

INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI[®]

Bell & Howell Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600

C*-Algebras of Sofic Shifts

by

Jonathan Niall Samuel

B.Sc. University of Saskatchewan, 1993

M.Sc. Dalhousie University, 1995

A Dissertation Submitted in Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

in the Department of Mathematics and Statistics.

We accept this dissertation as conforming to

~~the~~ required standard

Dr. Ian Putnam, Supervisor (~~Department of Mathematics~~)

Dr. Chris Bose, Departmental Member (~~Department of Mathematics~~)

Dr. John Phillips, Departmental Member (~~Department of Mathematics~~)

Dr. William Wadge, Outside Member (~~Department of Computer Science~~)

Dr. David Pask, External Examiner (~~Department of Mathematics, University of Newcastle, Australia~~)

©Jonathan Niall Samuel, 1998
University of Victoria

All rights reserved. This dissertation may not be reproduced in whole or in part,
by photocopying or other means, without permission of the author.

Abstract

This Dissertation shows how the theory of C*-algebra of graphs relates to the theory of C*-algebras of sofic shifts. C*-algebras of sofic shifts are generalizations of Cuntz-Krieger algebras [8]. It is shown that if X is a sofic shift, then the C*-algebra of the sofic shift, \mathcal{O}_X , is isomorphic to the C*-algebra of a directed graph E , $C^*(E)$. The graph E is shown to be the well known past set presentation of X constructed in [13].

We focus on the consequences of this result: In particular uniqueness of the generators of \mathcal{O}_X , pure infiniteness, and ideal structure of the algebra \mathcal{O}_X . We show the existence of an ideal $I \subset \mathcal{O}_X$ such that when we form the quotient, \mathcal{O}_X/I , it is isomorphic to $C^*(F)$, and F is the left Krieger cover graph of X – a well known, canonical graph one can associate with a sofic shift. The dual cover, the right Krieger cover, can also be related to the structure of \mathcal{O}_X , and we illustrate this relationship.

Chapter 6 shows what happens when we label a directed graph E in a left resolving way. When the graph E and the labeling satisfy certain technical conditions, we can generate a C*-algebra $\mathcal{L}_X \subset C^*(E)$, with $\mathcal{L}_X \cong \mathcal{O}_X$ provided that X an irreducible sofic shift.

Examiners:

Dr. Ian Putnam, Supervisor (Department of Mathematics)

Dr. Chris Bose, Departmental Member (Department of Mathematics)

Dr. John Phillips, Departmental Member (Department of Mathematics)

Dr. William Wadge, Outside Member (Department of Computer Science)

Dr. David Pask, External Examiner (Department of Mathematics, University of Newcastle, Australia)

Table of Contents

Abstract	ii
List of Figures	vii
Acknowledgments	viii
Dedication	ix
Chapter 1. Shift Spaces and Sofic Shifts	1
1.1. Introduction	1
1.2. Shift Spaces: Definitions and Examples	6
1.3. The Category of Shift Spaces	8
1.4. Directed Graphs and SFT's	10
1.5. Labeled Graphs and Sofic Shifts	12
1.6. The Follower Set and Predecessor Set Presentations	15
1.7. The Future and Past Set Presentations	18
1.8. Follower and Predecessor sets of Labeled Graphs	21
1.9. Properties of Follower Sets	23
Chapter 2. C^* -algebras of a Shift Spaces	27
2.1. The Fock Space Construction for a Shift Space C^* -algebra	27
2.2. Summary of the Structure of \mathcal{O}_X	30
2.3. Exploring the AF-Core	32
2.4. The Universal Property of \mathcal{O}_X	34

2.5.	The Failure of the Groupoid Constructions	39
Chapter 3. \mathcal{O}_X and Graph C*-algebras		42
3.1.	The Universal C*-algebra of a Graph	42
3.2.	Uniqueness and Pure Infiniteness of $C^*(E)$	45
3.3.	The Connecting Maps as Matrices	46
3.4.	That \mathcal{O}_X Equals $C^*(E_A)$	50
3.5.	The Perron-Frobenius Operator as a Cover for a Sofic Shift	54
3.6.	Uniqueness for \mathcal{O}_X in Terms of $C^*(E)$	56
3.7.	Pure Infiniteness of \mathcal{O}_X in Terms of $C^*(E)$	59
Chapter 4. The Ideal Structure of \mathcal{O}_X		61
4.1.	The Gauge Invariant Ideals for \mathcal{O}_X	61
4.2.	The Gauge Invariant Ideals for Sofic Shifts	64
4.3.	Equality of the AF-subalgebras	67
Chapter 5. The Structure of \mathcal{O}_X for Sofic Shifts		70
5.1.	The Presentation Σ_A and the Past Set Cover	71
5.2.	Reducing Σ_A to the Left Krieger Cover	74
5.3.	The Minimal Projections and the Right Krieger Cover	78
5.4.	The Structure of \mathcal{O}_X for AFT's	87
5.5.	The Structure of \mathcal{O}_X for Reducible X	91
Chapter 6. Generating \mathcal{O}_X from Left Resolving Labelings		93
6.1.	Left-Resolving Labelings and Graph C*-algebras	93
6.2.	More on the Left Krieger Cover Algebra	96
6.3.	The \mathcal{O}_X to \mathcal{L}_X Isomorphism	97
6.4.	The Factor Property and the Left Krieger Cover	101

Table of Contents

vi

6.5. The Follower Set and the Range Map	102
Chapter 7. Examples of \mathcal{O}_X for Sofic X	105
7.1. The Even shift	105
7.2. The Degree 3 Charge Constrained Shift	109
7.3. The AFT of Figure 1.5.2	114
7.4. A Non-AFT	116
7.5. The Failure of the Converse of Proposition 5.4.4	118
Bibliography	120

List of Figures

1.5.1	Two different presentations of the even shift	13
1.5.2	A non-left resolving Krieger cover of a sofic shift	15
1.6.1	The follower set presentation of the even shift	17
1.6.2	The right (and left) Krieger cover of the even shift	18
1.6.3	The right Krieger cover of the degree 3 charge constrained shift	18
1.9.1	A graph that presents only a one-sided sofic shift	24
6.5.1	A left resolving presentation of the even shift.	104
7.1.1	A modified even shift	108
7.3.1	The Perron-Frobenius presentation of a degree 2 AFT	115
7.4.1	A non-AFT presentation	116
7.5.1	A counter-example to proposition 5.4.4	119

Acknowledgments

The author wishes to thank the following people for their helpful discussions. Alex Kumjian, Iain Raeburn, David Pask, Teresa Bates. Thanks to Ian Putnam for all his help and suggestions. In addition, thanks to the staff at the Department of Mathematics at the University of Newcastle, Australia, where I spent some time doing this research.

Thanks to my family and friends for all their support these years. Special thanks to my brother, Adrian, who was always willing to take a break from the rat-race and head out with a backpack into the back country of Beautiful British Columbia. Finally I wish to thank Paul Wilson because he wished to be thanked, and he was a good friend when the going got tough.

Dedication

For my wife, Teresa.

CHAPTER 1

Shift Spaces and Sofic Shifts

1.1. Introduction

Symbolic Dynamics can be described as studying infinite strings of symbols obtained from a finite alphabet. Under certain conditions, collections of these infinite strings of symbols form a mathematical class called a shift space. The study of shift spaces has long been of interest to mathematicians, computer scientists, physicists and engineers. It's the study of "coding", and can be used to "code" more difficult problems into simpler ones, or as a practical use, to code data for more efficient means of storage, transmission, and retrieval. In the realm of symbolic dynamics, the sofic shifts have been a very manageable class in this study ("sofic" is derived from the Hebrew word for "finite"). Introduced by Weiss in [25], a sofic shift is a quotient of a even simpler shift space, a shift of finite type, or SFT.

The SFT is one of the simplest shift spaces. Suppose one had a finite set of "states" and certain allowable transitions from one state to another. One forms an infinite string of these states with the rule that the sub-string 'IJ' can appear in the infinite string if and only if the transition from state 'I' to state 'J' is allowed. The collection of all allowable infinite strings forms an SFT. As a topological space, it is a closed subspace of the infinite product of all states. To construct such a space, you do need the assumption there are no "dead ends", i.e., every state has a transformation to another state. To construct a sofic shift, one "labels" each of the distinct states with an alphabet, and rather than looking at infinite strings

of the states, look at the infinite string of the corresponding labels for the states. Certainly, if all states are labeled distinctly, you still have an SFT, however, if two states share a common label, there is a chance you no longer have an SFT. In terms of topological spaces, a sofic shift is a quotient space of an SFT, with the “labeling” map forming the quotient map.

There is an intimate relationship between SFT’s, matrices, and directed graphs; giving more power to analyse these shift spaces. Roughly speaking, if you have a SFT with n states, you form an $n \times n$ matrix A with the property that $A(i, j) = 1$ if there is a transition from state ‘i’ to state ‘j’. Otherwise $A(i, j) = 0$. As a directed graph, you have n vertices, and an edge from vertex i to vertex j if and only if $A(i, j) = 1$ (equivalently, the transition from state i to state j is allowed). The directed graph gives a visual representation of the allowable transitions. The structure of the graph can tell one much about the structure of the shift space one is studying. Furthermore, as directed graphs are associated with $n \times n$ matrices with entries in the positive integers, one could easily obtain information about the shift space by looking at the corresponding matrix.

In the mid 1970’s and 1980, Fischer [10] and Krieger [13] made the discovery that sofic shifts have a canonical SFT that has several nice properties. Much of the information about the sofic shift can be obtained by looking at the structure of this SFT.

It was not long after that Cuntz and Krieger [8] discovered that for shifts of finite type there was a C^* -algebra (a norm-closed $*$ -sub-algebra of bounded linear operators on a Hilbert space) one could obtain from the SFT based on the structure of its corresponding matrix. A very simple view of how this is done can be described as follows. If A is an $n \times n$ matrix with entries in $\{0, 1\}$, and \mathcal{H} is an infinite dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=1}^{\infty}$, partition this basis into

n disjoint subsets of infinite cardinality, say $\{B_i\}_{i=1}^n$. For each $1 \leq i \leq n$, look at all those j with the property that $A(i, j) = 1$, and form the direct sum

$$B_j^i := \bigoplus_{\{j : A(i,j)=1\}} \text{span}(B_j).$$

Because $\text{span}(B_i)$ and the above direct sum are both infinite dimensional, there is a linear isomorphism between the two spaces. Call this isomorphism $S_i : B_j^i \rightarrow \text{span}(B_i)$.

The isomorphism S_i as an operator on \mathcal{H} is a special kind of operator known as a partial isometry. When S_i is restricted to its domain and its adjoint S_i^* is restricted to the range of S_i ; the adjoint becomes the inverse of S_i . This means that both $S_i^* S_i$ and $S_i S_i^*$ are projections as operators on \mathcal{H} ; the first projecting onto B_i^j the second projecting onto $\text{span}(B_i)$. Cuntz and Krieger generated a C*-algebra using all the constructed isometries S_i , and related them back to the matrix A with the now famous Cuntz-Krieger relations:

$$S_i^* S_i = \sum_{j=1}^n A(i, j) S_j S_j^*.$$

What is surprising is that the algebra generated is independent of the choice of how one partitions the orthonormal basis (as long as each partition is infinite dimensional), and is independent of the choice of partial isometry chosen, provided that certain technical conditions on the matrix are satisfied (one being that your matrix is not a permutation matrix). Because these partial isometries are related to an $n \times n$ matrix A , which codes transitions between “states” (in this case, the “states” are closed subspaces of \mathcal{H}), it seemed natural to view this algebra as “the C*-algebra of the SFT generated by the matrix A ”. That is exactly what they did, showing many relationships between the structure of the C*-algebra and the corresponding SFT.

Because $n \times n$ matrices related well with directed graphs, in the 1990's A. Kumjian *et. al* [2, 15, 14] generalized the Cuntz-Krieger algebras to make a C*-algebra of a directed graph; the graph was allowed to be infinite, and/or was allowed to have vertices that emit no edges. In a different direction, K. Matsumoto [18, 20] generalized the Cuntz-Krieger algebras to algebras of general shift spaces. This allowed one to build a C*-algebra of a sofic shift. In spite of the difference in philosophy, the underlying strategy of constructing both C*-algebras was the same: that is, try to find a set of partial isometries on a Hilbert space that somehow captured the structure of the graph (or shift space) in which one was interested, and generate a C*-algebra from these partial isometries.

So if all these intimate relationships hold between SFT's, matrices, sofic shifts, labeled graphs, etc., what happens at the level of C*-algebras? The purpose of this dissertation is to explore these relationships. First, we show that the C*-algebra of a sofic shift, \mathcal{O}_X is also a C*-algebra of a certain directed graph E , denoted $C^*(E)$. We then use this fact to get theorems regarding the uniqueness of the generators of the sofic shift C*-algebra, and also whether or not this C*-algebra is purely infinite.

Both C*-algebras of sofic shifts, and C*-algebras of graphs admit a natural gauge action of the unit circle \mathbb{T} . We prove that the gauge invariant ideals of \mathcal{O}_X are precisely the same as the gauge invariant ideals of $C^*(E)$. Hence, the gauge invariant ideal structure of \mathcal{O}_X is entirely dependent on the structure of the the graph E .

The graph E is shown to be the past set cover of the sofic shift (the past set cover was constructed in [13]). This is one of the natural, left resolving graphs one can associate with a sofic shift. The past set cover always has a sub-graph that is minimal in the sense that it has the fewest number of vertices of all graphs that present the sofic shift (see [10, 13, 16]). We show the analogy of this fact in the theory of the C*-algebras: Every sofic shift C*-algebra has an ideal I such that when

you form the quotient C*-algebra, \mathcal{O}_X/I , you get the C*-algebra of this minimal graph. This ideal is trivial precisely when the past set cover and left Krieger cover coincide.

The past set graph has a dual graph called the future set cover, we show how the structure of \mathcal{O}_X relates to this future set cover also. We then apply this theory to a specific sub-class of sofic shifts called the shifts of Almost Finite Type (AFT) – introduced by B. Markus in [17]. When the sofic shift is an AFT, it is proved that \mathcal{O}_X always has a non-trivial ideal.

In the final part of this dissertation, we try to show a “converse” to all the work demonstrated so far. Instead of starting with a C*-algebra of a sofic shift and getting a C*-algebra of a graph, what happens if we take a C*-algebra of a graph, E , and try to make a C*-algebra of a sofic shift? We attempt to do this by labeling the edges on the graph. How we label the edges allows us to use the partial isometries that generate $C^*(E)$ to make other partial isometries that behave more like the C*-algebra of the sofic shift that the labeling presented. Under the right conditions, we can show that the C*-algebra of the sofic shift can be recovered from the C*-algebra of the graph with which we started (possibly as a sub-algebra).

This dissertation is divided into seven main chapters. Chapter one introduces the reader to shift spaces (SFT's and sofic shifts in particular) and their relationship with directed graphs and matrices. Chapter two covers the constructions of Matsumoto for a C*-algebra of a sofic shift. Chapter three covers the constructions of A. Kumjian *et. al* of a C*-algebra of a directed graph, and also establishes the relationship between the two algebras. Chapter four covers the ideal structure of the algebras. Chapter five shows the structure of the directed graph, and the labeled graph which the C*-algebra of the sofic shift is related. In chapter six, we look at what happens if you start with a labeled graph and attempt to generate a

C*-algebra of a sofic shift from it. Finally chapter seven shows many examples that illustrate the theory developed.

1.2. Shift Spaces: Definitions and Examples

Here, we give a brief introduction to the theory of shift spaces. More details can be found in any book on symbolic dynamics, for example [16].

Let $\Sigma = \{1, \dots, n\}$ be a finite symbol set. We let $Y = \prod_{i=-\infty}^{+\infty} \Sigma$ and $Y^+ = \prod_{i=1}^{\infty} \Sigma$ with the product topology. Both Y and Y^+ are compact, totally disconnected (or zero-dimensional) topological spaces and are referred to as the two-sided (or one-sided) full shift on n symbols respectively.

Define a map $\sigma : Y \rightarrow Y$, by $(\sigma(x))_i = x_{i+1}$ for $x = (x_i)_{i=-\infty}^{+\infty}$. The map σ is defined similarly on the one-sided shift. In both cases, σ is continuous. It is a homeomorphism on the two-sided shift.

Call a subspace $X \subseteq Y$ σ -invariant if $\sigma(X) = X$. A (two-sided) shift space X is a closed, σ -invariant subspace of Y (if the space is Y^+ , X is referred to as a one-sided shift space).

Assume X is a shift space on a finite symbol set $\Sigma = \{1, \dots, n\}$. We say $\mu = \mu_1\mu_2 \cdots \mu_l$ with $\mu_i \in \Sigma$ is a word in X if there is an $x \in X$ with $x_i = \mu_1, \dots, x_{i+l-1} = \mu_l$ for some i . If $\mu = \mu_1 \dots \mu_l$, let by $|\mu| = l$ denote the number of symbols in μ , which we will refer to as the *length* of μ . Let $B_l(X)$ denote the set of all words of length l , and $B(X) = \cup_{l=1}^{\infty} B_l(X)$. If $\mu, \nu \in B(X)$, then we denote their concatenation as $\mu\nu$. Note that $\mu\nu$ must be an allowable word in X in order to allow a concatenation. We denote by μ^n the word μ repeated n times.

For every $\mu = \mu_1\mu_2 \cdots \mu_n \in B(X)$ define the cylinder set of μ at coordinate k as

$$U_{\mu}^k = \{x \in X : x_k = \mu_1, \dots, x_{k+n} = \mu_n\}$$

It is well known that the cylinder sets are open, and form a basis for the product topology on X (inherited from the full shift Y). The convention is taken that the cylinder set of the empty word is all of X . The map σ sends cylinder sets to cylinder sets. If the shift is one-sided, and $|\mu| > 1$, σ will map U_μ^1 to the cylinder set V_ν^1 , where ν is the word obtained from μ by deleting the first symbol. If $|\mu| = 1$, there is a chance that $\sigma(U_\mu^1)$ is not an open set, and we shall investigate this further in chapter 2.

We say that a shift space X is *irreducible* if and only if for every $\mu, \nu \in B(X)$, there is a $\omega \in B(X)$ with $\mu\omega\nu \in B(X)$. For a shift space X we say $x \in X$ is periodic if $\sigma^n(x) = x$ for some $n \in \mathbb{N}$.

It can be shown that all shift spaces occur as follows. Let F be a collection of words in the full-shift Y . Define a subshift X as follows: if $y \in Y$, then $y \in X$ if and only if no word occurring in y occurs in F . If one can describe the shift space, X , with a finite set F , then X is referred to as a shift of finite type, or SFT.

Here are a few examples of two-sided shift spaces.

Example 1.2.1: Let $A = A(i, j)$ be an $n \times n$ matrix with entries in zero and one. Define a subshift X as follows

$$X = \left\{ x \in \prod_{i \in \mathbb{Z}} \{1, \dots, n\} : A_{x_i, x_{i+1}} = 1 \text{ for all } i \right\}.$$

This is a special class of SFT called a Markov Shift.

Example 1.2.2: Let $E = (E^0, E^1)$ be a directed graph with E^0 the vertex set and E^1 the edge set. Suppose E^0 is finite. Define a subshift X with symbol set E^0 as follows:

$$X = \left\{ x \in \prod_{i \in \mathbb{Z}} E^0 : \forall i \in \mathbb{Z}, \exists e \in E^1, \text{ connecting } x_i \text{ to } x_{i+1} \right\}.$$

Then X is a SFT called the vertex shift of a graph.

We will discuss these examples in more detail in the final section of this chapter.

Example 1.2.3: Let X be the infinite product (using either \mathbb{N} or \mathbb{Z} as an index set) on the symbols 1 and 2 with the following condition: Between any two successive 1's, there is always an even number of 2's. This shift space is called the even shift, and it is not a SFT.

Remark 1.2.4: If X is a two-sided shift space, there is a natural way of defining a one-sided space from it. Let

$$X^+ = \{(x_i)_{i \in \mathbb{N}} : \text{there exists } y \in X \text{ with } y_i = x_i \text{ for every } i \in \mathbb{N}\}.$$

Then X^+ is a one sided shift space. For notational convenience, we can do a similar thing for the negative integers and zero (but its not a shift space) by defining

$$X^- = \{(x_i)_{i \in \mathbb{Z} \setminus \mathbb{N}} : \text{there exists } y \in X \text{ with } y_i = x_i \text{ for every } i \in \mathbb{Z} \setminus \mathbb{N}\}.$$

So $x^+ \in X^+$ has the form $x_1 x_2 \dots$, and $x^- \in X^-$ has the form $\dots x_{-2} x_{-1} x_0$, and are sometimes referred to as the “heads” and “tails” of an element $x = x^- x^+ \in X$.

1.3. The Category of Shift Spaces

The category of shift spaces has objects (X, σ_X) with σ_X the shift map and X a shift space. The arrows $f : (X, \sigma_X) \rightarrow (Y, \sigma_Y)$ are continuous functions from X to Y such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \sigma_X \downarrow & & \downarrow \sigma_Y \\ X & \xrightarrow{f} & Y \end{array}$$

Although the definition is abstract, many morphisms in the shift space category occur as n -block maps, for some $n \in \mathbb{N}$. An n -block map $f : X \rightarrow Y$ is defined first

as a map

$$f : B_n(X) \rightarrow B_1(Y),$$

and then for $x = (x_i)_{i \in \mathbb{Z}}$, $f(x)$ is defined as

$$\dots f(x_0 x_1 \dots x_n) f(x_1 x_2 \dots x_{n+1}) \dots,$$

where $f(x_0 x_1 \dots x_n) = y_0$. A *factor map* is an n block map that is onto. Isomorphisms in the category of shift spaces are called conjugacies, and the two spaces are said to be conjugate. Not all conjugacies arise from n -block maps (see [16, Exercise 1.3.5]).

The following is well known for SFT's

Theorem 1.3.1: (see [16, Chapter 2]) *Every SFT is conjugate (through some n -block map) to a Markov shift.*

The SFT's form a sub-category of the category of shift spaces. It was shown by Weiss in [25] that the largest enveloping category of the sub-category of SFT's that is closed under quotients was the sofic shifts. We will use this as our definition.

Definition 1.3.2: A shift (X, σ_X) is sofic if there exists a SFT (Y, σ_Y) and a quotient map $\pi : Y \rightarrow X$; the space Y is called a *cover* for X .

Based on the above definition, SFT's are trivially sofic (take π to be the identity). We say a sofic shift X is *purely sofic* if it is not an SFT. Some examples of (purely) sofic shifts we will use throughout this dissertation include the even shift defined in example 1.2.3, and the degree 3 charge constrained shift, defined below.

Example 1.3.3: Let $n \in \mathbb{N}$ and $\Sigma = \{+1, -1\}$ be a symbol set. Define the degree n charge-constrained shift X as the (one or two sided) subshift with the property that every word in $B(X)$ has algebraic sum on its symbols no less than $-n$ and no more than n .

An example of a non-sofic shift is the so called context free shift. It is a shift on 3 symbols, 1, 2, 3, with the property that the word 12^m3^k1 occurs only if $m = k$. There are more non-sofic shifts than sofic. In fact, up to conjugacy, there are uncountably many shift spaces, but only countably many sofic shifts (see [16, Chapter 1]).

1.4. Directed Graphs and SFT's

There is a relationship between directed graphs and shifts of finite type that can be stated as follows. Theorem 1.3.1 tells us that every SFT is conjugate to a Markov shift. A Markov shift can be represented on a finite graph, and we shall see this construction in this section. From a graph theory point of view, this Markov shift is nothing but an “edge shift” on a directed graph.

Definition 1.4.1: A (finite) directed graph $E = (E^0, E^1)$ consists of a finite set of E^0 of vertices, and a set E^1 of edges. There are two maps $r, s : E^1 \rightarrow E^0$ (called the range and source maps respectively) defined as $r(e) = v$ if and only if v is the terminal vertex of the edge e , and $s(e) = v$ if and only if v is the initial vertex of the edge e .

We present an important example of an SFT obtained from a graph E .

Example 1.4.2: Let $E = (E^0, E^1)$ be a directed graph. Define a two sided shift

$$X := \left\{ e \in \prod_{i \in \mathbb{Z}} E^1 : r(e_{i-1}) = s(e_i) \right\}.$$

Then X is referred to as the *edge shift* of a directed graph E .

Given a graph $E = (E^0, E^1)$ one defines a *path* in the graph as a sequence of $e_i \in E^1$, $1 \leq i \leq n$ with the property that $r(e_{i-1}) = s(e_i)$ for $1 < i \leq n$. A *walk* is a path of infinite length, while a bi-infinite walk is a walk indexed by \mathbb{Z} rather than \mathbb{N} . The set of walks will form the one-sided edge shift of a graph E , and the set of

bi-infinite walks forms the two-sided edge shift for E . The set of all paths forms the words for the shift space.

Note that one could define paths and walks using the vertices of the graphs, rather than the edges. We will mainly use edges. A graph whose vertex or edge shift corresponds to a shift space X is called a *presentation* of X .

Remark 1.4.3: A vertex $v \in E^0$ is said to be a *source* if there is no edge e with $r(e) = v$. A *sink* is a vertex v with no edge e satisfying $s(e) = v$. A vertex is *isolated* if it is both a source and sink. When dealing with shift spaces on graphs, one can easily remove isolated vertices, and sinks without changing the presentation (see [16, Chapter 2]). For one-sided shifts, sources are possible (see figure 1.9.1). Since we will be working with two-sided shifts, we can assume our presentation has no sources or sinks.

A graph is said to be *irreducible* if and only if for every pair of vertices I, J there is a path from I to J . It is straightforward to check that a edge shift or vertex shift is irreducible if and only if it has an irreducible graph presenting it.

Edge shifts are Markov shifts, so they can be represented as an $n \times n$ matrix A with entries in $\{0, 1\}$. The construction is as follows. For a directed graph $E = (E^0, E^1)$ let m equal the number of vertices in E , n the number of edges in E , and let $E^1 = \{e_1, \dots, e_n\}$. Define an $n \times n$ matrix A whose (i, j) th entry is:

$$A(i, j) = \begin{cases} 1 & \text{if } r(e_i) = s(e_j), \\ 0 & \text{else.} \end{cases}$$

It is a straightforward exercise to check that the edge shift X (see example 1.4.2) is conjugate to the Markov shift generated by the matrix A (example 1.2.1). Conversely, one can go from a Markov shift to an edge shift of a directed graph. See [16, Chapter 2] for more.

Matrices that have entries in the non-negative integers can be used to form a directed graph, and vice versa. To go from a directed graph $E = (E^0, E^1)$ to an integer matrix, B , define a $m \times m$ (where m is the number of vertices) matrix B whose (i, j) th entry is:

$$B(i, j) := \text{the number of edges connecting vertex } i \text{ to vertex } j.$$

It may well be that B is not a zero-one matrix, but it can be converted to a zero-one matrix as follows: for each non-zero entry in $B(i, j)$, let $\{A^l(i, j)\}_{l=1}^{B(i, j)}$ be a set of “symbols”, $1 \leq i, j \leq m$. Define a transition from $A^l(i, j)$ to $A^p(k, l)$ if and only if $j = k$. One can check that this is a Markov shift, and is precisely the edge shift of the graph E .

To go from a matrix B with non-negative entries to a directed graph, suppose B is $m \times m$. Draw m vertices, and $B(i, j)$ edges from vertex i to vertex j . See [16] for more on these relationships.

Remark 1.4.4: Throughout this dissertation, we will be referring to a matrix with non-negative integer entries, and its corresponding edge shift. We will denote the edge shift graph corresponding to an non-negative integer matrix A as E_A . Conversely, given a directed graph E , we shall refer to the corresponding edge matrix as A_E .

1.5. Labeled Graphs and Sofic Shifts

A *labeling* for a directed graph $E = (E^0, E^1)$ is a (finite) collection of symbols Σ , and a labeling map $\pi : E^1 \rightarrow \Sigma$. We assume that π is onto. Given a labeled graph $E = (E^0, E^1, \Sigma, \pi)$ (denoted as (E, π) when the label set is known) we can define a sofic shift space X as follows.

$$X := \left\{ (\pi(e_i))_{i=-\infty}^{\infty} : (e_i)_{i=-\infty}^{\infty} \text{ is in the edge shift of } E \right\}.$$

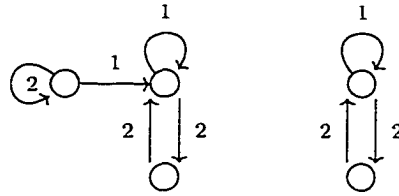


FIGURE 1.5.1. Two different presentations of the even shift

The symbol set for X is Σ , and the one block map $\pi : E^1 \rightarrow \Sigma$ makes the edge shift of the unlabeled graph a cover for X . As with SFT's, we shall refer to the labeled graph as a *presentation* of the sofic shift X . When X is irreducible, there is always an irreducible labeled graph that presents it.

Note that many different labeled graphs can represent the same sofic shift. For instance, figure 1.5.1 exhibits two different graph presentations of the even shift (example 1.2.3).

Definition 1.5.1: Let $E = (E^0, E^1, \Sigma, \pi)$ be a labeled graph:

1. E is said to be right resolving if for each vertex in the graph, all the edges leaving that vertex carry different labels. It is said to be left resolving if for each vertex in the graph, all the edges entering that vertex have distinct labels.
2. A vertex of a labeled graph is said to be right (left) resolving if every edge exiting (entering) the vertex has a distinct label.
3. A vertex of a labeled graph is said to be right (left) resolving with respect to label $I \in \Sigma$ if there is only one edge labeled I exiting (entering) that vertex.

A graph that presents a shift X is said to be *minimal* if it has the fewest number of vertices amongst all graphs that present X . Since sofic shifts are quotients of shifts of finite type, which are in turn conjugate to edge shifts of finite graphs, it

must be that the minimal graph for a sofic shift is a finite graph. For irreducible sofics, the minimal cover is unique:

Theorem 1.5.2: *(see [10]) If X is an irreducible sofic shift, then any two irreducible, minimal, right (left) resolving graph presentations of X have conjugate edge shifts via a 1-block map. Thus, up to a renaming of the labels, all such graph presentations are the same.*

A corollary to this is that there is a “canonical” SFT that represents the sofic shift; namely, the shift represented by the edge matrix of the minimal, right resolving graph. This is sometimes referred to as the (right) Krieger Cover ([13]), or the (right) Fischer Cover ([10]). The word “right” is used to indicate it is right resolving. The left Krieger (or Fischer) cover is the minimal, left resolving graph. The reason that both Fischer and Krieger get credit names is that both have given explicit (but slightly different) constructions of this cover. However, the above theorem tells us that as minimal covers, they are the same. Fisher [10] was the first to notice uniqueness of the minimal cover; Krieger [13] showed that any morphism between two sofic shifts could be “lifted” to the minimal cover. We will focus on the construction of the Krieger cover in this dissertation.

Note that a minimal, right resolving presentation need not be left resolving, as figure 1.5.2 shows. The graph in figure 1.5.2 presents a sofic shift X and is not left resolving at vertex b . However, we shall see later this is the minimal, right resolving graph for X .

A *path* on a labeled graph (E, π) is a finite sequence of labels, $\mu_1 \dots \mu_n$ such that there exists a path $w_1 \dots w_n$ on E with $\pi(w_i) = \mu_i$. We can extend this definition to walks and a bi-infinite walks for labeled graphs. The set of walks forms a one-sided sofic shift space, and the set of bi-infinite walks the two-sided sofic shift. When

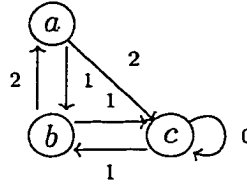


FIGURE 1.5.2. A non-left resolving Krieger cover of a sofic shift

we speak of a path on a labeled graph, we shall mean a path in the sense of this definition. A non-labeled graph can be thought of as a graph where every edge has a unique label, so these definitions are generalizations of paths and walks for directed graphs defined previously.

1.6. The Follower Set and Predecessor Set Presentations

The full details of follower set presentation can be found in [16, Chapter 3, Section 3]. For any $\mu \in B(X)$ we let

$$F_X(\mu) = \{x \in X^+ : \mu x \in X^+\}$$

be the follower set of μ (note that our definition is slightly different to that of [16], but is equivalent). We define an equivalence relation on $B(X)$, $\mu \sim \nu$ if and only if $F_X(\mu) = F_X(\nu)$. A well known result is that for sofic shifts, there are only finitely many follower sets [16, Theorem 3.2.10]. Thus there are finitely many equivalence classes.

If one prefers to look into the past, one can define the Predecessor set of μ as

$$P_X(\mu) = \{x \in X^- : x\mu \in X^-\}.$$

Like its follower set counterpart, a shift is sofic if and only if the number of predecessor sets is finite.

One way of presenting a shift space X on a labeled graph is the so called follower set graph. It is constructed as follows: The vertices of the graph are the $F_X(\mu)$, with $\mu \in B(X)$, and there is an edge labeled $i \in \Sigma$ from $F_X(\mu)$ to $F_X(\nu)$ if and only if $\mu i \in B(X)$ and $F_X(\mu i) = F_X(\nu)$. This is clearly a right resolving graph. The predecessor graph is defined with vertices $P_X(\mu)$ and an edge labeled $i \in \Sigma$ from $P_X(\mu)$ to $P_X(\nu)$ if and only if $i\mu \in B(X)$ and $P_X(\nu) = P_X(i\mu)$. This is clearly a left resolving graph.

Definition 1.6.1: A word $\mu \in B(X)$ is said to be magic (or intrinsically synchronizing [16], or finitary [13]) if for every $\nu, \lambda \in B(X)$ satisfying $\nu\mu, \mu\lambda \in B(X)$ the word $\nu\mu\lambda \in B(X)$.

Remark 1.6.2: Note that any word μ which contains a magic sub-word is also magic. A shift X is purely sofic if and only if for each $l \in \mathbb{N}$ there is a word μ with $|\mu| = l$ that is not magic [13, Proposition 4.3].

If a word μ is magic, then $F_X(\nu\mu) = F_X(\mu)$ for any $\nu \in B(X)$ with $\nu\mu \in B(X)$ (and $P_X(\mu\nu) = P_X(\mu)$ whenever $\mu\nu \in B(X)$). It can be shown that [16, Exercise 3.34, page 85], the sub-graph of the follower set graph with vertices consisting only of magic words forms the minimal right resolving presentation of X hence is the right Krieger cover. That the minimal right resolving presentation is a subgraph follows from remark 1.6.2.

Example 1.6.3: Figure 1.5.2 is the minimal, right resolving graph for the sofic shift presented by the labeled edge walks. The follower sets are the vertices $a = F_X(12)$, $b = F_X(01)$ and $c = F_X(0)$.

To obtain the left Krieger cover, we restrict ourselves to all those vertices in the predecessor set cover that are the predecessor sets of magic words.

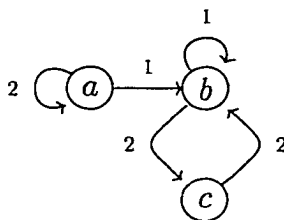


FIGURE 1.6.1. The follower set presentation of the even shift

Remark 1.6.4: One may believe the left Krieger cover could be obtained from the right Krieger cover just by reversing the range and source of the edges. If one does this to figure 1.5.2, one certainly gets a left resolving graph; note the one problem: if the left resolving graph must present the same shift, X , as the right, then a labeled path on the left resolving graph $e_1 e_2 \dots e_n$ would correspond to the word $e_n e_{n-1} \dots e_1 \in B(X)$. This is the main reason the left Krieger cover is defined “backwards”. There is a very special class of shifts where such a reversal will work. The degree zero AFT’s.

Definition 1.6.5:

1. A sofic shift is said to be almost finite type (AFT) if the right Krieger cover and left Krieger cover are conjugate as shifts of finite type.
2. An AFT is of degree zero if the right Krieger cover and the left Krieger cover are *equal* as labeled graphs.

Degree zero AFT’s have exactly the same right and left Krieger covers. Equivalently, the right Krieger cover graph is also left resolving (or the left Krieger cover graph is right resolving). A general AFT does not have this property. There are SFT’s that are not degree zero AFT’s. However, all SFT’s are AFT’s, and edge shifts are degree zero AFT’s.

Example 1.6.6: Figure 1.6.1 shows the follower set presentation for the even shift defined in example 1.2.3. The follower sets are $a = F_X(2)$, $b = F_X(1)$, $c = F_X(12)$.

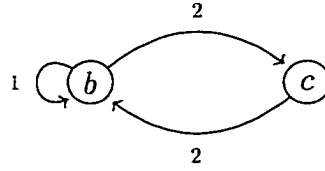


FIGURE 1.6.2. The right (and left) Krieger cover of the even shift

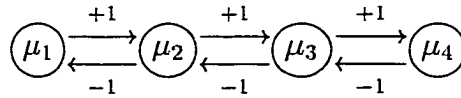


FIGURE 1.6.3. The right Krieger cover of the degree 3 charge constrained shift

The magic words are 1 and 12, and Figure 1.6.2 shows its right Krieger cover. It is a degree zero AFT (the graph is also left resolving), so figure 1.6.2 is also the left Krieger cover.

Example 1.6.7: The degree three charge constrained shift has Krieger cover as shown in figure 1.6.3. The magic words $\mu_1 = -1 - 1 - 1$, $\mu_2 = -1 - 1 - 1 + 1$, $\mu_3 = +1 + 1 + 1 - 1$ and $\mu_4 = +1 + 1 + 1$. The vertices are the follower sets of these words. Because this graph is also left resolving, it is a degree zero AFT.

1.7. The Future and Past Set Presentations

There is a slightly different view of the construction of the right and left Krieger covers which will be of more importance to us. If X^+ and X^- denote the one sided shifts defined in remark 1.2.4 of X , we can define, for $x \in X^-$, the future of x as

$$F_X(x) = \{y \in X^+ : xy \in X\},$$

and for $y \in X^+$, define the past of y .

$$P_X(y) = \{x \in X^- : xy \in X\}.$$

As outlined in [16, Exercise 3.2.8] (see also [13]) a shift space is sofic if and only if the number of futures (pasts) is finite. The future set graph of X , as a labeled graph has vertices $F_X(x)$ for $x \in X^-$, and there is an edge labeled $I \in \Sigma$ from $F_X(x_1)$ to $F_X(x_2)$, if and only if $F_X(x_1I) = F_X(x_2)$. Similarly, the past set graph has vertices $P_X(y)$ for $y \in X^+$ and there is an edge $I \in \Sigma$ from $P_X(y_1)$ to $P_X(y_2)$ if and only if $P_X(Iy_2) = P_X(y_1)$. Since the edge shift obtained from the future set graph is a cover of X , we will refer to it as the future cover. The past cover is defined similarly.

Like their follower and predecessor counterparts, the future cover is right resolving; the past cover is left resolving. In fact, both the future cover graph and follower set graph have a sub-graph that is exactly the same. Similarly, the past cover and predecessor set graph have a sub-graph that is exactly the same. Since we will be dealing mainly with the past cover graph later, we shall show it for the past.

If $(x_i)_{i=1}^\infty \in X^+$, we say that x is a magic walk if there exists $n \in \mathbb{N}$ with the word $x_1 \dots x_n$ a magic word. Because magic words have pasts independent of their futures, we see that if $x_1 \dots x_n$ is a magic word, and $m \geq n$ then

$$P_X(x_1 \dots x_n) = P_X(x_1 \dots x_m).$$

If we let $\mu = x_1 \dots x_n$, then we can conclude that $P_X(x) = P_X(\mu)$. Hence, a predecessor set of a magic walk is precisely the predecessor set of the first magic word obtained from its beginning. It is easily checked that if $P_X(x) = P_X(y)$ and x and y are magic walks, then the magic words $x_1 \dots x_n$ and $y_1 \dots y_m$ must have the same predecessor sets.

Conversely suppose μ is magic. Extend μ infinitely to the right to make $x = \mu x^+ \in X^+$. Once again, as magic words have pasts independent of their futures, we see that

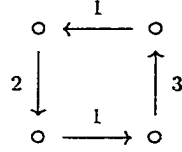
$$P_X(x) = P_X(\mu).$$

If μ, ν are magic words with the same predecessor sets, extending μ to the right to $x \in X^+$ and ν to the right to $y^+ \in X^+$ gives us $P_X(x) = P_X(y)$. This sets up a bijective correspondence between predecessor sets of magic words, and past sets of magic walks. Thus, in the past cover, we can regard the predecessor sets of magic walks as predecessor sets of magic words. Since the rules for drawing labeled edges from vertices are exactly the same regardless of whether you use past sets as vertices of predecessor sets of vertices we see that

Theorem 1.7.1: *For any sofic shift X , the left Krieger cover graph is a sub-graph of the past set graph. In particular, the left Krieger cover graph can be obtained from the past set graph by restricting to those vertices that coincide with pasts of magic walks.*

The past cover is non other than Krieger's past state chain [13]. When working in the world of C*-algebras of sofic shifts, it is the past cover that will be of interest. However, we will also establish a relationship between the follower sets of words and our C*-algebra; this is why both definitions have been given. Using follower sets and predecessor sets is more in the flavour of Fischer [10], while Krieger [13] used future and past sets. Although the two covers have a same subgraph, they are not quite the same, as the next example shows.

Example 1.7.2: This example shows that the past cover and the predecessor set cover are not the same. Suppose X is the following sofic shift



Then X^+ consists of four points

$$\{12131213\dots, 21312131\dots, 13121312\dots, 31213121\dots\}.$$

However, one can check directly that the predecessor set of the word '1' cannot be a past set for any of the points $x \in X^+$.

1.8. Follower and Predecessor sets of Labeled Graphs

When dealing with labeled graphs, one can define source and range maps. However, unlike the case for unlabeled graphs, the source and range maps will take values in the power set of E^0 (so they are set valued maps). We present the formal definition. To prevent confusion, we shall denote the range and source maps on an *unlabeled* graph as r_E and s_E respectively.

Definition 1.8.1: 1. Let ω be a path on a labeled graph $E = (E^0, E^1, \Sigma, \pi)$ (so $\omega \in B(X)$ for the associated sofic shift space X). Define the range of ω as

$$r(\omega) = \{v \in E^0 \mid \text{there is a path, } e \text{ on } (E^0, E^1) \text{ with } \pi(e) = \omega \text{ and } r_E(e) = v\}.$$

We say ω is a right synchronizing word for G if $r(\omega)$ is a single vertex. If that vertex is I , we say ω focuses to I .

2. Notation as above. We define the the source of ω as

$$s(\omega) = \{v \in E^0 \mid \text{there is a path, } e \text{ on } (E^0, E^1) \text{ with } \pi(e) = \omega \text{ and } s_E(e) = v\}.$$

We say ω is a left synchronizing word for G if $s(\omega)$ is a single vertex. If that vertex is J , we say ω radiates from J .

The above definitions of r and s clearly coincide with the range and source maps when the labels on the graph are all unique. Note further that sources, sinks, and isolated vertices are defined similar to the non-labeled graphs and remark 1.4.3 can be modified appropriately to apply to labeled graphs.

Given a labeled graph $E = (E^0, E^1, \Sigma, \pi)$, let X denote the sofic shift that is presented by E ; one defines the follower set of a vertex $v \in E^0$ as

$$F_E(v) = \{x \in X^+ : v \in s(x)\}.$$

Thus the follower set of a vertex is precisely the set of all walks that originate from that vertex. If the E is the follower set presentation of a sofic shift X , then it is clear that the follower set of a vertex $F_E(v) = F_X(\mu)$ for some word $\mu \in B(X)$. One defines a predecessor set of a vertex similarly.

Definition 1.8.2: A labeled graph, (E, π) is said to be follower (predecessor) separated if $v_1 \neq v_2 \in E^0$, then $F_E(v_1) \neq F_E(v_2)$ ($P_E(v_1) \neq P_E(v_2)$).

The importance of follower separation is the following theoretical description of the minimal right (left) resolving cover of a sofic shift X .

Theorem 1.8.3: (see [16, Corollary 3.3.19]) *Let X be an irreducible sofic shift. Then a right (left) resolving labeled graph (E, π) is the minimal right (left) resolving presentation if and only if E is irreducible and follower (predecessor) separated.*

Thus, the right Krieger cover is follower separated. Furthermore, when $F_X(\mu)$ is considered a vertex of the right Krieger cover, E , we have that $F_E(F_X(\mu)) = F_X(\mu)$. If one is given an arbitrary right (left) resolving labeled graph (E, π) , one

can “merge” vertices to produce the minimal right (left) resolving graph. See [16, Lemma 3.3.8].

Remark 1.8.4: Because the right Krieger cover is follower separated, magic words in the right Krieger cover have a singleton range (in the left Krieger cover they have singleton source). Furthermore, follower (resp. predecessor) separation allows one to take a non-magic word and extend it on the right (resp. left) to a magic word. Thus, every non-magic word is a sub-word of a magic word.

1.9. Properties of Follower Sets

In this section we shall show some properties of follower sets which will be important for structure theorems later. First, we need a few definitions regarding graph presentations of shift spaces.

Remark 1.9.1: 1. Note that if the labeled graph (E, π) is the right Krieger Cover for a sofic shift X , then

$$F_X(\nu) = \bigcup \{F_X(\mu) \mid F_X(\mu) \in r(\nu)\}.$$

In particular, $\mu \in B(X)$ is magic $\iff \mu$ is a right synchronizing word (remark 1.8.4).

2. If the shift is an AFT of degree zero, then ω is a right synchronizing word if and only if ω is a left synchronizing word because (E, π) is both right and left resolving.

We say a word $\nu = \nu_1 \dots \nu_n$, $n \in \mathbb{N}$ is in $B(F_X(\mu))$ if and only if $F_X(\mu) \in s(\nu)$. Since the follower set presentation has no sources or sinks, any such word can be extended to a walk νx on the graph satisfying $F_X(\mu) \in s(\nu x)$.

Lemma 1.9.2: *Let X be an irreducible sofic shift and $\mu \in B(X)$. If μ is magic then $B(F_X(\mu))$ contains magic words.*

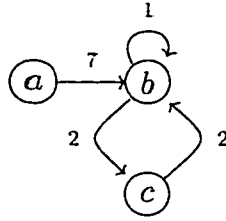


FIGURE 1.9.1. A graph that presents only a one-sided sofic shift

PROOF. Let μ be a magic word, then $\mu \in B(X)$, and thus must have a source on the follower set graph. Hence, $\mu \in F_X(\nu)$ for some magic word ν . If $\nu \sim \mu$, we are done. If not, then by irreducibility there is a walk from $F_X(\mu)$ to $F_X(\nu)$ call it γ . Thus $F_X(\mu\gamma) = F_X(\nu)$. Hence $\mu\gamma\mu \in B(X) \Rightarrow \gamma\mu \in B(X)$, and $\gamma\mu \in B(F_X(\mu))$ is magic by remark 1.6.2. \square

Remark 1.9.3: Although the proof of above lemma requires an irreducible sofic shift, one can modify the proof to include the reducible sofics as follows. Every sofic shift can be broken down into irreducible components; so one can just look at each irreducible component of the sofic shift. If the shift is two-sided, each component will not have sources or sinks, so cases like figure 1.9.1 will not happen.

Proposition 1.9.4: *Let X be a sofic shift with μ_1, \dots, μ_m its distinct magic words. Then X is an AFT of degree zero if and only if the intersection*

$$B(F_X(\mu_i)) \cap B(F_X(\mu_j)), \quad i \neq j$$

contains no magic words. Thus, there is no $x \in F_X(\mu_i) \cap F_X(\mu_j)$ with a magic word as a sub-word of x .

PROOF. To prove the ‘if’ part suppose $\omega \in B(F_X(\mu_i)) \cap B(F_X(\mu_j))$, $i \neq j$ is a magic word. Then ω is right synchronizing, hence left synchronizing by remark 1.9.1.

However, by assumption we must have $\{F_X(\mu_i), F_X(\mu_j)\} \subseteq s(\omega)$. This contradicts left synchronizing.

To prove the ‘only if’ part, suppose X is not an AFT of degree zero. Then the Krieger Cover is not left resolving. Suppose this non left resolving occurs at vertex $F_X(\mu_1)$. So there is an I in the symbol set with $|s(I)| > 1$, $F_X(\mu_1) \in r(I)$. By lemma 1.9.2 there is a magic word $\nu \in B(F_X(\mu_1))$. Then $I\nu$ is a magic word in X such that $|s(I\nu)| \geq 2$ in the right Krieger cover. Thus we have at least two magic words, say $F_X(\mu_i), F_X(\mu_j)$, with $I\nu \in B(F_X(\mu_i)) \cap B(F_X(\mu_j))$. This proves the theorem. \square

Corollary 1.9.5: *If X is an AFT of degree zero and X is also an SFT then $B(F_X(\mu_i)) \cap B(F_X(\mu_j))$ is finite for distinct magic words μ_i, μ_j (so $F_X(\mu_i) \cap F_X(\mu_j) = \emptyset$).*

PROOF. By proposition 1.9.4, the intersection of any two follower sets of magic words contains only non-magic words. Since for an SFT, every large enough word is magic, the intersection of two such sets can only be finite. \square

Remark 1.9.6: Note that AFT of degree zero is necessary for the proof of proposition 1.9.4. One can check the sofic shift shown in figure 1.5.2 has the magic word ‘21’ $\in B(F_X(12)) \cap B(F_X(01))$.

We also have a lemma that will be important for our calculations later. The proof is a direct consequence of lemma 1.9.2 and proposition 1.9.4.

Lemma 1.9.7: *If X is an AFT of degree zero then for distinct magic words μ_i, μ_j , $F_X(\mu_i) \not\subseteq F_X(\mu_j)$.*

Before closing this section, we will prove the converse of corollary 1.9.5; this gives us an equivalent definition for an AFT of degree zero to be an SFT. To do so, we need one lemma.

Lemma 1.9.8: *If X is sofic and ν is non-magic then there is at least two distinct magic words μ_1, μ_2 with $\nu \in B(F_X(\mu_1)) \cap B(F_X(\mu_2))$.*

PROOF. If ν is non-magic then by remark 1.9.1

$$F_X(\nu) = \bigcup_{i=1}^n \{F_X(\mu_i) \mid F_X(\mu_i) \in r(\nu)\}.$$

As ν is non-magic, we must have $|r(\nu)| > 1$. If $|s(\nu)| = 1$, then there exists a magic word μ with $s(\nu) = F_X(\mu)$. As the follower set graph is right resolving, this is impossible. Thus, there exists distinct magic words, μ_1, μ_2 , with $\{F_X(\mu_1), F_X(\mu_2)\} \subseteq s(\nu)$. Thus $\nu \in B(F_X(\mu_1)) \cap B(F_X(\mu_2))$. \square

Theorem 1.9.9: *Let X be a sofic shift. If $B(F_X(\mu_i)) \cap B(F_X(\mu_j))$ is finite for all distinct magic words μ_i, μ_j , then X has finite type. If X is an AFT of degree zero, then the converse also holds by corollary 1.9.5.*

PROOF. Suppose all the intersections are finite. By lemma 1.9.8 this means that there are $k < \infty$ many non-magic words, say ν_i , $1 \leq i \leq k$. Let

$$M = \sup_{1 \leq i \leq k} \{|\mu_i|\}$$

Then any word of length greater than M must be magic. This occurs if and only if X has finite type (see [16]). \square

CHAPTER 2

C^* -algebras of a Shift Spaces

This chapter covers the constructions of the C^* -algebras of K. Matsumoto [18]. Matsumoto's algebras are based on the realization of a C^* -algebra of a shift of finite type (or Cuntz-Krieger algebra) as creation operators on a sub-Fock space. Using this notion, he generalizes such a construction to an arbitrary shift space. We will explore this construction in some detail, because it is essentially this algebra we wish to explore in earnest. The full details can be found in the papers of Matsumoto [18, 19, 20].

2.1. The Fock Space Construction for a Shift Space C^* -algebra

We outline the construction of the C^* -algebra of a shift. The reader is referred to [18] for the details. The K -theory is worked out in [20].

We assume X is a shift space on a finite symbol set $\Sigma = \{1, \dots, n\}$. We say $\mu = \mu_1\mu_2 \cdots \mu_l$ with $\mu_i \in \Sigma$ is a word in X if there is an $x \in X$ with $x_i = \mu_1, \dots, x_{i+l-1} = \mu_l$. Denote by $|\mu|$ the number of symbols (length) in μ . Let $B^l(X)$ denote the set of all words of length l , and $B(X) = \cup_{l=1}^{\infty} B^l(X)$. Let e_0, e_1, \dots, e_n be an orthonormal basis for a Hilbert space, \mathcal{H} . If $\mu = \mu_1\mu_2 \cdots \mu_l \in B^l(X)$, let $e_\mu = e_{\mu_1} \otimes e_{\mu_2} \cdots \otimes e_{\mu_l} \in \otimes_{i=1}^l \mathcal{H}$ where $\mu_i \in \Sigma$.

Now let

$$\mathfrak{F}_0 = \mathbb{C}e_0 \text{ (Vacuum Vector)}$$

$$\mathfrak{F}_l = \text{Vector Space Spanned by vectors } e_\mu, \mu \in B^l(X)$$

$$\mathfrak{F} = \bigoplus_{i=0}^{\infty} \mathfrak{F}_i \text{ (Hilbert space direct sum).}$$

We define the (left) creation operator on \mathfrak{F} for $\mu \in B(X)$ as

$$T_\mu e_0 = e_\mu$$

$$T_\mu e_\nu = \begin{cases} e_\mu \otimes e_\nu & \text{if } \mu\nu \in B(X) \\ 0 & \text{otherwise} \end{cases}$$

The adjoint operator T_μ^* can be thought of as the left annihilation operator

$$T_\mu^* e_0 = 0$$

$$T_\mu^* e_\mu = e_0$$

$$T_\mu^* e_\nu = \begin{cases} e_\lambda & \text{if } \nu = \mu\lambda \in B(X) \\ 0 & \text{otherwise} \end{cases}$$

By letting P_0 be the rank one projection onto the vacuum vector e_0 we get that $\sum_{i=1}^n T_i T_i^* + P_0 = 1_{\mathcal{H}}$. Thus, the operator $T_\mu P_0 T_\nu^*$ is the rank one partial isometry from the vector e_ν to e_μ . This means that the C^* -algebra generated by the elements of the form $T_\mu P_0 T_\nu^*$ with $\mu, \nu \in B(X)$ is isomorphic to the compact operators on \mathfrak{F} . The C^* -algebra \mathcal{O}_X is defined to be the subalgebra of $B(\mathfrak{F})$ generated by the operators T_μ , $\mu \in B(X)$ modulo the compact operators. We will denote by S_μ the image of T_μ in this quotient. One notes that $S_i^* S_\mu^* S_\mu S_i = S_{\mu i}^* S_{\mu i}$ for $i \in \Sigma$, and is non-zero if $\mu i \in B(X)$. In fact $S_\mu S_\nu = S_{\mu\nu}$ whenever $\mu\nu \in B(X)$.

Recall the definition of $B(F_X(\mu))$ given in chapter 1: We say that $\nu = \nu_1 \dots \nu_n \in B(F_X(\mu))$ if and only if there exists an $(x_i)_{i=1}^\infty \in F_X(\mu)$ with $x_1 = \nu_1, \dots, x_n = \nu_n$. When $F_X(\mu)$ is viewed as a vertex in the follower set graph of X , we see that $\nu \in B(F_X(\mu))$ if and only if $F_X(\mu) \in s(\nu)$. Observe the following:

Lemma 2.1.1: *Let X be a sofic shift, \mathfrak{F}_X the Fock space constructed in this section, and $T_\mu, \mu \in B(X)$ the left creation operators. Then*

1. For each $\mu \in B(X)$

$$\begin{aligned} T_\mu^* T_\mu \mathfrak{F}_X &= \text{span}\{e_\nu : \mu\nu \in B(X)\} \\ &= \text{span}\{e_\nu : \nu \in B(F_X(\mu))\} \end{aligned}$$

and

$$(1 - T_\mu^* T_\mu) \mathfrak{F}_X = \text{span}\{e_\nu : \nu \notin B(F_X(\mu))\}$$

2. $F_X(\mu) = F_X(\nu)$ if and only if $S_\mu^* S_\mu = S_\nu^* S_\nu$.

PROOF. (1) follows directly from the construction of the operators T_μ . To show (2), if $F_X(\mu) = F_X(\nu)$ then certainly $B(F_X(\mu)) = B(F_X(\nu))$ thus $T_\mu^* T_\mu = T_\nu^* T_\nu$ by (1). Conversely, if $S_\mu^* S_\mu = S_\nu^* S_\nu$ then $T_\mu^* T_\mu - T_\nu^* T_\nu \in \mathcal{K}$. Therefore, $B(F_X(\mu)) \cap B(F_X(\nu))$ must be an infinite set. Thus $F_X(\mu) \cap F_X(\nu)$ is non-empty. Equality follows from the fact that if $(x_i)_{i=1}^\infty \in F_X(\mu)$ but not in $F_X(\nu)$, then any word of the form $x_1 \dots x_n$ will be in $B(F_X(\mu))$ but there will be infinitely many n for which $x_1 \dots x_n$ is not in $B(F_X(\nu))$. Using (1), this would mean that $T_\mu^* T_\mu - T_\nu^* T_\nu$ will not be compact, a contradiction. \square

2.2. Summary of the Structure of \mathcal{O}_X

We summarize the results from [18, 20]. We let $k, l, n \in \mathbb{N}$, $k \leq l$.

A_n = The C^* -subalgebra of \mathcal{O}_X generated by $S_\mu^* S_\mu$, $\mu \in B_n(X)$

A_X = The C^* -subalgebra of \mathcal{O}_X generated by $S_\mu^* S_\mu$, $\mu \in B(X)$

\mathcal{F}_k^l = The C^* -subalgebra of \mathcal{O}_X generated by $S_\mu a S_\nu^*$

$$\mu, \nu \in B^k(X), a \in A_l$$

\mathcal{F}_k^∞ = The C^* -subalgebra of \mathcal{O}_X generated by $S_\mu a S_\nu^*$

$$\mu, \nu \in B^k(X), a \in A_X$$

\mathcal{F}_X^∞ = The C^* -subalgebra of \mathcal{O}_X generated by $S_\mu a S_\nu^*$

$$\mu, \nu \in B(X), |\mu| = |\nu|, a \in A_X$$

We have the following results from [18, Section 3].

- Lemma 2.2.1:**
1. A_l is finite dimensional and commutative.
 2. A_l is contained in A_{l+1} so that $A_X = \overline{\bigcup_l A_l}$ is a commutative AF-algebra.
 3. Each element of \mathcal{F}_k^l is a finite linear combination of elements of the form $S_\mu a S_\nu^*$, $\mu, \nu \in B^k(X)$, $a \in A_l$. So \mathcal{F}_k^l is finite dimensional.
 4. There are two embeddings in $\{\mathcal{F}_k^l\}_{k \leq l}$:
 - (a) $\mathcal{F}_k^l \subset \mathcal{F}_k^{l+1}$ since $A_l \subset A_{l+1}$.
 - (b) $\mathcal{F}_k^l \subset \mathcal{F}_{k+1}^{l+1}$ through the identity:

$$S_\mu a S_\nu^* = \sum_{j=1}^n S_{\mu_j} S_j^* a S_j S_{\nu_j}^*, \quad \mu, \nu \in B_k(X), a \in A_l$$

5. Both $\mathcal{F}_k^\infty = \varinjlim \mathcal{F}_k^l$ and $\mathcal{F}_X^\infty = \varinjlim \mathcal{F}_k^\infty$ are AF algebras.

With these at our disposal, one can calculate the K-theory of the C*-algebra \mathcal{O}_X . The calculation is similar to the Cuntz-Krieger algebra case [8]. Define $\lambda_X : A_X \rightarrow A_X$ by

$$(2.1) \quad \lambda_X(a) = \sum_{i=1}^n S_i^* a S_i, \quad a \in A_X.$$

As in [18, 20], we refer to λ_X as the Perron-Frobenius operator on A_X . It is a *-homomorphism on A_X . From [20] we have the following about the K-theory.

- Proposition 2.2.2:**
1. [20, Proposition 3.5] $K_0(A_X) \cong \varinjlim (A_i, i_*)$ where i is the natural inclusion $A_i \rightarrow A_{i+1}$.
 2. [20, Proposition 3.11] $K_0(\mathcal{F}_X^\infty) \cong \varinjlim (K_0(A_X), \lambda_{X*})$ where λ_X is the Perron-Frobenius operator (2.1)

Like Cuntz-Krieger algebras, \mathcal{F}_X^∞ can be realized as stably isomorphic to a crossed product of a \mathbb{T} -action on \mathcal{O}_X [20, Corollary 4.2]. Thus, one can employ the Pimsner-Voiculescu exact sequence to get the K-theory of \mathcal{O}_X . Moreover, by [18, Proposition 8.2], the algebra A_X is finite dimensional if and only if the shift X is sofic. In this case $A_X \cong \mathbb{C}^n$ thus $K_0(A_X) \cong \mathbb{Z}^n$. Hence, the Perron-Frobenius operator can be viewed as a map on K-theory is an $n \times n$ matrix with entries in \mathbb{Z} ; the K-theory of \mathcal{O}_X can thus be described as

Theorem 2.2.3: (see [20, Theorem 4.9]) *Let X be a sofic shift and A the $n \times n$ matrix representing the Perron-Frobenius operator $\lambda_{X*} : K_*(A_X) \rightarrow K_*(A_X)$. Then*

$$K_0(\mathcal{O}_X) \cong \mathbb{Z}^n / (1 - A)\mathbb{Z}^n \quad K_1(\mathcal{O}_X) \cong \ker(1 - A)\mathbb{Z}^n.$$

We remark that the matrix representation for the Perron-Frobenius operator we will use in later sections will actually be the transpose of A in theorem 2.2.3. This is because we will be working with graphs, and our definition of A will be easier to work with in this context.

2.3. Exploring the AF-Core

From the last section, we know that $A_X = \overline{\bigcup_l A_l}$, with each A_l a finite dimensional, commutative algebra. Because A_X is commutative, it is isomorphic to the continuous functions on a compact space. We shall describe this space as done in [19, Section 2].

For the one-sided subshift X^+ put

$$P_l(x) = \{\mu \in B_l(X) : \mu x \in X^+\},$$

for $x \in X^+, l \in \mathbb{N}$, and define an equivalence relation on X^+ as follows $x \sim_l y$ if $P_l(x) = P_l(y)$. We say that $x, y \in X^+$ are l -past equivalent if $x \sim_l y$. Denote by $\Omega_l = X^+ / \sim_l$, the set of l -past equivalences on X^+ .

It is straightforward to check [19, Lemma 2.1], that if $x \sim_l y$ then $x \sim_m y$ for every $m \leq l$. Furthermore if $\mu \in B_k(X)$, and $x \sim_l y$ with $\mu x \in X^+$, then $\mu y \in X^+$ and $\mu x \sim_{l-k} \mu y$ for $l > k$, and for any shift space X , the number of l -past equivalence classes is finite.

Because of this, we get an inverse sequence of surjections

$$\Omega_1 \leftarrow \Omega_2 \leftarrow \cdots \leftarrow \Omega_l \leftarrow \cdots .$$

Let

$$\Omega_X = \varprojlim \Omega_l$$

be the projective limit as a topological space. As each Ω_l is finite, we put the discrete topology on Ω_l , and the inverse limit topology on Ω_X .

For $x, y \in X^+$, we say that x is past equivalent to y (denoted $x \sim_\infty y$) if $P_\infty(x) = P_\infty(y)$ where

$$(2.2) \quad P_\infty(x) = \{(\dots x_{-2}x_{-1}x_0) \in X^- : (\dots x_{-2}x_{-1}x_0)x \in X\}.$$

As mentioned in [19, Section 2], Ω_X is nothing but the quotient X^+ / \sim_∞ .

We have the following:

Proposition 2.3.1: [19, Lemma 4.6, Corollary 4.7] *Let X be a shift space.*

1. *For each $l \in \mathbb{N}$, the quotient space X^+ / \sim_l , endowed with the discrete topology, is compact and Hausdorff. Moreover the maps sending $S_\mu^* S_\mu$, $\mu \in B_l(X)$ to the characteristic function of*

$$\{[x]_l : x \in X^+, \mu x \in X^+\}$$

extends to an isomorphism between A_l and $C(X^+ / \sim_l)$.

2. *The map sending $S_\mu^* S_\mu$, $\mu \in B(X)$, to the characteristic function of*

$$\{[x]_\infty : x \in X^+, \mu x \in X^+\}$$

extends to an isomorphism between A_X and $C(X^+ / \sim_\infty)$; the topology on X^+ / \sim_∞ is the inverse limit topology.

Proposition 2.3.1 has some important implications for sofic shifts. First, recall the definition of a past set of an $x \in X^+$ from section 1.7. We see that $P_\infty(x)$ defined in (2.2) is equal to the past set $P_X(x)$. So $x \sim_\infty y$ if and only if $P_X(x) = P_X(y)$. Thus, as the number of past sets is finite if and only if X is a sofic shift we have.

Corollary 2.3.2: [18, Proposition 8.2] *The algebra A_X is finite dimensional if and only if X is a sofic shift.*

In view of proposition 2.3.1 for any shift X , we can regard any element of A_X as a continuous function on X^+ which is constant on a past equivalence class. In particular, elements of A_l will be associated to functions constant on an l -past equivalence class. Henceforth, we suppress this in our notation. For μ in $B(X)$, we

regard $S_\mu^* S_\mu$ as the characteristic function of

$$\{x : x, \mu x \in X^+\}.$$

When X is a sofic shift, X^+ / \sim_∞ is finite; if we choose a complete set of representatives for X^+ / \sim_∞ , say $x_1, \dots, x_n \in X^+$, then for each x_i we have a minimal projection P_i in A_X ,

$$P_i(y) = \begin{cases} 1 & \text{if } y \sim_\infty x_i \\ 0 & \text{otherwise.} \end{cases}$$

As each minimal projection can be regarded as a characteristic function on a past equivalence class, define:

$$(2.3) \quad \begin{aligned} \chi(P_i) &= \{x \in X^+ : x \sim_\infty x_i\} \\ &= \{x \in X^+ : P_X(x) = P_X(x_i)\}. \end{aligned}$$

Note the above equation will be independent of the choice of equivalence class representatives x_i . The correspondence above tells us that $x \in \chi(P_i)$ if and only if $P_i(x) = 1$ (when viewed as a characteristic function).

Remark 2.3.3: As a final remark, we mention that the algebra A_X is unital. This can be seen because

$$\sum_{I \in \Sigma} T_I^* T_I$$

is a positive, invertible operator on the Fock space. So the identity operator is in the algebra generated by $\{T_I^* T_I : I \in \Sigma\}$.

2.4. The Universal Property of \mathcal{O}_X

We state here the universal property of the C*-algebra \mathcal{O}_X .

Theorem 2.4.1: [18, Theorem 4.9] *Let \mathcal{A} be a unital C^* -algebra. Suppose that there is a unital $*$ -homomorphism π from A_X to \mathcal{A} , and there are n partial isometries $s_i \in \mathcal{A}$, $1 \leq i \leq n$ satisfying the following relations:*

1. $\sum_{j=1}^n s_j s_j^* = 1$, $s_\mu^* s_\mu s_\nu = s_\nu s_{\mu\nu}^* s_{\mu\nu}$, $\mu, \nu, \mu\nu \in B(X)$,
2. $s_\mu^* s_\mu = \pi(S_\mu^* S_\mu)$, $\mu \in B(X)$,

where $s_\mu = s_{\mu_k} \cdots s_{\mu_1}$, $\mu = (\mu_k \cdots \mu_1) \in B(X)$. Then there exists a homomorphism $\tilde{\pi}$ from \mathcal{O}_X to \mathcal{A} such that $\tilde{\pi}(S_i) = s_i$ for $1 \leq i \leq n$, and its restriction to A_X coincides with π .

Since our \mathcal{O}_X was constructed from concrete Hilbert space operators, one can ask if a different set of partial isometries satisfying (1) and (2) of theorem 2.4.1 give you an isomorphism between \mathcal{O}_X and \mathcal{A} . The theorem does not guarantee that the above homomorphism is injective. Injectivity can be obtained two ways. The first way is based on the structure of \mathcal{F}_X^∞ and is described in detail in [18]. We sketch the concepts here.

Let (X^+, σ) be the one-sided sofic shift. Denote by \mathcal{D}_X the C^* -algebra generated by projections $D_\mu = S_\mu S_\mu^*$. It is an abelian C^* -algebra that commutes with A_X . Define

$$\phi(D) = \sum_{i=1}^n S_i D S_i^*, \quad D \in \mathcal{D}_X.$$

We have the following from [18].

Proposition 2.4.2: *There exists an isomorphism $\omega : \mathcal{D}_X \rightarrow C(X^+)$, the continuous functions on X^+ such that $\omega\phi = \sigma^*\omega$, and $\omega(D_\mu) = \chi_\mu$ where χ_μ is the characteristic function of the cylinder set of μ , and $\sigma^* : C(X^+) \rightarrow C(X^+) : f \mapsto f \circ \sigma$.*

Definition 2.4.3: We say a shift X satisfies condition (I_X) if for every $l, k \in \mathbb{N}$ with $l \geq k$ there exists a projection q_k^l in \mathcal{D}_X such that:

1. $q_k^l a \neq 0$ for any nonzero $a \in A_l$,
2. $q_k^l \phi^m(q_k^l) = 0$, $1 \leq m \leq k$.

Remark 2.4.4: Since $\omega(S_\mu S_\mu^*)$ is the characteristic function on U_μ , the cylinder set of μ , one gets from proposition 2.4.2 and by induction on n that

$$(2.4) \quad \omega \circ \phi^n(S_\mu S_\mu^*) = 1_{\sigma^{-n}(U_\mu)}.$$

Here $1_{\sigma^{-n}(U_\mu)}$ is the characteristic function on the set $\sigma^{-n}(U_\mu)$. It is straightforward to show that the function $1_{\sigma^{-n}(U_\mu)}$ is a finite sum of characteristic functions on cylinder sets.

A matrix A satisfies condition (I) of [8] if and only if the edge shift, X , obtained from the matrix A satisfies condition (I_X) (see [18]). Although condition (I_X) does not just involve the subalgebra A_X , we shall see later that for sofic shifts, (I_X) is ultimately determined by the algebra A_X and the Perron-Frobenius operator A (c.f. theorem 3.6.1).

Theorem 2.4.5: [18, Theorem 5.2] *Assumptions as in Theorem 2.4.1. If in addition X satisfies condition (I_X) of definition 2.4.3 then the extension $\tilde{\pi}$ is injective.*

Based on the construction of the Fock Space, the transformation $e_\mu \mapsto z^k e_\mu$, where $\mu \in B^k(X)$, $z \in \mathbb{T}$ yields a unitary representation U_z of $z \in \mathbb{T}$ on \mathfrak{F} . The unitaries satisfy $U_z T_\mu U_z^* = z^k T_\mu$ and leave the compact operators invariant. Thus, there is a gauge action α_z on \mathcal{O}_X satisfying $\alpha_z(S_\mu) = z^k S_\mu$ whenever $\mu \in B^k(X)$. This yields a norm one projection E from \mathcal{O}_X to its fixed point algebra (under α) \mathcal{O}_X^α defined by:

$$E(x) = \int_{z \in \mathbb{T}} \alpha_z(x) dz$$

for $x \in \mathcal{O}_X$.

This fixed point algebra \mathcal{O}_X^α is precisely \mathcal{F}_X^∞ [18, Proposition 3.11]. The expectation E is faithful on the positive elements. With this gauge action, we can show uniqueness of \mathcal{O}_X similar to uniqueness of \mathcal{AO}_A defined in [2] for Cuntz-Krieger algebras.

Theorem 2.4.6: *Let \mathcal{A} be a unital C^* -algebra. Suppose that there is a unital $*$ -homomorphism π from A_X to \mathcal{A} , and there are n partial isometries $s_i \in \mathcal{A}$, $1 \leq i \leq n$ that generate \mathcal{A} and satisfy the following relations:*

1. $\sum_{j=1}^n s_j s_j^* = 1$, $s_\mu^* s_\mu s_\nu = s_\nu s_{\mu\nu}^* s_{\mu\nu}$, $\mu, \nu \in B(X)$,
2. $s_\mu^* s_\mu = \pi(S_\mu^* S_\mu)$, $\mu \in B(X)$,

where $s_\mu = s_{\mu_k} \cdots s_{\mu_1}$, $\mu = (\mu_k \cdots \mu_1) \in B(X)$. Suppose further there exists a gauge action $\beta_z : \mathbb{T} \rightarrow \text{Aut}(\mathcal{A})$ satisfying $\beta_z(s_i) = z s_i$ for all $z \in \mathbb{T}$, and $\beta_z(s_\mu^* s_\mu) = s_\mu^* s_\mu$. Then there exists an isomorphism $\tilde{\pi}$ from \mathcal{O}_X to \mathcal{A} such that $\tilde{\pi}(S_i) = s_i$ for $1 \leq i \leq n$, and its restriction to A_X coincides with π .

PROOF. Theorem 2.4.1 certainly shows us that $\tilde{\pi}$ exists and is surjective. Let $\rho : \mathcal{A} \rightarrow B(\mathcal{H})$ be a faithful representation of \mathcal{A} on a separable, infinite dimensional Hilbert space. Denote by T_i the partial isometry $\rho(s_i)$. Then $\tilde{\rho} = \rho \circ \tilde{\pi}$ is a representation of \mathcal{O}_X on $B(\mathcal{H})$ satisfying $\tilde{\rho}(S_i) = T_i$.

It suffices to show that $\tilde{\rho}$ is injective, hence $\tilde{\pi}$ must also be injective as ρ is. We can do this by showing $\tilde{\rho}$ is faithful on

$$\mathcal{O}_X^\alpha = \mathcal{F}_X^\infty = \overline{\bigcup_{l \geq k} \mathcal{F}_k^l}.$$

Using [1, Lemma 1.3] we have

$$\ker(\tilde{\rho}|_{\mathcal{F}_X^\infty}) = \overline{\bigcup_{l \geq k} \mathcal{F}_k^l \cap \ker(\tilde{\rho}|_{\mathcal{F}_k^\infty})}.$$

So it suffices to show that $\bar{\rho}$ is faithful on each \mathcal{F}_k^l for $l \geq k$. We know from [18] that \mathcal{F}_k^l is generated by the elements $S_\mu P_i S_\nu$ with $|\nu| = |\mu| = k$ and $P_i \in A_l$ is a minimal projection. Suppose that $\bar{\rho}(S_\mu P_i S_\nu^*) = 0$, then certainly

$$(2.5) \quad \bar{\rho}(S_\mu^*[S_\mu P_i S_\nu^*]S_\nu) = 0.$$

But as $S_\mu^*[S_\mu P_i S_\nu^*]S_\nu \in A_X$, and $\bar{\rho}$ is faithful on A_X (because $\bar{\pi}$ is faithful on A_X and ρ is faithful), we must have $\bar{\rho}(S_\mu P_i S_\nu^*) \neq 0$. Thus equation (2.5) is a contradiction. Thus $\bar{\rho}$ is faithful on each \mathcal{F}_k^l , hence on \mathcal{F}_X^∞ .

Furthermore, $\rho \circ \beta_z \circ \rho^{-1} \circ \bar{\rho}(S_I) = zT_I$ for every I in the symbol set Σ . We also must have $\rho \circ \beta_z \circ \rho^{-1} \circ \bar{\rho}(S_\mu^* S_\mu) = T_\mu^* T_\mu$ for every $\mu \in B(X)$. This means that $\rho \circ \beta_z \circ \rho^{-1} \circ \bar{\rho} = \pi \circ \alpha_z$ on \mathcal{O}_X . For any non-zero $x \in \mathcal{O}_X$ we get

$$\begin{aligned} 0 \neq \|\pi(\int_{\mathbb{T}} \alpha_z(x) dt)\| &= \|\int_{\mathbb{T}} \pi \circ \alpha_z(x) dt\| \\ &\leq \int_{\mathbb{T}} \|\pi \circ \alpha_z(x)\| dt \\ &= \int_{\mathbb{T}} \|\rho \circ \beta_z \circ \rho^{-1} \circ \bar{\rho}(x)\| dt \\ &= \int_{\mathbb{T}} \|\bar{\rho}(x)\| dt = \|\bar{\rho}(x)\|. \end{aligned}$$

So by using, for example [5, Lemma 2.2], we get that $\bar{\rho}$ is faithful on \mathcal{O}_X . Hence $\bar{\pi}$ must also be faithful, hence $\bar{\pi}$ is an isomorphism. \square

The above proof is an adaptation of the universal algebra \mathcal{AO}_A shown in [2]. Uniqueness of generators due to condition (I_X) is, in many cases, too strong a requirement; many of the sofic shifts will not have this uniqueness property (c.f. sections 7.1, 7.2). However, because of the natural gauge action on \mathcal{O}_X , we can still obtain uniqueness of the generators of \mathcal{O}_X .

Remark 2.4.7: When we speak of \mathcal{O}_X being universal, we mean in the sense of theorem 2.4.6. However, if X satisfies condition (I_X) , theorem 2.4.6 degenerates to theorem 2.4.5, which is a stronger uniqueness theorem.

2.5. The Failure of the Groupoid Constructions

Before we continue with the structure of \mathcal{O}_X , we will mention why the groupoid construction done in [14] does not work for sofic X , and thus cannot work for labeled graphs. There are many ways of stating this, mainly because there are many equivalent ways of expressing the same equivalence relation. We shall follow [3, 23] (see also [14, 22] for other views).

In order for the groupoid model to work, the one-sided shift map $\sigma : X^+ \rightarrow X^+$ must be a local homeomorphism; we shall show this is equivalent to saying that X^+ is a shift of finite type. For notational convenience, if $\pi : X \rightarrow Y$ is a finite to one map on shift spaces X, Y , we denote by $\#\pi^{-1}(y)$, $y \in Y$, the number of elements in $\pi^{-1}(y)$. If $y = \mu\mu\mu\dots$ is a periodic point in Y , then we denote by μ^n , $n \in \mathbb{N}$ the word in Y obtained by repeating μ n -times.

We need two technical lemmas; the first of which will also be important later.

Lemma 2.5.1: [13, Proposition 4.3] *If X is a two-sided purely sofic shift with Krieger cover $\pi : \Sigma_R \rightarrow X$, then there exists a periodic point $x \in X$ with $\#\pi^{-1}(x) > 1$.*

Remark 2.5.2: If the periodic point $x \in X$ obtained in lemma 2.5.1 is of the form $x = \dots\mu\mu\mu\dots$, then it is necessary that any word in the obtained by repeating μ any number of times must be non-magic. Thus, the pre-image of any such word will have cardinality larger than 1. Furthermore, if $y = \mu\mu\dots \in X^+$ then certainly y has more than one preimage in Σ_R^+ .

Lemma 2.5.3: *Let X be an irreducible purely sofic shift with right Krieger cover $\pi : \Sigma_R \rightarrow X$. Suppose that $x = \mu\mu\dots \in X$ with μ non-periodic, $\#\pi^{-1}(x) > 1$, and μ^k a non-magic word (remark 2.5.2) for every $k \in \mathbb{N}$. Then there exists magic words $\nu, \omega \in B(X)$ and $N_0 \in \mathbb{N}$ with $\nu\mu^n \in B(X)$ and $\mu^n\omega \in B(X)$, but $\nu\mu^n\omega \notin B(X)$ for every $n > N_0$.*

PROOF. Let $r(\mu)$ be the range of μ in the right Krieger cover graph from section 1.6 note that

$$r(\mu) \supseteq r(\mu\mu) \supseteq r(\mu\mu\mu) \supseteq \dots$$

As π is finite to one, there must exist an $N_0 \in \mathbb{N}$ with $r(\mu^n) = r(\mu^{n+1})$ for $n \geq N_0$. Suppose $r(\mu^n) = \{F_X(\mu_i)\}_{i=1}^k$ for some finite collection $k > 1$ of follower sets of magic words μ_i . As the right Krieger cover is follower separated we will assume without loss of generality that there exists a magic word $\omega \in B(F_X(\mu_1))$, $\omega \notin B(F_X(\mu_2))$.

By irreducibility, there must be a magic word ν with $F_X(\nu\mu^n) = F_X(\mu_2)$ (we can obtain such ν by “backtracking” on the Krieger cover graph of X from $F_X(\mu_2)$). The right resolving property ensures $F_X(\nu\mu^n) = F_X(\mu_2)$. Then both $\mu^n\omega$ and $\nu\mu^n \in B(X)$ by construction. However $\nu\mu^n\omega \notin B(X)$ because $\omega \notin B(F_X(\mu_2))$. \square

Theorem 2.5.4: *A one sided sofic shift, X , is a SFT if and only if the shift map $\sigma : X \rightarrow X$ is a local homeomorphism.*

THE PROOF OF THEOREM 2.5.4. The “if” will be proven by contrapositive. Suppose X is a purely sofic shift and $\pi : \Sigma_R \rightarrow X$ be the right Krieger cover. We choose a periodic point $x = \mu\mu\dots \in X$, $N_0 \in \mathbb{N}$, $\nu, \omega \in B(X)$ as in lemma 2.5.3. Let $z = \nu\mu\mu\dots \in X$.

If σ is a local homeomorphism, then so is $\sigma^{|\nu|}$ where $|\nu|$ denotes the length of ν . Suppose there exist open neighbourhoods U, V of z and $\sigma^{|\nu|}(z)$ respectively with

$\sigma^{|\nu|} : U \rightarrow V$ a homeomorphism. There must exist a cylinder set of z based at z_1 , such that $U_z^1 \subset U$ and $\sigma(U_z^1) \subset V$.

By choosing a small enough cylinder set, we can assume U_z^1 is a cylinder set of the form $\nu\mu^n$ for $n > N_0$. Thus, $W = \sigma^{|\nu|}(U_z^1)$ must be a cylinder set beginning with μ^n . Note that any $y \in X$ beginning with $\mu^n\omega$ will be in W , but y will have no preimage in U_z by lemma 2.5.3. This is a contradiction, so σ is not a local homeomorphism.

To show the “only if” part, recall that if X is a SFT, there exists a finite collection of forbidden words for X . Suppose the largest such word has length $N_0 \in \mathbb{N}$. Let $x \in X$ and U_x be a cylinder set of X specifying at least $N_0 + 2$ symbols; then $\sigma(U_x) = W_y$ where W_y is the cylinder set obtained from U_x by deleting the first symbol. Equality comes from the fact that both cylinder sets dictate at least N_0 symbols. One can then check σ is a homeomorphism on these neighbourhoods. This completes the proof. \square

Example 2.5.5: To illustrate theorem 2.5.4, let X be the one-sided even shift on the symbols $\{1, 2\}$. Thus between any two 1’s is an even number of 2’s (or zero 2’s). One can check directly that $x = 122222\dots \in X$, yet σ cannot be a local homeomorphism near x . This is because any cylinder set of x is of the form 12^n for some $n \in \mathbb{N}$; any $y \in X$ can be concatenated to 2^n (forming $2^n y$). However, depending on whether n is even or odd, y ’s beginning with an even or an odd number of 2’s respectively can be concatenated on 12^n . So no cylinder set of x maps onto a cylinder set of $\sigma(x)$. It is this technique that is generalized to prove lemma 2.5.3

Remark 2.5.6: For reducible sofics, theorem 2.5.4 can be used by breaking the sofic shift into its irreducible components and applying theorem 2.5.4 to each of

these components. For a reducible sofic shift, the restriction of σ to at least one of these components will not be a local homeomorphism.

CHAPTER 3

 \mathcal{O}_X and Graph C^* -algebras

In this chapter, we introduce the notion of a C^* -algebra of a graph constructed mainly by A. Kumjian, D. Pask, I. Raeburn, and J. Renault [14, 15]. Based on the fact the shifts of finite type are intimately related to finite directed graphs, they exploited this relationship to define a C^* -algebra of a (possibly infinite) graph. Rather than using the Fock space view of the Cuntz-Krieger algebra, they used a groupoid view; managing to show that the groupoid was built from the space of all right infinite paths on the graph (see [3] for related work). We will not look at the details of this construction, but instead will state the main theorems needed for our purposes.

We already know how sofic shifts relate to directed graphs from chapter 1. This chapter's aim is to prove that every \mathcal{O}_X is equal to $C^*(E_A)$, where E_A is the graph obtained from Perron-Frobenius operator A . This allows us to exploit the latest structure theorems for $C^*(E_A)$ and apply them to \mathcal{O}_X .

We will begin this chapter by looking at the construction of a C^* -algebra of a directed graph.

3.1. The Universal C^* -algebra of a Graph

We quickly outline the main theorems we need for the construction of the C^* -algebra of a directed graph. Most of this work can be found in [4, 14, 15].

For a directed graph $E = (E^0, E^1)$, one defines the C^* -algebra of the graph by defining some generating partial isometries on $B(\mathcal{H})$ (\mathcal{H} a Hilbert space) that

satisfy certain relations based on the structure of the graph. The power in the work of [14, 15], was that they could define such C*-algebras for infinite graphs, and/or graphs with sinks, thus extending the definition of a Cuntz-Krieger algebra. The only condition needed on E is that is that E is *row finite*, that is, every vertex only emits a finite number of edges; work for non row finite graphs can be found in [9].

Definition 3.1.1: (see [15]) Let $E = (E^0, E^1)$ be a row finite directed graph, then a Cuntz-Krieger E -family consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections, and a set $\{S_e : e \in E^1\}$ of partial isometries satisfying

$$(3.1) \quad \begin{aligned} S_e^* S_e &= P_{r(e)} \text{ for } e \in E^1 \\ P_v &= \sum_{\{e:s(e)=v\}} S_e S_e^* \text{ for each } v \in s(E^1). \end{aligned}$$

A similar definition is given for the matrices [14] (either the vertex or edge matrix) of the graph.

Definition 3.1.2: (see [2, 8, 14, 15]) If A is an $n \times n$ matrix with entries in $\{0, 1\}$, and Q_i a collection of n partial isometries acting on a Hilbert Space \mathcal{H} satisfying

$$(3.2) \quad Q_i^* Q_i = \sum_{j=1}^n A(i, j) Q_j Q_j^* \quad \sum_{i=1}^n Q_i Q_i^* = 1,$$

and $Q_i^* Q_j = 0$ for $i \neq j$, then the collection Q_i is called a Cuntz-Krieger A -family

In [2], the C*-algebra \mathcal{AO}_A is defined to be the universal C*-algebra generated by a Cuntz-Krieger A -family. The universal property of \mathcal{AO}_A [2, Theorem 2.3] is similar to the universal property of \mathcal{O}_X defined in theorem 2.4.6. This algebra was essentially a generalization of the universal algebra of a Cuntz-Krieger A -family, \mathcal{O}_A , defined in [8]. However, Cuntz and Krieger required a stronger condition on the matrix A in order to get their universal property: The condition on A is referred to as condition (I) (see [8]). We will study this further later.

Remark 3.1.3: If X and Y are SFT's, it should be noted there are always matrices associated with X and Y (see theorem 1.3.1, example 1.2.1). Denote these matrices as A and B respectively (see the proof of [8, Proposition 2.17]). Moreover, if the one-sided subshifts X^+ and Y^+ are topologically conjugate, then \mathcal{AO}_A is isomorphic to \mathcal{AO}_B . This proof is the same as the proof of [8, Proposition 2.17], except instead of invoking [8, Theorem 2.13], use [2, Theorem 2.3] (or theorem 2.4.6).

Remark 3.1.4: If A has non-negative integer entries, then one defines \mathcal{O}_A to be the C^* -algebra generated by the Cuntz-Krieger A_E family, where A_E is the corresponding edge matrix of the graph of A (see chapter 1 or [8, Remark 2.16]). If A was initially a 0-1 matrix, and one forms A_E , then A_E may not equal A . However, because A was initially a 0-1 matrix, the one-sided shift spaces obtained from A and E_A (example 1.2.1) will be conjugate [16, Section 2.3]. In this case, remark 3.1.3 tells us $\mathcal{AO}_A \cong \mathcal{AO}_{A_E}$.

Theorem 3.1.5: (see [15, Theorem 1.2]) *Let E be a directed graph. Then there is a C^* -algebra B generated by a Cuntz-Krieger E -family $\{s_e, p_v\}$ of non-zero elements such that, for every Cuntz-Krieger E -family $\{S_e, P_v\}$ of partial isometries on \mathcal{H} , there is a representation π of B on \mathcal{H} such that $\pi(s_e) = S_e$, and $\pi(p_v) = P_v$, for all $v \in E^0$ and $e \in E^1$.*

Remark 3.1.6: The C^* -algebra B in theorem 3.1.5 is unique up to isomorphism, and we will denote it as $C^*(E)$. When E has no sinks, the projections P_v become redundant, and $C^*(E)$ is a Cuntz-Krieger A_E -family for the edge matrix A_E . Furthermore, if E is finite and has no sinks, then $C^*(E)$ is isomorphic to \mathcal{AO}_{A_E} of [2] ([14, Proposition 4.1, Corollary 4.8] and [2, Theorem 2.1]); if A_E satisfies condition (I) of [8], then $C^*(E) \cong \mathcal{O}_{A_E}$.

3.2. Uniqueness and Pure Infiniteness of $C^*(E)$

Although theorem 3.1.5 shows the existence of $C^*(E)$, there is no guarantee that the representation π in theorem 3.1.5 is an injection. To ensure π is an injection, there is a generalized form of Cuntz-Krieger's condition (I) (based on the structure of E) which allows $C^*(E)$ to be generated independent of the choice of partial isometries satisfying (3.1) or (3.2) (so π is an injection). Condition (I) can be interpreted in terms of the structure of the graph, and will be more useful for us.

There is a possibility that $C^*(E)$ can be generated by different sets of partial isometries $\{S_e\}_{e \in E^1}$ and $\{T_e\}_{e \in E^1}$ such that the map $S_e \mapsto T_e$ is not an isomorphism. For instance, suppose one has a graph whose edge shift matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $C^*(E)$ is also generated by a Cuntz-Krieger A -family. Let

$$\begin{aligned} S_1 &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & S_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ T_1 &= S_1 & T_2 &= -S_2. \end{aligned}$$

Then both $\{S_1, S_2\}$ and $\{T_1, T_2\}$ form Cuntz-Krieger A -families. If a canonical isomorphism between these two families existed, it would have to send S_i to T_i for $i = 1, 2$. However $T_2 T_1 = -T_1^* T_1 = -S_1^* S_1 \neq S_2 S_1$, so no such isomorphism exists. Condition (I) on a matrix ensures a canonical isomorphism between two Cuntz-Krieger A -families without having to appeal to the gauge action of the unit circle \mathbb{T} (which is what is done to achieve an isomorphism in [2, Theorem 2.3]).

If $E = (E^0, E^1)$ is a directed graph, then we say a walk α is a *loop* based at $v \in E^0$ if $s(\alpha) = r(\alpha) = v$. We say a loop $\alpha = \alpha_1 \dots \alpha_n$ has an *exit* if there exists a path $\beta = \beta_1 \dots \beta_m$, with $\beta \neq \alpha$, $s(\beta) = s(\alpha)$, and $\beta_i \neq \alpha_i$ for some i .

Definition 3.2.1: We say that E satisfies condition (L) if and only if every loop in E has an exit.

Remark 3.2.2: (see [15, Lemma 3.3]) Let E be a directed graph with edge matrix A_E . If E satisfies (L) then A_E satisfies (I) . If E is a finite graph, (L) and (I) are equivalent.

Condition (L) ensures the existence of a canonical isomorphism between any two Cuntz-Krieger E -families without having to appeal to the gauge action of \mathbb{T} , as condition (I) does for Cuntz-Krieger algebras. As remark 3.2.2 states, (L) is a generalization of (I) .

Theorem 3.2.3: ([15, Theorem 3.7], [4]) *Let E be a row finite directed graph which satisfies condition (L) . Suppose B is a C^* -algebra generated by a Cuntz-Krieger E -family, $\{S_e, P_v\}$ with all S_e non-zero. Then there is an isomorphism π from $C^*(E)$ onto B such that $\pi(s_e) = S_e$.*

Finally, we state the theorem for $C^*(E)$ to be purely infinite.

Theorem 3.2.4: (see [4]) *Let E be a row finite directed graph which has no sinks. Then $C^*(E)$ is purely infinite if and only if E satisfies condition (L) and every vertex connects to a loop.*

Remark 3.2.5: If a matrix A has non-zero integer entries. We say that A satisfies condition (I) if and only if the corresponding edge shift matrix A_E (remark 3.1.4) satisfies condition (I) .

3.3. The Connecting Maps as Matrices

From this point on, we let X be a shift space and \mathcal{O}_X the C^* -algebra constructed in chapter 2. Define A_X and \mathcal{F}_X^∞ as in chapter 2. Recall from chapter 2 that there are two natural embeddings used to construct \mathcal{F}_X^∞ .

1. If $l > k$ then $\mathcal{F}_k^l \hookrightarrow \mathcal{F}_k^{l+1}$ via $A_l \hookrightarrow A_{l+1}$.
2. For $k \in \mathbb{N}$, $\mathcal{F}_k^\infty \hookrightarrow \mathcal{F}_{k+1}^\infty$ through the identification

$$S_\mu a S_\nu^* = \sum_{I \in \Sigma} S_\mu S_I S_I^* a S_I S_I^* S_\nu^*.$$

When the identifications above are restricted to the algebra A_X , we shall refer to the first map as the *inclusion operator*, and the second map the *Perron-Frobenius operator*. The Perron-Frobenius operator sends $a \in A_X$ to $\sum S_I^* a S_I$. First, we define the inclusion operator on the basic building blocks of A_X .

For each $l \in \mathbb{N}$ let $m = \dim(A_l)$ and $n = \dim(A_{l+1})$. Let P_i^l and P_j^{l+1} , $1 \leq i \leq m$, $1 \leq j \leq n$ denote the minimal projections for A_l and A_{l+1} respectively, and regard P_i^l and P_j^{l+1} as elements in A_X . Define an $n \times m$ matrix $I_{l,l+1}$ whose i th row and j th column are as follows:

$$(3.3) \quad I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } P_i^{l+1} \leq P_j^l \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check

$$P_i^l = \sum_{j=1}^m I_{l,l+1}(j, i) P_j^{l+1}$$

hence $I_{l,l+1}$ is the matrix that makes the following diagram commute

$$\begin{array}{ccc} A_l & \xrightarrow{i} & A_{l+1} \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{C}^m & \xrightarrow{I_{l,l+1}} & \mathbb{C}^n \end{array}$$

Matsumoto has defined a similar matrix in [20, Section 3], and it should be noted that the matrix defined in (3.3) is the transpose of Matsumoto's.

To define the Perron-Frobenius matrix takes a bit more work. First a wee lemma.

Lemma 3.3.1: *Suppose $P_s \neq P_t$, are minimal projections in A_l and both projections are non zero. Then for any $\mu \in B(X)$, $S_\mu^* P_s S_\mu$ is perpendicular to $S_\mu^* P_t S_\mu$.*

PROOF. Recall that $S_\mu S_\mu^*$ commutes with the elements in A_X , thus

$$S_\mu^* P_s S_\mu \cdot S_\mu^* P_t S_\mu = S_\mu^* P_s P_t S_\mu S_\mu^* S_\mu = 0$$

because P_s is perpendicular to P_t . □

Because of lemma 3.3.1, for each $I \in \Sigma$ we can define an $m \times n$ matrix $A_{l,l+1}^{(I)}$ whose (i, j) th entry is:

$$(3.4) \quad A_{l,l+1}^{(I)}(i, j) = \begin{cases} 1 & \text{if } S_I^* P_i^I S_I \geq P_j^{I+1} \\ 0 & \text{otherwise.} \end{cases}$$

Thus we define the $m \times n$ matrix

$$(3.5) \quad A_{l,l+1} = \sum_{I \in \Sigma} A_{l,l+1}^{(I)}.$$

Since $\dim(A_l) = m$ and $\dim(A_{l+1}) = n$, and both algebras are commutative and unital, we must have $A_l \cong \mathbb{C}^m$ and $A_{l+1} \cong \mathbb{C}^n$ (using as a basis $\{P_i^I\}_{i=1}^m$ and $\{P_j^{I+1}\}_{j=1}^n$ respectively). If we denote by $\lambda_{l,l+1} : A_l \rightarrow A_{l+1}$ as the Perron-Frobenius operator (restricted to A_l), then

$$\begin{array}{ccc} A_l & \xrightarrow{\lambda_{l,l+1}} & A_{l+1} \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{C}^m & \xrightarrow{A_{l,l+1}} & \mathbb{C}^n \end{array}$$

commutes. When doing calculations, it is best to keep track of the symbol I ; at most times we will use the $A_{l,l+1}^{(I)}$ when dealing with $A_{l,l+1}$

From this point on, let X be a sofic shift. We have A_X finite dimensional by corollary 2.3.2. Because of this, the inductive limit of A_X must stabilise, (so, for some $l_0 \in \mathbb{N}$, $A_l = A_{l+1}$ for every $l \geq l_0 \in \mathbb{N}$), and the inclusion operators become identity

matrices. The Perron-Frobenius operators, $\lambda_{l,l+1}$, will also stabilise whenever $l \geq l_0$, and each $A_{l,l+1}$ can be regarded as an $M \times M$ matrix, where $M = \dim(A_{l_0})$. Thus, λ_X can be regarded as an $M \times M$ matrix also. From here on, fix a sofic shift X , and assume $l \geq l_0$ for this X , and $\dim(A_l) = M$.

Let A be the $M \times M$ matrix representing the Perron-Frobenius operator for A_X using the orthonormal basis $\{P_1, \dots, P_M\}$ where P_i is a characteristic function on a past equivalence class. Recall the map χ defined in equation (2.3); we have $x \in \chi(P_i)$ if and only if $P_i(x) = 1$, for each minimal projection $P_i \in A_X$ (when P_i is viewed as a characteristic function).

Lemma 3.3.2: *Let $P_i \in A_X$.*

1. $S_I P_i \in \mathcal{O}_X$ is non zero if and only if for every $x \in \chi(P_i)$, $Ix \in X^+$.
2. $P_i S_I \in \mathcal{O}_X$ is non zero if and only if $Ix \in \chi(P_i)$ for some $x \in X$.

PROOF. We prove (1) as the other is similar.

$$\begin{aligned}
 S_I P_i \neq 0 &\iff \\
 S_I S_I^* S_I P_i \neq 0 &\iff (S_I \text{ is a partial isometry}) \\
 S_I P_i S_I^* S_I \neq 0 &\iff (\text{because } A_X \text{ is commutative}).
 \end{aligned}$$

Note that $P_i S_I^* S_I \neq 0$ if and only if $S_I^* S_I \geq P_i$; using part (2) of proposition 2.3.1 this is equivalent to saying that for every $x \in \chi(P_i)$, $Ix \in X^+$. \square

We give a condition when $A^{(I)}(i, j)$ is non-zero in terms of the one sided space X^+ .

Proposition 3.3.3: *The following are equivalent:*

1. $S_I^* P_i S_I \geq P_j$.
2. For every $x \in \chi(P_j)$, $Ix \in \chi(P_i)$.

PROOF. Follows from the proof of [19, Lemma 4.9], and Proposition 2.3.1. \square

The next proposition can be obtained using induction on the length of the word μ .

The case for $n = 1$ is proposition 3.3.3

Proposition 3.3.4: *Let $\mu \in B(X)$. The following are equivalent:*

1. $S_\mu^* P_i S_\mu \geq P_j$,
2. for every $x \in \chi(P_j)$, the word $\mu x \in \chi(P_i)$.

3.4. That \mathcal{O}_X Equals $C^*(E_A)$

For this section, suppose X is a sofic shift with $\dim(A_X) = M$. Let A be the $M \times M$ matrix representing the Perron-Frobenius operator for A_X using the orthonormal basis $\{P_1, \dots, P_M\}$ where P_i is a characteristic function on a past equivalence class. For each symbol $I \in \Sigma$, and $1 \leq i, j \leq M$ define the following partial isometry:

$$(3.6) \quad Q_{i,j}^{(I)} = P_i S_I P_j.$$

One can check that $Q_{i,j}^{(I)}$ is a partial isometry. Then:

Lemma 3.4.1: *The partial isometry $Q_{i,j}^{(I)}$ defined in equation (3.6) is non-zero if and only if $S_I^* P_i S_I \geq P_j$.*

Because each $A^{(I)}$ is a zero-one matrix we get:

Corollary 3.4.2: *The partial isometry $Q_{i,j}^{(I)}$ defined in equation (3.6) is non-zero if and only if $A^{(I)}(i, j) = 1$.*

THE PROOF OF LEMMA 3.4.1. Note that

$$(3.7) \quad P_i S_I P_j \neq 0 \iff$$

$$(3.8) \quad P_i S_I S_I^* S_I P_j \neq 0 \quad S_I \text{ is an partial isometry}$$

But as $S_I S_I^*$ commutes with A_X , line (3.8) is equivalent to saying

$$\begin{aligned} S_I S_I^* P_i S_I P_j \neq 0 &\iff \\ S_I \left(\sum_{k=1}^N A^{(l)}(i, k) P_k \right) P_j \neq 0 &\iff \\ A^{(l)}(i, j) S_I P_j \neq 0 &\Rightarrow \\ A^{(l)}(i, j) = 1 & \end{aligned}$$

To show the converse, we see whenever $A^{(l)}(i, j) = 1$, $S_I^* P_i S_I \geq P_j$. Multiplying both sides of the inequality on the right by P_j gives us $S_I^* P_i S_I P_j \geq P_j$ which means $P_i S_I P_j \neq 0$. \square

We remind the reader of the relationship between matrices with integer entries and edge shifts. Let $A = (A(i, j))_{1 \leq i, j \leq M}$ be an $M \times M$ matrix representing the Perron-Frobenius operator. Denote by E_A the graph obtained from A by drawing $A(i, j)$ edges from vertex i to vertex j .

Theorem 3.4.3: *Let X be a sofic shift with \mathcal{O}_X the corresponding C^* -algebra. Suppose that $l_0 \in \mathbb{N}$ satisfies $M = \dim(A_X) = \dim(A_l)$ for every $l \geq l_0$, and A is the $M \times M$ matrix representing the Perron-Frobenius operator. Then there exists a Cuntz-Krieger E_A -family of partial isometries in \mathcal{O}_X .*

PROOF. Let $\{P_i\}_{i=1}^M$, be the orthogonal family of minimal projections for A_X . Let E be the graph whose vertices $E^0 = \{P_i\}_{i=1}^M$. The edge set

$$E^1 = \{a_{i,j}^{(l)} : A^{(l)}(i, j) = 1\}.$$

Therefore, we see that there is an edge from P_i to P_j if and only if $P_i S_I P_j \neq 0$ (equivalently, if $A^{(l)}(i, j) = 1$). Let $Q_{i,j}^{(l)} = P_i S_I P_j$ be the corresponding partial

isometry for the edge $a_{i,j}^{(I)}$. Corollary 3.4.2 tells us $Q_{i,j}^{(I)} \neq 0$. Note that $s(a_{i,j}^{(I)}) = P_i$ and $r(a_{i,j}^{(I)}) = P_j$.

We must show the relations in equation (3.1) hold for our partial isometries $Q_{i,j}^{(I)}$. We will show first that, if non-zero, $Q_{i,j}^{(I)*} Q_{i,j}^{(I)} = P_{r(e)} = P_j$ whenever $Q_{i,j}^{(I)} = P_i S_I P_j$. Note that

$$\begin{aligned} Q_{i,j}^{(I)*} Q_{i,j}^{(I)} &= P_j S_I^* P_i P_i S_I P_j \\ &= P_j \left(\sum_{k=1}^M A^{(I)}(i, k) P_k \right) P_j \\ &= P_j \qquad \qquad \qquad \text{by corollary 3.4.2} \end{aligned}$$

The second identity needed,

$$(3.9) \qquad P_i = \sum_{\{e \in E^1 : s(e) = P_i\}} Q_e Q_e^*$$

can be seen as follows:

$$\begin{aligned} P_i &= P_i \left(\sum_{I \in \Sigma} S_I S_I^* \right) P_i && \text{because } \sum S_I S_I^* = 1 \\ &= P_i \left(\sum_{I \in \Sigma} S_I \left(\sum_{k=1}^M P_k \right) S_I^* \right) P_i && \text{because } \sum_{k=1}^M P_k = 1 \\ &= P_i \left(\sum_{I,k} S_I P_k P_k S_I^* \right) P_i && \text{as } P_k \text{ is a projection} \\ &= \sum_{I,k} (P_i S_I P_k) (P_k S_I^* P_i) \\ &= \sum_{I,k} Q_{i,k}^{(I)} Q_{i,k}^{(I)*} \\ &= \sum_{\{k, I : A^{(I)}(i,k) \neq 0\}} Q_{i,k}^{(I)} Q_{i,k}^{(I)*} && \text{by corollary 3.4.2} \\ &= \sum_{\{e = a_{i,j}^{(I)} : s(e) = P_i\}} Q_e Q_e^* \end{aligned}$$

Therefore, (3.9) holds. \square

For each $P_i \in A_X$, there exists a $I \in \Sigma$ with $S_I^* P_i S_I \neq 0$. Thus every vertex $P_i \in E^0$ must have an edge $e \in E^1$ whose source is P_i . So the graph E_A constructed from A cannot have any sinks. Since E_A is a finite graph, $C^*(E_A)$ is unique up to isomorphism ([15, Remark 1.3]).

Theorem 3.4.4: *Let X be a sofic shift. Then $\mathcal{O}_X \cong C^*(E_A)$ where E_A is the graph obtained from the edge matrix of the Perron-Frobenius operator A .*

PROOF. Theorem 3.4.3 certainly allows us to regard $C^*(E_A)$ as a subalgebra of \mathcal{O}_X . It suffices to show that for each $I \in \Sigma$, $S_I \in C^*(E_A)$. This follows immediately because

$$S_I = \sum_{i,j=1}^M P_i S_I P_j,$$

and each individual summand is in $C^*(E_A)$ \square

Because A is not necessarily a 0-1 matrix, we define $\mathcal{A}\mathcal{O}_A = \mathcal{A}\mathcal{O}_{A_E}$, where A_E is the edge shift matrix of the graph E_A (see remark 3.1.4). For the Perron-Frobenius operator A , $\mathcal{A}\mathcal{O}_A = C^*(E_A)$. Thus we have the following corollary.

Corollary 3.4.5: *If X is a sofic shift with Perron-Frobenius operator A , then $\mathcal{O}_X \cong \mathcal{A}\mathcal{O}_{A_E}$.*

Remark 3.4.6: It should be stressed that we have three ways of looking at a minimal projection $P_i \in A_X$ for sofic X .

1. As a minimal projection in a finite dimensional algebra, hence a diagonal matrix unit.
2. As a characteristic function of a past equivalence class.
3. As a vertex in a directed graph.

3.5. The Perron-Frobenius Operator as a Cover for a Sofic Shift

As we know, the Perron-Frobenius operator is an $M \times M$ matrix A with non-negative integer entries, so we can associate an edge shift with this matrix. In this section, we shall show that the graph E_A can be labeled to present the sofic shift X . This presentation is important for much of the theory on the structure of \mathcal{O}_X .

Suppose X is a sofic shift with \mathcal{O}_X its corresponding C^* -algebra. Let A be the $M \times M$ matrix representing the Perron-Frobenius operator for \mathcal{O}_X ; we know from a previous section that

$$A = \sum_{I=1}^N A^{(I)},$$

and each $A^{(I)}$ is a 0-1 matrix by lemma 3.3.1. Let E_A be the edge shift graph obtained from A . As we know from the proof of theorem 3.4.3, there is an edge from P_i to P_j (the minimal projections playing the role of vertices) for every $I \in \Sigma$ that satisfies $P_i S_I P_j \neq 0$. When such an edge exists, label it I .

By corollary 3.4.2, there is an edge labeled I from P_i to P_j if and only if $A^{(I)}(i, j) = 1$. So we can regard the edge shift, Y induced by the E_A having symbol set

$$\{a_{i,j}^{(I)} : A^{(I)}(i, j) = 1\},$$

and the transition from $a_{i,j}^{(I)}$ to $a_{k,l}^{(J)}$ is allowed if and only if $k = j$.

Proposition 3.5.1: *With the above construction, the one-block map $\pi : Y \rightarrow X$ given by $\pi : a_{i,j}^{(I)} \mapsto I$ is a onto cover of the sofic shift X that intertwines the shift map.*

PROOF. It suffices to show for every word $\mu \in B(X)$ there is a word $\xi \in B(Y)$ with $\pi(\xi) = \mu$. Suppose $\mu \in B(X)$; it must be that there exists minimal projections

$P_i, P_j \in A_X$ with

$$S_\mu^* P_i S_\mu \geq P_j$$

by proposition 3.3.4. If $\mu = \mu_1 \dots \mu_n$, we have that

$$S_{\mu_1}^* P_i S_{\mu_1} = \sum_{k \in K_1} P_k$$

where K_1 is a finite set. Furthermore

$$S_{\mu_2 \mu_1}^* P_i S_{\mu_1 \mu_2} = \sum_{k \in K_2} P_k$$

and K_2 is a finite set. We can thus conclude that there exists a $k \in K_2$, and $k' \in K_1$ with

$$S_{\mu_2} P_{k'} S_{\mu_2} \geq P_k,$$

thus there exists $1 \leq s \leq M$, such that $a_{i,k'}^{\mu_1} a_{k',s}^{\mu_2} \in B(Y)$. Clearly, $a_{i,k'}^{\mu_1} a_{k',s}^{\mu_2}$ has image under π equal to $\mu_1 \mu_2$. We continue this way inductively to find a path $\xi = a_{i,k}^{\mu_1} \dots a_{s,j}^{\mu_n}$ whose image under π is μ . □

The SFT Y may or may not be the Krieger cover; we shall investigate this further in subsequent chapters. We shall refer to this presentation of X as the *Perron-Frobenius presentation*. It is this shift Y where most of the algebraic information regarding \mathcal{O}_X lies.

Remark 3.5.2: Recall that $A^{(I)}(i, j) = 1 \iff S_I^* P_i S_I \geq P_j$. Hence $a_{i,j}^{(I)}$ is a symbol in Y if and only if $S_I^* P_i S_I \geq P_j$. This correspondence is important in decoding information about the sofic shift that Y presents.

A one-block map on shift spaces $\pi : Y \rightarrow X$, is said to be left resolving if whenever ab and $a'b$ are words of length two in Y , and $\pi(ab) = \pi(a'b)$ then $a = a'$. One can check that for sofic shifts, a left-resolving one-block map corresponds to a graph with a left resolving labeling.

Proposition 3.5.3: *The map $\pi : Y \rightarrow X$ defined in proposition 3.5.1 is left resolving; thus, the labeled graph (E_A, π) is a left resolving graph.*

PROOF. Suppose $a_{i',j}^{(I)}a_{j,k}^{(J)}$ and $a_{i,j}^{(I)}a_{j,k}^{(J)}$ are words of length two in Y with the same image under π . Then by remark 3.5.2 on the symbols $a_{i',j}^{(I)}$ and $a_{i,j}^{(I)}$ we have

$$S_I^* P_{i'} S_I \geq P_j \qquad S_I^* P_i S_I \geq P_j.$$

By lemma 3.3.1, $P_{i'} = P_i$, so $i = i'$. □

Because of the above and [16, Proposition 8.11] we have the following:

Corollary 3.5.4: *The above map $\pi : Y \rightarrow X$ is finite to one.*

3.6. Uniqueness for \mathcal{O}_X in Terms of $C^*(E)$

With the power of realizing \mathcal{O}_X as a $C^*(E_A)$ for a directed graph E_A , we can apply the theory developed by [4, 14, 15] to get some equivalent definitions for uniqueness of \mathcal{O}_X , and determine when \mathcal{O}_X is purely infinite. In [18, Corollary 7.4], Matsumoto states a condition for \mathcal{O}_X to be purely infinite. However, [18, Corollary 7.4] requires our AF-core to be simple; too strong a condition for many of our examples, as we shall see (cf. section 7.2). We shall find out exactly what the graph E_A is in chapter 5.

Recall condition (I_X) (definition 2.4.3) for a C^* -algebra \mathcal{O}_X . We have the following theorem.

Theorem 3.6.1: *Let X be a sofic shift, with Perron-Frobenius operator A . Then the following conditions are equivalent:*

1. X satisfies condition (I_X) ,

2. For every minimal projection $P \in A_X$ there is a non periodic point $x \in X^+$, (the one-sided sofic shift) with $x \in \chi(P)$,
3. A satisfies condition (I) of [8].

To prove this theorem, we will make use of some lemmas. We remind the reader of the map χ defined in equation (2.3) and the fact that $x \in \chi(P)$ if and only if $P(x) = 1$ when $P \in A_X$ is a minimal projection regarded as a characteristic function.

Lemma 3.6.2: *If a minimal projection P is such that $\chi(P)$ contains only periodic points in X^+ , then there is only a single point $x_0 \in \chi(P)$.*

PROOF. Suppose $\chi(P)$ contains only periodic points, and there are two such points $x = \mu\mu\mu\dots$ and $y = \nu\nu\nu\dots \in X^+$. We can assume the words μ and ν are non-periodic. By proposition 3.3.4 both $S_\mu^*PS_\mu \geq P$ and $S_\nu^*PS_\nu \geq P$. Proposition 3.3.4 once again says that $z = \nu\mu\mu\dots \in \chi(P)$ and z is non-periodic, a contradiction.

□

The following corollary makes use of proposition 3.3.4 again.

Corollary 3.6.3: *If $\chi(P)$ only contains a single periodic point $x = \mu\mu\dots$, then $S_\mu^*PS_\mu = P$.*

Remark 3.6.4: By the above corollary, if P only contains a periodic point $x = \mu\mu\dots$, where $\mu = \mu_1\dots\mu_n$, then $S_{\mu_1}^*PS_{\mu_1} = Q$, and if $y \in \chi(Q)$ then $y = \nu\nu\dots$ where $\nu = \mu_2\mu_3\dots\mu_n\mu_1$. We make use of proposition 3.3.4 once again.

THE PROOF OF THEOREM 3.6.1. (1) \Rightarrow (2) : Suppose (I_X) is satisfied, but (2) does not hold. This means there exists a minimal projection P_0 such that $\chi(P_0) = \{\mu\mu\dots\}$ (lemma 3.6.2). Because condition (I_X) is satisfied, [19, Lemmas 5.1, 5.3] tell us that for every $l \in \mathbb{N}$, and $y \in X^+$, there exists a $z \neq y$ with y l -past

equivalent to z . Choosing l large enough so that $\dim(A_l) = \dim(A_X)$, we must have a $z \in X^+$ that is l -past equivalent to, but not equal to x . However, this means that $z \in \chi(P_0)$, a contradiction. So (2) must hold.

(2) \Rightarrow (1) : The most involved part of the proof; we shall show by the contrapositive. If condition (I_X) is not satisfied, then by negating [19, Lemmas 5.1. 5.3], we get that there exists an $l \in \mathbb{N}$ and an $x \in X^+$ such that for every $y \in X^+$ with $y \neq x$, y is not l -past equivalent to x . Since $l + 1$ -past equivalence refines l past equivalence, we can assume $l \geq l_0$ without loss of generality. This means that there exists a minimal projection $P_i \in A_X$ such that $\chi(P_i)$ only contains $x \in X^+$.

If $x = x_1x_2\dots$, proposition 3.3.4 tells us that $S_{x_1}^*P_iS_{x_1} \geq P_j$, for some P_j . Proposition 3.3.4 and remark 3.6.4 tell us the inequality is an equality, and $\chi(P_j)$ only contains a single point. We can continue this way inductively to conclude that $S_{x_1\dots x_n}^*P_iS_{x_1\dots x_n} = P_k$, and $\chi(P_k)$ contains a single point.

Because there are only finitely many minimal projections, we are forced to conclude there exists a projection P_k that is repeated infinitely often as we continue to conjugate P_i by longer and longer words of x . Since $\chi(P_k)$ only has a single point in it, that point must be periodic. Thus the contrapositive.

(2) \Rightarrow (3) : Once again, we will prove this by the contrapositive. If condition (I) is not satisfied, we must have that the graph obtained from the edge shift of A has loop without an exit. Let $\xi = a_{i,j}^{(I)} \dots a_{i,i}^{(K)}$ be this loop, and let $\mu = \pi(\xi)$, where π is the quotient map defined in proposition 3.5.1. Since $a_{i,j}^{(I)}$ corresponds to $S_j^*P_iS_j \geq P_j$, and ξ has no exit, the inequality is an equality, and one can continue and get $S_\mu^*P_iS_\mu = P_i$. Furthermore, because ξ has no exit, there is no other $\nu \in B(X)$ satisfying $S_\nu^*P_iS_\nu \neq 0$. Thus as a characteristic function, $P_i(x) = 1$ if and only if $x = \mu\mu\dots$, and therefore we have a minimal projection P_i with $\chi(P_i)$ containing only a periodic point.

(3) \Rightarrow (2) : We show the contrapositive. Suppose $\chi(P_{i_0})$ contains only a periodic point $x = \mu\mu\mu\dots$, where $\mu = \mu_1\dots\mu_n$. By lemma 3.6.2, $\{x\} = \chi(P_{i_0})$. Let $\pi : Y \rightarrow X$ be the presentation defined in proposition 3.5.1. By lemma 3.6.2, and remark 3.6.4 there is only one j with $A^{(\mu_1)}(i_0, j_0) = 1$. This means, once we start at vertex P_{i_0} , there is only one edge, $a_{i_0, j_0}^{\mu_1}$, whose source is P_{i_0} . Thus $S_{\mu_1}^* P_{i_0} S_{\mu_1} = P_{j_0}$ and remark 3.6.4 tells us that $\chi(P_{j_0})$ has a single periodic point. Continuing this argument inductively, we are forced to conclude that the only n th edge we can end with once we start at vertex P_{i_0} is of the form $a_{k_0, i_0}^{(\mu_n)}$. Thus we have a loop on E_A , and furthermore, this loop has no exit. The graph E_A must not satisfy condition (L) of [15] and by remark 3.2.2 condition (I) of [8] cannot be satisfied. \square

3.7. Pure Infiniteness of \mathcal{O}_X in Terms of $C^*(E)$

All the work has been done for purely infinite now; its just a question of putting the puzzle pieces together. It is clear that \mathcal{O}_X will be purely infinite precisely when $C^*(E_A)$ is.

Remark 3.7.1: It should be noted that if A is a Perron-Frobenius operator for a sofic shift X , the corresponding graph for A does not have any sinks by the discussion following the proof of theorem 3.4.3. Because E is finite and has no sinks, every vertex must connect to a loop. So if A satisfies condition (I) also, $C^*(E_A)$ (and thus \mathcal{O}_X) is purely infinite [15, Theorem 3.9].

Combining the theory of the C^* -algebras of graphs with the theory on \mathcal{O}_X developed before, we have the following theorem. We will repeat the conditions of theorem 3.6.1 for we shall need them.

Theorem 3.7.2: *Let X be a sofic shift with E_A the directed graph corresponding to the Perron-Frobenius operator A . Then the following are equivalent:*

1. \mathcal{O}_X is purely infinite,
2. \mathcal{O}_X satisfies condition (I_X) ,
3. Every minimal projection $P \in A_X$ has a non-periodic point $x \in \chi(P)$.
4. A satisfies condition (I) of [8],
5. E_A satisfies condition (L) ,
6. \mathcal{O}_A is purely infinite (and isomorphic to $C^*(E_A)$).

Remark 3.7.3: That \mathcal{O}_A is isomorphic to $C^*(E_A)$ is in brackets in condition (6) is because if \mathcal{O}_A is purely infinite, A must satisfy condition (I) , since \mathcal{O}_A is defined to be \mathcal{O}_{AE} , and $\mathcal{O}_{AE} \cong C^*(E_A)$ (see remark 3.1.4).

PROOF. The equivalence of (2), (3), and (4) was theorem 3.6.1. (4) \iff (5) is remark 3.2.2; (5) \iff (6) is theorem 3.2.4 and remark 3.7.1; Condition (1) is equivalent to condition (5) by theorem 3.4.4 and remark 3.7.1. \square

CHAPTER 4

The Ideal Structure of \mathcal{O}_X

This chapter focuses on gauge invariant ideals for \mathcal{O}_X when X is sofic. As we know, $\mathcal{O}_X \cong C^*(E_A)$ and $C^*(E_A)$ also admits a gauge action (see [4]). We shall show that the gauge invariant ideals of $C^*(E_A)$ are in bijective correspondence with the gauge invariant ideals of \mathcal{O}_X (in fact, the ideals are equal). We do this by first showing the method for finding the gauge invariant ideals of \mathcal{O}_X for general X as worked out in [19, Sections 6,7]. We then apply the theory for the case when X is sofic to get our result. In our final section, we show that the AF-cores of $C^*(E_A)$ and \mathcal{O}_X are equal.

4.1. The Gauge Invariant Ideals for \mathcal{O}_X

In this section, we can assume X is any shift space with symbol set Σ . We follow [19] and include some sketch proofs for completeness. Recall the gauge action of \mathbb{T} on \mathcal{O}_X from chapter 2 defined as $\alpha_z(S_\mu) = z^{|\mu|} S_\mu$ for $S_\mu \in \mathcal{O}_X$ and $z \in \mathbb{T}$. A (closed two-sided) ideal I of \mathcal{O}_X is said to be *gauge invariant* if $\alpha_z(I) = I$ for every $z \in \mathbb{T}$.

If X is any two sided shift space, define the set

$$\Gamma_X = \{(i, l) : i = 1, \dots, \dim(A_l), l \in \mathbb{N}\}$$

with two kinds of partial orders \succ and \geq on Γ_X as follows:

1. $(i, l) \geq (j, l + 1)$ if $I_{l, l+1}(j, i) \neq 0$.
2. $(i, l) \succ (j, l + 1)$ if $A_{l, l+1}(i, j) \neq 0$

where $I_{l,l+1}$ and $A_{l,l+1}$ are the inclusion and Perron-Frobenius operator respectively (cf. section 2.3).

For (i, l) and (j, m) in Γ_X we say that $(i, l) \geq (j, m)$ if there exists $(i_1, l+1)$, $(i_2, l+2), \dots, (i_n, m-1)$ satisfying

$$(i, l) \geq (i_1, l+1) \geq (i_2, l+2) \geq \dots \geq (i_n, m-1) \geq (j, m);$$

and $(i, l) \succ (j, m)$ is defined similarly.

If $H \subseteq \Gamma_X$ we say that H is hereditary in \geq if $(i, l) \geq (j, m)$ and $(i, l) \in H$, then $(j, m) \in H$. We define hereditary in \succ similarly. If H is hereditary in both orders, we say H is hereditary.

Definition 4.1.1: Let H be a hereditary subset of Γ_X , then H is saturated in \geq if $(i, l) \notin H$ and $(i, l) \geq (j, l+1)$ with $(j, l+1) \in H$, then there exists $(k, l+1) \notin H$ with $(i, l) \geq (k, l+1)$. We define saturated in \succ similarly. If H is saturated in both orders, we say that H is saturated.

Given an hereditary subset $H \subset \Gamma_X$ we define its *saturation* as the smallest saturated subset containing H , and denote it \overline{H} ; i.e.

$$\overline{H} = \bigcap \{G : G \text{ is saturated and } H \subseteq G\},$$

We see that $H \subseteq \overline{H}$ with equality if and only if H is saturated.

Lemma 4.1.2: (see [19, Lemma 6.2]) For a gauge invariant ideal I in \mathcal{O}_X , put

$$H_I = \{(i, l) \in \Gamma_X : P_i^l \in A_X \cap I\}.$$

Then H_I is a saturated hereditary subset of Γ_X .

PROOF. See the proof of [19, Lemma 6.2] to see why H_I is hereditary. We need to show H_I is saturated in the sense of definition 4.1.1.

Suppose $(k, l) \geq (i, l + 1)$ and $(i, l + 1) \in H_I$, $(k, l) \notin H_I$. As $(k, l) \notin H_I$ this means that $P_k^l \notin I$. However, we know that

$$P_k^l = \sum_{s=1}^{\dim(A_l)} I_{l,l+1}(s, k) P_s^{l+1}.$$

And since $P_k^l \notin I$ there must exist an $1 \leq s_0 \leq \dim(A_{l+1})$ with $P_{s_0}^{l+1} \notin I$. This means that $(s_0, l + 1) \notin H_I$. However $(k, l) \geq (s_0, l + 1)$ so H_I is saturated in \geq .

To show H_I is saturated in \succ , suppose that $(k, l) \notin H_I$, and $(i, l + 1) \in H_I$ for every i satisfying $(k, l) \succ (i, l + 1)$. This means that $P_k^l \notin I$, but $\sum_{K \in \Sigma} S_K^* P_k^l S_K$ is in I . Because I is an ideal, and $S_K^* P_k^l S_K$ is just a linear combination of minimal projections in A_{l+1} , we can assume $S_K^* P_k^l S_K \in I$ for each K in the symbol set Σ . However, as I is a two sided ideal this means that $S_K S_K^* P_k^l S_K S_K^*$ is also in I , for each $K \in \Sigma$. Because $S_K S_K^*$ commutes with P_k^l , it forces us to conclude that $S_K S_K^* P_k^l$ is also in I , hence so is the sum

$$P_k^l = \sum_{K \in \Sigma} S_K S_K^* P_k^l \in I,$$

a contradiction. Thus, H_I is saturated in both orders. \square

Lemma 4.1.3: (see [19, Lemma 6.3]) For a saturated hereditary subset H of Γ_X put

$$I_H = \overline{\text{span}}\{S_\mu P_i^l S_\nu^* : (i, l) \in H\}.$$

Then I_H is a gauge invariant ideal of \mathcal{O}_X .

Theorem 4.1.4: (compare [19, Corollary 6.5]) There exists a bijection between the set of saturated hereditary subsets of Γ_X and the gauge invariant ideas of \mathcal{O}_X via the map $I \mapsto H_I$.

Remark 4.1.5: We note that [19, Lemma 6.3] does not ask for a saturated hereditary subset. However, if the subset H is not saturated, there is a chance that a

proper hereditary subset of H may generate the same ideal. We shall see such an example (example 4.1.6).

When the shift X is sofic, the order \geq becomes trivial for any $l \geq l_0$, where $l_0 \in \mathbb{N}$ is the number satisfying $\dim(A_l) = \dim(A_{l_0}) = \dim(A_X)$. To see this, note that if $l \geq l_0$, then $(i, l) \geq (j, l+1)$ if and only if $i = j$ because $I_{l, l+1}$ is an identity matrix whenever $l \geq l_0$. We shall use this fact for the next example, and the next section.

Example 4.1.6: As an example, suppose one has a sofic shift X whose Perron-Frobenius operator looks like

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(we will see in chapter 7 this is the Perron-Frobenius Operator for the even shift).

Now suppose $l_0 = 2$, then the set

$$H = \{(3, 100), (3, 101), (3, 102), \dots\}$$

is hereditary. However, because $P_3^l = P_3^{l_0}$ for every $l \geq l_0$, the set

$$G = \{(3, 2), (3, 2), (3, 4), \dots\}$$

will generate the same gauge invariant ideal as H ; this is the main reason one must saturate our hereditary subsets if one wants a bijective correspondence.

4.2. The Gauge Invariant Ideals for Sofic Shifts

For this section, assume X is a sofic shift with $l_0 \in \mathbb{N}$ satisfying $\dim(A_l) = \dim(A_{l_0})$ for every $l \geq l_0$. As remarked in the previous section, the order \geq becomes “trivial” in the sense that it just becomes an order on the non-zero integers whenever $l \geq l_0$. Thus, we only need to concern ourselves ultimately with \succ . Because this

order ties in with the Perron-Frobenius operator A , we can actually show that the gauge invariant ideals of \mathcal{O}_X are precisely those of $C^*(E_A)$. We shall do this from the view of the graph C^* -algebra.

To find the gauge invariant ideals of $C^*(E)$, we follow [2, 4]. Since we are not concerned with graphs with sinks, we assume that E has no sinks. Let $E = (E^0, E^1)$ be a directed graph. For $v, w \in E^0$ we say that $v \succcurlyeq w$ if there is a path α with $s(\alpha) = v$ and $r(\alpha) = w$.

Definition 4.2.1: A subset H^0 of E^0 is said to be *hereditary* if $v \in H^0$ and $v \succcurlyeq w$ implies $w \in H^0$. A hereditary set H^0 is said to be *saturated* if it contains all vertices v satisfying the condition that every edge e with $s(e) = v$ has $r(e) \in H^0$.

If $H^0 \subset E^0$, then we denote its *saturation* $\overline{H^0}$ as the smallest saturated set containing H . The following is an immediate consequence of [4, Lemma 4.2.1, Remark 4.2.2] or [2, Theorem 3.5] and its proof

Theorem 4.2.2: Let $E = (E^0, E^1)$ be a directed graph (with no sinks). Then there exists a bijective correspondence between saturated hereditary subsets $H^0 \subseteq E^0$, and closed, two-sided, gauge invariant ideals of $C^*(E)$.

If a graph E is irreducible, then there are only two saturated, hereditary subsets, E^0 and \emptyset . If E is not irreducible, then it has irreducible sub-graphs, and roughly speaking, the reducible parts of the graph will form proper, gauge invariant ideals of $C^*(E)$. See [2, 4, 14] for more.

Theorem 4.2.3: Let X be a sofic shift, with Γ_X the set defined in the previous section. Let $E_A = (E^0, E^1)$ be the graph obtained via the edge shift of the Perron-Frobenius operator A . Then there is a bijective correspondence between saturated hereditary subsets of E^0 and saturated hereditary subsets of Γ_X .

PROOF. If H^0 is a saturated hereditary subset of $E^0 = \{P_i\}_{i=1}^M$ let

$$H_0 = \{(i, l) : P_i \in H^0, \text{ and } l \geq l_0\}$$

Clearly, H_0 is hereditary in \geq . To show H_0 is hereditary in \succ , suppose $(i, l) \succ (j, l+1)$, and $(i, l) \in H_0$, then $S_K^* P_i S_K \geq P_j$ for some K in the symbol set Σ . Thus $A^{(K)}(i, j) = 1$, and P_i is connected to P_j . As H^0 is hereditary, $P_j \in H^0$. Hence, $(j, l+1) \in H_0$, so H_0 is hereditary in Γ_X .

It may not be the case that H_0 is saturated. However, we will show that when we form the saturation $\overline{H_0}$, we will only add elements of the form (i, l) where $l < l_0$. Ultimately, this means that different saturated subsets of E^0 will generate different saturated subsets of Γ_X ; this will give a one to one correspondence between the two sets.

Suppose that $\overline{H_0}$ adds elements of the form (i, l) with $l \geq l_0$. Then there exists an $(i, l) \in \overline{H_0}$ with $l \geq l_0$, and $(i, l) \notin H_0$. This means that for every $(j, l+1) \in \Gamma_X$ satisfying $(i, l) \succ (j, l+1)$, we must have $(j, l+1) \in H_0$. However, by reinterpreting this in terms of the graph Perron-Frobenius operator A , it must be the case that every edge $e \in E^1$ with $s(e) = P_i$ must have $r(e) \in H^0$. Because H^0 is saturated, we are forced to conclude that $P_i \in H^0$. Thus $(i, l) \in H_0$, a contradiction. Thus, different saturated subsets of E^0 will generate different saturated subsets of Γ_X . So the correspondence is one to one.

Conversely, if one has a saturated hereditary subset $H_0 \subseteq \Gamma_X$, let

$$H^0 = \{P_i : (i, l) \in H_0, l \geq l_0\}.$$

Clearly, $H^0 \subseteq E^0$; if $P_i \succ P_j$, and $P_i \in H^0$, then there is a walk connecting P_i to P_j by the definition if H_0 being hereditary in \succ , and the bijective correspondance

between minimal projections in A_X and vertices on E_A . Thus $P_j \in H^0$ so H^0 is a hereditary subset of E^0 .

We must show that H^0 is saturated. Suppose it is not. Then there exists a vertex $P_i \notin H^0$ with the property that every edge $e \in E^1$ with $s(e) = P_i$ has $r(e) \in H^0$. We denote by j_1, j_2, \dots, j_n the collection of indices satisfying $A(i, j_k) \neq 0$. As each P_{j_k} is in H^0 by assumption, it must be that $(j_k, l_0 + 1) \in H_0$. Furthermore, if $(i, l_0) \succ (k, l + 1)$ then $(k, l + 1) = (j_s, l_0 + 1)$ for some j_s . Because H_0 is saturated in \succ , we must have $(i, l_0) \in H_0$. This forces $P_i \in H^0$ – a contradiction. \square

Remark 4.2.4: Theorems 4.1.4, 4.2.2, and 4.2.3 gives us a bijective correspondence between gauge invariant ideals of \mathcal{O}_X and gauge invariant ideals of $C^*(E_A)$. It is a routine but rather long calculation to show that the gauge invariant ideals for $C^*(E_A)$ are *equal* to the gauge invariant ideals of \mathcal{O}_X . The result of the next section will reinforce this.

4.3. Equality of the AF-subalgebras

It is well known in the theory of Cuntz-Krieger algebras that $\mathcal{O}_A \otimes \mathcal{K} \cong \mathcal{F}_A \rtimes \mathbb{Z}$. Here \mathcal{F}_A is the AF-algebra generated by

$$\{Q_\gamma Q_\lambda^* : |\gamma| = |\lambda|\}.$$

We warn that γ and λ here are words in the edge shift generated by the matrix A . In terms of $C^*(E)$, one can add that $r(\lambda) = r(\gamma)$ for walks of the same length on the graph (if it is not the case, $Q_\gamma Q_\lambda^* = 0$). Like the case for \mathcal{O}_X , the AF-algebra is the fixed point algebra of a gauge action defined by

$$F(q) = \int_{\mathbb{T}} \beta_z(q) dt, \text{ for } q \in \mathcal{O}_A,$$

and $\beta_z(Q_i) = zQ_i$ for $z \in \mathbb{T}$ (the gauge action of general \mathcal{O}_X is a generalization of this). When the shift is a sofic shift, we can prove that the AF-algebras corresponding to \mathcal{O}_X and $C^*(E_A)$ are the same. We will denote by \mathcal{F}_{E_A} the underlying AF-algebra of $C^*(E_A)$.

Theorem 4.3.1: *If $\mathcal{O}_X = C^*(E_A)$, then $\mathcal{F}_X^\infty = \mathcal{F}_{E_A}$.*

Lemma 4.3.2: *Suppose γ is a walk in E_A (so $\gamma = \gamma_1\gamma_2 \dots \gamma_n$ with $\gamma_i = a_{j_i, k_i}^{(I_i)}$) for each i . Then*

$$Q_\gamma = \prod_{i=1}^n Q_i = P_{j_1} S_\mu P_{k_n}$$

where $\mu = I_1 I_2 \dots I_n$

PROOF. We do this by induction on n , the length of γ . If $n = 1$ the hypothesis is a tautology. Suppose the lemma holds for $m = n - 1$, and let $\gamma = \gamma_1 \dots \gamma_n$ be a path of length n on E_A . Then

$$(4.1) \quad Q_\gamma = \prod_{i=1}^n Q_i$$

$$(4.2) \quad = P_{j_1} S_\mu P_{k_{n-1}} P_{j_n} S_{I_n} P_{k_n} \quad \text{by induction hypothesis.}$$

and $\mu = I_1 \dots I_{n-1}$ is a word in X . For (4.2) to be non-zero, it must be that $k_{n-1} = j_n$. We thus get

$$\begin{aligned} (4.2) &= P_{j_1} S_\mu P_{j_n} S_{I_n} P_{k_n} \\ &= P_{j_1} S_\mu P_{j_n} S_{I_n} S_{I_n}^* S_{I_n} P_{k_n} && S_{I_n} \text{ is an isometry} \\ &= P_{j_1} S_\mu S_{I_n} S_{I_n}^* P_{j_n} S_{I_n} P_{k_n} && S_I S_I^* \text{ commutes with } P_i. \\ &= P_{j_1} S_\mu S_{I_n} P_{k_n} && \text{because } S_{I_n}^* P_{j_n} S_{I_n} \geq P_{k_n} \\ &= P_{j_1} S_{\mu I_n} P_{k_n}. \end{aligned}$$

Hence the statement is true for every n , and the lemma is proved. \square

THE PROOF OF THEOREM 4.3.1. Denote the fixed point algebra of the gauge action of $C^*(E_A)$ as \mathcal{F}_{E_A} . As in [14], \mathcal{F}_{E_A} is generated by $Q_\gamma Q_\lambda^*$, where $r(\gamma) = r(\lambda)$ and $|\gamma| = |\lambda|$. If $\gamma = \gamma_1 \dots \gamma_n$ then $Q_\gamma = Q_{\gamma_1} \dots Q_{\gamma_n}$, and lemma 4.3.2 tells us $Q_\gamma = P_{\gamma_1} S_\mu P_{\gamma_n}$, for some $\mu \in B(X)$ and minimal projections $P_{\gamma_1}, P_{\gamma_n} \in A_X$. Similarly $Q_\lambda^* = P_{\lambda_n} S_\nu^* P_{\lambda_1}$. As $|\gamma| = |\lambda|$ it must be that $|\mu| = |\nu|$.

Suppose $Q_\gamma Q_\lambda^* \neq 0$, then $P_{\gamma_n} = P_{\lambda_n}$ and we have

$$(4.3) \quad Q_\gamma Q_\lambda^* = P_{\gamma_1} S_\mu P_{\gamma_n} S_\nu^* P_{\lambda_1}.$$

Because P_{γ_1} and P_{λ_1} are in A_X , and $S_\mu P_{\gamma_n} S_\nu^*$ is in \mathcal{F}_X^∞ , we have $\mathcal{F}_{E_A} \subseteq \mathcal{F}_X^\infty$.

To show the other inclusion, it suffices to show for the generators of \mathcal{F}_X^∞ of the form $S_\mu P_i S_\nu^*$ with $|\mu| = |\nu|$. As $1 = \sum_j P_j$, we have

$$S_\mu P_i S_\nu^* = \sum_j (P_j S_\mu P_i) (P_i S_\nu^* \sum_j P_j);$$

which yields $\mathcal{F}_X^\infty \subseteq \mathcal{F}_{E_A}$ □

Remark 4.3.3: The work of Katayama *et. al.* on C^* -algebras of the β -shifts [11], has shown that when a β -shift is not sofic, it is isomorphic to the Cuntz-Krieger algebra \mathcal{O}_∞ (see the remark following Proposition 4.7 in [11]). The isomorphism is achieved using the classification theorem for nuclear, purely infinite C^* -algebras of Kirchberg [12] and Phillips [21]. In this case, the fixed point algebras are not isomorphic to each other. However, when a shift is sofic, \mathcal{O}_X 's fixed point algebra is always isomorphic to the fixed point algebra of its corresponding \mathcal{O}_A .

We will continue exploring sofic shifts and the structure of \mathcal{O}_X in subsequent chapters.

CHAPTER 5

The Structure of \mathcal{O}_X for Sofic Shifts

The theory presented in Chapters 3 and 4, especially theorem 3.4.4 tells us that for sofic X , there is a Cuntz-Krieger algebra that generates \mathcal{O}_X . This algebra is none other than $C^*(E_A)$, where E_A is the directed graph of the Perron-Frobenius operator A . We shall study this operator in more detail, as it contains much algebraic information about \mathcal{O}_X , including (as we know from chapters 3 and 4) uniqueness, pure infiniteness, and the gauge invariant ideal structure of the algebra.

From the last section of chapter 3, we know that the Perron-Frobenius operator actually codes a left-resolving cover of the sofic shift X . In fact this cover is the past set cover for X ; the matrix A being the vertex matrix for the graph of the past set cover. As we know from theorem 1.7.1, the past set cover, when viewed as a graph, has a minimal sub-graph that is the left Krieger cover. We shall see that the analogy for C^* -algebras is the existence of a gauge invariant ideal I of \mathcal{O}_X such that \mathcal{O}_X/I is isomorphic to the left Krieger cover graph C^* -algebra.

We can show how the matrix A is related to the right Krieger cover. This gives an alternative equivalence relation on X^+ based on the future of a word $\mu \in B(X)$, rather than as a past equivalence class.

Focusing on the subclass of shifts of almost finite type (AFT), we will see that when X is an irreducible AFT, \mathcal{O}_X always has a non-trivial, gauge invariant ideal. By modding out by this ideal, we can show that \mathcal{O}_X is isomorphic to the Cuntz-Krieger algebra of the minimal cover of the AFT described in [6] (we warn that in

[6], “minimal” means something different than fewest number of vertices, but both the left and right Krieger covers for AFT’s are minimal in the sense of [6]).

For most of this chapter, we focus on the irreducible sofic shifts. However, we mention what can be obtained for reducible sofic in the last section.

5.1. The Presentation Σ_A and the Past Set Cover

We now have the power to show that for any irreducible sofic shift X , there exists an ideal I (possibly equal to zero) such that \mathcal{O}_X/I is isomorphic to the C^* -algebra of the graph of the left Krieger cover [13]. From here on, we will let X be a (possibly reducible) sofic shift, and Σ_R , Σ_L , and Σ_A the right Krieger cover, the left Krieger cover, and Perron-Frobenius cover respectively. Let π_R , π_L , and π_A be the respective quotient maps onto X .

We saw how the past set cover was constructed in section 1.7. As a graph, its vertices are past sets of walks x in X^+ , and there is an edge labeled I from $P_X(x)$ to $P_X(y)$ if and only if $P_X(Iy) = P_X(x)$. Let $P_X(x_i)$, $1 \leq i \leq N$ be such that if $x \in X^+$ then there is an i with $P_X(x) = P_X(x_i)$. Let \mathcal{O}_X be the C^* -algebra of the sofic shift with A_X the AF-algebra generated by $S_\mu^* S_\mu$, $\mu \in B(X)$.

Let $E_A = (E^0, E^1)$ be the graph of the Perron-Frobenius presentation. Let $F = (F^0, F^1)$ be the graph of the past set presentation. We identify E^0 as the collection of all minimal projections $P_i \in A_X$, and we identify F^0 as the set of all past sets $P_X(x_i)$.

Using equation (2.3), we see that when X is sofic The map $\eta^0 : E^0 \rightarrow F^0$ defined by

$$(5.1) \quad \eta^0(P_i) = P(x_i) \text{ where } x_i \in \chi(P_i)$$

is a bijection. This means the number of vertices in the past set graph and the Perron-Frobenius graph is the same. We will show that there is an edge (labeled I) from $P_X(x_i)$ to $P_X(x_j)$, if and only if $A^{(I)}(i, j) = 1$. Thus, the past set cover and the Perron-Frobenius cover are the same.

To show that the past set cover and Σ_A have the same edges, we code the edge set of the past set cover, F^1 as follows: If there is an edge $f \in F^1$ labeled $I \in \Sigma$ from $P_X(x_i)$ to $P_X(x_j)$ then code the edge f as follows

$$(5.2) \quad f \leftrightarrow (I, P_X(x_i), P_X(x_j)).$$

We code the Perron-Frobenius cover obtained from the graph E_A similarly. Since the minimal projections form the vertices of the Perron-Frobenius cover, and there is an edge labeled I from vertex P_i to P_j if and only if $A^{(I)}(i, j) = 1$ (equivalently $a_{i,j}^{(I)}$ is in the symbol set of Σ_A), we code each edge, $a_{i,j}^{(I)} \in E^1$ in the following manner

$$(5.3) \quad a_{i,j}^{(I)} \leftrightarrow (I, P_i, P_j).$$

Using equations (5.2) and (5.3) we have the following lemma.

Lemma 5.1.1:

1. *The edge shift generated by the past set cover has symbol set F^1 defined in equation (5.2). The transition from $(I, P_X(x_i), P_X(x_j))$ to $(J, P_X(x_k), P_X(x_l))$ is allowed if and only if $j = k$.*
2. *The edge shift generated by the Perron-Frobenius cover's symbol set E^1 can be recoded to those symbols defined in equation (5.3). The transition from (I, P_i, P_j) to (J, P_k, P_l) is allowed if and only if $j = k$. This recoding gives a conjugacy between the the edge shift with symbol set $a_{i,j}^{(I)}$ and the edge shift with symbol set defined in equation (5.3), and we will refer to both shifts as Σ_A .*

We denote the edge shift in lemma 5.1.1 part 1 as Σ_L^p . Because each $P_X(x_i)$ can be identified with a minimal projection $P_i \in A_X$ via the map η^0 defined in equation (5.1), we get a well defined one block map $\eta^1 : \Sigma_A \rightarrow \Sigma_L^p$ by defining for each symbol in Σ_L^p :

$$(5.4) \quad \eta^1((I, P_i, P_j)) := (I, \eta^0(P_i), \eta^0(P_j)).$$

An immediate consequence of lemma 5.1.1, equations (5.1), (5.2), and (5.3) is:

Proposition 5.1.2: Σ_A is conjugate via a one block map, η^1 defined in equation (5.4), to the past set cover, Σ_L^p . Thus, if $\mathcal{XO}_X = C^*(E_A)$, then E_A is the past set cover of X .

Definition 5.1.3: If $E = (E^0, E^1)$, and $F = (F^0, F^1)$ are graphs, we say that $\Phi = (\phi^0, \phi^1) : E \rightarrow F$ is a *graph isomorphism* if and only if $\phi^0 : E^0 \rightarrow F^0$ and $\phi^1 : E^1 \rightarrow F^1$ satisfy

1. Both ϕ^0 and ϕ^1 are bijections (one to one and onto).
2. If e is an edge from $v_1 \in E^0$ to $v_2 \in E^0$ then $\phi^1(e)$ is an edge from $\phi^0(v_1)$ to $\phi^0(v_2)$.

Furthermore, if E and F are labeled by the same set, we say that Φ preserves the labels if both $e \in E^1$ and $\phi^1(e) \in F^1$ have the same label.

Remark 5.1.4: An example of a graph isomorphism is $\Phi = (\eta^0, \eta^1)$, defined in equation (5.1) and (5.4). Obviously, η^1 can be regarded as a map between edges. Isomorphic graphs have conjugate edge shifts. See [16, Chapter 2] for more on this. Certainly, if a graph isomorphism between E and F preserves labels, then the sofic shifts generated by the labeling of E and F are the same. Because the graphs of the Perron-Frobenius cover and past cover are isomorphic, and the labels in X are

preserved by η^0 and η^1 . We see that the following digram commutes:

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\eta^1} & \Sigma_L^p \\ \pi_A \downarrow & & \downarrow \pi_L^p \\ X & \xrightarrow{id} & X \end{array}$$

where π_L^p is the quotient map induced by the labeling of the past set cover.

5.2. Reducing Σ_A to the Left Krieger Cover

Obviously Σ_A is reducible precisely when Σ_L^p is. So one must remove the reducible part of A . As we know from theorem 1.7.1, the left Krieger Cover is obtained from the past set cover by restricting to only those walks $x \in X^+$ that are magic walks. Let

$$E_{\bar{A}}^0 := \{P_i : S_\mu^* P_i S_\mu \neq 0 \text{ for some magic word } \mu\},$$

and define $\Sigma_{\bar{A}}$ to be the edge shift whose vertices $P_i \in E_{\bar{A}}^0$. So we reduce the edge shift Σ_A to those symbols that correspond to $a_{i,j}^{(*)}$ (or $(*, P_i, P_j)$) whose P_i and P_j can be conjugated by magic words. The quotient map $\pi_{\bar{A}}$ is obtained by restricting π_A . We will denote by $E_{\bar{A}} = (E_{\bar{A}}^0, E_{\bar{A}}^1)$ the graph obtained from E whose edge set is those $a_{i,j}^{(l)}$ with $P_i, P_j \in E_{\bar{A}}^0$.

Lemma 5.2.1: *The map $\pi_{\bar{A}} : \Sigma_{\bar{A}} \rightarrow X$ is onto.*

PROOF. It suffices to show by every word in $B(X)$ has a pre-image in $B(\Sigma_{\bar{A}})$. Certainly, every magic word has a preimage. If one takes a non-magic word, then one can extend it to a magic word by remark 1.8.4, thus every non-magic word also has a pre-image in $B(\Sigma_{\bar{A}})$. \square

Proposition 5.2.2: *The edge shift $\Sigma_{\bar{A}}$ is conjugate via a one block map to Σ_L where Σ_L is the left Krieger Cover. The one block map is obtained by restricting η^1 to the edge shift generated by the graph $E_{\bar{A}}$.*

PROOF. Let $\tilde{\eta}^0 = \eta^0|_{E_A^0}$ and $\tilde{\eta}^1 = \eta^1|_{E_A^1}$. To prove this theorem, we will show that the tuple $\tilde{\Phi} = (\tilde{\eta}^0, \tilde{\eta}^1)$ is a graph isomorphism between $E_{\tilde{A}}$ and the left Krieger cover. Since Φ (and thus $\tilde{\Phi}$) preserves the labels for X , the result will follow.

We know that E_A is isomorphic to the past set cover, and theorem 1.7.1 tells us that the past set cover has the left Krieger cover as a subgraph. Thus, to prove this theorem, it is enough to show that the restriction of η^0 is a bijection between E_A^0 and the vertices in the left Krieger cover (past sets of magic walks – see section 1.7). That $\tilde{\eta}^1$ is a bijection follows from equation (5.4).

To show $\tilde{\eta}^0$ is a bijection, we first show that if $P_i \in E_A^0$, then there exists an $x \in \chi(P_i)$ with x a magic walk. Suppose $P_i \in E_A^0$, then there exists a magic word μ with

$$S_\mu^* P_i S_\mu \neq 0,$$

Hence $S_\mu^* P_i S_\mu \geq P_j$ for some $P_j \in A_X$. Let $y \in \chi(P_j)$. Proposition 3.3.4 tells us that $x = \mu y \in \chi(P_i)$. Furthermore equation (2.3) tells us that $\chi(P_i) = [x]$. Since x is a magic walk, and η^0 is an injection, we see that $\tilde{\eta}^0$ is an injection into the vertex set of the left Krieger cover.

We must show that $\tilde{\eta}^0$ is a surjection. If $P_X(x_i)$ is a vertex in the left Krieger cover, then $x_i \in X^+$ satisfies $x_i \sim_\infty y$ and $y \in X^+$ is a magic walk. So with no loss of generality, we can assume x_i is a magic walk. Write $x_i = \mu x'$ where μ is magic and $x \in X^+$. As $x \in X^+$, $x \in \chi(P_j)$ for some j . Let $P_i = (\eta^0)^{-1}(P_X(x_i))$. We must show $P_i \in E_A^0$. Using proposition 3.3.4 we see that

$$S_\mu^* P_i S_\mu \geq P_j,$$

and as μ is magic, $P_i \in E_A^0$. Thus $\tilde{\eta}^0$ is a bijection, and the proposition is proved. \square

Because the left Krieger cover is irreducible whenever X is, we have the following corollary.

Corollary 5.2.3: *If X is an irreducible sofic shift, then the edge shift $\Sigma_{\tilde{A}}$ constructed above is irreducible.*

If $E_A = (E^0, E^1)$ is the Perron-Frobenius graph of \mathcal{O}_X , let $H^0 = E^0 \setminus E_{\tilde{A}}^0$. We can show that H^0 is hereditary and saturated in the sense of definition 4.2.1. Thus by theorem 4.2.2, H^0 corresponds to a gauge invariant ideal I in $C^*(E_A)$.

Lemma 5.2.4: *The set H^0 defined above is hereditary and saturated (or the empty set).*

PROOF. If H^0 is the empty set, there is nothing to prove. We suppose $H^0 \neq \emptyset$ is not hereditary. This means there exists a $P_i \in E_{\tilde{A}}^0$, and $P_j \in H^0$ with $P_j \succcurlyeq P_i$, which meant there is a path ν connecting P_j to P_i . Thus,

$$S_\nu^* P_j S_\nu \geq P_i,$$

for some $\nu \in B(X)$. Furthermore, as $P_i \in E_{\tilde{A}}^0$, $S_\mu^* P_i S_\mu \neq 0$, for some magic word μ . However, this means that

$$S_\mu^*(S_\nu^* P_j S_\nu) S_\mu \neq 0$$

and as $\mu\nu \in B(X)$ must be magic, we are forced to conclude $P_j \in E_{\tilde{A}}^0$, a contradiction.

We must show H^0 is saturated. This follows because any $P_i \in E_{\tilde{A}}^0$ cannot have all its edges going into H^0 . If this was the case, $S_\mu^* P_i S_\mu = 0$ for any magic word μ . □

Theorem 5.2.5: *Let X be a sofic shift and \mathcal{O}_X the corresponding C^* -algebra. Then there exists a gauge invariant ideal I (possibly $I = \{0\}$) such that the quotient \mathcal{O}_X/I is isomorphic to the C^* -algebra of the graph of the left Krieger cover*

PROOF. By theorem 3.4.4 and proposition 5.1.2 $\mathcal{O}_X = C^*(E_A)$ and E_A is the past cover for X . Proposition 5.2.2 tells us that E_A has a sub-graph isomorphic to the left Krieger cover, namely $E_{\bar{A}} = (E_{\bar{A}}^0, E_{\bar{A}}^1)$. Let $H^0 = E^0 \setminus E_{\bar{A}}^0$.

If $H^0 = \emptyset$ then it must be that the past set cover equals the left Krieger cover. Let $I = \{0\}$, and the result follows by proposition 5.2.2. If $H^0 \neq \emptyset$ then lemma 5.2.4 shows H^0 is saturated and hereditary, and thus corresponds to a gauge invariant ideal I in $C^*(E_A)$. Remark 4.2.4 tells us that I is also gauge invariant in \mathcal{O}_X . Finally [2, Theorem 3.5] and [14, Proposition 4.1] tell us that $\mathcal{O}_X/I = C^*(E_A)/I$ is isomorphic to $C^*(E_{\bar{A}})$. As $E_{\bar{A}}$ is the left Krieger cover by proposition 5.2.2, this completes the proof. \square

Corollary 5.2.6: *Let X be an irreducible sofic shift with left Krieger Cover Σ_L . Then there exists a maximal (possibly zero) ideal $I \subset \mathcal{O}_X$ with \mathcal{O}_X/I isomorphic to the Cuntz-Krieger algebra corresponding to the edge shift Σ_L . Furthermore this Cuntz-Krieger algebra satisfies condition (I) of [8].*

PROOF. By theorem 5.2.5 there exists a gauge invariant ideal I with $\mathcal{O}_X/I = C^*(E_A)/I$ isomorphic to the left Krieger cover graph C*-algebra. Now since X is irreducible, $E_{\bar{A}}$ must satisfy condition (L), so $C^*(E_{\bar{A}})$ can be generated independent of the choice of isometries. As $E_{\bar{A}}$ is a finite graph, $C^*(E_{\bar{A}})$ is generated by a Cuntz-Krieger \tilde{A}_E family where \tilde{A}_E is the edge shift matrix of the graph $E_{\bar{A}}$ [14, Proposition 3.1]. Because $E_{\bar{A}}$ satisfies condition (L), \tilde{A}_E satisfies condition (I) of [8]. We know that $C^*(E_{\bar{A}}) \cong \mathcal{O}_{\tilde{A}_E}$, and irreducibility also allows us to conclude that $\mathcal{O}_{\tilde{A}_E}$ simple [8, Theorem 2.14]. Thus I must be a maximal ideal. \square

Remark 5.2.7: If we denote by \tilde{S}_μ , $\mu \in B(X)$ the image of S_μ in \mathcal{O}_X/I above, then \tilde{S}_μ clearly generates \mathcal{O}_X/I . It can also be shown that \tilde{S}_μ , \tilde{S}_μ^* , and their corresponding initial and final projections are non-zero in \mathcal{O}_X/I . The relations S_μ satisfies in

\mathcal{O}_X (see (1) of theorem 2.4.6) will also be satisfied by \tilde{S}_μ . Furthermore, if the left Krieger cover satisfies condition (L), then the algebra \mathcal{O}_X/I will satisfy Matsumoto's condition (I_X). This follows by theorem 3.6.1 applied to the generators of \mathcal{O}_X/I . In this case, the quotient algebra no longer will have those minimal projections in A_X that correspond only to periodic points.

5.3. The Minimal Projections and the Right Krieger Cover

This section is devoted to showing how the Perron-Frobenius operator is related to the right Krieger cover constructed in [13]. To do so, we shall give an equivalent definition of l -past equivalence in terms of the future of a word, rather than its past. The reason for this is twofold: first, one should be optimistic and look towards the future, rather than the past, and second, it shows how the left and right Krieger covers are related.

If $\mu \in B(X)$, then lemma 2.1.1 tells us $S_\mu^* S_\mu = S_\nu^* S_\nu$ if and only if

$$\{x \in X^+ : \mu x \in X^+\} = \{x \in X^+ : \nu x \in X^+\} \iff$$

$$F_X(\mu) = F_X(\nu)$$

Thus, $S_\mu^* S_\mu$ can be regarded as the characteristic function of the follower set of μ .

I.e. for $x \in X^+$

$$S_\mu^* S_\mu(x) = \begin{cases} 1 & \text{if } x \in F_X(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

As the collection of all follower sets is finite, we only have finitely many such $S_\mu^* S_\mu$'s.

Lemma 5.3.1: *Suppose X is a sofic shift.*

1. *If μ_i , $1 \leq i \leq n$ is a set of words of X with the property that for any word ν , $F_X(\nu) = F_X(\mu_i)$ for some i , then A_X is generated by $S_{\mu_i}^* S_{\mu_i}$.*

2. If μ_i , $1 \leq i \leq L$ are magic words with the property that if ν is a magic word, $F_X(\nu) = F_X(\mu_i)$, then A_X is generated by $S_{\mu_i}^* S_{\mu_i}$.

PROOF. (1) is clear from lemma 2.1.1. (2) follows because if ν is a non magic word, then $F_X(\nu)$ is a union of follower sets of magic words (part (1) of remark 1.9.1), say $\cup F_X(\mu_i)$. By taking the appropriate sums and products of the $S_{\mu_i}^* S_{\mu_i}$'s, one can obtain $S_\nu^* S_\nu$. \square

Let $E = (E_R^0, E_R^1, \Sigma, \pi_R)$ be the right Krieger cover graph for the sofic shift X . We can identify E_R^0 as $\{F_X(\mu_1), F_X(\mu_2), \dots, F_X(\mu_L)\}$, where each $F_X(\mu_i)$ is a follower set of a magic word μ_i . By extending each μ_i to the left, we can assume each μ_i satisfies $|\mu_i| = l_i \geq l_0$, where $\dim(A_{l_0}) = \dim(A_X)$.

We define an equivalence relation called *follower set equivalence* as follows. For $x, y \in X^+$, we say that x is follower set equivalent to y (denoted $x \sim_f y$) if and only if

$$x \in F_X(\mu_i) \iff y \in F_X(\mu_i), \text{ for all } 1 \leq i \leq L$$

(so x and y are in the same follower sets). Because we identify follower sets as vertices on the right Krieger cover graph of X , $F_X(\mu)$ is in $s(x)$ if and only if $x \in F_X(\mu)$. We therefore can conclude that

$$s(x) = s(y) \iff x \sim_f y.$$

Recall the l -past equivalence relation used in chapter 2 to obtain the minimal projections of A_X . We shall see that \sim_l and \sim_f are the same whenever $l \geq l_1$.

Proposition 5.3.2: *If X^+ is a one-sided sofic shift, and $l \geq l_1$, then the l -past equivalence classes of X^+ are the same as the follower set equivalence classes of X^+ .*

PROOF. Suppose $x \sim_f y$, then $s(x) = s(y)$ on the right Krieger cover graph. Suppose $\nu \in B(X)$ with $|\nu| \leq l$. If $\nu x \in X^+$, we must have

$$\emptyset \neq r(\nu) \cap s(x) = r(\nu) \cap s(y).$$

Thus, both $\nu x \in X^+$ and $\nu y \in X^+$. Since ν was arbitrary, we conclude

$$\{\nu : |\nu| \leq l, \nu x \in X^+\} = \{\nu : |\nu| \leq l, \nu y \in X^+\}$$

so $x \sim_l y$.

Conversely if $x \sim_l y$ and $l \geq l_1$ then

$$\{\nu : |\nu| \leq l, \nu x \in X^+\} = \{\nu : |\nu| \leq l, \nu y \in X^+\}.$$

We will show that $s(x) = s(y)$ on the right Krieger cover graph, hence $x \sim_f y$. Suppose not. With no loss of generality, this means there exists a $F_X(\mu_i) \in s(y)$ but $F_X(\mu_i) \notin s(x)$. This means that $\mu_i y \in X^+$, but $\mu_i x \notin X^+$ for some magic word μ_i . As $l_0 \leq |\mu_i| \leq l$, we are forced to conclude that $x \not\sim_l y$, a contradiction. \square

Corollary 5.3.3: *Each minimal projection $P \in A_X$ corresponds to an equivalence class of the follower set equivalence relation.*

Throughout this section, we will be referring to a minimal projection $P \in A_X$ as a characteristic function of a follower set equivalence class. We will return to the past equivalence relation in subsequent sections.

We now make an assumption we will use throughout the remainder of this section. Assume that for all magic words μ_i, μ_j with $F_X(\mu_i) \neq F_X(\mu_j)$, we have $F_X(\mu_i) \not\subseteq F_X(\mu_j)$. Lemma 1.9.7 shows this happens if X is an AFT of degree zero. We shall discuss what to do when one follower set is a subset of another in remark 5.3.10.

Let $M = \dim(A_X)$. Choose $x_1, x_2, \dots, x_M \in X^+$ a member from each follower set equivalence class. We also order the members as follows: we say x_i precedes

x_j if and only if the cardinality of the source of x_i is less than that of x_j . If the cardinalities are equal, use the lexicographical ordering on $F_X(\mu_1), \dots, F_X(\mu_L)$ (where $F_X(\mu_1) > F_X(\mu_2) > \dots > F_X(\mu_L)$). Because each x_i is in a different follower set equivalence class, it has a corresponding minimal projection $P_i \in A_X$ with the property that $P_i(x_i) = 1$, and $P_j(x_i) = 0$ for every $j \neq i$. We order the P_i 's based on the same order as their corresponding x_i 's.

Since for any two magic words μ_i, μ_j , $F_X(\mu_i) \not\subseteq F_X(\mu_j)$, it must be that for each magic word μ_i , there exists an $x \in F_X(\mu_i)$, and $x \notin F_X(\mu_j)$. Obviously, such x has its source a singleton set $\{F_X(\mu_i)\}$. Therefore, our first L representatives, x_1, \dots, x_L are precisely the elements in X^+ with the property that $s(x_i)$ is the singleton set $\{F_X(\mu_i)\}$. Thus, we regard the first L minimal projections, when viewed as characteristic functions on follower set equivalence classes as

$$P_i(x) = \begin{cases} 1 & \text{if } s(x) = s(x_i) = \{F_X(\mu_i)\} \\ 0 & \text{otherwise.} \end{cases}$$

for $1 \leq i \leq L$. Thus, as a characteristic function, each P_i is non zero only to those elements $x \in X^+$ whose source is $\{F_X(\mu_i)\}$.

We can then look at those x_i whose source has cardinality 2. For these x_i , we can regard the minimal projection P_i as the characteristic function of those $x \in X^+$ satisfying $s(x) = s(x_i)$, and the cardinality of the source is 2. Continue this way until we have exhausted all M equivalence class representatives. We shall denote the cardinality of the source of x as $\#s(x)$.

Remark 5.3.4: Recall how the map χ from equation (2.3) allowed us to identify a minimal projection $P \in A_X$ with a subset of X^+ . We have a similar idea from the

follower set point of view. It is clear that if $s(x_i) = \{F_X(\mu_{i_s})\}_{s=1}^k$ then

$$x_i \in \bigcap_{s(x_i)} F_X(\mu_{i_s}).$$

So we can regard the minimal projection P_i as

$$P_i = \prod_{F_X(\mu_{i_s}) \in s(x_i)} S_{\mu_{i_s}}^* S_{\mu_{i_s}} \prod_{F_X(\mu_{i_t}) \notin s(x_i)} (1 - S_{\mu_{i_t}}^* S_{\mu_{i_t}}).$$

Therefore, each minimal projection P_i corresponds to a characteristic function of those $x \in X^+$ in an intersection of follower sets. Furthermore, if $P_i(x) = 1$, then for any larger collection, S , of follower sets satisfying

$$s(x) \subsetneq S \text{ and } \bigcap_{F_X(\mu) \in S} F_X(\mu) \neq \emptyset,$$

x will not be in the above intersection. So the intersection of the follower sets in the source of x_i uniquely determines P_i . We will use this fact in the proof of some theorems, and will find it a useful correspondence when doing the examples in chapter 7.

Recall the map χ defined in (2.3). In light of the above remark, we can regard the map χ as a function from the minimal projections to the follower set equivalence class via

$$(5.5) \quad \chi(P_i) = \{x \in X^+ : s(x) = s(x_i)\}$$

We will use this definition of χ throughout this section.

We will see that the matrix A has a structure related to the right Krieger cover as follows. For each symbol $I \in \Sigma$, the matrix $A^{(I)}$ relates in the following way to the right Krieger cover of X . Based on our ordering of the minimal projections we

break $A^{(I)}$ into a block matrix

$$(5.6) \quad A^{(I)} = \begin{bmatrix} A_1^{(I)} & A_3^{(I)} \\ A_2^{(I)} & A_4^{(I)} \end{bmatrix},$$

where $A_1^{(I)}$ is $L \times L$. Roughly speaking (using remark 5.3.4), $A_1^{(I)}$ represents the transitions labeled I from follower sets to follower sets. $A_2^{(I)}$ represents transitions labeled I from intersections of follower sets to follower sets, $A_3^{(I)}$ represents transitions labeled I from follower sets to intersections of follower sets, and $A_4^{(I)}$ represents transitions labeled I from intersections of follower sets to intersections of follower sets.

Proposition 5.3.5: *Let X be sofic with symbol set $\Sigma = \{1, \dots, n\}$ and distinct magic words μ_1, \dots, μ_L . Order the minimal projections in A_X as above. If $E = (E^0, E^1, \pi, \Sigma)$ is the right Krieger cover of X , we identify E^0 as $\{F_X(\mu_i)\}_{i=1}^L$. Then for each $I \in \Sigma$ we have an $M \times M$ block matrix*

$$A^{(I)} = \begin{bmatrix} A_1^{(I)} & A_3^{(I)} \\ A_2^{(I)} & A_4^{(I)} \end{bmatrix}$$

where $A_1^{(I)}$ is $L \times L$ satisfying:

1. For $1 \leq i, j \leq L$, $A_1^{(I)}(i, j) = 1$ if and only if there is an edge e labeled I satisfying $s(e) = F_X(\mu_i)$, $r(e) = F_X(\mu_j)$, and e is the only edge labeled I whose range is $F_X(\mu_j)$.
2. For $1 \leq j \leq L$, and $L + 1 \leq i \leq M$ we have $s(x_i) = \{F_X(\mu_{i_1}), \dots, F_X(\mu_{i_s})\}$, $s > 1$. Then $A_2^{(I)}(i, j) = 1$ if and only if there are exactly s edges e_1, \dots, e_s satisfying $r(e_k) = F_X(\mu_j)$, $\pi(e_k) = I$, and $s(e_k) = F_X(\mu_{i_k})$ for $1 \leq k \leq s$.
3. If X is an edge shift, then $A^{(I)} = A_1^{(I)}$, and A is $L \times L$.

PROOF. If $1 \leq i, j \leq L$, then $A_{i,j}^{(I)} = 1$ if and only if $S_I^* P_i S_I \geq P_j$, which is equivalent to saying $Ix \in \chi(P_i)$ for every $x \in \chi(P_j)$ (theorem 3.3.3). We know

that $x_j \in \chi(P_j)$, and $x_i \in \chi(P_i)$. Furthermore, it must be that $s(x_j) = \{F_X(\mu_j)\}$, and $s(x_i) = \{F_X(\mu_i)\}$. Since $Ix_j \in \chi(P_i)$ (proposition 3.3.4), it must be that $s(Ix_j) = s(x_i) = \{F_X(\mu_i)\}$. As the right Krieger cover graph is right resolving, we conclude that $F_X(\mu_i I) = F_X(\mu_j)$. One argues similarly for part (2). Part (3) follows from the fact that if $x \in X^+$ and X^+ is an edge shift, then $\#s(x) = 1$ for every x . \square

Remark 5.3.6: In proposition 5.3.5, one can use similar techniques to those used in its proof to find transitions between pairwise intersections (remark 5.3.4), and pairwise intersections; then one could go to triple intersections, and so forth. The structure of $A_3^{(I)}$ from the follower set point of view can be calculated using proposition 3.3.3.

Because of proposition 5.3.5 we see that the matrix A is none other than

$$A = \sum_{I \in \Sigma} A^{(I)}$$

so we can regard A as having a block structure based on the follower set presentation.

This gives us the following theorem.

Theorem 5.3.7: *Let X be sofic with symbol set $\Sigma = \{1, \dots, n\}$, and L distinct follower sets. If the minimal projections are ordered as in proposition 5.3.5, then the Perron-Frobenius operator is an $M \times M$ matrix of the following form:*

$$\begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix}.$$

Where A_1 is $L \times L$, and $A_2 = 0$ if and only if the sofic shift is an AFT of degree zero. Also, if X is an edge shift, then $L = M$.

PROOF. The first part is shown in the discussion preceding the theorem.

For the second part of the theorem, we will use a lemma whose proof is similar to that of proposition 1.9.4.

Lemma 5.3.8: *Let X be a sofic shift. Then X is an AFT of degree zero if and only if for every $x \in X^+$ with $\#s(x) > 1$*

$$\bigcap_{F_X(\mu_i) \in s(x)} F_X(\mu_i)$$

contains only those $x \in X^+$ made up of non-magic walks.

We can now prove the second part of the theorem. Suppose $A_2 = 0$. This occurs if and only if $A_2^{(I)} = 0$ every symbol I . Using remark 5.3.4, and proposition 3.3.4 this is equivalent to saying that for each $x \in X^+$ with $\#s(x) > 1$

$$\bigcap_{F_X(\mu) \in s(x)} F_X(\mu)$$

has no magic walks. Use lemma 5.3.8. The final part follows from part (3) proposition 5.3.5. \square

Remark 5.3.9: We note that as an embedding map in $\mathcal{F}_k^\infty \hookrightarrow \mathcal{F}_{k+1}^\infty$ (see equation (2.1)), the transpose of A is used. This makes the K-theory formulae for \mathcal{O}_X consistent with theorem 2.2.3.

Remark 5.3.10: Throughout this section, we assumed that $F_X(\mu_i) \not\subseteq F_X(\mu_j)$ for $i \neq j$. If it happens to be the case that one follower set is a subset of another, then part (2) of proposition 5.3.5 will still hold. If $F_X(\mu_i) \subsetneq F_X(\mu_j)$, then any $x \in F_X(\mu_i)$ will also be in $F_X(\mu_j)$ so $\#s(x) > 1$. Thus, we generalize our order slightly to compensate for this. The first L projections are ordered as follows: If P is such that $P(x) = 1$ where $s(x) = \{F_X(\mu_{i_t})\}_{t=1}^s$ ($s \geq 1$) and

$$F_X(\mu_{i_1}) \subsetneq F_X(\mu_{i_2}) \subsetneq \dots \subsetneq F_X(\mu_{i_s})$$

then place P in the i_1 position. We do this because if $P(x) = 1$ and $s(x) = \{F_X(\mu_{i_t})\}_{t=1}^s$ defined above, then if $y \notin F_X(\mu_{i_1})$ but $y \in F_X(\mu_{i_t})$ for $t \geq 2$, then $P(y) = 0$. Order the remaining projections as before. With this modified order, part (1) of proposition 5.3.5 will hold. Part (2) is modified as follows

- 2' For $1 \leq j \leq L$, if $s(x_i) = \{F_X(\mu_{i_t})\}_{t=1}^s$, then $A^{(I)}(i, j) = 1$ if and only if $F_X(\mu_j)$ has s edges labeled I whose range is $F_X(\mu_j)$ and $F_X(\mu_{i_t}I) = F_X(\mu_j)$, for every $1 \leq t \leq s$.

Since there is no guarantee now that the first L minimal projections only correspond to $x \in X^+$ with $\#s(x) = 1$, we have to drop the condition that $L + 1 \leq i \leq M$. Part (3) is irrelevant in this case since edge shifts are degree zero AFT's, and will not have this problem.

Remark 5.3.11: Because the right Krieger cover is follower separated, we will always have at least one minimal projection $P_i \in A_X$ with the property that $P_i(x) = 1$ if and only if $x \in F_X(\mu_i)$ for some magic word μ_i , and $x \notin F_X(\mu_j)$ for any magic word μ_j whose follower set is not $F_X(\mu_i)$. The existence of such a projection plays an important role in the next section.

To finish this section, and to motivate the next, we see that the block A_2 in theorem 5.3.7 is the zero matrix precisely when X is a degree zero AFT. Using part (1) of proposition 5.3.5, and theorem 5.3.7 we see that A_1 is the adjacency matrix for the right (and left) Krieger cover. Because $A_2 = 0$, we must have a gauge invariant ideal I , generated by the minimal projections P_i , $L + 1 \leq i \leq M$, such that \mathcal{O}_X/I is isomorphic to the C*-algebra of the graph of the right Krieger cover. For general AFT's, this is also the case, and we shall explore this in the next section.

5.4. The Structure of \mathcal{O}_X for AFT's

As remarked in section 3.5 the edge shift Σ_A associated with the Perron-Frobenius operator may have more symbols than the presentation in the (left) Krieger cover. It could well be that Σ_A is not even irreducible, and we will show that when the purely sofic shift X is of Almost Finite Type (AFT), Σ_A is always a reducible edge shift.

When X is an AFT, both the left and right Krieger covers are conjugate, and furthermore have the property that any other presentation $\rho : Y \rightarrow X$ factors through them (see [6, Theorem 9]). Thus, there is a map $\bar{\rho}$ such that

$$\begin{array}{ccc} \Sigma_L & \xleftarrow{\bar{\rho}} & Y \\ \pi \downarrow & \swarrow \rho & \\ & & X \end{array}$$

commutes. In [6] it is proven that only AFT's have this property. As in [6], we shall refer to this cover as the *minimal cover*. Furthermore, when the ρ is one to one on an open set [6, Proposition 7, Corollary 10] then $\bar{\rho}$ is a conjugacy.

AFT's were first introduced by B. Markus [17]. They play an important role in the theory of sofic shifts as they are a nice intermediate class between SFT's and the class of all sofic shifts. There are many equivalent definitions for X to be a shift of almost finite type; we will use the following.

Definition 5.4.1: A sofic shift X is said to be of almost finite type (AFT) if there exists a cover, Y for X with $\pi : Y \rightarrow X$ one to one on an open set.

This definition is much different than the one given in chapter 1. However, it is equivalent (see theorem 5.4.3 and the discussion preceding it).

We will give the definition of an AFT in terms of the labeled graph presentation of X .

Definition 5.4.2: (see [16, Definition 5.1.4]) Let $E = (E^0, E^1, \pi, \Sigma)$ be a labeled graph. Then (E, π) is said to be left closing with delay D if ω, λ are paths on E with $|\omega| = |\lambda| > D$, and they end at the same vertex and have the same terminal label, they must have the same terminal edge.

Left closing presentations, Y , are “eventually left resolving”. By moving to a higher block presentation of Y , one can adjust π to be left resolving [16, Section 5.1]. A dual notion, right closing, is defined similarly.

Theorem 5.4.3: (see [6, Proposition 7]) *A sofic shift X is an AFT if and only if the minimal right resolving cover (i.e. the right Krieger cover) is left closing (equivalently the minimal left resolving cover is right closing).*

We can show for irreducible AFT's, the Perron-Frobenius cover is always reducible.

Proposition 5.4.4: *If X is an irreducible AFT that is not an SFT, then Σ_A is reducible.*

More important than this is the contrapositive corollary, which tells us when a purely sofic shift is not an AFT.

Corollary 5.4.5: *If X is a purely sofic shift with irreducible (as a edge shift) Perron-Frobenius operator A , then X is not an AFT.*

THE PROOF OF PROPOSITION 5.4.4. Let X be an AFT, with $\pi_R : \Sigma_R \rightarrow X$ the right Krieger cover. As π_R is n to one, for some $n \in \mathbb{N}$, we have by [24, Corollary 3.5], a periodic point $x = \dots\mu\mu\dots \in X$ with $\#\pi_R^{-1}(x) = n$. Denote also by $x = \mu\mu\mu\dots$ the corresponding point in the one-sided shift. We must have $\#\pi_R^{-1}(\mu\mu\dots) \geq \#\pi_R^{-1}(\dots\mu\mu\dots)$. From here on, x will be the point in the one-sided shift.

Because $\#\pi_R^{-1}(x) > 1$, it must be that $\#s(\mu^k) > 1$ for every $k \in \mathbb{N}$. Furthermore, because the periodic point in the two sided shift also has more than one preimage, we can conclude

$$\#(s(\mu^k) \cap r(\mu^k)) > 1$$

for every $k \in \mathbb{N}$. Let $P_i \in A_X$ the the minimal projection satisfying $P_i(x) = 1$, and let $P \in A_X$ be as in remark 5.3.11.

If the Perron-Frobenius cover is irreducible, it must be (proposition 5.2.2) that there exists a magic word ω with

$$S_\omega^* P_i S_\omega \geq P.$$

Thus, proposition 3.3.4, and equation (5.5) tell us that for each $y' \in \chi(P)$, $\omega y' \in \chi(P_i)$. Let $y = \omega y'$. As $x, y \in \chi(P_i)$, we have

$$(5.7) \quad s(y) = s(x) = s(\mu\mu \dots).$$

So $s(\omega) = s(y) = s(x)$, and as ω is magic, $\#r(\omega) = 1$. This allows us to conclude that $\mu\omega \in B(X)$. So we have, for each $k \in \mathbb{N}$,

$$(5.8) \quad \#(s(\omega) \cap r(\mu^k)) > 1.$$

This allows us to conclude that $\mu^k y \in X^+$ for every $k \in \mathbb{N}$. Because of (5.7) we must have $\#\pi^{-1}(\mu^k y) > 1$ for each k . Suppose $\{z^j\}_{j=1}^m = \pi^{-1}(\mu^k y)$. As X is an AFT, the right Krieger cover graph is left closing with degree D . Let k be chosen large enough so that $|\mu^k \omega| \gg D$, and let $D' = |\mu^k \omega|$. Because $\#r(\mu^k \omega) = 1$ any $z_1^i \dots z_{|\mu^k \omega|}^i \in \pi^{-1}(\mu^k \omega)$ must have the same terminal edge. Thus, each $z^j \in \pi^{-1}(\mu^k y)$

is of the form

$$\begin{aligned} z^1 &= z_1^1 z_2^1 \dots z_{D'}^1 z_+, \\ &\vdots \\ z^n &= z_1^1 z_2^1 \dots z_{D'}^1 z_+, \end{aligned}$$

So from $D' + 1$ on, all pre-images of $\mu^k y$ are the same. This forces us to conclude

$$\#(r(\mu^k) \cap s(\omega)) = 1$$

contradicting equation (5.8). Thus, Σ_A must be reducible. \square

Remark 5.4.6: We shall see later that the converse of proposition 5.4.4 does not hold in general (cf. section 7.5).

There are some interesting consequences to proposition 5.4.4 when combined with the theory of C^* -algebras of β -shifts [11]. Essentially, β -shifts can be of finite type, sofic, or neither; it depends on whether the β expansion of 1 terminates, is periodic, or is aperiodic respectively. It has been shown that C^* -algebras of β -shifts, are simple [11, Theorem 3.6]. As the Perron-Frobenius operator is reducible for purely sofic AFT's, it cannot be simple. Therefore we conclude:

Corollary 5.4.7: *If X is a purely sofic shift obtained from the β -expansion of 1 ($\beta > 1$), then X is not an AFT.*

We can now state our main structure theorem for AFT's. The proof is mainly putting together all the pieces of the puzzle developed so far.

Theorem 5.4.8: *Let X be an irreducible AFT whose left (and right) Krieger Cover graph satisfies condition (L). Then there exists a maximal ideal $0 \neq I \subset \mathcal{O}_X$ such that \mathcal{O}_X/I is isomorphic to the Cuntz-Krieger algebra of the minimal cover of the sofic shift X .*

PROOF. If X is an irreducible AFT, and A its Perron-Frobenius operator, then by proposition 5.4.4 there exists a reducible part of A . Using this fact and theorem 5.2.6, there is a non-zero, maximal ideal I with \mathcal{O}_X/I isomorphic to the left Krieger cover graph C*-algebra, $C^*(E_{\bar{A}})$, and this algebra satisfies condition (L). Since the left Krieger cover is minimal in the sense of [6], the theorem is proved. \square

Remark 5.4.9: Note that the algebra \mathcal{O}_X/I can be generated independent of the choice of isometries $S_I, I \in \Sigma$. This is because $\mathcal{O}_X/I = C^*(E_{\bar{A}})$, and $E_{\bar{A}}$ satisfies condition (L), thus any set of generators for \mathcal{O}_X are unique modulo I by theorem 3.6.1.

5.5. The Structure of \mathcal{O}_X for Reducible X

In light of section 5.1, any sofic X has \mathcal{O}_X equal to $C^*(E_A)$, where E_A is the graph of the past set cover for X . If a sofic shift is reducible, then certainly its past set cover is reducible. Thus by studying the “irreducible components” of X one obtains theorems about the ideal structure of X similar to those in [7, 2].

Suppose X is a sofic shift with Perron-Frobenius operator A . By reducing Σ_A as we did in section 5.1, we will clearly have a gauge invariant ideal I such that \mathcal{O}_X/I is the same as $C^*(E_{\bar{A}})$ and by theorem 5.2.5 $E_{\bar{A}}$ the left Krieger Cover. By reordering the minimal projections, we can assume the edge shift matrix \bar{A}_E can be arranged to have the following block structure.

$$\bar{A} = \begin{bmatrix} A_1 & * & \dots & * \\ 0 & A_2 & * & * \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & \dots & * \\ 0 & \dots & \dots & A_n \end{bmatrix}$$

One now can exploit the ideal structure theorems for $\mathcal{A}\mathcal{O}_{\tilde{A}_E}$ of [2] regarding this matrix. Specifically, for each A_i , $1 < i \leq n$, there is a gauge invariant ideal I_{A_i} such that \mathcal{O}_X/I_{A_i} is isomorphic to $\mathcal{A}\mathcal{O}_{\tilde{A}'}$ where

$$\tilde{A}' = \begin{bmatrix} A_1 & * & \dots & * \\ 0 & A_2 & * & \dots & * \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & * \\ 0 & \dots & \dots & \dots & A_{i-1} \end{bmatrix}$$

CHAPTER 6

Generating \mathcal{O}_X from Left Resolving Labelings

This chapter covers the converse of chapter 3. That is, if we have a sofic shift X , and we label a directed graph $E = (E^0, E^1)$ such that E presents X , can we use the generating partial isometries of $C^*(E)$ to make new partial isometries that behave like the generating partial isometries of \mathcal{O}_X ? Because of the vast possibilities of graph labelings, we restrict ourselves only to left resolving labelings. We will have to make more assumptions as the chapter progresses.

Throughout this chapter, fix a sofic shift X with symbol set Σ . Let \mathcal{O}_X be the C^* -algebra of X , and $E_A = (E^0, E^1, \pi, \Sigma)$ the Perron-Frobenius cover for X . Theorem 3.4.4 tells us that $\mathcal{O}_X = C^*(E_A)$. We will denote by $F = (F^0, F^1, \rho, \Sigma)$ a left-resolving graph that also presents the sofic shift X . We remind the reader that we suppress most of the notation, and denote each labeled graph as (E_A, π) and (F, ρ) respectively. The label maps π and ρ will also denote the corresponding one-block maps $\pi : \Sigma_{E_A} \rightarrow X$ and $\rho : Y \rightarrow X$ from the respective edge shifts, Σ_{E_A} and Y , to the sofic shift X .

6.1. Left-Resolving Labelings and Graph C^* -algebras

As always, for any shift X , X^+ is the corresponding one-sided shift, and $B(X)$ the set of all allowable words in X .

Whenever a labeling of a graph (F, ρ) is left resolving, one can define a set of partial isometries in $C^*(F)$ that behave similarly to the generating partial isometries of \mathcal{O}_X . We do so as follows: Suppose that $\{t_e\}_{e \in F^1}$ is a Cuntz-Krieger F -family. For

each $I \in \Sigma$ define

$$(6.1) \quad T_I := \sum_{\{e \in E^1 : e \in \rho^{-1}(I)\}} t_e.$$

A simple argument by induction on $|\mu|$ shows that for $\mu \in B(X)$

$$(6.2) \quad T_\mu = \sum_{\{\xi \in B(Y) : \xi \in \rho^{-1}(\mu)\}} t_\xi.$$

Lemma 6.1.1: *Suppose $\xi, \tau \in B(Y)$ and $\xi \neq \tau$, then*

1. $t_\xi^* t_\tau = 0$.
2. *If in addition $\rho(\xi) = \rho(\tau)$ then $t_\xi t_\tau^* = 0$.*

PROOF. (1) is in the proof of [14, Proposition 4.1]. To prove (2), one also has in the proof of [14, Proposition 4.1] that $t_\xi t_\tau^* \neq 0$ if and only if $r(\xi) = r(\tau)$ (remember that ξ, τ are paths on E). However, as ρ is a left resolving labeling, $r(\xi) = r(\tau)$ forces $\rho(\xi) \neq \rho(\tau)$. \square

Corollary 6.1.2: *If $\mu \in B(X)$ then*

$$(6.3) \quad T_\mu T_\mu^* = \sum_{\xi \in \rho^{-1}(\mu)} t_\xi t_\xi^*$$

$$(6.4) \quad T_\mu^* T_\mu = \sum_{\xi \in \rho^{-1}(\mu)} t_\xi^* t_\xi = \sum_{\substack{\{e \in E^1 : \xi \in \rho^{-1}(\mu) \\ r(\xi) = s(e)\}}} t_e t_e^*$$

$$(6.5) \quad = \sum_{\substack{\xi \in \rho^{-1}(\mu) \\ r(\xi) = v}} P_v.$$

PROOF. (6.3) and the first equality of (6.4) follow from lemma 6.1.1 and equations (6.1), (6.2). The last part of (6.4) is due to the extension of equation (3.1) as found in [15, Lemma 1.1]. Equation (6.5) is just rewriting the E -family relations (see equation (3.1)). \square

Lemma 6.1.3: *If $\mu, \nu \in B(X)$ and $|\mu| = |\nu|$ then $T_\mu^* T_\nu \neq 0$ if and only if $\mu = \nu$.*

PROOF. By direct calculation and [15, Lemma 1.1]. \square

Proposition 6.1.4: *If X is a sofic shift presented by a left resolving labeling of a graph F then for every $n \in \mathbb{N}$, and $\mu \in B(X)$*

$$(6.6) \quad \sum_{|\mu|=n} T_\mu T_\mu^* = 1.$$

PROOF.

$$\begin{aligned} \sum_{|\mu|=n} T_\mu T_\mu^* &= \sum_{|\mu|=n} \left(\sum_{\xi \in \rho^{-1}(\mu)} t_\xi t_\xi^* \right) && \text{equation (6.3)} \\ &= \sum_{|\xi|=n} t_\xi t_\xi^* && \text{as } \rho \text{ is onto} \\ &= 1 \end{aligned}$$

the last equality follows because $t_\xi t_\xi^*$ is precisely the characteristic function of the cylinder set of ξ , and these form a partition of the compact topological space Y^+ . \square

Theorem 6.1.5: *If X is a sofic shift presented by a left resolving labeling of a graph E with symbol set $\Sigma = \{1, \dots, N\}$ then*

1. $\sum_{I=1}^N T_I T_I^* = 1,$
2. $T_\mu^* T_\mu T_\nu = T_\nu T_{\mu\nu}^* T_{\mu\nu},$ where $\mu, \nu \in B(X).$

Hence the generators T_I satisfy the same relations as the generators S_I for \mathcal{O}_X .

PROOF. The first identity is a special case of proposition 6.1.4. We need to show the second identity.

$$\begin{aligned}
T_\mu^* T_\mu T_\nu &= \sum_{\substack{\alpha \in B(X) \\ |\alpha| = |\nu|}} T_\alpha T_\alpha^* T_\mu^* T_\mu T_\nu && \text{proposition 6.1.4} \\
&= \sum_{\substack{\alpha \in B(X) \\ |\alpha| = |\nu|}} T_\alpha T_{\mu\alpha}^* T_{\mu\nu} && \text{equation (6.2)} \\
&= T_\nu T_{\mu\nu}^* T_{\mu\nu} && \text{lemma 6.1.3.}
\end{aligned}$$

□

We shall denote by \mathcal{L}_X the C^* -subalgebra generated by the isometries T_I with $I \in \Sigma$.

6.2. More on the Left Krieger Cover Algebra

In light of theorem 6.1.5 two questions come into mind. First, we know that $\mathcal{O}_X = C^*(E_A)$. Thus, we can ask if the construction done in the previous section on $C^*(E_A)$ give us the original isometries S_I obtained in chapter 3. Second, we can ask for the general case when the map $S_I \mapsto T_I$ is an isomorphism.

Our first proposition answers the first question.

Proposition 6.2.1: *Suppose $\mathcal{O}_X = C^*(E_A)$ and E_A is the Perron-Frobenius cover for X . Let $\{S_I\}_{I \in \Sigma}$ generate \mathcal{O}_X . Then the construction done in section 6.1 gives $T_I = S_I$.*

PROOF. Let $\pi : E_A^1 \rightarrow \Sigma$ be the labeling map of the left Krieger Cover, $\{P_i\}_{i=1}^L$ a complete set of minimal projections for A_X , and $A = \sum A^{(I)}$ the Perron-Frobenius

operator, then

$$\begin{aligned}
T_I &= \sum_{e \in \pi^{-1}(I)} t_e \\
&= \sum_{e=a_{i,j}^{(I)}} t_e && \text{Section 3.5} \\
&= \sum_{A^{(I)}(i,j) \neq 0} P_i S_I P_j \\
&= \sum_{i,j=1}^L P_i S_I P_j && \text{by Corollary 3.4.2} \\
&= S_I.
\end{aligned}$$

Thus $\mathcal{L}_X = \mathcal{O}_X = C^*(E_A)$. □

The following lemma's proof is straightforward.

Lemma 6.2.2: *Let $F = (F^0, F^1)$ be a graph with left resolving labeling $\rho : F^1 \rightarrow \Sigma$. Let X be the sofic shift generated by this labeling. Then for each $z \in \mathbb{T}$ the gauge action β_z on $C^*(F)$ is inherited by \mathcal{L}_X and satisfies:*

$$\beta_z(T_I) = zT_I \text{ and } \beta_z(T_\mu^* T_\mu) = T_\mu^* T_\mu.$$

6.3. The \mathcal{O}_X to \mathcal{L}_X Isomorphism

In light of lemma 6.2.2 and theorem 6.1.5, to satisfy the hypotheses of theorem 2.4.6, all it seems that we need is a well defined *-isomorphism between $A_X = C^*(\{S_\mu^* S_\mu : \mu \in B(X)\})$ and $B_X = C^*(\{T_\mu^* T_\mu : \mu \in B(X)\})$ that sends $S_\mu^* S_\mu$ to $T_\mu^* T_\mu$. In general, this could be difficult to guarantee due to many factors; one being that E_A is not always an irreducible graph, even if X an irreducible sofic shift. A reasonable remedy is to ask for the following condition.

Definition 6.3.1: A left resolving labeling of a graph (F, ρ) has the factor property for Σ_{E_A} if and only if there exists an epimorphism (factor map) $\bar{\rho} : Y \rightarrow \Sigma_{E_A}$ such that

1. The following diagram commutes

$$\begin{array}{ccc} \Sigma_{E_A} & \xleftarrow{\bar{\rho}} & Y \\ \pi \downarrow & \swarrow \rho & \\ X & & \end{array} .$$

2. $\bar{\rho}$ is a one-block map.

We shall see when (F, ρ) has the factor property for Σ_{E_A} , we have the following theorem:

Theorem 6.3.2: *Let X is a sofic shift with Perron-Frobenius cover Σ_{E_A} . Suppose further that (F, ρ) is a left resolving labeling of an irreducible graph F with the factor property for Σ_{E_A} , and $\{S_I\}_{I \in \Sigma}$ generates \mathcal{O}_X and $\{T_I\}_{I \in \Sigma}$ generates \mathcal{L}_X . Then $\mathcal{O}_X \cong \mathcal{L}_X$ through the map that sends S_I to T_I .*

To prove theorem 6.3.2 we must show that the map $\theta : A_X \rightarrow B_X : S_\mu^* S_\mu \mapsto T_\mu^* T_\mu$ is a well defined $*$ -isomorphism. To do this, we shall see that both $S_\mu^* S_\mu$ and $T_\mu^* T_\mu$ can be regarded as elements of $C(Y^+)$, the space of continuous functions of Y^+ . Ultimately, we shall show that as functions on this space $S_\mu^* S_\mu$ and $T_\mu^* T_\mu$ are equal. The factor property of (F, ρ) is crucial for constructing such an isomorphism.

Denote by Σ_{E_A} the edge shift generated by the Perron-Frobenius cover E_A ; Y the edge shift generated by the graph F . Let X be the sofic shift presented by these

two graphs. As we know from corollary 6.1.2:

$$(6.7) \quad \begin{aligned} S_\mu^* S_\mu &= \sum_{\substack{\xi \in \pi^{-1}(\mu) \\ r(\xi) = s(f)}} s_f s_f^* \\ T_\mu^* T_\mu &= \sum_{\substack{\zeta \in \rho^{-1}(\mu) \\ r(\zeta) = s(e)}} t_e t_e^* \end{aligned}$$

By [8, Proposition 2.5], the C*-algebra generated by $t_\zeta t_\zeta^*$, $\zeta \in B(Y)$ is isomorphic to $C(Y^+)$, the continuous functions on the one sided space Y^+ . Similarly the C*-algebra generated by $s_\xi s_\xi^*$, $\xi \in B(\Sigma_{E_A})$ is isomorphic to $C(\Sigma_{E_A}^+)$. Equations (6.7) allow us to identify $S_\mu^* S_\mu \in C(\Sigma_{E_A}^+)$, and $T_\mu^* T_\mu \in C(Y^+)$. Clearly

$$(6.8) \quad S_\mu^* S_\mu(z) = \begin{cases} 1 & \text{if there exists } \xi \in \pi^{-1}(\mu), s(z) = r(\xi) \\ 0 & \text{otherwise,} \end{cases}$$

for $z \in \Sigma_{E_A}^+$. Similarly for $y \in Y^+$

$$(6.9) \quad T_\mu^* T_\mu(y) = \begin{cases} 1 & \text{if there exists } \zeta \in \rho^{-1}(\mu), s(y) = r(\zeta) \\ 0 & \text{otherwise.} \end{cases}$$

Remember, that z and y are walks on different graphs, so as similar as equations (6.8) and (6.9) look, they are very different. However we can use the factor property to rectify this.

Since both Y and Σ_{E_A} are left resolving presentations of X , and (F, ρ) has the factor property, the corresponding one sided shifts also must satisfy

$$\begin{array}{ccc} \Sigma_F^+ & \xleftarrow{\bar{\rho}} & Y^+ \\ \pi \downarrow & \swarrow \rho & \\ X^+ & & \end{array}$$

So $\bar{\rho}$, when restricted to the one sided shifts, is also onto (see for example [16, Page 461]). Since Y^+ is a compact, Hausdorff space, $\bar{\rho}$ induces an injection

$$\bar{\rho}^* : C(\Sigma_F^+) \rightarrow C(Y^+).$$

Defined as $\bar{\rho}^*(S_\mu^* S_\mu)(y) = (S_\mu^* S_\mu \circ \bar{\rho})(y)$. This gives the following lemma.

Lemma 6.3.3: *Each non-zero element $S_\mu^* S_\mu \in A_X$ can be regarded as a non-zero function $(\bar{\rho}^*(S_\mu^* S_\mu))$ in $C(Y^+)$ as follows:*

$$(6.10) \quad \bar{\rho}^*(S_\mu^* S_\mu)(y) = \begin{cases} 1 & \text{if there exists } \xi \in \pi^{-1}(\mu), s(\bar{\rho}(y)) = r(\xi) \\ 0 & \text{otherwise} \end{cases}$$

for each $y \in Y^+$.

In light of this, all we need to show is that as functions on $C(Y^+)$, $\bar{\rho}^*(S_\mu^* S_\mu) = T_\mu^* T_\mu$.

Lemma 6.3.4: $\bar{\rho}^*(S_\mu^* S_\mu) = T_\mu^* T_\mu$ when regarded as continuous functions on Y^+ .

PROOF.

$$\begin{aligned} T_\mu^* T_\mu(y) = 1 &\iff \text{there exists } \zeta, \rho(\zeta) = \mu, \text{ and } s(y) = r(\zeta) \\ &\iff \text{there exists } \zeta, \pi \circ \bar{\rho}(\zeta) = \mu, \text{ and } s(y) = r(\zeta). \end{aligned}$$

Let $\xi = \bar{\rho}(\zeta)$. Then $\pi(\xi) = \mu$ and $s(\bar{\rho}(y)) = r(\xi)$. Thus $\bar{\rho}^*(S_\mu^* S_\mu)(y) = 1$.

Now suppose

$$\begin{aligned} T_\mu^* T_\mu(y) = 0 &\iff \text{for all } \zeta \in \rho^{-1}(\mu), s(y) \neq r(\zeta) \\ &\iff \text{for all } \zeta \in (\bar{\rho}^{-1} \circ \pi^{-1})(\mu), s(y) \neq r(\zeta). \end{aligned}$$

As $\bar{\rho}$ and π are onto, it must be that for every $\xi \in \pi^{-1}(\mu)$, $s(\bar{\rho}(y)) \neq r(\xi)$. Which means that $\bar{\rho}^*(S_\mu^* S_\mu)(y) = 0$. □

Because of lemma 6.3.4, we can define $\theta : A_X \rightarrow B_X$ as $\theta(S_\mu^* S_\mu) = T_\mu^* T_\mu$; it is a *-isomorphism, and all the hypotheses of theorem 2.4.6 are satisfied. Thus $\mathcal{O}_X \cong \mathcal{L}_X$ via the map that sends S_I to T_I , and theorem 6.3.2 is proved.

Remark 6.3.5: In the proof of lemma 6.3.4, the fact that $\tilde{\rho}$ is a one-block map has been explicitly used since it is essential for $\tilde{\rho}(\mu)$ to be well defined for any word (block) of arbitrary length. And this subtle point is an essential ingredient for the proof.

6.4. The Factor Property and the Left Krieger Cover

There are cases (see, for example, section 7.4) when one is fortunate enough for $\mathcal{O}_X = C^*(E_A)$ and E_A is also the left Krieger cover for X . If in addition X is irreducible, one can appeal to the theory of the left Krieger cover to show that any irreducible left resolving graph presentation of X will have the factor property for the left Krieger cover.

Lemma 6.4.1: *If X is an irreducible sofic shift with left Krieger cover Σ_L , then any irreducible, left resolving presentation of X ($\rho : Y \rightarrow X$) has the factor property for the left Krieger cover.*

PROOF. The existence of a factor map $\tilde{\rho} : Y \rightarrow \Sigma_L$ follows from, for example [6, Proposition 4]. The proof of [6, Proposition 4] shows that the $\tilde{\rho}$ is an onto, one-block map. □

Because of this, we get a special case of theorem 6.3.2.

Theorem 6.4.2: *Suppose that X is an irreducible sofic shift and $\mathcal{O}_X = C^*(E_A)$ with E_A the left Krieger cover graph of X . Let (F, ρ) be any irreducible, left resolving presentation of X , and let $\mathcal{L}_X \subseteq C^*(F)$ be as before. Then $\mathcal{O}_X \cong \mathcal{L}_X$ through the map that sends S_I to T_I .*

In many cases, \mathcal{O}_X is not equal to the C^* -algebra of its left Krieger cover, but there exists a gauge invariant ideal I with \mathcal{O}_X/I isomorphic to the C^* -algebra of the left Krieger cover graph (theorem 5.2.5). If X is an irreducible sofic shift, theorem 6.4.2 will still hold if one replaces the \mathcal{O}_X in the statement with \mathcal{O}_X/I . One can see this as follows. If we denote by \tilde{S}_μ , $\mu \in B(X)$ as the image of S_μ in \mathcal{O}_X/I , then clearly $\{\tilde{S}_\mu\}_{\mu \in B(X)}$ generates \mathcal{O}_X/I . The relations S_μ has in \mathcal{O}_X will carry over to \mathcal{O}_X/I , and because I is gauge invariant, the gauge action can be used on \mathcal{O}_X/I .

The main difference between \mathcal{O}_X and \mathcal{O}_X/I is that the image of A_X in the quotient has been reduced by a few minimal projections, reducing its dimension. However, one can check that theorem 2.4.6 will hold with \mathcal{O}_X/I and $A_X/(I \cap A_X)$ in place of \mathcal{O}_X and A_X , and \tilde{S}_μ in place of S_μ . By modifying the results regarding \mathcal{O}_X in the first part of this chapter with \mathcal{O}_X/I in place of \mathcal{O}_X , one would obtain theorem 6.4.2 for \mathcal{O}_X/I . We leave these routine verifications as an exercise.

6.5. The Follower Set and the Range Map

Suppose that $\mathcal{O}_X = C^*(E_A)$, and E_A is the left Krieger cover. As mentioned in lemma 2.1.1, the projection $S_\mu^* S_\mu \in A_X$ can be identified with

$$\{x \in X^+ : x, \mu x \in X^+\}.$$

Thus using lemma 1.9.2 $S_\mu^* S_\mu = S_\nu^* S_\nu$ precisely when $F_X(\mu) = F_X(\nu)$.

This allows us to note the following special property in A_X and the graph E_A .

Proposition 6.5.1: *If $\mathcal{O}_X = C^*(E_A)$, and E_A is the left Krieger cover of X then the following are equivalent, for $\mu, \nu \in B(X)$.*

1. $S_\mu^* S_\mu = S_\nu^* S_\nu$,
2. $F_X(\mu) = F_X(\nu)$,
3. $r(\mu) = r(\nu)$ on the labeled graph (E_A, π) .

PROOF. (1) \iff (2) was proved in the discussion preceding the proposition. To show (1) \Rightarrow (3): If (1) holds, both $S_\mu^*S_\mu$ and $S_\nu^*S_\nu$ must be the sum of the same minimal projections, and since the minimal projections in A_X coincide with the vertices in the left Krieger cover, the result follows (equation (6.5)). (3) \Rightarrow (2) follows because if the $r(\mu) = r(\nu)$, then any $x \in X^+$ with $s(x) \in r(\mu) = r(\nu)$ means both μx and $\nu x \in X^+$. Since the left Krieger cover is onto, the result follows. \square

If one decides not to use the left Krieger cover E_A for X , does proposition 6.5.1 still hold for arbitrary irreducible left resolving labelings, with $T_\mu^*T_\mu$ in place of $S_\mu^*S_\mu$? The equivalence of (1) and (2) is just equation (6.5). To show (2) \Rightarrow (3), just note that if $r(\mu) = r(\nu)$ then as E presents X , their follower sets must be equal. Proving (3) \Rightarrow (2), or (3) \Rightarrow (1) is difficult to do directly from graph theory (it can be done with state splittings and left-synchronizing words), but the algebraic machinery developed in this chapter gives us a simple answer.

To show (3) \Rightarrow (1), we see that if $F_X(\mu) = F_X(\nu)$, then $S_\mu^*S_\mu = S_\nu^*S_\nu$; use lemma 6.3.4. Thus we also have:

Proposition 6.5.2: *Suppose $E = (E^0, E^1)$ is an irreducible graph with left resolving label map $\rho : E^1 \rightarrow \Sigma$. Then the following conditions are equivalent:*

1. $T_\mu^*T_\mu = T_\nu^*T_\nu$,
2. $r(\mu) = r(\nu)$ in (E, ρ) .
3. $F_X(\mu) = F_X(\nu)$.

Proposition 6.5.2 tells us there is no way to label an irreducible graph in a left resolving manner such that the follower sets of two words μ, ν are equal, but $r(\mu)$ does not equal $r(\nu)$.

Figure 6.5.1 shows a left resolving presentation of the even shift (example 1.2.3). Note that the graph E is irreducible. Thus the \mathcal{L}_X generated by the labeling is

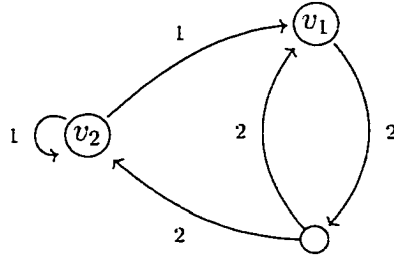


FIGURE 6.5.1. A left resolving presentation of the even shift.

isomorphic to \mathcal{O}_X/I , where I is maximal and gauge invariant. However, \mathcal{L}_X does not generate $C^*(E)$. One can see this because $P_1 = T_1^*T_1 - T_1^*T_1T_{12}^*T_{12}$ is a minimal projection in B_X (as $S_1^*S_1 - S_1^*S_1S_{12}^*S_{12}$ is a minimal projection in A_X), and $P_1 = P_{v_1} + P_{v_2}$. In spite of this, it does happen to be the case that $C^*(E)$ is stably isomorphic to \mathcal{O}_X because E was obtained via an out-splitting of the vertex $P_X(1)$ in the left Krieger Cover, and thus as (two-sided) edge shift spaces, they are conjugate (see [16, Chapter 2] for more on state splittings).

Remark 6.5.3: For reducible sofic, uniqueness of the left Krieger cover fails. A counter-example, due to N. Jonoska, can be found in [16, Example 3.3.21]. This example is for right resolving presentations, but also holds for left resolving by duality. Thus, there is no guarantee that any left resolving presentation of a reducible sofic shift will factor through the left Krieger cover. So although one could construct an \mathcal{L}_X for any left resolving presentation of a reducible sofic shift X , the techniques used in this chapter could not be carried over to find an isomorphism between \mathcal{L}_X and \mathcal{O}_X .

CHAPTER 7

Examples of \mathcal{O}_X for Sofic X

In this chapter, we present some examples that demonstrate the results from previous chapters. We focus first on the AFT's presented in chapter 1, and then give an example of a non-AFT where the past set cover and left Krieger cover are the same. In the last section, we show a counter example to the converse of proposition 5.4.4 as mentioned in remark 5.4.6.

We remind the reader that when a Perron-Frobenius operator is presented as a matrix in these examples, its transpose is used as a map on K-theory. This was done to make the K-theory formulae consistent with the Cuntz-Krieger algebras, and furthermore, it allows us to view the matrix as the vertex matrix of the past cover. I.e., there are n edges from vertex i to vertex j if and only if the (i, j) th entry of the matrix equals n .

7.1. The Even shift

We let X be the even shift (a degree zero AFT—see chapter 1) on two symbols defined in example 1.2.3. The follower set graph, and the right Krieger cover were shown in figures 1.6.1 and 1.6.2 respectively. The follower sets are shown below.

$$C_2 = F_X(2) = X^+$$

$$C_1 = F_X(1) = \{2_{\text{even}}1-, 22222\dots\}$$

$$C_{12} = F_X(12) = \{2_{\text{odd}}1-, 22222\dots\}$$

Here $2_{\text{even}}1-$ denotes a word with an even number of 2's, then a 1, followed by anything legal; $2_{\text{odd}}1-$ is defined similarly.

We now look at Matsumoto's construction done at the end of [18, Section 8]. He shows that:

$$S_{\mu}^* S_{\mu} = \begin{cases} 1 & \text{if } \mu = (2 \dots 2), \\ S_1^* S_1 & \text{if } \mu = (* \dots * 12_{\text{even}}), \\ S_2^* S_1^* S_1 S_2 & \text{if } \mu = (* \dots * 12_{\text{odd}}) \end{cases}$$

Here 2_{even} denotes an even block of 2's (zero is even), and 2_{odd} an odd number of two's.

To understand exactly how A_X looks takes a bit more work. We have to make the following modifications. To find the follower set equivalence classes we note we have two distinct magic words $\mu_1 = 1$, $\mu_2 = 12$, and they have a non trivial intersection. Thus we get.

$$(7.1) \quad \begin{aligned} P(\mu_1) &= S_1^* S_1 - P(\mu_1, \mu_2) \\ P(\mu_2) &= S_2^* S_1^* S_1 S_2 - P(\mu_1, \mu_2) \\ P(\mu_1, \mu_2) &= S_1^* S_1 \cdot S_2^* S_1^* S_1 S_2 \end{aligned}$$

Using the following order (which is the order explained in section 5.3), we define:

$$(7.2) \quad P_1 = P(\mu_1) \quad P_2 = P(\mu_2) \quad P_3 = P(\mu_1, \mu_2)$$

then $P_1 + P_2 + P_3 = 1$ and $A_X = CP_1 \oplus CP_2 \oplus CP_3$ (Note that this reverses P_2 and P_3 from [18]).

The key to finding out what the Perron-Frobenius operator will be as a matrix is to find out how the following map works:

$$a \mapsto \sum_{i=1}^n S_i^* a S_i.$$

So we have to see what this map does on our three projections. We will show the calculation for one projection. We will see what 'trick' has to be employed.

We calculate $S_1^*P_1S_1$. Note first that

$$\begin{aligned}
 S_1^*P(\mu_1)S_1 &= S_1^*S_1^*S_1S_1 - S_1^*P(\mu_1, \mu_2)S_1 \\
 &= S_1^*S_1 - 0 \\
 (7.3) \qquad &= [S_1^*S_1 - P(\mu_1, \mu_2)] + P(\mu_1, \mu_2) \\
 &= P(\mu_1) + P(\mu_1, \mu_2)
 \end{aligned}$$

Thus, (7.3) gives us

$$S_1^*P_1S_1 = P_1 + P_3.$$

We can do a similar calculation for $S_1^*P_2S_1$ and see it equals 0. We also have $S_1^*P_3S_1 = 0$. Thus we see that

$$(7.4) \qquad A^{(1)} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we do the same thing for $S_2^*P_kS_2$, $1 \leq k \leq 3$ and get the following:

$$\begin{aligned}
 S_2^*P_1S_2 &= P_2 \\
 (7.5) \qquad S_2^*P_2S_2 &= P_1 \\
 S_2^*P_3S_2 &= P_3
 \end{aligned}$$

From (7.5) we see that

$$(7.6) \qquad A^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

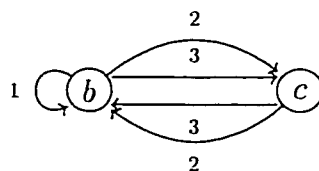


FIGURE 7.1.1. A modified even shift

The Perron-Frobenius operator as a matrix is defined as

$$\sum_{I=1}^2 A^{(I)};$$

adding (7.4) and (7.6), we get

$$(7.7) \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know by proposition 5.1.2 A is the adjacency matrix for the past set presentation (this can be verified directly). Note the top 2×2 left corner is the adjacency matrix for the left Krieger cover shown in figure 1.6.2. The algebra \mathcal{O}_X is not simple in this case, and the gauge invariant ideal generated by P_3 can be modded out leaving us with just the top 2×2 corner.

The matrix A does not satisfy condition (I) of [8]. We can see this also because the projection $P_3(x) \neq 0$ if and only if $x = 22222 \dots$. By modifying the even shift as in figure 7.1.1, we would have then that P_3 corresponded to non-periodic points (made up of combinations of 2's and 3's), and the Perron-Frobenius operator will satisfy condition (I). Although still non-simple (it is still an AFT), it is purely infinite.

7.2. The Degree 3 Charge Constrained Shift

The previous section shows a simple case. Here we present an AFT where the calculations get foul, so most will be skipped. The DC3-shift is explained in example 1.3.3, and its Krieger Cover was shown in Figure 1.6.3. The complete set of magic words for this shift are:

$$\begin{aligned}\mu_1 &= -1 - 1 - 1 & \mu_2 &= -1 - 1 - 1 + 1 \\ \mu_3 &= +1 + 1 + 1 - 1 & \mu_4 &= +1 + 1 + 1\end{aligned}$$

If we were going to do this like the calculations of the previous section, we would start at the bottom, and work up. We define:

$$P(\mu_1, \mu_2, \mu_3, \mu_4) = \prod_{i=1}^4 S_{\mu_i}^* S_{\mu_i} = 0$$

Hence we can ignore it. The next non-trivial projections are:

$$(7.8) \quad \begin{aligned}P(\mu_1, \mu_2, \mu_3) &= \prod_{i=1}^3 S_{\mu_i}^* S_{\mu_i} \\ P(\mu_2, \mu_3, \mu_4) &= \prod_{i=2}^4 S_{\mu_i}^* S_{\mu_i}\end{aligned}$$

Note that we have to subtract $P(\mu_1, \dots, \mu_4)$ from the equations in (7.8) but since it is trivial, we can ignore it.

Now we define the 2-way projections as follows

$$(7.9) \quad \begin{aligned}P(\mu_1, \mu_2) &= S_{\mu_1}^* S_{\mu_1} S_{\mu_2}^* S_{\mu_2} - P(\mu_1, \mu_2, \mu_3) \\ P(\mu_1, \mu_3) &= S_{\mu_1}^* S_{\mu_1} S_{\mu_3}^* S_{\mu_3} - P(\mu_1, \mu_2, \mu_3) = 0!! \\ P(\mu_2, \mu_3) &= S_{\mu_2}^* S_{\mu_2} S_{\mu_3}^* S_{\mu_3} - P(\mu_1, \mu_2, \mu_3) - P(\mu_2, \mu_3, \mu_4) \\ P(\mu_2, \mu_4) &= S_{\mu_2}^* S_{\mu_2} S_{\mu_4}^* S_{\mu_4} - P(\mu_2, \mu_3, \mu_4) = 0!! \\ P(\mu_3, \mu_4) &= S_{\mu_3}^* S_{\mu_3} S_{\mu_4}^* S_{\mu_4} - P(\mu_2, \mu_3, \mu_4)\end{aligned}$$

Two of these are zero because $F_X(\mu_1) \cap F_X(\mu_3)$ is actually equal to the three way intersection $F_X(\mu_1) \cap F_X(\mu_2) \cap F_X(\mu_3)$, and similarly for the other one. However, if we use the follower set equivalence relation of chapter 5 we see that $s(+1-1+1-1\dots) = \{F_X(\mu_i)\}_{i=1}^3$ and $s(-1+1-1+1\dots) = \{F_X(\mu_i)\}_{i=2}^4$. Thus, using the values of $s(x)$ will get the largest collection of follower sets of which x is in the intersection.

Finally, we get the projections for the follower sets. We have

$$\begin{aligned}
 P(\mu_1) &= S_{\mu_1}^* S_{\mu_1} - P(\mu_1, \mu_2) - P(\mu_1, \mu_2, \mu_3) \\
 P(\mu_2) &= S_{\mu_2}^* S_{\mu_2} - P(\mu_1, \mu_2) - P(\mu_2, \mu_3) \\
 &\quad - P(\mu_1, \mu_2, \mu_3) - P(\mu_2, \mu_3, \mu_4) \\
 P(\mu_3) &= S_{\mu_3}^* S_{\mu_3} - P(\mu_2, \mu_3) - P(\mu_3, \mu_4) \\
 &\quad - P(\mu_1, \mu_2, \mu_3) - P(\mu_2, \mu_3, \mu_4) \\
 P(\mu_4) &= S_{\mu_4}^* S_{\mu_4} - P(\mu_3, \mu_4) - P(\mu_2, \mu_3, \mu_4)
 \end{aligned}
 \tag{7.10}$$

That's them all!! Nine orthogonal projections (when one discounts the two that came up trivial). To do the calculation of the Perron-Frobenius operator, one also would start from the bottom up. We will do it for the +1, symbol. The -1 symbol is similar.

Let's first do the calculation for (7.8). We have

$$(7.11) \quad S_{+1}^* P(\mu_1, \mu_2, \mu_3) S_{+1} = \prod_{i=1}^3 S_{+1}^* S_{\mu_i}^* S_{\mu_i} S_{+1}$$

$$(7.12) \quad = \prod_{i=2}^4 S_{\mu_i}^* S_{\mu_i}$$

$$(7.13) \quad = P(\mu_2, \mu_3, \mu_4)$$

$$(7.14) \quad S_{+1}^* P(\mu_2, \mu_3, \mu_4) S_{+1} = \prod_{i=2}^4 S_{+1}^* S_{\mu_i}^* S_{\mu_i} S_{+1}$$

$$(7.15) \quad = 0$$

Next we do the calculations for (7.9). We will do one in detail, the others are similar. We make use of equations (7.13), (7.15).

$$(7.16) \quad S_{+1}^* P(\mu_1, \mu_2) S_{+1} = S_{\mu_2}^* S_{\mu_2} S_{\mu_3}^* S_{\mu_3} - S_{+1}^* P(\mu_1, \mu_2, \mu_3) S_{+1}$$

$$(7.17) \quad = S_{\mu_2}^* S_{\mu_2} S_{\mu_3}^* S_{\mu_3} + P(\mu_2, \mu_3, \mu_4) \\ - P(\mu_1, \mu_2, \mu_3) + P(\mu_1, \mu_2, \mu_3)$$

$$(7.18) \quad = P(\mu_2, \mu_3) + P(\mu_2, \mu_3, \mu_4)$$

$$(7.19) \quad S_{+1}^* P(\mu_2, \mu_3) S_{+1} = P(\mu_3, \mu_4)$$

$$(7.20) \quad S_{+1}^* P(\mu_3, \mu_4) S_{+1} = 0$$

Note that for (7.18) we had to employ a similar trick as in (7.3); we had to add then subtract appropriate projections (in this case $P(\mu_1, \mu_2, \mu_3)$), in order to get the equality in terms of other projections.

Now we will do the calculations for (7.10). They are similar to the above. so we won't do them in detail. We rely on the calculations from (7.13) to (7.20).

$$(7.21) \quad S_{+1}^* P(\mu_1) S_{+1} = P(\mu_2) + P(\mu_1, \mu_2) + P(\mu_1, \mu_2, \mu_3)$$

$$(7.22) \quad S_{+1}^* P(\mu_2) S_{+1} = P(\mu_3) + P(\mu_1, \mu_2, \mu_3)$$

$$(7.23) \quad S_{+1}^* P(\mu_3) S_{+1} = P(\mu_4)$$

$$(7.24) \quad S_{+1}^* P(\mu_4) S_{+1} = 0$$

Now with the following order

$$P_1 = P(\mu_1) \quad P_2 = P(\mu_2) \quad P_3 = P(\mu_3) \quad P_4 = P(\mu_4)$$

$$P_5 = P(\mu_1, \mu_2) \quad P_6 = P(\mu_2, \mu_3) \quad P_7 = P(\mu_3, \mu_4)$$

$$P_8 = P(\mu_1, \mu_2, \mu_3) \quad P_9 = P(\mu_2, \mu_3, \mu_4),$$

and calculations (7.13) to (7.24), one gets that $A^{(+1)}$ is the following 9×9 matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots\dots\dots & & & \\ 0 & \dots\dots\dots & & & & & & & \\ 0 & \dots\dots\dots & & & 1 & 0 & 1 & 0 & \\ 0 & \dots\dots\dots & & & 0 & 1 & 0 & 0 & \\ 0 & \dots\dots\dots & & & & & & & \\ 0 & \dots\dots\dots & & & & & & 1 & 0 \\ 0 & \dots\dots\dots & & & & & & & \end{bmatrix}$$

AF-algebra $\mathcal{F}_X^\infty / (I \cap \mathcal{F}_X^\infty)$ shows that the algebra $\mathcal{F}_X^\infty / (I \cap \mathcal{F}_X^\infty)$ is non-simple. but \mathcal{O}_X / I is, and is also purely infinite by theorems 3.2.4, 3.4.4, and 4.2.3.

7.3. The AFT of Figure 1.5.2

Our final AFT example is the sofic shift presented in figure 1.5.2. Although the graph is not left resolving, it is left closing (any word of length 2 or more that ends at the same vertex and has the same label has the same terminal edge). If $a = F_X(12)$, $b = F_X(01)$ and $c = F_X(0)$. One can check for each $x \in X^+$, $s(x)$ is one of the sets $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{a, b, c\}$. Thus we have the following minimal projections:

$$\begin{array}{lll}
 P_1 = P(a) & P_2 = P(b) & P_3 = P(c) \\
 P_4 = P(\{a, b\}) & P_5 = P(\{a, c\}) & P_6 = P(\{a, b, c\}).
 \end{array}$$

In this case, the intersection of follower sets b and c is the same as the intersection of a, b , and c .

One can check that for each symbol, the Perron-Frobenius operator is

$$A^{(0)} = \begin{bmatrix} 0 & \dots\dots\dots \\ 0 & \dots\dots\dots \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & \dots\dots\dots \\ 0 & \dots\dots\dots \\ 0 & \dots\dots\dots \end{bmatrix},$$

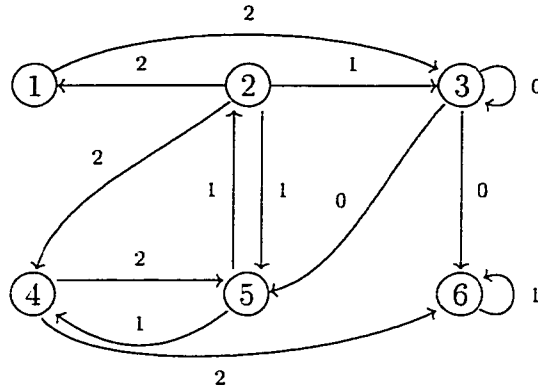


FIGURE 7.3.1. The Perron-Frobenius presentation of a degree 2 AFT

$$A^{(1)} = \begin{bmatrix} 0 & \dots\dots\dots & & & & & \\ 0 & 0 & 1 & 0 & 1 & 0 & \\ 0 & \dots\dots\dots & & & & & \\ 0 & \dots\dots\dots & & & & & \\ 0 & 1 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 1 & 0 & 0 & \\ 0 & \dots\dots\dots & & & & & \\ 0 & 0 & 0 & 0 & 1 & 1 & \\ 0 & \dots\dots\dots & & & & & \\ 0 & \dots\dots\dots & & & & & \end{bmatrix},$$

and the Perron-Frobenius operator is obtained by summing the above three matrices. As a labeled graph it looks like figure 7.3.1.

Because it is an AFT, it has a reducible part. By removing vertex 6 and all edges entering and leaving it, we are left with an irreducible graph. As a matrix, A will not satisfy condition (I), but once vertex 6 is removed, condition (I) is trivially satisfied.

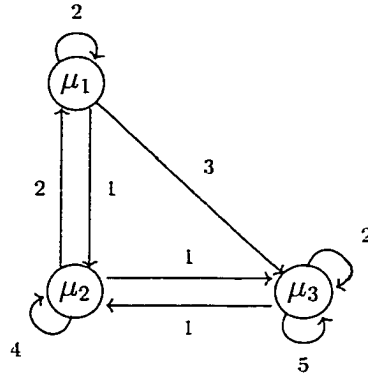


FIGURE 7.4.1. A non-AFT presentation

Figure 7.3.1 is the past set presentation of X , and with vertex 6 removed, is the left Krieger cover (propositions 5.1.2, and 5.2.2). Note how much different it looks from the right Krieger Cover (figure 1.5.2). However, the underlying edge shifts (obtained by ignoring the labels on the vertices) of figure 1.5.2 and the irreducible part of 7.3.1 are conjugate as shift spaces.

One other way of noticing that X has a reducible part is to note that P_6 only corresponds to the periodic point $111\dots$. Hence, the Perron-Frobenius operator cannot satisfy condition (I).

7.4. A Non-AFT

Consider the sofic shift X presented by figure 7.4.1. One can verify that figure 7.4.1 is not an AFT (there is no left closing property on left infinite strings of 2's). We shall see this also because when we obtain the past set cover using the methods of section 5.3 and proposition 5.1.2, we shall see that it is irreducible, hence the left Krieger cover.

We will not go into the details of calculating the Perron-Frobenius operator in this example. Rather, we shall investigate the follower set equivalence classes of X .

We will see right away that \mathcal{O}_X is a unique.

First, one can check by looking at the graph

$$s(1111\dots) = \{F_X(\mu_i)\}_{i=1}^3.$$

Thus

$$S_1 = \bigcap_{i=1}^3 F_X(\mu_i) = \text{the full 2-shift on symbols 1 and 2.}$$

The other sets that are sources of some $x \in X^+$ include $\{F_X(\mu_1), F_X(\mu_3)\} = s(23222\dots)$, and:

$$F_X(\mu_1) \cap F_X(\mu_2) = S_1 \cup 23F_X(\mu_3)$$

where $23F_X(\mu_3)$ denotes all $x \in X^+$ with the property that $x = 23x_+$ and $x_+ \in F_X(\mu_3)$. Furthermore $\{F_X(\mu_1), F_X(\mu_3)\} = s(144\dots)$, and:

$$F_X(\mu_1) \cap F_X(\mu_3) = S_1 \cup 1F_X(\mu_2).$$

And since no follower set is a subset of another (each vertex has a label exiting it that is exclusive to that vertex), the final sources are the sets $\{F_X(\mu_1)\}$, $\{F_X(\mu_2)\}$, and $\{F_X(\mu_3)\}$. As each of these follower set equivalence classes corresponds to non-periodic points, we can immediately conclude by theorem 3.7.2 that \mathcal{O}_X satisfies condition (I_X) , the Perron-Frobenius operator satisfies condition (I) , and \mathcal{O}_X is purely infinite.

With a bit of sweat, one can calculate the Perron-Frobenius operator as

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

The fact that A has 2's comes from the fact that there are two different symbols that conjugate a minimal projection to the same minimal projection. For example, P_3 has both $S_2^*P_3S_2 \geq P_3$ and $S_5^*P_3S_5 \geq P_3$. One can easily see this by looking at figure 7.4.1, and noting P_3 corresponds to the vertex μ_3 . This matrix is irreducible, and thus its edge shift must be the left Krieger cover of the sofic shift X .

7.5. The Failure of the Converse of Proposition 5.4.4

In light of the examples, and “seeing how its done” we can exhibit a counterexample to proposition 5.4.4 referred to in remark 5.4.6. Essentially, this can be achieved by taking an irreducible AFT, an irreducible non-AFT, and connecting them in an irreducible way. For example, consider the sofic shift X presented by figure 7.5.1. All that has been done is we have taken the even shift (and relabeled it), and tacked it to the non-AFT of figure 7.4.1.

Clearly, figure 7.5.1 is the follower set presentation of X . We still must have $s(BBB\dots) = \{a, b\}$, and only the periodic point $BBBB\dots$ in the intersection of $F_X(a)$ and $F_X(b)$. Thus, the Perron-Frobenius operator A will not satisfy condition (I), and is also reducible (it is definitely not a permutation matrix!).

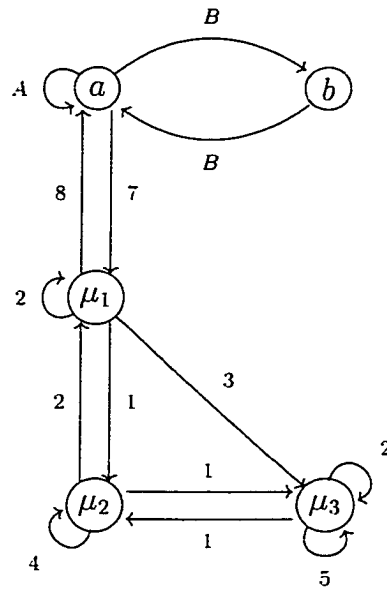


FIGURE 7.5.1. A counter-example to proposition 5.4.4

Bibliography

1. S. Adji, M. Laca, M. Nilsen, and I. Raeburn, *Crossed products by semigroups of endomorphisms and the toeplitz algebras of ordered groups*, Proc. Amer. Math. Soc. **122** (1994), 1133–1141.
2. A. an Huef and I. Raeburn, *The ideal structure of cuntz-krieger algebras*, Ergod. Th. & Dynam. Sys. **17** (1997), 611–624.
3. C Anantharaman-Delaroche, *Purely infinite c^* -algebras arising from dynamical systems*, Bull. Soc. Math. France **125** (1997), 199–225.
4. T. Bates, *Title pending*, PhD dissertation, University of Newcastle, Department of Mathematics, 1998, In Preparation.
5. S. Boyd, N. Keswani, and I. Raeburn, *Faithful representations of crossed products by endomorphisms*, Proc. Amer. Math. Soc. **118** (1993), 427–436.
6. M Boyle, B. Kitchens and B. Markus, *A note on minimal covers of sofic systems*, Proc. Amer. Math. Soc. **95** (1985), no. 3, 403–411.
7. J. Cuntz, *A class of c^* -algebras and topological chains ii: Reducible chains and the ext functor for c^* -algebras*, Invent. Math. **63** (1981), 25–40.
8. J. Cuntz and W. Krieger, *A class of c^* -algebras and topological markov chains*, Invent. Math. **51** (1980), 251–268.
9. R. Exel and M. Laca, *Cuntz-krieger algebras for infinite matrices*, Preprint.
10. R. Fischer, *Sofic systems and graphs*, Monats. Math. **80** (1975), 179–186.
11. Y. Katayama, K. Matsumoto and I. Watatani, *Simple c^* -algebras arising from β -expansion of real numbers*, preprint.

12. E. Kirchberg, *The classification of purely infinite c^* -algebras using kasparov's theory*, preprint.
13. W Krieger, *Sofic systems i*, Israel Journal of Math. **48** (1984), 305–330.
14. A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids and cuntz-krieger algebras*, Journal of Functional Analysis **144** (1997), 505–541.
15. A. Kumjian, D. Pask and I. Raeburn, *Cuntz-krieger algebras of directed graphs*. Pacific J. Math. (to appear).
16. D. Lind and B. Marcus, *Symbolic dynamics and coding*, Cambridge University Press, New York, 1995.
17. B. Markus, *Sofic systems and encoding data*, IEEE-Information Theory **31** (1985), 366–377.
18. K. Matsumoto, *On c^* -algebras associated with subshifts*, Internat. J. Math **8** (1997), 357–374.
19. ———, *Dimension groups for subshifts and simplicity for the corresponding c^* -algebras*, J. Math. Soc. Japan (in press).
20. ———, *K-theory for c^* -algebras associated with subshifts*, Math. Scand. (in press).
21. N.C. Phillips, *A classification theorem for nuclear purely infinite simple c^* -algebras*, preprint.
22. I. Putnam, *C^* -algebras from smale spaces*, Can. J. Math. **48** (1995), 175–195.
23. J Renault, *A groupoid approach to c^* -algebras*, Springer-Verlag, Berlin, Hiedelberg, New York, 1980.
24. K. Thomsen, *The defect of factor maps*, Ergod. Th. and Dynam. Sys. **17** (1997), 1233–1256.
25. B. Weiss, *Subshifts of finite type and sofic systems*, Monats. Math.. **77** (1973), 462–474.