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Article

Properties of Spiral-Like Close-to-Convex Functions Associated with Conic Domains

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Abstract: In this paper, our aim is to define certain new classes of multivalently spiral-like, starlike, convex and the varied Mocanu-type functions, which are associated with conic domains. We investigate such interesting properties of each of these function classes, such as (for example) sufficiency criteria, inclusion results and integral-preserving properties.

Keywords: analytic functions; multivalent functions; starlike functions; close-to-convex functions; uniformly starlike functions; uniformly close-to-convex functions; conic domains

MSC: Primary 05A30; 30C45; Secondary 11B65; 47B38

1. Introduction and Motivation

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (1)$$

which are analytic and p -valent in the open unit disk:

$$\mathbb{E} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

In particular, we write:

$$\mathcal{A}(1) = \mathcal{A}.$$

Furthermore, by $\mathcal{S} \subset \mathcal{A}$, we shall denote the class of all functions that are univalent in \mathbb{E} .

The familiar class of p -valently starlike functions in \mathbb{E} will be denoted by $\mathcal{S}^*(p)$, which consists of functions $f \in \mathcal{A}(p)$ that satisfy the following conditions:

$$\Re \left(\frac{z f'(z)}{f(z)} \right) > 0 \quad (\forall z \in \mathbb{E}).$$

One can easily see that:

$$\mathcal{S}^*(1) = \mathcal{S}^*,$$

where \mathcal{S}^* is the well-known class of normalized starlike functions (see [1]).

We denote by \mathcal{K} the class of close-to-convex functions, which consists of functions $f \in \mathcal{A}$ that satisfy the following inequality:

$$\Re \left(\frac{zf'(z)}{g(z)} \right) > 0 \quad (\forall z \in \mathbb{E})$$

for some $g \in \mathcal{S}^*$.

For two functions f and g analytic in \mathbb{E} , we say that the function f is subordinate to the function g and write as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w , which is analytic in \mathbb{E} with:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that:

$$f(z) = g(w(z)).$$

Furthermore, if the function g is univalent in \mathbb{E} , then it follows that:

$$f(z) \prec g(z) \quad (z \in \mathbb{E}) \implies f(0) = g(0) \quad \text{and} \quad f(\mathbb{E}) \subset g(\mathbb{E}).$$

Next, for a function $f \in \mathcal{A}(p)$ given by (1) and another function $g \in \mathcal{A}(p)$ given by:

$$g(z) = z^p + \sum_{n=2}^{\infty} b_{n+p} z^{n+p} \quad (\forall z \in \mathbb{E}),$$

the convolution (or the Hadamard product) of f and g is given by:

$$(f * g)(z) = z^p + \sum_{n=2}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z).$$

The subclass of \mathcal{A} consisting of all analytic functions with a positive real part in \mathbb{E} is denoted by \mathcal{P} . An analytic description of \mathcal{P} is given by:

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (\forall z \in \mathbb{E}).$$

Furthermore, if:

$$\Re \{h(z)\} > \rho,$$

then we say that h is in the class $\mathcal{P}(\rho)$. Clearly, one see that:

$$\mathcal{P}(0) = \mathcal{P}.$$

Historically, in the year 1933, Spaček [2] introduced the β -spiral-like functions as follows.

Definition 1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\beta)$ if and only if:

$$\Re \left(e^{i\beta} \frac{zf'(z)}{f(z)} \right) > 0 \quad (\forall z \in \mathbb{E})$$

for:

$$\beta \in \mathbb{R} \quad \text{and} \quad |\beta| < \frac{\pi}{2},$$

where \mathbb{R} is the set of real numbers.

In the year 1967, Libera [3] extended this definition to the class of functions, which are spiral-like of order ρ denoted by $\mathcal{S}_\rho^*(\beta)$ as follows.

Definition 2. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_\rho^*(\beta)$ if and only if:

$$\Re \left(e^{i\beta} \frac{zf'(z)}{f(z)} \right) > \rho \quad (\forall z \in \mathbb{E})$$

$$\left(0 \leq \rho < 1; \beta \in \mathbb{R} \quad \text{and} \quad |\beta| < \frac{\pi}{2} \right),$$

where \mathbb{R} is the set of real numbers.

The above function classes $\mathcal{S}^*(\beta)$ and $\mathcal{S}_\rho^*(\beta)$ have been studied and generalized by different viewpoints and perspectives. For example, in the year 1974, a subclass $\mathcal{S}_\beta^\alpha(\rho)$ of spiral-like functions was introduced by Silvia (see [4]), who gave some remarkable properties of this function class. Subsequently, Umarani [5] defined and studied another function class $SC(\alpha, \beta)$ of spiral-like functions. Recently, Noor et al. [6] generalized the works of Silvia [4] and Umarani [5] by defining the class $M(p, \alpha, \beta, \rho)$. Here, in this paper, we define certain new subclasses of spiral-like close-to-convex functions by using the idea of Noor et al. [6] and Umarani [5].

We now recall that Kanas et al. (see [7,8]; see also [9]) defined the conic domains Ω_k ($k \geq 0$) as follows:

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{2}$$

By using these conic domains Ω_k ($k \geq 0$), they also introduced and studied the corresponding class $k\text{-}\mathcal{ST}$ of k -starlike functions (see Definition 3 below).

Moreover, for fixed k , Ω_k represents the conic region bounded successively by the imaginary axis for ($k = 0$), for $k = 1$ a parabola, for $0 < k < 1$ the right branch of a hyperbola, and for $k > 1$ an ellipse. For these conic regions, the following functions $p_k(z)$, which are given by (3), play the role of extremal functions.

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \dots & (k = 0) \\ 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 & (k = 1) \\ 1 + \frac{2}{1-k^2} \sinh^2 \left\{ \left(\frac{2}{\pi} \arccos k \right) \arctan(h\sqrt{z}) \right\} & (0 \leq k < 1) \\ 1 + \frac{1}{k^2-1} \sin \left(\frac{\pi}{2K(\kappa)} \int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}} \right) + \frac{1}{k^2-1} & (k > 1), \end{cases} \tag{3}$$

where:

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa}z} \quad (\forall z \in \mathbb{E})$$

and $\kappa \in (0, 1)$ is chosen such that:

$$k = \cosh \left(\frac{\pi K'(\kappa)}{4K(\kappa)} \right).$$

Here, $K(\kappa)$ is Legendre's complete elliptic integral of the first kind and:

$$K'(\kappa) = K(\sqrt{1 - \kappa^2}),$$

that is, $K'(\kappa)$ is the complementary integral of $K(\kappa)$.

These conic regions are being studied and generalized by several authors (see, for example, [10–13]).

The class $k\text{-ST}$ is defined as follows.

Definition 3. A function $f \in \mathcal{A}$ is said to be in the class $k\text{-ST}$ if and only if:

$$\frac{zf'(z)}{f(z)} \prec p_k(z) \quad (\forall z \in \mathbb{E}; k \geq 0)$$

or, equivalently,

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right|.$$

The class of k -uniformly close-to-convex functions denoted by $k\text{-UK}$ was studied by Acu [14].

Definition 4. A function $f \in \mathcal{A}$ is said to be in the class $k\text{-UK}$ if and only if:

$$\Re \left(\frac{zf'(z)}{g(z)} \right) > k \left| \frac{zf'(z)}{g(z)} - 1 \right|,$$

where $g \in k\text{-ST}$.

In recent years, several interesting subclasses of analytic functions were introduced and investigated from different viewpoints (see, for example, [6,15–20]; see also [21–25]). Motivated and inspired by the recent and current research in the above-mentioned work, we here introduce and investigate certain new subclasses of analytic and p -valent functions by using the concept of conic domains and spiral-like functions as follows.

Definition 5. Let $f \in \mathcal{A}(p)$. Then, $f \in k\text{-K}(p, \lambda)$ for a real number λ with $|\lambda| < \frac{\pi}{2}$ if and only if:

$$\Re \left(\frac{e^{i\lambda} zf'(z)}{p \psi(z)} \right) > k \left| \frac{zf'(z)}{\psi(z)} - p \right| + \rho \cos \lambda \quad (k \geq 0; 0 \leq \rho < 1)$$

for some $\psi \in \mathcal{S}^*$.

Definition 6. Let $f \in \mathcal{A}(p)$. Then, $f \in k\text{-Q}(p, \lambda)$ for a real λ with $|\lambda| < \frac{\pi}{2}$ if and only if:

$$\Re \left(\frac{e^{i\lambda} zf'(z)}{p \psi'(z)} \right) > k \left| \frac{(zf'(z))'}{\psi'(z)} - p \right| + \rho \cos \lambda \quad (k \geq 0; 0 \leq \rho < 1)$$

for some $\psi \in \mathcal{C}$.

Definition 7. Let $f \in \mathcal{A}(p)$ with:

$$\frac{f'(z)f(z)}{pz} \neq 0$$

and for some real ϕ and λ with $|\lambda| < \frac{\pi}{2}$. Then, $f \in k\text{-Q}(\phi, \lambda, \eta, f, \psi)$ if and only if:

$$\Re (\mathcal{M}(\phi, \lambda, \eta, f, \psi)) > k |\mathcal{M}(\phi, \lambda, \eta, f, \psi) - p| + \rho \cos \lambda,$$

where

$$\begin{aligned} \mathcal{M}(\phi, \lambda, \eta, f, \psi) &= (e^{i\lambda} - \phi \cos \lambda) \frac{zf'(z)}{p\psi(z)} \\ &+ \frac{\phi \cos \lambda}{p - \eta} \left(\frac{(zf'(z))'}{\psi'(z)} - \eta \right) \quad \left(\frac{-1}{2} \leq \eta < 1 \right). \end{aligned} \tag{4}$$

2. A Set of Lemmas

Each of the following lemmas will be needed in our present investigation.

Lemma 1. (see [26] p. 70) Let h be a convex function in \mathbb{E} and:

$$q : \mathbb{E} \implies \mathbb{C} \text{ and } \Re(q(z)) > 0 \quad (z \in \mathbb{E}).$$

If p is analytic in \mathbb{E} with:

$$p(0) = h(0),$$

then:

$$p(z) + q(z)zp'(z) \prec h(z) \text{ implies } p(z) \prec h(z).$$

Lemma 2. (see [26] p. 195) Let h be a convex function in \mathbb{E} with:

$$h(0) = 0 \text{ and } A > 1.$$

Suppose that $j \geq \frac{4}{h'(0)}$ and that the functions $B(z)$, $C(z)$ and $D(z)$ are analytic in \mathbb{E} and satisfy the following inequalities:

$$\Re\{B(z)\} \geq A + |C(z) - 1| - \Re(C(z) - 1) + jD(z), \quad z \in \mathbb{E}.$$

If p is analytic in \mathbb{E} with:

$$p(z) = 1 + a_1z + a_2z^2 + \dots$$

and the following subordination relation holds true:

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z),$$

then:

$$p(z) \prec h(z).$$

3. Main Results and Their Demonstrations

In this section, we will prove our main results.

Theorem 1. A function $f \in \mathcal{A}$ is in the class $k\text{-}\mathcal{Q}(\phi, \lambda, \eta, f, \psi)$ if:

$$\sum_{n=2}^{\infty} \check{U}_n(p, \phi, \lambda, \eta, \xi) < p^2(p - \eta),$$

where:

$$\begin{aligned} \check{U}_n(p, \phi, \lambda, \eta, \xi) &= (k + 1) [(e^{i\lambda} - \phi \cos \lambda)(p - \eta)p + p^4\phi \cos \lambda \\ &+ (e^{i\lambda} - \phi \cos \lambda)(p - \eta)(n + p) |a_{n+p}| + (n + p)^2 |a_{n+p}| \\ &+ [(np\phi \cos \lambda + p^3(p - \eta))(n + p) |b_{n+p}| + np^2\phi \cos \lambda - p^3(p - \eta)]. \end{aligned} \tag{5}$$

Proof. Let us assume that the relation (4) holds true. It now suffices to show that:

$$k|\mathcal{M}(\phi, \lambda, \eta, f, \psi) - p| - \Re\{\mathcal{M}(\phi, \lambda, \eta, f, \psi) - p\} < 1. \tag{6}$$

We first consider:

$$\begin{aligned} &|\mathcal{M}(\phi, \lambda, \eta, f, \psi) - p| \\ &= \left| \left(e^{i\lambda} - \phi \cos \lambda \right) \frac{zf'(z)}{p\psi(z)} + \frac{\phi \cos \lambda}{(p - \eta)} \left(\frac{(zf'(z))'}{\psi'(z)} - \eta \right) - p \right| \\ &= \left| \frac{(e^{i\lambda} - \phi \cos \lambda)(p - \eta)f'(z)}{p(p - \eta)\psi'(z)} + \frac{p\phi \cos \lambda (zf'(z))'}{p(p - \eta)\psi'(z)} - \right. \\ &\quad \left. - \frac{\eta p \phi \cos \lambda \psi'(z)}{p(p - \eta)\psi'(z)} - \frac{p^2(p - \eta)\psi'(z)}{p(p - \eta)\psi'(z)} \right|. \end{aligned}$$

Now, by using the series form of the functions f and ψ given by:

$$f(z) = z^p + \sum_{n=2}^{\infty} a_{n+p}z^{n+p}$$

and:

$$\psi(z) = z^p + \sum_{n=2}^{\infty} b_{n+p}z^{n+p}$$

in the above relation, we have:

$$\begin{aligned} &|\mathcal{M}(\phi, \lambda, \eta, f, \psi) - p| \\ &= \left| \frac{(e^{i\lambda} - \phi \cos \lambda)(p - \eta)(pz^{p-1}) + p\phi \cos \lambda(p^2z^{p-1})}{p(p - \eta)(pz^{p-1} + \sum_{n=2}^{\infty}(n + p)b_{n+p}z^{n+p-1})} \right. \\ &\quad \left. + \frac{\sum_{n=2}^{\infty}(n + p)a_{n+p}z^{n+p-1}[(e^{i\lambda} - \phi \cos \lambda)(p - \eta) + (n + p)]}{p(p - \eta)(pz^{p-1} + \sum_{n=2}^{\infty}(n + p)b_{n+p}z^{n+p-1})} - \frac{n\phi \cos \lambda}{(p - \eta)} - p \right| \\ &\leq \frac{(e^{i\lambda} - \phi \cos \lambda)(p - \eta)(p) + p\phi \cos \lambda(p^2)}{p(p - \eta)(p + \sum_{n=2}^{\infty}(n + p)|b_{n+p}|)} \\ &\quad + \frac{\sum_{n=2}^{\infty}(n + p)|a_{n+p}| \{ (e^{i\lambda} - \phi \cos \lambda)(p - \eta) + (n + p) \}}{p(p - \eta)(p + \sum_{n=2}^{\infty}(n + p)|b_{n+p}|)} - \left\{ \frac{n\phi \cos \lambda}{(p - \eta)} + p \right\}. \end{aligned}$$

We now see that:

$$\begin{aligned} &k|\mathcal{M}(\phi, \lambda, \eta, f, \psi) - p| - \Re\{\mathcal{M}(\phi, \lambda, \eta, f, \psi) - p\} \\ &\leq (k + 1)|\mathcal{M}(\phi, \lambda, \eta, f, \psi) - p| \\ &\leq (k + 1) \left[\frac{(e^{i\lambda} - \phi \cos \lambda)(p - \eta)(p) + p\phi \cos \lambda(p^2)}{p(p - \eta)(p + \sum_{n=2}^{\infty}(n + p)|b_{n+p}|)} \right. \\ &\quad \left. + \frac{\sum_{n=2}^{\infty}(n + p)|a_{n+p}| [(e^{i\lambda} - \phi \cos \lambda)(p - \eta) + (n + p)]}{p(p - \eta)(p + \sum_{n=2}^{\infty}(n + p)|b_{n+p}|)} - \left[\frac{n\phi \cos \lambda}{(p - \eta)} + p \right] \right]. \end{aligned}$$

The above inequality is bounded above by one, if:

$$\begin{aligned}
 & (k + 1) \left[\left(e^{i\lambda} - \phi \cos \lambda \right) (p - \eta) p \right] + (p\phi \cos \lambda) p^2 \\
 & + \left(\sum_{n=2}^{\infty} (n + p) |a_{n+p}| \right) \left\{ \left(e^{i\lambda} - \phi \cos \lambda \right) (p - \eta) + (n + p) \right\} - \left[\frac{n\phi \cos \lambda}{(p - \eta)} - p \right] \\
 & \cdot \left\{ p(p - \eta) \left(p + \sum_{n=2}^{\infty} (n + p) |b_{n+p}| \right) \right\} \\
 & \leq p(p - \eta) p + \sum_{n=2}^{\infty} (n + p) |b_{n+p}|.
 \end{aligned}$$

Hence:

$$\sum_{n=2}^{\infty} \ddot{U}_n(p, \phi, \lambda, \eta, \xi) \leq p^2(p - \eta),$$

where $\ddot{U}_n(p, \phi, \lambda, \eta, \xi)$ is given by (5), which completes the proof of Theorem 1. \square

Theorem 2. A function $f \in \mathcal{A}(p)$ satisfies the condition:

$$\left| \frac{1}{e^{ij}F(z)} - \frac{1}{2\rho} \right| < \frac{1}{2\rho} \quad (0 \leq \rho < 1; j \in \mathbb{R}) \tag{7}$$

if and only if $f \in 0\text{-}\mathcal{K}(p, \lambda)$, where

$$F(z) = \frac{zf'(z)}{p\psi(z)}.$$

Proof. Suppose that f satisfies (7). We then can write:

$$\begin{aligned}
 & \left| \frac{2\rho - e^{ij}F(z)}{e^{ij}F(z)2\rho} \right| < \frac{1}{2\rho} \\
 & \iff \left(\left| \frac{2\rho - e^{ij}F(z)}{e^{ij}F(z)2\rho} \right| \right)^2 < \left(\frac{1}{2\rho} \right)^2 \\
 & \iff (2\rho - e^{ij}F(z)) (\overline{2\rho - e^{ij}F(z)}) < e^{-ij}\overline{F(z)}e^{ij}F(z) \\
 & \iff 4\rho^2 - 2\rho [e^{-ij}\overline{F(z)} + e^{ij}F(z)] + F(z)\overline{F(z)} < F(z)\overline{F(z)} \\
 & \iff 4\rho^2 - 2\rho [e^{-ij}\overline{F(z)} + e^{ij}F(z)] < 0 \\
 & \iff 2\rho - 2\Re [e^{ij}\overline{F(z)}] < 0 \\
 & \iff \Re [e^{ij}F(z)] > \rho \\
 & \iff \Re \left(e^{ij} \frac{zf'(z)}{p\psi(z)} \right) > \rho.
 \end{aligned}$$

This completes the proof of Theorem 2. \square

Theorem 3. For $0 \leq \varphi_1 < \varphi_2$, it is asserted that:

$$k\text{-}\mathcal{Q}(p, \varphi_2, \lambda, \eta) \subset 0\text{-}\mathcal{Q}(p, \varphi_1, \lambda, \eta).$$

Proof. Let $f(z) \in k\text{-}\mathcal{Q}(p, \varphi_2, \lambda, \eta)$. Then:

$$\begin{aligned} & \frac{1}{p-\eta} \left[\left(e^{i\lambda} - \varphi_1 \cos \lambda \right) (p-\eta) \frac{zf'(z)}{p\psi(z)} + \varphi_1 \cos \lambda \left(\frac{(zf'(z))'}{\psi(z)'} - \eta \right) \right] \\ &= \frac{\varphi_1}{\varphi_2} \left[\left(e^{i\lambda} - \varphi_2 \cos \lambda \right) \frac{zf'(z)}{p\psi(z)} + \frac{\varphi_2 \cos \lambda}{(p-\eta)} \left(\frac{(zf'(z))'}{p\psi(z)'} - \eta \right) \right] \\ &\quad - \left(\frac{\varphi_1 - \varphi_2}{\varphi_2} \right) e^{i\lambda} \frac{zf'(z)}{p\psi(z)} \\ &= \frac{\varphi_1}{\varphi_2} H_1(z) + \left(1 - \frac{\varphi_1}{\varphi_2} \right) H_2(z) = H(z), \end{aligned}$$

where:

$$H_1(z) = \left(e^{i\lambda} - \varphi_2 \cos \lambda \right) \frac{zf'(z)}{p\psi(z)} + \frac{\varphi_2 \cos \lambda}{(p-\eta)} \left(\frac{(zf'(z))'}{\psi'(z)} - \eta \right) \in \mathcal{P}(h_{k,\rho}) \subset \mathcal{P}(\rho)$$

and:

$$H_2(z) = e^{i\lambda} \frac{zf'(z)}{p\psi(z)} \in \mathcal{P}(\rho).$$

Since $\mathcal{P}(\rho)$ is a convex set (see [27]), we therefore have $H(z) \in \mathcal{P}(\rho)$. This implies that $f \in 0\text{-}\mathcal{Q}(p, \varphi_1, \lambda, \eta)$. Thus:

$$k\text{-}\mathcal{Q}(p, \varphi_2, \lambda, \eta) \subset 0\text{-}\mathcal{Q}(p, \varphi_1, \lambda, \eta).$$

The proof of Theorem 3 is now completed. \square

Theorem 4. Let $\phi > 0$ and $\lambda < \frac{\pi}{2}$. Then:

$$k\text{-}\mathcal{Q}(p, \phi, \lambda, \eta, \xi) \subset k\text{-}\mathcal{K}(p, 0, \xi).$$

Proof. Let $f \in k\text{-}\mathcal{Q}(p, \phi, \lambda, \eta, \xi)$, and suppose that:

$$\frac{f'(z)}{\psi'(z)} = p(z), \tag{8}$$

where $p(z)$ is analytic and $p(0) = 1$. Now, by differentiating both sides of (8) with respect to z , we have:

$$\frac{(zf'(z))'}{\psi'(z)} = zp'(z) + p(z)\varepsilon(z), \tag{9}$$

where:

$$\varepsilon(z) = \frac{(z\psi'(z))'}{\psi'(z)}.$$

By using (8) and (9) in (4), we arrive at:

$$\begin{aligned} \mathcal{M}(\phi, \lambda, \eta, f, \psi) &= \left(e^{i\lambda} - \phi \cos \lambda \right) \frac{p(z)}{p} + \frac{\phi \cos \lambda}{p-\eta} (zp'(z) + p(z)\varepsilon(z) - \eta) \\ &= \frac{\phi \cos \lambda}{p-\eta} zp'(z) + \left(\frac{e^{i\lambda}}{p} - \frac{\phi \cos \lambda}{p-\varepsilon(z)} \right) \left(\frac{\phi \cos \lambda}{p-\eta} \right) p(z) - \frac{\eta \phi \cos \lambda}{p-\eta} \\ &= B(z) zp'(z) + C(z) p(z) + D(z), \end{aligned} \tag{10}$$

where:

$$B(z) = \frac{\phi \cos \lambda}{p-\eta},$$

$$C(z) = \frac{e^{i\lambda}(p - \eta) - \phi \cos \lambda (p - \eta) + \phi \cos \lambda \varepsilon(z)p}{p(p - \eta)}$$

and:

$$D(z) = \frac{\eta\phi \cos \lambda}{p - \eta}.$$

Now, since $f \in k\text{-}\mathcal{Q}(p, \phi, \lambda, \eta, \xi)$, we have:

$$B(z)zp'(z) + C(z)p(z) + D(z) \prec p_k(z), \tag{11}$$

which, upon replacing $p(z)$ by:

$$p_*(z) = p(z) - 1,$$

and $p_k(z)$ by:

$$p_k^*(z) = p_k(z) - 1,$$

shows that the above subordination in (11) becomes as follows:

$$B(z)zp'_x(z) + C(z)p_x(z) + D_*(z) \prec p_k^*(z), \tag{12}$$

where:

$$D_*(z) = C(z) + D(z) - 1.$$

We now apply Lemma 2 with:

$$A = 0$$

and

$$p_*(z) \prec p_k^*(z).$$

We thus find that:

$$\frac{f'(z)}{\psi'(z)} = p(z) \prec p_k^*(z). \tag{13}$$

This complete the proof of Theorem 4. \square

For $f \in \mathcal{A}$, we next consider the integral operator defined by:

$$F(z) = I_m[f] = \frac{m+1}{z^m} \int_0^z t^{m-1} f(t) dt. \tag{14}$$

This operator was given by Bernardi [28] in the year 1969. In particular, the operator I_1 was considered by Libera [29]. We prove the following result.

Theorem 5. *Let $f(z) \in k\text{-}\mathcal{Q}(p, \phi, \lambda, \eta, \xi)$. Then, $I_m[f] \in \mathcal{K}(p, 0, \xi)$.*

Proof. Let the function $\psi(z)$ be such that:

$$\mathcal{M}(\phi, \lambda, \eta, f, \psi) = \left(e^{i\lambda} - \phi \cos \lambda \right) \frac{zf'(z)}{p\psi(z)} + \frac{\phi \cos \lambda}{(p - \eta)} \left(\frac{(zf'(z))'}{\psi'(z)} - \eta \right).$$

Then, according to [14], the function $G = I_m[f] \in \mathcal{CD}(k, \delta)$. Furthermore, from (14), we deduce that:

$$(1 + m)f(z) = (1 + m)F(z) + z(F(z))' \tag{15}$$

and:

$$(1 + m)g(z) = (1 + m)G(z) + z(G(z))'. \tag{16}$$

If we now put:

$$p(z) = \frac{F'(z)}{G'(z)}$$

and:

$$q(z) = \frac{1}{(m+1) + \left(\frac{zG''(z)}{G'(z)}\right)'}$$

then, by simple computations, we find that:

$$\frac{f(z)}{\psi(z)} = \frac{(1+m)F'(z) + zF''(z)}{(1+m)G'(z) + zG''(z)}$$

or, equivalently, that:

$$\frac{f'(z)}{\psi'(z)} = p(z) + zp'(z)q(z). \tag{17}$$

We now let:

$$\frac{f'(z)}{\psi'(z)} = p(z) + zp'(z)q(z) = h(z), \tag{18}$$

where the function $h(z)$ is analytic in \mathbb{E} with $h(0) = 1$. Then, by using (18), we have:

$$\frac{(zf(z))'}{\psi'(z)} = zh'(z) + \varepsilon(z)h(z), \tag{19}$$

where:

$$\varepsilon(z) = \frac{(z\psi'(z))'}{\psi'(z)}.$$

Furthermore, by using (18) and (19) in (4), we obtain:

$$\begin{aligned} \mathcal{M}(\alpha, \beta, \gamma, \lambda, \delta, f) &= \left(e^{i\lambda} - \theta \cos \lambda\right) \frac{zf'(z)}{\psi'(z)} + \frac{\phi \cos \lambda}{p - \eta} \left(\frac{(zf'(z))'}{\psi'(z)} - \eta\right) \\ &= \left(e^{i\lambda} - \theta \cos \lambda\right) + \frac{\phi \cos \lambda}{p - \eta} zh'(z) + [zh'(z) + \varepsilon(z)h(z) - \eta] \\ &= \frac{\phi \cos \lambda}{p - \eta} zh'(z) + \left(e^{i\lambda} - \phi \cos \lambda + \frac{\phi \cos \lambda}{p - \eta}\right) h(z) - \frac{\eta(\phi \cos \lambda)}{p - \eta} \\ &= B(z)zh'(z) + C(z)h(z) + D(z), \end{aligned}$$

where:

$$B(z) = \frac{\phi \cos \lambda}{p - \eta},$$

$$C(z) = \frac{((p - \eta)e^{i\lambda} - (p - \eta)\phi \cos \lambda + \phi \cos \lambda)}{p - \eta}$$

and:

$$D(z) = \frac{\eta(\phi \cos \lambda)}{p - \eta}.$$

Now, if we apply Lemma 1 with $A = 0$, we get:

$$\frac{f'(z)}{\psi'(z)} = h(z) \prec p_k(z). \tag{20}$$

Furthermore, from (18), we have:

$$p(z) + zp'(z)q(z) \prec p_k(z).$$

By using Lemma 2 on (20), we obtain the desired result. This completes the proof of Theorem 5. \square

4. Conclusions

Using the idea of spiral-like and close-to-convex functions, we have introduced Mocanu-type functions associated with conic domains. We have derived some interesting results such as sufficiency criteria, inclusion results, and integral-preserving properties. We have also proven that the our newly-defined function classes are closed under the famous Libera operator.

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