Balancing Polynomials Under Permutations

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Introduction: Polynomials and Their Permutations

A *polynomial* is a function of the variables $x_1, x_2, \ldots, x_k$ involving only the operations of addition, subtraction, multiplication, and raising the variables to positive integer powers. For example,

$$P(x,y,z) = 3x^2y + 4xyz + 7z + 1$$

is a polynomial in three variables.

A *permutation* of a polynomial is a function which "rearranges" the variables involved. As an example, all six possible permutations of the following polynomial in three variables are given below. Note the considerable amount of symmetry in the variables.

$$\begin{aligned} P(x,y,z) &= 2xyz + xy + 4z + x \\ P(z,x,y) &= 2xyz + xz + 4y + z \\ P(y,z,x) &= 2xyz + yz + 4x + y \\ P(x,z,y) &= 2xyz + xz + 4y + x \\ P(z,y,x) &= 2xyz + yz + 4x + z \\ P(y,x,z) &= 2xyz + yz + 4x + y \end{aligned}$$

Definitions

A polynomial is said to be *symmetric* if it is invariant under permutation. That is, the polynomial is perfectly symmetric in each of its variables. For example, the polynomial $P(x,y,z) = xyz + x + y + z$ is symmetric as all its permutations are identically equal.

Loosely speaking, the *balancing number* of a polynomial P, denoted $\mathrm{Bal}(P)$, is the smallest number of permuted copies of itself we could add together to obtain a symmetric polynomial. This is perhaps best illustrated with an example.

Consider the polynomial $P(x,y,z) = 2xy + yz$. Then,

$$\begin{aligned} P(x,y,z) + P(y,z,x) - P(z,y,x) = \; & (2xy + yz) \\ & + (2yz + xz) \\ & - (2yz + xy) \\ & = xy + xz + yz \end{aligned}$$

Since this is now symmetric, we simply add the coefficients for each of the permuted copies of P to find that $\mathrm{Bal}(P) = 1$. An immediate result of the above definition is the following proposition.

Proposition 1. *If the polynomial P is symmetric, then $Bal(P) = 1$.*

This result is clear - if P is already symmetric, then we only need one permuted copy to obtain a symmetric polynomial! As seen in the above example, the converse is **not** true in general for polynomials in higher numbers of variables. However, for a polynomial in only two variables we do have the converse as well.

Proposition 2. *If P is a bivariate polynomial, then $Bal(P) = 1$ if and only if P is symmetric. If P isn't symmetric, then $Bal(P) = 2$.*

The latter half holds because there only exist two permutations of a bivariate polynomial, so the worst we can do is add them.

Previous Results: Balancing Matrices

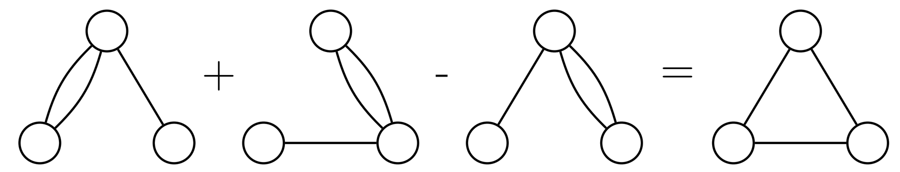
Balancing numbers of $n \times n$ matrices have previously been studied by del Valle and Dukes in [1]. We briefly outline their results below.

The balancing number of a matrix is defined similarly to that of a polynomial. It is the smallest possible number of permuted copies of a matrix which sum to a *completely symmetric matrix* - that is, with constant diagonal and constant off-diagonal entries.

We give a simple example. Take the following matrix. Then we have

$$\begin{bmatrix} 0&2&0\\2&0&1\\0&1&0 \end{bmatrix} + \begin{bmatrix} 0&0&1\\0&0&2\\1&2&0 \end{bmatrix} - \begin{bmatrix} 0&1&0\\1&0&2\\0&2&0 \end{bmatrix} = \begin{bmatrix} 0&1&1\\1&0&1\\1&1&0 \end{bmatrix}$$

which is completely symmetric, so $\mathrm{Bal}(A) = 1$. We can also visualize this differently:



Now consider an abitrary 3×3 matrix. It must have balancing number $1, 2, 3$, or 6. Thus, we can fully characterize the balancing numbers of the 3×3 matrices using the following results.

Lemma 1. *$Bal(A)$ is even if and only if $A_{12} + A_{23} + A_{31} \neq A_{21} + A_{32} + A_{13}$.*

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

This result connects the broken diagonal sums to the parity of the balancing number. We can see that if these sums are the same, then "cycling" along these broken diagonals would completely balance the matrix using only 3 copies.

We now introduce a new concept which will help further characterize the balancing number in the 3×3 case. A *ternary triple* is a set of 3 integers a, b, c such that the matrix with a, b, and c along the diagonal has balancing number 1. For example, the triple $(5, 8, 11)$ is a ternary triple as we can balance the matrix as follows:

$$\begin{bmatrix} 5&0&0\\0&8&0\\0&0&11 \end{bmatrix} + \begin{bmatrix} 8&0&0\\0&11&0\\0&0&5 \end{bmatrix} - \begin{bmatrix} 5&0&0\\0&11&0\\0&0&8 \end{bmatrix} = \begin{bmatrix} 8&0&0\\0&8&0\\0&0&8 \end{bmatrix}$$

When expressed in base 3, these numbers are all different in the first digit from the right where any of them differ. Expressed in base 3, the above triple becomes $(12, 22, 102)$. However, we can see that $(13, 16, 22)$ is not a ternary triple as it becomes $(111, 121, 211)$ in base 3. These numbers differ in the 2nd digit, but they're not all different! Now we can handle the remainder of the characterization.

Lemma 2. *Define $\Theta(A)$ by the following:*

$$\Theta(A) = \begin{bmatrix} 1 & 1 & 1 \\ A_{11} & A_{22} & A_{33} \\ d_1^+ & d_2^+ & d_3^+ \\ d_1^- & d_2^- & d_3^- \end{bmatrix}$$

Where d_i^+ is the sum of the entries in row i and d_j^- is the sum of the entries in column j. $Bal(A) \leq 2$ if and only if the following hold

1. $\Theta(A)$ does not have full rank,

2. All the rows of $\Theta(A)$ are ternary triples.

Combining this condition to have a small balancing number with the above parity condition allows a full characterization of the 3×3 matrices.

Results

Balancing Polynomials in 3 Variables

We generalize the results for matrices to polynomials. We present the results below.

The parity condition remains quite similar.

Lemma 3. *If f is a polynomial in x_1, x_2, and x_3, then we have that $f(x_1, x_2, x_3) = \sum a_{ijk} x_1^i x_2^j x_3^k$. $Bal(f)$ is even if and only if*

$$a_{ijk} + a_{kij} + a_{jki} \neq a_{jik} + a_{ikj} + a_{kji}$$

for all i, j, k distinct.

The matrix used in the second condition must be changed quite significantly. We construct $\Theta(f)$ by including rows which correspond to each of the different term shapes. For example:

$$f(x,y,z) = 2xy + yz + 2y^2 + z^2 \quad \longrightarrow \quad \Theta(f) = \begin{bmatrix} 1&1&1\\0&2&1\\1&0&2 \end{bmatrix}$$

Lemma 4. *$Bal(f) \leq 2$ if and only if the following two conditions hold:*

1. The matrix Θ does not have full rank,

2. All rows of $\Theta(f)$ are ternary triples.

The intuition for why this lemma holds is very similar to the previously stated version for matrices, with some small differences. The ternary triple condition ensures that each shape of term can be balanced easily, while the other condition still simply ensures that the polynomial is redundant enough to be balanced using very few copies of itself.

Thus, we see that the 3 variable polynomials can be characterized very similarly to the 3×3 matrices.

Future Work

It's possible to view polynomials as superpositions of multiple matrices when our terms are not too complicated. For example, we could represent the polynomial $f(x,y,z) = 2xy + 3xz + 2x^2 + y^2 + 4z^2$ as the following two matrices:

$$\begin{bmatrix} 2&0&0\\0&1&0\\0&0&4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0&2&3\\2&0&0\\3&0&0 \end{bmatrix}$$

Extending this idea to allow for more complicated terms could help characterize the balancing numbers for polynomials in higher numbers of variables.

References

[1] Coen del Valle and Peter Dukes, *Balancing permuted copies of multigraphs and integer matrices*, J. Comb. Theory Series A, 198, (2023), 105756.