A STUDY OF SOME NEW SOLUTIONS OF EINSTEIN AND EINSTEIN-MAXWELL EQUATIONS

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ABSTRACT

In this thesis we deal with static axially symmetric gravitational fields in vacuum and with static axially symmetric electrovacuum. The formalism of Weyl-Levi-Civita has been employed for obtaining solutions of Einstein's and Einstein-Maxwell equations. A solution representing the exterior field of a Curzon particle in combination with a general line mass is obtained. Through a suitable formalism we generate the charged metric representing a charged Curzon particle and a charged general line mass. We also examine some properties of the Bach and Weyl metric. Further we derive solutions of Einstein field equations representing point sources exhibiting multipole structure. Special cases of balance between multipole point sources in the general theory of relativity are also examined.

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TABLE OF CONTENTS

LIST OF FIGURES

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INTRODUCTION

One of the main features of Einstein's theory of gravitation is the geometrization of the gravitational field. Postulating an equivalence between gravitational fields and non-inertial systems of references, Einstein showed that the gravitational fields can be described by ten functions of the space-time coordinates. As we will see, these functions are the components of the metric tensor of the space-time which contains the gravitational fields. We should remember that the metric tensor is an object that determines the geometrical properties of a given space.

Furthermore, the components of the metric tensor and the sources of the gravitational field are connected through one of the most complicated systems of differential equations, the well known Einstein field equations. Despite the hard and persistent efforts of physicists and mathematicians, exact solutions of Einstein's field equations are not obtained very often. One way of getting solutions of Einstein's field equations is to examine special cases of gravitational fields, i.e. fields possessing various symmetries or fields produced by simple distributions of matter.

So far the concept of symmetry has played an important role in the formulation and simplification of various physical problems. Maximal exploitation of the symmetries, whenever they are present leads to beautiful and understandable results. H. Weyl (1952) noticed the following about the concept of symmetry:

> "Symmetry, as wide or as narrow as you may define it's meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection."

The first application of the concept of symmetry in the Einstein theory of gravitation has been done by K. Schwarzschild (1916) who obtained an exact solution of Einstein's field equations. He found the spherically (and, unique) symmetric vacuum solution of Einstein's field equations. Weyl (1917-1919) and Levi-Civita had been working on axially symmetric gravitational fields. As we will see, gravitational fields possessing axial symmetry lead to a new class of solutions of Einstein's field equations which contain the Schwarzschild solution as a special case.

This thesis deals with static axially symmetric gravitational fields. We will find and study some new static solutions of the Einstein and Einstein-Maxwell equations.

In chapter 2, a brief development of the special and general theory of relativity is discussed. In chapter 3 we define more precisely the static axially symmetric gravitational field. We will see that the concept of axial symmetry enables us to find the general form of a line element for a static axially symmetric gravitational field. Further, using Einstein's field equations it was proved that the static axially symmetric gravitational field in vacuum is described by only two functions λ and ν where λ satisfies Laplace's equation in a three dimensional flat space time (recall that the Newtonian gravitational potential satisfies the same equation) while the ν function is found through quadratures.

Hence, solutions of Laplace's equation can generate static axially symmetric solutions of Einstein's field equations. In chapter 4 we apply the formalism which is developed in chapter 3 and we obtain static solutions of Einstein's field equations. In section 4.1 we will see that the Newtonian potential of a point spherically symmetric source leads to a solution which exhibits a multipole structure while in 4.2 we prove that the Newtonian potential of a uniform line mass density with a special length, leads to the Schwarzschild solution.

So from those two examples we see that there is a mathematical one-to-one correspondence between the solutions of Laplace's equation and the solutions of Einstein's equations. However there is not a physical one as long as spherically symmetric solutions of the Laplace equation do not generate the spherically symmetric solutions of Einstein's equations.

Bach and Weyl (1922) found the solution which is generated by the gravitational potentials of two uniform density line masses, while Israel and Khan (1964), generalizing their work, obtained the solution which is generated by N uniform density line masses.

Curzon (1924) obtained a class of solutions which is generated by the gravitational potential of point masses. We consider it natural to find and study the solution of Einstein's field equations which is generated by the Newtonian potential of a point mass and a line mass separated by a distance different from zero. The properties of the above new solution of Einstein's field equations are examined. Applying the criterion of asymptotic flatness we found that the metric reduces to the Galilean one at spatial infinity. The problem of the regularity of the metric which is manifested by the breakdown of the elementary flatness is analyzed and a qualitative explanation is given.

Finally the horizon of the Bach and Weyl metric is examined. Certainly according to Israel's theorem (1967) the horizon of the above mentioned metric is singular. However, examining the horizon as the parameter which represents the separation distance between the two line masses goes to zero, we found that the horizon can be nonsingular depending on the special densities of the two-line masses.

In chapter 5 we briefly developed the Einstein-Maxwell equations which describe the gravitational field in the presence of the electromagnetic field. We know that Coulomb's law and Newton's law are identical apart from a sign. Hence in classical physics we can achieve balance between given distributions of matter when the charges and the masses are suitably chosen. Authors have examined the problem of balance in the theory of relativity. For example Cooperstock and de la Cruz (1978) examined the relativistic condition for balance between two Curzon particles. Szekeres(1968), by introducing negative mass, examined the balance between multipole particles (for definition of multipole particles, see chapter 6) Assuming a functional relation between the gravitational and the electrostatic potential, we work out the relativistic condition for balance between a charged line mass and a charged Curzon particle. We also develop the classical divergence theorem such that it is applicable in the case of curved space-time. Asymptotic expansion of the electrostatic potential enables us to determine the total charge of the system. However the detailed distribution of charges is found by an application of the divergence theorem (Papapetrou, 1978).Also in that chapter we examine the limit of the charged Bach and Weyl metric as the separation distance goes to zero.

In chapter 6 we formulate the equation which determines the V function using Szekeres complex variables. Using elliptical coordinates, Erez-Rosen (1959) found the metric corresponding to a Schwarzschild line mass superimposed with a quadrupole moment. In our work we will obtain a solution which is generated by a Curzon particle superimposed with a dipole moment. Further we obtained the metric which is generated by the Newtonian potential of a monopole and a dipole which are separated. That metric exhibits a singularity which is manifested by the breakdown of elementary flatness and expresses the necessity of a supporting strut to maintain balance between the monopole and the dipole when they are apart is examined. It is found that balance can be maintained only when the mass dipole and the charged dipole as well as the mass and the charge satisfy special conditions.

CHAPTER 2

FOUNDATION OF THE THEORY OF RELATIVITY

2.1 Galilean Relativity

It is well known that the structure of Newtonian or classical mechanics has as a foundation stone four laws including the law of universal gravitation, introduced by Newton.

Classical mechanics assumes the existence of an inertial system of reference i.e. system of reference in which the laws of Newton are valid. The transition from one system of reference to another can be done using the following coordinate transformation:

$$\vec{x}' = R\vec{x} + \vec{V}t + \vec{d} \qquad (2.1.1)$$

$$t' = t + \tau \qquad (2.1.2)$$

where $\vec{x}$ and $\vec{x}'$ are respectively the old and new coordinates of a point in the two systems, $\vec{V}$ is the velocity of the new system with respect to the old, $\vec{d}$ represents the original displacement of the new system's origin relative to the old, and R is an orthogonal matrix. The second equation connects the time between the two systems of reference and τ is usually taken to be zero. From a mathematical viewpoint the above set of transformations constitute a group (Galileo Group) and the invariance of the laws of classical mechanics under such a transformation is called Galilean invariance, or the principle of Galilean Relativity. This transformation expresses the absoluteness of time.

Further, classical mechanics assumes that the geometry of the three dimensional space is Euclidean, in which the square of distance ds^2 between two points is given by

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

Here and in what follows we make use of the Einstein summation convention where repeated indices are summed. If x^α represents Cartesian coordinates, then $g_{\alpha\beta}$ are given by

$$g_{\alpha\beta} = \delta^\alpha_\beta$$

where δ^α_β is the Kronecker symbol defined by

$$\delta^\alpha_\beta = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}$$

It is obvious from (2.1.1), and (2.1.2) that ds^2 is invariant under the Gallilean transformations.

2.2 Failure of Classical Mechanics

Classical mechanics has been quite successful in the study of motion of planets and the general study of motion of massive bodies moving with small velocities. However as experimentalists and theoreticians started using more precise equipment and new mathematical tools, the predictions of classical mechanics came in conflict with the experimental data and theoretical calculations. One of the most remarkable ones was due to the development of electromagnetic theory by J. C. Maxwell in 1864. Maxwell's equations of electromagnetism predicted that light in free space is propagated with a finite velocity

$$C \simeq 3 \cdot 10^{10} \ cm/sec$$

But if this is true in one inertial system of reference then it will not be true in another inertial system of reference due to the Galileo transformations. Maxwell himself postulated that the electromagnetic waves were carried by a highly tenuous medium called ether, that filled all the space so that his equations could hold only for a limited class of Galilean inertial frames, which are at rest with respect to the ether.

However, all the attempts to measure the velocity of the earth with respect to the ether failed. The most important experiment was due to A Michelson and E. W. Morley (1901) which showed that the velocity of light is the same for light traveling along the direction of the earth's motion as well as for that travelling perpendicular to it. Further, the failure of the experimentalists to discover properties of ether or even to detect it had led theoreticians to endow the ether with some peculiar properties and general physics was in chaos. While all this was going on, in 1905 Albert Einstein proposed a new theory in order to reconcile classical mechanics and electrodynamics. He wrote...

"The unsuccessful attempts to discover

any motion of the earth relative to the light medium, suggests that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that...the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (The Relativity) to the status of a postulate, and also introduce the former, namely that light is always propagated in empty space with a definite velocity C which is independent of the state of motion of the emitting body," Einstein (1972)

So the constancy of the velocity of light as well as invariance of the physical laws with respect to all inertial systems of reference became the new principles called the principles of the Special Theory of Relativity. He proposed that the transformation from one inertial system of reference to another should be done by the Lorentz group of transformations instead of the Galilean group. This Lorentz transformation is given as follows.

$$\vec{X}' = \vec{X} + \vec{V}\left[\frac{\vec{X}\cdot\vec{V}}{U^2}\left\{\left(1-\frac{U^2}{c^2}\right)^{-\frac{1}{2}} - 1\right\} - t\left(1-\frac{V^2}{c^2}\right)^{-\frac{1}{2}}\right]$$

$$t' = \left\{t - \frac{\vec{V}\cdot\vec{X}}{c^2}\right\}\left(1-\frac{U^2}{c^2}\right)^{-\frac{1}{2}}$$

(Møller 1972)

where $\vec{x}$ and $\vec{x}'$ are respectively the old and new coordinates of the point, $\vec{V}$ is the velocity of the new system with respect to the old, c and v are respectively the velocity of light and the magnitude of $\vec{V}$.

This transformation makes Maxwell's equations and the speed of light invariant. However, the equations of classical mechanics have to be modified so that they would be invariant under the Lorentz group. Thus the new physics consisted of Maxwell equations and these modified laws of classical mechanics (both of which now satisfied the above mentioned principles of the special theory of relativity.)

In 1908 Minkowski found a geometric framework of the special theory of relativity and in doing so, he introduced the four dimensional space called space-time. Let $x^0 = ct$ and if x^1, x^2, x^3, are the usual coordinates of three dimensional space, then for any two points

$$x = (x^0, x^1, x^2, x^3)$$

and

$$x + dx = (x^0 + dx^0, x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$$

the quantity

$$ds^2 = g_{ik}\, dx^i\, dx^k$$

remains invariant under the Lorentz transformations. The g_{ik} for Cartesian Coordinates are given by

$$g_{0\alpha} = 0$$

$$g_{\alpha\beta} = -\delta^{\alpha}_{\beta} \qquad (2.2.1)$$

$$g_{00} = +1$$

We see that the feature of the new transformation is the connection between space coordinates x^1, x^2, x^3 and the time coordinate x^0 such the ds^2 remains invariant. About the connection of space and time, H. Minkowski wrote:

> ..."The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therin lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." (Einstein 1972)

In other words space-time has an absolute meaning in the new theory unlike classical physics, where the space and time were completely distinct entities as we have discussed earlier.

2.3 Limitation of Newtonian Gravitation Theory

So far we have briefly seen how Einstein reconciled classical mechanics and the electromagnetic theory. In the following paragraph we will see another important difference between Maxwell's equations and the equations of Newtonian Gravitation.

If A^i and J^i are the four vector potential and four current density respectively, then the differential equations connecting those vectors are

$$\Box A^i = -\frac{4\pi}{c} J^i \qquad \frac{\partial A^i}{\partial x^i} = 0$$

where

$$\Box = g^{ik} \frac{\partial^2}{\partial x^i \partial x^k} =$$

$$= \frac{1}{c^2} \frac{\partial^2}{(\partial x^0)^2} - \frac{\partial^2}{(\partial x^1)^2} - \frac{\partial^2}{(\partial x^2)^2} - \frac{\partial^2}{(\partial x^3)^2}$$

is the d'Alembert operator in Cartesian coordinates. The general solution of the above hyperbolic equation is given by (Landau-Lifshitz 1962)

$$\phi = A^0 = \int \frac{J^0 (\vec{r}', t - \frac{R}{c}) \, dV}{R} \qquad (2.3.1)$$

$$\vec{A} = \frac{1}{c} \int \frac{\vec{J} (\vec{r}', t - \frac{R}{c}) \, dV}{R} \qquad (2.3.2)$$

The quantities $\vec{r}$, R etc. are explained in the figure number (1)

Further the well known elliptic differential equation of Newtonian gravitation

$$\nabla^2 \phi = - 4 \pi G \rho \qquad (2.3.3)$$

has the general solution

$$\phi(\vec{r}, t) = \int \frac{\rho(\vec{r}', t)}{R} \, dV \qquad (2.3.4)$$

where ϕ and ρ are respectively the gravitational potential and mass density.

Comparison between (2.3.1), (2.3.2) and (2.3.4) suggests that the electromagnetic disturbances are propagated with finite velocity in contrast to gravitational disturbances which are propagated with infinite velocity. So from this point of view, Newtonian gravitation had to be modified in order to be compatible with the principles of the special theory of relativity, which assumes finite propagation of disturbances. Further, Newtonian gravitation had also failed to explain the advance of the perihelion of the inner planets, especially that of Mercury, according to Newcombs (1880), Le Verrier (1850) etc. **

A natural generalization of the equation (2.3.3) is

$$\Box \phi = -4\pi G \rho$$

for a scalar potential function ϕ. We know that charge density is the time component of the four vector J^i (Landau-Lifshitz 1962), but the mass density does not have a corresponding property, but appears as the zeroth component of the second rank, energy-momentum tensor T_{ik}, which is used to describe continuous or discrete distribution of matter in the special theory of relativity. For this tensor

T_{oo} represents total non gravitational energy density.

T_{oi} represents momentum-density, or energy flux density

T_{ij} represents momentum flux density.

If T_{oo} is used as a source in a relativistic theory of gravitation then in another system of reference, T_{oo} will be a combination of mass density as well as of the other components of T_{ik} (This is due to the fact that tensors are transformed in a special way under coordinate transformation) (For example see Landau and Lifshitz 1962)

Another attempt to form a scalar theory was one by Nordstrom (1911) who used, as source, the following quantity:

** Papapetrou, Lectures on General Relativity

$$T^i_i = T^0_0 + T^1_1 + T^2_2 + T^3_3$$

This was a truly relativistic theory formulated in the Minkowski space-time of special relativity, and in the first approximation reduced to Newtonian theory, but failed to give the correct value for the advance of the perihelion of Mercury (Papapetrou 1974)

The next step then, would be the generalization of the above procedure to reconstruct a vector field theory, where both field variables and sources are vectors. Unfortunately, it is very difficult to form a vector from the second rank energy-momentum tensor. So the only possibility is a tensorial relativistic gravitational theory where both the field variables and sources are second rank tensors. This theory as formulated by Einstein (1915) is called the general theory of relativity. In what follows we will discuss its basic steps.

2.4 Principle of Equivalence

Let us consider an observer O $(\vec{x}, t)$ in a homogeneous and static gravitational field of strength g . Further, imagine an experimentalist $O'(\vec{x}', t')$ situated inside an elevator, moving under the influence of the above field, who tries to find out the equation of motion of N particles moving under their mutual gravitational attraction as well as under some other kind of forces depending on the position of the particles.

According to the observer O the equation of motion for any particles of the above system is.

$$m_{in} \frac{d^2\vec{x}}{dt^2} = m_{gr}\vec{g} + \vec{F}(\vec{x}_1, \ldots, \vec{x}_N)$$

where m_{in} and m_{gr} are respectively the inertial and the gravitational mass of the particle, and $m_{gr}\vec{g}$ and $\vec{F}(\vec{x}_1, \ldots, \vec{x}_N)$ describe the external gravitational force and the other forces respectively. The coordinates $(\vec{x}, t)$ and $(\vec{x}', t')$ are connected via the transformation

$$\vec{x}' = \vec{x} + \frac{1}{2}\vec{g}\,t^2 \qquad\qquad t = t'$$

(S. Weinberg 1972). Then the equation of motion for the observer O' will be

$$m_{in} \frac{d^2\vec{x}'}{dt^2} + m_{in}\vec{g} = m_{gr}\vec{g} + \vec{F}'(\vec{x}'_1, \ldots, \vec{x}'_N)$$

Now if the classical principle of equivalence is valid (ie. the gravitational and inertial mass of the bodies are equal), then the above equation yields

$$m \frac{d^2\vec{x}'}{dt^2} = \vec{F}'(\vec{x}'_1 \ldots \vec{x}'_N)$$

This equation suggests that the observer O' does not feel an external gravitational field, and the equations of motion of physical laws in his system of reference take the usual form given by special relativity. Thus, in the case of a homogeneous static field, it is possible to choose a system of reference so as to cancel the external gravitational field as well. It is natural to ask if the above result will be valid for an arbitrary gravitational field. Einstein (1907) postulated the following principle of equivalence.

> "At every space-time point in an arbitrary gravitational field it is possible to choose a system of reference such that within a sufficiently small region of the point under consideration, the laws of nature take the forms predicted by the special theory of relativity."

Henceforth we will refer to those special systems of reference as local inertial systems of reference. One can use the above principle in order to find the field variable in the new gravitation theory. Consider a particle moving freely under the influence of an arbitrary gravitational field. According to the Einstein principle of equivalence, in a local inertial system of reference ξ^i, the equation of motion of the particle is given by

$$\frac{d^2\xi^i}{ds^2} = 0 \qquad (2.4.1)$$

$$ds^2 = \eta_{ik}\, dx^i\, dx^k \qquad (2.4.2)$$

where η_{ik} are

$$\eta_{\alpha\beta} = -\delta^{\alpha}_{\beta}$$

$$\eta_{0\alpha} = 0$$

$$\eta_{00} = 1$$

Now, if x^i is a cartesian system of reference describing the external field, then the coordinates ξ^i and x^i are related via

$$\xi^i = \xi^i(x^0, x^1, x^2, x^3)$$

and the equation of motion of a particle in the x^i system will be

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0 \qquad (2.4.3)$$

where

$$\Gamma^i_{kl} = \frac{\partial x^i}{\partial \xi^a}\frac{\partial^2 \xi^a}{\partial x^k \partial x^l}$$

are termed the affine connection coefficients while ds^2 is given by

$$ds^2 = g_{ik}\,dx^i dx^k \qquad (2.4.5)$$

with

$$g_{ik} = \frac{\partial \xi^l}{\partial x^i}\frac{\partial \xi^m}{\partial x^k}\eta_{lm} \qquad (2.4.6)$$

Equation (2.4.5) suggests that the geometry of the space-time in the x^i system is not the usual Euclidean geometry nor the geometry of special relativity (Minkowski) but is a more general one, called Riemanian geometry, due to dependence of g_{ik} on x^i

The special transformation of η_{ik} from ξ^i to x^i implies that η_{ik} are the components of a second rank tensor. This tensor is termed as the metric tensor of the space-time. Moreover another important feature of the Riemanian Geometry is the dependence of $\Gamma^i_{k\ell}$ (affine connection) on the components of the metric tensor and its first derivatives with respect to the space-time coordinates, i.e.

$$\Gamma^i_{k\ell} = \frac{1}{2} g^{im} \left(g_{\ell m,k} + g_{km,\ell} - g_{k\ell,m} \right) \qquad (2.4.7)$$

(Adler-Bazin-Schiffer 1968)

From equation (2.4.3) we conclude now that quantities which determine the equation of motion of the particle in the gravitational field are the g_{ik} and its first derivatives with respect to space-time coordinates. i.e. the g_{ik} can completely determine the gravitational field. So in the new theory, g_{ik} are the field variables.

If we were to transform the equations (2.4.1), (2.4.2) into an arbitrary coordinate system z^i through the general transformation

$$z^i = z^i(\xi^0, \xi^1, \xi^2, \xi^3)$$

then they would be the same as (2.4.3) and (2.4.5). From the above it might seem that gravitational fields are equivalent to the noninertial system of reference but this is not true in general due to the following:

For any non inertial system of reference, through a suitable coordinate transformation we can bring the components of the metric tensor into their special values given by (2.2.1) over all the spacetime. But in the case of a general gravitational field it is not always possible to find such a coordinate transformation. We can do that only for an infinitesimal region of the space-time i.e. only locally gravitational fields are equivalent to non-inertial frames of reference.

2.5 Einstein's Field Equations

As we have discussed in 2.4 the Einstein theory of relativity is tensorial where the sources are the components of the energy-momentum tensor. Einstein's principle of equivalence leads us to adopt as the field variables, the g_{ik} components of the metric tensor. So the field equations have the form

$$F_{ik} \simeq T_{ik}$$

where F_{ik} is a second rank tensor containing the components and the partial derivatives of g_{ik}. Now an analytical expression for F_{ik} can be obtained if one assumes that for the case of weak fields and small velocities the new field equations reduce to the linear, second order Newtonian equation (2.3.3). This implies that F_{ik} contains partial derivatives of the g_{ik} only up to the second order and that F_{ik} is a linear function with respect to the second order partial derivatives. But the only tensors which can be constructed with the above properties are the Riemann tensor and its contractions. (Papapetrou, 1972). The Riemann tensor is defined as follows:

$$R^i_{jkl} = -\Gamma^i_{jk,l} + \Gamma^i_{jl,k} - \Gamma^n_{jk}\Gamma^i_{nl} + \\ + \Gamma^n_{jl}\Gamma^i_{nk}$$

The possible contractions of the Riemann tensor are:

Ricci tensor: $R_{jl} \equiv R^{i}_{jil}$

Curvature Scalar: $R = g^{jl} R_{jl}$

So the analytical form of F_{ik} and the field equations become

$$\alpha R_{ij} + \beta g_{ij} R + \gamma g_{ij} = - \kappa T_{ij}$$

with α, β, γ, and κ constants.

Einstein's initial field equations were

$$R_{ij} = - \kappa T_{ij} \qquad (2.5.1)$$

In a region of space containing no matter we have $T_{ij} = 0$ and consequently equation (2.5.1) reduces to $R_{ij} = 0$. But it was soon discovered that this needed to be modified as the law of conservation of energy, in the special relativity, requires that the divergence of the energy momentum tensor should vanish ie.

$$T^{ik}_{\ ;k} = 0$$

In a general coordinate system, the above equation becomes

$$T^{ik}_{\ ;k} = 0$$

By the principle of equivalence this must hold in a Riemann space which describes the gravitational field. So from (2.5.1) $R^{ik}_{\ ;k} = 0$, which represents four additional equations. Thus we would have 14 equations for ten unknowns g_{ik} . So in the place of R_{ik} we have to choose a tensor such that the covariant divergence is identically zero. That turns out to be

$$R_{ik} - \frac{1}{2} g_{ik} R$$

so the final form of the field equations is

$$R_{ik} - \frac{1}{2} g_{ik} R + \Lambda g_{ik} = -\kappa T_{ik} \qquad (2.5.2)$$

One should notice that the appearance of the term Λg_{ik} does not contradict the vanishing divergence form of the left hand side of (2.5.2). This is due to the fact that

$$g^{ik}{}_{;k} = 0$$

(Landau and Lifshitz 1962). The constant Λ is called the cosmological constant and is either zero or very small. It is usually taken to be zero except for cosmological problems.

2.6 Some Remarks on Einstein's Equations

The Einstein field equations determine the metric of the space from a given expression of T_{ik} . Thus we are in a completely new situation. The metric is no longer given a priori but has to be determined from the material distribution which we are considering. More generally, we can say that the distribution of matter in the universe determines the metrical properties of the space-time. On the other hand, the metric determines the geometrical properties, like curvature of the space, as well the motion of test particles. The equations (2.5.2) are not linear, containing products of the g_{ik} and their derivatives so the principle of superposition for solutions is not generally valid as in the Newtonian theory.

CHAPTER THREE

3.1 Static Axially Symmetric Gravitational Fields

We define an axially symmetric gravitational field as the gravitational field in which the g_{ik} are independent of the azimuthal angle $x^3 = \varphi$ in an appropriate system of cylindrical coordinates. In addition, if the field under consideration is static, then the components of the metric are independent of the time coordinate, $x^0 = t$

Moreover if we suppose that the distribution of matter does not rotate, then the metric will be invariant under the transformations,

$$t \longrightarrow -t \quad \text{(or)} \quad \varphi \longrightarrow -\varphi$$

The invariance of the metric under those transformations imply

$$g_{0a} = 0$$

$$g_{13} = g_{23} = 0$$

Synthesizing the assumptions thus far, the line element for a static axially symmetric gravitational field can be written as

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2$$

$$+ g_{33}(dx^3)^2 + 2\, g_{12}\, dx^1 dx^2$$

The number of the unknown functions in the above line element are 5 but we can reduce them to three by performing a transformation such that in the new system of reference, we have $g_{11} = g_{22}$ and $g_{12} = 0$. Such a transformation is always possible to find. More details can be seen in Synge, (1966), Weyl (1917, 1919), Levi-Civita (1919). Finally, the line element in the new coordinate system takes the form

$$ds^2 = -\alpha^2 [(d\bar{x}^1)^2 + (d\bar{x}^2)^2] - \beta^2 (d\bar{x}^3)^2 + \gamma^2 (dx^0)^2$$

where $\alpha \; \beta \; \gamma$ are functions of $\bar{x}^1$ and $\bar{x}^2$. In what follows we will, however, drop the bar on the x^i for simplicity. The above line element expresses the most general line element corresponding to a static axially symmetric gravitational field. We should notice that so far, we have not used the field equations. This will be the next step in order to find the differential equations satisfied by $\alpha \; \beta \; \gamma$. Calculating the Ricci tensor from (2.5.1), one gets the following values:

$$-R_{11} = \left(\frac{\alpha_1}{\alpha}\right)_1 + \left(\frac{\alpha_2}{\alpha}\right)_2 + \frac{\beta_{11}}{\beta} + \frac{\gamma_{11}}{\gamma} + \frac{\alpha_2}{\alpha}\left(\frac{\beta_2}{\beta} + \frac{\gamma_2}{\gamma}\right) - \frac{\alpha_1}{\alpha}\left(\frac{\beta_1}{\beta} + \frac{\gamma_1}{\gamma}\right)$$

$$-R_{22} = \left(\frac{\alpha_1}{\alpha}\right)_1 + \left(\frac{\alpha_2}{\alpha}\right)_2 + \frac{\beta_{22}}{\beta} + \frac{\gamma_{22}}{\gamma} + \frac{\alpha_1}{\alpha}\left(\frac{\beta_1}{\beta} + \frac{\gamma_1}{\gamma}\right) \qquad (3.1.1)$$

$$R_{12} = \frac{\beta_{12}}{\beta} + \frac{\gamma_{12}}{\gamma} - \frac{\alpha_2}{\alpha}\left(\frac{\beta_1}{\beta} + \frac{\gamma_1}{\gamma}\right) - \frac{\alpha_1}{\alpha}\left(\frac{\beta_2}{\beta} + \frac{\gamma_2}{\gamma}\right)$$

$$-R_{33} = \frac{\beta}{\alpha_2}\left\{\Delta\beta + \frac{1}{\gamma}(\beta_1\gamma_1 + \beta_2\gamma_2)\right\}$$

$$-R_{00} = -\frac{\gamma}{\alpha}\left\{\Delta\gamma + \frac{1}{\beta}(\beta_1\gamma_1 + \beta_2\gamma_2)\right\}$$

where the subscripts on the right denote partial derivatives with respect to x^1 and x^2, where

$$\Delta \beta = \beta_{11} + \beta_{22}$$

$$\Delta \gamma = \gamma_{11} + \gamma_{22}$$

From R_{33} and R_{00} we find

$$R^3_3 + R^0_0 = \beta^{-2} R_{33} + \gamma^{-2} R_{00} = \frac{1}{a^2 \beta\gamma} \Delta(\beta\gamma)$$

In the next pages we will study those fields in matter-free space time. As we have seen, that means $T_{ik} = 0$ and the field equations take the following simple form:

$R_{ik} = 0$ From the last equation, in conjunction with the equations (3.1.1), we get

$$\Delta(\beta\gamma) = 0 \qquad (3.1.2)$$

This equation implies that $\beta\gamma$ is a harmonic function of x^1 x^2, so there exists a conjugate harmonic function $z(x^1, x^2)$ such that

$$r + iz = f(x^1 + ix^2) \qquad (3.1.3)$$

$$r = \beta\gamma \qquad (3.1.4)$$

where f is an analytic function. We now make the transformation

$$(x^1, x^2) \longrightarrow (r, z)$$

$$x^0 \longrightarrow x^0$$

$$x^3 \longrightarrow x^3$$

Then

$$a^2\left[(dx^1)^2+(dx^2)^2\right] = A\,(dr^2+dz^2)$$

with A a function of, r z. Further, from (3.1.4) we put $\beta = \frac{r}{\gamma}$ and so the original line element becomes a form with only two arbitrary functions. Summarizing, if

$$x^1 = r \qquad x^2 = z \qquad x^3 = \varphi \qquad x^0 = t$$

then

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}(dr^2+dz^2) - r^2e^{-2\lambda}d\varphi^2.$$

where we put (3.1.5)

$$\alpha = e^{\nu-\lambda} \qquad \beta = re^{-\lambda} \qquad \gamma = e^{\lambda}$$

Then the above line element suggests that for any static, axially symmetric gravitational field which satisfies the relation

$$R^3_3 + R^0_0 = 0$$

the line element takes the form of (3.1.5) where ν λ are functions of r z. Using (3.1.5) and (3.1.4), the equations (3.1.1) take the form

$$\frac{1}{2}(R_{11}+R_{22}) = \Delta\nu - \left(\Delta\lambda + \frac{\lambda_1}{r}\right) + \lambda_1^2 + \lambda_2^2$$

$$\frac{1}{2}(R_{11}-R_{22}) = \lambda_1^2 - \lambda_2^2 - \frac{\nu_1}{r}$$

$$R_{12} = 2\lambda_1\lambda_2 - \frac{\nu_2}{r}$$

$$R^3_3 + R^0_0 = 0$$

$$R^3_3 - R^0_0 = -\frac{2}{a^2}\left(\Delta\lambda + \frac{\lambda_1}{r}\right)$$

From these equations, using the field equations, $R_{ik} = 0$ we get

$$\Delta\lambda + \frac{\lambda_1}{r} = 0 \qquad (3.1.7)$$

$$\nu_1 = r(\lambda_1^2 - \lambda_2^2) \qquad \nu_2 = 2r\lambda_1\lambda_2 \qquad (3.1.8)$$

$$\Delta\nu + \lambda_1^2 + \lambda_2^2 = 0 \qquad (3.1.9)$$

So the interesting result from the above analysis is that the vacum equations for λ and ν satisfy the above system of differential equations. Considering that

$$\Delta\lambda = \lambda_{11} + \lambda_{22}$$

The first equation (3.1.7) becomes

$$\frac{\partial^2\lambda}{\partial r^2} + \frac{1}{r}\frac{\partial\lambda}{\partial r} + \frac{\partial^2\lambda}{\partial z^2} = 0 \qquad (3.1.10)$$

which is the Laplace equation in cylindrical coordinates r, z, φ in Euclidian 3 space for a function independent of the azimuthal angle φ . From (3.1.8)

$$\nu = \int_C r\left[(\lambda_1^2 - \lambda_2^2)\,dr + 2\cdot\lambda_1\lambda_2\,dz\right] \qquad (3.1.11)$$

where the integration is extended over any regular curve C which lies entirely in vacuum space. Since λ

satisfies (3.1.8), the integral in (3.1.11) is independent of the path of integration.

We should emphasize that the above analysis is exact so all the above equations are valid for gravitational fields of arbitrary strength. Further the field equations are nonlinear, but the surprising fact is that the equation (3.1.7) is simply Laplace's equation. We should remember that the Newtonian potential satisfies the same equation which is a linear equation. So solutions of Laplace's equation can be used to generate static, axially symmetric gravitational fields in vacuum. For any solution of Laplace's equation, there corresponds a function ν given by (3.1.11) or (3.1.8). The above prescription for getting static, axially symmetric gravitational fields is applicable only in vacuum. In the case where matter is present, the ν function does not satisfy the equation (3.1.11) or (3.1.8), nor does λ satisfy the equation (3.1.10). Also in general the number of unknown functions in the line element increases to three. (see Synge (1965), Robertson and Noonan (1968)). In the following chapters we will work exclusively with axially symmetric gravitational fields in vacuum. So we will use equations (3.1.7) and (3.1.8) Before we finish the chapter, we will examine a property of the system of the differential equations which determines ν and λ. Let ν_i and λ_i $i = 1, \cdots, N$ be solutions of (3.1.7), (3.1.8). Then obviously

$$\lambda = \sum_{i=1}^{N} \lambda_i$$

is a solution of (3.1.7) If we put λ into (3.1.8) then the resulting ν has the following form:

$$\nu = \sum_{i=1}^{N} \sum_{j=1}^{N} \nu_{ij}$$

where ν_{ij} for $i=j$ is exclusively due to the term λ_i and expresses the field arising only from λ_i while the ν_{ij} $i \neq j$ parts express the field contributions due to the λ_i and λ_j .

3.2 The Criteria of Asymptotic and Elementary Flatness

So far we have derived the form of the line element which describes the static axially symmetric gravitational fields in vacuum. We have seen that the functions λ and γ satisfy the equations (3.1.7) and (3.1.8) respectively while the components of the metric tensor are given as follows:

$$g_{00} = e^{2\lambda}$$

$$g_{11} = g_{22} = - e^{2(\nu - \lambda)}$$

$$g_{33} = - r^2 e^{-2\lambda}$$

However λ and ν describe the metric of a real gravitational field only if they satisfy additional conditions.

It is assumed that an observer located at very large distances from the source of a gravitational field would not feel the influence of the field. i.e. the space-time around him would be a flat space time. That implies the line element would be given as follows

$$ds^2 = dt^2 - dr^2 - dz^2 - r^2 d\varphi^2 \qquad (3.2.1)$$

Comparison between (3.2.1) and (3.1.5) implies that λ and ν have to satisfy the condition

$$\lim_{\substack{r \to \infty \\ z \to \infty}} \lambda = 0 \qquad \lim_{\substack{r \to \infty \\ z \to \infty}} \nu = 0 \qquad (3.2.1a)$$

Those relations are two additional conditions which have to be satisfied by λ and ν . That argument can be generalized for any metric describing a real gravitational field. i.e. the limit of the metric at spatial infinity corresponds to a flat-space time metric.

That statement is a criterion for testing the physical acceptability of a given metric. This criterion is called the criterion of asymptotic flatness.

Besides the above criterion there is another criterion which was initiated by Einstein-Rosen (1937), called the criterion of elementary flatness. The history of that criterion is as follows:

In 1936 Silberstein (1936) using Weyl's formalism had obtained a static solution representing exterior solutions of two bodies at rest. Specifically he started out with the function λ having the following form:

$$\lambda = \frac{L_1}{r_1} + \frac{L_2}{r_2} \qquad (3.2.2)$$

with L_1 and L_2 constants and

$$r_1^2 = r^2 + (z - \alpha)^2$$

$$r_2^2 = r^2 + (z + \alpha)^2$$

with α and $-\alpha$ constants (see figure 8) His ν function was as follows:

$$\nu = -\frac{r^2}{2}\left(\frac{L_1^2}{r_1^4} + \frac{L_2^2}{r_2^4}\right) + \tag{3.2.3}$$
$$+ 2\frac{L_1 L_2}{D^2}\left[\left(1 - \frac{D^2 r^2}{r_1^2 r_2^2}\right)^{-\frac{1}{2}} - 1\right]$$

where D is the separation distance between the two masses. He noticed that the only singularities in his solution occured at $r_1 = 0$ and $r_2 = 0$. That implies that the two bodies can remain in a static configuration without any additional supporting strut. So Silberstein concluded that either the Einstein field equations are incorrect or we cannot consider material particles as singularities of the field. However Einstein and Rosen (1937) had noticed that in Silberstein's solution there is another singularity occuring in the interval $[a, -a]$. In order to examine the singularity carefully let us draw an infinitesimal circle perpendicular to z axis and its center on the z axis. Further let us suppose we are on the hypersurface $t = const.$ Corresponding to the measured circumference and radius of that circle are the invariants ds_{cir} , ds_{rad} respectively.

According to the principle of equivalence we can introduce at the point under consideration a local inertial system of reference in which

$$ds^2 = dt^2 - dr^2 - dz^2 - r^2 d\varphi^2$$

With respect to that local inertial system of reference the circumference ds and the radius dr of the above circle satisfy the relation

$$\frac{ds}{dr} = 2\pi$$

However since ds_{cir} and ds_{rad} are invariant we have to have as well

$$\frac{ds_{cir}}{ds_{rad}} = 2\pi$$

But from the line element (3.1.5) in connection with (3.2.3) we have

$$\frac{ds_{cir}}{ds_{rad}} = 2\pi\, e^{-\nu}$$

But $\nu \neq 0$ on the closed interval $[-a, a]$. So we see a singularity in the metric occuring on the interval $[-a, a]$. The only way that we can make the metric consistent with the principle of equivalence is to suppose that there is a kind of stress along the z axis between the two bodies. In that case the ν function would be different from (3.2.3). From the above example we see that the metric is regular if and only if the ratio of the circumference of an infinitesimal circle to it's radius is equal to 2π .

That statement is the criterion of elementary flatness which assures the regularity of a given metric in vacuum. It is easy now to see that the criterion of elementary flatness, in the case of the line element requires the vanishing of ν function along the z axis.

This is another additional condition which has to be satisfied by ν function besides those of (3.2.1 a) (3.1.7) and (3.1.8)

CHAPTER FOUR

STATIC AXIALLY SYMMETRIC SOLUTIONSOF EINSTEIN EQUATIONS

4.1 Introduction

In this chapter, we will apply the procedure of chapter 3 to obtain metrics for static axially symmetric gravitational fields. In particular, we will adopt the form of the function which corresponds, in the Euclidean chart, to the potential of a point mass and a line mass separated by a distance.

4.2 Curzon Metric

As the first example of the procedure described in chapter 3 we will consider the case where

$$\lambda = -\frac{L}{R}$$

with $L = Gm$ a constant and $R^2 = r^2 + z^2$. As is immediately recognized, the above expression for λ corresponds to the gravitational potential of a point mass located at the origin of the coordinates. Substituting the partial derivatives of λ into (3.1.8) and after an elementary integration of (3.1.8), the ν function turns out to be

$$\nu = -\frac{L^2 r^2}{4R^4}$$

The above λ and ν functions completely determine the metric. This metric will be refered to as the Curzon single particle metric while we will refer to the real distrubution acting as a source of the above metric as a Curzon particle.

4.3 General Line Mass Metric

As a second application we consider the λ function corresponding to the Newtonian potential of a line mass of length 2ℓ and uniform density

$$\rho = \frac{m}{2\ell}$$

located along the Z axis. Analytically

$$\lambda = \frac{K}{2} \ln \frac{r_1 + r_2 - 2\ell}{r_1 + r_2 + 2\ell} \tag{4.3.1}$$

where

$$r_i^2 = r^2 + (z - z_i)^2 \qquad i = 1, 2$$

$$K = \frac{Gm}{\ell}$$

(see figure 2)

Again after a simple integration of (3.1.8) the function ν turns out to be

$$\nu = \frac{K^2}{2} \ln \frac{(r_1 + r_2)^2 - 4\ell^2}{4 r_1 r_2} \tag{4.3.2}$$

(see also Robertson and Noonan 1968). In the special case of a line mass with $K = 1$ the line element takes the form

$$ds^2 = \frac{r_1 + r_2 - 2\ell}{r_1 + r_2 + 2\ell} dt^2 - \frac{(r_1 + r_2)^2 - 4\ell^2}{4 r_1 r_2} \left(dr^2 + dz^2 \right) +$$

$$+ \frac{r_1 + r_2 + 2\ell}{r_1 + r_2 - 2\ell} r^2 d\varphi^2$$

where for convenience, we choose units such that the velocity of light is unity.

Transforming to another coordinate system of reference via the transformation

$$\rho = \frac{r_1 + r_2 - 2\ell}{2}$$

$$\cos\theta = \frac{r_2 - r_1}{2}$$

$$\varphi = \varphi \qquad t = t$$

(Robertson and Noonan 1968),
the above line element is transformed into

$$ds^2 = \left(1 - \frac{2\ell}{\rho}\right) dt^2 - \frac{d\rho^2}{1 - \frac{2\ell}{\rho}} - \rho^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

where ρ is defined only for $\rho > 2\ell$. But the above line element is recognized as the spherically symmetric Schwarzschild exterior solution and is unique according to Birkoff's theorem (Papapetrou, Lectures on General Relativity page 70). Voorhees (1970) has presented arguments to support the contention that the case $K \neq 1$ represents the exterior field of a family of spheroids and hence we consider it natural to refer to this general line mass metric for $K \neq 1$ as a "spheroidal metric".

4.4 Metric for Curzon Particle and General Line Mass

Above we have found the metric corresponding to Curzon particles and the general line mass separately. It is natural to obtain and examine the metric considering the Curzon particle and a general line mass together. This implies that the λ function has the form

$$\lambda = -\frac{L}{r_3} + \frac{k}{2} \ln \frac{r_1 + r_2 - 2\ell}{r_1 + r_2 + 2\ell} \qquad (4.4.1)$$

The ν function can be obtained from (3.1.8) by substituting the partial derivatives of λ and integrating the system of differential equations (3.1.8) For the present case, the above procedure involves extensive calculations due mainly to the complicated form of λ function. Thus in order to find ν in this case we first examine the metric due to the λ function corresponding to the two line mass potentials. Bach and Weyl (1922), in their well known paper, started out with the λ function as

$$\lambda = \frac{k_1}{2} \ln \frac{r_1 + r_2 - 2\ell_1}{r_1 + r_2 + 2\ell_1} + \frac{k_2}{2} \ln \frac{r_3 + r_4 - 2\ell_2}{r_3 + r_4 + 2\ell_2} \qquad (4.4.2)$$

where

$$r_i^2 = r^2 + (Z - Z_i)^2 \qquad i = 1, 2, 3, 4$$

$$2\ell_1 = Z_1 - Z_2 \qquad 2\ell_2 = Z_3 - Z_4 \qquad k_J = \frac{G m_J}{\ell_J} \quad J = 1, 2$$

(See also figure number 3)

After extensive calculations the ν function is given by the following lengthy form

$$\nu = \frac{k_1^2}{2} \ln \frac{(r_1 + r_2)^2 - 4\ell_1^2}{4 r_1 r_2} + \frac{k_2^2}{2} \ln \frac{(r_3 + r_4)^2 - 4\ell_2^2}{4 r_3 r_4} +$$

$$+ k_1 k_2 \ln \frac{\ell_1 r_4 - (\ell_1 + d) r_1 - (\ell_1 + \ell_2 + d) r_2}{\ell_1 r_3 - d r_1 - (\ell_1 + d) r_2} + \qquad (4.4.3)$$

$$+ k_1 k_2 \ln \frac{d}{\ell_1 + d}$$

(Robertson and Noonan (1968)). We should notice that in the expression (4.4.3), there are terms proportional to K_1^2, K_2^2 and $K_1 K_2$ respectively. The terms proportional to K_1^2 and K_2^2 according to (4.3.2) express the individual fields due exclusively to the presence of the two line masses while the term proportional to $K_1 K_2$ expresses the interaction between the two line masses. All the above remarks are in agreement with what we have seen at the end of Chapter Three. The Bach and Weyl metric contains the following parameters: K_1, K_2, ℓ_1, ℓ_2, d
We will prove next that when $\ell_2 \to 0$ keeping the rest of the parameters constant, the Bach and Weyl metric goes to the metric corresponding to Curzon and general line mass metric. First we will take the limit of λ when $2\ell_2 = (z_3 - z_4) \longrightarrow 0$ For simplicity during the calculations we suppose that $z_3 = 0$. Then an easy way of making $2\ell_2 \to 0$ is to allow $z_4 \to 0$. Hence

$$\lim_{z_4 \to 0} \lambda = \lim_{z_4 \to 0} \left[\frac{K_1}{2} \ln \frac{r_1 + r_2 - 2\ell_1}{r_1 + r_2 + 2\ell_1} + \frac{K_2}{2} \ln \frac{r_3 + r_4 - 2\ell_2}{r_3 + r_4 + 2\ell_2} \right]$$

The limit of the first term remains invariant under that limiting process, but the limit of the second term yields the indeterminate $\frac{0}{0}$ form. In order to find the real value of the λ function, the L'Hospital rules are employed.
Let

$$\lambda = \frac{f(z_4)}{\phi(z_4)}$$

where

$$f(z_4) = Gm \ln \frac{r_3 + r_4 - 2\ell_2}{r_3 + r_4 + 2\ell_2} \qquad \phi(z_4) = -z_4$$

Then

$$\lim_{z_4 \to 0} \frac{f(z_4)}{\phi(z_4)} = \lim_{z_4 \to 0} \frac{\dfrac{df(z_4)}{dz_4}}{\dfrac{d\phi(z_4)}{dz_4}}$$

The above procedure, after some elementary algebra gives

$$\lim_{z_4 \to 0} \frac{f(z_4)}{\phi(z_4)} = -\frac{L}{r_3} \qquad \text{where} \qquad L = Gm_2$$

So finally the limit of λ takes the form

$$\lambda = \frac{K_1}{2} \ln \frac{r_1 + r_2 - 2l_1}{r_1 + r_2 + 2l_1} - \frac{L}{r_3} \qquad (4.4.5)$$

The same procedure will be applied for the ν function. Writing the ν function as the sum of ν_{11}, ν_{22} and ν_{12} where ν_{11} and ν_{22} are proportional to K_1^2 and K_2^2 respectively while ν_{12} is proportional to $K_1 K_2$ Then

$$\lim_{z_4 \to 0} \nu = \lim_{z_4 \to 0} \nu_{11} + \lim_{z_4 \to 0} \nu_{22} + \lim_{z_4 \to 0} \nu_{12}$$

The limit of ν_{11} remains invariant as we can see from (4.4.3). However the limit of ν_{22} yields the indeterminate form $\frac{0}{0}$. Again L'Hospital's rule is applied.

If $\quad f(z_4) = Gm_2 \ln \dfrac{(r_3 + r_4)^2 - 4l_2^2}{4 r_3 r_4} \qquad \phi(z_4) = z_4^2$

then

$$\lim_{z_4\to 0} \frac{f(z_4)}{\phi(z_4)} = \lim_{z_4\to 0} \frac{\dfrac{d^2 f(z_4)}{d z_4^2}}{\dfrac{d^2\phi(z_4)}{d z_4^2}} = -\frac{G^2 m_2^2 r^2}{2 r_3^4}$$

Similarily for ν_{12}, by applying L'Hospital's rule, the limiting procedure gives

$$\nu_{12} = K_1 G m_2 \frac{2 l_1 z - r_3(r_1 + r_2)}{r_3\left[l_1(r_3 - r_2) - d(r_1 + r_2)\right]} - \frac{K_1 G m_2}{d}$$

Putting the results together, the form of the ν function is

$$\nu = \frac{K^2}{2} \ln \frac{(r_1 + r_2)^2 - 4l^2}{4 r_1 r_2} - \frac{L^2 r^2}{2 r_3^4} +$$
$$+ KL \frac{2lz - r_3(r_1 + r_2)}{r_3\left(l(r_3 - r_2) - d(r_1 + r_2)\right)} - \frac{LK}{d} \qquad (4.4.7)$$

where for simplicity we wrote K L l instead of K_1 Gm_2 l_1. Because the functions λ ν have been found without directly solving the equations (3.1.8), (3.1.7) it is useful to verify that they satisfy the Einstein field equations or the equivalents (3.1.7), (3.1.8). The equation (4.4.5) obviously satisfies (3.1.7). In order to verify the form of ν function we form

$$\nu_r \equiv \frac{\partial \nu}{\partial r} = \frac{K^2}{2} \frac{4 r_2 r_1}{(r_1 + r_2)^2 - 4l^2} \cdot \left[\frac{8 r_1 r_2 (r_1 + r_2)(r_1 + r_2) r}{16 r_1^2 r_2^2} - \right.$$

$$- \frac{4((r_1+r_2)^2 - 4\ell^2)(r_1 r_2)_\kappa}{16 r_1^2 r_2^2} \Bigg] - \frac{L^2 r^2 (z^2 - r^2)}{r_3^6} +$$

$$+ KL \Bigg[\frac{[(-(r_1+r_2) r_3)_\kappa] \cdot [r_3 \ell(r_3 - r_2) - d(r_1+r_2)]}{r_3^2 [\ell_1(r_3 - r_2) - d(r_1+r_2)]^2} -$$

$$- \frac{(2\ell z - r_3(r_1+r_2)) \cdot (r_3(\ell_1(r_3-r_2) - d(r_1+r_2))_\kappa}{r_3^2 [\ell_1(r_3 - r_2) - d(r_1+r_2)]^2} \Bigg] \quad (4.4.8$$

$$\nu_\zeta \equiv \frac{\partial \nu}{\partial z} = \frac{K^2}{2} \frac{4 r_1 r_2}{(r_1+r_2)^2 - 4\ell^2} \cdot \Bigg[\frac{8 r_1 r_2 (r_1+r_2)(r_1+r_2)_\zeta}{16 r_1^2 r_2^2} -$$

$$- \frac{4((r_1+r_2)^2 - 4\ell^2)(r_1 r_2)_\zeta}{16 r_1^2 r_2^2} \Bigg] - \frac{L^2 r^2 z}{r_3^6} +$$

$$+ KL \Bigg[\frac{[((r_1+r_2) r_3)_\zeta] \cdot [r_3 \ell(r_3 - r_2) - r_3 d(r_1+r_2)]}{r_3^2 [\ell_1(r_3 - r_2) - d(r_1+r_2)]^2} -$$

$$- \frac{(2\ell z - r_3(r_1+r_2))\cdot(r_3(\ell(r_3-r_2) - d(r_1+r_2)))_\zeta}{r_3^2[\ell(r_3-r_2) - d(r_1+r_0)]^2} \Bigg] \qquad (4.4.9)$$

$$\lambda_\kappa \equiv \frac{\partial\lambda}{\partial r} = \frac{2k\ell\left(\frac{r}{r_1} + \frac{r}{r_2}\right)}{(r_1+r_2)^2 - 4\ell^2} - \frac{Lr}{r_3^3} \qquad (4.4.10)$$

$$\lambda_\zeta \equiv \frac{\partial\lambda}{\partial z} = \frac{2k\ell\left(\frac{z-z_1}{r_1} + \frac{z-z_2}{r_2}\right)}{(r_1+r_2)^2 - 4\ell^2} - \frac{Lz}{r_3^3} \qquad (4.4.11)$$

where we use the letters κ and ζ to denote partial derivatives with respect to r and z respectively. Then using (4.4.10) (4.4.11) the right handside of equations (3.1.8) are of the form

$$r(\lambda_r^2 - \lambda_z^2) = K^2\varphi_1(r,z) + L^2\varphi_2(r,z) + \\ + K'L\varphi_3(r,z) \qquad (4.4.12)$$

$$2r\lambda_r\lambda_z = K^2 f_1(r,z) + L^2 f_2(r,z) + \\ + KL f_3(r,z) \qquad (4.4.13)$$

where $\varphi_i(r,z)$ and $f_i(r,z)$ $i=1,2,3$ are functions of r, z.

From (4.4.8),(4.4.9),(4.4.10) and (4.4.13) we can easily verify that the terms proportional to L^2 are equal to each other. The terms which are proportional to K^2 are found to be equal after an elementary calculation and using the following identities:

$$z - z_1 = \frac{r_2^2 - r_1^2 - 4\ell^2}{4\ell}$$

$$z - z_2 = \frac{r_2^2 - r_1^2 + 4\ell^2}{4\ell}$$

However, the verification of the equality of terms proportional to KL is not so easy as they are very lengthy expressions. We therefore used numerical analysis in order to do so. The numerical calculations (see appendix 1) show that those terms also are equal to each other. We can now claim that (4.4.5) and (4.4.7) determine the metric for a Curzon particle and a general line mass in superposition.

4.5 Regularity of the Metric

The line element of the above space-time is

$$ds^2 = e^{2\lambda} dt^2 - e^{2(\nu-\lambda)}(dr^2 + dz^2) - e^{-2\lambda} r^2 d\varphi^2$$

where λ and ν respectively are given by (4.4.5) and (4.4.7). This metric will be physically acceptable if it satisfies (as we have discussed) the criterion of asymptotic flatness and that of elementary flatness. The asymptotic flatness criterion requires that λ and ν satisfy the conditions

$$\lim_{\substack{r \to \infty \\ z \to \infty}} \lambda = 0 \qquad \lim_{\substack{r \to \infty \\ z \to \infty}} \nu = 0$$

A straight-forward calculation shows this to be true. Further, the elementary flatness requires the vanishing of ν function on the Z axis (see discussion at the end of chapter 3) except that portion of the Z axis, which is occupied by the singular sources or line stresses. The behaviour of ν along the z axis is expressed as follows:

$$\nu = \begin{cases} \dfrac{KL}{(l+d)d} & 0 < Z < Z_2 \\ 0 & Z < 0 \quad Z > Z_1 \end{cases}$$

(see figure number 4). We can then conclude that the metric is irregular on the portion of Z axis between the "line mass" and the "point mass". The irregularity of the metric along the Z axis seems to be bizzare. However as we will see, this is not the case. For example for the Bach and Weyl metric the behaviour of the metric is completely analogous to the behaviour of the metric in our case. The corresponding value of ν function in the case of Bach and Weyl is

$$\nu = \begin{cases} K_1 K_2 \ln \dfrac{d(l_1 + l_2 + d)}{(l_1 + d)(l_2 + d)} & Z_3 < Z < Z_2 \\ 0 & Z > Z_1 \quad Z_4 > Z \end{cases}$$

(see figure number 3) It will be helpful to explain the irregularity of the metric if we first look at the physical problem. We began by examining the problem assuming the static axially symmetric gravitational field produced by two separated distrubutions of matter. But physically that is impossible since two masses must have been moving toward each other or rotating about their common center of mass, obviously a situation which is not static. This result is an example of the wonderful physical consistency of Einstein's field equations. The field equations do not allow two separated distrubutions of matter (in the absence of charge) to remain permanently at rest in vacuum because of their mutual gravitational attraction. Bach and Weyl explained the quantity

$$\frac{\nu(r=0)}{4G}$$

up to the first order, as the force between the two masses. In their case

$$\frac{\nu(r=0)}{4G} = \frac{G m_1 m_2}{4 d^2} + o(b^3)$$

and in our case

$$\frac{\nu(r=0)}{4G} = \frac{G m_1 m_2}{4(\ell + d) d} \simeq \frac{G m_1 m_2}{4 b^2} + o(b^3)$$

if $d \gg \ell$.

These examples suggest that the irregularity of the metric expresses the necessity of a supporting strut between the distribution of matter in order to keep them in static configuration. That strut is often called the Weyl strut According to W. Israel (1977), that strut makes no contribution to the function λ and ν.

4.6 Properties of the Metric

In the metric described by (4.4.5) and (4.4.7) there are the following independent parameters: K, L and ℓ Letting $\ell \longrightarrow 0$ the metric should go to the metric corresponding to two Curzon particles. Taking the limit of λ as $\ell \rightarrow 0$ (this can be done by letting $z_1 \rightarrow z_2$) (see figure 4) we get

$$\lambda = -\frac{Gm_1}{r_2} - \frac{Gm_2}{r_3} \qquad (4.6.1)$$

where we wrote Gm_2 instead of L . The limit of ν function yields into the following form, after successive applications of L'Hospital's rules:

$$\nu = -\frac{G^2 m_1^2 r^2}{2 r_2^4} - \frac{G^2 m_2^2 r^2}{2 r_3^4} + $$

$$+ 2\frac{G^2 m_1 m_2}{4 d^2}\left[\frac{r^2 + z^2 - z z_2}{r_2 r_3} - 1\right]$$

which is recognized as the Curzon metric (Synge 1965, Curzon 1918)

4.7 The Limit of Bach and Weyl Metric as the Separation Distance goes to Zero

Earlier, we have examined the limit of the Bach and Weyl metric as the parameter ℓ_2 goes to zero. Secondly, we performed the calculation when both ℓ_1 and ℓ_2 tend to zero while keeping the other parameters constant. In this paragraph first we will examine the limit of Bach

and Weyl metric as the separation distance goes to zero.
Second, we will study how the horizon changes during that quasi static approach of the two line masses. It is known that the horizon or infinite-red-shift surface, is characterized by the equation

$$g_{00} = 0$$

For example in the Schwarzschild solution ($g_{00} = 1 - 2m\rho^{-1}$) the horizon corresponds to a surface of a sphere ($\rho = 2m$) According to Israel's theorem (Israel 1967), among all static, asymptotically flat vacuum space-times with closed simply connected equipotential surfaces g_{00} = constant, the Schwarzschild solution is the only one which has a nonsingular infinite-red-shift surface. That theorem suggests that the Bach and Weyl solution has a singular horizon. We are interested in seeing if the horizon which corresponds to the Bach and Weyl solution with separation distance zero will be singular or not. For the first step we will obtain the metric which corresponds to the two line masses metric together. In doing so we start from Bach and Weyl metric (separation distance different than zero) and we assume that $z_3 = 0$ and let $z_1 \rightarrow 2\ell_1 + z_2$ and $z_2 \longrightarrow 0$. Then the limits of λ and ν are found after applying L'Hospital's rules. The results are:

$$\lambda = \frac{k_1}{2} \ln \frac{r_1 + r_2 - 2\ell_1}{r_1 + r_2 + 2\ell_1} + \frac{k_2}{2} \ln \frac{r_2 + r_3 - 2\ell_2}{r_2 + r_3 + 2\ell_2} \qquad (4.7.1)$$

$$\nu = \frac{k_1^2}{2} \ln \frac{(r_1 + r_2)^2 - 4\ell_1^2}{4 r_1 r_2} + \frac{k_2}{2} \ln \frac{(r_2 + r_3)^2 - 4\ell_2^2}{4 r_2 r_3} +$$

$$+ k_1 k_2 \ln \frac{l_1(r_3 - r_2) - l_2(r_1 + r_2)}{l_2[2l_1 z - r_2(r_1 + r_2)]} \qquad (4.7.2)$$

(see figure number 5)

As in the case of the metric corresponding to a Curzon particle and a general line mass we have to test if the above functions are physically and mathematically acceptable solutions of Einstein's field equations.

The criterion of elementary flatness as we have seen requires the vanishing of the ν function on the Z axis outside the interval (z_1, z_3) (see figure number 4). On the Z axis we have $r_i = |z - z_i|$ $i=1,2,3$. Using this, one gets the value zero for the ν function. Also the asymptotic flatness criterion is satisfied by ν, λ as we can easily determine from the forms of (4.7.1) and (4.7.2)

Before one claims the λ and ν functions describe the metric, one has to prove that (4.7.1) and (4.7.2) are solutions of the differential equations (3.1.8) and (3.1.10) As we have seen in the case of (4.4.5) and (4.4.7) an attempt to analytically verify this, is rather a difficult task involving lengthy calculations. The verification of the equations (4.7.1) (4.7.2) will hence be done here using a numerical approach. Numerically it has been proved that the following identities between r_i, l_1, l_2 & $i=1,2,3$ are valid (see numerical results at the end of the thesis).

$$\frac{r_1 + r_2 - 2l_1}{r_1 + r_2 + 2l_1} \cdot \frac{r_2 + r_3 - 2l_2}{r_2 + r_3 + 2l_2} = \frac{r_1 + r_3 - 2(l_1 + l_2)}{r_1 + r_3 + 2(l_1 + l_2)} \qquad (4.7.3)$$

$$\frac{(r_1+r_2)^2-4l_1}{4r_1r_2}\cdot\frac{(r_2+r_3)^2-4l_2^2}{4r_2r_3}\cdot\left(\frac{[l_1(r_3-r_2)-l_2(r_1+r_2)]r_2}{l_2[2l_1z-r_2(r_1+r_2)]}\right)^2=$$

$$=\frac{(r_1+r_3)^2-4(l_1+l_2)^2}{4r_1r_3} \qquad (4.7.4)$$

From (4.7.4) and (4.7.3) we conclude after taking the logarithm of both sides and if $k_1=k_2=1$ that λ, ν are solutions of Einstein's equation because they are equal to

$$\lambda=\frac{1}{2}\ln\frac{r_1+r_3-2l}{r_1+r_3+2l} \qquad \nu=\frac{1}{2}\ln\frac{(r_1+r_3)^2-4l^2}{4r_1r_3}$$

respectively with $l=l_1+l_2$

The above functions suggest that when two Schwarzschild line masses (note that a Schwarzschild line mass is characterized by the fact $k=1$) come together, the geometry of the space-time becomes the geometry of one Schwarzschild line mas with the parameter $l=l_1+l_2$ If we take the logarithm of both sides of equations (4.7.2) and (4.7.3) and multiply both sides by k , then λ and ν yield

$$\lambda=\frac{k}{2}\ln\frac{r_1+r_3-2(l_1+l_2)}{r_1+r_3+2(l_1+l_2)} \qquad (4.7.5)$$

$$\nu=\frac{k^2}{2}\ln\frac{(r_1+r_3)^2-4(l_1+l_2)^2}{4r_1r_3} \qquad (4.7.6)$$

which obviously satisfy the Einstein field equations. (see discussion at chapter 2.2). Physically the equations (4.7.5) and (4.7.6) imply that the geometry of space-time, when two generalized line masses come together, is the same as if we had a single spheroid with the parameter $\ell = \ell_1 + \ell_2$ Finally the verification in the case where $k_1 \neq k_2$ can be proved using the fact that when $k_1 = k_2 = 1$ then (4.7.1) and (4.7.2) are solutions of (3.1.8) and (3.1.10) Writing λ and ν as ($k_1 = k_2 = 1$)

$$\lambda = \lambda_1 + \lambda_2 \qquad \nu = \nu_{11} + \nu_{22} + \nu_{12}$$

where

$$\lambda_1 = \lambda_1 (r_1, r_2) \qquad \lambda_2 = \lambda_2 (r_2, r_3)$$

$$\nu_{11} = \nu_{11} (r_1, r_2) \qquad \nu_{22} = \nu_{22} (r_2 r_3) \qquad \nu_{12} = \nu_{12}(r_1, r_2, r_3)$$

then

$$\lambda_r = \lambda_{1r} + \lambda_{2r}$$

$$\lambda_z = \lambda_{1z} + \lambda_{2z}$$

$$\nu_r = \nu_{11r} + \nu_{22r} + \nu_{12r} \qquad \nu_z = \nu_{11z} + \nu_{22z} + \nu_{12z}$$

From

$$\nu_z = 2r \lambda_r \lambda_z$$

$$\nu_r = 2r (\lambda_z^2 - \lambda_r^2)$$

we get

$$\nu_{11r} + \nu_{22r} + \nu_{12} = 2r (\lambda_{1r}^2 - \lambda_{1z}^2) \quad +$$

$$g_{oob} = \frac{k_1}{2} \ln \frac{r_1 + r_2 - 2l_1}{r_1 + r_2 + 2l_1} + \frac{k_2}{2} \ln \frac{r_3 + r_4 - 2l_2}{r_3 + r_4 + 2l_2}$$

$$g_{ooA} = \frac{k_1}{2} \ln \frac{r_1 + r_2 - 2l_1}{r_1 + r_2 + 2l_1} + \frac{k_2}{2} \ln \frac{r_2 + r_3 - 2l_2}{r_2 + r_3 + 2l_2}$$

a) if $\quad k_1 = k_2 = 1 \quad$ (i.e. two Schwarzschild line masses) then according to (4.7.3) we have

$$g_{oo\,A} = \frac{1}{2} \; \frac{r_1 + r_2 - 2(l_1 + l_2)}{r_1 + r_2 + 2(l_1 + l_2)}$$

Therefore the equation for the horizon is given by

$$r_1 + r_2 - 2(l_1 + l_2) = 0$$

i.e. the equation of the horizon for a Schwarzschild line mass with $l = l_1 + l_2$, a horizon which is obviously non-singular. So we see that the state of the horizon changes from a singular situation to a non-singular situation.

b) if $\quad k_1 \neq k_2 \quad$ then $\quad g_{oo\,A} \quad$ takes the form

$$g_{oo\,A} = \left(\frac{r_1 + r_2 - 2l_1}{r_1 + r_2 + 2l_1}\right)^{k_1} \cdot \left(\frac{r_2 + r_3 - 2l_2}{r_2 + r_3 + 2l_2}\right)^{k_2}$$

so $\quad g_{oo\,A} = 0 \quad$ if and only if

$$r_1 + r_2 - 2l_1 = 0 \qquad \text{or} \qquad r_2 + r_3 - 2l_2 = 0$$

But in that case the horizon would be singular as long as it does not correspond to the horizon of Schwarzschild line mass.

$$+ 2r(\lambda_{2r}^2 - \lambda_{2z}^2) + 2r(\lambda_{1r}\lambda_{2r} - \lambda_{1z}\lambda_{2r})$$

$$\nu_{11z} + \nu_{22z} + \nu_{12z} = 2r\lambda_{1r}\lambda_{1z} + 2r\lambda_{2r}\lambda_{2z} +$$

$$+ 2r(\lambda_{1r}\lambda_{2z} + \lambda_{1z}\lambda_{2r})$$

Remembering that individually the pairs λ_1, ν_{11} and λ_2, ν_{22} are solutions of the equations (3.1.8) (3.1.10) then we have to have the following equalities:

$$\nu_{11r} = 2r(\lambda_{1r}^2 - \lambda_{1z}^2) \qquad (4.7.7)$$

$$\nu_{22r} = 2r(\lambda_{2r}^2 - \lambda_{2z}^2) \qquad (4.7.8)$$

$$\nu_{12r} = 2r(\lambda_{1r}\lambda_{2r} - \lambda_{1z}\lambda_{2r}) \qquad (4.7.9)$$

$$\nu_{12z} = 2r(\lambda_{1r}\lambda_{2z} + \lambda_{2z}\lambda_{1z}) \qquad (4.7.10)$$

multiplying equations (4.7.7) and (4.7.8) by K_1^2 K_2^2 respectively and (4.7.9) (4.7.10) by $K_1 K_2$ and adding then together we conclude that the functions (4.7.1) and (4.7.2) are solutions of Einstein's equations in the general case where $K_1 \neq K_2$.

We are now able to work out the second step i.e. to see what will be the final state of the horizon under the above quasi static approach of the two line masses. To start with we write the g_{00} component of the metric tensor before (g_{00b}) and after (g_{00A})the approach of the two line masses.

The above analysis suggests to us that generally the Bach and Weyl solution possesses a singular horizon. However considering Bach and Weyl metric as the separation distance goes to zero then the horizon can change from a singular to a non-singular one depending on the values of K_1 and K_2.

CHAPTER FIVE

STATIC AXIALLY SYMMETRIC ELECTROVACUUM

5.1 Maxwell's Equations in the Presence of a Gravitational Field

Maxwell's equations of the electromagnetic field in a flat space-time in the theory of relativity have the following form:

$$\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} J^i \qquad (5.1.1)$$

$$\frac{\partial F^{ik}}{\partial x^l} + \frac{\partial F^{kl}}{\partial x^i} + \frac{\partial F^{li}}{\partial x^k} = 0 \qquad (5.1.2)$$

where x^1, x^2, x^3 are Cartesian coordinates, and

$$F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k} \qquad (5.1.3)$$

is a second rank tensor with $A^i = (\phi, \vec{A})$ the four potential, and $J^i = (c\rho, \rho\vec{v})$ is the current four-vector acting as the source of the electromagnetic field. Equations (5.1.1) and (5.1.2) can be easily generalized so they are applicable in any arbitrary system of reference. This could be achieved by using the principle of General Covariance which states that all the systems of reference are equivalent for the formulation of the laws of nature. Thus physical laws have to be written in a form which is invariant under any coordinate transformation. ie. they should conform to covariant form. This could be done by writing the various quantities appearing in the physical laws in a tensor form, since tensor equations have the important property of being invariant under any coordinate transformation. Equations (5.1.1) and (5.1.2) obviously are not written in a covariant form since the partial derivative of a tensor is not a tensor. However, from the tensor calculus, it is known that the covariant derivative of a tensor is also a tensor. So the covariant form of (5.1.1) and (5.1.2) can be achieved upon replacing the partial derivative by a covariant one. We should remember that if A^{ik} is a second rank tensor then the covariant derivative with respect to the x^{ℓ} coordinate is given by

$$A^{ik}_{;\ell} = \frac{\partial A^{ik}}{\partial x^{\ell}} + \Gamma^{i}_{m\ell} A^{mk} + \Gamma^{k}_{m\ell} A^{im} \qquad (5.1.4)$$

while

$$A_{ik;\ell} = \frac{\partial A_{ik}}{\partial x^{\ell}} - \Gamma^{m}_{i\ell} A_{mk} - \Gamma^{m}_{k\ell} A_{im} \qquad (5.1.4a)$$

with $\Gamma^{i}_{m\ell}$ the affine connection of the space-time given by (2.4.7). So the covariant form of (5.1.1) and (5.1.2) is

$$F^{ik}_{;k} = -\frac{4\pi}{c} J^{i} \qquad (5.1.5)$$

$$F^{ik}_{;\ell} + F^{k\ell}_{;i} + F^{\ell i}_{;k} = 0 \qquad (5.1.6)$$

and

$$F^{ik} = A^{i}_{;k} - A^{k}_{;i} \qquad (5.1.7)$$

using (5.1.3) (5.1.4) and the fact

$$A^{i}_{;k} = \frac{\partial A^{i}}{\partial x^{k}} - \Gamma^{i}_{kl} A^{l}$$

Equations (5.1.7) and (5.1.6) are respectively reduced to

$$\frac{\partial F^{ik}}{\partial x^{l}} + \frac{\partial F^{kl}}{\partial x^{i}} + \frac{\partial F^{li}}{\partial x^{k}} = 0$$

$$F^{ik} = \frac{\partial A^{i}}{\partial x_{k}} - \frac{\partial A^{k}}{\partial x^{i}}$$

Exploiting the fact that $F^{ik} = - F^{ki}$, (5.1.5) reduces to

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{k}}\left(\sqrt{-g}\, F^{ik} \right) = - \frac{4\pi}{c} J^{i} \qquad (5.1.9)$$

We also have to determine the current four vector in curvilinear coordinates. Keeping in mind that

1. The invariant spatial volume element in curvilinear coordinates is given by

$$dV = \sqrt{\gamma}\, dx^{1} dx^{2} dx^{3}$$

where γ is the determinant of the three space metric given by

$$\gamma_{\alpha\beta} = - g_{\alpha\beta} + \frac{g_{0\alpha}\, g_{0\beta}}{g_{00}}$$

(Landau and Lifshitz 1962) and,

2. The charge density ρ satisfies the equation $de = \rho\sqrt{\gamma}\, dV$ where de is the charge located within the volume element dV, then

$$J^i = \frac{\rho c}{\sqrt{g_{00}}} \frac{dx^i}{dx^0} \tag{5.1.11}$$

Eqs. (5.1.5), (5.1.6), (5.1.7) in conjunction with (5.1.11) are the covariant equations of the electromagnetic field valid in any system of reference. According to the principle of equivalence, the above equations have the same form when a gravitational field is present where the influence of the gravitational field on the electromagnetic field is expressed by the presence of g_{ik} in the equations (5.1.5) (5.1.6), (5.1.7) and (5.1.11)

We will now deal with the reverse problem. i.e. Does the electromagnetic have any influence on the gravitational field and how will we describe it? We already know that a charged material distribution in the special theory of relativity is described by the energy-momentum tensor $T_{ik} = E_{ik} + M_{ik}$ where E_{ik} is the energy-momentum tensor of the electromagnetic field

$$T^{ik} = \frac{1}{4\pi}\left(-F^{il}F^{k}_{\ l} + \frac{1}{4} g^{ik} F_{lm} F^{lm}\right)$$

and E_{ik} is the energy-momentum tensor describing the material of the system

The tensor T^{ik} obeys the conservation law

$$\frac{\partial T^{ik}}{\partial x^k} = 0$$

In the theory of general relativity we have to use the energy-momentum tensor T_{ik}, describing the sources of the gravitational field, as $M_{ik} + E_{ik}$ where E_{ik} is the energy-momentum tensor of the electromagnetic field in curved space-time.

$$E_{ik} = \frac{1}{4\pi}\left(-F_{il}F^{l}_{k} + \frac{1}{4}F_{lm}F^{lm}g_{ik}\right)$$

The energy-momentum tensor satisfies the conservation laws

$$(M^{ik} + E^{ik})_{;k} = 0$$

So the equation of the gravitational field is

$$R_{ik} - \frac{1}{2}g_{ik}R = -\kappa\,(M_{ik} + E_{ik})$$

The above field equations are called the Einstein-Maxwell equations. In a region of space-time where $J^i = 0$ and $M_{ik} = 0$, the equations are reduced to

$$R_{ik} - \frac{1}{2}g_{ik}R = -\kappa E_{ik} \qquad (5.1.13)$$

The space-time domain which is characterized by $J^i = 0$ and $M_{ik} = 0$ will be referred to as electrovacuum. However the Einstein-Maxwell equations for electrovacuum can take a simpler form using the fact that $E^{i}_{i} = 0$. An equivalent representation of (5.1.13) is

$$R^{i}_{k} - \frac{1}{2}\delta^{i}_{k}R = -\kappa E^{i}_{k}$$

or for $i = k$

$$R^{i}_{i} - \frac{1}{2}\delta^{i}_{i}R = -\kappa E^{i}_{i}$$

or $$R^{i}_{i} - 2R = 0$$

which implies

$$R = 0$$

So equations (5.1.3) take the form

$$R_{ik} = -\kappa E_{ik}$$

which is the final form of Einstein-Maxwell equations in electro-vacuum.

The above equations have been used by Weyl (1918) who showed that for any static, axially-symmetric electrovacuum the line element can be written as

$$ds^2 = e^{w} dt^2 - e^{w-v}(dr^2 + dz^2) - r^2 e^{-w} d\varphi^2$$

(Here the units have been choosen such that $c = G = 1$) where r, z, φ are cylindrical polar coordinates with $w = w(r, z)$ $v = v(r, z)$ and the line element has precisely the same form as in the absence of electromagnetic field, (3.1.5).

Weyl also obtained a particular class of solutions for an axially-symmetric electrovacuum that involves a functional relation between the electrostatic potential Φ and g_{00} component of the metric tensor.

Majumdar (1947) and Papapetrou (1948) working on this independently, showed that if there is any functional relation between g_{00} and Φ , then it must be

$$g_{00} = 1 + A\Phi + \Phi^2 \qquad (5.1.14)$$

where A and B are constants. The above relation is a general one and does not assume any special symmetry.

Authors who have developed methods of getting solutions of Einstein-Maxwell equations are Cooperstock-de la Cruz, (1978), Misner-Rainch-Wheeler (1957), Gautreau-Hoffman, (1970). etc. In the next pages we will describe briefly the method due to Cooperstock and de La Cruz (1978) and will apply their procedure in order to obtain the charged metric for the case of a Curzon particle and a generalized line mass.

5.2 The Formalism

We adopt the Weyl line element ie.

$$ds^2 = e^{w} dt^2 - e^{w-v}(dr^2 + dz^2) - e^{-w} r^2 d\varphi^2 \qquad (5.2.1)$$

and assume an asymptotic flat space which implies that

$$g_{00} = 1 - \frac{2m}{r} + O(\tfrac{1}{r^2})$$
$$\phi = \frac{q}{r} \qquad (5.2.2)$$

with m and q the total mass and charge on the sources. Comparison between (5.1.14) and (5.2.2) implies

$$A = \frac{-2m}{q}$$

Weyl showed that the function defined as

$$\chi = \int \left(1 - \frac{2m}{q}\phi + \phi^2\right)^{-1} d\phi \qquad (5.2.3)$$

satisfies the Laplace equation ie. $\chi_{rr} + \chi_{zz} + \frac{\chi_r}{r} = 0$
Integration of (5.2.3) gives

$$\chi = \frac{a}{1-a^2} \ln\left[\frac{\phi - a}{\phi - a^{-1}}\right] \qquad (5.2.4)$$

where a is a parameter defined by

$$\frac{1+a^2}{2a} = \frac{m}{q} \tag{5.2.5}$$

Introducing a function

$$f = \frac{1}{a^2} e^{\frac{a^2-1}{a} \chi} \tag{5.2.6}$$

and eliminating χ between (5.2.6) and (5.2.4) gives

$$\phi = a \frac{f-1}{a^2 f - 1} \tag{5.2.7}$$

then from (5.2.2)

$$e^{w} = (1-a^2) \frac{f-1}{(a^2 f - 1)^2} \tag{5.2.8}$$

From (5.2.6) we get

$$\nabla^2 (\ln f) = 0 \tag{5.2.9}$$

Hence solutions of the Laplace equation can be used to generate the e^{w} and ϕ . Moreover the field equations for γ yield the following system of differential equations:

$$\begin{aligned} \gamma_r &= \frac{\partial \gamma}{\partial r} = \frac{r}{2}\left[(\ln f)_r^2 - (\ln f)_z^2\right] \\ \gamma_z &= \frac{\partial \gamma}{\partial z} = 2r(\ln f)_r (\ln f)_z \end{aligned} \tag{5.2.10}$$

which are identical in form with equations (3.1.8). So according to that formalism eqs. (5.2.10), (5.2.7), (5.2.8), in conjunction with (5.2.5) determine the static axially symmetric electrovacuum.

5.3 Gauss Theory in Curved Space-Time

From the equation (5.1.9), we get

$$\frac{1}{\sqrt{g_{oo}}} \cdot \frac{1}{\sqrt{\gamma}} \cdot \frac{\partial}{\partial x^{\alpha}} \left(\sqrt{\gamma} \sqrt{g_{oo}}\, F^{o\alpha} \right) = - \frac{4\pi \rho c}{\sqrt{g_{oo}}}$$

or putting

$$\varepsilon^{\alpha} = \sqrt{g_{oo}}\, F^{o\alpha}$$

then

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^{\alpha}} \left(\sqrt{\gamma}\, \varepsilon^{\alpha} \right) = - 4\pi \rho c$$

or

$$\varepsilon^{\alpha}_{;\alpha} = - 4\pi \rho c \qquad (5.3.1)$$

where $\varepsilon^{\alpha}_{;\alpha}$ represents the covariant derivative of the three vector ε^{α} with respect to the spatial metric

$$\gamma_{\alpha\beta} = - g_{\alpha\beta} + \frac{g_{o\alpha} g_{o\beta}}{g_{oo}}$$

integrating equation (5.3.1) over a volume V we get

$$\int_V \varepsilon^{\alpha}_{;\alpha} \sqrt{\gamma}\, dV = - 4\pi \rho c \int_V \rho \sqrt{\gamma}\, dV \qquad (5.3.2)$$

The Gauss divergence theorem in Euclidean geometry states that for any vector field $\vec{X}$ the following identity is true

$$\int_V \vec{\nabla} \cdot \vec{X}\, dV = \oint_{S(V)} \vec{X} \cdot \vec{\eta}\, ds \qquad (5.3.3.a)$$

where $S(V)$ is a simple surface bounding the volume V and $\vec{\eta}$ is the normal unit vector on S while $\vec{\nabla}$ is the gradient operator.

In the case of a curved three dimensional space the above integral identity should be

$$\int X^{a}_{;a} \sqrt{\gamma}\, dV = \oint X^{a} \sqrt{\gamma}\, ds_{\alpha} \tag{5.3.3}$$

where dv and ds respectively are the coordinate volume and surface elements while $\sqrt{\gamma}\, dV$ and $\sqrt{\gamma}\, ds$ express the invariant volume and surface element respectively. Combining (5.3.2) and (5.3.3) we obtain

$$\int_{V} \varepsilon^{a}_{;\alpha} \sqrt{\gamma}\, dV = \oint_{S(V)} \varepsilon^{a} \eta_{a} \sqrt{\gamma}\, ds \tag{5.3.4}$$

Next we will give a simpler expression of the Gauss divergence theorem in the case of static axially symmetric electrovacuum. For the line element (5.2.1) equation (5.3.4) gives

$$\oint_{S(V)} \varepsilon^{a} \eta_{a} \sqrt{\gamma}\, ds = \oint_{S(V)} \sqrt{g_{oo}}\, F^{oa} \eta_{a} \sqrt{\gamma}\, ds \tag{5.3.5}$$

Now

$$\sqrt{g_{oo}} \sqrt{\gamma} = r e^{v-w}$$

From $F^{kl} = g^{kn} g^{lm} F_{nm}$, we get

$$F^{oa} = g^{oo} g^{\alpha m} F_{om}$$

The only non-zero components of F^{oa} are

$$F^{o1} = -e^{-w} e^{w-v} \frac{\partial \phi}{\partial x^1}$$

$$F^{o2} = -e^{-w} e^{w-v} \frac{\partial \phi}{\partial x^2}$$

where ϕ is the electrostatic potential. Hence (5.3.5) yields

$$\oint_{S(V)} \varepsilon^a \eta_a \sqrt{\gamma}\, ds = \oint r e^{-w} \left[\frac{\partial \phi}{\partial x^1} \eta^1 ds + \frac{\partial \phi}{\partial x^2} \eta^2 ds \right] \tag{5.3.6}$$

But according to the formalism 5.2 the electrostatic potential is given by

$$\phi = a \frac{f-1}{a^2 f - 1}$$

so (5.3.6) yields

$$\oint \varepsilon^a \eta_a \sqrt{\gamma}\, ds = -\frac{a}{1-a^2} \oint r e^{-w} \frac{1-a^2}{(a^2 f - 1)^2} \cdot \frac{1}{f}$$

$$\cdot \left(\frac{\partial f}{\partial x^1} \eta^1 ds + \frac{\partial f}{\partial x^2} \eta^2 ds \right)$$

Finally (5.3.3) gives.

$$4\pi \int \rho \sqrt{\gamma}\, dv =$$

$$= \frac{a}{1-a^2} \oint r e^{-w} \frac{1-a^2}{f(a^2 f - 1)^3} \left(\frac{\partial f}{\partial x^1} \eta^1 ds + \frac{\partial f}{\partial x^2} \eta^2 ds \right) \quad (5.3.7)$$

In the following we will be writing the above expression for finding the relativistic charge density ρ for different charge distributions.

5.4 Metric for a Charged Curzon Particle and a General Line Mass

In chapter 4 we obtained the metric for a combination of the Curzon particle and a generalized line mass. We see that the function

$$\lambda = \frac{K}{2} \ln \frac{r_1 + r_2 - 2\ell}{r_1 + r_2 + 2\ell} - \frac{L}{r_3}$$

satisfies the Laplace equation. If

$$\ln f = 2 C \lambda$$

where C is a constant, then $\nabla^2 \ln f = 0$. Hence the function

$$\ln f = C_1 \ln \frac{r_1 + r_2 - 2\ell}{r_1 + r_2 + 2\ell} - \frac{2 C_2}{r_3} \quad (5.4.1)$$

with C_1 and C_2 constants generates the metric corresponding to the charged case. Then equations (5.2.10) take the forms

$$V_r = \frac{\partial V}{\partial r_1} = \frac{r}{2}\left[C^2 \lambda_r - C^2 \lambda_z \right] = C^2 \gamma_r$$

$$V_z = \frac{\partial V}{\partial z} = \frac{r}{2}\left[C^2 \lambda_r \lambda_z \right] = C^2 \gamma_z$$

so

$$V = C_1^2 \ln \frac{(r_1+r_2)^2 - 4\ell^2}{4 r_1 r_2} - \frac{C_2^2 r^2}{2 r_3^4} - \frac{C_1 C_2}{d} + \qquad (5.4.2)$$

$$+ C_1 C_2 \frac{2\ell z - r_3(r_1+r_2)}{r_3\left[\ell(r_3 - r_2) - d(r_1+r_2)\right]}$$

The other component of the metric tensor and the electrostatic potential are respectively given by the relations

$$e^{w} = (1-a^2)^2 \frac{\left(\frac{x-2\ell}{x+2\ell}\right)^{C_1} e^{-\frac{2C_2}{r_3}}}{\left(a^2\left(\frac{x-2\ell}{x+2\ell}\right)^{C_1} e^{-\frac{2C_2}{r_3}} - 1\right)^2} \qquad (5.4.3)$$

$$\phi = a \frac{\left(\frac{x-2\ell}{x+2\ell}\right)^{C_1} e^{-\frac{2C_2}{r_3}} - 1}{a^2\left(\frac{x-2\ell}{x+2\ell}\right)^{C_1} e^{-\frac{2C_2}{r_3}} - 1} \qquad (5.4.4)$$

where $\quad X = r_1 + r_2$

Equations (5.4.2), (5.4.3) and (5.4.4) determine the metric if the constants C_1 and C_2 are known. The asymptotic expansions of e^w and ϕ are of the form

$$e^w = 1 - 2\,\frac{1+a^2}{1-a^2}\,\frac{C_1 l + C_2}{r} + O\left(\frac{1}{r^2}\right)$$

$$\phi = \frac{2a}{1-a^2}\,\frac{C_1 l + C_2}{r} + O\left(\frac{1}{r^2}\right)$$

On comparing these with (5.2.2) one finds that the total (effective)mass, and total charge of the system are given by

$$m = \frac{1+a^2}{1-a^2}\,(C_1 l + C_2) \qquad (5.4.7)$$

$$q = \frac{2a}{1-a^2}\,(C_1 l + C_2) \qquad (5.4.8)$$

These formulas give the relation between the constants C_1 C_2 and the total mass and charge. It may be noted that m and q are respectively the total mass and charge as viewed by an observer at infinity.

The final distribution of charge is found below by an application of Gauss's theorem, while for the distribution of mass we assume the following

$$m_1 = \frac{1+a^2}{1-a^2}\,C_1 l \qquad (5.4.9)$$

$$m_2 = \frac{1+a^2}{1-a^2}\,C_2 \qquad (5.4.10)$$

where m_1 and m_2 will be respectively the mass on the "line mass" and on the "Curzon particle". Such a distribution of the total mass has been adopted since for the case that $q \rightarrow 0$ the results have to be in agreement with the results in the uncharged case.

5.5 Distribution of Charge

Substituting (5.4.1) in (5.3.7) we get

$$4\pi \int \rho \sqrt{\gamma}\, dV = \oint \frac{a}{1-a^2}\, r\, e^{-w} \frac{1-a^2}{(a^2 f-1)^2} \cdot \frac{1}{f} \cdot$$

$$\left[\frac{\partial}{\partial x^1}\left(c_1 \ln \frac{r_1+r_2-2l}{r_1+r_2+2l} - \frac{2c_2}{r_3}\right)\eta^1 ds + \right.$$

$$\left. + \frac{\partial}{\partial x^2}\left(c_1 \ln \frac{r_1+r_2-2l}{r_1+r_2+2l} - \frac{2c_2}{r_3}\right)\eta^2 ds \right]$$

or

$$4\pi \int \rho \sqrt{\gamma}\, dV = \frac{a}{1-a^2} \oint \frac{\partial}{\partial x^1}\left(c_1 \ln \frac{r_1+r_2-2l}{r_1+r_2+2l}\right) \cdot \qquad (5.5.1)$$

$$\cdot\, \eta^1 r\, d\varphi\, dz + \frac{\partial}{\partial x^2}\left(c_2 \ln \frac{r_1+r_2-2l}{r_1+r_2+2l}\right)\eta^2 r\, d\varphi\, dz +$$

$$+ \oint_{S(r)} \frac{a}{1-a^2}\left[\frac{\partial}{\partial x^1}\left(-\frac{2c_2}{r_3}\right)\eta^1 r\,d\varphi\,dz + \frac{\partial}{\partial x^2}\left(-\frac{2c_2}{r_3}\right)\eta^2 r\,d\varphi\,dr\right]$$

From the right hand side of the latter equation we can see that $r\,dr\,d\varphi$ and $r\,dz\,d\varphi$ express the surface elements in the three dimensional Euclidean space so equation (5.5.1) can be rewritten as follows

$$4\pi\int \rho\sqrt{\gamma}\,dV = \frac{a}{1-a^2}\oint \vec{\nabla}\left(-\frac{2c_2}{r_3}\right)\cdot\vec{\eta}\,ds +$$

$$+ \frac{a}{1-a^2}\oint \vec{\nabla}\left(c_1 \log\frac{r_1+r_2-2l}{r_1+r_2+2l}\right)\cdot\vec{\eta}\,ds$$

Applying Gauss's divergence theorem (5.3.3a) we get

$$4\pi\int \rho\sqrt{\gamma}\,dV = \frac{a}{1-a^2}\int \nabla^2\left(-\frac{2c_2}{r_3}\right)dV_{Eucl.} +$$

$$+ \frac{a}{1-a^2}\int \nabla^2\left(c_1 \ln\frac{r_1+r_2-2l}{r_1+r_2+2l}\right)dV_{Eucl.} \quad (5.5.1a)$$

where

$$dV_{Eucl} = r\,dr\,dz\,d\varphi$$

But from classical electrodynamics

$$\nabla^2\left(c_1 \log\frac{r_1+r_2-2l}{r_1+r_2+2l}\right) = 4\frac{c_1 k(z)\,\delta(r)}{r}$$

where $k(z)$ is a step function defined as follows

$$K(Z) = \begin{cases} 0 & \text{if} \quad Z > Z_1 \text{ and } Z < Z_2 \\ 1 & \text{if} \quad Z_2 < Z < Z_1 \end{cases}$$

and

$$\nabla^2 \left(- \frac{2 C_2}{r_3} \right) = 4\pi \, \delta(r_3)$$

where δ , in the above formulas, represent the Dirac Delta function. Then (5.5.1a) yields:

$$4\pi \rho = 4 \frac{C_1 K(Z) \delta(r)}{\sqrt{\gamma}} + 4\pi \frac{\delta(r_3)}{\sqrt{\gamma}}$$

From the above expression for the relativistic charge density we conclude that the charges are located at the point $r=0$ $Z=0$ and along the part of the Z axis between Z_1 and Z_2

The total charge at the origin is given by integrating the density ρ over a closed surface which intersects the Z axis at $Z < Z_2$ and encloses the origin. Then

$$q_2 = \frac{2 a C_2}{1 - a^2} \qquad (5.5.2)$$

Proceeding in a similar way, the charge between Z_1 and Z_2 is given by

$$q_1 = \frac{2 a C_1 \ell}{1 - a^2} \qquad (5.5.3)$$

5.6 The Regularity of the Metric

Equations (5.4.2), (5.4.3) and (5.4.4) determine the metric of the electrovacuum space-time. As we have discussed before, the criteria of asympotic flatness and elementary flatness have to be employed in order to accept the above solutions as physically consistent. Also in chapter 4 we saw that the corresponding uncharged case exhibits a singularity on the portion of Z axis between the line mass and the Curzon particle respectively. In the next paragraph, we will try to get rid of the singularity by a proper choice of the parameters m_1 m_2 and q_1 q_2.

The new asympotic flatness criterion requires $f \to 1$ asymptotically, which is satisfied here as can be seen by an inspection of (5.4.3), (5.4.4) and (5.4.2) The limits of e^w, V and Φ are as follows:

$$\lim_{\substack{r\to\infty \\ z\to\infty}} e^{w} = 1 \qquad \lim_{\substack{r\to\infty \\ z\to\infty}} V = 0$$

$$\lim_{\substack{r\to\infty \\ z\to\infty}} \Phi = 0$$

Hence the line element (5.2.1) becomes

$$ds^2 = dt^2 - dr^2 - dz^2 - r^2 d\varphi^2$$

That implies that at very large distances from the sources the geometry of the space-time tends to the usual geometry of the flat space-time of the special theory of relativity.

Elementary flatness criterion requires V to vanish along the Z axis. From (5.4.2) we have the following values for V :

$$V = \begin{cases} 0 & \text{for } z > z_1 \text{ or } z < 0 \text{ and } r = 0 \\ -\dfrac{c_1 c_2}{d(d+\ell)} & 0 < z < z_2 \text{ and } r = 0 \end{cases}$$

The values of the V function suggest that even in the charged case we generally need a strut in order to keep the system in static configuration. However the charged case allows the possibility of removing the strut by choosing carefully m_1 m_2 and q_1 q_2 . From (5.4.9), (5.4.10) we have the following expression for V :

$$V(r=0,\ 0<z<z_2) = -\frac{(1-a^2)}{(1+a^2)} \cdot \frac{m_1 m_2}{\ell} \cdot \frac{1}{d(d+\ell)} =$$

$$= \left[\frac{m_1 m_2}{\ell} - \left(\frac{2a}{(1-a^2)}\right)^2 \cdot \left(\frac{1-a^2}{1+a^2}\right)^2 \cdot \frac{m_1 m_2}{\ell} \right] \cdot \frac{1}{d(d+\ell)} =$$

$$= \frac{m_1 m_2 - q_1 q_2}{d(d+\ell)} \cdot \frac{1}{\ell}$$

So the possibility of making V zero occurs when

$$m_1 m_2 - q_1 q_2 = 0 \qquad (5.6.1)$$

Furthermore,

$$\frac{m_1}{m_2} = \frac{q_1}{q_2} \qquad (5.6.2)$$

So these two equations together yield

$$m_1^2 = q_1^2 \qquad m_2^2 = q_2^2 \qquad (5.6.3)$$

The above relation is the relativistic condition for balance between the "line mass" and the "point mass". Note that the equation (5.6.1) is the necessary condition for balance between a charged line mass and a charged point mass separated by a distance in Newtonian classical physics.

5.7 Evaluation of the Constants

The metric for the charged Curzon particle and a generalized line mass is described by equations (5.4.2) (5.4.3), (5.4.4) where the constants C_1 C_2 are related to the masses and charges m_i q_i $i=1,2$ via (5.4.7)(5.4.8). Further, in the expressions (5.4.7), (5.4.8) there is the parameter a defined by (5.2.5). In this section we will relate C_1 C_2 only with m_i q_i $i=1,2$ The roots of the equation

$$\frac{1+a^2}{2a} = \frac{m}{q}$$

can be real, or complex, depending on the ratio $\frac{m}{q}$. So we have the following cases:

1. $m^2 > q^2$, undercharged case, equation (5.2.5) has two real roots one of those between -1 and 1. Restricting a between -1 and 1 then from (5.4.7) and (5.4.8)

$$C_1 l = m_1 \sqrt{1 - \frac{q_1^2}{m_1^2}} \qquad (5.7.1)$$

$$C_2 = m_2 \sqrt{1 - \frac{q_2^2}{m_2^2}} \qquad (5.7.2)$$

2. $q^2 > m^2$ (overcharged case) we can define an angle β by the equation $\frac{m}{q} = \cos\beta$ and $0 < \beta < \pi$ Then equation (5.2.5) has the following roots:

$$\alpha = e^{\pm i \beta}$$

Then C_1 C_2 are given by

$$i C_1 l = q_1 \sqrt{1 - \frac{m_1^2}{q_1^2}} \qquad (5.7.3)$$

$$i C_2 = q_2 \sqrt{1 - \frac{m_2^2}{q_2^2}} \qquad (5.7.4)$$

3. $q^2 = m^2$, critically charged case, the C_1 C_2 can be obtained from (5.7.1) and (5.7.2) by taking the limits as $q_i^2 \longrightarrow m_i^2$ $i = 1, 2$ then

$$C_1 l = \lim_{q_1^2 \to m_1^2} m_1 \sqrt{1 - \frac{q_1^2}{m_1^2}} = 0$$

$$C_2 l = \lim_{q_2^2 \to m_2^2} m_2 \sqrt{1 - \frac{q_2^2}{m_2^2}} = 0$$

For those values of c_1 and c_2 , V vanishes while e^w and ϕ yield $\frac{0}{0}$ indeterminate form, and the equation

$$\frac{1+a^2}{2a} = \frac{m}{q}$$

has as roots $a = \pm 1$. The real values of e^w and ϕ can be found by taking the limit of (5.4.3) and (5.4.2) and by applying the following identity

$$a^x \equiv 1 + x \ln a + O(x^2)$$

then

$$e^w = \frac{1}{\left(\frac{m_1}{2\ell} \ln \frac{r_1 + r_2 - 2\ell}{r_1 + r_2 + 2\ell} - \frac{m_2}{r_3} + 1 \right)^2}$$

$$\phi = \frac{\frac{q_1}{2\ell} \ln \frac{r_1 + r_2 - 2\ell}{r_1 + r_2 + 2\ell} - \frac{q_2}{r_3}}{\frac{q_1}{2\ell} \ln \frac{r_1 + r_2 - 2\ell}{r_1 + r_2 + 2\ell} - \frac{q_2}{r_3} + 1}$$

5.8 The Limit of the Charged Bach and Weyl Metrics as the Separation Distance goes to Zero

In chapter four we have discussed the limit of Bach and Weyl metric as the separation distance goes to zero. In this chapter we will develop the charged Bach and Weyl metric and find under what conditions the metric for two charged "line masses" (as the separation distance between them goes to zero) can be replaced by one charged "line mass".

As we have seen the function (4.4.2) satisfies the Laplace equation. Thus the charged Bach and Weyl metric can be generated by the following function:

$$f = \left(\frac{r_1 + r_2 - 2l_1}{r_1 + r_2 + 2l_1}\right)^{c_1} \cdot \left(\frac{r_3 + r_4 - 2l_2}{r_3 + r_4 + 2l_2}\right)^{c_2} \qquad (5.8.1)$$

with c_1 and c_2 constants

Then e^W ϕ and V are given by

$$e^W = (1-a^2)^2 \frac{\left(\frac{r_1+r_2-2l}{r_1+r_2+2l}\right)^{c_1}\left(\frac{r_3+r_4-2l_2}{r_3+r_4+2l_2}\right)^{c_2}}{\left[a^2\left(\frac{r_1+r_2-2l_1}{r_1+r_2+2l_1}\right)^{c_1}\left(\frac{r_3+r_4-2l_2}{r_3+r_4+2l_2}\right)^{c_2} - 1\right]^2}$$

$$\phi = a \frac{\left(\frac{r_1+r_2-2l_1}{r_1+r_2+2l_1}\right)^{c_1}\left(\frac{r_3+r_4-2l_2}{r_3+r_4+2l_2}\right)^{c_2} - 1}{a^2\left(\frac{r_1+r_2-2l_1}{r_1+r_2+2l_1}\right)^{c_1}\left(\frac{r_3+r_4-2l_2}{r_3+r_4+2l_2}\right)^{c_2} - 1} \qquad (5.8.3)$$

$$V = c_1 \ln \frac{(r_1+r_2)^2 - 4l_1^2}{4r_1r_2} + c_1 \ln \frac{(r_3+r_4)^2 - 4l_2^2}{4r_3r_4} +$$

$$+ c_1c_2 \ln \frac{d}{d+l_1} + c_1c_2 \ln \frac{l_1r_4 - (l_1+d)r_1 - (l_1+l_2+d)r_2}{l_1r_3 - d\,r_1 - (l_1+d)r_2}$$

The asymptotic expansions of e^W and ϕ yield

$$e^W = 1 - 2\frac{1+a^2}{1-a^2}(l_1c_1 + l_2c_2) \qquad (5.8.4)$$

$$\phi = \frac{2a}{1-a^2}(l_1c_1 + l_2c_2) \qquad (5.8.5)$$

These equations imply

$$m = \frac{1+a^2}{1-a^2}\left(c_1 l_1 + c_2 l_2\right) \tag{5.8.6}$$

$$q = \frac{2a}{1-a^2}\left(c_1 l_1 + c_2 l_2\right) \tag{5.8.7}$$

Substituting (5.8.1) in (5.3.7) we get the following expression for the relativistic charge density

$$\rho = \frac{a}{1-a^2}\frac{c_1 \delta(r) k_1(z)}{\sqrt{\gamma}} + \frac{a}{1-a^2}\frac{c_2 \delta(r) k_2(z)}{\sqrt{\gamma}}$$

which implies the following charge distrubution

$$q_1 = \frac{2a}{1-a^2} l_1 c_1 \tag{5.8.8}$$

$$q_2 = \frac{2a}{1-a^2} l_2 c_2 \tag{5.8.9}$$

while the mass distribution is assumed to be

$$m_1 = \frac{1-a^2}{1+a^2} l_1 c_1 \qquad m_2 = \frac{1-a^2}{1+a^2} l_2 c_2 . \tag{5.8.10}$$

Assuming $z_3 = 0$ with $z_2 \to 0$ $z_1 \to 2l_1 + z_2$ and letting the separation distance $d \to 0$, the quantities e^w, ϕ, V take the following form

$$e^w = (1-a^2)^2 \frac{\left(\frac{r_1+r_2-2l_1}{r_1+r_2+2l_1}\right)^{c_1}\left(\frac{r_2+r_3-2l_2}{r_2+r_3+2l_2}\right)^{c_2}}{\left[a\left(\frac{r_1+r_2-2l_1}{r_1+r_2+2l_1}\right)^{c_1}\left(\frac{r_2+r_3-2l_2}{r_2+r_3+2l_2}\right)^{c_2}-1\right]^2} \tag{5.8.11}$$

$$\phi = a\,\frac{\left(\frac{r_1+r_2-2\ell_1}{r_1+r_2+2\ell_1}\right)^{c_1}\left(\frac{r_2+r_3-2\ell_2}{r_2+r_3+2\ell_2}\right)^{c_2}-1}{a^2\left(\frac{r_1+r_2-2\ell_1}{r_1+r_2+2\ell_1}\right)^{c_1}\left(\frac{r_2+r_3-2\ell_2}{r_2+r_3+2\ell_2}\right)^{c_2}-1} \tag{5.8.12}$$

$$V = c_1^2 \ln \frac{(r_1+r_2)^2-4\ell_1^2}{4r_1r_2} + c_2^2 \ln \frac{(r_2+r_3)^2-4\ell_2^2}{4r_2r_3} +$$

$$+ c_1c_2 \ln \frac{[\ell_1(r_3-r_2)-\ell_2(r_1+r_2)]\,r_2}{\ell_2[2\ell_1 z - r_2(r_1+r_2)]} \tag{5.8.13}$$

However, if $c_1 = c_2 = c$ then using the identities (4.7.3)(4.7.4) e^W, ϕ and V give

$$e^W = (1-a^2)^2 \frac{\left(\frac{r_1+r_3-2\ell}{r_1+r_3+2\ell}\right)^c}{\left[a^2\left(\frac{r_1+r_3-2\ell}{r_1+r_3+2\ell}\right)^c-1\right]^2}$$

$$\phi = a\,\frac{\left(\frac{r_1+r_3-2\ell}{r_1+r_3+2\ell}\right)^c-1}{\left[a^2\left(\frac{r_1+r_3-2\ell}{r_1+r_3+2\ell}\right)^c-1\right]^2}$$

$$V = c^2 \ln \frac{(r_1+r_3)^2-4\ell^2}{4r_1r_3}$$

with $\ell = \ell_1 + \ell_2$

The above functions exactly describe the metric of one charged line mass, (Cooperstock and de la Cruz (1978)). Summarizing, the limit of the charged Bach and Weyl metric tends to a single charged line mass metric if and only if $C_1 = C_2$ Further, following a procedure similar to one in 5.6, we can express C_1 and C_2 in terms of m_i q_i $i = 1, 2$ Thus C_1 and C_2 are given as follows:

$$c_i \ell_i = m_i \sqrt{1 - \frac{q_i^2}{m_2}} \qquad m^2 > q^2$$

$$c_i \ell_i = 0 \qquad m^2 = q^2$$

$$i\, c_i \ell_i = q_i \sqrt{1 - \frac{m_i^2}{q_i^2}} \qquad m^2 < q^2$$

$$i = 1, 2$$

If $C_1 = C_2$ then

$$\frac{m_1^2}{\ell_1^2} - \frac{q_1^2}{\ell_1^2} = \frac{m_2^2}{\ell_2^2} - \frac{q_2^2}{\ell_2^2} \qquad (5.8.14)$$

from (5.8.8), (5.8.9) and (5.8.10) we get

$$\frac{m_1}{q_1} = \frac{m_2}{q_2} \qquad (5.8.15)$$

or a combination of (5.8.14) and (5.8.15) yields

$$\frac{\dfrac{m_1^2}{l_1^2}}{\dfrac{q_1}{l_1^2}} = \frac{\dfrac{m_2^2}{l_2^2}}{\dfrac{q_2^2}{l_2^2}}$$

which implies

$$\frac{m_1^2}{l_1^2} = \frac{m_2^2}{l_2^2} \qquad \frac{q_1^2}{l_1^2} = \frac{q_2^2}{l_2^2} \qquad (5.8.16)$$

So $C_1 = C_2$ implies that the mass and charge densities of one line mass is equal to that of the other, in order that the limit of the metric be equal to that of Weyl line mass.

If $C_1 \neq C_2$ then the metric is described by (5.3.3) in conjunction with (5.2.5)

CHAPTER 6

METRICS FOR MULTIPOLE PARTICLES

6.1 Introduction

In chapter 4 we found the solutions of the field equations which are generated by the functions corresponding either to the Newtonian potential of a point mass or the potential of a uniform density line mass or a combination of these. However another class of solutions can be generated by using as λ function the Newtonian potential of a point mass which exhibits multipole structure. i.e.

$$\lambda = - \sum_{\eta=1}^{N} \frac{C_\eta}{R^{\eta+1}} P_\eta (\cos\Theta) \qquad (6.1.1)$$

Where $R^2 = r^2 + z^2$, C_η are constants while $P_\eta(\cos\Theta)$ represent the Legendre polynonial of η^{th} order and Θ is the angle between the z axis and the direction of R. So far a general solution of the field equations generated by (6.1.1) has not been found. The main difficulty of doing so is on one hand the complicated form of λ and on the other hand the problem of the integration of the system (3.1.8)

Geroch (1968, 1970) has proved that the metrics which are generated by (6.1.1) for every value of N possess multipole moments which generally are different from those multipole moments of classical mechanics corresponding to (6.1.1)

That implies that the sources of those solutions exhibit multipole structure.

Szekeres (1968) has obtained solutions which are generated by a point mass possessing a dipole moment and a quadrupole moment while Rosen (1959) discovered a solution which was obtained by Schwarzschild line mass superimposed with a quadrupole moment. In our work we will obtain a solution which is generated by a Curzon particle superimposed with a dipole moment. In our work, in order to simplify the integration of (3.1.8) we will use the Szekeres (1968) coordinates. We will also find and examine the metric generated by a Curzon particle and a point dipole where now it is assumed that the two particles are separated. In order to comply with the results of the Geroch works we will refer to the metrics which are generated by (6.1.1) as metrics corresponding to multipole particles.

6.2 Szekeres Formulation

Adopting Szekeres independent variables $\bar{\jmath}$ and $\jmath$ as

$$\jmath = r + i z$$

$$\bar{\jmath} = r - i z$$

and noticing that

$$\frac{\partial}{\partial J} = \frac{1}{2}\left(\frac{\partial}{\partial r} - i\frac{\partial}{\partial z}\right)$$

$$\frac{\partial}{\partial \bar{J}} = \frac{1}{2}\left(\frac{\partial}{\partial r} + i\frac{\partial}{\partial z}\right),$$

the equations defining the γ function

$$\gamma_r = r(\lambda_r^2 - \lambda_z^2)$$

$$\gamma_z = 2r\lambda_r\lambda_z$$

may be replaced by the following complex equation

$$\frac{\partial \gamma}{\partial J} = (J + \bar{J})\left(\frac{\partial \lambda}{\partial J}\right)^2 \qquad (6.2.1)$$

The advantage of this method is that if we are given λ as a function of J and $\bar{J}$, the γ function can be obtained from (6.2.1) by simple integration. Szekeres (1968) applied the above formalism and derived the metric that corresponds to a dipole potential i.e.

$$\lambda = -\frac{D\cos\Theta}{R^2} = \frac{1}{2}Di(J-\bar{J})(J\bar{J})^{-3/2} \qquad (6.2.2)$$

where D is the dipole moment of the point dipole (for the definition of the point dipole see discussion in the next paragraph) $R^2 = r^2 + z^2$ and Θ is the angle between the z axis and the direction of the unit vector along R (see fig. 6) Substituting (6.2.2) into (6.2.1), the γ

function turns out to be

$$\gamma = \frac{D^2 r^2 (r^2 - 8z^2)}{4R^8}$$

6.3 Metric When Monopole-Dipole are Situated at the Same Point

In this paragraph we will obtain the metric for the function

$$\lambda = -\frac{m}{R} - \frac{D\cos\theta}{R^2}$$

corresponding to a Newtonian potential of a point mass with dipole moment. In classical electrodynamics we define a dipole as a system of two equal and opposite charges q separated by a distance d . The quantity $D = qd$ is called the dipole moment of the system. From the above we can construct the point-dipole under the following assumptions.

$$q \longrightarrow \infty \qquad d \longrightarrow 0$$

with D remaining finite. If we want to carry this definition in the case of a mass distribution, we have to introduce the concept of negative mass. The above assumption seems to be artificial and nonphysical as we are familiar with only one sign of mass. However, we will not deal with the problem of whether or not negative mass exists in the universe. (note that some authors e.g. (Bondi 1967) have considered that problem).

So far we are taking soluti ons of the Laplace equation and from these, generate exact solutions of the Einstein equations. We now consider the potential of a point mass dipole as solution of Laplace's equation disregarding whether or not there is negative mass in nature. This point mass dipole is defined in a similar way as the point charge dipole, and is characterized by the quantity

$$D = \lim_{\substack{d \to 0 \\ m \to \infty}} m \cdot d$$

where d is the separation between the positive mass m and the negative mass $-m$. It is easy to see that the gravitational potential of the above system is given by

$$\lambda = \frac{D \cos \Theta}{R^2}$$

Taking

$$\lambda = -\frac{m}{R} - \frac{D \cos \Theta}{R^2}$$

we see that $\nabla^2 \lambda = 0$. Changing r z coordinates to $\mathfrak{J}$ $\bar{\mathfrak{J}}$ coordinates, the above equation becomes

$$\lambda = -\frac{m}{(\mathfrak{J}\bar{\mathfrak{J}})^{\frac{1}{2}}} - \frac{1}{2} D i (\mathfrak{J} - \bar{\mathfrak{J}})(\mathfrak{J}\bar{\mathfrak{J}})^{-\frac{3}{2}} =$$

$$= \lambda_1 + \lambda_2$$

where λ_1 is proportional to m, and λ_2 to D. Then the equation (6.2.1)yields

$$\frac{\partial \nu}{\partial J} = (J+\bar{J})\left[\left(\frac{\partial \lambda_1}{\partial J}\right)^2 + \left(\frac{\partial \lambda_2}{\partial J}\right)^2 + 2\left(\frac{\partial \lambda_1}{\partial J}\right)\left(\frac{\partial \lambda_2}{\partial J}\right)\right]$$

where the terms

$$\left(\frac{\partial \lambda_1}{\partial J}\right) \qquad \left(\frac{\partial \lambda_2}{\partial J}\right)$$

and

$$\left(\frac{\partial \lambda_1}{\partial J}\right)\left(\frac{\partial \lambda_2}{\partial J}\right) \qquad \text{generate}$$

respectively the monopole, dipole, and the interaction part of the ν function. On substituion of the partial derivatives of λ into (6.2.1) we get

$$\frac{\partial \nu}{\partial J} = (J+\bar{J})\left[\frac{m^2}{4\bar{J}J^3} - \frac{1}{16} D^2 \frac{(3\bar{J}-J)^2}{\bar{J}^5 J^3} + \frac{1}{4} m D i (3\bar{J}-J)\right]$$

or

$$\nu = \frac{1}{4}\int \frac{m^2(J+\bar{J})\,dJ}{\bar{J}J^3} - \frac{1}{16}\int \frac{D^2(\bar{J}+J)(3\bar{J}-J)^2}{J^5\bar{J}^3}\,dJ + \frac{1}{4}\int \frac{(J+\bar{J})(3\bar{J}-J)m}{dJ^{-1}}$$

A straightforward evaluation of the above integrals gives

$$\nu = -\frac{m^2}{4}\left(\frac{1}{J\bar{J}} + \frac{1}{2\bar{J}^2}\right) + f_1(\bar{J}) +$$

$$+ \frac{1}{16} D^2 \left(\frac{1}{J\bar{J}^3} - \frac{5}{2J^2\bar{J}^2} + \frac{1}{J^3\bar{J}} + \frac{9}{4J^4}\right) + f_2(\bar{J}) +$$

$$+ \frac{mDi}{4}\left(-\frac{1}{\mathfrak{J}^2\bar{\mathfrak{J}}} + \frac{1}{\mathfrak{J}^2\bar{\mathfrak{J}}} + \frac{1}{\mathfrak{J}\bar{\mathfrak{J}}^2} - \frac{1}{\mathfrak{J}^3}\right) + f_3(\bar{\mathfrak{J}})$$

where $f_i(\bar{\mathfrak{J}})$ $i = 1, 2, 3$ are the complex functions of the variable, $\bar{\mathfrak{J}}$. These functions are found by utilizing the fact that ν has to be a real function. From inspection of ν , the $f_i(\bar{\mathfrak{J}})$ functions are given as follows :

$$f_1(\bar{\mathfrak{J}}) = -\frac{m^2}{8\bar{\mathfrak{J}}^2}$$

$$f_2(\bar{\mathfrak{J}}) = \frac{9}{16}\frac{D^2}{4\bar{\mathfrak{J}}^4}$$

$$f_3(\bar{\mathfrak{J}}) = \frac{mDi}{4\bar{\mathfrak{J}}^3}$$

rewriting the function ν in r, z variables one finds

$$\nu = -\frac{m^2 r^2}{2R^4} + \frac{D^2 r^2 (r^2 - 8z^2)}{4R^8} - \frac{2mDr^2 z}{R^6}$$

The metric which is described by the functions λ and ν obviously satisfies both the elementary flatness and asymptotic flatness criteria and it is physically acceptable.

6.4 Metric Where a Monopole and Dipole are Separated

In this paragraph we will find and examine the metric which is derived from a λ function due to the gravitational potential of a monopole and dipole which are separated from each other.

For this case, λ is given by

$$\lambda = -\frac{m}{R_{-a}} - \frac{D\cos\theta}{R_a^2}$$

where

$$R_a^2 = r^2 + (z-a)^2$$

$$R_{-a}^2 = r^2 + (z+a)^2$$

(see figure 7)

In order to find the ν function we can use the differential equation (6.2.1). Szekeres has found the metric corresponding to the following form of λ function:

$$\lambda = -\frac{M_1}{R_a} - \frac{D\cos\theta}{R_a^2} - \frac{m}{R_{-a}}$$

and the ν function is given by the following lengthy expression:

$$\nu = -\frac{M_1^2 r^2}{2R_a^4} + \frac{D^2 r^2 (r^2 - 8(z-a)^2)}{4R_a^8} - \frac{m^2 r^2}{2R_{-a}^4} -$$

$$-\frac{2mM_1(R^2 - a^2)}{4a^2 R_a R_{-a}} + \frac{mD(a^4 - 2za^3 - 2r^2a^2 + 2zR^2a - R^4)}{2a^3 R_a^3 R_{-a}} -$$

$$-\frac{2M_1 D r^2 (z-a)}{R_a^6}$$

with

$$R^2 = r^2 + z^2$$

If we take the limit of the above metric as $M \to 0$ then the new metric corresponds to the case of separated monopole dipole. The limits of λ and ν are given as follows

$$\lambda = -\frac{m}{R_{-a}} - \frac{D\cos\theta}{R_a^2}$$

$$\nu = -\frac{m^2 r^2}{2R_{-a}^4} + \frac{D^2 r^2[r^2 - 8(z-a)^2]}{4R_a^8} + \frac{mD(a^4 - 2za^3 - 2r^2a^2 + 2zR_{-a}^2 - R^4)}{2a^3 R_a^3 R_{-a}}$$

these are solutions of (6.2.1) , as one can easily verify.

6.5 Behaviour of the Metric at Infinity

The asympotic flatness condition requires

$$\lim_{\substack{r\to\infty \\ z\to\infty}} \lambda = 0 \qquad \lim_{\substack{r\to\infty \\ z\to\infty}} \nu = 0$$

The limit of the λ function satisfies the above relation but not the limit of the ν function. The limit of the ν function is as follows :

$$\lim_{\substack{r\to\infty \\ z\to\infty}} \nu = -\frac{mD}{2a^3}$$

which is a constant. Thus, in order to satisfy the asymptotic flatness condition we add to the expression of the ν

function, the constant $\frac{mD}{2a^3}$. This new function $\nu + \frac{mD}{2a^3}$ also satisfies equation (3.1.8) as we can easily verify. Elementary flatness requires $\nu = 0$ on the Z axis but

$$\nu = \begin{cases} 0 & |z| > a \\ \frac{mD}{a^3} & |z| < 0 \end{cases}$$

Thus we again face the same problem that we had in the case of a Curzon particle and a generalized line mass. (Chapter 4) Thus as before, the non-zero value of the function ν in the interval ($-a$, a) expresses the necessity of a supporting strut in order to keep the bodies in a static configuration. Further, from classical gravitation the gravitational force on a monopole located on the Z axis at distance r due to point dipole is given by

$$\vec{F} = -m \nabla \Phi = -m \nabla \left(-\frac{D\cos\theta}{r^2}\right)$$

or

$$\vec{F} = \frac{2Dm}{r^3}$$

Thus even in the case of multipole potential the non-zero value of ν function gives the force between the two distributions of matter. In the next section we will find the charged metric of the above system and we examine the possibility of getting rid of the singularity.

6.6 Transition to the Charged Case

In chapter 5 we developed the metric for a charged Curzon particle and a generalized line mass. We also found the necessary condition for balance between the charged "point" and the "charged line mass". In the last section, we saw that for the case of a monopole and a multipole particle there exists a singularity manifested by the breakdown of the elementary flatness criterion. Authors have gotten rid of that singularity by introducing negative mass (Israel and Khan 1964) (Synge 1964) or by a superposition of multipole particles (Szekeres 1968) In the next section we will develop the charged metric having as a main purpose, the removal of the singularity. As we discussed before, the removal of the singularity implies balance between the monopole and dipole without any supporting strut. Cooperstock and de la Cruz (1978) found the necessary condition for balance between charged Curzon particles while Szekeres (1968) found the condition for balance between a monopole dipole together and a monopole.

Further we are interested in finding a corresponding condition for balance between a "monopole" and a "dipole".

The formalism developed in 5.2 is applicable in the case of the metric corresponding to the multipole particle since it does not impose any restriction on the form of function. Then, according to the previous paragraph the function

$$f \equiv e^{-\frac{C_1}{R_{-a}} - \frac{C_2 \cos\theta}{R_a^2}} \qquad (6.6.1a)$$

will generate the charged metric. And the quantities e^{w}, ν, ϕ are given as follows:

$$e^{w} = (1-a^{2})^{2}\,\frac{e^{\frac{C_1}{R_{-a}} - \frac{C_2\cos\theta}{R_a^{2}}}}{\left(a^{2}e^{-\frac{C_1}{R_{-a}} - \frac{C_2\cos\theta}{R_a^{2}}} - 1\right)^{2}}$$

$$V = -\frac{C_1^{2}r^{2}}{2R_{-a}^{4}} + \frac{C_2^{2}r^{2}\left[r^{2} - 8(z-a)^{2}\right]}{4R_a^{8}} + \frac{C_1C_2\,(a^{4} - 2za^{3} - 2r^{2}a^{2})}{2a^{3}R_a^{3}R_{-a}}$$

$$\phi = a\,\frac{e^{-\frac{C_1}{R_{-a}} - \frac{C_2\cos\theta}{R_a^{2}}} - 1}{a^{2}e^{-\frac{C_1}{R_{-a}} - \frac{C_2\cos\theta}{R_a^{2}}} - 1}$$

A straightforward asymptotic expansion of e^{w} and ϕ yields

$$e^{w} = 1 - 2\,\frac{1+a^{2}}{1-a^{2}}\,\frac{C_1}{r} - 2\,\frac{1+a^{2}}{1-a^{2}}\,\frac{C_1 s\cos\theta}{r^{2}} - 2\,\frac{1+a^{2}}{1-a^{2}}\,\frac{C_2\cos\theta}{r^{2}} + 2\,\frac{1-2a^{4}}{(1-a^{2})^{2}}\,\frac{C_1^{2}}{r^{2}} + O\!\left(\frac{1}{r^{3}}\right) \qquad (6.6.1)$$

$$\phi = \frac{2a}{1-a^{2}}\,\frac{C_1}{r} + \frac{2a}{1-a^{2}}\,\frac{C_1 s\cos\theta}{r^{2}} + O\!\left(\frac{1}{r^{3}}\right) \qquad (6.6.2)$$

Further the asympotic behaviour of e^{w} and ϕ are respectivly given by

$$e^{\nu} = 1 - \frac{2m}{r} - \frac{2 D_m \cos\theta}{r^2} - \frac{2mS\cos\theta}{r^2} + O(\frac{1}{r^3}) \qquad (6.6.3)$$

and

$$\phi = \frac{q}{r} + \frac{qS\cos\theta}{r^2} + \frac{D_q\cos\theta}{r^2} + O(\frac{1}{r^3}) \qquad (6.6.4)$$

a comparison between (6.6.1),(6.6.2), (6.6.3), (6.6.4) gives

$$q = \frac{2a}{1-a^2} C_1 \qquad D_q = \frac{2a}{1-a^2} C_2$$

$$m = \frac{1+a^2}{1-a^2} C_1 \qquad D_m = \frac{1+a^2}{1-a^2} C_2$$

Applying the Gauss divergence theorem to the function f given by (6.6.1), one finds

$$4\pi \int \rho \sqrt{\gamma}\, dV \equiv \frac{a^2}{1-a^2} \int \nabla^2 \left(- \frac{2C_1}{R_{-a}} - \frac{C_2 \cos\theta}{R_a} \right) r\,dr\,dz\,d\varphi$$

The second volume integral of the above relation vanishes everywhere. Hence

$$\rho = \frac{2a}{1-a^2} \frac{\delta(R_{-a})}{\sqrt{\gamma}}$$

and the total charge

$$q = \frac{2aC_1}{1-a^2}$$

This implies that the charge is located at the point $r=0$ $z=-a$ and we assume that the dipole is located at the point $r=0$ $z=a$ From the values of the effective mass m and charge dipole we can relate the constants c_1 and c_2 with m, q D_m D_q (by a similar procedure as in chapter 5) We have the following cases

1) $m^2 > q^2$ $D_m^2 > D_q^2$ (undercharged case)

$$c_1 = m\sqrt{1-\frac{q^2}{m^2}} \qquad c_2 = D_m\sqrt{1-\frac{D_q^2}{D_m^2}}$$

2) $m^2 < q^2$ $D_m^2 < D_q^2$ (undercharged case)

$$i\,c_1 = q\sqrt{1-\frac{m^2}{q^2}} \qquad i\,c_2 = D_q\sqrt{1-\frac{D_m^2}{D_q^2}}$$

3) $m^2 = q^2$ $D_m^2 = D_q^2$ (critically charged case)

$$c_1 = c_2 = 0$$

and

$$V = 0$$

and

$$e^W = \frac{1}{\left[\frac{m}{R_{-a}} + \frac{D_m \cos\theta}{R_{+a}^2} + 1\right]^2}$$

$$\phi = \frac{\frac{q}{R_{-a}} + \frac{D_q \cos\theta}{R_a}}{\frac{m}{R_{-a}} + \frac{D_m \cos\theta}{R_a^2} + 1}$$

we shall also determine the general relativistic equation connecting m q D_m D_q such that the metric is regular everywhere (except at the point $r = 0$ $z = \pm a$) As before,

$$V(r=0, |z|<a) = \frac{c_1 c_2}{a^3} \equiv$$

$$= \frac{m D_m - q D_q}{a^3}$$

So $V = 0$ if and only if $m D_m = q D_q$ but also

$$\frac{m}{q} = \frac{D_m}{D_q} \qquad (6.6.5)$$

those two equations together yield

$$m^2 = q^2 \qquad D_m^2 = D_q^2 \qquad (6.6.6)$$

which is the equation for the balance of a monopole and a dipole without any additional strut. A comparison with the classical case can be made after calculating the forces acting on a monopole mass being in the gravitational field of a point dipole and secondly the force acting on a charged monopole due to a dipole charge. The gravitational force on the monopole is given by

$$F_g = \frac{D_m m}{r^3}$$

while

$$F_e = \frac{D_q q}{r^3}$$

those two equations imply that balance in the classical sense can be achieved if $D_m m = D_q q$ which is the same equation as in the relativistic case.

SUMMARY AND CONCLUSION

In this thesis we have obtained some new static axially symmetric solutions of the Einstein and the Einstein-Maxwell field equations. In chapter four we analyzed all the the steps leading to the exterior solution of a Curzon particle in combination with a generalized line mass. By a slightly different procedure in chapter 6 we have found solutions of the field equation representing the exterior field of multipole points.

Although the exterior fields of the above mentioned solutions are known, our knowledge about the distribution of matter acting as sources of those solutions are very limited. As we discussed in chapter four the field of a general line mass might represent the exterior field of a family of spheroids. However, for the Curzon metric the knowledge about its source is very limited. The only thing that is known about its source is the fact that it exhibits multipole structure. We should point out the following situation in the General Theory of Relativity. The Schwarzschild solution does not exhibit multipole structure (see Voorhes 1970), while as we said before the Curzon metric which is generated by the Newtonian potential of a monopole does exhibit multipole structure. So we can say that in the theory of relativity the relativistic monopole corresponds to the source of the Schwarzschild solution. Therefore we can say now that the sources for the exterior metric of a Curzon particle and a generalized line mass might be spheroidal in a combination with a point source exhibiting multipole structure.

We also have obtained a charged metric corresponding to a charged line mass and a charged Curzon particle. This has been done by assuming a functional relation between the g_{00} and the electrostatic potential and by application of

Weyl's axially symmetric electrovacuum formalism. Under these assumptions we also eliminate the singularity which occurs on the portion of the z axis between the line mass and the point mass. We found that the necessary condition for balance between the point mass and the line mass is equ. (5.6.3) which suggests that the line mass and the point mass have to be critically charged. However, we should remember that this conclusion has been obtained under the assumption that the charge to mass ratios of the point and line components in the superposed charged point-line mass metric have to be equal to eachother (see 5.6.2). This equation is simply a consequence of the Weyl electrovacuum formalism in conjunction with the functional relation between g_{oo} and electrostatic potential. A more general equation for balance between the gravitational attraction and the electrostatic repulsion should be obtainable but not within the framework of the Weyl formalism.

Almost the same results have been obtained when in chapter six we have looked at the relativistic condition for balance between a charged Curzon particle and a charged point dipole. We have found that the balance can occur when the Curzon particle is critically charged and when the point dipole is characterized by the equation $D_m^2 = D_q^2$ where D_m is the mass dipole moment and D_q is the charge dipole moment. Again these equations have been obtained under the restriction (6.6.5)

Further in chapter 4 we have examined the Bach and Weyl metric as the parameter d (separation distance) goes to zero. Studying the horizon of the above metric before and after the limit $d \rightarrow 0$ we proved that the horizon of the Bach and Weyl metric during that quasi-static approach can emerge as singular or nonsingular depending on the values of k_1 and k_2. If $k_1 = k_2 = 1$ then the singular horizon after the contact emerges as nonsingular while for $k_1 \neq k_2$ the final state of the horizon is always singular.

Finally we generated the charged Bach and Weyl metric.

Again by letting $d \longrightarrow 0$ we have seen that the geometry of the space time can be replaced by the geometry of one charged line mass as long as the mass density and charge density on one of the line masses are respectively equal to the mass density and charge density of the other one.

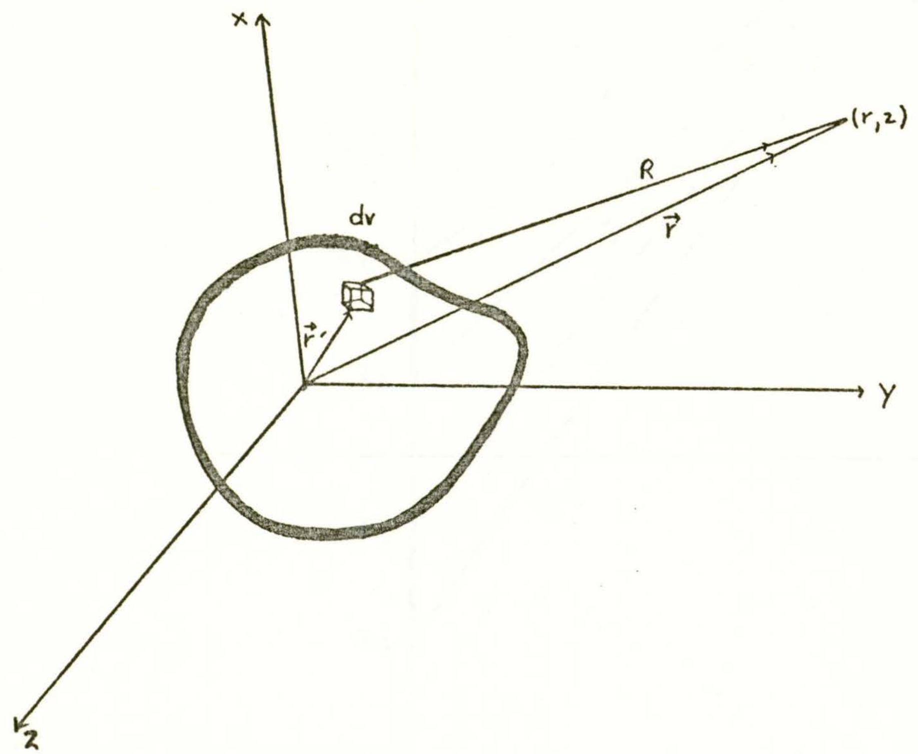

Figure 1. Distrubution of Charge

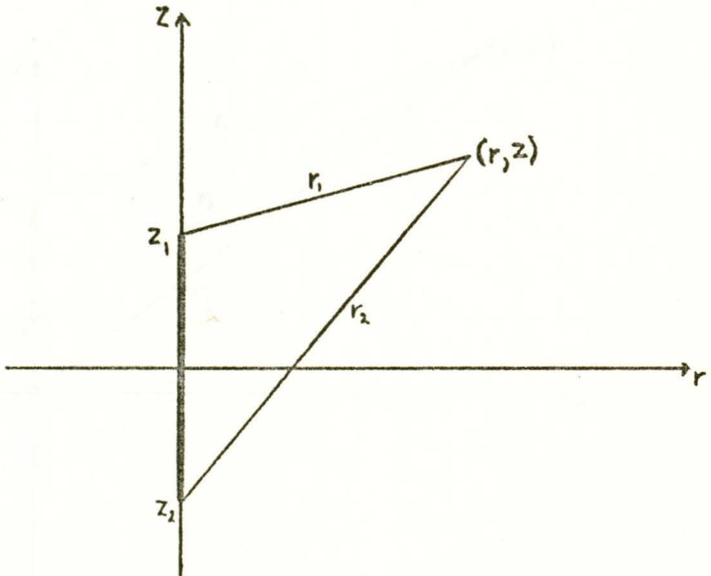

Figure 2. 1 Line Mass

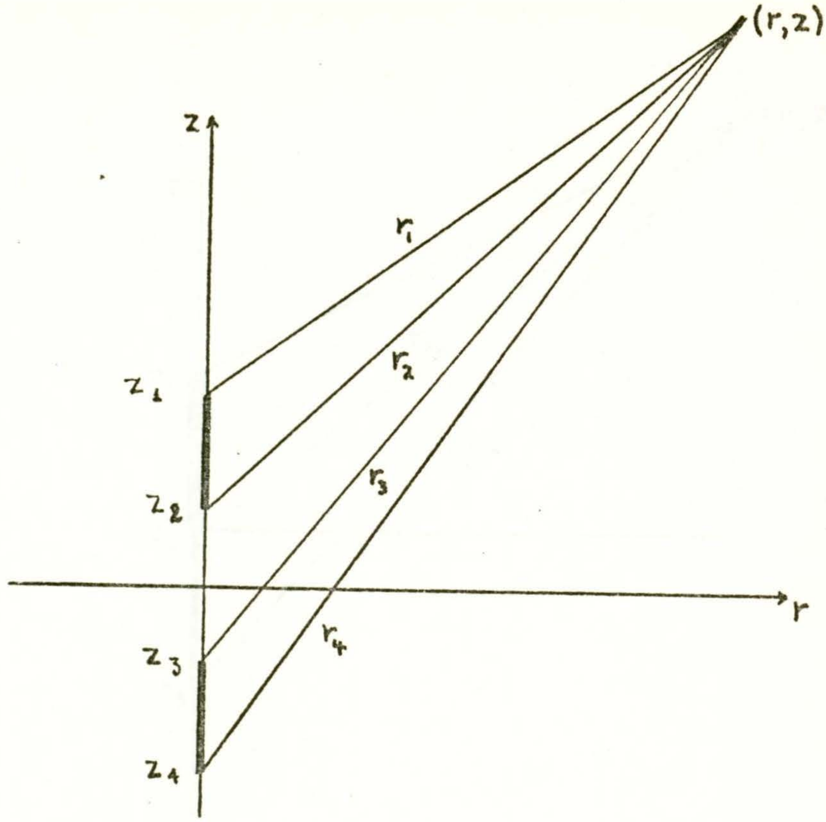

Figure 3. 2 Line Masses

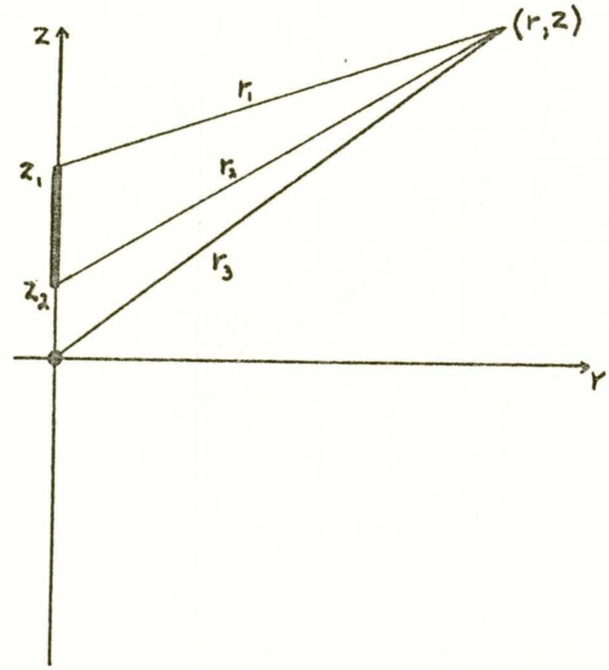

Figure 4. Curzon Particle and Line Mass

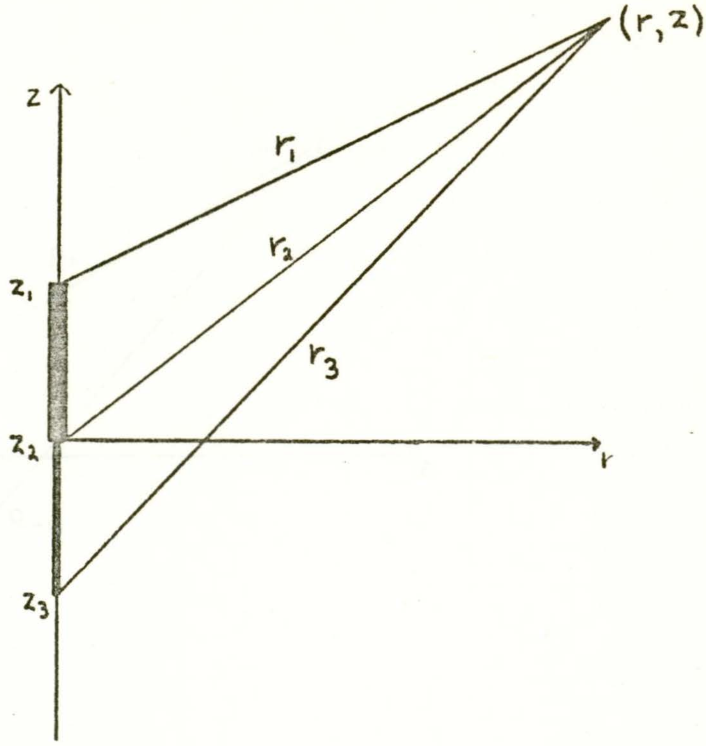

Figure 5. 2 Line Masses Together

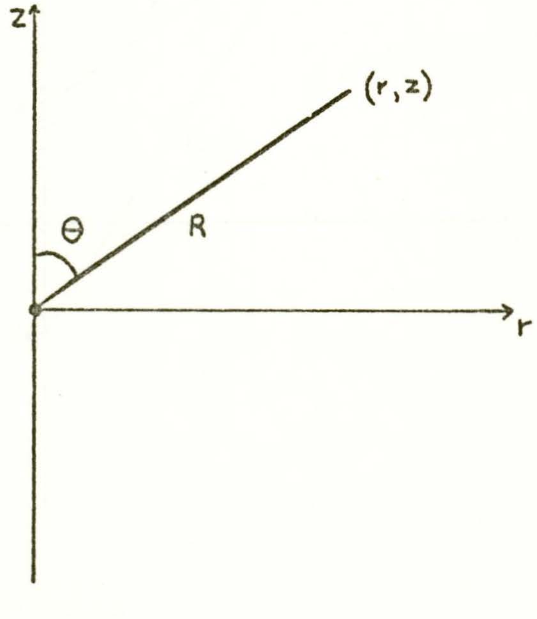

Figure 6. 1 Monopole

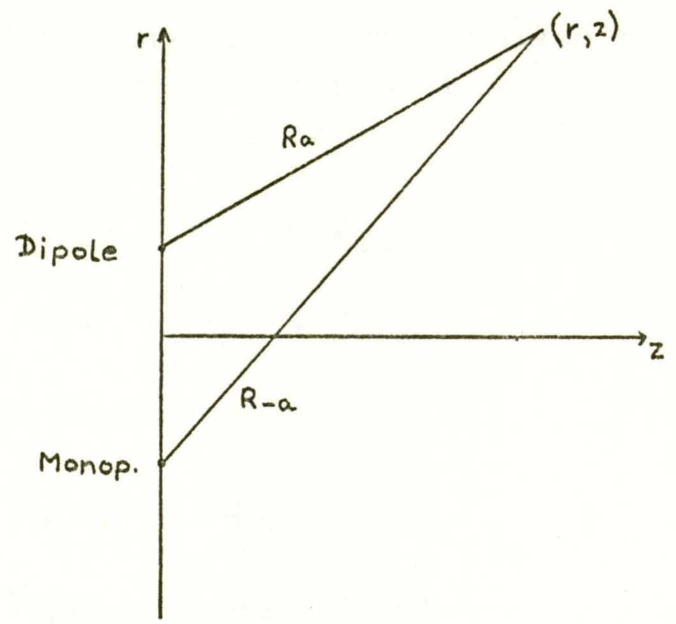

Figure 7. Monopole Dipole Separated

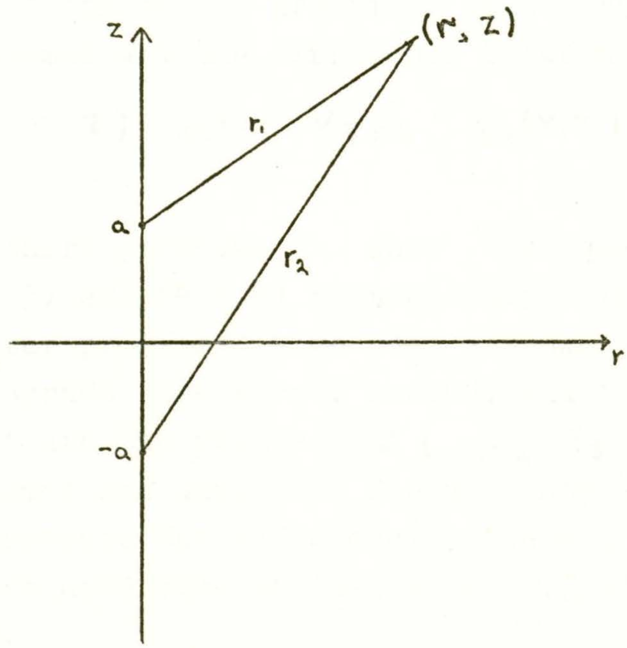

Figure 8. Silberstein Solution

APPENDIX

In the following pages there are three computer programes accompanied with numerical results. Writing (4.4.8) and (4.4.9) as follows:

$$\nu_K = \nu_{KK^2} + \nu_{KL} + \nu_{KKL}$$

$$\nu_\zeta = \nu_{\zeta K^2} + \nu_{\zeta L} + \nu_{\zeta KL}$$

Then the first program shows that

$$\nu_{KKL} = \varphi_3(r, z)$$

$$\nu_{\zeta KL} = f_3(r, z)$$

where $\varphi_3(r, z)$ and $f_3(r, z)$ are defined by (4.4.12) and (4.4.13). The numerical results from the computer are as follows. The first, second and third columns from the left represent the variables z_1 z_2 and z_3 , (see fig. 4) While the fourth and fifth column are field points and the sixth and seventh columns are the difference between

$$\nu_{KKL},\ \varphi_3(r, z) \quad \text{and} \quad \nu_{\zeta KL},\ f_3(r, z)$$

respectively.

The second and third programs show the equality of the relation (4.7.3) and (4.7.4) respectively. The results of the computer programs have been arranged as follows. In both programs the first, second, and third columns from the left are the points z_1 z_2 z_3. The fourth and fifth columns are again the field points and the sixth column represents the difference between the right hand side and the left hand side of relations (4.7.3) and (4.7.4) respectively.

```
$WATFIV (TZANNIAS/00243/224/CCOPERFI)
      DIMENSION X(10),Y(10),Z(10),H(10),E(10), R(10)
      READ(5,1) (X(I),Y(I),Z(I),I=1,10 )
    1 FORMAT(3F10.3)
      DO 7 I=1,10
      WRITE (6,4) X(I),Y(I),Z(I)
    4 FORMAT(3F10.3)
    7 CONTINUE
      READ(5,3)(H(J),E(J),J=1,10)
    3 FORMAT(2F10.3)
          DO 8 J=1,10
      WRITE (6,5) H(J), E(J)
    5 FORMAT(2F10.3)
    8 CONTINUE
      DO 10 J=1,10
      DO 10 I=1,10
      R(1)=SQRT(H(J)**2+(E(J)-X(I))**2)
      R(2)=SQRT(H(J)**2+(E(J)-Y(I))**2)
      R(3)=SQRT(H(J)**2+(E(J)-Z(I))**2)
      CD=Y(I)-Z(I)
      C1=X(I)-Y(I)
      D1=(E(J)-X(I))/R(1)
      D2=(E(J)-Y(I))/R(2)
      D3=(E(J)-Z(I))/R(3)
      B1=H(J)/R(1)
      B2=H(J)/R(2)
      B3=H(J)/R(3)
      S1=((R(1)+R(2))**2)-((C1)**2)
      AL1R=(C1*(B1+B2))/S1
      AL1Z=(C1*(D1+D2))/S1
      AL2R=(B3)/(R(3)**2)
      AL2Z=(D3)/(R(3)**2)
      AN1=C1*E(J)-(R(3)*(R(1)+R(2)))
      AN2=R(3)*(C1*(R(3)-R(2))-(CD*(R(1)+R(2))))
      PDAN1Z=(-D3*(R(1)+R(2)))-(R(3)*(D1+D2))
      PDAN1R=(-B3*(R(1)+R(2)))-(R(3)*(E1+E2))
      PDAN2Z=D3*(AN2/R(3))+(R(3)*((C1*(D3-D2))-(CD*(D1+D2))))
      PDAN2R=B3*(AN2/R(3))+(R(3)*(C1*(B3-B2)-(CD*(B1+B2))))
      ALFA=(2.*(PDAN1R*AN2-(AN1*PDAN2R)))/(AN2**2)
      BITA=(2.*(PDAN1Z*AN2-(PDAN2Z*AN1)))/(AN2**2)
      SHELZ=2.*H(J)*(AL2Z*AL1Z-(AL1R*AL2R))
      SHELR=-2.*H(J)*(AL1R*AL2Z+(AL2R*AL1Z))
      RES1=ALFA-SHELZ
      RES2=BITA-SHELR
      WRITE (6,2) X(I),Y(I),Z(I),H(J),E(J),RES1,RES2
    2 FORMAT (' ',3F9.2, 2F12.3, 2F10.5)
   10 CONTINUE
      STOP
      END
```

```
        $ENTRY
  180.000   110.000     0.000
  415.000    63.000     0.000
  285.000    99.000     0.000
  626.000   614.000     0.000
  225.000   210.000     0.000
  500.000   290.000     0.000
  825.435   675.217     0.000
-1578.000 -3545.000     0.000
```

25278.400	11524.300	0.000				
55000.000	150000.000	0.000				
15.000	22.000					
14.000	25.000					
24.000	39.000					
5.000	29.000					
80.000	-23.000					
-80.000	-45.000					
-19.000	35.000					
70.000	-60.000					
145.000	563.000					
952.000	563.000					
180.00	110.00	0.00	15.000	22.000	0.00019	0.00007
415.00	68.00	0.00	15.000	22.000	0.00071	0.00037
285.00	99.00	0.00	15.000	22.000	0.00034	0.00015
626.00	614.00	0.00	15.000	22.000	0.00000	0.00000
225.00	210.00	0.00	15.000	22.000	0.00001	0.00001
500.00	250.00	0.00	15.000	22.000	0.00008	0.00003
825.44	675.22	0.00	15.000	22.000	0.00001	0.00000
-1578.00	-3545.00	0.00	15.000	22.000	-0.00001	-0.00000
125278.40	11524.30	0.00	15.000	22.000	0.00000	0.00000
855000.00	150000.00	0.00	15.000	22.000	0.00000	0.00000
180.00	110.00	0.00	14.000	25.000	0.00017	0.00009
415.00	68.00	0.00	14.000	25.000	0.00064	0.00046
285.00	99.00	0.00	14.000	25.000	0.00030	0.00019
626.00	614.00	0.00	14.000	25.000	0.00000	0.00000
225.00	210.00	0.00	14.000	25.000	0.00001	0.00001
500.00	250.00	0.00	14.000	25.000	0.00007	0.00004
825.44	675.22	0.00	14.000	25.000	0.00001	0.00001
-1578.00	-3545.00	0.00	14.000	25.000	-0.00001	-0.00001
125278.40	11524.30	0.00	14.000	25.000	0.00000	0.00000
855000.00	150000.00	0.00	14.000	25.000	0.00000	0.00000
180.00	110.00	0.00	24.000	39.000	0.00014	0.00007
415.00	68.00	0.00	24.000	39.000	0.00058	0.00050
285.00	99.00	0.00	24.000	39.000	0.00026	0.00015
626.00	614.00	0.00	24.000	39.000	0.00000	0.00000
225.00	210.00	0.00	24.000	39.000	0.00001	0.00000
500.00	250.00	0.00	24.000	39.000	0.00005	0.00003
825.44	675.22	0.00	24.000	39.000	0.00001	0.00000
-1578.00	-3545.00	0.00	24.000	39.000	-0.00001	-0.00000
125278.40	11524.30	0.00	24.000	39.000	0.00000	0.00000
855000.00	150000.00	0.00	24.000	39.000	0.00000	0.00000
180.00	110.00	0.00	5.000	29.000	0.00007	0.00015
415.00	68.00	0.00	5.000	29.000	0.00026	0.00070
285.00	99.00	0.00	5.000	29.000	0.00012	0.00030
626.00	614.00	0.00	5.000	29.000	0.00000	0.00000
225.00	210.00	0.00	5.000	29.000	0.00000	0.00001
500.00	250.00	0.00	5.000	29.000	0.00003	0.00007
825.44	675.22	0.00	5.000	29.000	0.00000	0.00001
-1578.00	-3545.00	0.00	5.000	29.000	-0.00000	-0.00001
125278.40	11524.30	0.00	5.000	29.000	0.00000	0.00000
855000.00	150000.00	0.00	5.000	29.000	0.00000	0.00000
180.00	110.00	0.00	80.000	-23.000	0.00001	-0.00001
415.00	68.00	0.00	80.000	-23.000	0.00003	0.00001
285.00	99.00	0.00	80.000	-23.000	0.00001	-0.00002
626.00	614.00	0.00	80.000	-23.000	-0.00000	-0.00000
225.00	210.00	0.00	80.000	-23.000	0.00000	-0.00000
500.00	250.00	0.00	80.000	-23.000	-0.00000	-0.00001
825.44	675.22	0.00	80.000	-23.000	-0.00000	-0.00000
-1578.00	-3545.00	0.00	80.000	-23.000	0.00000	0.00000
125278.40	11524.30	0.00	80.000	-23.000	-0.00000	-0.00000
855000.00	150000.00	0.00	80.000	-23.000	-0.00000	-0.00000
180.00	110.00	0.00	-80.000	-45.000	0.00000	-0.00000
415.00	68.00	0.00	-80.000	-45.000	0.00001	0.00003

285.00	99.00	0.00	-80.000	-45.000	0.00001	0.00000
626.00	614.00	0.00	-80.000	-45.000	0.00000	-0.00000
225.00	210.00	0.00	-80.000	-45.000	0.00000	-0.00000
500.00	250.00	0.00	-80.000	-45.000	0.00001	-0.00001
825.44	675.22	0.00	-80.000	-45.000	0.00000	-0.00000
-1578.00	-3545.00	0.00	-80.000	-45.000	-0.00000	0.00000
125278.40	11524.30	0.00	-80.000	-45.000	0.00000	-0.00000
855000.00	150000.00	0.00	-80.000	-45.000	0.00000	-0.00000
180.00	110.00	0.00	-19.000	35.000	-0.00014	0.00008
415.00	68.00	0.00	-19.000	35.000	-0.00058	0.00030
285.00	99.00	0.00	-19.000	35.000	-0.00025	0.00017
626.00	614.00	0.00	-19.000	35.000	-0.00000	0.00000
225.00	210.00	0.00	-19.000	35.000	-0.00001	0.00001
500.00	250.00	0.00	-19.000	35.000	-0.00005	0.00003
825.44	675.22	0.00	-19.000	35.000	-0.00001	0.00000
-1578.00	-3545.00	0.00	-19.000	35.000	0.00001	-0.00000
125278.40	11524.30	0.00	-19.000	35.000	-0.00000	0.00000
855000.00	150000.00	0.00	-19.000	35.000	-0.00000	0.00000
180.00	110.00	0.00	70.000	-60.000	-0.00001	0.00000
415.00	68.00	0.00	70.000	-60.000	-0.00003	0.00005
285.00	99.00	0.00	70.000	-60.000	-0.00002	0.00001
626.00	614.00	0.00	70.000	-60.000	-0.00000	-0.00000
225.00	210.00	0.00	70.000	-60.000	-0.00000	-0.00000
500.00	250.00	0.00	70.000	-60.000	-0.00001	-0.00000
825.44	675.22	0.00	70.000	-60.000	-0.00000	-0.00000
-1578.00	-3545.00	0.00	70.000	-60.000	0.00000	0.00000
125278.40	11524.30	0.00	70.000	-60.000	-0.00000	-0.00000
855000.00	150000.00	0.00	70.000	-60.000	-0.00000	-0.00000
180.00	110.00	0.00	145.000	563.000	-0.00000	0.00000
415.00	68.00	0.00	145.000	563.000	-0.00000	0.00005
285.00	99.00	0.00	145.000	563.000	-0.00000	0.00001
626.00	614.00	0.00	145.000	563.000	0.00000	0.00000
225.00	210.00	0.00	145.000	563.000	-0.00000	0.00000
500.00	250.00	0.00	145.000	563.000	-0.00000	0.00002
825.44	675.22	0.00	145.000	563.000	0.00000	0.00000
-1578.00	-3545.00	0.00	145.000	563.000	-0.00000	-0.00000
125278.40	11524.30	0.00	145.000	563.000	0.00000	0.00000
855000.00	150000.00	0.00	145.000	563.000	0.00000	0.00000
180.00	110.00	0.00	952.000	563.000	0.00000	0.00000
415.00	68.00	0.00	952.000	563.000	0.00000	0.00001
285.00	99.00	0.00	952.000	563.000	0.00000	0.00000
626.00	614.00	0.00	952.000	563.000	0.00000	0.00000
225.00	210.00	0.00	952.000	563.000	0.00000	0.00000
500.00	250.00	0.00	952.000	563.000	0.00000	0.00000
825.44	675.22	0.00	952.000	563.000	0.00000	0.00000
-1578.00	-3545.00	0.00	952.000	563.000	-0.00000	0.00000
125278.40	11524.30	0.00	952.000	563.000	0.00000	-0.00000
855000.00	150000.00	0.00	952.000	563.000	0.00000	-0.00000

CORE USAGE OBJECT CODE= 3560 BYTES, ARRAY AREA= 240 BYTES, TOTAL A

DIAGNOSTICS NUMBER OF ERRORS= 0, NUMBER OF WARNINGS= 0, NUMBER OF

COMPILE TIME= 0.17 SEC, EXECUTION TIME= 0.45 SEC, WATFIV - JUL 1973 V1L4

```
$WATFIV (TZANNIAS/00248/224/CCCPERFI)
      DIMENSION  X(10),Y(10),Z(10),H(10),E(10),R(10)
      READ(5,1) (X(I),Y(I),Z(I), I=1,10 )
    1 FORMAT(3F10.3)
      DO 7 I=1,10
      WRITE (6,4) X(I),Y(I),Z(I)
    4 FORMAT(3F10.3)
    7 CONTINUE
      READ(5,3)(H(J),E(J),J=1,10)
    3 FORMAT(2F10.3)
          DO 8 J=1,10
      WRITE (6,5) H(J), E(J)
    5 FORMAT(2F10.3)
    8 CONTINUE
      DO 10 I=1,10
      DO 10 J=1,10
      R(1)=SQRT(H(J)**2+(E(J)-X(I))**2)
      R(2)=SQRT(H(J)**2+(E(J)-Y(I))**2)
      R(3)=SQRT(H(J)**2+(E(J)-Z(I))**2)
      ALPA=((R(1)+R(2)-(X(I)-Y(I)))*(R(2)+R(3)-(Y(I)-Z(I))))
      ALMA=(R(1)+R(2)+(X(I)-Y(I)))*(R(2)+R(3)+(Y(I)-Z(I)))
      ALFA=ALPA/ALMA
      BITA=(R(1)+R(3)-(X(I)-Z(I)))/(R(1)+R(3)+(X(I)-Z(I)))
      GAMA=ALFA-BITA
      WRITE (6,2) X(I),Y(I),Z(I),H(J),E(J),GAMA
    2 FORMAT(3F15.3,2F10.3,F20.6)
   10 CONTINUE
      STOP
      END

       $ENTRY
    2.000      0.000     -2.000
    3.000      0.000     -3.000
   18.000      0.000    -18.000
   50.000      0.000    -50.000
    2.000      0.000     -1.000
    2.000      0.000     -9.000
    3.000      0.000    -25.000
   52.000      0.000    -59.000
  150.000      0.000    -55.000
  850.000      0.000   -320.000
    1.000      1.000
    2.000      2.000
   10.000     10.000
    6.000     -3.000
   25.000     -9.000
  -70.000     65.000
  150.000    200.000
  355.000    652.000
-5344.125    853.000
  845.000   -354.000
          2.000          0.000         -2.000     1.000     1.000            0.0000
          2.000          0.000         -2.000     2.000     2.000            0.0000
          2.000          0.000         -2.000    10.000    10.000            0.0000
          2.000          0.000         -2.000     6.000    -3.000           -0.0000
          2.000          0.000         -2.000    25.000    -9.000           -0.0000
          2.000          0.000         -2.000   -70.000    65.000            0.0000
          2.000          0.000         -2.000   150.000   200.000           -0.0000
          2.000          0.000         -2.000   355.000   652.000            0.0000
          2.000          0.000         -2.000 -5344.125   853.000           -0.0000
          2.000          0.000         -2.000   845.000  -354.000           -0.0000
          3.000          0.000         -3.000     1.000     1.000            0.0000
          3.000          0.000         -3.000     2.000     2.000            0.0000
```

3.000	0.000	-3.000	10.000	10.000	0.0000
3.000	0.000	-3.000	6.000	-3.000	-0.0000
3.000	0.000	-3.000	25.000	-9.000	-0.0000
3.000	0.000	-3.000	-70.000	65.000	0.0000
3.000	0.000	-3.000	150.000	200.000	-0.0000
3.000	0.000	-3.000	355.000	652.000	0.0000
3.000	0.000	-3.000	-5344.125	853.000	0.0000
3.000	0.000	-3.000	845.000	-354.000	-0.0000
18.000	0.000	-18.000	1.000	1.000	0.0000
18.000	0.000	-18.000	2.000	2.000	0.0000
18.000	0.000	-18.000	10.000	10.000	0.0000
18.000	0.000	-18.000	6.000	-3.000	0.0000
18.000	0.000	-18.000	25.000	-9.000	0.0000
18.000	0.000	-18.000	-70.000	65.000	0.0000
18.000	0.000	-18.000	150.000	200.000	-0.0000
18.000	0.000	-18.000	355.000	652.000	-0.0000
18.000	0.000	-18.000	-5344.125	853.000	0.0000
18.000	0.000	-18.000	845.000	-354.000	0.0000
50.000	0.000	-50.000	1.000	1.000	0.0000
50.000	0.000	-50.000	2.000	2.000	0.0000
50.000	0.000	-50.000	10.000	10.000	0.0000
50.000	0.000	-50.000	6.000	-3.000	0.0000
50.000	0.000	-50.000	25.000	-9.000	0.0000
50.000	0.000	-50.000	-70.000	65.000	-0.0000
50.000	0.000	-50.000	150.000	200.000	0.0000
50.000	0.000	-50.000	355.000	652.000	-0.0000
50.000	0.000	-50.000	-5344.125	853.000	0.0000
50.000	0.000	-50.000	845.000	-354.000	0.0000
2.000	0.000	-1.000	1.000	1.000	0.0000
2.000	0.000	-1.000	2.000	2.000	0.0000
2.000	0.000	-1.000	10.000	10.000	0.0000
2.000	0.000	-1.000	6.000	-3.000	-0.0000
2.000	0.000	-1.000	25.000	-9.000	0.0000
2.000	0.000	-1.000	-70.000	65.000	0.0000
2.000	0.000	-1.000	150.000	200.000	0.0000
2.000	0.000	-1.000	355.000	652.000	-0.0000
2.000	0.000	-1.000	-5344.125	853.000	-0.0000
2.000	0.000	-1.000	845.000	-354.000	-0.0000
2.000	0.000	-9.000	1.000	1.000	0.0000
2.000	0.000	-9.000	2.000	2.000	-0.0000
2.000	0.000	-9.000	10.000	10.000	0.0000
2.000	0.000	-9.000	6.000	-3.000	0.0000
2.000	0.000	-9.000	25.000	-9.000	0.0000
2.000	0.000	-9.000	-70.000	65.000	0.0000
2.000	0.000	-9.000	150.000	200.000	-0.0000
2.000	0.000	-9.000	355.000	652.000	-0.0000
2.000	0.000	-9.000	-5344.125	853.000	-0.0000
2.000	0.000	-9.000	845.000	-354.000	0.0000
3.000	0.000	-25.000	1.000	1.000	0.0000
3.000	0.000	-25.000	2.000	2.000	0.0000
3.000	0.000	-25.000	10.000	10.000	0.0000
3.000	0.000	-25.000	6.000	-3.000	0.0000
3.000	0.000	-25.000	25.000	-9.000	0.0000
3.000	0.000	-25.000	-70.000	65.000	0.0000
3.000	0.000	-25.000	150.000	200.000	0.0000
3.000	0.000	-25.000	355.000	652.000	0.0000
3.000	0.000	-25.000	-5344.125	853.000	0.0000
3.000	0.000	-25.000	845.000	-354.000	0.0000
52.000	0.000	-59.000	1.000	1.000	0.0000
52.000	0.000	-59.000	2.000	2.000	0.0000
52.000	0.000	-59.000	10.000	10.000	0.0000
52.000	0.000	-59.000	6.000	-3.000	0.0000
52.000	0.000	-59.000	25.000	-9.000	0.0000
52.000	0.000	-59.000	-70.000	65.000	0.0000

52.000	0.000	-59.000	150.000	200.000	-0.00000
52.000	0.000	-59.000	355.000	652.000	0.00000
52.000	0.000	-59.000	-5344.125	853.000	0.00000
52.000	0.000	-59.000	845.000	-354.000	0.00000
150.000	0.000	-55.000	1.000	1.000	0.00000
150.000	0.000	-55.000	2.000	2.000	0.00000
150.000	0.000	-55.000	10.000	10.000	0.00000
150.000	0.000	-55.000	6.000	-3.000	0.00000
150.000	0.000	-55.000	25.000	-9.000	0.00000
150.000	0.000	-55.000	-70.000	65.000	0.00000
150.000	0.000	-55.000	150.000	200.000	0.00000
150.000	0.000	-55.000	355.000	652.000	0.00000
150.000	0.000	-55.000	-5344.125	853.000	-0.00000
150.000	0.000	-55.000	845.000	-354.000	-0.00000
850.000	0.000	-320.000	1.000	1.000	0.00000
850.000	0.000	-320.000	2.000	2.000	0.000000
850.000	0.000	-320.000	10.000	10.000	0.000000
850.000	0.000	-320.000	6.000	-3.000	0.000000
850.000	0.000	-320.000	25.000	-9.000	0.000000
850.000	0.000	-320.000	-70.000	65.000	0.000000
850.000	0.000	-320.000	150.000	200.000	0.000000
850.000	0.000	-320.000	355.000	652.000	0.000000
850.000	0.000	-320.000	-5344.125	853.000	0.000000
850.000	0.000	-320.000	845.000	-354.000	-0.000000

```
CORE USAGE      OBJECT CODE=    2216 BYTES,   ARRAY AREA=     240 BYTES,   TOTAL A-
DIAGNOSTICS        NUMBER OF ERRORS=       0, NUMBER OF WARNINGS=        0, NUMBER OF
COMPILE TIME=    0.10 SEC,   EXECUTION TIME=    0.33 SEC,   WATFIV - JUL 1973 V1L4
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```
$WATFIV (TZANNIAS/U0243/224/COUPERF1)
      DIMENSION X(10),Y(10),Z(10),H(10),E(10), R(10)
      READ(5,1) (X(I),Y(I),Z(I),I=1,10 )
    1 FORMAT(3F10.3)
      DO 7 I=1,10
      WRITE (6,4) X(I),Y(I),Z(I)
    4 FORMAT(3F10.3)
    7 CONTINUE
      READ(5,3)(H(J),E(J),J=1,10)
    3 FORMAT(2F10.3)
        DO 8 J=1,10
      WRITE (6,5) H(J), E(J)
    5 FORMAT(2F10.3)
    8 CONTINUE
      DO 10 J=1,10
      DO 10 I=1,10
      R(1)=SQRT(H(J)**2+(E(J)-X(I))**2)
      R(2)=SQRT(H(J)**2+(E(J)-Y(I))**2)
      R(3)=SQRT(H(J)**2+(E(J)-Z(I))**2)
      C1=X(I)-Y(I)
      C2=Y(I)-Z(I)
      C3=X(I)-Z(I)
      S1=((R(1)+R(2))**2)-((C1)**2)
      S2=((R(2)+R(3))**2)-((C2)**2)
      DITI=(C1*(R(3)-R(2))-C2*(R(1)+R(2)))*R(2)
      ZITA=C2*(C1*E(J)-R(2)*(R(1)+R(2)))
      ALMA=(((R(1)+R(2))**2)-((X(I)-Y(I))**2))/(4.*R(1)*R(2))
      ALPA=(((R(2)+R(3))**2)-((Y(I)-Z(I))**2))/(4.*R(2)*R(3))
      ALKA=(((R(1)+R(3))**2)-((X(I)-Z(I))**2))/(4.*R(1)*R(3))
      T3=ALMA*ALPA*((DITI/ZITA)**2)-ALKA
      WRITE (6,2) X(I),Y(I),Z(I),H(J),E(J),T3
    2 FORMAT(3F15.3,2F10.3,F20.6)
   10 CONTINUE
      STOP
      END
```

```
          $ENTRY
   2.000      0.000     -2.000
   3.000      0.000     -3.000
  18.000      0.000    -18.000
  50.000      0.000    -50.000
   2.000      0.000     -1.000
   2.000      0.000     -9.000
   3.000      0.000    -25.000
  52.000      0.000    -59.000
 150.000      0.000    -55.000
 850.000      0.000   -320.000
   1.000      1.000
   2.000      2.000
  10.000     10.000
   6.000     -3.000
  25.000     -9.000
 -70.000     65.000
 150.000    200.000
 355.000    652.000
-5344.125   853.000
 845.000   -354.000
          2.000          0.000         -2.000     1.000     1.000          0.0000
          3.000          0.000         -3.000     1.000     1.000          0.0000
         18.000          0.000        -18.000     1.000     1.000          0.0000
         50.000          0.000        -50.000     1.000     1.000          0.0000
          2.000          0.000         -1.000     1.000     1.000         -0.0000
          2.000          0.000         -9.000     1.000     1.000          0.0000
```

3.000	0.000	-25.000	1.000	1.000	0.0000
52.000	0.000	-59.000	1.000	1.000	0.0000
150.000	0.000	-55.000	1.000	1.000	0.0000
850.000	0.000	-320.000	1.000	1.000	0.0000
2.000	0.000	-2.000	2.000	2.000	-0.0000
3.000	0.000	-3.000	2.000	2.000	0.0000
18.000	0.000	-18.000	2.000	2.000	0.0000
50.000	0.000	-50.000	2.000	2.000	0.0000
2.000	0.000	-1.000	2.000	2.000	-0.0000
2.000	0.000	-9.000	2.000	2.000	-0.0000
3.000	0.000	-25.000	2.000	2.000	0.0000
52.000	0.000	-59.000	2.000	2.000	0.0000
150.000	0.000	-55.000	2.000	2.000	0.0000
850.000	0.000	-320.000	2.000	2.000	0.0000
2.000	0.000	-2.000	10.000	10.000	0.0000
3.000	0.000	-3.000	10.000	10.000	0.0000
18.000	0.000	-18.000	10.000	10.000	0.0000
50.000	0.000	-50.000	10.000	10.000	0.0000
2.000	0.000	-1.000	10.000	10.000	-0.0000
2.000	0.000	-9.000	10.000	10.000	0.0000
3.000	0.000	-25.000	10.000	10.000	-0.0000
52.000	0.000	-59.000	10.000	10.000	0.0000
150.000	0.000	-55.000	10.000	10.000	0.0000
850.000	0.000	-320.000	10.000	10.000	0.0000
2.000	0.000	-2.000	6.000	-3.000	-0.0000
3.000	0.000	-3.000	6.000	-3.000	0.0000
18.000	0.000	-18.000	6.000	-3.000	0.0000
50.000	0.000	-50.000	6.000	-3.000	0.0000
2.000	0.000	-1.000	6.000	-3.000	0.0000
2.000	0.000	-9.000	6.000	-3.000	0.0000
3.000	0.000	-25.000	6.000	-3.000	-0.0000
52.000	0.000	-59.000	6.000	-3.000	0.0000
150.000	0.000	-55.000	6.000	-3.000	0.0000
850.000	0.000	-320.000	6.000	-3.000	0.0000
2.000	0.000	-2.000	25.000	-9.000	-0.0000
3.000	0.000	-3.000	25.000	-9.000	0.0000
18.000	0.000	-18.000	25.000	-9.000	-0.0000
50.000	0.000	-50.000	25.000	-9.000	-0.0000
2.000	0.000	-1.000	25.000	-9.000	-0.0000
2.000	0.000	-9.000	25.000	-9.000	-0.0000
3.000	0.000	-25.000	25.000	-9.000	-0.0000
52.000	0.000	-59.000	25.000	-9.000	-0.0000
150.000	0.000	-55.000	25.000	-9.000	0.0000
850.000	0.000	-320.000	25.000	-9.000	0.0000
2.000	0.000	-2.000	-70.000	65.000	-0.0000
3.000	0.000	-3.000	-70.000	65.000	-0.0000
18.000	0.000	-18.000	-70.000	65.000	0.0000
50.000	0.000	-50.000	-70.000	65.000	-0.0000
2.000	0.000	-1.000	-70.000	65.000	0.0000
2.000	0.000	-9.000	-70.000	65.000	0.0000
3.000	0.000	-25.000	-70.000	65.000	-0.0000
52.000	0.000	-59.000	-70.000	65.000	-0.0000
150.000	0.000	-55.000	-70.000	65.000	0.0000
850.000	0.000	-320.000	-70.000	65.000	0.0000
2.000	0.000	-2.000	150.000	200.000	0.0000
3.000	0.000	-3.000	150.000	200.000	-0.0000
18.000	0.000	-18.000	150.000	200.000	0.0000
50.000	0.000	-50.000	150.000	200.000	0.0000
2.000	0.000	-1.000	150.000	200.000	-0.0000
2.000	0.000	-9.000	150.000	200.000	-0.0000
3.000	0.000	-25.000	150.000	200.000	-0.0000
52.000	0.000	-59.000	150.000	200.000	0.0000
150.000	0.000	-55.000	150.000	200.000	0.0000
850.000	0.000	-320.000	150.000	200.000	0.0000

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