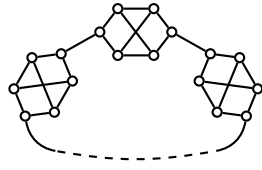
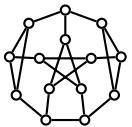
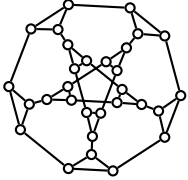
Bridgeless?	Class 2?	Δ -free?	Example with $c_2(G) = \lfloor \frac{n+2}{4} \rfloor$	Example with $c_2(G) > \lfloor \frac{n+2}{4} \rfloor$
No	Yes	No		
No	Yes	Yes		Any triangle-free cubic graph of the form $\textcircled{H}-\textcircled{H}$ where H has order $n \equiv 1 \pmod{4}$
Yes	No	No		
Yes	No	Yes	Q_3 , Generalized fullerenes with $r \geq 4$	
Yes	Yes	No		
Yes	Yes	Yes	All strong snarks	Discussion will follow ¹

Table 4.1: Combinations of snark properties that permit equality/inequality in the lower bound on $c_2(G)$.

¹ Examples and discussion are given in Section 4.3.

H_1 , H_2 , H_3 and H_4 shown in Figure 4.6.

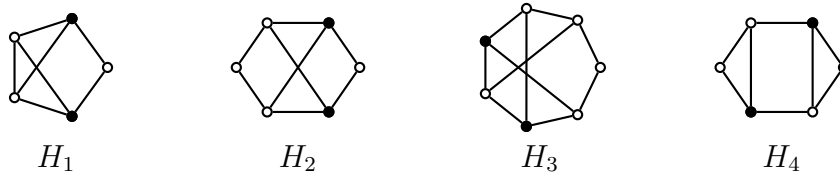


Figure 4.6: Building blocks for graphs that exceed the bound.

Lemma 4.11. *Let H_1 , H_2 , H_3 and H_4 be as shown in Figure 4.6, and let G be a graph containing H_i as an induced subgraph, for some $1 \leq i \leq 4$. Then any minimum 2-conversion set of G contains exactly 2 vertices from each copy of H_i .*

Proof. Figure 4.6 gives a 2-conversion set of size 2 for each graph H_i . On the other hand, no vertex is on every cycle of H_i , so there is no 2-conversion set of G containing fewer than two vertices from any copy of H_i . \square

In the next proposition Propositions 4.12 and 4.13 we construct bridged, class 2 cubic graphs with and without triangles, respectively, that exceed the bound.

Proposition 4.12. *Let $m \geq 2$ and let G be the cubic graph constructed from P_m by replacing each leaf with a copy of H_1 and each internal vertex with a copy of H_2 , where H_1 and H_2 are as shown in Figure 4.6. Then*

(a) G is a bridged, class 2 cubic graph with triangles, and

(b) $c_2(G) - \lceil \frac{|V(G)|+2}{4} \rceil = \lfloor \frac{m}{2} \rfloor$.

Proof. For (a), the class 2 property follows from the bridged property by Theorem 4.9. For (b), $|V(G)| = 6m - 2$ and by Lemma 4.11, $c_2(G) = 2m$. \square

Proposition 4.13. *Let $m \geq 2$ and let G be the cubic graph constructed from P_m by replacing each leaf with a copy of H_3 and each internal vertex with a copy of H_2 , where H_2 and H_3 are as shown in Figure 4.6. Then*

(a) G is a bridged, class 2, triangle-free cubic graph, and

(b) $c_2(G) - \lceil \frac{|V(G)|+2}{4} \rceil = \lfloor \frac{m}{2} \rfloor - 1$.

Proof. For (a), the class 2 property follows from the bridged property by Theorem 4.9. For (b), $|V(G)| = 6m + 2$ and by Lemma 4.11, $c_2(G) = 2m$. \square

In Proposition 4.14 we construct bridgeless, class 1 cubic graphs with and without triangles that exceed the bound.

Proposition 4.14. *Let $m \geq 3$ and let H_2 and H_4 be as shown in Figure 4.6. Let G_1 be the cubic graph constructed from C_m by replacing each vertex with a copy of H_4 , and let G_2 be the cubic graph constructed from C_m by replacing each vertex with a copy of H_2 . Then*

- (a) G_1 is a bridgeless, class 1 cubic graph with triangles,
- (b) G_2 is a bridgeless, class 1, triangle-free cubic graph, and
- (c) for $i = 1, 2$, $c_2(G_i) - \lceil \frac{|V(G_i)|+2}{4} \rceil = \lfloor \frac{m-1}{2} \rfloor$.

Proof. Parts (a) and (b) can be easily verified, using Theorem 4.8 for (a). For part (c), it is clear that $|V(G_i)| = 6m$ and by Lemma 4.11, $c_2(G_i) = 2m$, for $i = 1, 2$. \square

We have presented cubic graphs with an arbitrary difference between c_2 and the lower bound for each of the first four categories defined in Table 4.1. We now describe a construction that produces graphs in the fifth category—bridgeless, class 2 cubic graphs of girth 3—with an arbitrary difference between c_2 and the bound (4.1). In fact, the same construction can be used to produce additional examples for any of the girth 3 categories.

To construct girth 3 graphs (which can be bridged or bridgeless and class 1 or class 2) with an arbitrary difference between c_2 and the bound (4.1), we begin with a cubic graph G and replace each vertex with a triangle. We call this operation *triangle replacement* of G and we call the resulting girth 3 graph the *triangle-replaced graph* of G , and denote it by $T(G)$, as in [114]. Lemma 4.15 guarantees that the bridged/bridgeless properties and the class 1/class 2 properties are preserved under triangle replacement. Therefore in order to produce a bridgeless, class 2 cubic graph with triangles, for example, we take the triangle replacement of any bridgeless, class 2 cubic graph. Figure 4.7 shows the triangle-replaced graph of the Petersen graph.

Since the Petersen graph is bridgeless and class 2, so is its triangle-replaced graph (by Lemma 4.15).

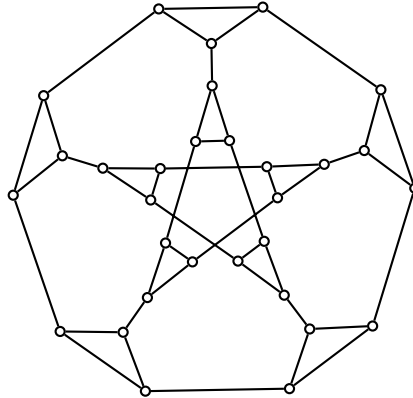


Figure 4.7: The triangle-replaced graph of the Petersen graph.

Lemma 4.16 gives a lower bound on $c_2(T(G))$, from which we deduce in Proposition 4.17 that there are triangle-replaced graphs $T(G)$ with arbitrary difference between $c_2(T(G))$ and the bound (4.1).

Lemma 4.15. *For any cubic graph G , G and $T(G)$ have the same number of bridges and the same chromatic index.*

Proof. The first statement is obvious. For the second statement, let $G' = T(G)$ and let $T(v)$ denote the triangle in G' arising from v , for each vertex v of G . We consider $E(G)$ to be a subset of $E(G')$. We show that $\chi'(G) = 3$ if and only if $\chi'(G') = 3$; the result then follows by Vizing's Theorem.

Suppose first that G' has a proper 3-edge colouring $f : E(G') \rightarrow \{1, 2, 3\}$. Consider three edges incident with a vertex v in G . In a proper 3-edge colouring of G' , these edges all have different colours, since each is incident with two of the three edges of $T(v)$. Therefore the colouring of the edges of G obtained by restricting f to $E(G)$ is a proper 3-edge colouring of G .

Now suppose G has a proper 3-edge colouring. For each $v \in V(G)$ we extend the colouring f to $T(v)$ such that the edge e of $T(v)$ gets the same colour as the edge of $E(G)$ that is incident with the other two edges of $t(v)$. \square

Lemma 4.16. *Let G be a $(k+1)$ -regular graph with a collection of d pairwise disjoint cycles. Then $c_k(G) \geq d$ for all k .*

Proof. The result follows immediately from Proposition 2.23. \square

We are now ready to show that the difference between the 2-conversion number and the bound (4.1) for triangle-replaced graphs $T(G)$ grows with the order of G . Since there are arbitrarily large graphs G for each feasible category of cubic graphs defined in Table 4.1, there are arbitrarily large differences between the 2-conversion number and the bound for each category with triangles.

Proposition 4.17. *Let H be a cubic graph of order m and let $G = T(H)$. Then $c_2(G) - \lceil \frac{|V(G)|+2}{4} \rceil \geq \lfloor \frac{m-2}{4} \rfloor$. Moreover, G has the same number of bridges and the same chromatic index as H .*

Proof. By Proposition 4.16, $c_2(G) \geq m$. The first statement follows, with $|V(G)| = 3m$. The second statement follows from Lemma 4.15. \square

For each of the first five categories of cubic graphs defined in Table 4.1, we have given a construction to produce a graph G with an arbitrarily large difference between $c_2(G)$ and the lower bound $\lceil \frac{|V(G)|+2}{4} \rceil$. However, for all of the triangle-free graphs, while the difference may be large, the ratio $\frac{c_2(G)}{|V(G)|}$ approaches $\frac{1}{4}$, and hence the ratio $\frac{c_2(G)}{\lceil \frac{|V(G)|+2}{4} \rceil}$ approaches 1, as $|V(G)|$ becomes large. By contrast, for the girth 3 graphs we have constructed in this section, $\frac{c_2(G)}{|V(G)|}$ approaches $\frac{1}{3}$ as $|V(G)|$ becomes large.

In the next section we determine whether this ratio can be greater than $\frac{1}{4}$, asymptotically, for triangle-free graphs.

4.3 3-edge connected cubic graphs

In the previous section we constructed infinite families of graphs for which the difference between the 2-conversion number and the lower bound (4.1) could be made arbitrarily large. All of these examples—in fact, all examples we have seen so far that do not meet the lower bound—contain triangles or have connectivity at most 2. We also saw infinite families of graphs for which the ratio $\frac{c_2(G)}{|V(G)|}$ exceeds $\frac{1}{4}$ asymptotically

(in $|V(G)|$), but all of these examples have girth 3. These observations lead us to the following two questions.

Question 4.18. Is there a family of 3-connected, triangle-free cubic graphs G such that $c_2(G) > \left\lceil \frac{|V(G)|+2}{4} \right\rceil$?

Question 4.19. Is there a family of triangle-free cubic graphs such that

$$\frac{c_2(G)}{|V(G)|} \rightarrow r > \frac{1}{4} \text{ as } |V(G)| \rightarrow \infty?$$

In this section we answer both questions in the affirmative. In fact, for Question 4.18 we describe a construction for an infinite family of 3-connected graphs of arbitrary girth such that the difference between c_2 and the lower bound (4.1) increases with order. The same family of graphs provides an answer to Question 4.19. The graphs produced by our construction are of particular interest in the study of snarks because, as noted, many modern definitions of snarks require higher connectivity (rather than simply bridgeless) and higher girth (rather than simply triangle-free).

We begin by defining a graph product that produces an r -regular graph from two smaller r -regular graphs. In this section we use this product with $r = 3$; we use it again in Chapter 7 with $r > 3$.

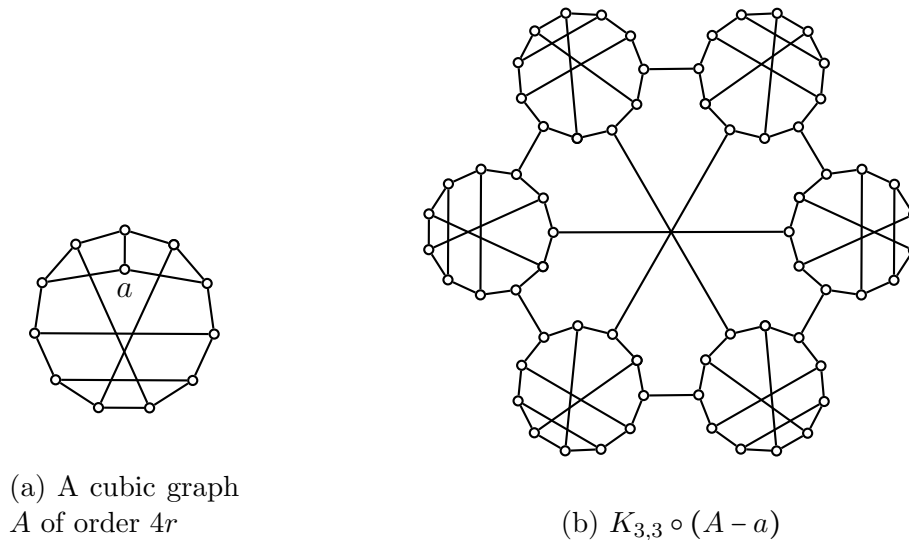
Definition 4.20. Let G and A be r -regular graphs, $r \geq 2$, and define $A^- = A - a$, for any vertex a . Let \mathcal{C} be the class of graphs that can be obtained by replacing each vertex v of G by a copy A_v^- of A^- and joining a degree $r - 1$ vertex of A_u^- to a degree $r - 1$ vertex of A_v^- if and only if $uv \in E(G)$ ¹. We denote by $G \circ A^-$ any graph in \mathcal{C} .

We will not need to differentiate between different elements of \mathcal{C} , as our results hold for any such graph.

Figure 4.8 shows an example of a cubic graph A with vertex a identified, and a graph $K_{3,3} \circ (A - a)$.

Proposition 4.22 asserts that if A is a cubic graph of order $4r$ then $G \circ A^-$ exceeds the bound (4.1). To answer Question 4.18 we then show that the construction can

¹Note that this construction can yield non-isomorphic graphs depending on how the different copies of A^- are joined.



(a) A cubic graph
 A of order $4r$

(b) $K_{3,3} \circ (A - a)$

Figure 4.8: An example of the construction of a cubic graph $G \circ A^-$.

yield 3-edge connected— and therefore 3-connected²— graphs of arbitrary girth; this is achieved in Propositions 4.23 and 4.24. We begin with a lemma which guarantees that any 2-conversion set of $G \circ A^-$ contains at least r vertices from each copy of A^- .

Lemma 4.21. *If A is a cubic graph of order $4r$ and $A^- = A - a$ is an induced subgraph of a cubic graph H , then any 2-conversion set of H contains at least r vertices of A^- .*

Proof. Suppose H has a 2-conversion set S such that $|S \cap V(A^-)| < r$. Then $(S \cap V(A^-) \cup \{a\})$ is a 2-conversion set of A of cardinality at most r . However, by (4.1), $c_2(A) \geq \left\lceil \frac{4r+2}{4} \right\rceil = \frac{4r+4}{4} = r + 1$. \square

Proposition 4.22. *For any cubic graphs G of order $n \geq 6$ and A of order $4r$,*

$$c_2(G \circ A^-) - \left\lceil \frac{|V(G \circ A^-)| + 2}{4} \right\rceil \geq \left\lfloor \frac{n - 2}{4} \right\rfloor.$$

Proof. Let S be a 2-conversion set of $G \circ A^-$. By Lemma 4.21, S contains at least r vertices of each copy of A^- , hence $|S| \geq nr$. The result follows because $V(G \circ A^-)$ has order $(4r - 1)n$. \square

Proposition 4.23. *Let A and G be cubic graphs. Then $G \circ A^-$ has girth at least $g(A)$.*

²It is an easy exercise to show that the connectivity of any cubic graph is equal to its edge connectivity [29].

Proof. Let $g(A) = g$ and let C be any cycle in $G \circ A^-$. If C is contained in any copy of A^- , then C has length at least $g(A)$. If C is not contained in a copy of A^- , then for any copy A_v^- of A^- , $C \cap A_v^- = \emptyset$ or $C \cap A_v^-$ is a single path, since each copy of A^- is joined by only three edges to the rest of $G \circ A^-$. Therefore C consists of segments Q_1, Q_2, \dots, Q_s of paths in distinct copies of A^- , together with edges e_i joining Q_i to Q_{i+1} , $i = 1, \dots, s-1$, and e_s joining Q_s to Q_1 . Each Q_i has length at least $g-2$, otherwise Q_i and the vertex a that was removed from A to form A^- produce a cycle of length less than g in A . Therefore C has length at least $s(g-2)$. Since G has no multiple edges, $s \geq 3$, and the result follows. \square

Proposition 4.24. *Let A and G be 3-connected cubic graphs. Then $G \circ A^-$ is 3-connected.*

Proof. Let x and y be any distinct vertices of $G \circ A^-$, say $x \in V(A_u^-)$ and $y \in V(A_v^-)$, for $u, v \in V(G)$. Let u_i and v_i , $i = 1, 2, 3$, be the vertices of degree 3 in A_u^- and A_v^- , respectively. First, suppose $u = v$. Since A is 3-connected, A contains three internally disjoint $x - y$ paths, at most one of which contains a . These correspond to three internally disjoint $x - y$ paths in $G \circ A^-$: at least two are contained in A_v^- and the third may contain the vertices v_1 and v_2 , say, and a $v_1 - v_2$ path in $(G \circ A^-) - A_v^-$. Now suppose $u \neq v$. Then in A , x is connected to a by three internally disjoint paths; therefore in A^- , x is connected to the u_i 's by three internally disjoint paths. Similarly, in A_v^- , y is connected to the v_i by three internally disjoint paths. Since G is 3-connected, there are, without loss of generality, three internally disjoint paths $u_i - v_i$, $i = 1, 2, 3$. Therefore x is connected to y in $G \circ A^-$ by three internally disjoint paths. \square

Together, Lemma 4.22 and Propositions 4.23 and 4.24 imply that if A is a 3-connected cubic graph of order $4r$ and girth g , and G is a 3-connected cubic graph of order $n \geq 6$, then $G \circ A^-$ is a 3-connected cubic graph of girth at least g such that $c_2(G \circ A^-)$ exceeds the bound (4.1) by at least $\lfloor \frac{n-2}{4} \rfloor$. We note that for $g = 3$, we may use $A = K_4$, and then the graph $G \circ A^-$ is the triangle-replaced graph of G . That is, the 3-connected cubic graphs of girth 3 that we presented in Proposition 4.17 are obtainable from the construction presented in this section.

It remains to show that there exist appropriate cubic graphs A and G for $g \geq 4$. For G , we simply require a 3-connected cubic graph of order at least 6. There are

many such graphs; we highlight one example, which will also help us find A . For $k \geq 2$ and $g \geq 3$, a (k, g) -cage is a graph that has the least number of vertices among all k -regular graphs with girth g . Erdős and Sachs [41], as cited in [29], proved that (k, g) -cages exist for all $k \geq 2$ and $g \geq 3$, and Daven and Rodger [32] showed that all (k, g) -cages are 3-connected. Therefore a $(3, g)$ -cage is an appropriate choice for G , and if the number of vertices in such a graph is a multiple of 4 then we may use it for A as well. (In fact, we may use a $(3, g_1)$ -cage for G , for any $g_1 \geq 3$, and a $(3, g_2)$ -cage for A , provided that this graph has order $4r$. The girth of $G \circ A^-$ will then be at least g_2 , as shown in Proposition 4.23.) If, for the specified girth $g \geq 4$, a $(3, g)$ -cage B has order $m \equiv 2 \pmod{4}$, we can obtain a 3-connected cubic graph of order $4r$ and girth at least g by modifying and joining together two copies of any 3-connected cubic graph of order $4r + 2$ and girth at least g (such as B).

Proposition 4.25. *For every $g \geq 3$ there exists a 3-connected cubic graph of order $4r$ and girth at least g .*

Proof. For every $g \geq 3$ there exists a 3-connected cubic graph with girth g , for example a $(3, g)$ -cage. The $(3, 3)$ -cage is K_4 , so the statement is true for $g = 3$. Let $g \geq 4$ and suppose B is a 3-connected cubic graph of girth g and order $n \equiv 2 \pmod{4}$. Let u and v be two adjacent vertices of B . Since $g \geq 4$, u and v have no common neighbour. Let a and b be the neighbours of u in $B - v$ and let c and d be the neighbours of v in $B - u$. Consider two copies H and H' of $B - \{u, v\}$; for each vertex v in H , we denote its counterpart in H' by v' . Let A be the cubic graph obtained from H and H' by adding edges aa' , bb' , cd' and dc' . We show that A is 3-edge connected and has girth at least g .

Clearly, any cycle in H has length at least g , since it is also a cycle in B . Let C be a cycle in A containing vertices from both H and H' and suppose C has length ℓ . Then, since the vertices a' , b' , c' and d' are all distinct, $C \cap H$ is a path P of length at most $\ell - 3$ whose endpoints are two of a, b, c and d . If the endpoints of P are a and b then $P + au + ub$ is a cycle in B of length at most $\ell - 1$ in B , so $\ell - 1 \geq g$. If the endpoints of P are a and c , then $P + au + uv + vc +$ is a cycle in B of length at most ℓ , so $\ell \geq g$.

It remains to show that A is 3-connected. Let x be any vertex of H . To see that there are three edge-disjoint $x - x'$ paths in A , consider three edge-disjoint $x - v$

paths in B . Without loss of generality, we may assume that one contains the edge au , another contains the edge cv and the third contains the edge dv . Therefore there are paths $x - a$, $x - c$ and $x - d$ in H and paths $a' - x'$, $c' - x'$ and $d' - x'$ in H' which are all edge-disjoint. Adding the edges aa' , cd' and dc' produces three edge-disjoint $x - x'$ paths in A .

Now let x and y be any two vertices of H . Since B is 3-connected, H is connected. There are two cases to show that there are three edge-disjoint $x - y$ paths in A .

Case 1: Suppose there is only one $x - y$ path P in H . Then u and v are contained in distinct $x - y$ paths of B , one of which contains the subpath $a - u - b$ and the other contains the subpath $c - v - d$. Then H contains edge-disjoint paths $x - a$, $b - y$, $x - c$, $d - y$, each of which is disjoint from P , and these paths are copied in H' . Therefore A contains three edge-disjoint $x - y$ paths, $(x - a) + aa' + (a' - x') + (x' - c') + c'd + (d - y)$, $(x - c) + cd' + (d' - y') + (y' - b') + b'b + (b - y)$, and P .

Case 2: Suppose there are exactly two edge-disjoint $x - y$ paths P_1 and P_2 in H . Then a third such path in B contains u or v (maybe both), and therefore it contains two of a, b, c and d , say a and b (the other cases are similar). Since H' is connected there is a path in H' between any two of a', b', c', d' . Then there is a path $(x - a) + aa' + (a' - b') + b'b + (b' - y)$ in A which is edge-disjoint from P_1 and P_2 .

Finally, we must show that for any two vertices x, y of H , there are three edge-disjoint $x - y'$ paths in A . Let X be any 2-edge cut in A . Since there are three edge-disjoint $x - y$ paths in A , x and y are in the same component of $A - X$. Likewise, since there are three edge-disjoint $y - y'$ paths in A , y and y' are in the same component of $A - X$. Therefore x and y' are in the same component of $A - X$. Since X is any 2-edge cut, there are three edge-disjoint $x - y'$ paths in A . \square

We are now ready to answer Question 4.18 by proving the existence of 3-connected cubic graphs of arbitrarily large girth that fail to meet the lower bound (4.1). However, chromatic index (either 3, corresponding to class 1, or 4, corresponding to class 2) was central to our discussion in the previous section, and we have not yet discussed the chromatic index of the graphs we have constructed to answer Question 4.18. In Proposition 4.27 we show that the construction produces a class 1 graph if and only if G and A are both class 1. We begin with a lemma which is used in the proof of

Proposition 4.27.

Lemma 4.26. *If H is a cubic class 2 graph, then any 4-edge colouring of H contains at least two edges of each colour.*

Proof. Consider a 4-edge colouring of H in which uv is the only edge coloured 4, and consider $H - u$, which is 3-edge colourable. Let v, x, y be the vertices of $H - u$ of degree 2. Since H is class 2, in any 3-edge colouring of $H - u$, two of v, x, y are incident with edges with the same set of two colours (otherwise H could be 3-edge coloured). Suppose without loss of generality that x and y are incident with edges coloured 1 and 2. Then in $(H - u) + xy$, the edge xy (which may be a multiple edge) can be coloured with colour 3. Therefore $(H - u) + xy$ is 3-edge colourable. On the other hand, $(H - u) + xy$ has odd order, say $2r + 1$, and $3r + 1$ edges. Since $3r + 1 > 3 \left\lfloor \frac{2r+1}{2} \right\rfloor$, $(H - u) + xy$ is in fact not 3-edge colourable, which is a contradiction. \square

Proposition 4.27. *For any cubic graphs G and A , the graph $G \circ A^-$ is class 1 if and only if G and A are class 1.*

Proof. If A is class 2, then A^- is class 2, by Lemma 4.26, and therefore $G \circ A^-$ is class 2. Hence assume A is class 1. Say $A^- = A - a$ and let a_1, a_2, a_3 be the vertices of A adjacent to a . Arguing as in the proof of Lemma 4.26, we see that in any 3-edge colouring of A^- , a_1, a_2 and a_3 are incident with edges coloured with three different pairs of colours.

Assume G is class 1 and consider any 3-edge colourings of G and A^- in the same colours. Colouring the edges $A_u^- A_v^-$ of $G \circ A^-$ the same colour as uv in G and suitably permuting the colours in the copies of A^- produces a 3-edge colouring of $G \circ A^-$.

Now assume G is class 2 and suppose for a contradiction that $G \circ A^-$ has a 3-edge colouring. For any copy A_v^- of A^- , let xa_1, ya_2 and za_3 be the three edges that join A_v^- to the rest of $G \circ A^-$. Since a_1, a_2 and a_3 are incident with edges coloured with three different pairs of colours, xa_1, ya_2 and za_3 have three different colours. Contracting each copy of A^- to a single vertex yields G as well as a 3-edge colouring of G , which is a contradiction. \square

Theorem 4.28. *For any $g \geq 3$ and $m \in \mathbb{N}$, there exists a 3-connected cubic graph $H = G \circ A^-$ of girth at least g such that $c_2(H) - \left\lceil \frac{|V(H)|+2}{4} \right\rceil \geq m$. Moreover, H is class 1 if and only if G and A are class 1.*

Proof. Proposition 4.25 guarantees the existence of a 3-connected cubic graph of order $4r$ and girth at least g . Let A be such a graph and let G be any 3-connected cubic graph of order at least $4m+2$. Then by Propositions 4.23 and 4.24, $H = G \circ A^-$ is a 3-connected cubic graph of girth at least g , and by Proposition 4.22, $c_2(H)$ exceeds the lower bound (4.1) by at least m . The chromatic index of H is given by Proposition 4.27. \square

Any class 2, girth $g \geq 4$ graph $G \circ A^-$ produced by our construction is a Gardner snark. For example, taking A to be the flower snark J_5 (pictured in Figure 4.5), a Gardner snark of order 20 and girth 5, and any 3-connected cubic graph G , $G \circ A^-$ is class 2 (by Proposition 4.27), 3-connected and has girth at least 5. Therefore it is a Gardner snark (in fact it satisfies a more restricted definition of snarks, since it has girth greater than 4 and connectivity greater than 2).

We now turn our attention to Question 4.19. Consider a 3-connected cubic graph G of order n and a triangle-free 3-connected cubic graph A of order $4r$, as required for our construction of the graph $G \circ A^-$. In Lemma 4.21 we showed that any minimum 2-conversion set of $G \circ A^-$ contains at least r vertices from each copy of A^- . Therefore $\frac{c_2(G \circ A^-)}{|V(G \circ A^-)|} \geq \frac{rn}{(4r-1)n} = \frac{r}{4r-1} > \frac{1}{4}$. For example, taking A to be the graph shown in Figure 4.8(a), and G any 3-connected cubic graph, $G \circ A^-$ has $\frac{c_2(G \circ A^-)}{|V(G \circ A^-)|} = \frac{3}{11}$.

In fact, it follows from the proof of Lemma 4.21 that any 2-conversion set of $G \circ A^-$ contains at least $c_2(A) - 1$ vertices from every copy of A^- , with $c_2(A) \geq r + 1$ by (4.1). Therefore, if $c_2(A) = r + 1 + s$, $s \geq 0$, then every 2-conversion set of $G \circ A^-$ contains at least $r + s$ vertices from each copy of A^- . Therefore $\frac{c_2(G \circ A^-)}{|V(G \circ A^-)|} = \frac{r+s}{4r-1}$. That is, by choosing A to be a cubic graph of order $4r$ that does not meet the lower bound (4.1), we can increase the ratio $\frac{c_2(G \circ A^-)}{|V(G \circ A^-)|}$.

Choosing smaller values of r also increases the ratio. For example, if A is a cubic graph of order 8, then $c_2(A) = 3$ (all cubic graphs of order 8 meet the lower bound (4.1)) and for any cubic graph G , any 2-conversion set of $G \circ A^-$ contains at least two vertices from each copy of A^- . Then $\frac{c_2(G \circ A^-)}{|V(G \circ A^-)|} = \frac{2}{7}$. Examples of 3-connected cubic graphs of order 8 with girth 4—suitable choices for A in the construction of triangle-free 3-connected cubic graphs with ratio $\frac{2}{7}$ —are shown in Figure 2.8.

Together, Proposition 3.7 (specifically, equation (4.1)) and Theorem 2.29 bound

the value of $c_2(G)$ between $\lceil \frac{n+2}{4} \rceil$ and $\lfloor \frac{3n+2}{8} \rfloor$ for cubic graphs G of order $n > 4$. Therefore the ratio $\frac{c_2(G)}{|V(G)|}$ cannot exceed $\frac{1}{3}$ for any triangle-free cubic graph. It also follows from Theorem 2.29 that this ratio is bounded asymptotically by $\frac{3}{8}$ for all cubic graphs, and that the asymptotic bound is attained by the infinite family \mathcal{G} , defined on page 29. The graphs of \mathcal{G} all have girth 3, so the following question remains open.

Question 4.29. What is the largest ratio $\frac{c_2(H)}{|V(H)|}$ achievable by an infinite family of 3-connected triangle-free cubic graphs H ?

Chapter 5

A lower bound on $c_k(G)$ for graphs of maximum degree $k + 1$

In this chapter we use the lower bound of Proposition 3.7, which applies to k -conversion in $(k + 1)$ -regular graphs, to derive lower bounds on the k -conversion number of graphs of maximum degree $k + 1$. (In general, the graphs considered in this chapter contain cycles; we study the k -conversion number of trees in Chapter 6.) We will return to the study of k -conversion in $(k + 1)$ -regular graphs in Chapter 7, where we focus on the structure of minimum k -conversion sets rather than their size. We show that certain subgraphs can always be avoided in such sets.

5.1 Definitions and preparation

In this section we lay the groundwork for our lower bound on the k -conversion number of graphs of maximum degree $k + 1$. We introduce necessary definitions, prove several lemmas and useful results, and discuss the strategy we will use to obtain the bound.

In Proposition 3.7 we presented a lower bound on the k -conversion number of a $(k + 1)$ -regular graph. To extend this lower bound to graphs G of maximum degree $k + 1$, we wish to construct a $(k + 1)$ -regular graph G' whose k -conversion number is no more than the k -conversion number of G . We then use the bound on $c_k(G')$ given by Proposition 3.7 to deduce a bound on $c_k(G)$. Recall that in Definition 3.4 we defined

the Δ -deficiency of a vertex v to be $\Delta - \deg(v)$. We now define the Δ -deficiency of a graph.

Definition 5.1. *Let G be a graph of maximum degree at most Δ . We define the Δ -deficiency of G by $\text{def}_\Delta(G) = \sum_{v \in V(G)} \text{def}_\Delta(v)$.*

We note that for a graph G with n vertices, m edges and maximum degree Δ ,

$$\text{def}_\Delta(G) = n\Delta - \sum_{v \in V(G)} \deg(v) = n\Delta - 2m. \quad (5.1)$$

Clearly, the deficiency of G quantifies how close G is to being regular. Specifically, the Δ -deficiency of G is the number of additional incidences required to make every vertex of G have degree Δ . If our goal was simply to obtain a $(k+1)$ -regular graph from G , we would first try to do so by simply adding edges between existing vertices of G . However, we wish to obtain a $(k+1)$ -regular graph whose k -conversion number is no larger than the k -conversion number of G , so that the lower bound on the k -conversion number of the new (regular) graph will also give us a lower bound on the k -conversion number of G . Since adding edges to a graph may decrease its k -conversion number, we wish to obtain a $(k+1)$ -regular graph from G without adding new edges between the vertices of G . Therefore, we must add new vertices; in fact, we add a graph F to obtain a $(k+1)$ -regular graph of which G is an induced subgraph. Furthermore, in order to ensure that the new (regular) graph does not have a larger k -conversion number than G , we must construct the new graph in such a way that the vertices of F are guaranteed to convert once the vertices of G have converted. Lemma 5.2 provides necessary conditions on the structure of F .

Lemma 5.2. *Let G be a graph of order n_G and maximum degree $k+1$. Suppose there exists a graph F such that it is possible to obtain a simple $(k+1)$ -regular graph $G \oplus F$ by adding edges between vertices of G and vertices of F , with no new edges between vertices of G or between vertices of F . Then*

- (a) $\text{def}_{k+1}(F) = \text{def}_{k+1}(G)$, and
- (b) every k -conversion set of G is a k -conversion set of $G \oplus F$ if and only if F is a forest.

Proof. The first part is clear from the definition of deficiency. For the second part, we note that every k -conversion set of G is a k -conversion set of $G \oplus F$ if and only if $V(G)$ is a k -conversion set of $G \oplus F$. The result then follows from Corollary 2.23, with $S = V(G)$. \square

Proposition 5.2 does not guarantee that it is possible to construct a simple $(k+1)$ -regular graph from G whose k -conversion number is at most the k -conversion number of G . However, it tells us that *if* it is possible, then the graph F that we add to G must be a forest. From now on we use the notation $G \oplus H$ and rG as follows.

Definition 5.3. For graphs G and H , we denote by $G \oplus H$ any graph constructed from G and H by adding edges between G and H , with no new edges between vertices of G or between vertices of H . We denote the disjoint union of r copies of G by rG .

We note that the graph $G \oplus H$ is not necessarily unique for a given G and H . In light of Lemma 5.2, we would like to know when there exists a simple, $(k+1)$ -regular graph $G \oplus F$ such that F is a forest.

Figure 5.1 displays a graph G of maximum degree 4 from which it is impossible to construct a simple 4-regular graph by adding a forest F . The vertex deficiencies are 0, 1, 1, 1 and 3, as shown in the figure, giving G a 4-deficiency of 6. The vertex with deficiency 3 must be adjacent to three distinct vertices of F ; however, the only forest with 4-deficiency 6 is K_2 .

We note that nr is the degree sum of an r -regular graph on n vertices, and a forest F of order n and maximum degree $\Delta \leq r$ with y components has degree sum $2(n - y)$. Therefore F has r -deficiency $nr - 2(n - y)$.

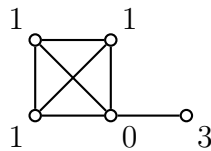


Figure 5.1: A graph of maximum degree 4, with vertex deficiencies indicated, from which it is impossible to obtain a simple 4-regular graph by adding a forest.

For the next lemma, we note that a *linear forest* is a forest in which every component is a path.

Lemma 5.4. *If there exists a forest F such that $G \oplus F$ is simple and $(k+1)$ -regular, then there exists a linear forest F' which satisfies the same conditions.*

Proof. Any two trees of the same order have the same degree sum, and therefore the same Δ -deficiency (provided they both have maximum degree at most Δ , so that their Δ -deficiency is defined). Therefore if T is a tree of order n in F , we may replace it with P_n . \square

We can now use the bound of Proposition 3.7 (for the k -conversion number of $(k+1)$ -regular graphs), to get a lower bound on the k -conversion number of graphs with maximum degree $k+1$.

Proposition 5.5. *Let G be a graph of order n_G and maximum degree $k+1$, and suppose there exists a forest F such that it is possible to obtain a simple $(k+1)$ -regular graph $G \oplus F$ by adding edges between vertices of G and vertices of F (with no new edges between vertices of G or between vertices of F). Let $y \geq 1$ be the number of components of F . Then*

$$c_k(G) \geq \frac{n_G(k-1) + \text{def}(G) - 2y + 2}{2k}.$$

Proof. Since $G \oplus F$ is $(k+1)$ -regular, Proposition 3.7 gives

$$c_k(G \oplus F) \geq \frac{(n_G + n_F)(k-1) + 2}{2k}.$$

Since F is a forest, Lemma 5.2(5.2) guarantees that a minimum k -conversion set of G is a k -conversion set of $G \oplus F$, so $c_k(G) \geq c_k(G \oplus F)$. Therefore we have

$$c_k(G) \geq \frac{(n_F + n_G)(k-1) + 2}{2k}. \quad (5.2)$$

Let m_F be the number of edges in F . By Lemma 5.2 (5.2), $\text{def}_{k+1}(G) = \text{def}_{k+1}(F)$, where

$$\begin{aligned} \text{def}_{k+1}(F) &= n_F(k+1) - 2m_F \\ &= n_F(k+1) - 2(n_F - y) \\ &= n_F(k-1) + 2y. \end{aligned}$$

Therefore $n_F = \frac{\text{def}(G)-2y}{k-1}$. Substituting this into (5.2) gives the result. \square

The bound of Proposition 5.5 is better when y , the number of components of F , is smaller. This naturally leads to the question, “when can we take F to be a single tree?” If such a tree exists, we get the bound

$$c_k(G) \geq \frac{n_G(k-1) + \text{def}(G)}{2k}. \quad (5.3)$$

In fact, by Lemma 5.4, we can always take F to be a linear forest, and if F is a single tree then we can assume it is a path.

In Proposition 5.6 we show that for graphs of maximum degree 3 (ie. $k = 2$), we can always obtain a simple cubic graph by adding a single path, unless G has very low deficiency.

Proposition 5.6. *Let G be a connected graph of maximum degree at most 3 and 3-deficiency at least 4. Then it is possible to add edges between G and $P_{\text{def}(G)-2}$ such that the resulting graph is simple and cubic.*

Proof. Let $d \geq 4$ be the 3-deficiency of G . Clearly $P = P_{d-2}$ has 3-deficiency d as well, so it is possible to obtain a cubic graph by adding d edges between G and P . We must show that this can be done without any parallel edges. If G has no vertices of degree 1 (that is, no vertices with deficiency 2), then none of the added edges are incident with the same vertex of G , so there are no parallel edges. If G has one vertex x of degree 1, then add the edges xu and xv , where u and v are the leaves of P . Clearly G and P still have the same deficiency (now $d-2$), and neither G nor P has any remaining vertices of deficiency 2. This means G and P both have $d-2$ vertices of deficiency 1; we obtain a simple cubic graph by adding a perfect matching between these vertices. If G has two vertices x and y of deficiency 2, then add edges xu, xv, yu, yv . Again, G and P still have the same deficiency (now $d-4$), so it is possible to add $d-4$ more edges between G and P to get a cubic graph. Since P has no remaining vertices of deficiency 2, none of these additional edges are incident with the same vertex of P , so no two are parallel. \square

Proposition 5.6 establishes that the bound (5.3) holds for graphs G of maximum

degree at most 3 and deficiency at least 4 by showing that there is a simple cubic graph $G \oplus P$, where P is a path. In Corollary 5.7 we show that in fact (5.3) holds for all graphs G of maximum degree at most 3. For the low deficiency cases (which aren't covered by Proposition 5.6), we construct a simple cubic graph G' from multiple copies of G and a path P . We will use this technique again in Proposition 5.9 to construct simple 4-regular graphs, in order to prove the bound (5.3) for $k = 3$. We note that, while the bound of Corollary 5.7 holds for cubic graphs (ie. $\text{def}(G) = 0$), it is weaker than the bound previously given in Proposition 3.7 because Proposition 5.5, which yields (5.3), assumes that $y \geq 1$, or in other words, that G is not regular.

Corollary 5.7. *Let G be a graph of order n and maximum degree 3. Then*

$$c_2(G) \geq \frac{n + \text{def}(G)}{4}.$$

Proof. If G has 3-deficiency $d \geq 4$ then by Proposition 5.6, there is a cubic graph $G \oplus P_{d-2}$ such that $c_2(G) \geq c_2(G \oplus P_{d-2})$. Therefore by the lower bound of Proposition 3.7, $c_2(G) \geq \frac{(n+d-2)+2}{4}$, as desired.

It remains to show that the bound holds when $d \leq 3$. For these cases, we construct a graph G' from some number r of copies of G and the path $P = P_{rd-2}$, which has the same deficiency as r copies of G . We add rd edges between P and the copies of G (and no other edges). By Lemma 5.2 (b), P converts once all vertices in all copies of G have converted, so $rc_2(G) \geq c_2(G')$. Then by Proposition 3.7, $c_2(G) \geq \frac{rn+(rd-2)+2}{4r} = \frac{n+d}{4}$.

If $d = 1$ then G has one vertex of degree 2 and all other vertices have degree 3. Take $r = 3$ copies of G , so $P = P_1$.

If $d = 2$ then G has either one vertex of degree 1 or two vertices of degree 2, and all other vertices have degree 3. In the first case, construct the cubic graph G' from $r = 3$ copies of G and the path P_4 . In the second case, where G has two vertices of degree 2, use $r = 2$ copies of G and the path P_2 .

If $d = 3$ then G has either one vertex of degree 1 and one of degree 2, or three vertices of degree 2, and all other vertices have degree 3. In the first case, take $r = 2$ copies of G and the path P_4 . In the second case, $r = 1$ copy of G suffices, so $P = P_1$. \square

For graphs of maximum degree $\Delta = 3$ and 3-deficiency $d \geq 3$, there is always a path

with 3-deficiency d , namely P_{d-2} . In general, the path with s vertices has Δ -deficiency

$$\text{def}_\Delta(P_s) = s(\Delta - 2) + 2. \quad (5.4)$$

Given a graph G with maximum degree Δ and Δ -deficiency d , we wish to find a path with the same Δ -deficiency. However, for a given Δ and d , (5.4) gives $s = \frac{d-2}{\Delta-2}$, which is not necessarily an integer. Therefore, in many cases, there is no path P such that $G \oplus P$ is simple and Δ -regular. As suggested by the proof of Corollary 5.7, we may overcome this obstacle by constructing a regular graph with r disjoint copies of G and an appropriate path P_s . The disjoint union of r copies of G has deficiency rd , so we wish to find integers r and s such that

$$rd = (\Delta - 2)s + 2. \quad (5.5)$$

However, for $\Delta \geq 5$, Equation (5.5) reveals that it is not always possible to find such integers r and s . To see this, suppose d is divisible by $\Delta - 2$, and consider Equation (5.5) modulo $(\Delta - 2)$: we get $0 \equiv 2 \pmod{\Delta - 2}$, which is only possible when $\Delta = 3$ or 4 . (The case $\Delta = 3$ has already been addressed in Corollary 5.7, and the case $\Delta = 4$ will be addressed in Proposition 5.9.) For larger values of Δ there is always a graph of maximum degree Δ for which it is impossible to construct a Δ -regular graph from disjoint copies of G and a path P by adding edges between P and the copies of G . For example, let G be a graph whose vertices all have degree Δ or $\Delta - 2$, with at least one vertex of each type. For this graph, as observed above, there are no integers r and s satisfying (5.5).

In Proposition 5.9 we prove that the bound (5.3) holds for graphs of maximum degree $\Delta = 4$ (i.e. $k = 3$). The proof of Proposition 5.9 uses the following lemma. For a graph G of maximum degree Δ , we let $d_{\max}(G)$ denote the largest Δ -deficiency of any vertex in G .

Lemma 5.8. *Let G be a graph of maximum degree Δ . If P is a path with at least $d_{\max}(G)$ vertices and $\text{def}_\Delta(P) = \text{def}_\Delta(G)$ then there exists a simple Δ -regular graph $G \oplus P$.*

Proof. Let $P = P_s$. Since $\text{def}_\Delta(G) = \text{def}_\Delta(P)$, we can add edges between G and P until all vertices in G and P have degree Δ . We claim that if $s \geq d_{\max}$ then it is

possible to do this while avoiding parallel edges. Before we add any edges between P and G , the deficiencies of vertices in P are $\Delta - 1$ and $\Delta - 2$, so they are all within 1 of each other. Let v be a deficient vertex of G . Join v to $\text{def}_\Delta(v) \leq d_{\max} \leq s$ vertices of P , starting with those of highest deficiency (lowest degree). Repeat this process for each deficient vertex of G , always using the highest deficiency vertices of P first. Eventually, all deficient vertices of P will have deficiency 1, at which point it is easy to see that the remaining edges will all be incident with different vertices of P . \square

Proposition 5.9. *Let G be a graph of order n and maximum degree 4. Then*

(a) *there are integers r and s such that it is possible to construct a simple 4-regular graph $G' = rG \oplus P_s$, or it is possible to construct a simple 4-regular graph by adding edges between two copies of G , and*

(b) $c_3(G) \geq \frac{2n + \text{def}_4(G)}{6}$.

Proof. Take $r = 2$ and let $d = \text{def}_4(G)$. We have $\text{def}_4(2G) = 2d$ and $d_{\max}(2G) = d_{\max}(G)$. The path with deficiency $2d$ has $s = \frac{2d-2}{\Delta-2} = d-1$ vertices. Suppose first that G has more than one deficient vertex, so that $d_{\max} \leq d-1$. It follows from Lemma 5.8 that it is possible to construct a simple 4-regular graph $G' = 2G \oplus P_{d-1}$. This proves (a) for all cases where G has at least two deficient vertices. For these cases, we have

$$2c_3(G) \geq c_3(G') \geq \frac{n_{G'}(k-1) + 2}{2k} = \frac{(2n + d - 1)2 + 2}{6}.$$

Therefore

$$c_3(G) \geq \frac{4n + 2d}{3 \cdot 4} = \frac{2n + d}{6} = \frac{n(k-1) + \text{def}(G)}{2k},$$

which is the bound (5.3). This proves (b).

We now consider the case where the total deficiency of G is concentrated in a single vertex. In that case $\text{def}_4(G) = 1, 2$ or 3 . If $\text{def}_4(G) = 1$, construct G' from $r = 2$ copies of G by adding an edge between the degree 3 vertices. If $\text{def}_4(G) = 2$, construct G' from $r = 3$ copies of G and a P_2 . If $\text{def}_4(G) = 3$, construct G' from $r = 4$ copies of G and a P_5 . In all cases, it is easy to verify that

$$c_3(G) \geq \frac{1}{r} \frac{2n_{G'} + 2}{6} = \frac{2n_G + \text{def}_4(G)}{6},$$

and this completes the proof. \square

For $\Delta \geq 5$ we have shown that, given a graph G of maximum degree Δ and deficiency d , it may not be possible to construct a simple Δ -regular graph $rG \oplus P$ from r disjoint copies of G and some path P . However, the bound (5.3) still holds for $k \geq 5$. We just need to modify our proof technique.

In our previous strategy, we aimed to find a suitable tree T from which to construct a simple $(k+1)$ -regular graph $G \oplus T$. The advantage of that strategy was that a k -conversion set of G was also a conversion set of $G \oplus T$, so $c_k(G) \geq c_k(G \oplus T)$. A disadvantage, as we have seen, is that not all deficiencies are realizable by a tree, or even by a forest. Even when we take multiple copies of G , there is not necessarily a tree with the same total deficiency. Another irritation is that trees are not regular, so their vertices don't have uniform deficiency. This makes it difficult, or at least annoying, to come up with a general construction for adding edges between G (or copies of G) and T such that $G \oplus T$ is simple and $(k+1)$ -regular. We can overcome both of these obstacles by taking, say, r copies of G and adding a cycle C instead of a tree to form a simple $(k+1)$ -regular graph $G' = rG \oplus C$. The tradeoff is that a conversion set of G (copied r times) no longer converts G' . However, adding one additional seed vertex on C fixes this problem, so we have

$$c_k(G') \leq rc_k(G) + 1. \quad (5.6)$$

Suppose that $\text{def}_\Delta(G) = d$ and note that r disjoint copies of G have a total Δ -deficiency of rd . The cycle C_s has Δ -deficiency $(\Delta - 2)s$. Therefore, to determine how many copies of G we should use and which cycle we should add to them to construct G' , we wish to find integers r and s such that $rd = (\Delta - 2)s$. An obvious choice is $r = \Delta - 2$ and $s = d$. This works as long as $d \geq 3$; for now we will assume this is the case.

To construct a simple Δ -regular graph $G' = (\Delta - 2)G \oplus C_d$, let v be a vertex of C_d (so $\text{def}_\Delta(v) = \Delta - 2$). Choose a deficient vertex u of G and join v to each copy of u . The deficiency of v becomes 0 and the deficiency of each copy of u decreases by 1. Repeat this process until there are no more deficient vertices. Since no vertex of C_d gets joined to two vertices in the same copy of G , there are no parallel edges.

Figure 5.2 illustrates the result of this construction for a graph G of maximum degree $\Delta = 5$ with vertex deficiencies 1, 1 and 3 (for a total of $d = 5$).

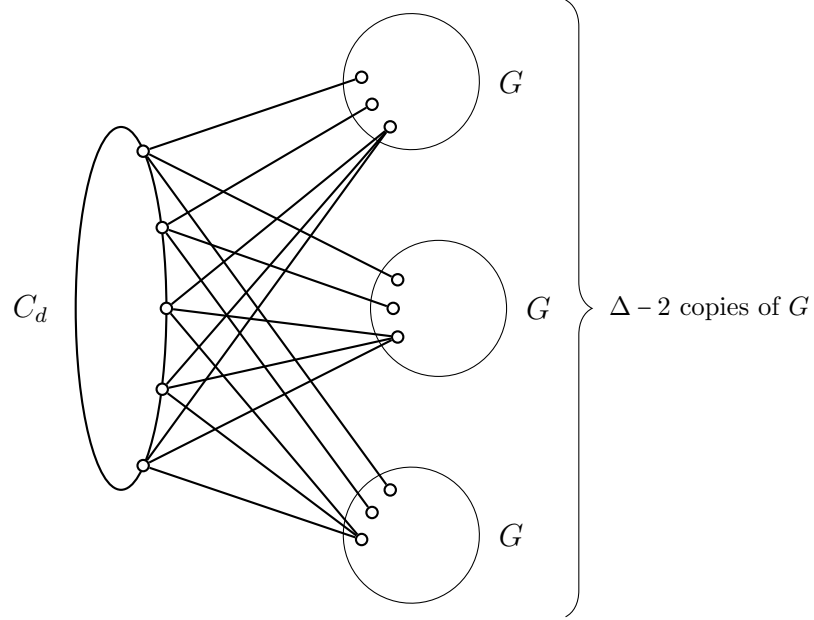


Figure 5.2: Illustration of the construction of a Δ -regular graph $(\Delta - 2)G \oplus C_d$ from a graph G with Δ -deficiency d .

Setting $r = \Delta - 2$ in (5.6) and rearranging, we see that

$$c_k(G) \geq \frac{c_k(G') - 1}{\Delta - 2}. \quad (5.7)$$

Since G' is a regular graph, the bound of Proposition 3.7 applies (with $k + 1 = \Delta$). That is,

$$c_k(G') \geq \frac{n_{G'}(k - 1) + 2}{2k}. \quad (5.8)$$

Combining (5.7) and (5.8) gives

$$c_k(G) \geq \frac{n_{G'}(k - 1) - 2k}{2k(\Delta - 2)}. \quad (5.9)$$

Substituting $n_{G'} = (\Delta - 2)n_G + d$ and $\Delta - 2 = k - 1$ into (5.9) then gives the following bound on $c_k(G)$ where G is a graph of order n_G , maximum degree $k + 1$ and

deficiency $d \geq 3$:

$$\begin{aligned} c_k(G) &\geq \frac{(n_G(k-1) + d)(k-1) - 2k}{2k(k-1)} \\ &= \frac{n_G(k-1) + d}{2k} - \frac{1}{k-1}. \end{aligned} \tag{5.10}$$

The bound of (5.10) is similar to the bound of (5.3), which we were aiming for, but not identical. First, (5.10) only applies when $d \geq 3$. Second, only the first term of (5.10) is present in the bound of (5.3). The second term of (5.10) represents the collected effects of the vertex v of C_d , which we had to add to the conversion set of G' . The addition of v to the conversion set means that $c_k(G')$ is one off from a multiple of $c_k(G)$. This observation suggests a possible strategy for eliminating this “error term” from the bound: by using more copies of G and a larger cycle, we can reduce v 's impact on $c_k(G')$, making $c_k(G')$ closer to a multiple of $c_k(G)$. Happily, using a larger cycle than C_d also sidesteps the problem with small deficiencies, allowing us to eliminate the condition on d . In Section 5.2, we use this “scaled up” version of our previous strategy to prove Theorem 5.10, which establishes the bound (5.3) for all $k \geq 2$ and all deficiencies.

5.2 The lower bound

Theorem 5.10. *Let G be a graph of order n and maximum degree $\Delta = k + 1$, with $k \geq 2$. Then*

$$c_k(G) \geq \frac{n(k-1) + \text{def}_\Delta(G)}{2k}. \tag{5.11}$$

Proof. Let $\text{def}_\Delta(G) = d$ and let N be any integer satisfying $Nd \geq 3$. We construct a Δ -regular graph G' from $N(\Delta - 2)$ disjoint copies of G and a cycle C_{Nd} . The graphs $N(\Delta - 2)G$ and C_{Nd} both have Δ -deficiency $N(\Delta - 2)d$. We construct the graph $G' = N(\Delta - 2)G \oplus C_{Nd}$ by adding edges between the copies of G and the cycle C_{Nd} as follows. (The construction is illustrated in Figure 5.3.)

Partition the vertices of C_{Nd} into N intervals P^1, \dots, P^N of d vertices each, and arrange the $N(\Delta - 2)$ copies of G into N groups S_1, \dots, S_N of $\Delta - 2$. For each $i = 1, \dots, N$, let v be a vertex in P^i (so $\text{def}_\Delta(v) = \Delta - 2$). Choose a deficient vertex u

of G , and join v to the copy of u in each of the $\Delta - 2$ copies of G in S_i . The deficiency of v becomes 0 and the deficiency of each copy of u in S_i decreases by one. Repeat this process until there are no more deficient vertices in P^i (and, automatically, there will be no more deficient vertices in S_i either). Since no vertex of P_i gets joined to two vertices in the same copy of G , there are no parallel edges.

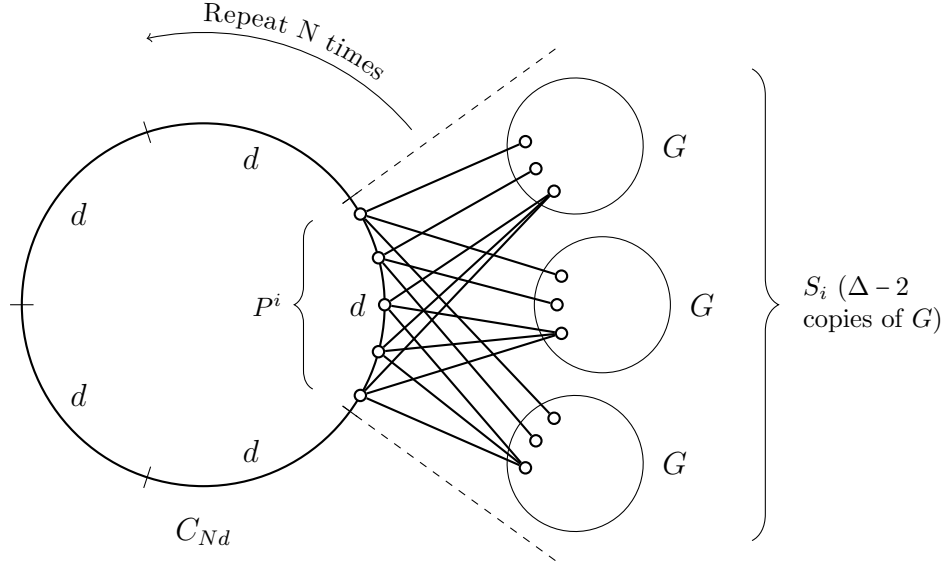


Figure 5.3: The construction of the Δ -regular graph $G' = N(\Delta - 2)G \oplus C_{Nd}$ in the proof of Theorem 5.10.

Let S be a minimum k -conversion set of G . Then the $N(\Delta - 2)$ copies of S (one for each copy of G in G'), together with one vertex v from the cycle C_{Nd} , form a k -conversion set of G' , so $c_k(G') \leq N(\Delta - 2)c_k(G) + 1$. This gives

$$c_k(G) \geq \frac{c_k(G') - 1}{N(\Delta - 2)}. \quad (5.12)$$

Since G' is a regular graph, the bound of Proposition 3.7 applies, with $k + 1 = \Delta$. That is,

$$c_k(G') \geq \frac{n_{G'}(k - 1) + 2}{2k}. \quad (5.13)$$

Substituting (5.13) into (5.12) (and dropping a constant term) gives

$$c_k(G) \geq \frac{n_{G'}(k - 1) - 2k}{2kN(\Delta - 2)}, \quad (5.14)$$

and substituting $n_{G'} = N(\Delta - 2)n_G + Nd$ and $\Delta - 2 = k - 1$ into (5.14) then gives

$$\begin{aligned} c_k(G) &\geq \frac{(N(k-1)n_G + Nd)(k-1) - 2k}{2k(k-1)N} \\ &= \frac{n_G(k-1) + d}{2k} - \frac{1}{(k-1)N}. \end{aligned} \quad (5.15)$$

The bound 5.15 holds for any $N \in \mathbb{N}$ sufficiently large such that $Nd \geq 3$. Taking the limit as N approaches infinity gives the desired bound. \square

The bound (5.11) of Theorem 5.10 can be rewritten in terms of the number of edges in G , since

$$\text{def}_\Delta(G) = \sum_{v \in V(G)} (\Delta - \deg(v)) = n\Delta - 2|E(G)| = n(k+1) - 2|E(G)|.$$

This gives the bound

$$c_k(G) \geq n - \frac{|E(G)|}{k}$$

for $k \geq 2$.

In Section 5.2.1 we prove that the bound of Theorem 5.10 is sharp by presenting an infinite family of graphs of maximum degree $k+1$ for which $c_k(G) = \frac{n(k-1) + \text{def}_\Delta(G)}{2k}$. Then, in Section 5.2.2 we present infinite families of graphs whose true k -conversion numbers are much larger than the lower bound.

5.2.1 Graphs for which the lower bound is sharp

A *perfect k -ary tree* with $\ell + 1$ levels (labelled 0 to ℓ) is a rooted tree in which every internal vertex has exactly k children and all leaves are on level ℓ . For $k \geq 2$, every perfect k -ary tree with at least three levels meets the lower bound of Theorem 5.10 for graphs of maximum degree $k+1$. Such a tree T has $\frac{k^{\ell+1}-1}{k-1}$ vertices, of which k^ℓ are leaves. The leaves form a k -conversion set, so $c_k(T) \leq k^\ell$. On the other hand, the $(k+1)$ -deficiency is $k(k^\ell) + 1$, so the lower bound from Theorem 5.10 is $c_k(T) \geq \frac{\frac{k^{\ell+1}-1}{k-1}(k+1) + k(k^\ell) + 1}{2k} = \frac{2k^{\ell+1}}{2k} = k^\ell$.

5.2.2 Graphs of maximum degree $k+1$ with large k -conversion number

For $k = 2$ there is an infinite family of subcubic graphs for which $c_2(G)$ is far from the bound $\frac{n+\text{def}_3(G)}{4}$. In Section 2.3.2 we defined the set \mathcal{G} to be the set of cubic graphs obtained from a cubic tree¹ by replacing each internal vertex with a K_3 and each leaf with a copy of the graph H obtained by subdividing an edge of K_4 . Let $G \in \mathcal{G}$ and let G' be the subcubic graph obtained from G by deleting from one of the copies of H an edge incident with two vertices whose degree in H is 3. (An example of such a graph is depicted in Figure 5.4, with a minimum 2-conversion set shown in black.) The 3-deficiency of G' is 2, so by Proposition 5.7, the lower bound on the 2-conversion number of G' is $c_2(G') \geq \frac{n+2}{4}$. However, the true 2-conversion number of G' is approximately 3/2 of this number: $c_2(G') = c_2(G) = \frac{3n+2}{8}$, where the last equality is given by Theorem 2.29.

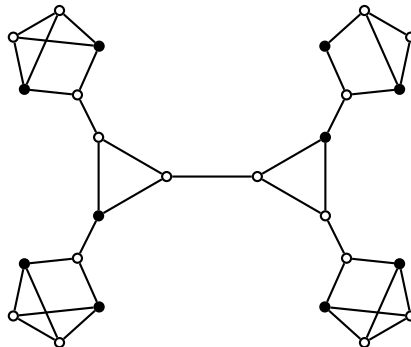


Figure 5.4: A subcubic graph for which $c_2(G)$ exceeds the lower bound.

Below we present a construction that yields infinite families of graphs for which the true k -conversion number is much larger than the lower bound given by Theorem 5.10, for $k \geq 3$.

Proposition 5.11. *Let T be a tree with r internal vertices and ℓ leaves, and suppose that every internal vertex of T has degree $k+1 \geq 4$. Let H be K_{k+2} with an edge subdivided. Let G be the graph obtained by replacing each internal vertex of T with a copy of K_{k+1} and each leaf v of T with a copy of H , identifying v with the degree 2 vertex of H . Then*

¹Recall from Section 2.3.2 that a cubic tree is a tree whose internal vertices all have degree 3.

- (a) G has maximum degree $k + 1$ and $(k + 1)$ -deficiency $(k - 2)\ell$,
- (b) $c_k(G) = (k - 1)r + k\ell$, and
- (c) the difference between $c_k(G)$ and the lower bound given by Theorem 5.10 is $\frac{k^2-1}{2k}(\ell + r) + \frac{3}{2k}\ell$.

Proof. For a given copy of H , let uv be the edge that is subdivided, and let w be the vertex of degree 2 in H . Every vertex of G has degree $k + 1$ except for the vertices w , which have degree 3, and hence $(k + 1)$ -deficiency $k - 2$. This proves (a).

For (b), let S be a minimum k -conversion set of G . We claim that, for each copy of H , $|V(H) \cap \overline{S}| \leq 3$. To see this, let $X = V(H) - \{u, v, w\}$ and note that any three vertices of X form a triangle, so \overline{S} contains at most two vertices of X . Any two vertices of X together with either u or v form a cycle, and any one vertex of X together with $\{u, v, w\}$ forms a cycle. This proves the claim, and therefore $V(H) \cap S \geq k$. In K_{k+1} any three vertices form a cycle, so $|S \cap V(K_{k+1})| \geq k - 1$. Finally, we note that the set containing every copy of X and any $k - 1$ vertices from each copy of K_{k+1} is a k -conversion set of G , so $c_k(G) = (k - 1)r + k\ell$, as desired.

Finally, for (c),

$$\begin{aligned} c_k(G) - \frac{n(k-1) + \text{def}_{k+1}(G)}{2k} &= (k-1)r + k\ell - \frac{(k-1)((k-1)r + (k+3)\ell) + (k-2)\ell}{2k} \\ &= \frac{k^2-1}{2k}(\ell+r) + \frac{3}{2k}\ell, \end{aligned}$$

as claimed. □

For the graphs G described in Proposition 5.11, the difference between $c_k(G)$ and the lower bound given by Theorem 5.10 is positive for all $k \geq 2$ and increases without bound as the order of T (the tree from which G is constructed) increases. (We note that r and ℓ are related by the equation

$$(k+1)r + \ell = 2(r + \ell - 1),$$

which is given by the degree sum formula.)

Chapter 6

k -conversion in trees

In this Chapter we determine the exact values of $c_k(G)$ for caterpillars (Section 6.1), spiders and double spiders (Section 6.2). We begin with a characterization of minimal k -immune sets in trees.

Proposition 2.6 asserts that a connected barely k -immune set in a graph G is a minimal k -immune set. In Proposition 6.1 we prove that a minimal k -immune set of a tree is either a single vertex of low degree (which we consider to be a trivial k -immune set), or a connected barely k -immune set. That is, the converse of Proposition 2.6 holds for nontrivial k -immune sets in trees. From this, we characterize the nontrivial minimal k -immune sets in trees (Corollary 6.2).

Proposition 6.1. *Let T be a tree and let U be a minimal k -immune set of T . Then $U = \{v\}$ where $\deg v < k$, or U is a barely k -immune set of size at least 2 and $T[U]$ is connected.*

Proof. Let U be a minimal k -immune set of T of size at least two. Clearly, $T[U]$ is connected. To see that U is a barely k -immune set, suppose for a contradiction that U contains a vertex y that has fewer than $k - 1$ neighbours in \overline{U} . Since T is a tree, $T[U - \{y\}]$ is disconnected. Let W be a component of $T[U - \{y\}]$ and let $X = U - W$. We claim that X is a k -immune set. Indeed, for each vertex $v \neq y$ in X , $|N(v) \cap \overline{X}| = |N(v) \cap \overline{U}| \leq k - 1$, and $|N(y) \cap \overline{X}| = |N(y) \cap \overline{U}| + 1 \leq k - 1$. This contradicts the minimality of U . Therefore every vertex of U has exactly $k - 1$ neighbours in \overline{U} , so U is barely k -immune. \square

Corollary 6.2. *Let T be a tree and let U be a nontrivial k -immune set of T . Then U is minimal k -immune if and only if it is barely k -immune and $T[U]$ is connected.*

6.1 Caterpillars

A *caterpillar* is a tree whose internal vertices form a path. By Proposition 6.1, the minimal k -immune sets of a caterpillar T come in two types, as described below.

Type 1: The sets $\{v\}$ where $\deg v < k$.

Type 2: The paths P consisting of internal vertices of T each of which has exactly $k - 1$ neighbours outside P . That is, the paths v_1, \dots, v_r of internal vertices of T such that $r \geq 2$, $\deg_T v_1 = \deg_T v_r = k$ and for $1 < i < r$, $\deg_T v_i = k + 1$.

Definition 6.3. *Let $k \geq 2$, let T be a caterpillar, and let \mathcal{K} be the set of all maximal subpaths P of T such that the endpoints of P (which may not be distinct) have degree k in T and the internal vertices of P have degree k or $k+1$ in T . We refer to the elements of \mathcal{K} as $(k, k+1)$ -paths, and we call a $(k, k+1)$ -path with an odd (even) number of degree k vertices an odd (even) $(k, k+1)$ -path.*

Note that $(k, k+1)$ -paths contain only internal vertices of T and the nontrivial $(k, k+1)$ -paths are the components of the subgraph of T induced by the union of the Type 2 minimal k -immune sets. Moreover, the degree k vertices of a non-trivial $(k, k+1)$ -path P are the endpoints of the individual minimal k -immune sets that make up P . The trivial (i.e. order 1) $(k, k+1)$ -paths are not k -immune sets, but it is convenient to include them in \mathcal{K} so that all degree k vertices are in a $(k, k+1)$ -path.

Figure 6.1 illustrates the minimal k -immune sets of type 2 and the maximal $(k, k+1)$ -paths (odd and even) of a caterpillar, for $k = 2$. The $(k, k+1)$ -path on the right is even, and the others are odd.

Proposition 6.4. *Let T be a caterpillar. For each $i \geq 1$, let n_i be the number of vertices of degree i in T and let s be the number of odd $(k, k+1)$ -paths of T , as defined in Definition 6.3. Then $c_k(T) = n_1 + \dots + n_{k-1} + \frac{n_k - s}{2}$.*

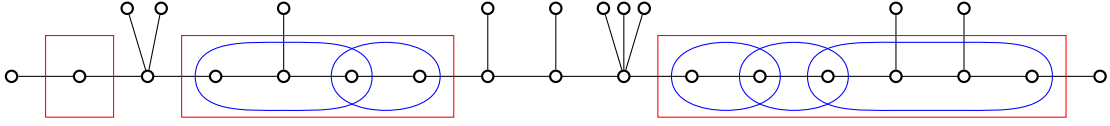


Figure 6.1: Type 2 minimal 2-immune sets (blue) and $(2,3)$ -paths (red) in a caterpillar.

Proof. We begin by constructing a conversion set of the given size. Let P_1, \dots, P_r be the $(k, k+1)$ -paths of T (which are disjoint, since they are maximal by definition), and for each P_i , let $v_{i1}, \dots, v_{i\ell}$ be the degree k vertices in P_i , in order. Let

$$S = \{v \in T : \deg(v) < k\} \cup \left(\bigcup_{i=1}^r \{v_{ij} \in P_i : j \text{ is even}\} \right),$$

so S contains all the vertices of degree less than k and the degree k vertices which have even index within their respective $(k, k+1)$ -paths. Therefore, if P_i is an odd $(k, k+1)$ -path then S does not contain the last degree k vertex in P_i , and contains half of the remaining degree k vertices in P_i . If P_i is an even $(k, k+1)$ -path then S contains half of the degree k vertices in P_i . This gives a total of $\frac{n_k - s}{2}$ degree k vertices in S .

To see that S is a minimum k -conversion set for T , we will show that S contains exactly one vertex from each minimal k -immune set of T . This is clear for the k -immune sets of type 1, which are all disjoint from each other and from the type 2 immune sets. To see that it is also true for the type 2 minimal k -immune sets of T , let P be such a set. Then P is a path with at least two vertices, the endpoints of P are vertices of degree k in T and the internal vertices of P have degree $k+1$ in T . Therefore P is contained in a k -immune path of T . Since no two consecutive degree k vertices in any k -immune path are omitted from S , S contains an endpoint of P . In fact, since no two consecutive degree k vertices are included in S , S contains exactly one endpoint of P . Finally, since S does not contain any degree $k+1$ vertices, S contains exactly one vertex from P . \square

The proof of Proposition 6.4 demonstrates that there's no advantage in including a vertex of degree greater than k in a k -conversion set of a caterpillar— they always have a minimum k -conversion set with no high-degree vertices. In the next section, we see that for some trees (specifically, some spiders and double spiders), there is no

minimum k -conversion set containing only vertices of degree at most k .

6.2 Spiders and double spiders

A *spider* is a tree with one vertex of degree at least 3 and all other vertices of degree 1 or 2. A *double spider* is a tree with two adjacent vertices of degree at least 3 and all other vertices of degree 1 or 2. A vertex of degree at least 3 in a spider or double spider is called a *head* vertex, and maximal paths induced by non-head vertices are called *legs*. We call a leg *even* if it has an even number of vertices, and *odd* otherwise.

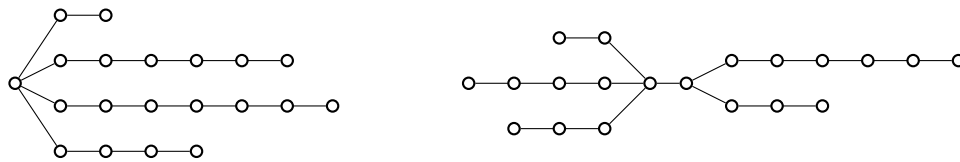


Figure 6.2: A spider and a double spider.

For $k \geq 3$, any k -conversion set of a spider or double spider must contain all leg vertices. Therefore the k -conversion number of a spider is $n - 1$ or n , depending on the degree of the head, and the k -conversion number of a double spider is between $n - 2$ and n . The following two propositions make these observations precise.

Proposition 6.5. *Let T be a spider with n vertices, and let $k \geq 3$. Then*

$$c_k(T) = \begin{cases} n - 1 & \text{if the head of } T \text{ has degree at least } k, \\ n & \text{otherwise.} \end{cases}$$

Proof. The seed set consists of all vertices of degree less than k . □

Proposition 6.6. *Let T be a double spider with n vertices, and let $k \geq 3$. Let q and r be the degrees of the head vertices, with $q \leq r$. Then*

$$c_k(T) = \begin{cases} n & \text{if } r < k, \\ n - 1 & \text{if } q < k \text{ and } r \geq k, \text{ or if } q = r = k, \\ n - 2 & \text{if } q \geq k \text{ and } r > k. \end{cases}$$

Proof. Let S be a minimum k -conversion set of T . Every leg vertex of T has degree less than k and therefore belongs to S . If both head vertices have degree less than k , they must be in S as well. If both head vertices have degree k , then each of them is adjacent to exactly $k - 1$ leg vertices; that is, the head vertices form a k -immune set of size 2. Therefore S contains one of the head vertices, so $|S| = n - 1$. If one head vertex has degree less than k then it must be in S ; in that case, if the other head vertex has degree at least k , it converts at $t = 1$. Finally, if one head vertex has degree at least k and the other has degree greater than k then the head vertices do not form an immune set, so S contains only the leg vertices. \square

Propositions 6.5 and 6.6 indicate that $k = 2$ is the only interesting case, for both spiders and double spiders.

Proposition 6.7. *Let T be a spider with s odd legs. Then*

$$c_2(T) = \begin{cases} \frac{n-1+s}{2} & \text{if } s \geq 2, \\ \frac{n+1+s}{2} & \text{if } s < 2. \end{cases}$$

Proof. Let h be the head vertex. By Proposition 6.1, the minimal 2-immune sets of T are the sets I of vertices that each have exactly 1 neighbour outside I . Therefore, there are three possible types of minimal 2-immune sets of T :

Type 1: The sets $\{v\}$ where v is a leaf vertex.

Type 2: The sets $\{x, y\}$ where x and y are adjacent vertices of degree 2.

Type 3(a): If h has no leaf neighbours, then the sets $N[h] - \{v\}$ are minimal 2-immune sets, where v is a degree 2 neighbour of h .

Type 3(b): If h has exactly one leaf neighbour v , the set $N[h] - \{v\}$ is a minimal 2-immune set.

For each leg of T , label the vertices $1, 2, \dots$ in order, starting with the leaf vertex, and let S' be the set of leg vertices with odd labels. Then $|S'| = \frac{n-1+s}{2}$. It is easy to see that S' intersects all 2-immune sets of Type 1 and Type 2, and that no smaller set does. For each neighbour v of h , $v \in S'$ if and only if v belongs to an odd leg. Therefore, if there are at least 2 odd legs, S' contains a vertex from each 2-immune set of Type 3 as well, so S' is a minimum 2-conversion set of T .

If T has $s < 2$ odd legs, then S' does not intersect all Type 3 immune sets. Since h is in all Type 3 immune sets, $S' \cup \{h\}$ is a minimum 2-conversion set of T . Therefore, when $s = 0$ or 1 , $c_2(T) = |S'| + 1 = \frac{n+1+s}{2}$. \square

The analysis is very similar for double spiders.

Proposition 6.8. *Let T be a double spider with head vertices h_1 and h_2 , and let s_1 and s_2 be the number of odd legs adjacent to h_1 and h_2 , respectively, with $s_1 \leq s_2$. Then*

$$c_2(T) = \begin{cases} \frac{n+2}{2} & \text{if } s_1 = s_2 = 0, \\ \frac{n+s_1+s_2}{2} & \text{if } s_1 = 0 \text{ and } s_2 \geq 1 \text{ or } s_1 = s_2 = 1, \\ \frac{n-2+s_1+s_2}{2} & \text{if } s_1 \geq 1 \text{ and } s_2 \geq 2. \end{cases}$$

Proof. The proof is very similar to the proof of Proposition 6.7, so we provide only a sketch. For each leg of T , label the vertices as in the proof of Proposition 6.7, and let S' be the set of odd-numbered leg vertices, as in that proof. Then $|S'| = \frac{n-2+s_1+s_2}{2}$. If $s_1 = s_2 = 0$ then two vertices must be added to S' (for example, h_1 and h_2) to form a 2 conversion set, so $c_2(T) = |S'| + 2$. If $s_1 = 0$ and $s_2 \geq 1$ or $s_1 = s_2 = 1$, then $S' \cup \{h_1\}$ is a minimum 2-conversion set, so $c_2(T) = |S'| + 1$. If $s_1 \geq 1$ and $s_2 \geq 2$, then S' is a minimum 2-conversion set. \square

Chapter 7

Subgraph-avoiding minimum k -conversion sets in $(k + 1)$ -regular graphs

In Proposition 3.7 we saw that a $(k + 1)$ -regular graph G achieves the lower bound on $c_k(G)$ if and only if it has a minimum k -conversion set S such that S is independent and $G - S$ is connected (that is, $G - S$ is a tree). The notion that a k -conversion set S performs more efficiently if there are as few edges as possible between its vertices is intuitive, since adjacency within S is obviously wasteful. We saw in Section 3.2.2 that for each k there are $(k + 1)$ -regular graphs that meet the bound, and therefore have independent k -conversion sets. In Section 7.1, we prove (for the case $k = 2$) that, in fact, *every* non-complete cubic graph has an independent minimum 2-conversion set.

For larger values of k , there is no guarantee that a given $(k + 1)$ -regular graph has an independent minimum k -conversion set. We determine appropriate generalizations for larger values of k , and prove a number of results of the form

“Every $(k + 1)$ -regular graph G has a minimum k -conversion set S such that $G[S]$ does not contain the subgraph H ,”

for various combinations of k and specified subgraph H . (In fact, we prove stronger statements than the above.)

In Section 7.6, we apply these results to derive upper bounds on $c_k(G)$ for $(k+1)$ -regular graphs, as well as bounds and structural results for various vertex colouring problems, including a new proof of Brooks's Theorem.

We begin by describing a move that we use to destroy unwanted subgraphs H in conversion sets. This move is used repeatedly throughout Section 7, for various definitions of H .

Let G be a $(k+1)$ -regular graph, $k \geq 2$, and let S be a minimum k -conversion set of G . Suppose $x \in S$ has exactly $k-1$ neighbours in S (this is the largest possible number of S -neighbours for a vertex in S , by the minimality of S). Let $v \in \overline{S}$ be one of the two nonseed neighbours of x . We call the operation $(S - \{x\}) \cup \{v\}$, denoted by $x \mapsto v$, a *seed shuffle* from x to v . Figure 7.1 illustrates a seed shuffle in the case $k = 2$. If x belongs to a copy of H , where H is the subgraph we wish to avoid in $G[S]$, we call $x \mapsto v$ a *restricted seed shuffle*. Lemma 7.1 guarantees that the new seed set obtained by performing a seed shuffle is still a k -conversion set.



Figure 7.1: Illustration of seed shuffling for $k = 2$, with seed vertices shown in black.

Lemma 7.1 (The Seed Shuffle Lemma). *Let G be a $(k+1)$ -regular graph, $k \geq 2$, with k -conversion set S , and suppose that $G[S]$ contains a vertex x of degree $k-1$ in $G[S]$. Let v be a neighbour of x not in S . Then $S' = S - \{x\} \cup \{v\}$ is a k -conversion set of G with the same size as S . In particular, if S is minimum, then so is S' .*

Proof. The set $S - \{x\} \cup \{v\}$ contains k neighbours of x , so it converts x at $t = 1$. The set of converted vertices at $t = 1$ contains the k -conversion set S , so it is a k -conversion set. \square

Our first choice of structure H to avoid arises from our study of the lower bound on $c_k(G)$ for $(k+1)$ -regular graphs, given by Proposition 3.7.

7.1 Independent minimum 2-conversion sets in cubic graphs

In this section we prove that every non-complete cubic graph has an independent minimum 2-conversion set. In fact, we prove the following stronger theorem.

Theorem 7.2. *Let $G \neq K_4$ be a cubic graph. For any minimum 2-conversion set S of G there exist an independent minimum 2-conversion set S' of G and a sequence of restricted seed shuffles that transforms S into S' .*

A slightly weaker restatement of Theorem 7.2, namely that every non-complete cubic graph has an independent minimum 2-conversion set, follows from a result of Catlin and Lai [25] on vertex arboricity¹. We give a new proof in this section to prepare the reader for the proofs of our more general results.

We prove Theorem 7.2 at the end of Section 7.1, after a series of lemmas. The first lemma, Lemma 7.3, guarantees that for any minimum 2-conversion set S of $G \neq K_4$, there exists a minimum 2-conversion set S' with at most one edge which can be obtained from S by a sequence of restricted seed shuffles. For the proof of Lemma 7.3, we define the distance between two edges uv and xy in a graph G to be $d_G(uv, xy) = \min\{d_G(u, x), d_G(u, y), d_G(v, x), d_G(v, y)\}$. The distance between an edge uv and a vertex x is similarly defined to be $d_G(uv, x) = \min\{d_G(u, x), d_G(v, x)\}$.

Lemma 7.3. *Let $G \neq K_4$ be a cubic graph and let S be any minimum 2-conversion set of G . Then there exists a minimum 2-conversion set S' that induces at most one edge, and a sequence of restricted seed shuffles that transforms S into S' .*

Proof. Suppose for a contradiction that every sequence of restricted seed shuffles applied to S produces a 2-conversion set of G that induces at least two edges. Among all minimum 2-conversion sets that can be obtained from S by restricted seed shuffles, restrict to those that induce the minimum number of edges. Among those, let S' be a 2-conversion set in which the minimum distance between two edges is as small as possible. (That is, choose S' from the restricted collection of 2-conversion sets such that $\min\{d_G(e_1, e_2) : e_1, e_2 \in E(G[S']), e_1 \neq e_2\}$ is as small as possible.) Let uv

¹Defined on page 125.

and xy be two edges in $G[S']$ at minimum distance from each other. If $d(uv, xy) = 0$ or 1 then one of the vertices u, v, x, y can be removed from S' . Therefore, by the choice of S' , $d_G(uv, xy) \geq 2$. Without loss of generality, let $P = (u, v, w, \dots, x, y)$ be a shortest path between uv and xy that includes both edges, and note that $w \in \overline{S}$. Define $S'' = (S - \{v\}) \cup \{w\}$. By Lemma 7.1, S'' is a 2-conversion set, and $|S''| \leq |S'|$. There are three cases, each leading to a contradiction of our choice of S' .

Case 1: w has exactly one neighbour in S' (namely v). Then S'' induces fewer edges than does S' .

Case 2: w has exactly two neighbours in S' (including v). If these two neighbours are v and x , then $(S' - \{v, x\}) \cup \{w\}$ is a smaller 2-conversion set than S' . Otherwise, S'' induces the same number of edges as does S' , but the minimum distance between two edges in $G[S'']$ is less than that in $G[S']$.

Case 3: all neighbours of w are in S' . Then w and two of its neighbours are in S'' , and so $S'' - \{w\}$ is a smaller 2-conversion set than S' . \square

For Theorem 7.2 we wish to show that for any minimum 2-conversion set S , there is a sequence of restricted seed shuffles that yields a minimum 2-conversion set with no edges. Lemma 7.3 guarantees that there is always a sequence of restricted seed shuffles that yields a minimum 2-conversion set with at most one edge. Therefore, for the rest of Section 7.1, we may restrict our attention to the minimum 2-conversion sets S that have exactly one edge. For a given such 2-conversion set S , we denote the unique edge by $e(S)$.

Definition 7.4. For any cubic graph G , let $\mathcal{S}(G)$ denote the set of all minimum 2-conversion sets of G that induce exactly one edge and for which no sequence of restricted seed shuffles applied to S yields an independent minimum 2-conversion set.

To prove Theorem 7.2, we will show that $\mathcal{S}(G)$ is empty for $G \neq K_4$. First, we establish some properties of $G[S]$ and $G[\overline{S}]$ for $S \in \mathcal{S}(G)$. The first is that the components of $G[\overline{S}]$ are paths.

Lemma 7.5. Let S be a minimum 2-conversion set in a graph G such that $G[S]$ has exactly one edge. If $G[\overline{S}]$ has a vertex of degree 3 then there exists an independent minimum 2-conversion set S' of G and a sequence of restricted seed shuffles that transforms S into S' . Equivalently, for all $S \in \mathcal{S}(G)$, $G[\overline{S}]$ is a collection of paths.

Proof. Let $e(S) = uv$. We use induction on the shortest distance d to a vertex y of degree 3 in $G[\overline{S}]$. Assume without loss of generality that $d_G(e(S), y) = d_G(v, y)$.

Since $\deg_{\overline{S}}(y) = 3$, $d \geq 2$. Suppose $d = 2$ and let x be the common neighbour of v and y . If the restricted seed shuffle $v \mapsto x$ does not yield an independent 2-conversion set of G , perform the additional shuffle $x \mapsto y$. No neighbour of y is in the resulting 2-conversion set, so it is independent.

For the induction hypothesis, suppose $d \geq 3$ and assume that whenever the shortest distance from $e(S)$ to a vertex of degree 3 in \overline{S} is less than d , it is possible to obtain an independent 2-conversion set S' from S by a sequence of restricted seed shuffles.

Now suppose the shortest distance from $e(S)$ to a vertex y of degree 3 in \overline{S} is d . Assume without loss of generality that $d(e(S), y) = d(v, y)$, and let x be adjacent to v on a shortest path from v to y . Let S' be the minimum 2-conversion set resulting from the restricted seed shuffle $v \mapsto x$. Since $d \geq 3$, x is not adjacent to y , hence y has degree 3 in \overline{S}' . If S' is independent, we are done. Otherwise, S' has exactly one edge, $e(S')$, which is incident with x , and the shortest distance from $e(S')$ to a vertex of degree 3 in \overline{S}' is at most $d(x, y) < d$. Therefore, by the induction hypothesis, there exists a sequence of restricted seed shuffles from S' that yields an independent minimum 2-conversion set. Since $v \mapsto x$ is itself a restricted seed shuffle, the same is true for S . \square

Lemma 7.6. *Let $G \neq K_4$ be a cubic graph and let $S \in \mathcal{S}(G)$. Let $e(S) = x_1x_2$. For each $i \in \{1, 2\}$, the non-seed neighbours of x_i are the endpoints of a component path $P(x_i)$ of \overline{S} . Moreover, $P(x_1) \cap P(x_2) = \emptyset$.*

Proof. Since S is a minimum 2-conversion set, $\overline{S} \cup \{x_i\}$ contains a cycle, for $i = 1, 2$. Therefore the two nonseed neighbours of x_i are in the same component of \overline{S} . If x_i is adjacent to an internal vertex w of $P(x_i)$ then $x_j, j \neq i$ is not adjacent to w , so the restricted seed shuffle $x_i \mapsto w$ produces an independent minimum 2-conversion set, contradicting our assumption that $S \in \mathcal{S}(G)$. Finally, suppose for a contradiction that $P(x_1) \cap P(x_2) \neq \emptyset$. In that, case, by the previous assertion that each x_i is adjacent to the endpoints of the component path $P(x_i)$, we have $P(x_1) = P(x_2) = P$. Let u and v be the endpoints of this path, so each x_i is adjacent to u and to v . Since $G \neq K_4$, P has length at least 2. Therefore the two restricted seed shuffles $x_1 \mapsto u$ and $x_2 \mapsto v$,

performed one after the other, produce a minimum 2-conversion set S' with no edge. This contradicts our assumption that $S \in \mathcal{S}(G)$. \square

For a given minimum 2-conversion set $S \in \mathcal{S}(G)$ with $e(S) = xy$, we denote the component path of \bar{S} whose endpoints are adjacent to x by $P(x)$. More specifically, $P(x)$ denotes the set of vertices in G which form the component path of \bar{S} whose endpoints are adjacent to x . (We continue to denote this set of vertices by $P(x)$ even if we perform a restricted seed shuffle that results in a 2-conversion set S' for which x is not an endpoint of $e(S')$.)

We now restate and prove Theorem 7.2.

Theorem 7.2 (again). Let $G \neq K_4$ be a cubic graph. For any minimum 2-conversion set S of G there exist an independent minimum 2-conversion set S' of G and a sequence of restricted seed shuffles that transforms S into S' .

Proof. We wish to show that $\mathcal{S}(G) = \emptyset$, so we assume for a contradiction that S is in $\mathcal{S}(G)$. We define a sequence of minimum 2-conversion sets S_0, S_1, \dots of G , each of which is obtained from the last by a restricted seed shuffle. A priori, the sequence may be infinite. The 2-conversion sets S_i are defined as follows.

Let $S_0 = S$. By definition, S_0 contains exactly one edge, $e(S_0) = x_0y_0$. By Lemma 7.6, $P(x_0) \cap P(y_0) = \emptyset$. Let x_1 be an endpoint of $P(y_0)$, and let S_1 be the minimum 2-conversion set obtained by shuffling $y_0 \mapsto x_1$. Note that x_1 is adjacent to at most one vertex in $S_0 \setminus \{y_0\}$, and non-adjacent to x_0 . For $i \geq 1$, we obtain S_{i+1} as follows.

By assumption $S_i \in \mathcal{S}(G)$, so S_i induces exactly one edge $e(S_i)$, which contains the vertex x_i . Let $e(S_i) = x_iy_i$. By Lemma 7.6, $P(x_i) \cap P(y_i) = \emptyset$. If $P(y_i) \cap P(x_j) = \emptyset$ for all $0 \leq j < i$ as well, let x_{i+1} be an endpoint of $P(y_i)$ and define S_{i+1} to be the minimum 2-conversion set resulting from the restricted seed shuffle $y_i \mapsto x_{i+1}$.

The sequence S_0, S_1, \dots terminates if the stopping condition

$$P(y_i) \cap P(x_j) \neq \emptyset \text{ for some } 0 \leq j < i$$

is met.

In order to show that the stopping condition is met for some i (that is, the sequence S_0, S_1, \dots is in fact finite), we show that $P(x_0), P(x_1), \dots, P(x_i)$ are disjoint for all i .

Claim. For each $i \geq 0$, $P(x_0), P(x_1), \dots, P(x_i)$ are disjoint components of $\overline{S_i}$.

Proof of claim. We prove the claim by induction on i . The statement is trivial for $i = 0$. For some $i \geq 0$, assume that $P(x_0), P(x_1), \dots, P(x_{i-1})$ are disjoint components of $\overline{S_{i-1}}$. For the induction step, we show

1. $P(x_i)$ is disjoint from $P(x_0), \dots, P(x_{i-1})$,
2. $P(x_0), \dots, P(x_i)$ are contained in $\overline{S_i}$, and
3. $P(x_0), \dots, P(x_i)$ are components of $\overline{S_i}$.

For the first part, we note that $P(x_i) = P(y_{i-1}) + y_{i-1} - x_i$ and by the stopping condition (which has not been met), $P(y_{i-1})$ is disjoint from $P(x_0), \dots, P(x_{i-1})$. Therefore, to show that $P(x_i)$ is disjoint from $P(x_0), \dots, P(x_{i-1})$ it suffices to show that $y_{i-1} \notin P(x_j)$ for $0 \leq j < i$. This follows from the observation that $y_{i-1} \in S_{i-1}$ and $P(x_j) \in \overline{S_{i-1}}$ for $0 \leq j < i$.

For the second part, we note that $\overline{S_i} = \overline{S_{i-1}} + y_{i-1} - x_i$ and $P(x_i) = P(y_{i-1}) + y_{i-1} - x_i$. Since $P(y_{i-1}) \subseteq \overline{S_{i-1}}$ it follows that $P(x_i) \subseteq \overline{S_i}$.

For the third part, $P(x_i)$ is a component of $\overline{S_i}$ by definition. We must show that $P(x_j)$ is a component of $\overline{S_i}$ for all $0 \leq j < i$. We have already established that $P(x_j)$ is contained in $\overline{S_i}$, and it follows that $P(x_j)$ is connected in $G[\overline{S_i}]$. Therefore if $P(x_j)$ is *not* a component of $\overline{S_i}$ then y_i is adjacent to $P(x_j)$, that is, the component of $\overline{S_i}$ containing $P(x_j)$ is $P(x_j) \cup \{y_i\}$. However, this contradicts the assumption that $P(y_i) \cap P(x_j) = \emptyset$ for all $0 \leq j \leq i$. This concludes the proof of the claim.

Since G is finite and the sets $P(x_0), \dots, P(x_i)$ are disjoint for all i , there exists some $\ell \geq 1$ such that $P(y_\ell) \cap P(x_j) \neq \emptyset$ for some $0 \leq j < \ell$. In fact, by ignoring S_0, \dots, S_{j-1} (and their associated $x_i, y_i, P(x_i)$, etc.) and reindexing, we may assume that $j = 0$. That is $P(y_\ell)$ intersects $P(x_0)$. Moreover, since $P(y_\ell)$ and $P(x_0)$ are components of $\overline{S_\ell}$, this implies $P(y_\ell) = P(x_0)$. Figure 7.2 illustrates the sequence of seed shuffles performed between S_0 and S_ℓ . The black vertices represent S_ℓ .

Let u and v be the endpoints of $P(x_0) = P(y_\ell)$ and let $S_{\ell+1}$ be the minimum

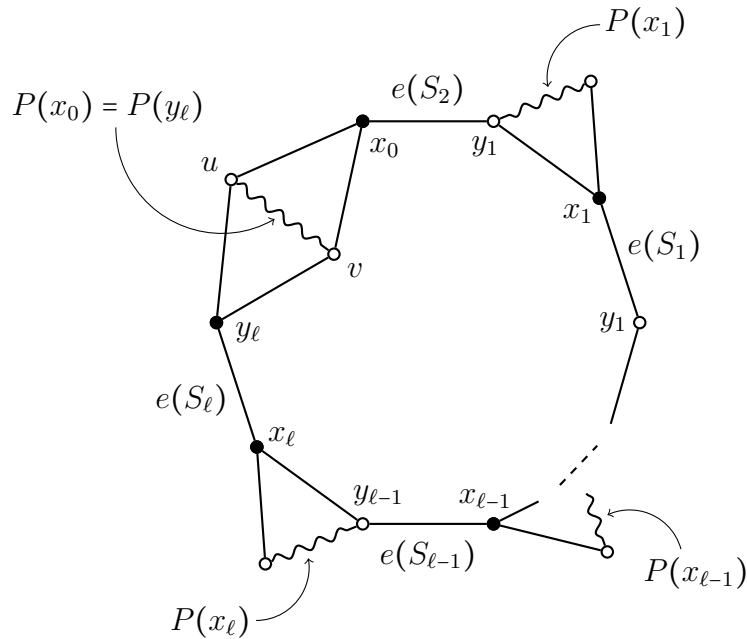


Figure 7.2: The set S_ℓ (black) and the paths $P(x_0), \dots, P(x_\ell), P(y_\ell)$, illustrating the stopping condition for the sequence S_0, S_1, \dots

2-conversion set obtained by shuffling $y_\ell \mapsto u$. Then $S_{\ell+1}$ has exactly one edge, x_0u . Since $S_{\ell+1} \in \mathcal{S}(G)$ by assumption, each of u and x_0 is adjacent to the endpoints of some component path of $\overline{S_{\ell+1}}$. Denote these component paths by P and P' , respectively. Since $S_{\ell+1} \in \mathcal{S}(G)$, P and P' are disjoint, by Lemma 7.6. On this other hand, we observe that P and P' both contain v to arrive at the desired contradiction. \square

In general, we cannot guarantee the existence of an independent minimum k -conversion set. In Proposition 7.7 we show that for every $k \geq 3$, there exists an arbitrarily large $(k+1)$ -regular graph with no independent k -conversion set of any size. Proposition 7.8 gives an additional family of counterexamples for the case where $k \geq 3$ is odd (that is, for graphs with even regularity). In Definition 4.20 we introduced the notation $G \circ A^-$ to denote any r -regular graph obtained by replacing each vertex of an r -regular graph G by a copy of $A - a$, where A is an r -regular graph and a is any vertex of A . If $A = K_{r+1}$ then $A^- = K_r$; in that case we simply write $G \circ K_r$. These graphs provide the counterexamples we need in Proposition 7.7 to show that Theorem 7.2 does not hold for $k \geq 3$.

Proposition 7.7. *Let $k \geq 3$, and let G be a $(k+1)$ -regular graph of order n . Then*

every k -conversion set of $G \circ K_{k+1}$ contains at least $(k-1)n$ edges.

Proof. For $k \geq 3$ the decycling number of K_{k+1} is at least $k-1 \geq 2$, so any k -conversion set of $G \circ K_{k+1}$ must contain at least $k-1$ vertices from each copy of K_{k+1} . \square

Proposition 7.8. *Let L be the line graph of an r -regular graph G , with $r \geq 3$. Then L is a $(2r-2)$ -regular graph with no independent $(2r-3)$ -conversion set.*

Proof. We show that any independent seed set in L fails to convert any non-seed vertices under a $(2r-3)$ -conversion process. Let e be an edge of G , corresponding to vertex v in L . Since e is incident with $r-1$ other edges at each of its endpoints, the closed neighbourhood of v in L consists of two cliques of size r , joined at v (and nowhere else). Therefore, between any three neighbours of v there is at least one edge. That is, no independent set of L contains more than two neighbours of v , so v does not convert at $t=1$ from any independent seed set. Since this holds for all $v \in L$, the result follows. \square

7.2 Possible generalizations of Theorem 7.2

Since Theorem 7.2 does not hold for larger values of k , we seek in this section an interpretation of that theorem that generalizes to $k \geq 3$. We can rephrase the result of Theorem 7.2 in various ways, two of which are below.

1. Every cubic graph G has a minimum 2-conversion set S such that $G[S]$ has zero edges.
2. Every cubic graph G has a minimum 2-conversion set S such that $G[S]$ has maximum degree zero.

One may imagine that a $(k+1)$ -regular graph G must have a k -conversion set S such that $G[S]$ has at most $k-2$ edges, or perhaps has maximum degree at most $k-2$. However, both of these proposed generalizations are false for $k \geq 3$. We define an infinite class \mathcal{G}_k of $(k+1)$ -regular graphs below, and demonstrate in Corollary 7.11 that these graphs are counterexamples to the proposed generalizations.

Definition 7.9. Let $k \geq 3$. Consider r copies of $K_{k+2} - e$, and for $i = 0, \dots, r-1$, let $x_i y_i$ be the missing edge in the i^{th} copy. Define \mathcal{G}_k to be the family of $(k+1)$ -regular graphs constructed from these copies by adding the edges $x_i y_{i+1}$ for all i , where addition is performed modulo r . Figure 7.3 gives an example of a graph in this class for $k = 3$.

Proposition 7.10 implies that for every $G \in \mathcal{G}_k$ and for every minimum k -conversion set S of G , the maximum degree of $G[S]$ is $k-1$. It follows immediately from this result that the generalizations of Theorem 7.2 proposed above are false.

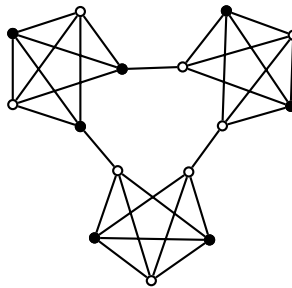


Figure 7.3: An example of a graph in \mathcal{G}_3 , and a minimum 3-conversion set.

Proposition 7.10. Let $k \geq 3$ and let $G \in \mathcal{G}_k$. Then for every minimum k -conversion set S of G , one of the copies of K_{k+2} in G contains k vertices of S .

Proof. Let $G \in \mathcal{G}_k$ and let S be a minimum k -conversion set of G . We argue that some copy of K_{k+2} contains k seed vertices. Let v be a vertex that converts at $t = 1$. If v belongs to a copy of K_{k+2} with fewer than k seed vertices then $v = x_i$ or y_i for some i . Assume, without loss of generality, that $v = y_1$. Then $x_0 \in S$. Let H be the 0^{th} copy of K_{k+2} , and assume every vertex outside H has converted by some time t . Let w be a first vertex of $V(H) - S$ to convert. Then w has k converted neighbours (including seed vertices) at time t . If $w = y_0$ then at least $k-1$ of these converted neighbours are in H , and by definition of w they are all seed vertices. In that case, H contains $k-1$ seed vertices adjacent to w , as well as x_0 . On the other hand, if $w \neq y_0$ then all neighbours of w are in H , and therefore k of them are seed vertices. \square

Corollary 7.11. Let $G \in \mathcal{G}_k$ and let S be a minimum k -conversion set of G . Then $G[S]$ contains at least $k-2$ vertices of degree $k-1$.

Proof. The result follows immediately from Proposition 7.10. \square

The graphs in the class \mathcal{G}_k are the only known $(k+1)$ -regular graphs G with the property that every minimum k -conversion set of G has at least $k-2$ vertices of degree $k-1$. This leads to the following open problem.

Question 7.12. For which classes of $(k+1)$ -regular graphs, other than \mathcal{G}_k , does every minimum k -conversion set of G induce at least $k-2$ vertices of degree $k-1$?

A third restatement of Theorem 7.2, which leads to a weaker generalization than the previous two, is that every cubic graph G has a minimum 2-conversion set S such that $G[S]$ is K_2 -free. The proof of Proposition 7.10 shows that if $G \in \mathcal{G}_k$ and S is a minimum k -conversion set of G then $G[S]$ contains a copy of $K_k - e$, but not K_k . This raises the question of whether every $(k+1)$ -regular graph has a minimum k -conversion set that avoids K_k . Section 7.3 is devoted to proving that the answer is “yes”, except in the case where G is itself a complete graph.

7.3 Avoiding K_k in minimum k -conversion sets of $(k+1)$ -regular graphs

The main result of this section is Theorem 7.13. The proof appears at the end of the section, after a series of lemmas.

Theorem 7.13. *Let $k \geq 2$ and let $G \neq K_{k+2}$ be a $(k+1)$ -regular graph. For any minimum k -conversion set S of G there exist a minimum k -conversion set S' of G such that $G[S']$ is K_k -free and a sequence of restricted seed shuffles that transforms S into S' .*

A slightly weaker restatement of Theorem 7.13, namely that every non-complete $(k+1)$ -regular graph G has a minimum k -conversion set S such that $G[S]$ is K_k -free, follows from a result of Catlin and Lai [25]. We give a new proof, in the context of k -conversion sets, to prepare the reader for our more general results.

The result of Theorem 7.13 is already established for $k=2$ by Theorem 7.2. For $k \geq 3$, we start by proving an important property of any K_k that is induced by a

minimum k -conversion set S of G , namely that any K_k in $G[S]$ can be identified as the “seed neighbourhood” of any of its vertices (Lemma 7.15).

Definition 7.14. Let G be a graph and let H_1 and H_2 be subgraphs of G . We define the distance (in G) between H_1 and H_2 to be

$$d_G(H_1, H_2) = \min\{d_G(x, y) : x \in V(H_1) \text{ and } y \in V(H_2)\}.$$

We say H_1 and H_2 are adjacent if they are disjoint and $d_G(H_1, H_2) = 1$.

Lemma 7.15. Let S be a minimum k -conversion set of a $(k + 1)$ -regular graph G with $k \geq 2$. Then any copy of K_k induced by S is a component of $G[S]$. In particular, any two distinct copies of K_k in $G[S]$ are disjoint and non-adjacent.

Proof. Since S is a minimum k -conversion set, no vertex of S has k seed neighbours. Since v has $k - 1$ seed neighbours in K_k , it has no other seed neighbours. \square

Let v be a vertex in a copy of K_k induced by a k -conversion set S of G . By Lemma 7.15, this copy of K_k is the subgraph induced by the (closed) S -neighbourhood of v . In what follows, it is often convenient to view a copy of K_k in this way— as being induced by the seed neighbourhood of one of its vertices, rather than being induced by the whole seed set. For that reason, we use the notation $K_k(v, S)$ to denote the copy of K_k induced by $N_S[v]$ — that is, the component of $G[S]$ containing v . When it is known that the conversion set S induces exactly one copy of K_k , we may denote that copy of K_k by $K_k(S)$.

By Lemma 7.15, the distance (in G) between any two copies of K_k in $G[S]$ is at least 2. We prove in Lemma 7.16 that in fact it is always possible to reduce the number of K_k 's in a minimum k -conversion set of G to at most one.

Lemma 7.16. Let $G \neq K_{k+2}$ be a $(k + 1)$ -regular graph and let S be any minimum k -conversion set of G . Then there exist a minimum k -conversion set S' of G that induces at most one K_k , and a sequence of restricted seed shuffles that transforms S into S' .

Proof. Assume for a contradiction that every sequence of restricted seed shuffles applied to S yields a minimum k -conversion set of G that induces at least two copies

of K_k . Among all minimum k -conversion sets of G that can be obtained from S by a sequence of restricted seed shuffles, restrict to those that induce the least number of K_k 's, and among those, let S' be one with the smallest possible distance between two copies of K_k . Let H_1 and H_2 be two copies of K_k in $G[S']$ at that minimum distance from each other. Let u and v be vertices in H_1 and H_2 , respectively, such that $d_G(u, v) = d_G(H_1, H_2)$. Consider a shortest path $P = (u, a, \dots, v)$ between u and v ; by Lemma 7.15 it has length at least 2, so a, u and v are all distinct.

By the Seed Shuffle Lemma, $S'' = (S' - \{u\}) \cup \{a\}$ is a k -conversion set of G and $|S''| \leq |S'|$. By the minimality of S' , $|S''| = |S'|$, so S'' is also a minimum k -conversion set of G . Removing u from S' eliminates the K_k defined by the S' -neighbourhood of u . Since S' was chosen to have the minimum number of K_k 's, we may assume the S'' -neighbourhood of a defines a new K_k . In fact, we may assume that by shuffling the seed to a we have introduced exactly one new K_k , since all new K_k 's must contain a . Therefore S'' is a second minimum k -conversion set with the fewest possible number of K_k 's. Let H_3 be the new copy of K_k . Since H_3 contains a and H_2 does not, H_3 and H_2 are distinct and therefore non-adjacent, by Lemma 7.15. But $d_G(H_3, H_2) \leq d_G(a, v) < d_G(u, v) = d_G(H_1, H_2)$. This contradicts our choice of S' . \square

For the proof of Theorem 7.13, we will show that in fact for any minimum k -conversion set S it is possible to reduce the number of copies of K_k to zero. We argue by contradiction, assuming there exists some minimum k -conversion set S for which this is impossible. In light of Lemma 7.16, we may assume S has exactly one K_k . This observation prompts the following definition.

Definition 7.17. *For a given $(k + 1)$ -regular graph G , with $k \geq 3$, let $\mathcal{S}(G)$ denote the set of minimum k -conversion sets S of G that induce exactly one copy of K_k and for which no sequence of restricted seed shuffles from S yields a K_k -free minimum k -conversion set.*

The proof of Theorem 7.13 amounts to showing that for $G \neq K_{k+2}$, $\mathcal{S}(G)$ is empty. Starting with a minimum k -conversion set $S \in \mathcal{S}(G)$, we perform a restricted seed shuffle to destroy $K_k(S)$, but this (necessarily) introduces a new one. We repeat this procedure until the newly created K_k intersects one of the earlier ones. This provides the conditions necessary to derive a contradiction. The next lemma (Lemma 7.18) asserts that at each step, for each x in the current K_k there is a shuffle $x \mapsto v$ such that

the new K_k does not intersect the current one (unless G is a complete graph). This will be convenient for obtaining the desired contradiction in the proof of Theorem 7.13 in the proof of the following lemma.

For any minimum k -conversion set S containing a copy of K_k , a restricted seed shuffle cannot increase the number of K_k s. If $S \in \mathcal{S}(G)$, then any restricted seed shuffle $x \mapsto v$ produces another k -conversion set in $\mathcal{S}(G)$. This allows us to use the notation $K_k((S - \{x\}) \cup \{v\})$ without ambiguity.

Lemma 7.18. *Let G be a $(k+1)$ -regular graph, $k \geq 3$, and let $S \in \mathcal{S}(G)$. If $G \neq K_{k+1}$ then for every $x \in K_k(S)$, x has a neighbour $v \in \overline{S}$ such that $K_k(S) \cap K_k((S - \{x\}) \cup \{v\}) = \emptyset$.*

Proof. We prove the contrapositive. Suppose there exists a vertex $x \in K_k(S)$ such that for both neighbours v_1 and v_2 of x in \overline{S} , the shuffle $x \mapsto v_i$ produces a minimum k -conversion set S^i such that $K_k(S^i) \cap K_k(S) \neq \emptyset$. We claim that in that case, for each i , $K_k(S^i) = (K_k(S) - \{x\}) \cup \{v_i\}$. To see this, let $y \in K_k(x, S) \cap K_k(v_i, S^i)$. Then $K_k(x, S) = K_k(y, S)$ and $K_k(v_i, S^i) = K_k(y, S^i)$, so the only difference between $K_k(S)$ and $K_k(S^i)$ is the replacement of x by v_i .

The above observation implies that for $i = 1, 2$, v_i is adjacent to every vertex of $K_k(S)$ and therefore $K_k(S) \cup \{v_i\}$ forms a copy of K_{k+1} . Let $y \neq x$ be a vertex of $K_k(S)$. If v_1 and v_2 are not adjacent, then the seed shuffles $x \mapsto v_1$ and $y \mapsto v_2$ together produce a minimum k -conversion set that does not induce a copy of K_k , which contradicts our choice of S . Therefore $v_1 \sim v_2$, and therefore $G = K_{k+2}$. \square

We now restate and prove Theorem 7.13.

Theorem 7.13 (again). Let $k \geq 2$ and let $G \neq K_{k+2}$ be a $(k+1)$ -regular graph. For any minimum k -conversion set S of G there exist a minimum k -conversion set S' of G such that $G[S']$ is K_k -free and a sequence of restricted seed shuffles that transforms S into S' .

Proof. For $k = 2$ the result is established by Theorem 7.2, so assume $k \geq 3$. Arguing by contradiction, we will show that $\mathcal{S}(G)$ is empty. By Lemma 7.18, for every $S \in \mathcal{S}(G)$, every $x \in K_k(S)$ has a neighbour $v \in \overline{S}$ such that minimum k -conversion set S' resulting from the restricted seed shuffle $x \mapsto v$ is also in $\mathcal{S}(G)$ and $K_k(S) \cap K_k(S') = \emptyset$.

We recursively define a sequence S_0, S_1, \dots, S_ℓ of minimum k -conversion sets of G , each of which is in $\mathcal{S}(G)$. At each step, we obtain a new k -conversion set from the current one by shuffling in such a way that the K_k induced by the new k -conversion set is disjoint from the K_k induced by the current k -conversion set. We repeat this process until the new K_k intersects (what remains of) one of the previous ones (but not the current one: we've defined things so this can't happen).

Assume for a contradiction that there exists some $S_0 \in \mathcal{S}(G)$. Let x_0 be any vertex of $K_k(S_0)$ and let $v_1 \in \overline{S_0}$ be a neighbour of x_0 such that

$$K_k(S_0) \cap K_k((S_0 - \{x_0\}) \cup \{v_1\}) = \emptyset.$$

The existence of v_1 is guaranteed by Lemma 7.18. Define $S_1 = (S_0 - \{x_0\}) \cup \{v_1\}$. By definition, $K_k(S_1) \cap K_k(S_0) = \emptyset$.

We use the same process to obtain the subsequent k -conversion sets S_2, S_3, \dots , but with the added constraint that $x_i \neq v_i$. That is, the vertex we remove from S_i to obtain S_{i+1} must not be the vertex we just added. This ensures that the process of repeated seed shuffling does not degenerate into shuffling back and forth between the same two vertices, v_i and x_{i-1} , which would not lead to the desired contradiction.

For $i \geq 1$, if $K_k(S_i) \cap (K_k(S_j) - \{x_j\}) = \emptyset$ for all $0 \leq j < i$, let $x_i \neq v_i$ be a vertex of $K_k(S_i)$ and let $v_{i+1} \in \overline{S_i}$ be a neighbour of x_i such that $K_k(S_i) \cap K_k((S_i - \{x_i\}) \cup \{v_{i+1}\}) = \emptyset$. The existence of such a vertex v_{i+1} is guaranteed by Lemma 7.18. Define

$$S_{i+1} = (S_i - \{x_i\}) \cup \{v_{i+1}\}.$$

The process terminates when $K_k(S_\ell)$ intersects $K_k(S_j) - \{x_j\}$ for some earlier j . In fact, by ignoring S_0, \dots, S_{j-1} and reindexing, we may assume that $j = 0$. That is, we may assume $K_k(S_\ell)$ intersects $K_k(S_0) - \{x_0\}$. Moreover, since $K_k(S_\ell) \cap (K_k(S_{\ell-1}) - \{x_{\ell-1}\}) = \emptyset$ by definition, $\ell \geq 2$.

By definition, for $0 \leq i < \ell$, S_ℓ contains all vertices of $S_i - \{x_i\}$. In particular, all vertices of $K_k(S_0) - \{x_0\}$ are still seed vertices (that is, they are in S_ℓ). By applying Lemma 7.15 to a vertex $y \in K_k(S_\ell) \cap K_k(S_0) - \{x_0\}$, we see that $K_k(S_0) - \{x_0\} \subset K_k(S_\ell)$. That is, $K_k(S_\ell) = (K_k(S_0) - \{x_0\}) \cup \{v_\ell\}$. The sequence of K_k 's is depicted

in Figure 7.4, with vertices of S_ℓ in black.

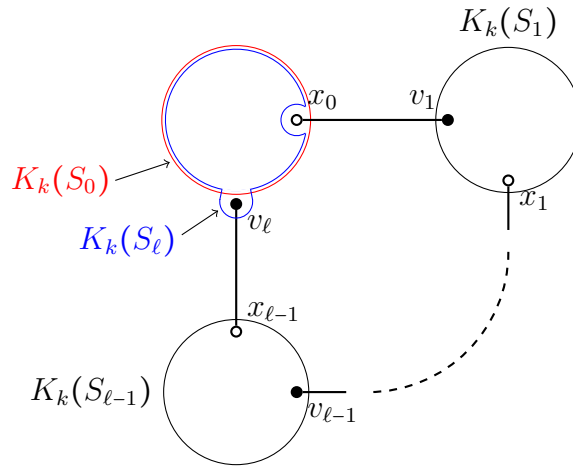


Figure 7.4: The sequence of K_k 's in the proof of Theorem 7.13.

It is possible that $v_\ell = x_0$. This gives rise to two cases.

Case 1. Suppose $v_\ell = x_0$. Then this vertex is adjacent to k vertices in S_ℓ , specifically the $k - 1$ other vertices of $K_k(S_\ell)$, and v_1 . This contradicts the minimality of S_ℓ .

Case 2. Suppose $v_\ell \neq x_0$. We shuffle once more, again producing a k -conversion set of the same size. Let y be a vertex in $K_k(S_\ell) - \{v_\ell\}$. Then $y \in K_k(S_0)$, so y is adjacent to x_0 . Define $S_{\ell+1} = (S_\ell - \{y\}) \cup \{x_0\}$. That is, shuffle from y to x_0 .

We claim that x_0 is adjacent to v_ℓ . To see this, let $z \neq v_\ell, x_0$ be a vertex in $K_k(S_{\ell+1})$. The only difference between $K_k(z, S_\ell)$ and $K_k(z, S_{\ell+1})$ is that y has been replaced with x_0 . Therefore, $v_\ell, x_0 \in K_k(z, S_{\ell+1})$, so v_ℓ and x_0 are adjacent. Now x_0 has k neighbours in $S_{\ell+1}$, specifically the $k - 1$ other vertices of $K_k(z, S_{\ell+1})$, and v_1 . This contradicts the minimality of $S_{\ell+1}$. \square

7.4 Acyclic minimum 3-conversion sets in 4-regular graphs

For $k = 3$, Theorem 7.13 states that every 4-regular graph (except K_5) has a minimum 3-conversion set that does not induce any K_3 's. In this section we strengthen this

result (for $k = 3$) to show that every 4-regular graph (except K_5) has a minimum 3-conversion set that does not induce any cycles. The proof is an adaptation of the proof of Theorem 7.13, including its technical lemmas.

Theorem 7.19. *Let $G \neq K_5$ be a 4-regular graph. For any minimum 3-conversion set S of G there exists a minimum 3-conversion set S' of G such that $G[S']$ is a linear forest, and a sequence of restricted seed shuffles that transforms S into S' .*

The following lemma is analogous to Lemma 7.15.

Lemma 7.20. *Let S be a minimum 3-conversion set of a 4-regular graph G . Then each component of $G[S]$ is either a path or a cycle. In particular, any two distinct cycles in $G[S]$ are disjoint and non-adjacent.*

Proof. By the minimality of S , $G[S]$ has maximum degree at most 2. □

We denote the component of $G[S]$ containing the vertex v by $K(v, S)$. The following lemma, which is analogous to Lemma 7.15, follows immediately from Lemma 7.20.

We now prove that for every 4-regular graph $G \neq K_5$ and every minimum 3-conversion set S of G , there is a sequence of restricted seed shuffles that yields a minimum 3-conversion set with at most one cycle. This is analogous to Lemma 7.16 from Section 7.3.

Lemma 7.21. *Let $G \neq K_5$ be a 4-regular graph and let S be any minimum 3-conversion set of G . Then there exists an acyclic minimum 3-conversion set S' of G which induces at most one cycle, and a sequence of restricted seed shuffles that transforms S into S' .*

Proof. The proof is identical to the proof of Lemma 7.16 for $k = 3$, but with “ K_k ” replaced by “cycle” and referring to Lemma 7.20 instead of Lemma 7.15. □

For convenience, if v is a vertex in a cycle component of $G[S]$, we denote that cycle by $C(v, S)$. If S induces exactly one cycle, we denote it by $C(S)$.

We will use the same proof technique in the proof of Theorem 7.19 as we did in the proof of Theorem 7.13. That is, we will assume for a contradiction that a 4-regular

graph $G \neq K_5$ has a minimum 3-conversion set S such that no sequence of restricted seed shuffles applied to S yields an acyclic minimum 3-conversion set. Lemma 7.21 guarantees that the number of cycles *can* be reduced to at most 1, so we may assume that $G[S]$ has exactly one cycle, $C(S)$. For any 4-regular graph G , we let $\mathcal{S}(G)$ denote the set of minimum 3-conversion sets S of G with exactly one cycle such that no sequence of restricted seed shuffles eliminates the cycle.

As in Sections 7.1 and 7.3, we wish to show that $\mathcal{S}(G)$ is empty whenever $G \neq K_5$ is a 4-regular graph. Starting with a minimum 3-conversion set $S_0 \in \mathcal{S}(G)$, we obtain a sequence of minimum 3-conversion sets S_0, S_1, \dots, S_ℓ , each of which is in $\mathcal{S}(G)$, and by analysing the structure of this sequence we derive a contradiction. At each step we obtain S_i from S_{i-1} by shuffling a seed vertex from some $x \in C(S_i)$ to one of its neighbours in $\overline{S_i}$. As in the proof of Theorem 7.13, some care must be taken in the choice of the vertex x that we remove from S_i to create S_{i+1} . Specifically, x must be chosen to avoid “undoing” the shuffle that created S_i , while also ensuring that $C(S_{i+1})$ is disjoint from $C(S_i)$. Lemma 7.23, which is analogous to Lemma 7.18 from the proof of Theorem 7.13, guarantees that, at each step, there is a vertex $x \in C(S_i)$ that satisfies these requirements. Lemma 7.22, below, is used in the proof of Lemma 7.23 and in the proof of Theorem 7.19.

Lemma 7.22. *Let G be a 4-regular graph and let $S \in \mathcal{S}(G)$. Let $x \in C(S)$ and $v \in N(x) - S$, and define $S' = (S - \{x\}) \cup \{v\}$. Then*

(a) $S' \in \mathcal{S}(G)$, and

(b) if $C(S) \cap C(S') \neq \emptyset$ then $C(S) \cap C(S') = C(S) - \{x\} = C(S') - \{v\}$.

Proof. For (a), we note that S' has the same cardinality as S , so S' is a minimum 3-conversion set. Therefore, by Lemma 7.20, any two cycles in $G[S']$ are disjoint. Since $S \in \mathcal{S}(G)$, S' contains at least one cycle. Since $x \notin S'$, $C(S)$ is not a cycle in $G[S']$, and so every cycle in $G[S']$ contains v . Therefore $G[S']$ contains exactly one cycle, since distinct cycles are disjoint.

For (b), it is clear that $C(S) \cap C(S')$ is a union of paths. We first show that it does not contain a trivial path. Suppose for a contradiction that $C(S) \cap C(S')$ contains an isolated vertex y . Then y has two S -neighbours in $C(S)$ and at least one S -neighbour in $C(S') - C(S)$, since every vertex of $C(S')$ except v is also in S . This

contradicts the minimality of S .

Therefore $C(S) \cap C(S')$ consists of paths of order at least 2. Suppose for a contradiction that for some leaf y of $C(S) \cap C(S')$, $\{x, v\} \notin N(y)$. We note that y has a neighbour in $C(S) \cap C(S')$ and a neighbour in $C(S') - C(S)$, which are both in S' . It also has a neighbour in $C(S) - C(S')$, and if y is not adjacent to x then this neighbour is in S' . This contradicts the minimality of S' . In a similar way, we obtain a contradiction to the minimality of S if y is not adjacent to v . We conclude that every leaf of $C(S) \cap C(S')$ is adjacent to both x and v .

Since only two vertices of $C(S)$ are adjacent to x , these are the only leaves of $C(S) \cap C(S')$, which implies that $C(S) \cap C(S')$ is a single path. In particular, it is the path $C(S) - \{x\}$. Similarly, since only two vertices of $C(S')$ are adjacent to v , we conclude that $C(S) \cap C(S') = C(S') - \{v\}$. \square

Lemma 7.23. *Let $G \neq K_5$ be a 4-regular graph and let $S \in \mathcal{S}(G)$. Then for every $x \in C(S)$ there is a vertex $v \in N(x) - S$ such that $C(S) \cap C((S - \{x\}) \cup \{v\}) = \emptyset$.*

Proof. Let $C(S) = (x_0, x_1, \dots, x_{g-1})$ and, without loss of generality, suppose for a contradiction that for every $v \in N(x_0) - S$, $C(S) \cap C((S - \{x_0\}) \cup \{v\}) \neq \emptyset$. Let $N(x_0) - S = \{u, v\}$. Then by Lemma 7.22, u and v are both adjacent to x_1 and x_{g-1} . This implies that $N(x_1) - S = N(x_{g-1}) - S = \{u, v\}$.

Let $(S - \{x_1\}) \cup \{v\} = S'$ and $(S - \{x_1\}) \cup \{u\} = S''$. By Lemma 7.22 (a), $G[S']$ and $G[S'']$ each contains exactly one cycle. Now $C(S') = \{x_0, x_{g-1}, v\}$, so $C(S) \cap C(S')$ is nonempty. Similarly, $C(S) \cap C(S'') \neq \emptyset$. Therefore Lemma 7.22 applies to x_1 , so u and v are adjacent to both neighbours of x_1 in $C(S)$. It follows inductively that u and v are adjacent to every vertex of $C(S)$. Therefore, since G is 4-regular, $C(S)$ has order at most 4.

If $C(S) = C_4$ then, by regularity, $G = C_4 \vee \overline{K_2}$. However, any three vertices of C_4 form a minimum 3-conversion set in $C_4 \vee \overline{K_2}$, contradicting our assumption that G does not have an acyclic minimum 3-conversion set. Therefore $C(S) = K_3$. Since $G \neq K_5$, u and v are nonadjacent. We have shown that if $S \in \mathcal{S}(G)$ then $C(S) = K_3$ and every vertex of $C(S)$ is adjacent to two independent vertices, u and v . In Case 1 of the proof of Lemma 7.18 we showed that these conditions contradict the minimality of S . \square

We now restate and prove Theorem 7.19.

Theorem 7.19 (again). Let $G \neq K_5$ be a 4-regular graph. For any minimum 3-conversion set S of G there exist a minimum 3-conversion set S' of G such that $G[S']$ is a linear forest, and a sequence of restricted seed shuffles that transforms S into S' .

Proof. Let $G \neq K_5$ be a 4-regular graph and assume for a contradiction that for some minimum 3-conversion set S of G , no sequence of restricted seed shuffles applied to S yields an acyclic minimum 3-conversion set. That is, assume $S \in \mathcal{S}(G)$. By Lemma 7.23, every vertex $x \in C(S)$ has a neighbour $v \in \overline{S}$ such that $C(S) \cap C((S - \{x\}) \cup \{v\}) = \emptyset$. By Lemma 7.22 (a), $S' = (S - \{x\}) \cup \{v\} \in \mathcal{S}(G)$.

We define a sequence S_0, S_1, \dots, S_ℓ of minimum 3-conversion sets of G , each in $\mathcal{S}(G)$. At each step, we obtain the new 3-conversion set S_{i+1} from S_i by shuffling in such a way that the cycle induced by the new 3-conversion set is disjoint from the cycle induced by the current 3-conversion set. That is, we define S_{i+1} such that $C(S_{i+1}) \cap C(S_i) = \emptyset$. We repeat this process until the new cycle $C(S_{i+1})$ intersects (what remains of) one of the previous ones (but not the current one; the definition of S_{i+1} prevents that). The sequence S_0, S_1, \dots, S_ℓ is defined as follows.

Let $S_0 = S$, and let x_0 be a vertex of $C(S_0)$. By Lemma 7.23, x_0 has a neighbour $v_1 \in \overline{S_0}$ such that $C(S_0) \cap C((S_0 - \{x_0\}) \cup \{v_1\}) = \emptyset$. Define $S_1 = (S_0 - \{x_0\}) \cup \{v_1\}$. By definition, $C(S_0) \cap C(S_1) = \emptyset$.

We repeat the same process to obtain the subsequent 3-conversion sets S_2, S_3, \dots, S_ℓ , but with the added constraint that $x_i \neq v_i$. That is, the vertex we remove from S_i to obtain S_{i+1} must not be the vertex that was just added. This ensures that the process of repeated seed shuffling does not degenerate into shuffling back and forth between the same two vertices v_i and x_{i-1} , which would not lead to the desired contradiction.

For $i \geq 1$, if $C(S_i) \cap (C(S_j) - \{x_j\}) = \emptyset$ for all $0 \leq j < i$, let $x_i \in C(S_i) - \{v_i\}$. By Lemma 7.23, x_i has a neighbour $v_{i+1} \in \overline{S_i}$ satisfying $C(S_i) \cap C((S_i - \{x_i\}) \cup \{v_{i+1}\}) = \emptyset$. Define $S_{i+1} = (S_i - \{x_i\}) \cup \{v_{i+1}\}$.

The process terminates when $C(S_\ell)$ intersects $C(S_j) - \{x_j\}$ for some $j < \ell$. In fact, by ignoring S_0, \dots, S_{j-1} and reindexing, we may assume that $j = 0$. That is, we may assume that $C(S_\ell)$ intersects $C(S_0) - \{x_0\}$. Moreover, $C(S_\ell) \cap C(S_{\ell-1}) - \{x_{\ell-1}\} = \emptyset$

by definition, so $\ell \geq 2$.

By definition, for $0 \leq j < i \leq \ell$, $S_i = (S_j - \{x_j, \dots, x_{i-1}\}) \cup \{v_{j+1}, \dots, v_i\}$. That is, S_i and S_j are identical except for the x_r 's and the v_r 's, and these are all distinct by the condition on our choice of x_i . We use this observation to determine the structure of $C(S_\ell) \cap (C(S_0) - \{x_0\})$. In particular, $C(S_0) - \{x_0\} \subseteq S_\ell$, so, since the cycle component $C(S_\ell)$ contains some of these vertices, it contains all of them. That is, $C(S_\ell) = (C(S_0) - \{x_0\}) \cup \{v_\ell\}$. This implies that v_ℓ is adjacent to both neighbours of x_0 in $C(S_0)$. Denote these vertices by y_1 and y_2 , and let z be the other neighbour of y_2 in $C(S_0)$. Then $z \in C(S_\ell)$ as well, since $z \neq x_0$.

There are two cases.

Case 1. Suppose $v_\ell = x_0$. Then v_ℓ is adjacent to three vertices in S_ℓ , namely y_1 , y_2 and v_1 . Since $v_\ell \in S_\ell$, this contradicts the minimality of S_ℓ .

Case 2. Suppose $v_\ell \neq x_0$. The sequence of cycles, including $C(S_0) \cap C(S_\ell)$, is depicted in Figure 7.5, with vertices of S_ℓ shown in black. Define $S_{\ell+1} = (S_\ell - \{y_1\}) \cup \{x_0\}$. By Lemma 7.22 (a), $S_{\ell+1}$ is a minimum 3-conversion set of G and $G[S_{\ell+1}]$ contains exactly one cycle.

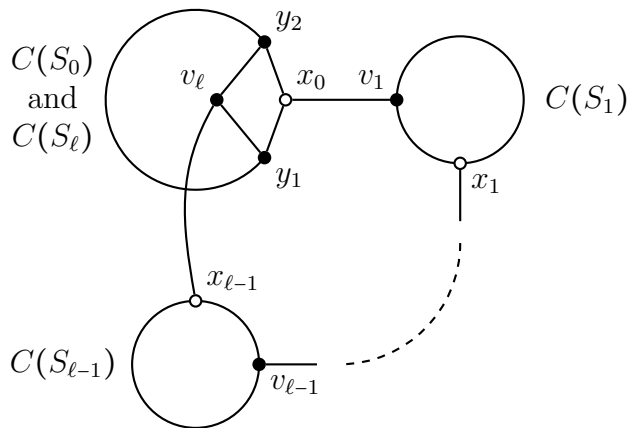


Figure 7.5: The sequence of cycles in the proof of Theorem 7.19 (Case 2), with vertices of S_ℓ in black.

By the minimality of $S_{\ell+1}$, x_0 is not adjacent to v_ℓ , since x_0, y_2, v_1 and v_ℓ are in $S_{\ell+1}$ and x_0 is adjacent to y_2 and v_1 . Similarly, y_2 is adjacent to x_0, v_ℓ and z . Since y_2, x_0 and v_ℓ are in $S_{\ell+1}$, we have $z \in \overline{S_{\ell+1}}$. This implies that $z = y_1$, since y_1 is the only

vertex in $(C(S_0) \cap C(S_\ell)) - S_{\ell+1}$. Therefore $C(S_0)$ and $C(S_\ell)$ are copies of C_3 .

By our choice of G , $G[S_{\ell+1}]$ contains a cycle, $C(S_{\ell+1})$, and that cycle contains x_0 . In fact, by Lemma 7.20, $C(S_{\ell+1}) = K(x_0, S_{\ell+1})$.

The neighbourhood of v_ℓ consists of y_2 (which is in $S_{\ell+1}$), y_1 and $x_{\ell-1}$ (which are in $\overline{S_{\ell+1}}$), and a fourth vertex, w . Since $v_\ell \in K(x_0, S_{\ell+1})$, a 2-regular subgraph of $G[S_{\ell+1}]$, $w \in S_{\ell+1}$. We note that $w \neq x_0$ (the only vertex which is in $S_{\ell+1} - S_\ell$), which implies that $w \in S_\ell$ as well. Therefore $N(v_\ell) \cap S_\ell = \{w, y_1, y_2\}$. Since v_ℓ is itself in S_ℓ , this contradicts the minimality of S_ℓ . \square

Theorem 7.19 does not hold for $k > 3$. In general, there are $(k+1)$ -regular graphs G for which every minimum k -conversion set S induces a cycle. For example, the graphs in \mathcal{G}_k , defined in Section 7.2, have this property, as proved in Proposition 7.10. However, we show in the next section that for every $(k+1)$ -regular graph $G \neq K_{k+2}$, with $k \geq 3$, there is a minimum k -conversion set S that does not induce any $(k-1)$ -regular subgraphs (and from any minimum k -conversion set there is a sequence of restricted seed shuffles that yields such a minimum k -conversion set S). The proof given in Section 7.5 holds for all $k \geq 3$. The idea and structure of the proof are similar to Theorem 7.19 but, unlike in the $k = 3$ case, for $k \geq 4$ a $(k-1)$ -regular component of $G[S]$ may have a cut vertex.

7.5 Avoiding $(k-1)$ -regular subgraphs in k -conversion sets of $(k+1)$ -regular graphs

In Section 7.4 we proved that from any minimum 3-conversion set of a 4-regular graph $G \neq K_5$, it is possible to obtain, via a sequence of restricted seed shuffles, a minimum 3-conversion set that does not induce any cycles—that is, a minimum 3-conversion set that does not induce any 2-regular subgraphs. In this section, we generalize this result to larger values of k by proving that in a $(k+1)$ -regular graph we can always obtain a minimum k -conversion set that does not induce any $(k-1)$ -regular subgraphs. We state the result precisely in Theorem 7.24, and prove the theorem at the end of the section, after some lemmas.

Theorem 7.24. *If $G \neq K_{k+2}$ is a $(k+1)$ -regular graph then, from any minimum k -conversion set S of G , it is possible to obtain, via a sequence of restricted seed shuffles, a new minimum k -conversion set with no $(k-1)$ -regular subgraphs.*

We note that, for a minimum k -conversion set S of G , any $(k-1)$ -regular subgraph of $G[S]$ is a whole component of $G[S]$, since $G[S]$ has maximum degree at most $k-1$. Therefore our goal for the rest of the section is to show that we can obtain a minimum k -conversion set S' such that $G[S']$ has no $(k-1)$ -regular components.

For the rest of Section 7.5, let $G \neq K_{k+2}$ be a $(k+1)$ -regular graph, where $k \geq 3$, and let S be a minimum k -conversion set of G . We prove the main result (Theorem 7.24) in several steps. We first show in Lemma 7.26 that any minimum k -conversion set S can be transformed into a minimum k -conversion set S' with at most one $(k-1)$ -regular component, $C(S')$. We then determine a number of conditions that guarantee the unique remaining $(k-1)$ -regular component $C(S')$ can be eliminated and we constrain the structure of $G[S']$ and $G[\overline{S'}]$ when S' does not satisfy those conditions.

The proof that we can always reduce the number of $(k-1)$ -regular components to one follows from the simple observation (stated in Lemma 7.25) that restricted seed shuffling never increases the number of $(k-1)$ -regular components.

Lemma 7.25. *If S is a minimum k -conversion set of G and S' is obtained from S by restricted seed shuffling, then the number of $(k-1)$ -regular components of $G[S']$ is less than or equal to the number of such components of $G[S]$.*

Proof. Suppose S' is obtained from S by the restricted seed shuffle $x \mapsto v$. Since $x \notin S'$, the $(k-1)$ -regular component of $G[S]$ containing x is not a component of $G[S']$. On the other hand, any $(k-1)$ -regular component of $G[S']$ that is not a component of $G[S]$ contains v , so there is at most one. \square

Lemma 7.26. *Starting from any minimum k -conversion set S we can, by repeated restricted seed shuffling, obtain a minimum k -conversion set S' such that at most one component of $G[S']$ is $(k-1)$ -regular.*

Proof. Among all minimum k -conversion sets of G that are obtainable from S by a sequence of restricted seed shuffles, restrict to those that induce the minimum

number c of $(k-1)$ -regular components. If $c \leq 1$, there is nothing to prove, so assume $c > 1$; we will derive a contradiction. From the restricted set of minimum k -conversion sets of G , choose S' such that the smallest number arising as the distance (in G) between any two $(k-1)$ -regular components of $G[S']$ is as small as possible. Call this distance $d \geq 2$, and let C_1 and C_2 be two $(k-1)$ -regular components of $G[S']$ such that $d_G(C_1, C_2) = d$. Let x_1 and x_2 be two vertices of C_1 and C_2 , respectively, such that $d_G(x_1, x_2) = d$. If $d = 2$, the restricted seed shuffle sending x_1 to a common neighbour of x_1 and x_2 makes x_2 into a vertex of degree k in the resulting seed set, contradicting the minimality of S' . Thus $d \geq 3$. Let x_1, a, b, \dots, x_2 be a shortest path from x_1 to x_2 . By our choice of S' , the restricted seed shuffle $x_1 \mapsto a$ decreases the number of $(k-1)$ -regular components, or else we end up with two that are closer than before. Therefore $c \leq 1$, as desired. \square

Given an arbitrary minimum k -conversion set S , Lemma 7.26 guarantees that it is possible to transform it into a new minimum k -conversion set with at most one $(k-1)$ -regular component. Therefore, for the rest of the section, we may restrict our attention to the minimum k -conversion sets S that have exactly one $(k-1)$ -regular component. We denote that unique $(k-1)$ -regular component of S by $C(S)$.

Definition 7.27. *For any $(k+1)$ -regular graph G , let $\mathcal{S}(G)$ be the set of all minimum k -conversion sets S of G that induce exactly one $(k-1)$ -regular component, and such that no sequence of restricted seed shuffles applied to S yields a minimum k -conversion set S' with no $(k-1)$ -regular component.*

We prove Theorem 7.24 by showing that $\mathcal{S}(G)$ is empty for all $G \neq K_{k+2}$. Through a series of lemmas, we establish a number of properties that must be satisfied by any k -conversion set in $\mathcal{S}(G)$. Then, in the proof of Theorem 7.24, we derive a contradiction from this collection of properties. The first property we prove about $S \in \mathcal{S}(G)$ is that \overline{S} induces a collection of paths.

Lemma 7.28. *Let S be a minimum k -conversion set of G with exactly one $(k-1)$ -regular component, $C(S)$. If the forest $G[\overline{S}]$ is not a linear forest (that is, a collection of paths), then we can, by a sequence of restricted seed shuffles, get a minimum conversion set with no $(k-1)$ -regular components. Equivalently, for all $S \in \mathcal{S}(G)$, $G[\overline{S}]$ is a linear forest.*

Proof. If \bar{S} is not a collection of paths then there is at least one vertex $y \in \bar{S}$ with $\deg_{\bar{S}}(y) \geq 3$. Therefore y has at most $k - 2$ neighbours in S . We use induction on the distance from $C(S)$ to y . If $d(C(S), y) = 1$ then some vertex $x_0 \in C(S)$ is adjacent to y . The minimum k -conversion set S' obtained by the restricted seed shuffle $x_0 \mapsto y$ does not contain $C(S)$, and since y now has at most $k - 3$ seed neighbours, the component of $G[S']$ containing y is not $(k - 1)$ -regular. Furthermore, components of $G[S']$ that do not contain y are components of $G[S]$ distinct from $C(S)$, so they are not $(k - 1)$ -regular either. Therefore S' does not induce any $(k - 1)$ -regular components, so $S' \notin \mathcal{S}(G)$.

For the induction hypothesis, assume the statement is true whenever S is a minimum k -conversion set of G that has exactly one $(k - 1)$ -regular component $C(S)$ and the distance from $C(S)$ to the nearest vertex of degree at least 3 in $G[\bar{S}]$ is at most $d - 1$, with $d \geq 2$. Consider a minimum k -conversion set S for which this distance equals d . Let x_0, x_1, \dots, y be a shortest path from $C(S)$ to y , where $x_0 \in C(S)$. If the seed set S' produced by shuffling $x_0 \mapsto x_1$ does not contain a $(k - 1)$ -regular component, we're done. On the other hand, if it does contain a $(k - 1)$ -regular component $C(S')$ then $d(C(S'), y) \leq d - 1$. Therefore by the induction hypothesis, it is possible to obtain a minimum k -conversion set S'' from S' by a sequence of restricted seed shuffles. Since the shuffle $x_0 \mapsto x_1$ was itself a restricted seed shuffle, the statement is true for S as well. \square

Lemma 7.29. *Let $G \neq K_{k+2}$ be a $(k + 1)$ -regular graph. If $S \in \mathcal{S}(G)$ then for every vertex $v \in C(S)$, the nonseed neighbours of v are the leaves of a component path of $G[\bar{S}]$.*

Proof. By Lemma 7.28, $G[\bar{S}]$ is a linear forest. If some $x \in C(S)$ is adjacent to a vertex $y \in \bar{S}$ with $\deg_{\bar{S}}(y) \geq 2$, then the restricted shuffle $x \mapsto y$ results in a minimum k -conversion set S' that does not induce any $(k - 1)$ -regular components. Indeed, after the shuffle y is adjacent to at least three non-seed vertices, so it is not in a $(k - 1)$ -regular component of S' and therefore S' does not have any such components. This implies that $S \notin \mathcal{S}(G)$. Therefore, for all $S \in \mathcal{S}(G)$, the neighbours of any vertex $x \in C(S)$ are leaves of $G[\bar{S}]$. To see that the neighbours of x are leaves of the same component path of $G[\bar{S}]$, note that $\bar{S} \cup \{x\}$ contains a cycle, by the minimality of S . \square

In light of Lemma 7.29, we may associate with each vertex $x \in C(S)$ a component path of $G[\overline{S}]$.

Definition 7.30. *Let $S \in \mathcal{S}(G)$. For any vertex x of $C(S)$ we denote by $P(x)$ the component path of $G[\overline{S}]$ whose leaves are neighbours of x .*

In the next lemma we constrain the structure of $C(S)$ for $S \in \mathcal{S}(G)$.

Lemma 7.31. *Let $G \neq K_{k+2}$ and let $S \in \mathcal{S}(G)$. If x and y are distinct vertices of $C(S)$ such that $P(x) = P(y)$, then x and y are adjacent.*

Proof. Let u be a leaf of $P(x) = P(y)$. If x and y are nonadjacent then the shuffle $x \mapsto u$ makes y into a seed vertex of degree k , contradicting the minimality of S . \square

Thus, if $S \in \mathcal{S}(G)$, we have the following picture of the unique $(k-1)$ -regular component $C(S)$ of $G[S]$. Let P^1, P^2, \dots, P^r be the distinct component paths of $G[\overline{S}]$ occurring as $P(x)$ for some $x \in C(S)$. Then by Lemma 7.31, for each i , the set of vertices $\{x \in C(S) : P(x) = P^i\}$ induces a complete subgraph of $G[S]$, say of order $r_i < k$. Therefore we have a covering of $C(S)$ by disjoint complete graphs K_{r_i} such that all of the vertices x in any one of the K_{r_i} have $P(x) = P^i$.

Lemma 7.32. *Let $G \neq K_{k+2}$ be a $(k+1)$ -regular graph and let $S \in \mathcal{S}(G)$. If P is a path in $G[\overline{S}]$ such that $V_P = \{x \in C(S) : P(x) = P\}$ has at least two elements, then*

- (a) V_P is a clique in S ,
- (b) P is an edge, and
- (c) every $x \in V_P$ is a cut vertex of $C(S)$.

Proof. The first statement follows immediately from Lemma 7.31.

For the second statement, we show that if P has length at least 2 then there is a restricted seed shuffle $S \mapsto S' \notin \mathcal{S}(G)$, so $S \notin \mathcal{S}(G)$. Write $P = u_0 u_1 \dots u_\ell$, and assume that $\ell \geq 2$. By assumption there exist two (adjacent) vertices $x, y \in C(S)$ with $P(x) = P(y) = P$. Let S' be the minimum k -conversion set obtained from S by performing the restricted seed shuffle $x \mapsto u_0$. By assumption, $S' \in \mathcal{S}(G)$, so $G[S']$ has a $(k-1)$ -regular component $C(S')$, specifically, the component of $G[S']$ containing u_0 .

Denote by P' the path $u_1 \dots u_\ell x$, which is the path associated with u_0 in $G[\overline{S'}]$. Since y and u_0 are adjacent, $y \in C(S')$, and since y is adjacent to x , the path associated with y in $G[\overline{S'}]$ is also P' . Therefore y is adjacent to the other endpoint u_1 of P' . Since $\ell \geq 2$, u_0, u_1 and u_ℓ are all distinct vertices of \overline{S} , and we now see that y is adjacent to all of them. This contradicts the assumption that y is in a $(k-1)$ -regular component of $G[S]$.

For the third statement, again start with two vertices $x, y \in C(S)$ such that $P(x) = P(y) = P = uv$ and suppose x is not a cut vertex. We derive a contradiction by showing that, in this case, $G = K_{k+2}$. Let S' be the minimum k -conversion set of G obtained by the restricted seed shuffle $x \mapsto u$. By assumption $S' \in \mathcal{S}(G)$. Since x is not a cut vertex of $C(S)$, every vertex of $C(S) - \{x\}$ belongs to $C(S')$, because u is connected to y and $y \in C(S') - \{x\}$. In particular, the neighbourhood N of x in $C(S)$ belongs to $C(S')$ (including y). Therefore every vertex of N still has $k-1$ seed neighbours after the shuffle $x \mapsto u$, so they are all adjacent to u . Thus, for every $z \in N \cup \{x\}$, $P(z) = P$. Hence, by Lemma 7.31, all of the $k-1$ vertices in N are also adjacent to each other. We have shown that the $k+2$ vertices in $N \cup \{x\} \cup \{P\}$ are pairwise adjacent, so $G = K_{k+2}$. \square

For $x \in C(S)$, we say x is a *sharing vertex* if $P(x) = P(y)$ for some $y \in C(S), y \neq x$, and in this case we call $P(x)$ a *shared path*.

Lemma 7.32 (b) and (c) imply that for all $S \in \mathcal{S}(G)$, all shared paths (if there are any) are merely edges and all sharing vertices (if there are any) are cut vertices of $C(S)$. For the proof of Theorem 7.24 we require $C(S)$ to have at least two non-cut vertices. This is guaranteed by Lemma 7.33.

Lemma 7.33. *Let $G \neq K_{k+2}$ and suppose $S \in \mathcal{S}(G)$. Then $C(S)$ contains at least two non-cut, non-sharing vertices.*

Proof. The leaves of any spanning tree of $C(S)$ are not cut vertices of $C(S)$. By Lemma 7.32 (3), they are not sharing vertices. \square

Lemma 7.34. *Let $S \in \mathcal{S}(G)$, let x be a non-cut (hence non-sharing) vertex of $C(S)$ and let v be a leaf of $P(x)$. Let $S' \in \mathcal{S}(G)$ be the minimum k -conversion set of G obtained from the restricted seed shuffle $x \mapsto v$. Then $C(S') \cap C(S) = \emptyset$.*

Proof. Since x is a non-cut vertex of $C(S)$, it is also a non-sharing vertex, by Lemma 7.32 (c). Suppose for a contradiction that some vertex $y \in C(S) - \{x\}$ is in $C(S')$. Then, since x is not a cut vertex, every vertex of $C(S) - \{x\}$ is in the component $C(S')$ (that is, $C(S') = (C(S) - \{x\}) \cup \{v\}$). This implies that every neighbour u of x in $C(S)$ is adjacent to v , since they must have degree $k-1$ in $C(S')$, and therefore $P(u) = P(x)$. This makes x a sharing vertex, which is a contradiction. \square

We have established a large set of properties that must be satisfied by any minimum k -conversion set $S \in \mathcal{S}(G)$. In the proof of Theorem 7.24 we will show that in fact $\mathcal{S}(G)$ is empty by showing that these properties are contradictory.

Theorem 7.24 (again). Let $G \neq K_{k+2}$ be a $(k+1)$ -regular graph. Then $\mathcal{S}(G) = \emptyset$. That is, if S_0 is any minimum k -conversion set of G then, by a sequence of restricted seed shuffles, S_0 can be transformed into a minimum k -conversion set S_ℓ of G with no $(k-1)$ -regular component.

Proof. Assume for a contradiction that $S_0 \in \mathcal{S}(G)$. We define a sequence $S_0 \mapsto S_1 \mapsto \dots \mapsto S_\ell$ of minimum k -conversion sets of G , each of which is obtained from the last by a restricted seed shuffle. By assumption, $S_i \in \mathcal{S}(G)$ for all i .

By Lemma 7.33, there is a non-cut vertex $x_0 \in C(S_0)$ (in fact there are two). Let v_1 be a leaf of $P(x_0)$ (a non-shared path) and denote by S_1 the minimum k -conversion set obtained by shuffling $x_0 \mapsto v_1$.

For $1 \leq i \leq \ell - 1$, the set S_{i+1} is obtained from S_i as follows.

Since $S_i \in \mathcal{S}(G)$, $G[S_i]$ contains a unique $(k-1)$ -regular component, $C(S_i)$, namely the component containing the vertex v_i . Furthermore, by Lemma 7.33, $C(S_i)$ has at least two non-cut, non-sharing vertices. If v_i is not adjacent to any vertex of $C(S_j) - \{x_j\}$ for any $j < i$ (that is, if the component $C(S_i)$ does not contain any remaining vertices of a previous $(k-1)$ -regular component), then let $x_i \neq v_i$ be a non-cut (and hence non-sharing) vertex of $C(S_i)$. Let v_{i+1} be an endpoint of (the nonshared path) $P(x_i)$, and shuffle $x_i \mapsto v_{i+1}$ to obtain the minimum k -conversion set S_{i+1} .

The process terminates when some v_ℓ is adjacent to a vertex of $C(S_j) - \{x_j\}$ for some $j < \ell$. This is illustrated in Figure 7.6.

By Lemma 7.34, $j \neq \ell - 1$. In fact, by ignoring S_0, \dots, S_{j-1} and reindexing, we may assume that $j = 0$. That is, we may assume that v_ℓ is adjacent to some vertex of $C(S_0) - \{x_0\}$. Since x_0 is not a cut vertex of $C(S_0)$, every vertex of $C(S_0) - \{x_0\}$ is therefore in the same component of $G[S_\ell]$ as v_ℓ . By assumption (since $S_\ell \in \mathcal{S}(G)$), this component is $(k-1)$ -regular, so we call it $C(S_\ell)$.

Let N denote the $(k-1)$ -element set of neighbours of x_0 in $C(S_0)$. We claim that v_ℓ is adjacent to every vertex $w \in N$. First, note that the only difference between S_0 and S_ℓ is that all of the x_i 's have been removed from the seed set and all of the v_i 's have been added, and these are all distinct vertices. When x_0 was removed from the seed set, the number of seed-neighbours of every $w \in N$ was reduced to $k-2$. In $G[S_\ell]$, these vertices all have degree $k-1$, since they are in $(C(S_0) - \{x_0\}) \subseteq C(S_\ell)$, so each one must be adjacent to exactly one vertex from $\{v_1, \dots, v_\ell\}$. However, for $1 \leq i < \ell$, v_i is not adjacent to any vertex of $S_0 - \{x_0\}$, or the algorithm would have terminated sooner. Thus, v_ℓ is adjacent to all $k-1$ vertices of N , and therefore, by regularity, $C(S_\ell) = (C(S_0) - \{x_0\}) \cup \{v_\ell\}$.

Our next claim is that $v_\ell \neq x_0$. Indeed, x_0 is adjacent to k vertices in S_ℓ , namely v_1 and the $k-1$ vertices of N , so by the minimality of S_ℓ , $x_0 \notin S_\ell$.

Since $x_0 \neq v_\ell$, $x_0 \notin S_\ell$. Let $y \in N$. Then performing the shuffle $y \mapsto x_0$ followed by the shuffle $v_\ell \mapsto y$ results in a k -conversion set S'_ℓ containing x_0 , N and v_1 . This contradicts the minimality of S_ℓ , since $|S'_\ell| = |S_\ell|$ but $S_\ell - \{x_0\}$ is still a k -conversion set. \square

7.6 Consequences of Theorems 7.2, 7.13, 7.19 and 7.24

In this section we present some corollaries of our results on subgraph-avoiding minimum k -conversion sets in $(k+1)$ -regular graphs. We give a new proof of Brooks's Theorem as a corollary of Theorem 7.24, and present bounds on various types of

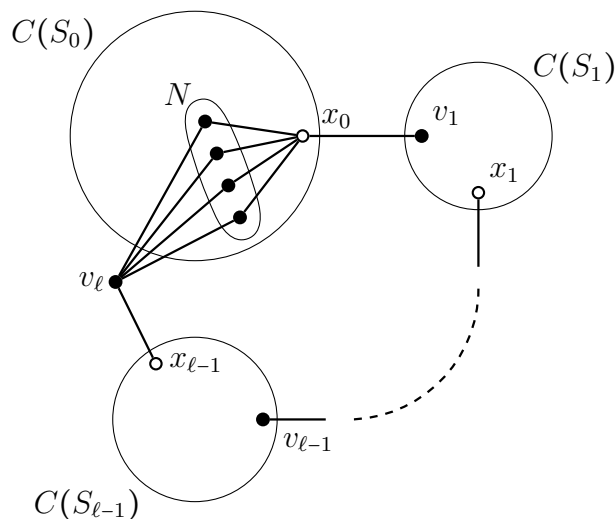


Figure 7.6: The outcome of the algorithm in the proof of Theorem 7.24, with vertices of S_ℓ shown in black.

forest partitions of G , which can be viewed as vertex colourings in which each colour class induces a forest. Finally, we derive upper bounds on the k -conversion number of $(k + 1)$ -regular graphs.

Corollary 7.35. *The vertex set of every graph $G \neq K_4$ of maximum degree 3 can be partitioned into two sets that induce a forest and an independent set, respectively.*

Proof. Let H be a cubic graph that contains G as an induced subgraph. By Theorem 7.2, the vertices of H can be partitioned into an independent set S and a maximum forest \bar{S} . The result follows by restricting S and \bar{S} to G . \square

Corollary 7.36. *The vertex set of every 4-regular graph $G \neq K_5$ can be partitioned into two sets that induce a maximum forest and a linear forest, respectively.*

Proof. By Theorem 7.19, G has an acyclic minimum 3-conversion set S . By the minimality of S , $G[S]$ has maximum degree 2, hence it is a linear forest, and $G[\bar{S}]$ is a maximum induced forest. \square

Corollary 7.37. *The vertex set of every graph $G \neq K_5$ of maximum degree 4 can be partitioned into two sets that induce a forest and a linear forest, respectively.*

Proof. The result follows from Corollary 7.36 by embedding G in a 4-regular graph. \square

Corollary 7.38. *Let $G \neq K_{r+1}$ be a graph of maximum degree $r \geq 3$. Then $V(G)$ can be partitioned into sets X and \bar{X} such that $G[X]$ has maximum degree at most $r - 2$ but does not contain an $(r - 2)$ -regular subgraph, and $G[\bar{X}]$ is a forest.*

Proof. Let H be an r -regular graph that contains G as an induced subgraph. By Theorem 7.24, H has a minimum $(r - 1)$ -conversion set S such that $H[S]$ contains no $(r - 2)$ -regular subgraphs. By the minimality of S , $H[S]$ has maximum degree at most $r - 2$, and since S is an $(r - 1)$ -conversion set, $H[\bar{S}]$ is a forest. The result follows by taking X and \bar{X} to be the restriction of S and \bar{S} , respectively, to G . \square

In Corollary 7.40 we provide an alternative proof of Brooks's Theorem, using Corollary 7.38 and the following lemma, from [29]. Recall that a *proper vertex colouring* of G is a partition of $V(G)$ into independent sets, and the *chromatic number* of G , denoted by $\chi(G)$, is the minimum number of sets in a proper vertex colouring of G . We write $H \trianglelefteq G$ if H is an induced subgraph of G .

Lemma 7.39. [29] *For every graph G , $\chi(G) \leq 1 + \max\{\delta(H) : H \trianglelefteq G\}$.*

Corollary 7.40 (Brooks's Theorem). *If $r \geq 3$ and $G \neq K_{r+1}$ is a graph of maximum degree r , then $\chi(G) \leq r$.*

Proof. By Corollary 7.38, $V(G)$ can be partitioned into sets X and \bar{X} such that $G[X]$ has maximum degree at most $r - 2$ but does not have an $(r - 2)$ -regular subgraph, and $G[\bar{X}]$ is acyclic. Therefore $G[\bar{X}]$ can be 2-coloured.

Let H be any induced subgraph of $G[X]$. Then $\Delta(H) \leq r - 2$ and H is not $(r - 2)$ -regular, hence $\delta(H) < r - 2$. Therefore $\max\{\delta(H) : H \trianglelefteq G\} \leq r - 3$. By Lemma 7.39, it follows that $\chi(G[X]) \leq r - 2$.

Any $(r - 2)$ -colouring of $G[X]$ and any 2-colouring of $G[\bar{X}]$ now give an r -colouring of G . \square

One generalization of proper vertex colourings involves partitioning $V(G)$ into sets such that each set induces a forest. We call such a partition a *forest partition*

of G . The minimum number of sets in a forest partition is called the *vertex arboricity* of G and denoted by $a(G)$. Vertex arboricity, which is studied in [25] and [64], is a variation on the *arboricity* of a graph G , defined as the minimum number of sets needed to partition the edge set of G such that each set induces a forest. Arboricity was first studied in the early 1960s by Nash-Williams and Tutte [80, 81, 110]. The following bound on $a(G)$ is due to Kronk and Mitchem [64]. We give a new proof using our previous results.

Corollary 7.41 (The Kronk-Mitchem Bound). *If G is neither a cycle nor an odd clique then G has vertex arboricity at most $\lceil \frac{\Delta(G)}{2} \rceil$.*

Proof. We first prove that the bound holds if G is an even clique. For any K_n , $a(K_n) = \lceil \frac{n}{2} \rceil$, since any three vertices induce a cycle. If G is an even clique then $G = K_{\Delta+1}$ for some odd Δ , so $a(G) = \frac{\Delta+1}{2} = \lceil \frac{\Delta}{2} \rceil$.

Now assume G is neither a cycle nor a clique. In that case, if $\Delta(G) = 2$ then G is a path, hence $a(G) = 1$, as required. If $\Delta(G) = 3$ or 4 then by Corollaries 7.35 and 7.37, respectively, $a(G) = 2$, as required.

Let $r \geq 5$ and assume that $a(G) \leq \lceil \frac{\Delta}{2} \rceil$ for all graphs G of maximum degree $\Delta < r$ that are neither cycles nor odd cliques. Now let G be a graph of maximum degree r and assume G is neither a cycle nor an odd clique. Let $\{X, \bar{X}\}$ be a partition of $V(G)$ as described in Corollary 7.38. Since $G[X]$ has maximum degree at most $r - 2$ and does not contain K_{r-1} as a subgraph, the induction hypothesis guarantees that $G[X]$ has a forest partition into at most $\lceil \frac{r-2}{2} \rceil$ sets. The result follows since $G[\bar{X}]$ is itself a forest. \square

The following result strengthens Corollary 7.41 by specifying that a forest partition of size at most $\lceil \frac{\Delta(G)}{2} \rceil$ can be chosen to include a linear forest, or in many cases, an independent set.

Corollary 7.42. *Suppose that G is neither a cycle nor a clique and that $\Delta(G) \geq 2$. If $\Delta(G)$ is odd then G has a forest partition into at most $\frac{\Delta(G)+1}{2}$ sets, one of which is independent, and if $\Delta(G)$ is even then G has a forest partition into at most $\frac{\Delta(G)}{2}$ sets, one of which induces a linear forest.*

Proof. If $\Delta = 2$ then G is itself a linear forest, since it is not a cycle. If $\Delta = 3$ or $\Delta = 4$ the statement follows from Corollaries 7.35 and 7.37, respectively. Let $r \geq 5$ and assume the statement holds all non-complete graphs of maximum degree $3 \leq \Delta < r$. Let $G \neq K_{r+1}$ be a graph of maximum degree r . Let $V(G) = X \cup \overline{X}$ be a partition of $V(G)$ as described in Corollary 7.38. Then $G[\overline{X}]$ is a forest and $G[X]$ has maximum degree $D \leq r - 2$ which does not contain K_{r-1} , since it does not contain any $(r - 2)$ -regular subgraph. By the induction hypothesis, if D is odd then $G[X]$ has a forest partition into at most $\frac{D+1}{2} \leq \frac{r+1}{2} - 1$ sets, one of which is independent, and if D is even then $G[X]$ has a forest partition into at most $\frac{D}{2} \leq \frac{r}{2} - 1$ sets, one of which induces a linear forest. If r is even or D is odd then we are done (noting that an independent set is a linear forest, for the case where r is even and D is odd). If r is odd and D is even then $D \leq r - 3$, so G has a forest partition into at most $\frac{r-3}{2} + 1 = \frac{r-1}{2}$ sets, one of which induces a linear forest. Then, by further partitioning one of the forests, we may obtain a forest partition into at most $\frac{r-3}{2} + 2 = \frac{r+1}{2}$ sets, one of which is independent. \square

From Corollary 7.42 we obtain the following two corollaries. Corollary 7.43 strengthens Corollary 7.42 for graphs with odd $\Delta \geq 5$, showing that they have a forest partition satisfying the bound of Corollary 7.42 in which one of the sets is not only independent but maximal independent. Corollary 7.44 strengthens the result of Corollary 7.42 in the case where G is regular, stating that one of the sets induces a maximum forest.

Corollary 7.43. *Let G be a non-complete graph with $\Delta(G) = 2r + 1 \geq 5$ and let A be any maximal independent set of G such that $G - A$ does not contain a copy of K_{2r+1} . Then $V(G)$ can be partitioned into A and at most r sets, each of which induces a forest and one of which induces a linear forest.*

Proof. Since A is maximal independent, it is dominating, hence $\Delta(G - A) \leq 2r$. Let H be the union of all components of $G - A$ that are not cycles or cliques. By Corollary 7.42, H has a forest partition into at most r sets, one of which induces a (possibly independent) linear forest. Now, since $G - A$ does not contain K_{2r+1} , any clique component or cycle component of $G - A$ has a forest partition into at most r sets, each of which induces a linear forest. Since the union of any one of these sets with any one of the sets in a forest partition of H still induces a forest, the $G - A$ has forest partition with the required properties. \square

Corollary 7.44. *Let $G \neq K_{r+1}$ be an r -regular graph, for $r \geq 3$. If r is odd then G has a forest partition into at most $\frac{r+1}{2}$ sets of which one set is independent and another induces a maximum forest, and if r is even then G has a forest partition into at most $\frac{r}{2}$ sets of which one set induces a maximum forest and another induces a linear forest.*

Proof. If G is r -regular then Theorem 7.24 states that $V(G)$ can be partitioned into sets X and \bar{X} where $G[\bar{X}]$ is a maximum forest and $G[X]$ has maximum degree at most $(r - 2)$ and does not contain any $(r - 2)$ -regular subgraphs. The proof is otherwise identical to the proof of Corollary 7.42 for $\Delta(G) \geq 3$. \square

Catlin and Lai [25] have proved that for odd r , Corollary 7.44 holds for graphs of maximum degree r , and for even r a relaxed version of Corollary 7.44 holds for graphs of maximum degree r . We state their results below.

Theorem 7.45. [25, Theorems 1 and 2(a)] *Let $G \neq K_{\Delta+1}$ be a graph of maximum degree Δ . If Δ is odd then G has a forest partition into at most $\frac{\Delta+1}{2}$ sets, of which one induces an independent set and another induces a maximum forest. If Δ is even then G has a forest partition into at most $\frac{\Delta}{2}$ sets, one of which induces a maximum forest.*

Catlin and Lai also prove that for non-complete graphs G of odd maximum degree, we may choose a forest partition of G in which one of the sets is a maximum independent set, rather than one that contains a maximum forest and a (not necessarily maximum) independent set.

Theorem 7.46. [25, Theorem 2(b)] *Let $G \neq K_{\Delta+1}$ be a graph of maximum degree Δ , for odd Δ . Then G has a forest partition into at most $\frac{\Delta+1}{2}$ sets, one of which induces a maximum independent set.*

Theorem 7.46 can also be obtained as a corollary of Corollary 7.43.

In Section 2.3.2 we presented a sharp upper bound on $c_2(G)$ for cubic graphs and the upper bound $\phi(G) \leq \frac{r-2}{r}n$ for r -regular graphs of order n , with $r \geq 4$, due to Punnim [91]. This corresponds to the bound $c_k(G) \leq \frac{k-1}{k+1}n$ for $(k+1)$ -regular graphs, $k \geq 3$. In our final corollary of Theorems 7.2, 7.13, 7.19 and 7.24, we derive Punnim's

bound for odd k and prove that it is sharp in this case, and we prove a slightly weaker bound for even k . Specifically, these bounds follow from Corollary 7.41 of Section 7.6, which is itself a consequence of Theorem 7.24 of Section 7.5 (though the original proof is due to Kronk and Mitchem [64]). For a $(k+1)$ -regular graph $G \neq K_{k+2}$, Corollary 7.41 states that there exists a forest partition of G into at most $\lceil \frac{k+1}{2} \rceil$ sets.

Proposition 7.47. *Let G be a $(k+1)$ -regular graph of order n , $k \geq 2$. If k is odd and $G \neq K_{k+2}$, then $c_k(G) \leq \frac{k-1}{k+1}n$, and this bound is sharp. If k is even or G is any clique then $c_k(G) \leq \frac{k}{k+2}n$.*

Proof. We first prove the bound when G is a clique; that is, $G = K_{k+2}$. If S consists of any k vertices of K_{k+2} then \bar{S} induces P_2 , so S is a k -conversion set of G . On the other hand, if S consists of any $k-1$ vertices of K_{k+2} then \bar{S} contains a triangle, so S is not a k -conversion set. Therefore $c_k(K_{k+2}) = k = \frac{k}{k+2}n$, as required.

Now suppose $k = 2\ell + 1$ and $G \neq K_{k+2}$. By Corollary 7.41, $a(G) \leq \lceil \frac{k+1}{2} \rceil = \ell$. Let V_1, \dots, V_ℓ be a forest partition of G , where the labelling is such that $|V_1| \leq \dots \leq |V_\ell|$. Then $X = \cup_{i=1}^{\ell-1} V_i$ is a k -conversion set of G , since $V - X$ is a forest, and $|X| \leq \frac{\ell-1}{\ell}n = \frac{k-1}{k+1}n$. The result follows since $c_k(G) \leq |X|$.

Finally, suppose $k = 2\ell$. We use the same argument as in the odd case, but here $a(G) \leq \ell + 1$. We define $V_1, \dots, V_{\ell+1}$ as before, and set $X = \cup_{i=1}^{\ell} V_i$. Then $|X| \leq \frac{\ell}{\ell+1}n = \frac{k}{k+2}n$. The result follows as before.

To prove sharpness in the case where k is odd, consider the $(k+1)$ -regular graphs $G = K_{k+2} \circ K_{k+1}$, as defined in Definition 4.20. Any k -conversion set of G must contain $k-1$ vertices from each copy of K_{k+1} , so $c_k(G) \geq \frac{k-1}{k+1}n$. \square

Punnim's result, which we state in Theorem 2.30, establishes that the bound for even k given in Theorem 7.47 is not sharp. This suggests the following question, which remains open.

Question 7.48. For even $k \geq 2$, is the largest ratio $\frac{c_k(G)}{n}$ achieved by a $(k+1)$ -regular graph $G \neq K_{k+2}$ of order n ?

Dross *et al.* [39] have proved lower bounds on the size of a largest induced forest in planar graphs with certain girth restrictions (their work corrects an earlier result

of the same nature by Kowalik *et al.*). If the graphs are also $(k + 1)$ -regular, this translates to an upper bound on the k -conversion number for this class of graphs, which we state in Theorem 7.49. We then compare this bound with the bound of Proposition 7.47.

Theorem 7.49 (From [39], Theorem 10). *Let G be a triangle-free, planar $(k + 1)$ -regular graph of order n . Then $c_k(G) \leq \frac{n(7k+19)}{88}$.*

Proof. By Theorem 10 of [39], the forest number of a triangle-free planar graph G with n vertices and m edges is at most $\max\left\{\frac{38n-7m}{44}, n - \frac{m}{4}\right\}$. If G is $(k + 1)$ -regular then a maximum induced forest is the complement of a minimum k -conversion set, so

$$\begin{aligned} c_k(G) &\leq n - \max\left\{\frac{38n-7m}{44}, n - \frac{m}{4}\right\} \\ &= \min\left\{\frac{6n+7m}{44}, \frac{m}{4}\right\} \\ &= \min\left\{\frac{n(7k+19)}{88}, \frac{n(k+1)}{8}\right\}, \end{aligned}$$

where the last equality is obtained by setting $m = \frac{n(k+1)}{2}$, since G is $(k + 1)$ -regular, and simplifying. Finally, we note that for all $k \geq 2$, $\frac{(7k+19)}{88} \leq \frac{(k+1)}{8}$. \square

For $k = 2$, the strongest upper bound is that of Theorem 2.29, from Section 2.3.2. For $k \leq 6$ the bound $c_k(G) \leq \frac{(7k+19)}{88}n$ of Theorem 7.49 (for planar, triangle-free graphs) is stronger than the bound

$$c_k(G) \leq \begin{cases} \frac{k-1}{k+1}n & \text{if } k \text{ is odd,} \\ \frac{k}{k+2}n & \text{if } k \text{ is even,} \end{cases}$$

of Proposition 7.47, but this is reversed for $k \geq 7$. This suggests that—at least for $k \geq 7$ —regularity constrains the k -conversion number more tightly than planarity or girth. On the other hand, Dross *et al.* [40] go on to prove stronger bounds on (what amounts to) $c_k(G)$ for planar $(k + 1)$ -regular graphs of larger girth. In particular, they prove that the decycling number of any planar graph is at most $\frac{4m}{3g}$, where m is the number of edges and g is the girth. If G is $(k + 1)$ -regular we replace m with $\frac{(k+1)n}{2}$ (and “decycling number” with “ k -conversion number”) to obtain the bound

$$c_k(G) \leq \frac{4(k+1)}{6g}n. \tag{7.1}$$

We now note that the bound (7.1) is stronger than the bound of Proposition 7.47 when $g > \frac{4(k+1)^2}{3(k-1)}$ for odd k , and when $g > \frac{2(k+1)(k+2)}{3k}$ for even k . We conclude that the combination of planarity and large girth seems to constrain the k -conversion number more tightly than regularity.

Chapter 8

Conclusion and open problems

Throughout this dissertation we have proved new results on both the size and the structure of minimum k -conversion sets in regular graphs, trees, and graphs of maximum degree $k + 1$.

In Chapter 3 we began with the trivial lower bound $c_k(G) \geq k$, and asked which graphs meet this bound. For k -regular graphs, we completely characterized the structure of those graphs. For $(k + 1)$ -regular graphs we constrained the order n of the graphs that meet the trivial lower bound, to $n \leq 2k + 2$, and for each $k \geq 2$ we constructed graphs of this order that meet the trivial bound.

For $(k+1)$ -regular graphs of order greater than $2k+2$, the trivial lower bound is not sharp, but there is an existing non-trivial lower bound on $c_k(G)$ for all $(k+1)$ -regular graphs G . We derived this lower bound and proved a new structural characterization of the k -conversion sets that meet it in Proposition 3.7. This characterization uses the well known equivalence between k -conversion sets and decycling sets in $(k+1)$ -regular graphs.

In Section 3.3 we considered k -conversion sets in $(k + r)$ -regular graphs G , for $r \geq 0$ (with a focus on $r \geq 1$). As mentioned, for $r = 1$, S is a k -conversion set of G if and only if it is a decycling set. We began the section by generalizing this characterization to $r \geq 0$ in terms of degeneracy, noting that 1-degenerate simply means acyclic. We proved that, for $r \geq 0$, a set S of vertices in a $(k + r)$ -regular graph is a k -conversion set if and only if $V - S$ is r -degenerate. We then used this result

to prove, in Proposition 3.11, a lower bound on $c_k(G)$ for $(k+r)$ -regular graphs G and (for $r \geq 1$) a characterization of the k -conversion sets that meet it, generalizing Proposition 3.7. Our bound is sharp for $0 \leq r < k$ and improves upon earlier bounds on the k -conversion number of regular graphs of degree greater than $k+1$.

In Chapter 4 we returned our focus to k -conversion in $(k+1)$ -regular graphs. Specifically, we considered the case $k=2$, that is, 2-conversion in cubic graphs, and asked which graphs achieve equality in the lower bound (4.1) of Proposition 3.7, and which do not. It is known that all cyclically 4-edge connected cubic graphs meet the bound, which led us to study fullerenes and snarks. We gave a constructive proof that all (r, ℓ) -generalized fullerenes meet the bound.

Since all strong snarks are cyclically 4-edge connected, they meet the bound as well. However, we determined that Gardner snarks (which are defined as bridgeless, class 2 and triangle-free cubic graphs) may or may not meet the bound. In fact, we showed that every combination of the three Gardner snark properties admits graphs that meet the bound as well as graphs for which the difference between c_2 and the lower bound is arbitrarily large.

On the other hand, the ratio $\frac{c_2(G)}{|V(G)|}$ is bounded asymptotically between $\frac{1}{4}$ and $\frac{3}{8}$ (by Proposition 3.7 and Theorem 2.29), with equality in the upper bound for certain girth 3 graphs, including the triangle-replaced graph of a cubic graph. Such a graph may be bridged and class 2 or bridgeless and class 1 or class 2.

We construct an infinite family of 3-connected graphs with arbitrarily large girth for which $\frac{c_2(G)}{|V(G)|} > \frac{1}{4}$. These may be class 1 or class 2; in the latter case, they are Gardner snarks. However, we still do not know how large this ratio can be for 3-connected triangle-free cubic graphs, except that it is bounded by the the general upper bound of $\frac{3}{8}$, which applies to all cubic graphs. Therefore, we have the following open problem.

Open Problem 8.1 (Question 4.29.). What is the largest ratio $\frac{c_2(G)}{|V(G)|}$ achievable by an infinite family of 3-connected triangle-free cubic graphs G ?

The broad question of which cubic graphs achieve equality in the lower bound (4.1), which motivated many of our results in Chapter 4, also remains open.

Open Problem 8.2. Characterize the cubic graphs G for which $c_2(G) = \left\lceil \frac{|V(G)|+2}{4} \right\rceil$.

The main result of Chapter 5 is Theorem 5.10, where we proved a new sharp lower bound on the k -conversion number of graphs of maximum degree $k + 1$. This result generalizes the bound of Proposition 3.7 to non-regular graphs. For each $k \geq 2$, we presented infinite families of graphs that meet the bound, and others for which the k -conversion number is much larger than the bound. Such investigations provide the first steps towards solving the following open problem.

Open Problem 8.3. Characterize the graphs G of maximum degree $\Delta = k + 1$ and order n for which $c_k(G) = \frac{n(k-1)+\text{def}_\Delta(G)}{2k}$.

In Chapter 6 we determined exact values of $c_k(T)$ for caterpillars, spiders and double spiders, T . While the k -conversion number of an arbitrary tree can be computed in linear time via an algorithm of Centeno *et al.* (see Section 2.4), our results provide exact formulas for the k -conversion number in terms of degree sequence and easily verified structural properties. Previously, such a formula only existed for the simplest trees, namely paths.

In Chapter 7 we began by proving that every cubic graph except for K_4 has an independent minimum 2-conversion set (Theorem 7.2). In fact, we proved that from any minimum 2-conversion set S it is possible to obtain an independent minimum 2-conversion set S' by performing a sequence of restricted seed shuffles. We then generalized this result to larger values of k , first proving that it is possible to obtain a minimum k -conversion set that does not induce a copy of K_k , and eventually showing that in fact we can eliminate all $(k - 1)$ -regular subgraphs (not just K_k) from the seed set by performing restricted seed shuffles (Theorem 7.24).

In the process of determining an appropriate generalization of Theorem 7.2 for $k \geq 3$, we found that a class \mathcal{G}_k of graphs (defined on page 103) provided counterexamples to two restatements of Theorem 7.2 for general k . This led us to Theorems 7.13 and 7.24, but we note that stronger results may hold for all but a relatively small number of graphs. This prompts the following question.

Open Problem 8.4. Which stronger results of the form, “Every $(k + 1)$ -regular graph $G \notin \mathcal{K}$ has a minimum k -conversion set S such that $G[S]$ does not contain the subgraph H ,” can be obtained by allowing \mathcal{K} to contain more than just K_{k+2} ?

In this vein, we encountered the following specific question in Section 7.2.

Open Problem 8.5 (Question 7.12). For which classes of $(k + 1)$ -regular graphs, other than \mathcal{G}_k , does every minimum k -conversion set of G induce at least $k - 2$ vertices of degree $k - 1$?

In Section 7.6, we considered the implications of our results from the earlier sections of Chapter 7, namely Theorems 7.2, 7.13, 7.19 and 7.24. We showed that these results (in particular the most general of them, Theorem 7.24) imply Brooks's Theorem and have strong connections with results on forest partitions of a graph G . Our results provide alternative proofs of several previous results on the vertex arboricity of G (that is, the minimum number of sets in a forest partition of G), and the structure of the sets in a minimum forest partition (Corollaries 7.41, 7.42, 7.43 and 7.44). Finally, in Proposition 7.47, we used Theorem 7.24 to derive an upper bound of $\frac{k-1}{k+1}n$ on the k -conversion number of $(k + 1)$ -regular graphs, for $k \geq 2$. This bound is sharp when k is odd, but not when k is even. Therefore, it remains an open problem to determine a sharp upper bound on $c_k(G)$ for $(k + 1)$ -regular graphs when k is even, as stated below.

Open Problem 8.6 (Question 7.48). For even $k \geq 2$, is the largest ratio $\frac{c_k(G)}{n}$ achieved by a $(k + 1)$ -regular graph $G \neq K_{k+2}$ of order n ?

Our last open problem arises from an observation we made in Chapter 2. After presenting known upper bounds on the k -conversion numbers of the Cartesian product¹, $G \square H$, and the tensor product, $G \times H$, of G and H , we noted that the k -conversion number of the strong product, $G \boxtimes H$, is bounded above by both $c_k(G \square H)$ and $c_k(G \times H)$. This results in the naïve bound

$$c_k(G \boxtimes H) \leq \min\{c_k(G)c_k(H), c_k(G)|V(H)|, c_k(H)|V(G)|\}.$$

However, this bound is not very good, since $G \boxtimes H$ is much more dense than $G \square H$ and $G \times H$, and has a larger maximum degree. (For example, for $G = H = P_5$, the bound above gives $c_2(P_5 \boxtimes P_5) \leq 9$ and $c_3(P_5 \boxtimes P_5) \leq 25$, but the true conversion numbers are $c_2 = 2$ and $c_3 = 3$.) Therefore we have the following open problem.

Open Problem 8.7. Determine a sharp upper bound on $c_k(G \boxtimes H)$, in terms of $c_k(G)$ and $c_k(H)$, or perhaps in terms of $c_\ell(G)$ and $c_m(H)$, for some $\ell, m \leq k$.

¹The Cartesian product, tensor product and strong product of G and H are defined on pages 18, 19 and 22, respectively.

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