PREDICTING THE FINANCIAL RETURN FROM A FOREST PLANTATION INVESTMENT

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PREDICTING THE FINANCIAL RETURN FROM A FOREST PLANTATION INVESTMENT.

by

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Abstract

In order to predict the financial return from a forestry plantation one must predict both the price of timber and the volume growth of the plantation many years into the future. In this article plausible statistical models based on stochastic differential equations are fitted to historical time-series data on stumpage prices and to data on volume growth. In particular geometric Brownian motion is used to model stumpage price evolution. Extrapolations into the future are made using these models and various cutting rules. The simplest forecasts of financial return are based on "certainty-equivalence" deterministic models. However because of the skewness of the price distribution, which arises as a consequence of the geometric Brownian motion specification, a distinction between a "mean certainty equivalence" and a "median certainty equivalence" arises. Feedback harvest rules based on stochastic dynamic programming and the heuristic "myopic-look-ahead" procedure are considered. However in order to include the statistical sampling error in parameter estimates along with the system error in price and volume evolution, simulation methods are required. It is demonstrated how the sampling error in parameter estimates results in a great deal of uncertainty concerning financial return. Also some consequences of the geometric Brownian motion specification for the price process are discussed. In particular it is demonstrated how this price model (which seems to fit the historical data very well) leads to an extremely skewed distribution of financial return, with the expected value often falling above the 90th. percentile of the distribution. The median and quartiles of the distribution of the present value provide a better indication of the possible financial return, than do the mean and variance.

1. Introduction

An important problem in forest management involves the decision of whether or not to plant a site, and if so, with what species and at what density *&c.* In order to make such decisions forest owners or managers need to have some idea of the financial return from a given planting action. In fact the determination of financial return is one of the oldest theoretical problems in forestry. Its solution (in a deterministic framework) was provided by Martin Faustmann in 1849, who sought to find the "correct" valuation of forest land for taxation purposes. Today Faustmann's work is more widely known for providing the solution to the problem of determining the optimal rotation age or determining what has been called the *age of financial maturity* (see *e.g.* Duerr, Fedkiw and Guttenburg 1956), even though Faustmann himself did not actually take the step of deriving the first-order condition (now known as the *Faustmann formula*) for this optimum.

The model employed by Faustmann in his analysis was deterministic. In order to use it to determine the age of financial maturity and the financial return one needs to specify a deterministic value-at-age function as well as various costs and an appropriate rate of discount. In practice this involves making a number of what Samuelson (1976) has referred to as "heroic assumptions" including those that future volume growth and future timber prices and costs are known. While it might be possible to predict with some precision future volume growth, using data from forest mensuration studies on similar sites, it seems a much more difficult task to predict timber prices, which are known to exhibit a considerable degree of fluctuation even in the short run.

In view of this fact the Faustmann model and all of the deterministic variants of it, seem inappropriate for practically predicting financial return. Instead what is needed is a stochastic model which explicitly includes uncertainty, especially in timber prices, but also ideally in costs, volume growth *&c.* Mathematically this increases the complexity of the problem considerably. In place of a simple single-variable calculus problem to determine a single optimal cutting age one is now faced with a problem of determining an optimal feedback harvest rule, since the decision to harvest or not to harvest at any point in time can depend on current timber price, timber volume *&c.* Such problems in *stochastic* *optimal control* can in principle be solved numerically by the methods of *stochastic dynamic programming* (SDP), and indeed there have been a number of papers (*e.g.* Norstrom 1975, Brazee & Mendelsohn 1988, Lohmander 1988, Teeter & Caulfield 1991, Thomson 1992, Haight & Holmes 1991) in recent years in which the methods of SDP are used to determine optimal harvest strategies and implicitly the *expected* financial return. Other authors (Clarke & Reed 1989 , Reed & Clarke 1990) have used the analytic methods of *optimal stopping*; and the methodology of *contingent claims* (Morck *et al.* 1989).

In most of these papers the main focus has been to determine the optimal harvesting policy. The valuation problem is solved more as less as a by-product, since all of the methods involve determining a *value function*, which provides the expected present value of the stand as a function of the current values of the state variables (price and sometimes growing stock). However financial return is a random variable and knowing its *expected value* falls a long way short of knowing its complete distribution. Nonetheless if the optimal harvest strategy is known it should be possible to determine empirically, via simulation, the probability distribution of the financial return. This paper considers such simulations in a broader context in order to determine the probability distribution of financial return.

In order to determine the distribution of financial return it is necessary to specify stochastic models for the way in which timber price and volume evolve over time. Three kinds of models have been used in the determination of optimal timber harvest policies. Some authors (Brazee & Mendelsohn 1988, and Lohmander 1988) assume that prices in successive periods are independent and identically distributed random variables. Others (Norstrom 1975, Teeter & Caulfield 1991, Haight & Holmes 1991) have used autoregressive (AR) time series models. It has been pointed out that models of both types are inconsistent with the informational efficiency of stumpage markets (Washburn & Binkley 1990). A model for stumpage prices which is consistent with informational efficiency is that of *geometric Brownian motion* (GBM), which has been widely used in other areas of economics, especially, although not exclusively, financial economics (see *e.g.* Pindyck 1991). Several authors have used GBM to model stumpage prices (Morck *et al.* 1989, Clarke & Reed 1989 , Reed & Clarke 1990, Thomson 1992). A major aim of this paper is to examine the fit of this model to a time series of timber price data and to explore the consequences of this model specification on the distribution of financial return.

While stochastic models for stumpage price have been quite widely employed, models which include *both* stochastic timber growth and stochastic price evolution are much less common. Timber growth has been modeled using GBM or a variant of it (Morck *et al.* 1989, Clarke & Reed 1989 , Reed & Clarke 1990), which requires the specification of relatively few parameters. Alternatively timber growth has been modeled using more complicated stochastic simulation programs (Taylor & Fortson 1991). In this paper we use an "age-dependent" stochastic differential equation variant of GBM and examine its fit to a time series of plantation volume observations.

Once stochastic models for price and volume evolution have been specified, it is necessary to obtain estimates of the models' parameters. This is, of course, a statistical problem. Because of their emphasis on the stochastic control aspects of the optimal harvesting most authors ignore parameter estimation and use either *ad hoc* plausible parameter values or simple point estimates. Here, we address the statistical problem of estimating parameters for GBM-type models for price and volume evolution. Moreover, we determine the impacts of both *sampling error* in the parameter estimates and *system noise* (inherent randomness in timber and price processes) on the distribution of the financial return.

The final element needed to determine the distribution of financial return is a harvest policy. We evaluate the performance characteristics of various procedures for determining when to cut, ranging from deterministic methods based on the Faustmann formulation to the use of SDP and other feedback harvest policies.

The paper is organized in the following way. Sections 2 & 3 deal with statistical analysis and model fitting for timber price and growth data respectively. Section 4 deals with various methods for predicting financial return. It starts (in 4(a)) considering certainty equivalent methods based on the Faustmann model and from there builds progressively, considering process stochasticity for fixed-age and feedback cutting rules (in 4(b) and (c)); sampling error and how it can be incorporated into a simulation study (in 4(d) and (e)); with finally the results of the simulation and how they can be used in the problem of predicting financial return presented in 4(f). Section 5 contains concluding remarks. A mathematical appendix deals with some issues related to geometric Brownian motion, in particular the specification of a stochastic integral and its relevance to the current problem; the behavior of the mean and median of a GBM process; and the explanation of some apparently paradoxical results in the simulations.

2. The Price Data and Model

Quarterly prices net of harvest costs, in dollars per thousand board feet (International 1/4 inch scale) for loblolly pine (*Pineas taeda* L.) stumpage in the Piedmont region of North Carolina for the period 1977-1988 were used to fit a model for the price process. They are the prices at the beginning of months January, April, June and September reported in *Timber Mart-South.* Figure 1 shows the natural logarithm of price plotted as a time series. There appears to be a slightly downward trend over the twelve year period. A model for the behavior of prices widely used in the economics literature (including forestry economics — see *e.g.* Pindyck 1991, Morck *et al.* 1989, Clarke & Reed 1989, Reed & Clarke 1990, Thomson 1992) is that of *geometric Brownian motion, (GBM)*,

$$dP = bPdt + \sigma_P P dw_P(t) \tag{1}$$

where $P(t)$ is the price at time t (measured in years from the beginning of 1977), and $\{w_P(t)\}$ is a *standard Wiener process*—see *e.g.* Karlin & Taylor,1981, p.342. This model assumes that the incremental *proportional* change in price at any time comprises a fixed component (bdt) and a random component ($\sigma_P dw_P(t)$). If this model is correct then it can easily be shown using the Stratonovich calculus [1] that the first differences in the log-price series,

$$y_i = \ell n(P_i) - \ell n(P_{i-1}) = \ell n(P_i/P_{i-1}), \qquad \text{for i=2,3,}\ldots\text{,n} \tag{2}$$

(where i indexes the quarter and P_i is the observed price at the beginning of that quarter) should be independent, normally distributed (*n.i.d.*) random variables with mean $b/4$ and

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[1] The choice of stochastic calculus (Itô or Stratonovich) is somewhat arbitrary. From the point of view of statistical estimation of the model and predictions made with it, it is irrelevant which calculus is chosen. Each will give the same result. See Appendix 1 for a further discussion of this issue.

variance $\sigma_P^2/4$. The *maximum likelihood (ML)* estimates of the parameters b and σ_P^2 of the GBM model (1) are thus simply:

$$\hat{b} = 4\bar{y} = -0.01188 \tag{3}$$

and

$$\hat{\sigma}_P^2 = 4\tilde{s}^2 = 0.07281 \tag{4}$$

where $\tilde{s}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$.

To check the model's validity the residuals

$$e_i = y_i - \bar{y} \tag{5}$$

can be checked for normality and independence. Figure 2 displays a histogram and a Q-Q plot (*i.e* a plot of the empirical quantiles against those of a standard normal distribution) of the standardized residuals. They indicate a remarkable degree of closeness of fit to a normal form; for example the tail areas beyond ± 1.95 are each of size 2.25 percent. The Shapiro-Wilk (see *e.g.* Bickel & Doksum 1977) test for lack of normality produced a value of $W = 0.986$ with a corresponding P-value of .93. Thus there appears to be no evidence to contradict the normality property. Of four tests for serial correlation (Box-Pierce, Turning Point, Difference-Sign and Rank— see *e.g* Brockwell & Davis 1991), none produced a significant result (P-values 0.92, 0.46 ,0.07 and 0.97 respectively). As a further check on the model the sample autocorrelation and partial autocorrelation functions of the residuals were computed. None of the lagged autocorrelations or partial autocorrelations fell outside of the 95 percent confidence bounds, indicating compatibility with white noise (i.e. with the residuals being n.i.d. random variables). Thus the quarterly price data appear to be compatible with GBM model[2].

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[2] Washburn & Binkley (1990) and Haight & Holmes (1991) found evidence of autocorrelation in the *monthly* series of loblolly pine stumpage prices. The latter authors also found that one test (the unit root test) gave evidence of autocorrelation in the *quarterly* series. Thus there appears to be some short order autocorrelation present. We shall ignore

3. The Volume Growth Data and Model

Observations on timber volume (in board feet International 1/4 inch scale) in four test plantations of loblolly pine grown in South Carolina (Buford 1991) were used to fit a model for the volume growth process. The plantations grow on sites with similar production capacities (site index 80 ft. base age 25) and had initial 12×12 foot spacing. The data were collected at five-year intervals for 30 years (Fig. 3). For convenience in separating the four plots the points have been joined by straight line segments[3].

A possible continuous-time model for the volume growth is given by the following stochastic differential equation

$$dX = g(t)X\,dt + \sigma_X X\,dw_X(t) \tag{6}$$

where $X(t)$ is the volume at age t (measured in years from the planting date). The function $g(t)$ reflects an *age-dependent* deterministic component of proportional volume growth, while $\sigma_X dw_X(t)$ reflects a random component of that growth ($\{w_X(t)\}$ is another standard Wiener process, assumed to be independent of the $\{w_P(t)\}$ process).

If this model is correct then the first differences in the logarithms of volumes at 5-year intervals

$$x_j = \ell n(X_j) - \ell n(X_{j-1}) \qquad \text{for j=2,3,...} \tag{7}$$

should satisfy:

$$x_j = \int_{5(j-1)}^{5j} g(s)\,ds + \epsilon_j \tag{8}$$

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this in this paper because we wish to concentrate on the consequences of GBM on the distribution of financial return. It is well known that a Stratonovich stochastic differential equation can be used to approximate a continuous stochastic process with short order autocorrelation (see *e.g.* Mortensen, 1969). Thus the fitted GBM could be regarded as an approximation to the true price process. However further work needs to be done to investigate the effects of alternative price models (*e.g.* AR models), and the consequences of model misspecification.

[3] We are grateful to Marilyn A. Buford, USDA Forest Service, Southeastern Experiment Station, for providing the plantation data.

where the ϵ_j are n.i.d. random variables with mean 0 and variance $5\sigma_X^2$. The normality of the error terms, ϵ_j, allows for the maximum likelihood estimates of the parameters of a parametrically specified growth function $g(t)$ and of σ_X^2 to be obtained using non-linear least squares

Many possible parametric forms could be used for the function $g(t)$. One would expect $g(t)$ to be decreasing at least for t suitably large (reflecting diminishing proportional growth as a stand ages). The simplest form would be a linear one ($g(t) = \alpha - \beta t$). Unfortunately when this model was fitted the residuals exhibited a pattern indicating model lack of fit.

An alternative 3-parameter form

$$g(t) = \frac{Ke^{-K(t-\gamma)}}{1+(\theta-1)e^{-K(t-\gamma)}} \tag{9}$$

was considered instead. This corresponds to the *Richards Curve* (see *e.g.* Seber & Wild 1989, p332) with

$$X(t) = X(15)exp\{\int_{15}^{t} g(s)\,ds + \sigma_X w_X(t-15)\} = \alpha Z\Big[\frac{1+(\theta-1)e^{-K(t-\gamma)}}{1+(\theta-1)e^{-K(15-\gamma)}}\Big]^{\frac{1}{1-\theta}} \tag{10}$$

where α is a parameter corresponding to the volume at age 15, and Z is a log-normally distributed random variable ($=exp[\sigma_X w_X(t-15)]$). This form is quite general and includes as special cases:

the *von Bertalanffy* model ($\theta = 2/3$)
the *logistic* model ($\theta = 2$)
the *monomolecular* model ($\theta = 0$)
the *Gompertz* model ($\theta \to 1$)

To fit the model the integral in (8) needs to be evaluated. It can readily be shown to be

$$\int_{5(j-1)}^{5j} g(s)\,ds = \frac{1}{1-\theta}\ell n\{\frac{1+(\theta-1)e^{K(5j-\gamma)}}{1+(\theta-1)e^{K(5(j-1)-\gamma)}}\} = \phi_j(\gamma,\theta,K), \tag{11}$$

say. Thus the successive first differences of $\ell n(volume)$, on a given plot, should satisfy

$$x_j = \phi_j(\gamma,\theta,K) + \epsilon_j \tag{12}$$

However because the four test plots were grown over the same time period the values of x_j between plots are likely to be correlated. To incorporate this fact first let

$$x_{ij} = \ell n(X_j^i) - \ell n(X_{j-1}^i) \tag{13}$$

denote the first difference in $\ell n(volume)$ in period j on plot i ($i = 1, ...4$). We can then fit the model

$$x_{ij} = \phi_j(\gamma, \theta, K) + \epsilon_{ij} \tag{14}$$

where the ϵ_{ij} are normal random variables with zero mean and covariances zero except for within a period *i.e.*

$$\begin{aligned} cov(\epsilon_{ij}, \epsilon_{kj}) &= \rho\sigma_{\cdot}^2 \qquad i \neq k \\ &= \sigma_{\cdot}^2 \qquad i = k \end{aligned} \tag{15}$$

The parameter ρ represents the correlation between the random components of growth during a given 5-year period on different plots; also $\sigma_{\cdot}^2 = 5\sigma_X^2$.

To find the ML estimates of ρ, θ, γ, K and $\sigma_{\cdot}^2$, *weighted non-linear least squares* can be used, or more simply the x_{ij} can be transformed (see *e.g* Draper & Smith, 1981, p108) to yield n.i.d. "error" terms, and then *ordinary* non-linear least squares employed.

By using this procedure it was found that the ML estimate of θ was zero (*N.B.* θ is constrained to be non-negative in the Richards model). Thus the fitted form of the Richards curve was in fact of the monomolecular form with the *median*[4] volume following

$$m\{X(t)\} = \alpha\left[\frac{1 - e^{-K(t-\gamma)}}{1 - e^{-K(15-\gamma)}}\right] \tag{16}$$

The ML estimate of the correlation parameter ρ was $\hat{\rho} = 0.089$ which is not significantly different from zero. Thus between-plot correlation was ignored. The other ML parameter estimates were (with standard errors in brackets)

$$\hat{\gamma} = 13.471\,(0.1131), \quad \hat{K} = 0.0831\,(0.00553), \quad \hat{\sigma}_{\cdot}^2 = 0.08376\,(0.001861) \tag{17}$$

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[4] Because the log-normal distribution is not symmetric the mean and median do not coincide. (See Appendix 1). The expected value of $X(t)$ is equal to the median times $exp[0.5\sigma_X^2(t - 15)]$. This factor is the expected value of the log-normal random variable Z in (10).

Also the asymptotic correlation between the estimates $\hat{\gamma}$ and $\hat{K}$ was computed as 0.811.

Residual plots showed no sign of model misspecification. Also an F-test for lack of fit was not significant ($P = 0.28$).

Because of the use of first differences in $\ell n(volume)$, the parameter α does not appear in the equation (14) used for estimating the parameters. Instead of estimating the parameter α in (10), we can think of the problem as being one of estimating an initial condition for the stochastic differential equation (6) (Seber & Wild, 1989 p335). The initial age at which observations on volume were made was 15 years. If we assume that the trees were growing according to (6) before that time, the volume at age 15 would follow a *log-normal* distribution. We can thus regard the four observed volumes at age 15 as coming from such a distribution. The parameters μ_0 and σ_0^2 of this distribution can be estimated by the mean and sample variance of the logarithms of the four volumes at age 15 (*i.e* $\hat{\mu}_0 = 7.910$ and $\hat{\sigma}_0^2 = 0.1382$). A single point estimate for the initial (age 15) volume would be given by $\hat{X}_0 = exp(\hat{\mu} + 0.5\hat{\sigma}_0^2) = 2919.30$.

Integrating (6) for a given initial condition $X(15) = X_0$ one obtains

$$X(t) = X_0 exp\{\int_{15}^{t} g(s)\,ds + \sigma_X w_X(t-15)\} \tag{18}$$

which for the monomolecular model has expected value

$$E(X(t)) = X_0 e^{0.5\sigma_X^2(t-15)} \frac{1 - e^{-K(t-\gamma)}}{1 - e^{-K(15-\gamma)}}, \qquad \text{for } t \geq 15 \tag{19}$$

Using the ML estimates of K, γ and $\sigma_X^2 (= \sigma_{\cdot}^2/5)$ along with the point estimate $\hat{X}_0 = exp(\hat{\mu}_0 + 0.5\hat{\sigma}_0^2)$ provides an estimated "mean value curve" for volume growth. This is plotted in Figure 4.

4. Determining financial return.

In this section the problem of predicting financial return, using the data and models fitted in the previous two sections, is discussed. The analysis builds progressively. To begin simple deterministic methods are employed, ignoring uncertainty. Next stochastic effects in future price and growth are incorporated and more sophisticated cutting rules considered. Finally the sampling error in parameter estimates is coupled with stochasticity in future price and growth.

(a) Using deterministic methods.

The traditional way of determining the financial return from an investment in planting trees is via use of the *Faustmann formula* for determining the optimal rotation period (see *e.g.* Samuelson 1976). The Faustmann rotation period T, which maximizes present value, is given by the solution to

$$V'(T) = \frac{\delta(V(T) - C)}{1 - e^{-\delta T}} \tag{20}$$

where $V(t)$ is the *value* (=price x volume) net of harvest costs of the stand at age t; C is the cost of preparing and replanting the site, and δ is the *instantaneous per annum discount rate.* The present value (PV) of (or financial return from) a newly planted site is then determined as:

$$PV = e^{-\delta T} \frac{(V(T) - C)}{1 - e^{-\delta T}} \tag{21}$$

Of course the validity of this formula is dependent on the underlying assumptions of the Faustmann model, among which are those of:

- a known (deterministic) volume-at-age relationship;
- known (deterministic) prices in the future;
- known harvest and replanting costs;
- a constant discount rate;
- that future rotations will exhibit identical volume growth to the current one, and that
- there is no possibility of catastrophic loss through fire, hurricane, pest infestation *etc.*

Of course many, if not all, of these assumptions are violated in practice. Evidence of the violation of the first two is presented in the Secs. 2 and 3. Thus in order to use the Faustmann method we need to replace the stochastic processes for $P(t)$ and $X(t)$ by *deterministic* processes. Such a procedure is known as a *certainty equivalence* procedure. However there is more than one way of making such a replacement. If we simply set the variance parameters all equal to zero and then use the estimated values of the other parameters we would get $\tilde{V}(t) = P_0\hat{X}_0 e^{\hat{b}t}\frac{1-e^{-\hat{K}(t-\hat{\gamma})}}{1-e^{-\hat{K}(15-\hat{\gamma})}}$ where P_0 is the 1988 price of \$124.60 per thousand board feet, and the hatted parameter values are the estimates obtained in Sections 2 and 3. In fact $\tilde{V}(t)$ is the estimated *median*[5] of the random variable $V(t) = P(t)X(t)$. We shall refer to this substitution procedure as a *median certainty equivalence procedure.* The more usual use of certainty equivalence is to substitute (estimated) *expected* values for future stochastic variables. In this case this would result in a different substitution *viz.* $\bar{V}(t) = P_0\hat{X}_0 exp[0.5\hat{\sigma}_0^2 + (\hat{b} + 0.5(\hat{\sigma}_P^2 + \hat{\sigma}_X^2))t]\left(\frac{1-e^{-\hat{K}(t-\hat{\gamma})}}{1-e^{-\hat{K}(15-\hat{\gamma})}}\right)$. We shall refer to this as a *mean certainty equivalence procedure.* Note that the estimated mean, $\bar{V}(t)$, is bigger than the estimated median, $\tilde{V}(t)$, by a factor depending on the three variances. The reason for this is the skewness of the log-normal distribution (see Appendix 1). This characteristic of the log-normal distribution (which arises from the specification of geometric Brownian motion) will be seen to play a very important role in all of the approaches to evaluating financial return discussed in this paper.

In order to compute the present value for either certainty-equivalence procedure using (20) and (21) , values of the discount rate δ and of the cost parameter C must be specified. These have been set at $\delta = 0.05$ *per annum* and $C=$ \$300 per acre. The latter is about the current price for site preparation and replanting for loblolly pine. With these parameter values using median certainty equivalence the Faustmann cutting age is 23.1 years and the corresponding present value \$450 per acre. This is the estimate of financial return using the simplest deterministic analysis. Subtracting the planting cost gives an estimate of the *site value* (or *land expectation value*) of \$150 per acre.

In contrast when using mean certainty equivalence the Faustmann cutting age is 30.1 years. The corresponding present value, which provides an estimate of financial return, is \$1461 per acre, with the site value estimated at \$1161 per acre.

These two estimates are *very* different. Which of the two provides the better estimate of financial return? To answer this question we need to consider the various sources of uncertainty which are present. Both of the above calculations ignore all uncertainty. They

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[5] See Appendix 1, equation (A.13) for the median price; the median volume at age t is $X_0\frac{1-e^{-K(t-\gamma)}}{1-e^{-K(15-\gamma)}}$; see equation (16)

differ in the way in which an uncertain future is replaced by a hypothetical certain one.

(b) Using fixed-age cutting rules.

In actual fact the financial return is very uncertain, since future prices and volume growth are uncertain. Rather than attempting to describe the financial return by a single number, as is done with the Faustmann analysis, it would be more appropriate to attempt to describe its *probability distribution.* To do this one needs to make assumptions concerning the future evolution of prices and future timber growth. In the previous two sections stochastic differential equation models were fitted to *past* data of prices and growth. In order to predict future prices and growth, one could assume that the same stochastic differential equation models would continue to govern their behavior in the future. Of course in view of the fact that predictions may need to be made up to fifty years or more into the future, this assumption is indeed heroic. Nonetheless it is better than having no model at all and hopefully the results obtained under this assumption will provide a baseline against which other projections can be compared.

Even with knowledge of how, stochastically, timber prices and growth will behave in the future, it is not possible to determine the probability distribution of financial returns without some assumptions concerning the nature of the harvesting strategy employed by the stand owner.

The simplest assumption would be that a cutting age was determined in advance by a certainty-equivalence Faustmann analysis. This is not very realistic, but nonetheless it is worth computing for the sake of making comparisons with more sophisticated harvest strategies, which we shall discuss later. Furthermore it will reveal an important phenomenon, readily understood in this simple situation, which is present in more complex models.

If a harvest takes place at a pre-set age T, the net value of that harvest will be given by the random variable $P(T)X(T)$. By integrating the stochastic differential equations (1) and (6), it is easily shown that

$$\begin{aligned} P(T)X(T) &= P_0X_0exp\{bT + \int_{15}^{T} g(s)\,ds + \sigma_P w_P(T) + \sigma_X w_X(T-15)\} \\ &= P_0X_0\left[\frac{1-e^{-K(t-\gamma)}}{1-e^{-K(15-\gamma)}}\right]exp\{bT + \sigma_P w_P(T) + \sigma_X w_X(T-15)\} \qquad (22) \end{aligned}$$

where P_0 is the (assumed known) price of timber at the planting date; X_0 is a random variable denoting the volume of the stand at age 15 and w_P and w_X are standard Wiener processes. If as was assumed in Section 3, the volume X_0 at age 15 has a log-normal distribution with parameters μ_0 and σ_0^2, then it will follow that $P(T)X(T)$ will also be log-normally distributed, with mean and variance parameters (for $\ell n(P(T)X(T))$ $[\ell n(P_0) + \mu_0 + bT + \int_{15}^{T} g(s)\,ds]$ and $[\sigma_0^2 + \sigma_P^2 T + \sigma_X^2(T-15)]$.

Thus the median of $P(T)X(T)$ is (see Appendix 1)

$$m[P(T)X(T)] = P_0 \left[\frac{1-e^{-K(t-\gamma)}}{1-e^{-K(15-\gamma)}} \right] exp\{\mu_0 + bT\}, \tag{23}$$

while the mean or expected value of $P(T)X(T)$is

$$\begin{aligned} E[P(T)X(T)] &= P_0 \left[\frac{1-e^{-K(t-\gamma)}}{1-e^{-K(15-\gamma)}} \right] exp\{\mu_0 + bT + 0.5(\sigma_0^2 + \sigma_P^2 T + \sigma_X^2(T-15))\} \\ &= m[P(T)X(T)] exp\{0.5(\sigma_0^2 + \sigma_P^2 T + \sigma_X^2(T-15))\} \end{aligned} \tag{24}$$

Thus the mean exceeds the median by a factor depending on the several variances.

Using the estimates of the various parameters, the estimated mean, standard deviation, median and quartiles of the present value of a harvest at the median certainty-equivalence Faustmann age ($T = 23.1$) and at the mean certainty-equivalence Faustmann age ($T = 30.1$) are as given in Table 1 (middle 2 lines). Subtracting the cost of replanting (C = \$300) will provide estimates of the *site value.* These calculations ignore ongoing rotations. A crude adjustment [6] to the mean and median to include future rotations can be made by adding $e^{-\delta T}$ times the expected (or median) site value to the single rotation present values. The bottom two lines of Table 1 gives the estimated mean and median present value with this adjustment for ongoing rotations.

One thing that is immediately apparent in Table 1 is the large discrepancy between the *mean* present value and the *median* present value, for either of the fixed age harvest rules. For example for a single rotation with cutting age 30.1 years the mean present value is \$1203 while the median is only \$331. Also the mean falls in the upper quartile of the

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[6] A more sophisticated adjustment could be obtained by summing a series of weighted non-independent log-normal random variables.

distribution of present value. The reason for this is the high degree of skewness in the log normal distribution, which results from the geometric Brownian motion (GBM) model specification for the price process and from the similar model specification for biological growth. (See Appendix 1.)

Another aspect of the results in Table 1 is that higher mean PV (with cutting age 30.1 yrs.) is associated with the lower median PV, and *vice versa* (at cutting age 23.1 yrs.). One explanation of this fact could be that higher median is associated with the median certainty-equivalence cutting age (23.1 yrs.) while the higher mean is associated with the mean certainty-equivalence age (30.1 yrs.). However there is another explanation which is discussed in detail in Appendix 1. There it is shown that for the values of the parameters of the GBM price process estimated from the data, the mean price *grows* exponentially, while the median price *decays* exponentially. Thus cutting at a later age will result in a higher mean price, but a lower median price. This apparent paradox can be explained by the fact that while after any period of time fifty percent of all sample paths of the GBM will lie above the median value and fifty percent will lie below, there is an asymmetry in that those sample paths that end up above the median may exceed it by a very large amount (several times the median), while those that end up below the median can differ from it at most by a factor of one (since they must remain positive). This results in the mean exceeding the median with the degree of discrepancy increasing with the duration of time elapsed and also increasing with the variance of the GBM process. If the parameter b (representing the systematic component of the growth rate) is negative, and the variance parameter σ_P^2 (representing the randomness or uncertainty in the growth rate) suitably large the median price will decay over time while the mean price will increase over time.

These aspects of the fitted model for price and biological growth may be of considerable importance to forest owners. Predictions of the mean financial return, using for example the standard (mean) certainty equivalence procedure, could over-estimate the actual financial return by a considerable amount. While it might be true that the expected financial return is of the order of \$1200 per acre the probability of the owner on any given rotation actually achieving that return could be quite small (considerably less than 0.25, since the mean PV lies well above the upper quartile). One might expect a typical riskneutral forest owner to be more interested in the median return than in the mean return, since the latter is heavily influenced by the small probability of a big bonanza resulting from a dramatic (but unlikely) increase in price.

The discrepancy between mean and median financial return (and the phenomenon of larger means being associated with smaller medians and *vice versa*) will be seen later in the paper, where more realistic harvest rules and other sources of uncertainty are considered. In spite of the added complexity of these later models, many of the results can be readily understood as consequences of the basic properties of GBM.

These consequences of GBM appear not to have been widely recognized in spite of the fact that GBM has been extensively used in the economics literature to model prices and has a basis in economic theory[7]. Since GBM appears to fit the quarterly price data so well, and is a theoretically plausible model for prices, at least over reasonable periods of time, it is important that the consequences of the model be well understood.

(c) Using feedback cutting rules.

It is well known that when there is stochasticity in the volume growth of trees and in the price of timber, feedback harvest rules can perform better (in the sense of increasing expected return) than fixed age harvest rules. (See *e.g* Clarke & Reed 1989, Reed & Clarke 1990 for a theoretical discussion; and Norstrom 1975, Lohmander 1988, Brazee & Mendelsohn 1988, Haight & Holmes 1991 Teeter & Caulfield 1991, Thomson 1992 for numerical results.) Thus the calculations based on a fixed cutting age will underestimate the expected financial return. The adaptive cutting rule to maximize expected present value can be found by the use of *stochastic dynamic programming (SDP).*

This has been done assuming price and growth dynamics follow the models of Sections 2 & 3 , using the estimated parameter values from the historic data. The objective used

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[7] Although see Samuelson (1965) where the possibility of price going to zero with probability one, with at the same time the expected price increasing exponentially, is recognized in a result called the "Theorem of virtual certainty of (relative) ruin." Samuelson was applying the GBM model to the price of traded stocks and claimed that for such the theorem made very good economic sense.

was to maximize the expected present value

$$EPV = E\{e^{-\delta\tau}[P(\tau)X(\tau) + L - C]\} \tag{25}$$

over all *stopping rules*, where τ is a random variable denoting the first time the state of the system enters the *stopping region* (*i.e* the first time the cutting condition is met), and L represents the site value of a clear-cut site.[8]

To solve the SDP problem numerically the time variable and *both* the price and volume variables were discretized.[9] The time step was set at 0.25 years and a time-horizon of 60 years employed; it was assumed that all stands reaching age 60 would be harvested automatically. The price and volume intervals were defined on the logarithm scale as 0.075 and 0.005, respectively. In real terms, prices ranged from \$3 to \$2,300 per thousand board feet and volumes range from 2,919 to 127,000 board feet per acre. Land value L was set at \$500 per acre, which is near to the current market price of forest land in rural locations in North Carolina. The volume at age 15 was fixed at 2,919 board feet per acre, the estimated expected value.

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[8] To include ongoing rotations in an SDP formulation is difficult. Instead here we have chosen to represent the value of future rotations (which after all might involve a different species of tree *etc.*) by a single non-stochastic variable L. It is well-known that the optimal cutting rule is not very sensitive to the precise value of this variable, which will be small relative to to the value of a timber harvest

[9] In fact for this problem it is not necessary to consider a two-dimensional state variable, $(P(t), X(t))$ since these variables enter the objective only through their product and since a single stochastic differential equation can be written down for the evolution of this product. Specifically letting $V(t) = P(t)X(t)$, the evolution of $V(t)$ (which represents *stand value*) is governed by the stochastic differential equation

$$dV = [b + g(t)]V dt + \sigma_V V dw_V(t)$$

where $\sigma_V^2 = \sigma_P^2 + \sigma_X^2$ and $\{w_V(t)\}$ is a standard Wiener process. However for models in which volume growth has a *size-dependent* component, such a simplification is not possible.

The optimal cutting rule turns out to have a fairly simple form. At any age, there is a volume–dependent reservation price. The stand is harvested when the current price drops below the reservation price associated with the current stand volume. This rule can be simplified further by computing an age–dependent *reservation value* and cutting when the the product of the current price and volume is less than the reservation value. Optimal reservation values using the reservation prices associated with the expected volume trajectory were computed. Although the reservation value at a given age varies somewhat across the range of reservation prices and volumes, the reservation value computed with the expected volume provides a good estimate of the true reservation value.

The lower curve in Figure 5 shows the reservation value plotted against age for an age-15 volume of 2919 board feet. The expected present value of returns net of costs, using this optimal cutting rule, with an initial price of $124.60 per thousand board feet, is $1265. This is a little larger than the expected present value using a fixed age cutting rule determined via mean certainty equivalence (for which a comparable value can be obtained by adding $(L-C)e^{-\delta T}$, or $44, to $1203 to yield $1247).

A stopping (or cutting) rule which is known in many optimal stopping problems to provide a good approximation to the optimal stopping rule is the so-called *myopic look ahead, (MLA)* rule (see *e.g.* Reed & Clarke 1990), in which the process is stopped (stand harvested) when the net harvest value is growing in expectation at the discount rate; or in other words when the expected growth in net present value is zero. This translates into a harvest rule qualitatively similar to the optimal harvest rule described above. As for that rule there is a reservation value, with a harvest carried out when the current value (price x volume) drops below the reservation value. The MLA reservation value at age t is given by:

$$V_M = \frac{\delta(L-C)}{\delta - b - g(t) - 0.5(\sigma_P^2 + \sigma_X^2)} \tag{26}$$

This reservation value is plotted against age in Figure 5. It can be seen that the harvest prescribed by the MLA rule will always be earlier than that prescribed by the optimal rule [10] However the difference between the two rules does not appear large, except at older

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[10] A theorem of Mirosnichenko (See Reed & Clarke 1990 for references), establishes that

ages. However there is a low probability of the stand reaching these higher ages where the difference is significant. Unfortunately it is not possible to calculate analytically the expected present value using the MLA rule. However it can be done by simulation.

A simulation based on 10,000 runs using the MLA rule above with the updating done according to equations (1) and (6) resulted in an average cutting age of 30.4 years and an average present value of the returns net of costs of $1220. This is comparable to the value $1265 obtained from SDP, both SDP nad MLA methods including land value L and planting costs C. The corresponding standard deviations were 4.48 years and $3,490. The median cutting age was 31.8 yrs. and the median present value $306. The distribution of present values is highly skewed, with the mean falling between the 95*th.* and 99*th.* percentiles. The lower quartile of the distribution of present values was $157 and the upper quartile $846. As one might expect the standard deviation in the present value is quite large. At $3490 it is almost three times the estimated expected value. This is similar to the magnitude of the standard deviation in present value using fixed age cutting rules and the reason for it is the same *viz* the highly skewed nature of the distribution of the present value, which arises as a consequence of the GBM specification.

The mean values of the cutting age and present value in the simulations can be regarded as estimates of their *expected* values through using the MLA rule. The standard errors of these estimates are 0.02 years and $34.9. The expected present value of returns net of harvest costs from using the MLA rule thus appears to be close to that obtained from the optimal rule obtained by SDP. The estimated difference is of the order of 3.5 percent. The two standard error band for the difference is 0 to 9 percent. Thus the MLA rule seems to provide a good approximation to the optimal cutting rule.

(d) Including parameter uncertainty

In using the optimal cutting rule (via SDP) or the MLA rule, as described in the previous section it was assumed that parameter values of the price and growth processes (along with sundry other parameters, such as cost, land value *etc.*) were known. The

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the stopping set of the MLA rule always *contains* the stopping set of the optimal rule. The MLA rule will thus always prescribe a premature harvest.

point estimates for the parameters obtained from the statistical analysis in Sections 2. & 3. were used. In fact there is a considerable degree of uncertainty with respect to actual parameter values. For example the standard error of $\hat{b}$, the estimate of the mean growth rate in timber price, is 0.0468. Since the sampling distribution of $\hat{b}$ is approximately normal it follows that the range of values $\hat{b} \pm 2 * s.e.(\hat{b})$, or -0.0932 to 0.0695 should all be plausible values for the parameter b (it is an approximate 95% confidence interval for b). In other words the time series data on timber prices *are compatible with everything from an average nine percent per annum downward trend to a seven percent per annum upward trend.* This of course is a tremendous range, and implies a great deal of uncertainty with respect to financial return. It is equivalent to having b fixed but the discount rate δ varying between 13 percent ($.05 + 2 \times .0468$) and a negative value ($.05 - 2 \times .0468$). As is well known the present value of a forestry investment is very sensitive to the discount rate. In consequence the *sampling error* in the parameter estimates implies a great deal of uncertainty in predicting financial return. In the language of control theory, it is not only *system error* (*i.e* the random terms in the equations for the evolution of price and growth) that generates uncertainty in predicting financial return, but also the *sampling error* in parameter estimation. Indeed the latter source of uncertainty is probably larger than the former.[11] As far as we are aware, this source of uncertainty in predicting financial return has received little attention in the forest economics literature.

The uncertainty in parameter estimates is present for all of the parameters estimated statistically. In principle one could consider a region in parameter space of plausible values (say a 95% confidence region) and for each of the parameter combinations on a grid within this region run the SDP algorithm. In practice however this is not really feasible, since there are seven parameters estimated ($b, \sigma_P^2, \gamma, K, \sigma_X^2, \mu_0$ & σ_0^2). Even if only three values were selected for each (a maximum, a mean and a minimum) one would be looking at $3^7 = 2187$ runs of the SDP algorithm. Furthermore even if one were to carry out this exercise, it would be difficult to draw conclusions from the output, except perhaps an upper and lower bound on the financial return. However for the data analyzed in Section

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[11] Of course another important source of uncertainty could be in model specification. We shall discuss this later in the closing section

2, the upper bound would be determined by the time horizon set for the SDP problem. This is because with the parameter b at the upper end of its 95% confidence interval, the price would be growing in expectation at a rate (of 6.95% p.a.) greater than the discount rate (of 5% p.a.). In consequence it would pay to defer a harvest until the last possible moment, since the discounted value would be growing in expectation at a positive rate. Since the specification of the time horizon is quite arbitrary, so would be the upper bound on the expected present value. Simply by pushing back the time horizon one could increase the expected present value. Theoretically any harvest rule which *ever* prescribed a harvest would be sub-optimal[12]. From a practical forestry point of view, permanently deferring a harvest would make no sense since eventually the trees would die or burn or suffer some other catastrophe. Thus following this approach it would not even be possible to put a plausible upper bound on the expected present value.

It is however possible to put an approximate lower bound on the *expected* financial return, simply by running the SDP algorithm with the parameters set at the lower end of their plausible ranges. This was done yielding a value of $771, which is about sixty percent of that obtained with parameters set at their point estimates. It should be emphasized that this estimate is for the *expected* financial return. If the parameters were really all at the lower ends of their plausible ranges and stands were grown many times over, each one being cut according to the optimal rule as determined by SDP, then the *long run average* financial return would be around $771. For any particular planting the financial return could, and likely would, be less than $771. This is because the financial return using the optimal harvest rule for the given parameter values would be a realization of a random variable with a very skewed distribution. As with fixed-age cutting rules the *median* of the distribution of financial return would be considerably less than the expected value.

In order to get a better idea of the range and distribution of possible *actual* levels of financial return (rather than simply of its expected value), one needs to incorporate the system error in price and volume growth with the sampling error in parameter estimates. While this is difficult to do theoretically it can be done by simulation. A description of a

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[12] In technical language the optimal stopping problem outlined in equation (23) would not be *stable*. (See *e.g* Ross 1983.)

method of performing such simulation is described in Section 4(e) and the results of the simulations presented in Section 4(f).

(e) Using simulation to include sampling error and system error

In order to include the large amount of sampling error in the parameter estimates, one needs a method of describing the relative plausibilities of various combinations of parameter values. One way of accomplishing this is to consider a *probability distribution* of possible parameter values. This requires a *Bayesian*, or at least a *fiducial* (see *e.g* Kalbfleisch 1985) approach to statistical estimation. Rather than try to specify a joint *prior* probability distribution for the parameters, which a full-blown Bayesian analysis would require, we have chosen instead to use a fiducial approach. Fiducial probability although not strictly interpretable in a frequentist sense does provide a measure of the degree of uncertainty concerning parameter values.[13] Essentially here the fiducial approach involves regarding the asymptotic normal pivot involving a group of parameters and their maximum likelihood estimators in an inverse sense, *i.e.* regarding the estimates as fixed values and the parameters as random variables with (fiducial) normal distributions centered at the estimated values. Thus since the asymptotic distribution of the ML estimators $\hat{b}$ and $\hat{\sigma}_P^2$ of the price process (1) is bivariate normal centered at b and σ_P^2, and with zero covariance, the fiducial distribution of b and σ_P^2 will be bivariate normal with zero covariance and centered at the ML point estimates ($\hat{b} = -0.01188$ and $\hat{\sigma}_P^2 = 0.07281$). The fiducial variances of b and σ_P^2 are 0.00162 and 0.000236 obtained from inverting the *observed information matrix.* Similarly the parameters K, γ and σ_X^2 of the growth process can be regarded as approximately jointly normal random variables with means given by the point estimates, and variances obtained from the inverse of the observed information matrix. However unlike the parameters of the price process the parameters K and γ will be correlated with a correlation of 0.811 (the asymptotic correlation of $\hat{K}$ and $\hat{\gamma}$ obtained from

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[13] The notion of fiducial probability, first proposed by R. A. Fisher, has always been somewhat controversial in the discipline of Statistics. An alternative way of justifying the use of the likelihood as a probability distribution of parameter values would be to adopt a Bayesian point of view with uniform prior distributions for all of the parameters.

the ML estimation). In a similar fashion the parameters μ_0 and σ_0^2 of the age-15 volume can be regarded as jointly normal random variables, with distributions independent of those of the other parameters.

Also to take into account the fact that land value might change over time, a stochastic variable $L(t)$ was included. It was assumed that $L(t)$ followed a geometric Brownian motion process similar to that for price $P(t)$. The rationale for this is that land whose most productive use is for growing trees, should have value closely related to the price of timber. A simple assumption would be that land value is directly proportional to timber price. A model which allows some (random) deviation from this assumption is that $\{L(t)\}$ is a stochastic process (GBM) governed by:

$$dL = b_L L dt + \sigma_L L dw_L(t) \tag{27}$$

with b_L and σ_L set equal to the mean growth rate and variance parameters for the price process (b and σ_P), and with the Wiener process $\{w_L(t)\}$ (which reflects future unpredictable changes) closely correlated to that for price $\{w_P(t)\}$. For the simulations the correlation coefficient between the two Wiener processes was set at 0.70.

In running the simulations parameter values generated from the fiducial distributions described above were generated. A single run in the simulation consisted of:

(i) generating a set of parameter values from the fiducial distributions of b, σ_P^2, K, γ, σ_X^2,μ_0 and σ_0^2 using a pseudo-random number generator;

(ii) generating an age-15 volume, X_0 from a log normal distribution with the parameter values μ_0 and σ_0^2 generated in step (i);

(iii) generating a sample path of the price process (1) with initial (time zero) price ($124.60 per thousand cu. ft.);

(iii) generating a sample path of the land value process (27) with initial (time zero) value ($500 per acre) and parameters b_L and σ_L^2 equal to the values b and σ_P^2 generated for the price process in step (i) above ;

(v) generating a sample path of the volume growth process (6) with starting value X_0 (determined in step (ii)) at time 15 years;

(vi) determining the cutting age and resulting net return for various cutting rules and recording them.

This process was repeated 20,000 times.

The cutting rules used were of two types:

I. Fixed Age Rules. For these rules the cutting age was set in advance; in the language of control theory such rules are of an *open loop* nature. Information on timber prices and volume growth is not used in determining the harvest date.

II. Feedback Rules. These cutting rules were derived using a myopic-look-ahead (MLA) procedure. Essentially a harvest is made when the net harvest value is growing (in some "average" sense) at the discount rate. Since the growth rate of net harvest value depends on current price volume and land value, rules of this type are of a feedback or *closed loop* nature.

Rules of both types can be thought of in a certainty-equivalence sense. For fixed age rules, the age of harvest was determined assuming that the actual price, growth and land value processes were deterministic rather than stochastic. However as was discussed in Section 4(b), there is more than one way to substitute a deterministic process for a stochastic one; two obvious choices are to use the *mean-value processes* and the *median-value processes*, leading to mean certainty equivalence and median certainty equivalence. Fixed harvest ages were determined using both methods.

The use of certainty equivalence in the feedbawck rules is a little more subtle. The myopic-look-ahead discussed in Section 4(c) (which seemed to provide a very good approximation to the "optimal" harvest rule obtained numerically by SDP) involves harvesting when the net value of trees and land is growing *in expectation* at the discount rate. Another way of expressing this would be by reference to the mean certainty-equivalence deterministic model, since for a deterministic model a harvest is made at the (fixed) age at which the net harvest value is growing at the discount rate (when the first derivative of present value is zero). Computing this age for the mean certainty equivalence process yields the equation

$$[g(t) + b + 0.5(\sigma_P^2 + \sigma_X^2 - \delta)]\bar{V}(t) + [b_L + 0.5\sigma_L^2]\bar{L}(t) + \delta C = 0 \qquad (28)$$

(where $\bar{V}(t) = \bar{P}(t)\bar{X}(t)$ and $\bar{L}(t)$ are known functions of t - see equation (24)) which must be solved for t to obtain the (fixed) harvest age. However if one regards (28) as an equation in the *stochastic* variables $V(t)$ ($=P(t)X(t)$) and $L(t)$ (rather than as an equation in t for which $\bar{V}(t)$ and $\bar{L}(t)$ are known functions), one obtains a feedback rule which is exactly the MLA rule discussed above.[14] Thus one obtains the feedback harvest rule: harvest as soon as

$$[g(t)+b+0.5(\sigma_P^2+\sigma_X^2-\delta)]V(t)+[b_L+0.5\sigma_L^2]L(t)+\delta C\leq 0 \tag{29}$$

Another way of deriving a feedback harvest rule would be to follow the same procedure using the median certainty-equivalence deterministic process. This results in the feedback harvest rule: harvest as soon as

$$[g(t)+b-\delta)]\ V(t)+b_L L(t)+\delta C\leq 0 \tag{30}$$

For convenience we shall call this latter rule the *median feedback rule* and the former (MLA) rule the *mean feedback rule.*

Thus there are four distinct harvest rules *viz.* the mean and median fixed-age rules and the mean and median feedback rules.

In the simulations each of these four harvest rules was employed in two distinct ways. In the first it was assumed that the actual parameter values generated from the fiducial distribution were known, and the fixed-age and feedback harvest rules were based upon these known values. In the second, in contrast, it was assumed that the parameter values were not known exactly, and the harvest rules were based upon the point estimates of the parameter values obtained from the data in Sections 2 & 3 (or put another way the means of the fiducial distributions of the parameters were used).

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[14] This differs from the MLA rule discussed in Section 4(c) (equation (26)) because of the inclusion of the stochastic land-value process. The MLA rule does not now involve a simple reservation value for timber but rather involves harvesting when a (time-dependent) linear combination of timber value and land value drops below a certain level. In the simulations an additional obvious condition for this MLA rule was imposed *viz.* that the net value of the harvest $P(t)X(t)+L(t)-C$ should be positive.

The first case represents a scenario in which *more* information than would be available in practice is assumed, while the second case corresponds to one in which *less* information is assumed. In practice one would start at the time of planting with information on parameter values based solely on historical data (as in the second case above). As time passed more observations on the price time series would become available and parameter estimates could be updated. Similarly estimates of the parameters of the volume growth and land value processes could be improved. If observations were to continue indefinitely the updated parameter estimates would, in theory, converge to the true parameter values in operation. However in practice, observations would continue only for a finite time and therefore parameter estimates, although based on more information than the historical data, would still not coincide exactly with the actual parameter values. Thus the former case above corresponds to a situation with more information than would be available in practice. One would expect that the performance of a harvesting rule under this scenario would do better than the corresponding rule in which parameter estimates were updated. In contrast the latter case, in which point estimates based on historical data are used, corresponds to a situation with less information than would be available in practice, and one would therefore expect the harvesting rule under this scenario to do worse than the corresponding rule with parameter estimates updated. Thus the performance of the four fixed-age and feedback rules under the two scenarios, which we shall refer to as **(A) Parameters known** and **(B) Parameters estimated**, should provide upper and lower bounds of the actual performance of these rules in practice.

Thus in the simulations eight different harvest scenarios were employed, corresponding to the 2 x 2 x 2 classification:

(fixed age *or* feedback) x (mean *or* median certainty equivalence)

x (parameters known *or* estimated).

The results of the simulations and their interpretation are described in the following section.

(f) Simulation results and interpretation.

In order to run the simulations of the continuous-time processes for price, volume growth and land value (equations (1), (6) & (27)) the time scale had to be discretized. A time step of 0.02 years was used, corresponding approximately to one week.

In addition a time horizon, T_U, had to be specified. The value T_U =50 years, was used initially. Any stands not harvested by this time under any cutting rule were harvested at this age. In some cases this resulted in a negative net present value (when $P(T_U)X(T_U) + L(T_U) < C$); in others it resulted in *extremely large* net returns. For the most part these large values occurred on runs when the parameters for the growth of price or volume were large. However large values could also result from either feedback rule with parameter values near their means but with, through chance, the sample paths growing very large, and never dropping below the stopping boundary. To investigate the sensitivity of the results to the time horizon the simulations were re-run with T_U =60 years. In each case twenty thousand runs were made.

The results (with T_U =50 years) are presented in Tables 2 & 3. We consider first the feedback rules (Table 2). Figs. 6 and 7 show respectively density histograms[15] Thus of cutting ages and present values using the **mean feedback (MLA) rule** and the **median feedback (MLA) rule** , each under the two scenarios **(A) Parameters known** and **(B) Parameters estimated** described in Section 4(d). One outstanding feature for the mean feedback rule, when parameter values are known (top left hand panel of Fig. 6) is the large number of runs (about 50 percent) in which the stand was not cut until the time horizon of 50 years was reached. A consequence of this is that the median of the cutting age is 46.5 years. These non-terminating runs correspond, for the most part, to cases when the parameter values selected from the fiducial distributions are large and there is no solution to the optimal stopping problem (see discussion in Section 4(c)) because the net timber value is growing in expectation faster than the rate of discount. This results in the harvest condition(29) never being met because the coefficients of $V(t)$ & $L(t)$ on left-hand side are positive at all times up to the horizon. For the mean feedback rule, when parameters are

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[15] For a density histogram the vertical scale is chosen so that the total area under the histogram equals unity; it thus represents probabilty density. As with a probability density function the probability of falling in any cell (or group of cells) is given by the area of the cells.

estimated, there are fewer (but still a considerable number of) runs in which the stand is not cut before the time horizon is reached; the median cutting age in this case is 36.0 years. For the median feedback rule there are considerably fewer non-terminating runs (see two left panels of Fig.7).

The right hand panels of Figs. 6 and 7 are histograms of present value for the two feedback rules under the two scenarios (A) and (B). All distributions are *extremely long-tailed*, with a significant frequency of present values of \$10,000 or higher (N.B the last cell in the histograms is a collector cell including all present values at or above \$8,750)). In Section 4(b) the skewness of the log-normal distribution resulting from a fixed-age cutting rule was discussed. A similar phenomenon is operative for the feedback rules. However the uncertainty in parameter values increases the variance, thereby stretching, to an enormous extent, the right hand end of the distribution of present value. One consequence of this extreme skewness, is that the mean exceeds the median by a very large factor. For example for the mean feedback rule with parameters known (scenario(A)) the mean present value is \$7581 while the median is only \$404; for the same rule under scenario (B) the mean is \$3366 and the median \$422. A similar situation prevails with the median feedback rule. Note also (in Table 2) how the larger mean values tend to correspond to the smaller median values. The reason for this is probably the property of GBM discussed in Section 4(b) and in the Appendix *viz.* that for certain parameter values (including the estimated values in Section 2) the median value of GBM can decay exponentially, while at the same time the mean value grows exponentially.

The large discrepancy between mean and median for skewed distributions is due to the fact that the mean is very sensitive to extreme observations, while the median is not. The very large values of present value by and large correspond to non-terminating runs in which the parameter values generated from the fiducial distribution cause the timber value to be growing faster than the discount rate. They therefore depend very much on the value of the time horizon parameter T_U, which was arbitrarily set. Changing the value of T_U would thereby be expected to change the value of the mean. In fact it does so to a very marked degree; for example with T_U=60 yrs. the mean present value for the mean feedback rules increases to over \$20,000 under scenario (A). This fact alone renders the mean present value an extremely unreliable measure of overall performance of a given harvest rule, and an extremely unreliable predictor of financial return.

Even if the mean were not sensitive to specification of a time horizon (in fact it is much less sensitive under scenario (B) where there are far fewer non-terminating run), there is another problem with using it as a predictor of performance. This is because it lies far out in the tail of the distributions of present value. For example for all harvest rules under scenario (A) the mean falls between the 90th and 95th. percentiles of the distribution of present values, while for all cases under scenario(B) it falls between the 75th. and 90th. percentiles. Thus there would be a probability of less than 0.25 (and quite likely less than 0.1) of attaining a financial return as large as the mean present value. We shall therefore not consider the mean further as a predictor of financial return. Instead we shall concentrate on the median and quartiles which are insensitive to outlying values. This can be seen by their relative stability over all cases in Table 2. Using the median as a predictor has the property that there is a fifty-fifty chance of attaining at least that level of financial return; furthermore there is a similar chance of the return being within the interquartile range.

On comparing medians and interquartile ranges under the two scenarios (A) and (B), one can see that there is little difference. In other words there is little to be gained through updating parameter values. In consequence predictions made using the means of the fiducial distributions of parameter values will probably give reliable results, even though in practice further information would become available.

Comparing the fixed-age rules (Table 3) with the feedback rules (Table 2), one sees that in terms of the median and quartiles of present value, the feedback rules perform somewhat better. Of the two feedback rules the median one seems to perform better than the mean one. Thus we shall base predictions of financial return on the median feedback rule under scenario (B). Based on this a point prediction of financial return would be $540 per acre. There should be an approximately fifty-fifty chance of exceeding this value; and also a fifty-fifty chance of the financial return lying in the range $150 to $1500 per acre.

In summary the following conclusions can be drawn from the results of the simulation studies:

(1) There is a great deal of uncertainty concerning the present value of the returns from a given planting option. The uncertainty concerning parameter values (sampling error) is a major reason for this, although intrinsic uncertainty (process error) in future prices and volume growth also plays an important role.

(2) The distributions of present values are all extremely skewed.

(3) The mean present value provides a poor descriptor of the central location of the distribution of possible values, falling in all cases in the upper quartile and in some cases above the 95*th.* percentile of the distribution. The dependence of the mean present value on the specification of a time horizon renders it an unreliable predictor of financial return. The median and quartiles seem to provide better summaries of the distributions.

(4) A comparison of medians and quartiles indicates that the median feedback rule , on the whole, performs somewhat better than the fixed-age rules and the mean feedback rule. This effect does not show up in a comparison of means, in large part because the magnitude of the mean is determined by the non-terminating runs with extremely high present values.

(5) Information concerning parameter values does not appear to have much positive effect on the median nor on the middle half (between quartiles) of the distribution of present values.

(6) Using the median feedback harvest rule and a discount rate of five percent *per annum* one can conclude that there is a probability of around 0.5 that the financial return will lie somewhere in the range of $150 to $1500 per acre with a median value in the $540 range. While there is a probability of around 0.25 of the financial return being below $50 and even of being negative it will never be very negative. In contrast while there is a probability of around 0.25 of financial return exceeding $1500 per acre there is the possibility of it exceeding $1500 by a very large amount. Thus there is the possibility of reaping *very high returns,* if timber prices grow rapidly. The fact that potential losses are bounded while potential gains are almost unlimited is a fact in favor of investment in forestry plantations.

5. Concluding remarks.

In this paper we have looked at the problem of predicting the financial return from a forestry plantation, using actual data on timber prices and volume growth. The first obvious conclusion is that there is a great deal of uncertainty in the prediction process. In the preceding sections we have identified two of the sources of uncertainty, *viz.*

1. *System noise. i.e.* the inherent randomness in timber growth and in timber prices. Evidence of randomness of this type is provided in the data used in Sections 2 & 3. System noise has long been recognized as important in the problem of determining optimal cutting strategies, and thereby determining financial return, whether numerically via stochastic dynamic programming (see *e.g.* Norstrom 1975, Lohmander 1988, Brazee & Mendelsohn 1988, Haight & Holmes 1991 Teeter & Caulfield 1991, Thomson 1992) or analytically (see *e.g.* Clarke & Reed 1989 and Reed & Clarke 1990). Given that growing trees for timber is, by its nature, a long-term operation, the uncertainty in volume growth and timber prices perhaps up to fifty or more years into the future, is inevitably of considerable magnitude. In consequence predictions of financial return will involve much uncertainty from this source.

2. *Sampling error. i.e.* the statistical uncertainty in parameter estimates. This source of uncertainty has received much less attention in the literature. However for the data examined in this paper, sampling error in parameter estimates seems to be relatively large and therefore of considerable importance. For example from the time-series data on net timber prices we have seen that the mean annual growth rate in timber price could plausibly lie anywhere between −9% and +7%. Other parameter estimates exhibit a similar degree of sampling error. The consequences of this on possible financial return are very large indeed. For example if prices were dropping at a fixed 9% *per annum* rate the drop over a, say, thirty year rotation period would be of the order of 94% (*i.e* price reduced to 6% of its initial value); in contrast if prices were rising at a fixed 7% *per annum* rate the increase over the same 30 year period would be of the order of 660%. Other parameters exhibit similar degrees of sampling error and when this uncertainty is incorporated in extrapolations into the future the resulting predictions of financial return will reflect this high degree of uncertainty.

Another important point is that the models used for price and volume evolution in this paper are specified parsimoniously in terms of the number of parameters used (seven in all). In contrast to our rather simple models other studies have used very complicated stochastic simulators to predict timber volume growth. For example the simulator used by Taylor & Fortson (1991) to estimate the variability in loblolly pine volume yield includes more than 20 estimated parameters. If the sampling error associated with all of these parameters were accounted for then the cumulative effect on the distribution of financial return would likely be extremely large indeed.

A natural question to ask is whether there is any way of reducing the sampling error? Since the magnitude of sampling error is inversely related to the information available for estimating parameters, the simple answer is that one should be able to reduce sampling error through observing a longer time series of prices or through having more replicated test plots to estimate timber growth. However there are limits to the extent to which this can be done. If one uses price data which are very old, one runs the risk that the underlying parameters may have changed in value since the data were collected, because of major underlying changes in the market for timber products. Similarly, although in principle one should be able to conduct more experiments on timber growth on test plots thereby reducing sampling error in growth parameter estimates, there is a limit to the degree to which the sampling error can be reduced, simply because of the geographic variability in growing conditions from plot to plot.

Of course the above argument concerning timber prices, could be applied to the projection of timber prices into the future. We have no guarantee that the underlying parameters of timber price dynamics will remain unchanged during a rotation period. In fact it seems quite likely that they will change, as the cheap supply wood in natural forests around the world is liquidated, and the market for timber is supplied more and more from plantation forests. Similarly with the possibility of global climate change there is a possibility that biological growth will be affected. These problems, which are inherent in any kind of prediction, give rise then to another source of uncertainty which can be called *structural uncertainty*. We have made no attempt to deal with structural uncertainty – essentially we have assumed that statistically, at least, the future will be similar to the fairly recent past – simply because there seems to be very little solid ground on which to make predictions concerning structural changes in timber markets *&c.* In essence one person's guess is as good as another's. The results presented in this paper are probably best regarded as "base-line" cases *i.e.* under the assumption that no major structural changes take place. Of course owners of forest land will use their own judgments concerning the future when making planting decisions. The techniques developed in this paper may be of some use at least in providing crude bounds. For example if a forest owner believes that in the future increased scarcity will cause timber prices to rise more quickly than in the past, then the predictions made using the methods of this paper should provide a lower bound on financial return; or if contrarily the owner believes that the price of timber will drop because for example substitutes for wood products will be found, then they would provide an upper bound, albeit a very fuzzy one.

Another source of uncertainty, related to some extent to the notion of structural uncertainty, concerns the possibility of *model specification error.* The predictions of financial return made in Section 4 were made assuming that the models fitted to the data in Sections 2 & 3, were correctly specified and would hold over the planning horizon. Apart from the possibility of the process structure changing in the future as discussed in the previous paragraph, another potential source of uncertainty lies in the fact that one can never know for certain that the model specified is in fact the "true" one, or perhaps more pragmatically that it constitutes a reasonable approximation to the underlying reality. For example the specification of geometric Brownian motion for the price process has some very important implications (of which more will be said later) but there must necessarily remain some doubt as to the validity of this model specification. It is true that the GBM model seems to fit the quarterly time series of prices very well, passing all statistical tests to which is was subjected, and also that there is some basis in economic theory for the GBM model. Nonetheless there could be other models which would fit the data equally well and which might result in very different predictions. Thus some uncertainty must remain concerning the extrapolations from this and the age-dependent volume growth model specified.

Other sources of uncertainty concern the possibility of catastrophic loss. These include destruction by fire, windstorm and pest infestation. We have not included these factors here because to a large extent they can be handled by an adjustment to the discount rate (see *e.g.* Reed 1984, Reed & Errico 1985). Thus the parameter δ should be thought of as a *risk adjusted* discount rate, obtained by adding a *risk premium* to the risk-free rate.

One of the most interesting aspects of this paper concerns the consequences of the widely used GBM model for timber price and a similar "age-dependent GBM" for volume growth. It has been seen how with all the harvesting rules considered the resulting distribution of present values is very highly skewed, with an extremely long right-hand tail. One effect of this skewness is to push the mean of the distribution a long way to the right of the median, in some cases beyond the upper tenth percentile. This would imply a probability of less than 0.10 in some cases of the present value achieving its expected value. This, and the fact that in some cases the mean present value depends heavily on on the arbitrarily set time horizon makes the mean present value a very unreliable predictor of the financial yield. The median and quartiles of the present value distribution are much more stable than the mean and provide a better idea of the range of plausible financial returns. Another curious fact concerning GBM established theoretically in the Appendix, and borne out numerically in the simulations is that, over time, the median value of a GBM process can decrease, while the mean value increases. This can only occur if the mean growth rate parameter is negative, with the variance parameter more than twice as large in absolute value. It just so happens that this condition is met for the parameter values estimated from the price data. The difficulty here lies more with the notion of the mean (or expected value) characterizing the center of a skewed distribution than with the GBM model itself. The mean, unlike the median, is very sensitive to extreme values. Thus the possibility of the price rising to a very high level (albeit with small probability) has the effect of inflating the value of the mean relative to that of the median. For most forest owners the median and quartiles of the distribution would probably provide a better indication of the possible financial return than would the mean and standard deviation.

Much of the literature on contingent claims methodology, both in financial economics and for real assets such as a forest plantation (see Morck *et al.* 1989), is based on the assumption of GBM, and uses expectations to determine the "correct" value of the asset. However the reasons listed above suggest that this valuation procedure may be very misleading, and there may be only a relatively small probability of actually achieving the nominal value of the resource.

It would appear from the simulations that, when judged in terms of median and quartiles, the median feedback rule performs best among the harvest rules considered. Under this harvest rule the distribution of present values has its median around \$540 per acre with the quartiles around \$150 and \$1500 per acre. Thus with probability one half the present value should be \$540 per acre or more; and again with probability one half should lie in the range \$150 to \$1500 per acre. The possibility of the present value being below \$150 per acre (or even of being negative) is more than offset by the possibility of it being larger than \$1500 per acre (both with probability 0.25). While the losses are strictly limited the gains are almost unbounded with the possibility of a huge bonanza being realized.

Finally if we consider how established methods of determining financial return compare with the results of the simulations, we see that the Faustmann method, using median certainty equivalence provides a fair rough and ready estimate (\$450 per acre) of the median present value when system error and sampling error are taken into consideration. However it provides no indication of the tremendous range of possible returns.

The estimate obtained from the Faustmann method using mean certainty equivalence (\$1461 per acre) appears to lie near the 75th. percentiles of the distribution of present values and does not appear to be particularly useful for prediction of the actual return. The expected present value (\$1265 per acre) which results from the use of stochastic dynamic programming is of similar magnitude and again appears to be of limited usefulness. The reason for this is that the SDP approach (and of course the mean certainty equivalence Faustmann method) ignores sampling error. The inclusion of sampling error does not have a great deal of effect on the median present value, but greatly increases the spread and skewness of the distribution thereby increasing the mean. While both SDP and the mean certainty equivalence Faustmann method provide good estimates of the mean present value when sampling error is ignored, they fail to do so when sampling error is included, because of the fact that the inclusion of sampling error greatly inflates the mean. If SDP is to be used it would probably be better to consider the objective of maximizing the expected present value of the *natural logarithm* of net return (or of net return plus some small constant to prevent negative values) since such a transformation would have the effect of reducing the skewness of the distribution thereby causing the mean and median to be closer to one another. The inverse transformation of the value function would then provide an estimate of median financial return. In fact this is essentially equivalent to using a "risk-averse" logarithmic utility function and maximizing expected discounted utility. While such a procedure would likely give a reasonable prediction of the median financial return, it would not by itself, give any indication of the possible variability in the return. The possible degree of variabilty is likely to be very important to those making investments in forestry plantations, especially in view of the fact that risk cannot be diversified through holding many such forestry investments, since the prices of primary forest products are likely to be highly correlated.

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Appendix 1.

On Geometric Brownian Motion and the Choice of Stochastic Calculus.

The stochastic differential equation of Geometric Brownian Motion (GBM)

$$dP = bPdt + \sigma_P P dw_P(t) \tag{A.1}$$

can be integrated using either the Itô or the Stratonovich calculus. Which of the two forms is more appropriate for modeling real-life phenomena has been widely discussed elsewhere. (See *e.g* Mortensen 1969, Turelli 1977, Clarke & Reed 1989). We shall not enter further into this discussion here since, from the point of view of statistical estimation and prediction, it is irrelevant which of the two forms is used. To see this observe firstly that the time series of first differences in the natural log of prices $\{y_i\}$

$$y_i = \ell n(P_i) - \ell n(P_{i-1}) = \ell n(P_i \div P_{i-1}), \qquad \text{for i=2,3,...} \tag{A.2}$$

will under the either calculus be a series of independent, normallly distributed (*n.i.d.*) random variables. The only difference is in the *mean value* of the normal distribution. Using Itô's Lemma (see *e.g* Karlin & Taylor 1981) we have that under the Itô calculus

$$y_i \sim N([b - \frac{\sigma_P^2}{2}]/4, \frac{\sigma_P^2}{4}) \tag{A.3}$$

while under the Stratonovich calculus

$$y_i \sim N(b/4, \frac{\sigma_P^2}{4}) \tag{A.3}$$

Thus the maximum likelihood estimate (ML) of σ_P^2 is

$$\hat{\sigma}_P^2 = \tilde{s}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 \qquad ((A.4)$$

under either calculus, while the ML estimate of b is under the Stratonovich calculus

$$\hat{b} = 4\bar{y}, \tag{A.5}$$

whereas under the Itô calculus, from the invariance property of ML estimators, it is

$$\hat{b} = 4(\bar{y} + \frac{\tilde{s}^2}{2}) \tag{A.6}$$

Under either calculus the random variable $P(t)$ will be log-normally distributed. However while for the Itô calculus the mean and median of $P(t)$ will be

$$E[P(t)] = P_0 e^{bt} \tag{A.7}$$

and

$$m[P(t)] = P_0 e^{(b - \frac{\sigma_P^2}{2})t} \tag{A.8}$$

respectively, under the Stratonovich calculus they will be

$$E[P(t)] = P_0 e^{(b + \frac{\sigma_P^2}{2})t} \tag{A.9}$$

and

$$m[P(t)] = P_0 e^{bt}. \tag{A.10}$$

The variance of $P(t)$ is (in both cases)

$$V[P(t)] = E^2[P(t)](e^{\sigma_P^2 t} - 1). \tag{A.11}$$

Substituting in the ML estimates of b and σ_P^2 gives, *for both calculi*, the ML estimates of the mean and median of $P(t)$ as

$$E[\widehat{P(t)}] = P_0 e^{4(\bar{y} + \frac{\bar{s}^2}{2})t} \tag{A.12}$$

and

$$m[\widehat{P(t)}] = P_0 e^{4\bar{y}t} \tag{A.13}$$

The ML estimate of $V[P(t)]$ is also the same under both calculi, since the ML estimate $\hat{\sigma}_P^2$ has this property. Since the log-normal distribution is completely characterized by its mean and variance, it follows that the ML estimate of the distribution of $P(t)$ is the same under both calculi.

Note that for both calculi the expected value exceeds the median by a factor $e^{\frac{\sigma_P^2 t}{2} t}$ *i.e*

$$\frac{E[P(t)]}{m[P(t)]} = e^{\frac{\sigma_P^2 t}{2} t} \tag{A.14}$$

Since $\sigma_P^2 > 0$ this factor grows exponentially with time t. This phenomenon, along with a similar one for the volume growth, explains the large difference between the mean and median present values for fixed-age cutting rules encountered in Section 4(a), and for other cutting rules elsewhere in Section 4.

Also under both calculi the *coefficient of variation*

$$CV = \frac{st.devn.[P(t)]}{E[P(t)]} = \left(e^{\frac{\sigma_P^2 t}{2} t} - 1\right)^{1/2} \qquad (A.15)$$

increases with time t. This aspect of GBM, along with a similar one for the volume growth, explains the large standard deviation of the present value relative to its mean encountered for various harvesting rules in Section 4.

An interesting and rather strange property of GBM is that it is possible for the mean $E[P(t)]$ to *grow* exponentially, while the median *decays* exponentially. This would happen for the Itô formulation (see (A.7) and (A.8)) if

$$0 < b < \frac{\sigma_P^2}{2} \qquad (A.16)$$

or, for the Stratonovich formulation (see (A.9) and (A.10)), if

$$-\frac{\sigma_P^2}{2} < b < 0 \qquad (A.17)$$

Replacing parameter values by their ML estimates in (A.14) and (A.15) results in the single condition

$$-\frac{\tilde{s}^2}{2} < \bar{y} < 0 \qquad (A.18)$$

for the estimated mean to grow exponentially, while the estimated median decays exponentially.

For the time series of quarterly timber prices, analyzed in Section 2, with $\bar{y} = -0.00297$ and $\tilde{s}^2 = .018203$ the condition (A.18) is met. Thus predictions of future price will grow "in expectation" but decay "in median." This may be the explanation for the somewhat bizarre results from the simulations in Section 4(e), in which the cases with the larger mean present values are associated with smaller median present values.

In fact if the condition (A.16) or (A.17) were met (depending on which stochastic calculus were employed) it would not just be the median of the distribution of $P(t)$ which decayed exponentially, but *any percentile* of the distribution. It follows that under these conditions $P(t)$ would tend "in probability" to zero (as $t \rightarrow \infty$), even though $E[P(t)]$ would tend to infinity! To see this consider for any $\epsilon > 0$ the probability that $P(t)$ is less than ϵ, which can be written (using the Stratonovich calculus)

$$Pr(P(t) < \epsilon) = Pr(\ell n(P(t)) < \epsilon) = \Phi(\frac{\ell n(\epsilon) - bt}{\sigma_P \sqrt{t}}) \qquad (A.19)$$

where $\Phi(.)$ is the cumulative distribution function of a standard normal random variable. With b negative the argument of Φ tends to infinity as $t \rightarrow \infty$, and so $Pr(P(t) < \epsilon) \rightarrow 1$, or in other words, $P(t)$ tends to zero *in probability.* Under condition (A.17) $P(t)$ would tend to zero in probability, while $E[P(t)]$ would tend to infinity. A similar argument using (A.16) would give a similar result using the Itô calculus.

The reason behind this apparent paradox is that while after a finite time most sample paths will be below any arbitrarily small $\epsilon > 0$, of the few that are not, some will have extremely large positive values. These few very large positive values are enough to ensure that the mean value is very high. As one might expect in this case, the variance of $P(t)$ increases with t, tending to infinity in the limit.

Table 1. Statistics for the financial return of a plantation investment computed using mean and median certainty equivalent (CE) procedures.

Procedure	Rotation age (years)	Mean ($/ac)	Median ($/ac)	Standard deviation ($/ac)	Percentiles	
					25th ($/ac)	75th ($/ac)
A. Deterministic Faustmann						
Mean CE	30.1	1,461	1,461	—	—	—
Median CE	23.1	450	450	—	—	—
B. Stochastic, single rotation						
Mean CE	30.1	1,203	331	4,207	112	977
Median CE	23.1	997	375	2,456	146	962
C. Stochastic, ongoing rotation						
Mean CE	30.1	1,403	338	—	—	—
Median CE	23.1	1,217	399	—	—	—

Table 2. Statistics for the rotation age and financial return of a plantation investment in which the rotation age is determined using mean and median feedback rules with parameters either known or estimated.

	Rotation age		Present value	
Parameters	Mean (st.dev.)	Median (quartiles)	Mean (st.dev.)	Median (quartiles)
A. Mean feedback rule				
Known	39.2 (12.0)	46.5 (27.5–50.0)	7,581 (100,679)	404 (85–1,549)
Estimated	38.1 (9.3)	36.0 (33.0–50.0)	3,366 (32,210)	422 (16–1,666)
B. Median feedback rule				
Known	27.4 (10.3)	23.8 (20.5–30.5)	6,579 (96,106)	534 (147–1,665)
Estimated	24.9 (8.1)	23.8 (21.3–24.8)	1,956 (8,994)	540 (146–1,534)

Table 3. Statistics for the rotation age and financial return of a plantation investment in which the rotation age is determined using mean and median fixed-age rules with parameters either known or estimated.

	Rotation age		Present value	
Parameters	Mean (st.dev.)	Median (quartiles)	Mean (st.dev.)	Median (quartiles)
A. Mean fixed-age rule				
Known	36.7 (11.6)	34.1 (25.4–50.0)	7,560 (100,683)	364 (72–1,486)
Estimated	33.6 —	33.6 —	3,364 (19,622)	361 (50–1,625)
B. Median fixed-age rule				
Known	25.8 (8.7)	22.6 (20.3–27.1)	6,572 (96,076)	499 (127–1,646)
Estimated	23.0 —	23.0 —	1,946 (7,642)	517 (122–1,595)

Figure Captions.

Figure 1. The natural logarithm of quarterly sawlog prices for loblolly pine. For details of data source see text.

Figure 2. **(a)** Histogram of standardized residuals of the natural logarithm of price after geometric Brownian motion has been fitted; **(b)** Plot of the quantiles of the standardized residuals against the quantiles of a standard normal ($N(0,1)$) distribution.

Figure 3. Observed volumes on four test plots of loblolly pine. For details see text.

Figure 4. Fitted monomolecular mean growth curve and sample plot volumes.

Figure 5. The reservation value plotted against age for two feedback harvest rules. The lower curve was obtained by stochastic dynamic programming (SDP), while the upper one was obtained from the heuristic myopic-look-ahead (MLA) procedure. For either case a harvest does not take place until the observed value (=price x volume) drops below the reservation value.

Figure 6. Density histograms of the cutting age (left-hand panels) and present value (right-hand panels) for the mean feedback rule under two scenarios (A) parameters known and (B) parameters estimated, as discussed in the text. Note that the area under the histogram in any region represents the probability of the random variable (cutting age or present value) falling in the given region.

Figure 7. Density histograms of the cutting age (left-hand panels) and present value (right-hand panels) for the median feedback rule under two scenarios (A) parameters known and (B) parameters estimated, as discussed in the text. Note that the area under the histogram in any region represents the probability of the random variable (cutting age or present value) falling in the given region.

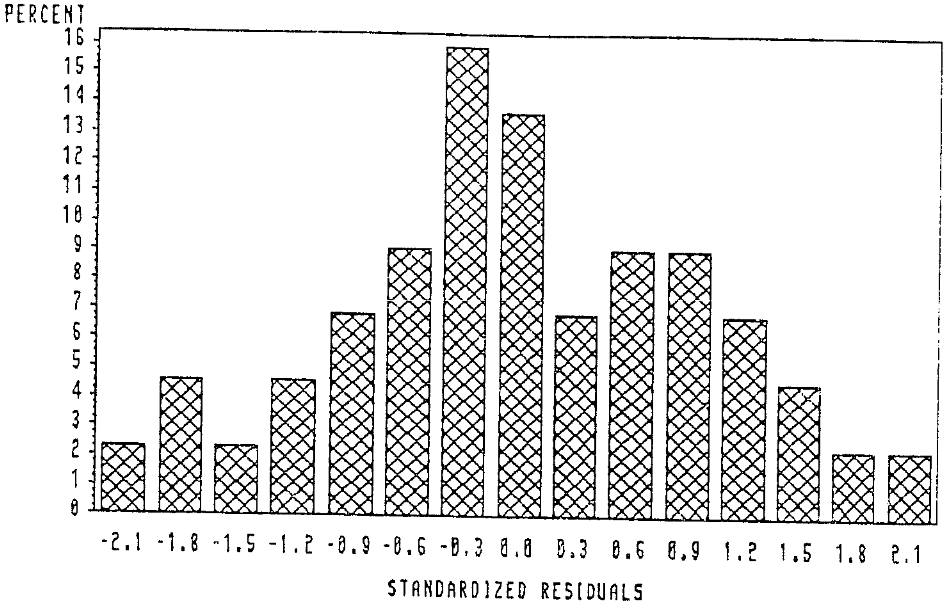
Fig 1

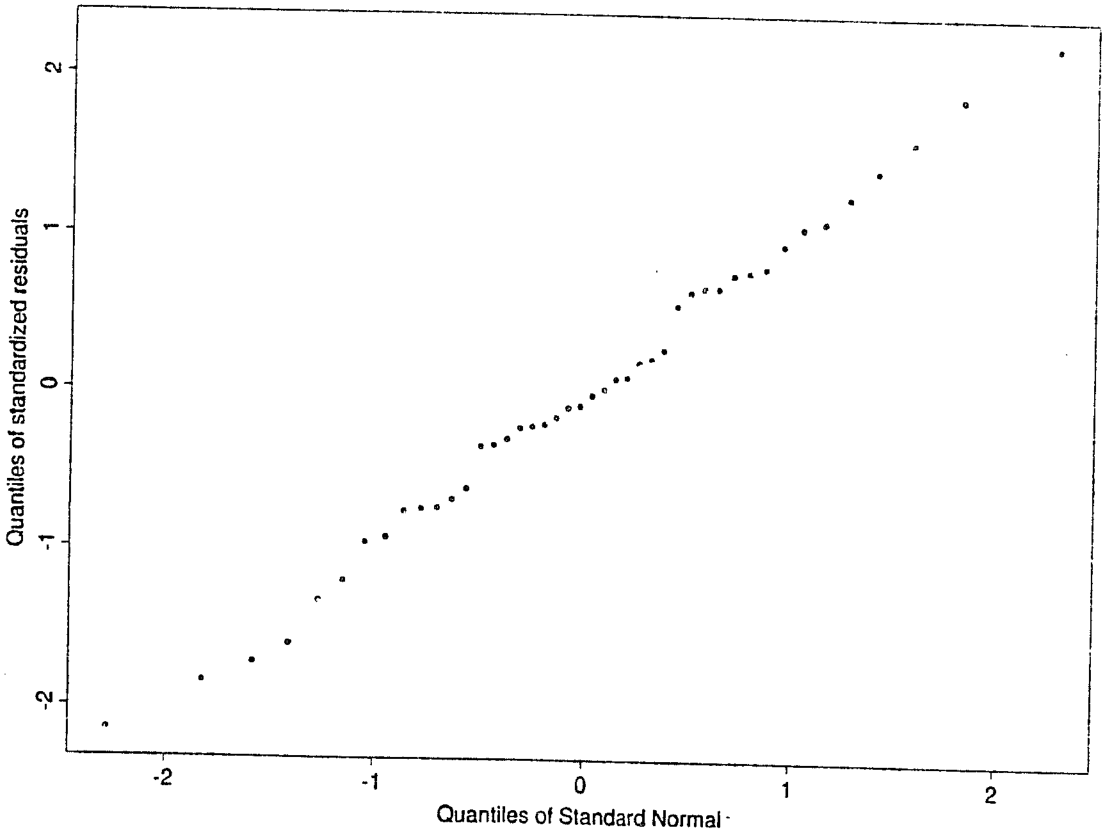
NATURAL LOGARITHM OF QUARTERLY SAWLOG PRICE

FOR YEARS 1977-1988

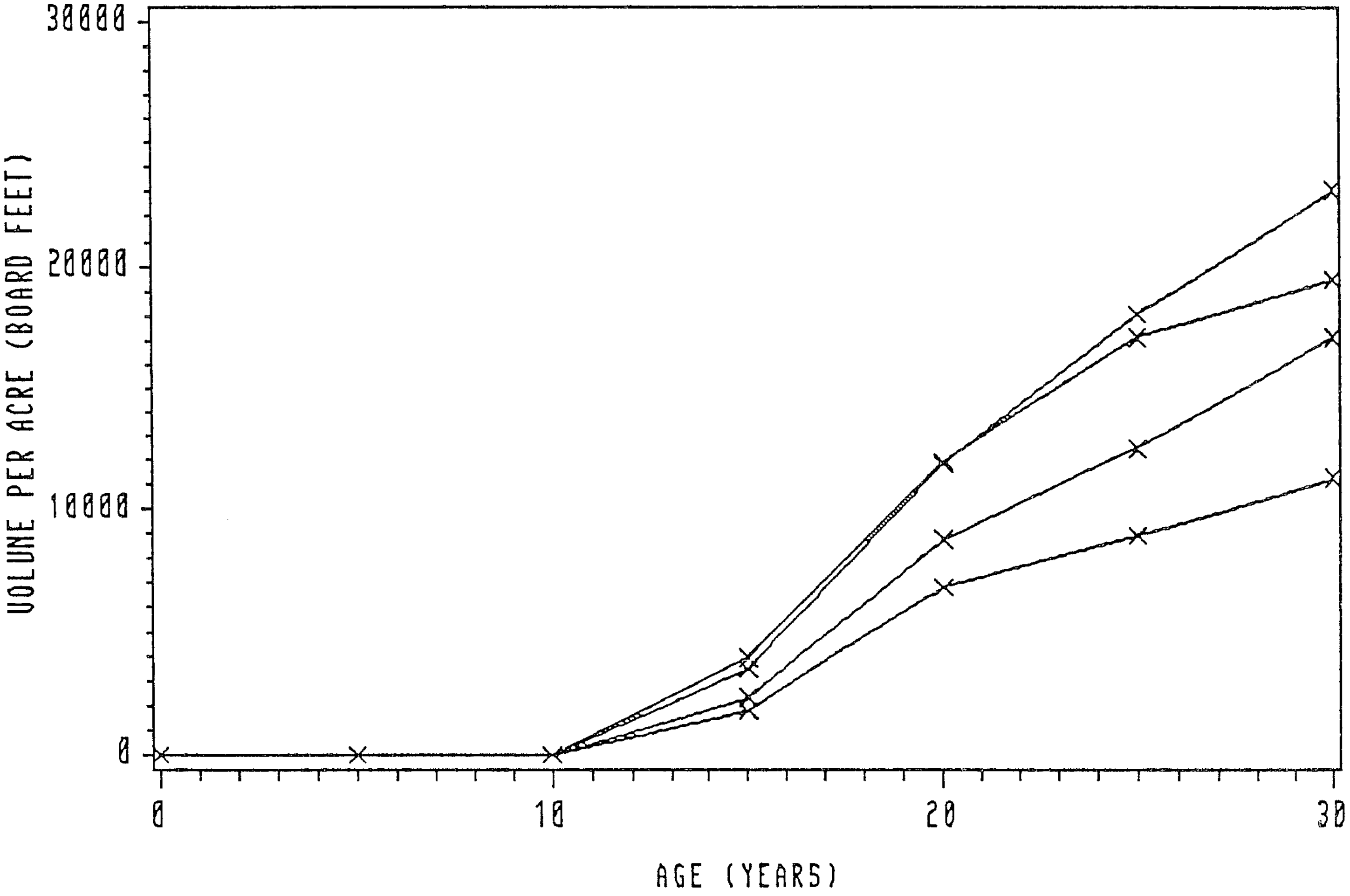
Fig 2

HISTOGRAM OF RESIDUALS FROM PRICE MODEL

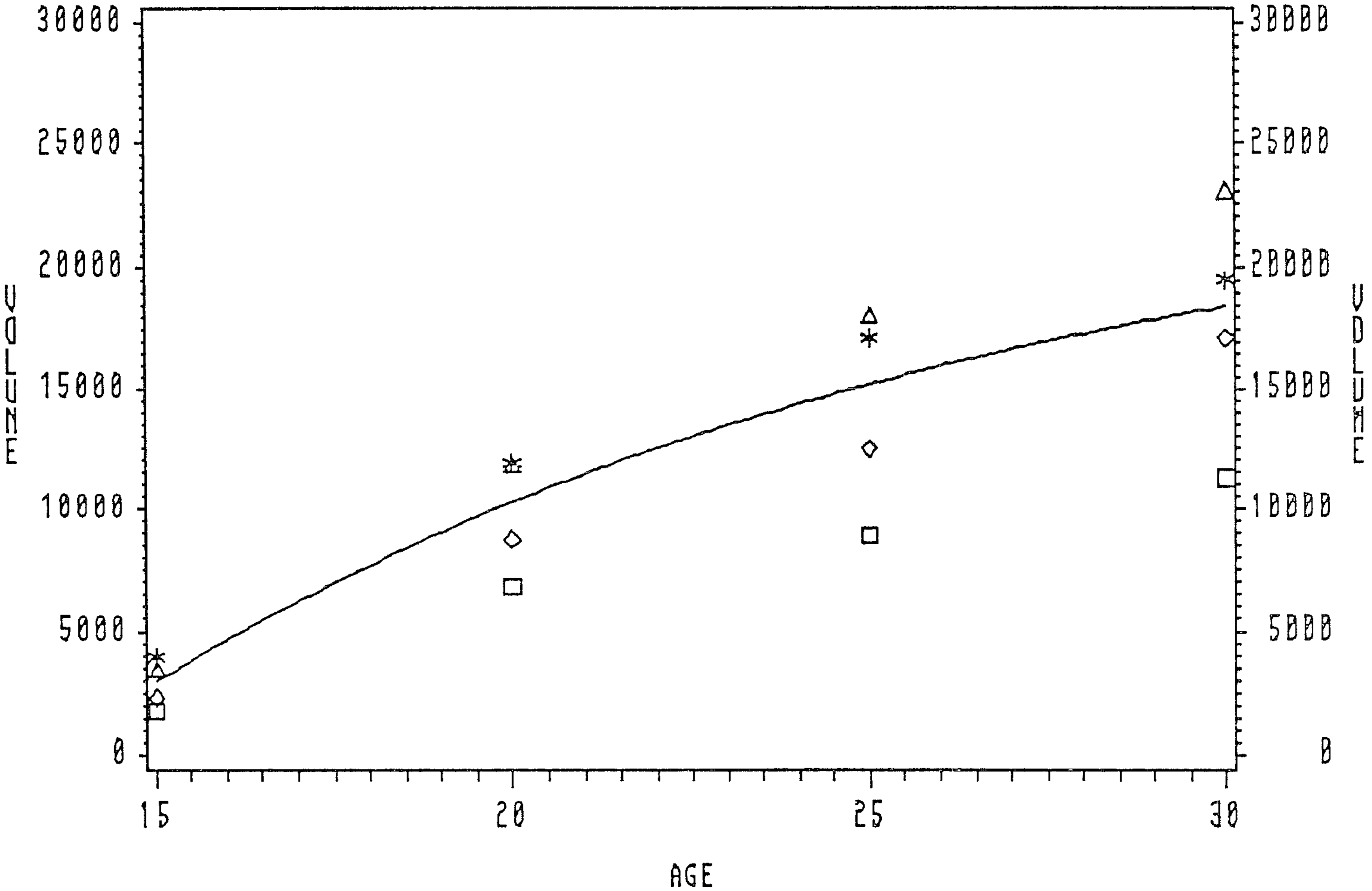


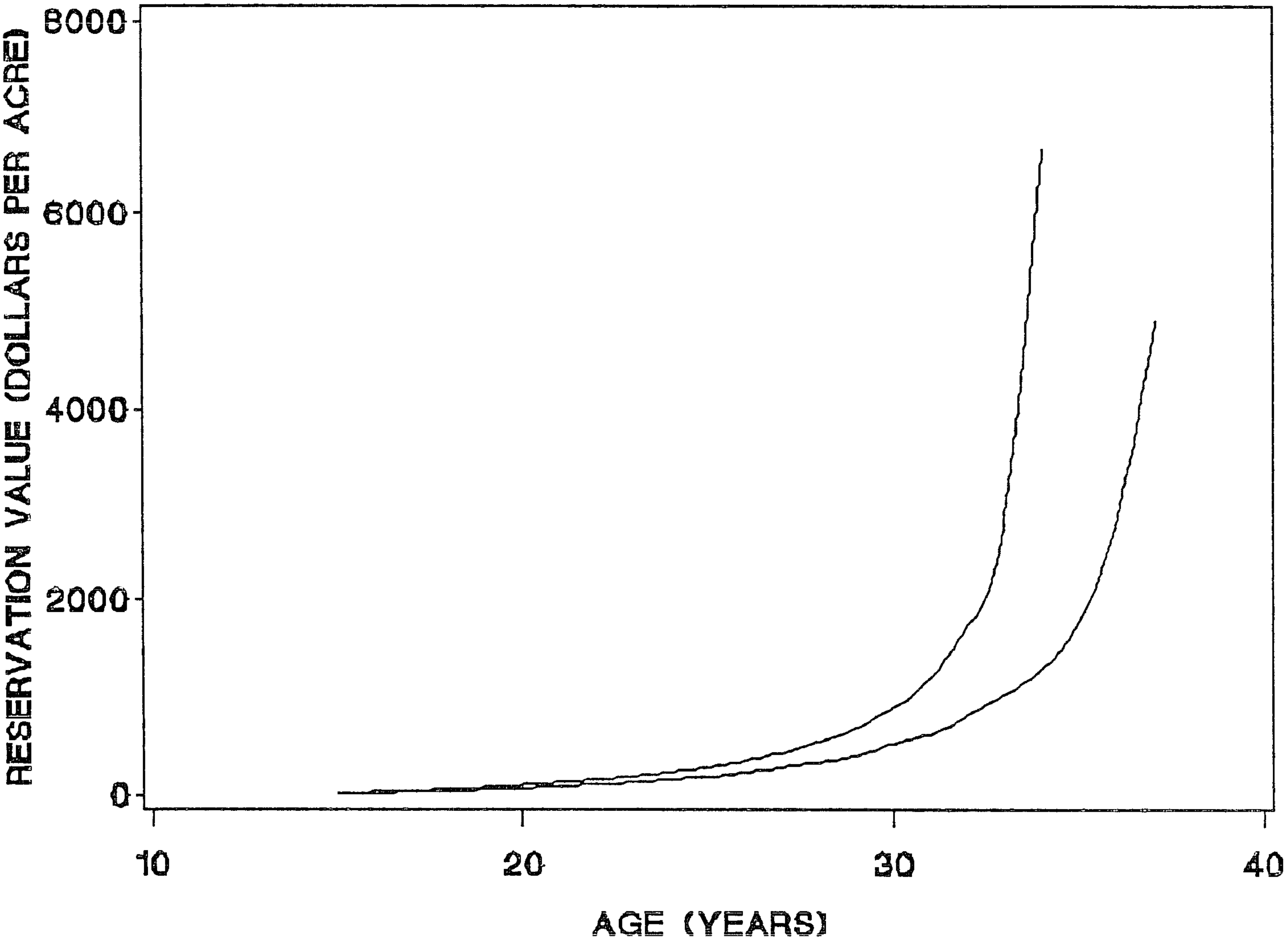


VOLUME GROWTH PER ACRE ON FOUR TEST PLOTS

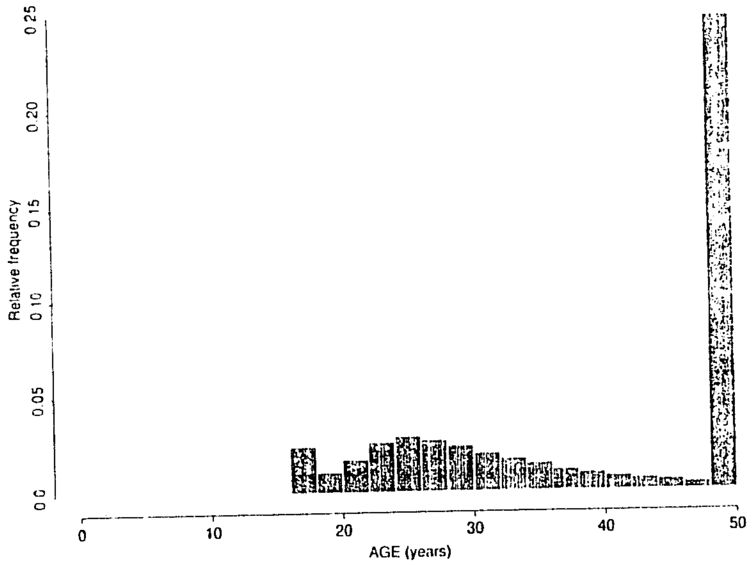


FITTED MEAN GROWTH CURVE AND SAMPLE PLOT VOLUMES

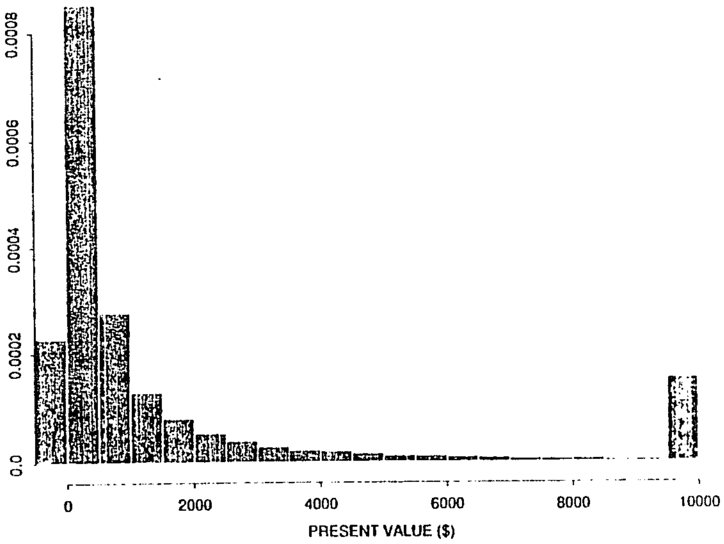



RESERVATION VALUE (DOLLARS PER ACRE)
8000
6000
4000
2000
0
10
20
30
40
AGE (YEARS)

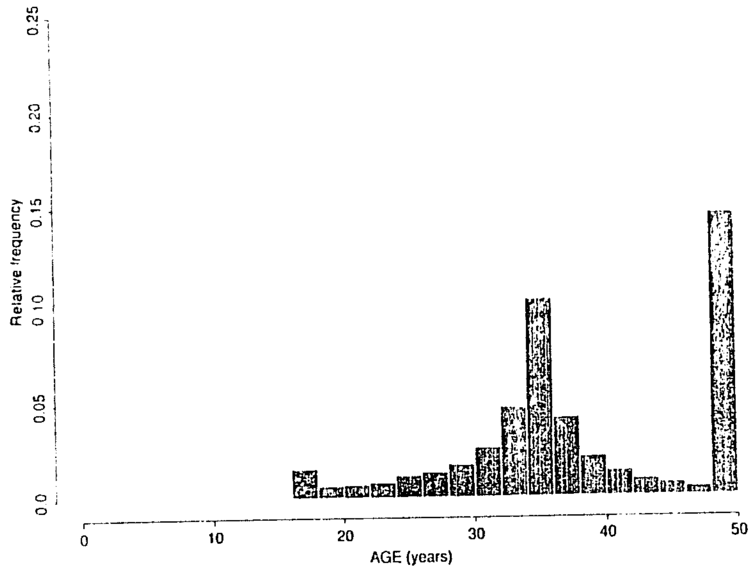
Histogram of cutting ages - mean feedback rule - parameters known



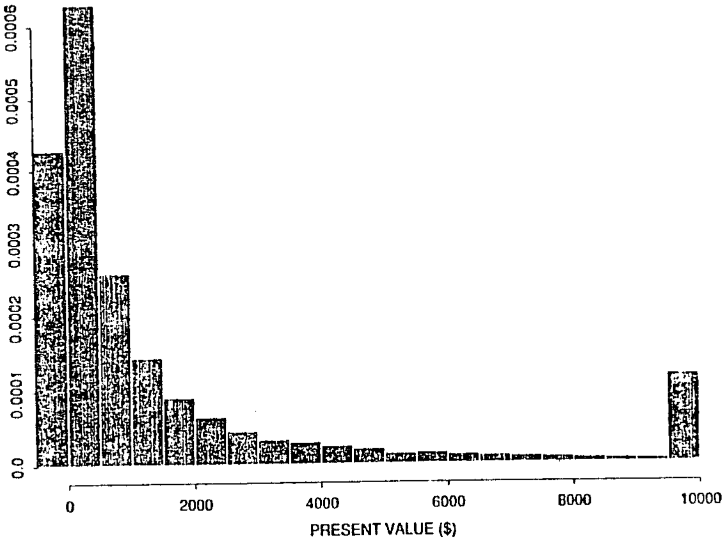
Histogram of present value ($) - mean feedback rule - parameters known

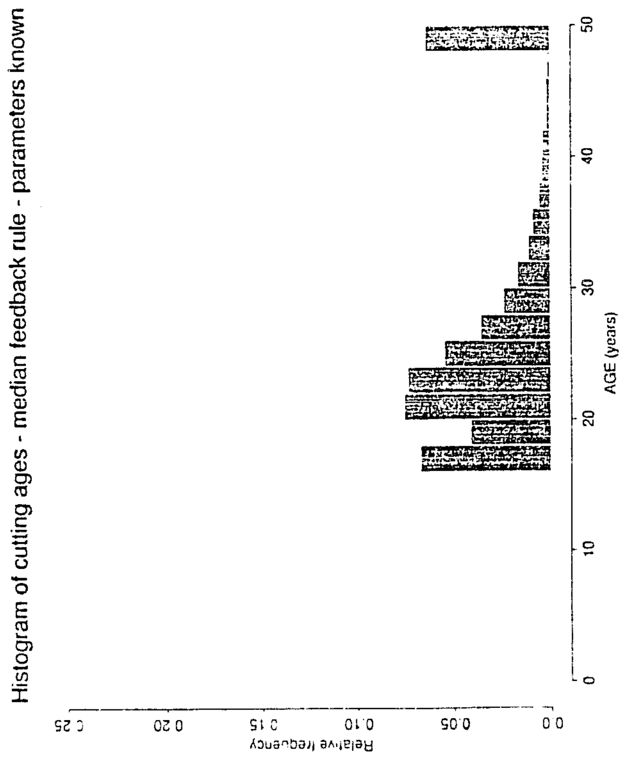


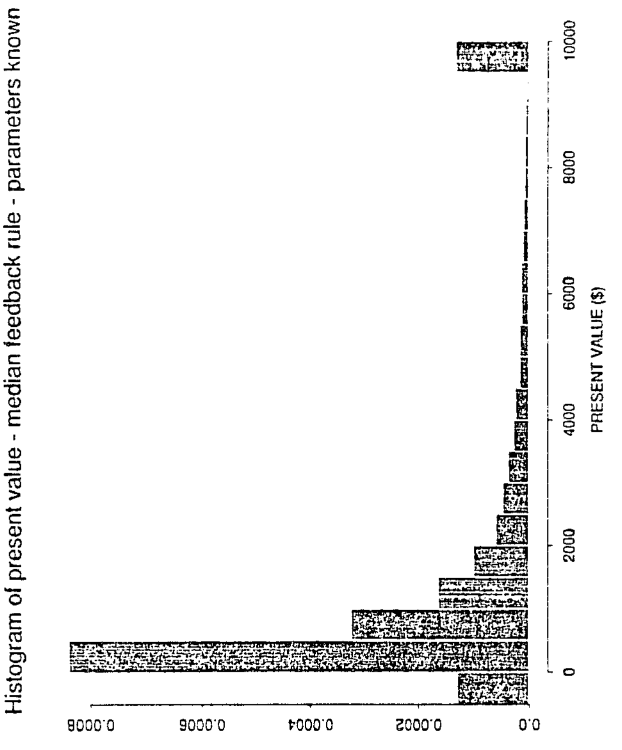
Histogram of cutting ages - mean feedback rule - parameters estimated

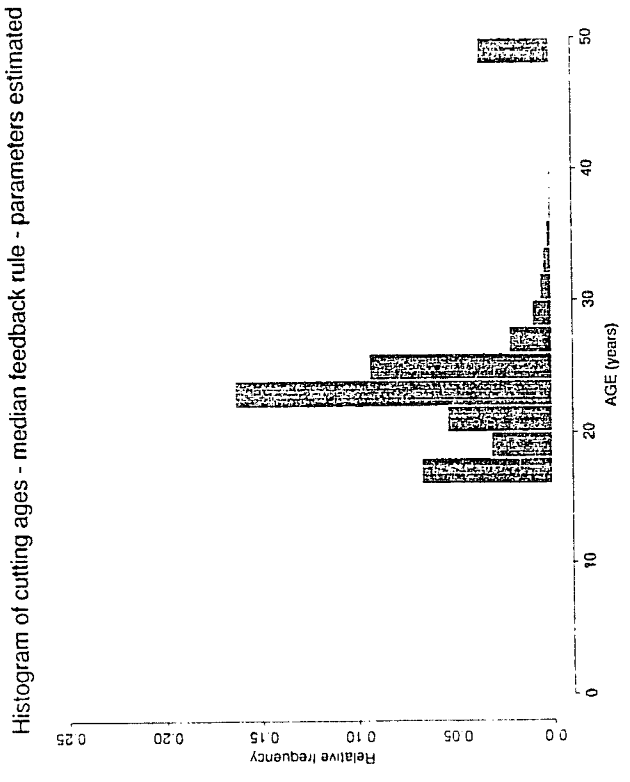


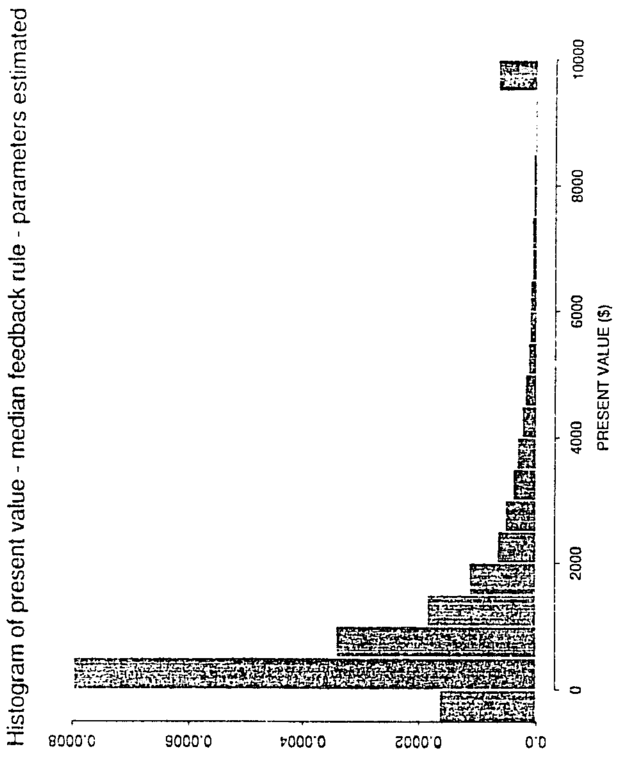
Histogram of present value - mean feedback rule -parameters estimated




Histogram of cutting ages - median feedback rule - parameters known


Histogram of present value - median feedback rule - parameters known


Histogram of cutting ages - median feedback rule - parameters estimated


Histogram of present value - median feedback rule - parameters estimated