Trees with Equal Broadcast and Domination Numbers

by

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B.Sc., University of Victoria, 2009

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

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Abstract

A *broadcast* is a function $f : V \to \{0, ..., \text{diam}(G)\}$ that assigns an integer value to each vertex such that, for each $v \in V$, $f(v) \leq e(v)$, the eccentricity of v. The *broadcast number* of a graph is the minimum value of $\sum_{v \in V} f(v)$ among all broadcasts f with the property that for each vertex x of V, $f(v) \geq d(x, v)$ for some vertex v having positive $f(v)$. This number is bounded above by both the radius of the graph and its domination number. Graphs for which the broadcast number is equal to the domination number are called 1-*cap graphs*. We investigate and characterize a class of 1-cap trees.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Consider a radio station wishing to transmit a broadcast across a large area. It must decide where to place the broadcast towers (and how big the towers should be) in order to minimize the number of towers while ensuring that the entire region hears the broadcast. We can model this scenario with a graph G, where the vertices represent geographic regions and two vertices are adjacent if their corresponding regions are close enough that a weak broadcast from one region can be heard from the other. If the towers can only broadcast to adjacent regions, then finding the optimal layout is equivalent to finding a minimum dominating set S of G, that is, a set of vertices of G where each vertex of G is either in S or adjacent to a vertex in S. If the station can use stronger towers (at a higher cost) then the goal is now to minimize the total cost of the towers. Placing the towers and determining their strength is equivalent to assigning a nonnegative integer to each vertex, where the regions corresponding to vertices with a zero do not have towers, and the strength of each tower on all other regions is proportional to the integer for that vertex. This thesis considers the case where the graphs representing regions are trees, and investigates a class of trees for which the use of arbitrarily strong transmitters does no better than using transmitters that only broadcast to adjacent regions.

1.1 Introduction

In order to formalize the above-mentioned problem some definitions are required. Consider a graph $G = (V, E)$. For any vertex $v \in V(G)$, the set of vertices adjacent to v, $N(v) = \{u \in V : uv \in E(G)\}$, is called the *open neighbourhood* of v. The *closed* *neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$. Open and closed neighbourhoods are defined similarly for sets of vertices: for a set $S \subset V(G)$ of vertices, the *open neighbourhood* of S is the set $N(S) = \{u : u \in N(v) \text{ for some } v \in S\}$, and the *closed neighbourhood* of S is the set $N[S] = N(S) \cup S$. For any $v \in S$, the *private neighbourhood of v with respect to S* is the set of vertices in $N[v]$ that are not contained in $N[s]$ for any $s \in S - \{v\}$ and is denoted by $\mathrm{PN}(v, S)$. That is, $\mathrm{PN}(v, S) = N[v] - N[S - \{v\}]$.

A set $S \subset V(G)$ is called a *dominating set* if $N[S] = V(G)$. The *domination number* $\gamma(G)$ of a graph G is the size of a minimum dominating set. A minimum dominating set of G is called a *γ-set*. A dominating set S of G is an *efficient dominating set* if $N[u] \cap N[v] = \varnothing$ for each $u, v \in S$, $u \neq v$. An efficient dominating set is necessarily a γ-set.

A *broadcast* on a connected graph G is a function $f : V(G) \to \{0, 1, 2, ..., \mathrm{diam}(G)\}$. The value $f(v)$ is referred to as the *strength* of the vertex v with respect to f. The *cost* of a broadcast is the value $\mathrm{cost}(f) = \sum_{v \in V} f(v)$. A *dominating broadcast* is a broadcast f for which each vertex of G is within distance $f(v)$ from some vertex v with $f(v) \geq 1$. The *broadcast number* $\gamma_b(G)$ of G is the minimum cost among all dominating broadcasts.

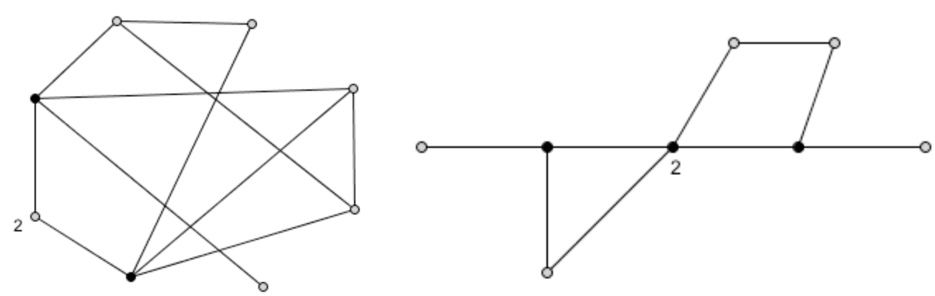
The concept of graph broadcasts was first investigated in 2001 by D.J. Erwin in his dissertation. As referenced in [10], he determined upper and lower bounds on the broadcast number of a graph:

Proposition 1.1. [8] *For every nontrivial connected graph G,*

$$\left\lceil \frac{\mathrm{diam}(G) + 1}{3} \right\rceil \leq \gamma_b(G) \leq \min\{\mathrm{rad}(G), \gamma(G)\}.$$

Graphs for which $\gamma(G) = \mathrm{rad}(G)$ are called Type 1 graphs or *radial* graphs, and graphs for which $\gamma(G) = \gamma_b(G)$ are called Type 2 or 1-*cap* graphs (see Figure 1.1 where the black vertices form a minimum dominating set and the numbers indicate non-zero broadcast strengths). The terminology “1-cap graph” is an abbreviation of 1-*capacity* graph, that is, a graph in which the cost of an arbitrary dominating broadcast is no less than that of a dominating broadcast for which each vertex is assigned a value of either 0 or 1. In the context of the described problem, a 1-cap graph is a graph for which each vertex broadcasts with a strength of 1 or not at all to form a dominating broadcast of minimum cost. Any graph that is not a Type 1 or Type 2 graph is called a Type 3 graph. Erwin also proved that the difference between the broadcast number and the radius or the broadcast number and the domination number can be made arbitrarily large [10].

This thesis investigates a class of 1-cap trees and is organized as follows. In the remainder of Chapter 1 we define relevant terminology and review background material on broadcasts. We investigate previous results on 1-cap trees in Chapter 2. In Chapter 3 we restrict our attention to a specific class of trees and characterize 1-cap trees within the class. Chapter 4 considers creating new 1-cap trees by joining and manipulating others. In Chapter 5 we summarize our results and list some open problems for future research. Algorithms for determining pertinent parameters of certain classes of trees can be found in the appendix.



(a) A *1-cap* graph G (b) A *non-1-cap* graph H

Figure 1.1: $\gamma_b(G) = \gamma(G) = 2$, whereas $\gamma_b(H) = 2$ and $\gamma(H) = 3$

1.2 Definitions and Background

For undefined concepts see [1]. The *eccentricity* of a vertex v of a graph G is the value $e(v) = \max\{d(u,v) : u \in V(G)\}$. The *radius* of a graph G is the minimum eccentricity amongst all vertices: $\operatorname{rad}(G) = \min\{e(v) : v \in V(G)\}$, and the diameter is the maximum eccentricity amongst all vertices: $\operatorname{diam}(G) = \max\{e(v) : v \in V(G)\}$. A *central vertex* of a graph G is a vertex v where $e(v) = \operatorname{rad}(G)$. The set of all central vertices of a graph G is the *centre* of G. A *diametrical path* of a tree T is a path of maximum length, $\operatorname{diam}(T)$. The parity of a path's length determines whether the path is *even* or *odd.* A tree is *central* if its centre consists of exactly one vertex, and *bicentral* otherwise, where the centre consists of exactly two adjacent vertices. Note that a tree T is central if and only if $\operatorname{diam}(T)$ is even. A *leaf* of a tree T is a vertex of degree one. A *stem* or *support vertex* of a tree is a vertex that is adjacent to a leaf, and a *branch vertex* of a tree is a vertex of degree at least three. A tree with exactly one branch vertex is called a *spider*. The spider $S(a_1, ..., a_k)$, $k \geq 3$, has one vertex c of degree k and for each $i \in 1, 2, ..., k$ there is a distinct, internally disjoint path of length a_i from c to a leaf in which each internal vertex has degree two. For example, the star $K_{1,t}$ is the spider $S(1, 1, ..., 1)$.

Let f be a broadcast on a graph G. A *broadcast vertex* is a vertex v for which $f(v) \geq 1$. The set of all broadcast vertices is denoted $V_f^+(G)$, or V_f^+ when the graph under consideration is clear. For $v \in V_f^+$, the *f-neighbourhood* $N_f[v]$ of v is the set $\{u : d(u, v) \leq f(v)\}$, while the *f-private neighbourhood* $\mathrm{PN}_f[v]$ of v consists of all vertices in $N_f[v]$ that are not also in $N_f[w]$ for any $w \in V_f^+ - \{v\}$. A vertex u *hears* a broadcast from $v \in V_f^+$, and v *broadcasts to* u, if $u \in N_f[v]$. A vertex v is *overdominated* if $f(u) - d(u, v) > 0$ for some $u \in V_f^+$.

A broadcast f of a tree T is *central* if $V_f^+(T) = \{v\}$ for some central vertex v with $f(v) = \mathrm{rad}(T)$, otherwise it is referred to as *non-central.*

A broadcast f is a *dominating broadcast* if every vertex hears at least one broadcast. The *cost* of a broadcast f is defined as $\mathrm{cost}(f) = \sum_{v \in V(G)} f(v)$. The *broadcast number* of G is denoted $\gamma_b(G)$, that is, $\gamma_b(G) = \min\{\mathrm{cost}(f) : f$ is a dominating broadcast of $G\}$. A dominating broadcast f of a graph G for which $\mathrm{cost}(f) = \gamma_b(G)$ is called a *γ_b-broadcast* (see Figure 1.2).

If f is a γ_b-broadcast of a tree T such that V_f^+ contains a leaf v and u is the stem adjacent to v, then define the broadcast g as follows: $g(u) = f(v)$, $g(v) = 0$, and $g(w) = f(w)$ for every other vertex w. Then g is also a γ_b-broadcast.

- We will only consider broadcasts where no leaf is a broadcast vertex.

The problem of finding $\gamma_b(G)$ for any given graph was initially thought to be NP-hard (as are many other varieties of domination), but, in 2006, Heggernes and Lokshtanov [9] showed that minimum broadcast domination is solvable in polynomial time for any graph. Their algorithm runs in $O(n^6)$ time for a graph with n vertices. In 2007 Dabney [4] showed that, for a tree T, $\gamma_b(T)$ can be determined by an algorithm that runs in $O(n)$ time (also see [5]). While the problem of determining the domination number of an arbitrary graph is NP-complete, it has long been known that the domination number of a tree can be determined in linear time (see [2]). Knowing that $\gamma(T) = \gamma_b(T)$ for some tree T (or for finitely many given trees) however does not adequately reveal the properties of all 1-cap trees, which merits investigation in its own right. This thesis forms part of this investigation.

1.3 Previous Results

In this section we mention previous results pertinent to our investigation.

While the broadcast number of a subgraph of a graph G can be greater than, smaller than, or equal to that of G, the situation for trees is much simpler, and of major importance in this investigation.

Theorem 1.2. [7] *If T' is a subtree of T then $\gamma_b(T') \leq \gamma_b(T)$.*

If f is a dominating broadcast such that $f(v) = 1$ for each $v \in V_f^+$, then V_f^+ is a dominating set of G, and the minimum cost of such a broadcast is the usual domination number $\gamma(G)$. Recall that a graph G with the property that $\gamma_b(G) = \gamma(G)$ is called a *1-cap* graph.

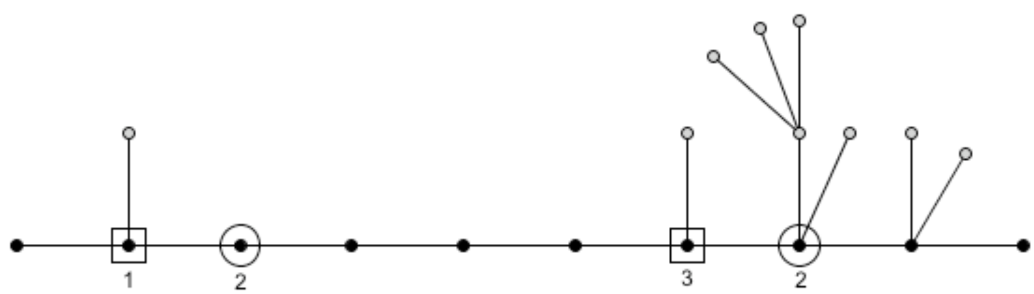


Figure 1.2: A tree with two γ_b-broadcasts

1.3.1 Split-sets

Let T be a tree and P a diametrical path of T. A set M of edges of P is a *split-P set* if each component T' of $T - M$ has even positive diameter, and $P' = T' \cap P$ is a diametrical path of T'. A *split-set* of T is a split-P set for some diametrical path P of T. Any edge in any split-set of T is called a *split-edge* (see Figure 1.3).

1.3.2 Radial and uniquely radial graphs

If f is a broadcast such that $f(v) = \text{rad}(G)$ for some central vertex v and $V_f^+ = \{v\}$, then f is a dominating broadcast. Recall that a graph G with the property that $\gamma_b(G) = \text{rad}(G)$ is called a *radial* graph. A *uniquely radial graph* is a graph G where the only γ_b-broadcast is a central broadcast, that is, a broadcast obtained by

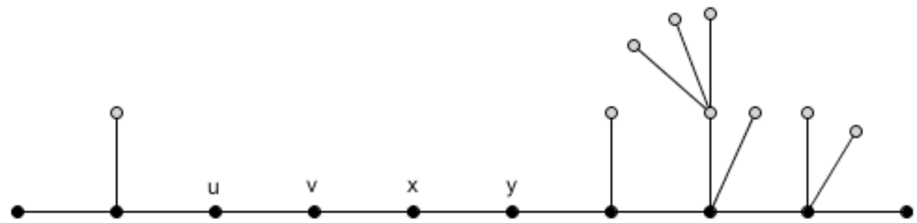


Figure 1.3: A tree with two maximum split-sets $\{uv\}$ and $\{xy\}$

broadcasting from a central vertex with strength $\text{rad}(G)$. In 2009 Herke characterized radial trees.

Theorem 1.3. [10, 11] *A tree T is radial if and only if it has no nonempty split-set.*

Corollary 1.4. [10] *For any tree T, let M be a maximum split-set of cardinality m. Then $\gamma_b(T) = \text{rad}(T) - \lceil \frac{m}{2} \rceil$.*

A characterization of uniquely radial trees is given in Section 1.4.

1.3.3 Shadow trees

Let $P = v_0, v_1, ..., v_d$ be a diametrical path of a tree T. The *shadow tree $S_{T,P}$ of T with respect to P* is defined as follows: For each $v_i \in V(P)$, let V_i be the set of all vertices of T that are connected to v_i by a (possibly trivial) path internally disjoint from P. Let u_i be a vertex in V_i at maximum distance from v_i, and let Q_i be the $v_i - u_i$ path. Then $S_{T,P}$ is the subtree of T induced by $\bigcup_{i=0}^{d} V(Q_i)$ (see Figure 1.4). It is possible for two different diametrical paths P and P' to yield different shadow trees $S_{T,P}$ and $S_{T,P'}$. If the choice of a diametrical path is irrelevant then the notation S_T is sufficient. Each path Q_i which has length at least one is referred to as a *branch*. The vertices on the i^{th} branch, starting with the branch vertex v_{c_i}, are denoted by $v_{c_i}, u_{i,1}, ..., u_{i,b_i}$, where b_i is the length of the i^{th} branch. Any tree T such that $S_T = T$ is referred to as a shadow tree.

A shadow tree with diametrical path $P = v_0, v_1, ..., v_d$ can be drawn on the Cartesian plane so that P lies on the x-axis with v_0 at the origin and each edge is one unit in length, where the edges not on P are drawn above the x-axis parallel to the y-axis. Thus a vertex v_i is described as being to the left of v_j, or v_j to the right of v_i, if $i < j$.

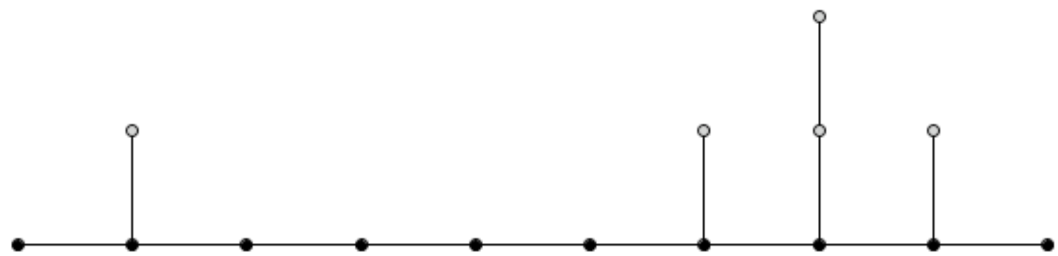


Figure 1.4: The shadow tree for the tree in Figure 1.3

A shadow tree drawn in this way is said to be in *standard representation* (see Figure 1.5).

Herke and Mynhardt demonstrated in [11] that the broadcast number of a tree T is equal to the broadcast number of any shadow tree S_T obtained from T.

Theorem 1.5. [11] *For any shadow tree S_T of T, $\gamma_b(S_T) = \gamma_b(T)$.*

Algorithms for determining S_T, $\gamma(S_T)$ and $\gamma_b(S_T)$ can be found in the appendix. Algorithm A.2 for determining $\gamma_b(S_T)$ is much simpler than the algorithm given in [5].

1.3.4 Triangles

Recall that $S(t,t,t)$ is the tree consisting of three paths of length t sharing one common vertex on the end of each path. When $S(t,t,t)$ appears as a subtree of a shadow tree S_T drawn in standard representation its leaves describe an isosceles right triangle Δ with hypotenuse of length $2t$ along the diametrical path P. We call Δ the triangle associated with $S(t,t,t)$ and say Δ has *radius* t.

For convenience, the vertices of each branch Q_i are labelled as follows, starting with the branch vertex v_{c_i} and ending with a leaf at distance t from v_i: $v_i, u_{i,1}, ..., u_{i,t}$. If a shadow tree S is drawn in standard representation, we can place an isosceles right triangle Δ_i on each branch of S where the radius of the triangle is equal to the length of the branch. The (geometric) vertices of Δ_i are the vertices v_{i-t}, v_{i+t} and $u_{i,t}$. We say that Δ_i is a *triangle* of S and that v_i is the *branch vertex* of Δ_i.

The following corollary to Theorem 1.3 is a geometric equivalent to the theorem stated in terms of the standard representation of the shadow tree.

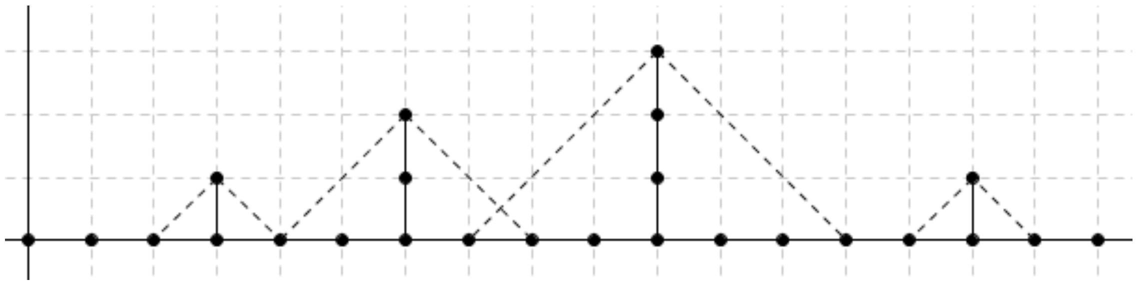


Figure 1.5: A shadow tree drawn in standard representation with triangles

Corollary 1.6. [10, 11] *A tree T is radial if and only if the vertices of the standard representation of S_T cannot be covered by isosceles right triangles where the hypotenuses have even integer lengths that sum to less than* $\operatorname{diam}(T)$.

1.3.5 Free edges

Let S be a shadow tree with diametrical path P. An edge v_iv_{i+1} on P is a *free* edge if it does not lie on the hypotenuse of any triangle of S. It is worth noting that any split-edge is necessarily a free edge, but a free edge is not necessarily a split-edge. If S has k triangles, $\Delta_1, ..., \Delta_k$, then free edges before Δ_1 are called *leading free edges* and free edges after Δ_k are called *trailing free edges*. Free edges which are neither leading nor trailing are called *internal free edges*.

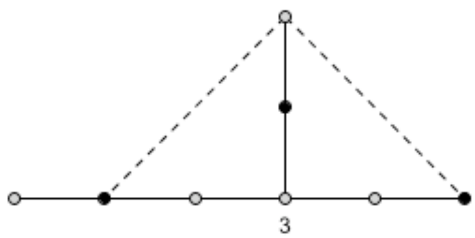
Theorem 1.7. *Suppose that T is a tree, and let T' be a tree obtained by adding three leading or trailing free edges to a diametrical path of T. Then $\gamma_b(T') - \gamma(T') = \gamma_b(T) - \gamma(T)$.*

Proof. Let T be a tree with diametrical path P and T' the tree obtained by adding a path u, v, w to P, joining u to an end-vertex t of P. Suppose D is a γ-set of T'. Then $D \cap \{v, w\} \neq \varnothing$ because D dominates w, and $D - \{v, w\}$ dominates T. Hence $\gamma(T) < \gamma(T')$. Let X be a γ-set of T and $X' = X \cup \{v\}$. Then X' dominates T' and therefore $\gamma(T') \leq \gamma(T) + 1 \leq \gamma(T')$. It follows that $\gamma(T') = \gamma(T) + 1$.

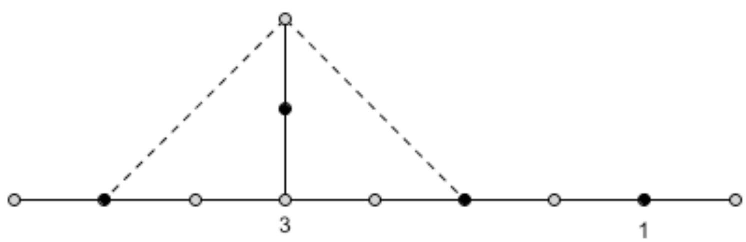
Suppose f' is a γ_b-broadcast of T'. Say w hears the broadcast from z. (Note that z does not necessarily lie on P.) Then $d(z, w) \leq f'(z)$. Let x be the vertex on P such that $d(x, w) = d(z, w)$; $x = z$ if and only if $z \in V(P)$. The function $g = (f' - \{(z, f'(z)\}) \cup \{(x, f'(z))\}$ is also a γ_b-broadcast of T'. Let y be the vertex on P adjacent to x such that $d(y, w) = d(x, w) + 1$. Now the function $g' = (g - \{(x, f'(z))\}) \cup \{(y, f'(z) - 1)\}$ is a dominating broadcast of T, hence $\gamma_b(T) < \gamma_b(T')$.

On the other hand, if f is a γ_b-broadcast of T, then $f \cup \{(v,1)\}$ is a dominating broadcast of T' and so $\gamma_b(T') \leq \gamma_b(T)+1 \leq \gamma_b(T')$. Therefore $\gamma_b(T') = \gamma_b(T)+1$ and the result follows (see Figure 1.6).

■



(a) A tree with $\gamma_b = \gamma$.



(b) The same tree with three trailing edges added.

Figure 1.6: Adding three free edges to a diametrical path of a tree does not affect the difference between γ and γ_b.

Corollary 1.8. *Let $k = 3\ell$ for some $\ell \in \{0,1,2,...\}$. Adding k leading or trailing free edges to a 1-cap tree yields another 1-cap tree.*

A triangle Δ of S is *nested* if it is contained within another triangle Δ' of S. Suppose that Δ is a nested triangle of S, and S' is the shadow tree obtained by removing the vertices on the branch of Δ. Then any edge is a split-edge of S if and only if it is a split edge of S', and it follows from Theorem 1.3 that $\gamma_b(S) = \gamma_b(S')$. The removal of nested triangles of S changes neither the broadcast number nor the radius. See Figure 1.7.

1.3.6 Branch length sequences and overlap sequences

Let $v_{c_1}, v_{c_2,}, ..., v_{c_k}$ be the branch vertices on a diametrical path $P = v_0, v_1, ..., v_d$ of a shadow tree S, and let B_i be the branch connected to v_{c_i}; denote the length of B_i

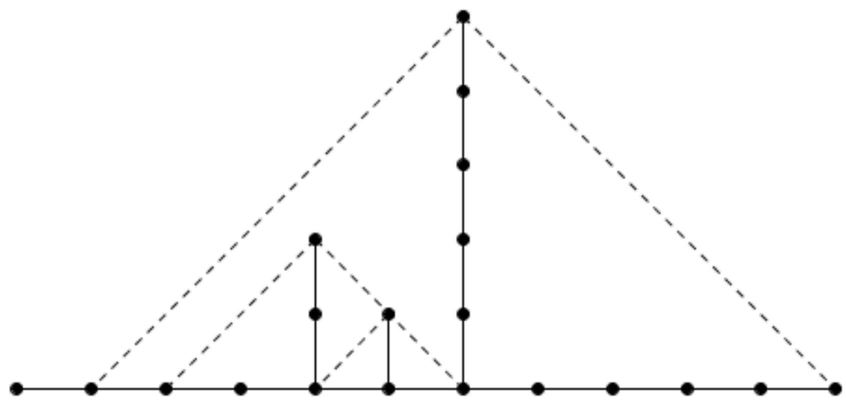


Figure 1.7: A shadow tree with nested triangles

as b_i. Furthermore, let Δ_i be the triangle with branch vertex v_{c_i}. The *branch length sequence* of S is the sequence $\overline{b} = (b_1, b_2, ..., b_k)$. Let v_{l_i} (v_{r_i} respectively) be the first (last, respectively) vertex of Δ_i on P. For $i \in \{2, 3, ..., k\}$, define $h_i = r_{i-1} - l_i$. Note that for $i \in \{2, ..., k\}, h_i$ may be positive, negative, or zero. If h_i is non-negative then it is the number of shared edges between Δ_i and Δ_{i+1}, and if h_i is negative it is the number of free edges between Δ_i and Δ_{i+1}. Also define $h_1 = -l_1$ and $h_{k+1} = r_k - d$; thus $h_1, h_{k+1} \leq 0$, $|h_1|$ is equal to the number of leading free edges, and $|h_{k+1}|$ is equal to the number of trailing free edges. The *overlap sequence* of S is the sequence $\overline{h} = h_1, h_2, ..., h_{k+1}$. It is worth noting that S is uniquely determined by its branch length sequence and overlap sequence, thus the notation $S = T(\overline{b}, \overline{h})$ can be used. See Figure 1.8.

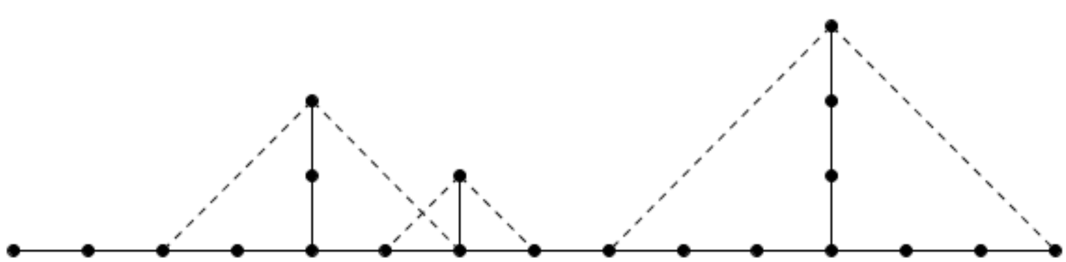


Figure 1.8: A shadow tree with branch length sequence $\overline{b} = (2, 1, 3)$ and overlap sequence $\overline{h} = (-2, 1, -1, 0)$ shown with its triangles

1.4 Characterization of Uniquely Radial Trees

In 2010 Mynhardt and Wodlinger [13] characterized uniquely radial trees. The following manipulation of a shadow tree S is necessary for the characterization of uniquely radial trees: Let S be a shadow tree with diametrical path $P = v_0, v_1, ..., v_d$ and triangles $\Delta_1, \Delta_2, ..., \Delta_k$. If $\deg v_1 = 2$ ($\deg v_{d-1} = 2$), join another leaf to v_1 (v_{d-1} respectively) to make the tree S^*. Note that the (possible) addition of these new leaves affects neither the radius nor the diameter of the tree. The triangles Δ_1 and Δ_k both have radius one and may or may not be nested. The *enhanced shadow tree* Z is obtained by removing any nested triangle Δ_i where $1 < i < k$ from S^*. That is, remove all nested triangles **except** Δ_1 and Δ_k. The choice of diametrical path is irrelevant in the construction of the enhanced shadow tree.

Let $P = v_0, ..., v_d$ be a diametrical path of a shadow tree T with enhanced shadow tree $Z = Z_{T,P}$. Whether or not T is uniquely radial depends on a number of conditions. The possibilities are listed below and the characterization is stated in Theorem 1.9.

A1 The branch B_i of Z of length $t_i \geq 4$ occurs at v_{c_i}, where $t_i \equiv c_i (\mathrm{mod}\, 2)$, and $h_{i-1}, h_i \leq 2$.

A2 The branch B_i of Z of length $t_i \geq 2$ occurs at v_{c_i}, where $t_i \not\equiv c_i (\mathrm{mod}\, 2)$, and $h_{i-1}, h_i \leq 1$.

A3 The branch B_i of Z of length $t_i \geq 3$ occurs at v_{c_i}, where $t_i \equiv c_i (\mathrm{mod}\, 2)$, and $h_{i-1} \leq 2$, $h_i \leq 2t_i - 3$.

A4 The branch B_i of Z of length $t_i \geq 3$ occurs at v_{c_i}, where $t_i \not\equiv c_i (\mathrm{mod}\, 2)$, and $h_{i-1} \leq 2t_i - 3$, $h_i \leq 2$.

B Let d be even.

B1 Z has no free edges.

B2 Z has no zero overlaps at vertices labelled with an even subscript.

B3 If A1 holds for some $i, 1 \leq i \leq k$, then T has a vertex w at distance $t_i - 2$ or $t_i - 1$ from v_{c_i} such that $w \notin V(Z)$.

B4 If A2 holds for some $i, 1 \leq i \leq k$, then T has a vertex w at distance $t_i - 1$ or t_i from v_{c_i} such that $w \notin V(Z)$ and the $v_{c_i} - w$ path is internally disjoint from b_i.

C Let d be odd.

C1 Z has no free edges and no zero overlaps.

C2 Z has no overlap $h = 1$ of the form $v_{2j-1}v_{2j}$, $1 \leq j \leq \frac{d}{2}$.

C3 If A3 holds for some $i, 1 \leq i \leq k$, then T has a vertex w at distance t_i or $t_i + 1$ from v_{c_i+2} such that $w \notin V(Z)$ and the $v_{c_i+2} - w$ path Q contains v_{c_i+1}; if v_{c_i+1} is the last vertex of P on Q then $d(w, v_{c_i+2}) = t_i$.

C4 If A4 holds for some $i, 1 \leq i \leq k$, then T has a vertex w at distance t_i or $t_i + 1$ from v_{c_i-2} such that $w \notin V(Z)$ and the $v_{c_i-2} - w$ path Q contains v_{c_i-1}; if v_{c_i-1} is the last vertex of P on Q then $d(w, v_{c_i-2}) = t_i$.

Theorem 1.9. [13] *A tree T is uniquely radial if and only if* B1 - B4 *or* C1 - C4 *hold.*

Chapter 2

Previous results on 1-cap trees

2.1 Introduction and Basic Results

As stated in Chapter 1, a graph G is *1-cap* if $\gamma(G) = \gamma_b(G)$. Erwin was the first to explore broadcasts in graphs and mention the idea of 1-cap graphs in his dissertation and referred to such graphs as Type 2 graphs. In 2003 and 2006 Dunbar et al. [6, 7] further investigated 1-cap graphs and broadcasts in general. In [7] they refer to Liu who discusses the idea of *dominance* in communication networks, where cities are represented as vertices and a dominating set represents a set of cities which have broadcast stations and can broadcast messages to every city in the network. Since it was assumed that a broadcast station can only reach adjacent cities, the optimal case will be a 1-cap graph.

One unsurprising result from Erwin [8] is the observation that all paths are 1-cap.

Proposition 2.1. [7] *For every integer $n \geq 2$,*

$$\gamma_b(P_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

As mentioned in Chapter 1, Herke and Mynhardt [11] explored radial trees, and in 2009 developed a characterization for such trees, that is trees of Type 1. Later with Cockayne [3] they discovered that 1-cap trees can be broken into radial components.

Theorem 2.2. [3] *A tree T is 1-cap if and only if it has a maximum split-set M such that each component T_i of $T - M$ is 1-cap.*

Note that this property is not required for every maximum split-set, just for at least one, as can be seen in Figure 2.1. In our figures we henceforth draw the vertices in a dominating set as black vertices.

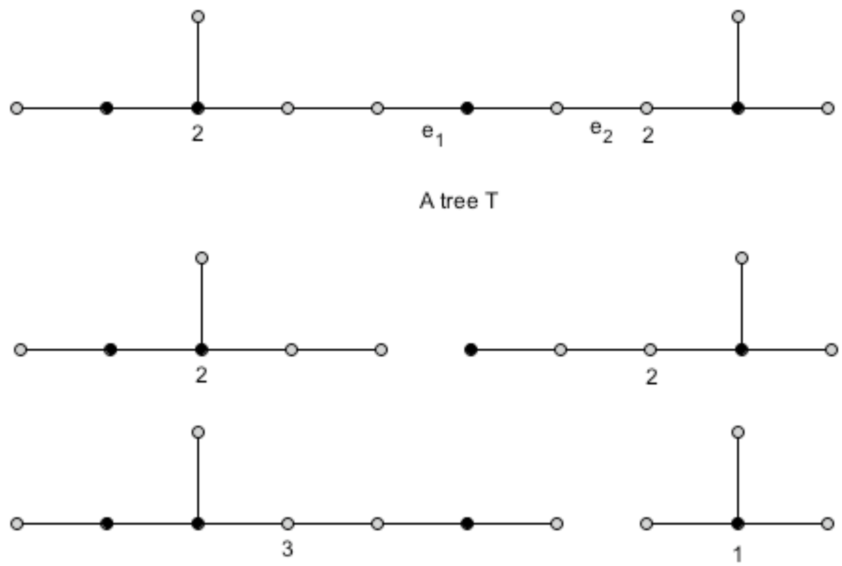


Figure 2.1: Here T is 1-cap, and both $\{e_1\}$ and $\{e_2\}$ are maximum split-sets, but only the components of $T - e_2$ are also 1-cap.

2.2 1-cap Shadow Trees

The following corollary to Theorem 1.5 demonstrates the importance of shadow trees to the class of all 1-cap trees.

Corollary 2.3. [3]

(*i*) *If T is 1-cap, then $\gamma(T) = \gamma(S_T)$.*

(*ii*) *If S_T is 1-cap, $\gamma(S_T) = k$, and $\gamma(T) = k$, then T is 1-cap.*

Due to the relatively simple structure of shadow trees, the following approach to studying 1-cap trees is useful.

Step 1 Find all 1-cap shadow trees S with $\gamma_b(S) = k$.

Step 2 If S is a 1-cap shadow tree with $\gamma_b(S) = k$, use Corollary 2.3 to find all 1-cap trees T with $\gamma_b(T) = k$ that have S as a shadow tree.

As demonstrated by Cockayne et. al. [3], whether or not a shadow tree is 1-cap depends not on the branch lengths of the triangles, but only on their least residues modulo 3 and the overlap sequence.

Theorem 2.4. [3] *If $T(\bar{b}, \bar{h})$ is 1-cap then any shadow tree $T' = T'(\bar{b}', \bar{h})$, where $\bar{b}' = b'_1, b'_2, ..., b'_k$ such that $b'_i \equiv b_i (\text{mod } 3)$ for each $i \in \{1, ..., k\}$ and such that T' contains no nested triangles is 1-cap.*

Due to this fact we first study the possibilities for branch lengths modulo 3 separately.

2.3 Branches with lengths congruent to $1 (\text{mod}\, 3)$

A *caterpillar* is a shadow tree where all branches have length equal to one. In 2008 Seager [14] studied dominating broadcasts of caterpillars, characterizing caterpillars of Type 1 and of Type 2 [14]. In 2010 Mynhardt and Wodlinger [12] extended Seager's results on caterpillars to the class of trees whose shadow trees have branch lengths congruent to $1 (\text{mod}\, 3)$. For the shadow tree T of such a tree, let $\sigma = \Delta_i, ..., \Delta_j, i \leq j$, be a sequence of consecutive triangles with branch vertices $v_{c_i}, ..., v_{c_j}$ such that $h_{i+1}, ..., h_j \geq 0$. Such a sequence is called a *nonnegative overlap sequence.* A nonnegative overlap sequence that is not contained in a larger nonnegative overlap sequence is a *maximal nonnegative overlap sequence (MNOS).* Denote by T_σ the subtree of T induced by σ. The subtree T_σ is referred to as the *subtree of T associated with σ.* It is clear that T_σ has no free edges and thus is radial. As demonstrated in [12], T_σ is 1-cap if and only if σ contains only overlaps of cardinality $0, 1, 2, 3$, or 5, at most one of which has odd cardinality. We now restrict the MNOS's to those from trees which could possibly be 1-cap. Thus, if σ is an MNOS containing only overlaps of size 0 or 2, then it has even diameter and is called an *even MNOS*, otherwise σ has odd diameter and is called an *odd MNOS.*

Consider a sequence $\sigma_i, ..., \sigma_j, i \leq j$ of consecutive MNOS's of T where the negative overlaps between two consecutive MNOS's σ_ℓ and $\sigma_{\ell+1}$ is exactly -1. Such a sequence of MNOS's is called a *tight sequence.* Let $S_{i,j}$ be the subtree of T associated with $\sigma_i, ..., \sigma_j$. As done in [12], for each $s = i, ..., j$ the notation T_{σ_s} to denote the subtree of T associated with σ_s can be simplified to T_s. A *maximal tight sequence (MTS)* is a tight sequence that is not contained within a larger tight sequence. Let T be a tree with MTS's $S_1, ..., S_r$. Again as is done in [12] the notation S_i is also used to represent the subtree of T associated with the MTS S_i.

Let Q_1 (Q_{r+1}, respectively) be the subpath of the diametrical path P induced by the free edges preceding S_1 (following S_r, respectively), and for $i = 2, ..., r$, let Q_i be the subpath of P induced by the free edges that join S_{i-1} to S_i. Say Q_i contains q_i vertices that do not lie on S_{i-1} or S_i. The 1-cap shadow trees with branches of length congruent to $1 (\bmod 3)$ are characterized as follows.

Theorem 2.5. [12] *Let T be a shadow tree without nested triangles and with branch lengths congruent to $1 (\bmod 3)$. Furthermore, let T have MTS's $S_1, ..., S_r$ and define $q_1, ..., q_{r+1}$ as above. For each $k \in 1, ..., r$, let $\sigma_{k,1}, ..., \sigma_{k,t_k}$ be the MNOS's of S_k. Then T is 1-cap if and only if the following conditions hold:*

1. Each $\sigma_{k,i}$ contains only overlaps of cardinality $0, 1, 2, 3$, or 5, at most one of which is odd.

2. If $\sigma_{k,1}, ..., \sigma_{k,t_k}$ are all odd, then $q_k \not\equiv 1 (\bmod 3)$ and $q_{k+1} \not\equiv 1 (\bmod 3)$.

3. If exactly one of $\sigma_{k,1}, ..., \sigma_{k,t_k}$ is even, then $q_k \not\equiv 1 (\bmod 3)$ or $q_{k+1} \not\equiv 1 (\bmod 3)$.

4. Suppose $k' \geq k + 1$ and consider the MTS's $S_k, S_{k+1}, ..., S_{k'}$. If exactly one of $\sigma_{i,1}, ..., \sigma_{i,t_i}$ is even for each i such that $k < i < k'$, and

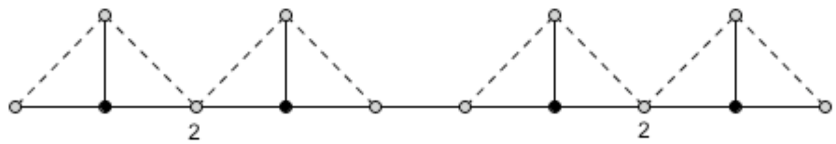
   (a) $\sigma_{k,1}, ..., \sigma_{k,t_k}, \sigma_{k',1}, ..., \sigma_{k',t_{k'}}$ are all odd, or

   (b) (without loss of generality) $\sigma_{k,1}, ..., \sigma_{k,t_k}$ are all odd, exactly one of $\sigma_{k',1}, ..., \sigma_{k',t_{k'}}$ is even and $q_{k+1} \equiv 1 (\bmod 3)$, or

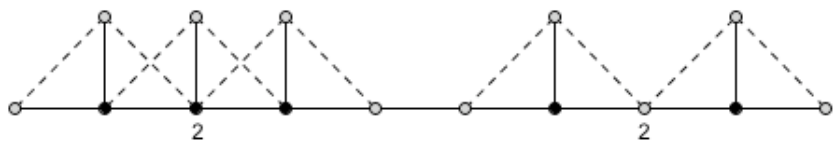
   (c) exactly one of $\sigma_{k,1}, ..., \sigma_{k,t_k}$ and exactly one of $\sigma_{k',1}, ..., \sigma_{k',t_{k'}}$ are even, and $q_k \equiv q_{k+1} \equiv 1 (\bmod 3)$,

   then $q_i \equiv 0 (\bmod 3)$ for at least one i such that $k < i \leq k'$.

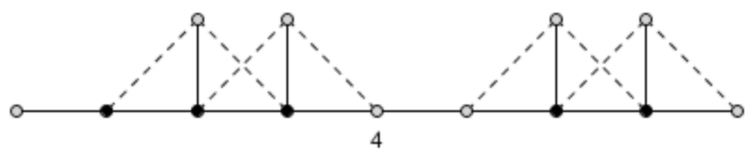
Examples of non-1-cap trees violating each condition can be seen in Figures 2.2 and 2.3.



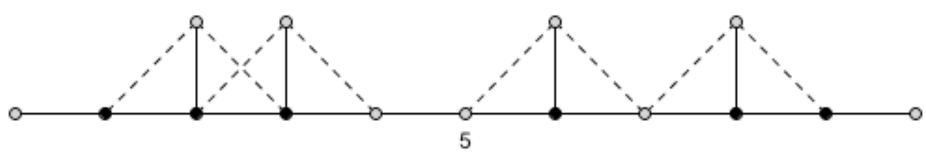
(a) A 1-cap tree



(b) A non-1-cap tree violating condition 1.

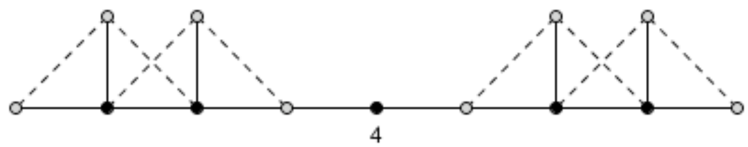


(c) A non-1-cap tree violating condition 2.

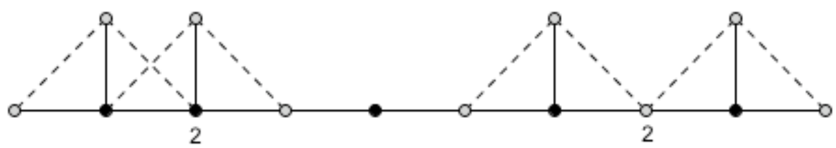


(d) A non-1-cap tree violating condition 3.

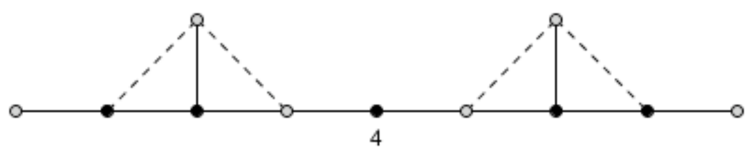
Figure 2.2: A 1-cap tree followed by non-1-cap trees violating conditions 1, 2, and 3.



(a) A non-1-cap tree satisfying condition 4(a).



(b) A non-1-cap tree satisfying condition 4(b).



(c) A non-1-cap tree satisfying condition 4(c).

Figure 2.3: Non-1-cap trees which satisfy the conditions in 4, but violate the conclusion.

Chapter 3

Trees with branches of length congruent to $2 \pmod 3$ and no internal free edges

3.1 Introduction

Henceforth, all trees in this chapter are shadow trees with branches of length congruent to $2 \pmod 3$ and no internal free edges. In Section 3.2 we determine six types of 1-cap trees, all with no interior free edges. We then show in Sections 3.3 and 3.4 that these trees are in fact the only 1-cap trees of this nature. We first mention a number of further assumptions we make throughout this chapter.

Each shadow tree, T, in this chapter is assumed to have a diametrical path $P = v_0, ..., v_d$ with branch vertices $v_{c_1}, ..., v_{c_k}$, $k \geq 1$. The branch $B_i = v_{c_i}, u_{i,1}, ..., u_{i,b_i}$ of T attached to v_{c_i} has length $b_i = 3m_i + 2$, $i = 1, ..., k$, and is covered by the triangle Δ_i, where Δ_i, Δ_{i+1} overlap by $h_{i+1} \geq 0$ edges, $i = 1, ..., k-1$. Only h_1 and h_{k+1} can be negative. Since the radius of Δ_i is b_i, the consecutive triangles Δ_i, Δ_{i+1} contain $2(b_i + b_{i+1}) - h_{i+1}$ edges of P.

Given a γ-set X of T, a branch vertex v_{c_i} may or may not be in X. In either case, X contains at least $\left\lceil \frac{3m_i+1}{3} \right\rceil = m_i + 1$ vertices of $B_i - v_{c_i}$. A γ-set X of T such that $X \cap (V(B_i - v_{c_i})) = \{u_{i,1}, u_{i,4}, ..., u_{i,3m_{i+1}}\}$ for each $i = 1, ..., k$ is called a *natural γ-set* of T. See Figure 3.1. Unless stated otherwise the γ-sets of trees in this chapter are all natural γ-sets.

As mentioned in Section 1.3.5, removing nested triangles does not change the broadcast number or the radius of a tree. Suppose T has branches of length congruent to 2 (mod 3) and a nested triangle Δ_i associated with the branch $B_i = v_{c_i}, u_{i,1}, ..., u_{i,b_i}$. As shown above, any γ-set of T contains at least one vertex of $B_i - v_{c_i}$. Let $T' = T - \{u_{i,1}, ..., u_{i,b_i}\}$. Then $\gamma_b(T) = \gamma_b(T') \leq \gamma(T') < \gamma(T)$ and so T is not a 1-cap tree. Therefore we henceforth assume that our trees do not have nested triangles.

By Theorem 2.4, if $T(\overline{b}, \overline{h})$ is 1-cap then any shadow tree $T' = T'(\overline{b'}, \overline{h})$, where $\overline{b'} = (b'_1, b'_2, ..., b'_k)$ such that $b'_i \equiv b_i \pmod 3$ for each $i \in \{1, 2, ..., k\}$ and such that T' contains no nested triangles is 1-cap. Therefore, if we consider trees with an overlap sequence $\overline{h} = (h_1, h_2, ..., h_{k+1})$ such that $h_i \leq 3$, $i \in \{2, 3, ..., k\}$ (i.e. each internal overlap is at most three), then no nested triangles result if we assume each branch to have length exactly two. We sometimes make this assumption for simplicity.

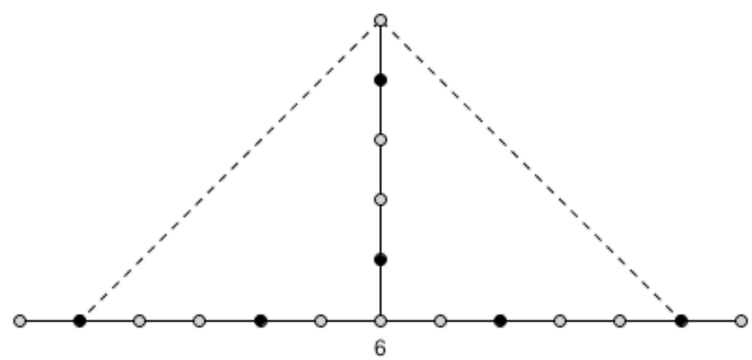


Figure 3.1: A tree with a natural dominating set consisting of the black vertices.

3.2 Six types of 1-cap trees

Six types of 1-cap trees are described in Theorems 3.1 to 3.4.

The reader is encouraged to verify these and investigate other 1-cap trees with the use of the website
`http://www.math.uvic.ca/~slunney/GraphGUI.html`.

Theorem 3.1. *Let T be a shadow tree with $s \geq 1$ branch vertices such that all $s-1$ internal overlaps are 0-overlaps, and T has x leading and y trailing free edges. Then T is 1-cap if and only if*

(a) $s = 1$ *and* $x \equiv y \equiv 2 \pmod 3$, *or*

(b) $s \geq 1$ *and at least one of x and y is congruent to* $1 \pmod 3$.

Proof. By Theorem 2.4 we may assume that each branch of T has length 2. We prove the result for $x, y \in \{0, 1, 2\}$; the theorem will then follow from Corollary 1.8. Let T' be the subtree of T induced by all edges of T except the leading and trailing free edges. Then $\text{diam}(T') = 4s$ and $\text{rad}(T') = \gamma_b(T') = 2s$. Let $P = v_0, v_1, ..., v_{4s}$ be a diametrical path of T'. Note that $v_2, v_6, ..., v_{4s-2}$ are the branch vertices. For each $i \in \{1, ..., s\}$, the branch B_i that starts at $v_{2+4(i-1)} = v_{4i-2}$ consists of the path $v_{4i-2}, u_{i,1}, u_{i,2}$. Define $D \subseteq V(T')$ by

$$D = \{u_{i,1} : i \in \{1, 2, ..., s\}\} \cup \{v_{4j} : j \in \{0, 1, ..., s\}\}.$$

Then $|D| = 2s+1$ and D is an efficient dominating set of T', hence $\gamma(T') = 2s+1$. Let X be any γ-set of T and let $X' = X \cap V(T')$. We consider three cases.

Case 1 X' dominates neither v_0 nor v_{4s}. See Figure 3.2a. Then $x, y \geq 1$, $|X| \geq |X'|+2$, and $X' \cap \{v_0, v_1, v_{4s-1}, v_{4s}\} = \varnothing$. Hence, in order to dominate v_1 and v_{4s-1}, $\{v_2, v_{4s-2}\} \subseteq X'$. To dominate v_{4i}, $i \in \{1, ..., s-1\}$, $\{v_{4i-1}, v_{4i}, v_{4i+1}\} \cap X' \neq \varnothing$. Hence $|X'| = 2$ if $s = 1$ and $|X'| \geq 2s+1$ if $s \geq 2$. For each $x, y = 1, 2$, let $T(x, y)$ be the tree obtained from T' by adding x leading and y trailing free edges to P. By Theorem 1.3 each $T(x, y)$ is radial, hence $\gamma_b(T(x, y)) = \text{rad}(T(x, y))$. Moreover, $\gamma(T(x, y)) = 4$ for each $x, y \in \{1, 2\}$ if $s = 1$ and $\gamma(T(x, y)) \geq 2s+3 > 2s+2 \geq \text{rad}(T(x, y))$ if $s \geq 2$. Therefore $T(x, y)$ is 1-cap in this case if and only if $x = y = 2$ and $s = 1$. Hence Corollary 1.8 implies that (a) holds.

Case 2 Without loss of generality X' dominates v_0 but not v_{4s}. See Figure 3.2b. Then $y \geq 1$, $|X| \geq |X'|+1$, $v_{4s-2} \in X'$ to dominate v_{4s-1} and $\{v_0, v_1\} \cap X' \neq \varnothing$ to dominate v_0. Hence $|X'| \geq 2s+1$ for each $s \geq 1$. For $x \in \{0, 1, 2\}$ and $y \in \{1, 2\}$, define $T(x, y)$ as in Case 1. Observe that under the assumption that $v_{4s-2} \in X' \subseteq X$, X is a γ-set of T if and only if $y = 2$. As shown in Table 3.1, $T(x, y)$ is 1-cap if and only if $x = 1$.

Case 3 X' dominates v_0 and v_{4s}. See Figure 3.2c. We may assume without loss of generality that $X' = D$ as defined above. The values of $\gamma(T(x, y))$ and $\gamma_b(T(x, y))$ are given in Table 3.1. Note that $\gamma(T(x, y)) = \gamma_b(T(x, y))$ if and only if $1 \in \{x, y\}$. By Corollary 1.8, (b) holds.

■

Free edges (x, y)	γ	γ_b	1-cap?
$T(0,0)$	$2s+1$	$2s$	NO
$T(0,1)$	$2s+1$	$2s+1$	YES
$T(0,2)$	$2s+2$	$2s+1$	NO
$T(1,1)$	$2s+1$	$2s+1$	YES
$T(1,2)$	$2s+2$	$2s+2$	YES
$T(2,2)$	$2s+3$	$2s+2$	NO

Table 3.1: Possibilities for $\gamma(T(x,y))$ and $\gamma_b(T(x,y))$ in Theorem 3.1.

Theorem 3.2. *Let T be a shadow tree with $s \geq 2$ branches, exactly one 1-overlap and $s-2$ 0-overlaps, and x leading and y trailing free edges. Then T is a 1-cap tree if and only if*

(*a*) $s = 2$ *and* $x \equiv y \equiv 2 \pmod 3$, *or*

(*b*) $s \geq 2$ *and* $x \equiv y \equiv 1 \pmod 3$.

Proof. By Theorem 2.4 we may assume that each branch of T has length 2. By Corollary 1.8 we may assume that $x, y \in \{0,1,2\}$. Define the tree T' as in the proof of Theorem 3.1. Say T and T' has $s = s_1 + s_2$ branches $B_1, B_2, ..., B_s$ with triangles $\Delta_1, \Delta_2, ..., \Delta_s$, where Δ_{s_1} and Δ_{s_1+1} overlap in one edge. Then $\text{diam}(T') = 4s-1$ and $\text{rad}(T') = 2s$. Let $P = v_0, v_1, ..., v_{4s-1}$ be a diametrical path of T'. Let T_1 and T_2 be the subtrees of T' formed by $\Delta_1, ..., \Delta_{s_1}$ and $\Delta_{s_1+1}, ..., \Delta_s$ respectively. For $i = 1, 2$, define the efficient dominating set D_i of T_i similar to the dominating set D of T' in the proof of Theorem 3.1. Then

$$D_1 = \{u_{i,1} : i \in \{1, 2, ..., s_1\}\} \cup \{v_{4j} : j \in \{0, 1, ..., s_1\}\}$$

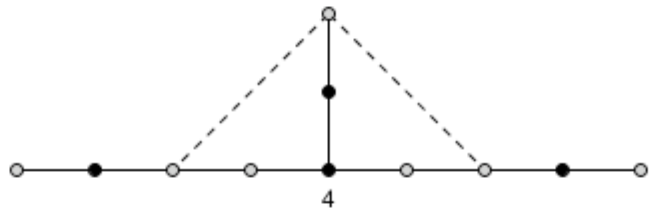
and

$$D_2 = \{u_{i,1} : i \in \{s_1+1, ..., s\}\} \cup \{v_{4j-1} : j \in \{s_1+1, ..., s\}\}.$$

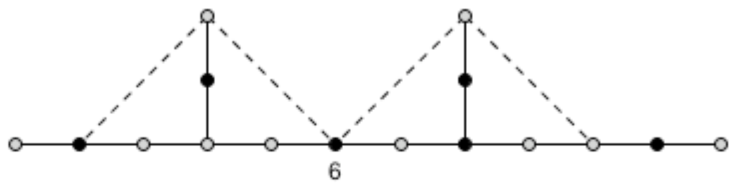
Now $D = D_1 \cup D_2$ is a minimum dominating set of T' of cardinality $2s+1$. Hence $\gamma(T') = 2s+1$.

Let X be a γ-set of T and let $X' = X \cap V(T')$. We consider three cases.

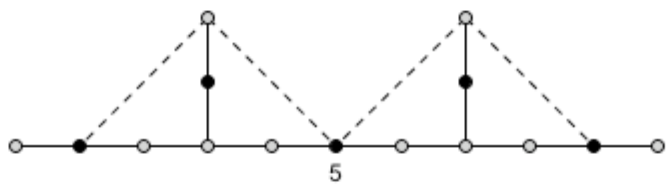
Case 1 X' dominates neither v_0 nor v_{4s-1}. Then $x, y \geq 1$, and $X' \cap \{v_0, v_1, v_{4s-2}, v_{4s-1}\} = \varnothing$. Hence, in order to dominate v_1 and v_{4s-2}, $\{v_2, v_{4s-3}\} \subseteq X'$. To dominate v_{4i}, $i \in \{1, ..., s_1 - 1\}$, $\{v_{4i-1}, v_{4i}, v_{4i+1}\} \cap X' \neq \varnothing$. To dominate v_{4i},



(a) Case 1 in Theorem 3.1 (two leading and two trailing free edges).



(b) Case 2 in Theorem 3.1 (one leading and two trailing free edges).



(c) Case 3 in Theorem 3.1 (one leading and one trailing free edge).

Figure 3.2: Trees from Theorem 3.1.

$i \in \{s_1+1, ..., s-1\}$, $\{v_{4i-1}, v_{4i}, v_{4i+1}\} \cap X' \neq \varnothing$. To dominate v_{4s_1-1} and v_{4s_1}, at least one more vertex is needed, unless $v_{4s-1} = v_3$ (since $v_2 \in X'$) and $v_{4s_1} = v_{4s-1}$ (since $v_{4s-3} \in X'$). In the latter case, $s_1 = 1$ and $s = 2$. For the trees $T(x, y)$, $x, y \geq 1$ as defined in the proof of Theorem 3.1, we see that $\gamma(T(x, y)) = \gamma_b(T(x, y))$ if and only if $x = y = 2$, see Figure 3.3a. Hence (a) holds. Now assume $s \geq 3$. By the above, $|X'| \geq 2+s_1-1+s_2-1+1+s = 2s+1$, so that $|X| \geq 2s+3$. However, for $x, y \in \{1, 2\}$, $T(x, y)$ is radial and has radius at most $2s+2$. Thus $T(x, y)$ is not 1-cap.

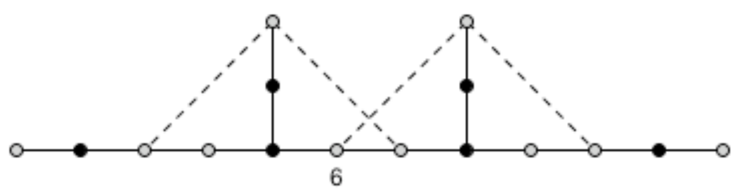
Case 2 Without loss of generality X' dominates v_0 but not v_{4s-1}. Then $y \geq 1$, $|X| \geq |X'|+1$. As above, we can show that $|X'| \geq 2s+1$. As in Case 3 in the proof of Theorem 3.1 we observe that X is a γ-set of T if and only if $y = 2$. The values of $\gamma(T(x, y))$ and $\gamma_b(T(x, y))$ are given in Table 3.2 and we see that $T(x, y)$ is not 1-cap.

Case 3 X' dominates v_0 and v_{4s-1}. Then $x, y \in \{0, 1, 2\}$ and we may assume without loss of generality that $X' = D$ as defined above. Hence $|X'| = 2s + 1, s \geq 2$. We deduce from Table 3.2 that $T(x, y)$ is 1-cap if and only if $x = y = 1$. See Figure 3.3b.

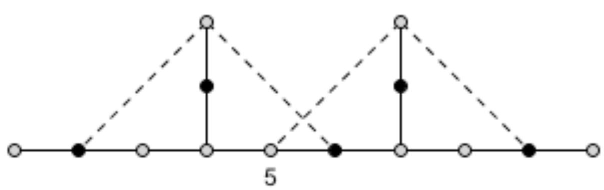
■

$T(x,y)$	γ	γ_b	1-cap?
$T(0,0)$	$2s+1$	$2s$	NO
$T(0,1)$	$2s+1$	$2s$	NO
$T(0,2)$	$2s+2$	$2s+1$	NO
$T(1,1)$	$2s+1$	$2s+1$	YES
$T(1,2)$	$2s+2$	$2s+1$	NO
$T(2,2)$	$2s+3$	$2s+2$	NO

Table 3.2: Possibilities for $\gamma(T(x,y))$ and $\gamma_b(T(x,y))$ in Theorem 3.2.



(a) Case 1 in Theorem 3.2 (two leading and two trailing free edges).



(b) Case 3 in Theorem 3.2 (one leading and one trailing free edge).

Figure 3.3: Trees from Theorem 3.2.

Theorem 3.3. *Let T be a shadow tree with any number of 0-overlaps and exactly one 2-overlap. Then T is a 1-cap tree if and only if T begins with x free edges, followed by all of the 0-overlaps, followed by the 2-overlap, followed finally by y free edges, where $x \equiv 1 (\text{mod } 3)$ and $y \equiv 2 (\text{mod } 3)$, or the reverse of such a tree.*

Proof. Say T has s branches. Proceed as in the proof of Theorem 3.2 to construct the trees T', T_1 and T_2 where this time Δ_{s_1} and Δ_{s_1+1} overlap in two edges; hence $\text{diam}(T') = 4s - 2$. If $P = v_0, ..., v_{4s-2}$ is a diametrical path of T', then defining D_i, $i = 1, 2$ as before,

$$D_1 = \{u_{i,1} : i \in \{1, 2, ..., s_1\}\} \cup \{v_{4j} : j \in \{0, 1, ..., s_1\}\}$$

and

$$D_2 = \{u_{i,1} : i \in \{s_1 + 1, ..., s\}\} \cup \{v_{4j-2} : j \in \{s_1 + 1, ..., s\}\},$$

and $D = D_1 \cup D_2$ is a minimum dominating set of T' of cardinality $2s + 1$. Hence $\gamma(T') = 2s + 1$ while $\text{rad}(T') = 2s - 1$. Therefore, for $x, y \in \{0, 1, 2\}$, if $T(x, y)$ is a 1-cap tree, then $\text{rad}(T(x, y)) \geq 2s + 1$, that is, $3 \leq x + y \leq 4$. By examining the possible cases we find that $T(x, y)$ is a 1-cap tree if and only if the following conditions hold: $s = s_1 + 1$, $x = 1$, and $y = 2$; that is, the 2-overlap is the last overlap and is followed by two free edges $v_{4s-2}v_{4s-1}$ and $v_{4s-1}v_{4s}$, and any minimum dominating set of T contains v_{4s-1}, v_{4s-4} and v_0, or the reverse of this situation. See Figure 3.4. ∎

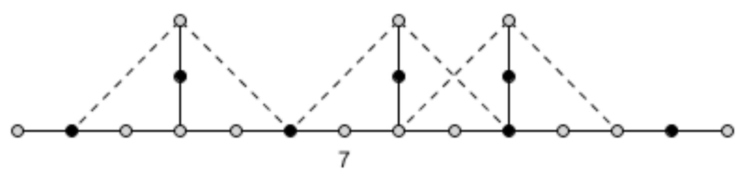


Figure 3.4: A tree from Theorem 3.3.

Theorem 3.4. *Let T be a shadow tree with any number of 0-overlaps, exactly one 3-overlap, no other overlaps, and x leading and y trailing free edges. Then T is a 1-cap tree if and only if $x \equiv y \equiv 1 \pmod 3$.*

Proof. Again, we assume that T has s branches, $B_1, B_2, ..., B_s$, each of length two and that $x, y \in \{0, 1, 2\}$. Define the trees T', T_1 and T_2 as in the proof of Theorem 3.2, where in this case Δ_{s_1} and Δ_{s_1+1} overlap in three edges. Then $\text{diam}(T') = 4s - 3$ and $\text{rad}(T') = 2s - 1$. Let $P = v_0, v_1, ..., v_{4s-3}$ be a diametrical path of T'. For $i = 1, 2$, define the efficient dominating set D_i of T_i as in the proof of Theorem 3.2. Then

$$D_1 = \{u_{i,1} : i \in \{1, 2, ..., s_1\}\} \cup \{v_{4j} : j \in \{0, 1, ..., s_1\}\}$$

and

$$D_2 = \{u_{i,1} : i \in \{s_1+1, ..., s\}\} \cup \{v_{4j-3} : j \in \{s_1, ..., s\}\}.$$

Now $D = D_1 \cup D_2 - \{v_{4s_1-3}, v_{4s_1}\}$ is an efficient dominating set of T' of cardinality $2s$. Hence $\gamma(T') = 2s$. Clearly if $x = y = 1$ then D is also a γ-set of $T(x, y)$ and $\text{rad}(T(x, y)) = 2s = \gamma(T(x, y))$. The efficiency of D also implies that if $x = 2$ or $y = 2$, then $\gamma(T(x, y)) = 2s + 2 > \text{rad}(T(x, y)) = 2s + 1$. Hence $T(x, y)$ is a 1-cap tree if and only if $x = y = 1$. See Figure 3.5. ■

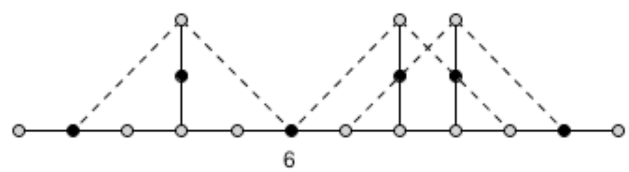


Figure 3.5: A tree from Theorem 3.4.

Six classes of 1-cap shadow trees were described in Theorems 3.1 – 3.4. In all cases the only free edges are leading or trailing free edges, and in all cases the 1-cap trees contain at least one free edge. In the next two sections we completely characterize 1-cap trees of this nature.

3.3 Clear shadow trees and pure minimum dominating sets

A shadow tree T that has at least one γ-set D that contains no branch vertices is called *clear* and D is called a *pure γ-set* of T. Recall that $T(\overline{b}, \overline{h})$ denotes the shadow tree with branch length sequence $\overline{b} = b_1, ..., b_k$ and overlap sequence $\overline{h} = h_1, ..., h_{k+1}$. We now show that the only clear 1-cap shadow trees whose only free edges are leading and trailing free edges are the trees $T(\overline{b}, \overline{h})$ mentioned in Theorems 3.1(b), 3.2(b) and 3.4, and the shadow trees $T(\overline{b'}, \overline{h})$ associated with these trees as explained in Theorem 2.4.

Theorem 3.5. *Let T be a clear shadow tree whose only free edges are x leading and y trailing free edges. Then T is a 1-cap tree if and only if*

(a) *without loss of generality $x \equiv 1 \pmod 3$ and all overlaps are zero, or*

(b) $x \equiv y \equiv 1 \pmod 3$ *and exactly one overlap is positive, this overlap being equal to* 1 *or* 3.

Proof. By Theorem 2.4 we may assume that $x, y \in \{0,1,2\}$ so that T is radial. Recall that the branch $B_i = v_{c_i}, u_{i,1}, ..., u_{i,b_i}$ of T attached to v_{c_i} has length $b_i = 3m_i + 2$, $i = 1, ..., k$, and is covered by the triangle Δ_i, where Δ_i, Δ_{i+1} overlap by h_i edges, $i = 1, ..., k-1$. Then

$$\begin{aligned}\operatorname{rad} T = \left\lceil \frac{d}{2} \right\rceil &= \left\lceil \frac{1}{2}\left(x + y + \sum_{i=1}^{k} 2b_i - \sum_{i=1}^{k-1} h_i \right) \right\rceil = \sum_{i=1}^{k} b_i + \left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil \\ &= 2k + 3\sum_{i=1}^{k} m_i + \left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil . \qquad (3.1)\end{aligned}$$

For each $i = 1, ..., k-1$, let Q_i be the path $v_{c_i+1}, ..., v_{c_{i+1}-1}$. Since $d(v_{c_i}, v_{c_{i+1}}) = c_{i+1} - c_i = b_i + b_{i+1} - h_i$, the length $\ell(Q_i)$ of Q_i is given by $\ell(Q_i) = b_i + b_{i+1} - h_i - 2$, hence Q_i contains $b_i + b_{i+1} - h_i - 1$ vertices. We determine $\gamma(T)$. Let D be a pure natural γ-set of T.

- Each bough B_i contains $\lceil \frac{b_i}{3} \rceil = m_i + 1$ vertices in D. The vertex $u_{i,1}$ is in D since D is a natural γ-set and dominates v_{c_i}.
- The path $v_0, ..., v_{c_1-1}$ contains v_{c_1} vertices, $\lceil \frac{c_1}{3} \rceil$ of which are in D.
- The path $v_{c_k+1}, ..., v_d$ contains $d - c_k$ vertices, $\lceil \frac{d-c_k}{3} \rceil$ of which are in D.
- Each path Q_i contains $\left\lceil \frac{b_i+b_{i+1}-h_i-1}{3} \right\rceil$ vertices in D.

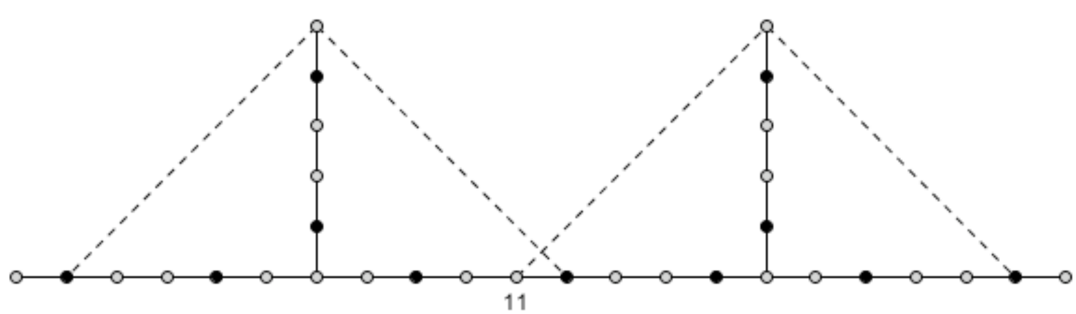


Figure 3.6: Here $d = 21$, $c_1 = 6$, $c_2 = 15$, $b_1 = 5 = b_2$ and $h_1 = 1$.

Since $c_1 = x + b_1$ and $d - c_k = y + b_k$, we obtain

$$\begin{aligned}\gamma(T) &= \sum_{i=1}^{k} \left\lceil \frac{b_i}{3} \right\rceil + \left\lceil \frac{x+b_1}{3} \right\rceil + \left\lceil \frac{y+b_k}{3} \right\rceil + \sum_{i=1}^{k-1} \left\lceil \frac{b_i + b_{i+1} - h_i - 1}{3} \right\rceil \\ &= \sum_{i=1}^{k} (m_i + 1) + \left\lceil \frac{x+3m_1+2}{3} \right\rceil + \left\lceil \frac{y+3m_k+2}{3} \right\rceil + \sum_{i=1}^{k-1} \left\lceil \frac{3m_i + 3m_{i+1} - h_i + 3}{3} \right\rceil \\ &= \sum_{i=1}^{k} m_i + k + m_1 + m_k + \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil + k - 1 + \sum_{i=1}^{k-1} (m_i + m_{i+1}) - \sum_{i=1}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor \\ &= 3\sum_{i=1}^{k} m_i + 2k - 1 + \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil - \sum_{i=1}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor . \qquad (3.2)\end{aligned}$$

If T is a 1-cap tree, then $\gamma_b(T) = \operatorname{rad} T = \gamma_b(T)$, hence from (3.1) and (3.2),

$$\left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil = \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil - 1 - \sum_{i=1}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor . \qquad (3.3)$$

Note that $\lceil \frac{x+y}{2} \rceil \leq \lceil \frac{x+2}{3} \rceil + \lceil \frac{y+2}{3} \rceil - 1$, with equality if and only if $x = 1$ and $y \in \{0, 1, 2\}$, or $x \in \{0, 1, 2\}$ and $y = 1$. Hence if $h_i = 0$ for all i, then without loss of generality $x = 1$ and $y \in \{0, 1, 2\}$. Therefore (a) holds.

Also, $\frac{1}{2} \sum_{i=1}^{k-1} h_i > \sum_{i=1}^{k-1} \lfloor \frac{h_i}{3} \rfloor$ for all $k \geq 2$ if $h_i > 0$ for at least one i. Therefore, if there is an even positive number of odd overlaps, then

$$\left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil = \left\lceil \frac{x+y}{2} \right\rceil - \frac{1}{2}\sum_{i=1}^{k-1} h_i < \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil - 1 - \sum_{i=1}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor ,$$

hence (3.3) does not hold and T is not a 1-cap tree. It follows that there is an odd number of odd overlaps. Assume therefore that $h_j = t$ for some j, where t is odd. Then $\sum_{i=1, i\neq j}^{k-1} h_i$ is even, hence

$$\left\lceil \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{k-1} h_i \right\rceil = \left\lceil \frac{x+y-t}{2} \right\rceil - \frac{1}{2}\sum_{i=1, i\neq j}^{k-1} h_i \leq \left\lceil \frac{x+2}{3} \right\rceil + \left\lceil \frac{y+2}{3} \right\rceil - 1 - \left\lfloor \frac{t}{3} \right\rfloor - \sum_{i=1, i\neq j}^{k-1} \left\lfloor \frac{h_i}{3} \right\rfloor ,$$

and equality holds if and only if $h_i = 0$ for all $i \neq j$ and $\lceil \frac{x+y-t}{2} \rceil = \lceil \frac{x+2}{3} \rceil + \lceil \frac{y+2}{3} \rceil - 1 - \lfloor \frac{t}{3} \rfloor$. But the latter equality holds if and only if $x = y = t = 1$ or $x = y = 1$ and $t = 3$ (see Table 3.3). Therefore (b) holds. ■

x	y	t	$\lceil \frac{x+y-t}{2} \rceil$	$\lceil \frac{x+2}{3} \rceil + \lceil \frac{y+2}{3} \rceil - 1 - \lfloor \frac{t}{3} \rfloor$
1	0	1	0	1
1	**1**	**1**	**1**	**1**
1	2	1	1	2
1	0	3	-1	0
1	**1**	**3**	**0**	**0**
1	2	3	0	1
1	0	5	-2	0
1	1	5	-1	0
1	2	5	-1	1

Table 3.3: Possibilities for x, y, and t in Theorem 3.5.

3.4 Tainted trees and stained minimum dominating sets

A shadow tree that is not clear is said to be *tainted* and its γ-sets are said to be *stained*. Among all γ-sets of a tainted shadow tree T, if D is one that contains the minimum number of branch vertices, then D is a *minimally stained γ-set* of T.

In this section we show that the only tainted 1-cap shadow trees whose only free edges are leading and trailing free edges are the trees $T(\overline{b}, \overline{h})$ mentioned in Theorems 3.1(a), 3.2 (a) and 3.3, and their associated shadow trees $T(\overline{b'}, \overline{h})$ as explained in Corollary 1.8. We need a few lemmas.

Lemma 3.6. *Let T be a tainted shadow tree with a minimally stained γ-set D and branch vertices $v_{c_1}, ..., v_{c_k}$, $k \geq 1$. Suppose $v_{c_\alpha} \in D$. Define the vertex z to the right of v_{c_α} as follows.*

- *If $\alpha \neq k$ and $v_{c_{\alpha+1}} \in D$, let $z = v_{c_{\alpha+1}}$; if $v_{c_{\alpha+1}} \notin D$ let $z = u_{\alpha+1,1}$. (See* Figure 3.7*).*
- *If $\alpha = k$, let $z = v_d$.*

Define the vertex z' to the left of v_i similarly. Let Q be the $z' - z$ subpath of T. Then $d(v_{c_\alpha}, q) \equiv 0 \pmod 3$ for each vertex $q \in V(Q) \cap D$.

Proof. Neither $v_{c_\alpha - 1}$ nor $v_{c_\alpha+1}$ is a branch vertex: if both were branch vertices, then $D - \{v_{c_\alpha}\}$ would be a dominating set of T, which is not the case, and if (say)

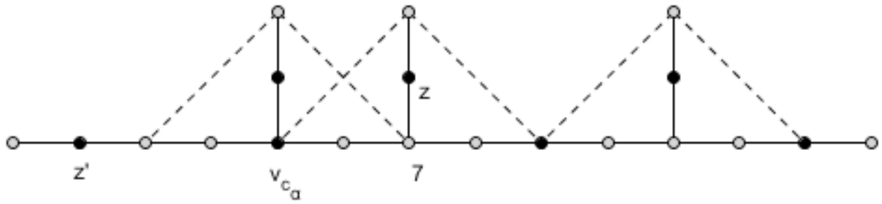


Figure 3.7: Here $v_{c_{\alpha+1}} \notin D$ so $z = u_{\alpha+1,1}$.

$v_{c_\alpha - 1}$ were a branch vertex but not $v_{c_\alpha + 1}$, then $(D - \{v_{c_\alpha}\}) \cup \{v_{c_\alpha+1}\}$ would be a γ-set containing fewer branch vertices than D, contrary to the choice of D.

Now suppose $d(v_{c_\alpha}, q') \not\equiv 0 \pmod 3$ for some vertex $q' \in V(Q) \cap D$ to the right of v_{c_α}. Let q be the first vertex on Q to the right of v_{c_α} such that $q \in D$ and $d(v_{c_\alpha}, q) \not\equiv 0 \pmod 3$. Let $v_{r_1}, ..., v_{r_j}$ be the vertices in $V(Q) \cap D$ that lie strictly between v_{c_α} and q. Then $D' = (D - \{v_{c_\alpha}, v_{r_1}, ..., v_{r_j}\}) \cup \{v_{c_\alpha - 1}, v_{r_1 - 1}, ..., v_{r_j - 1}\}$ is a γ-set of T containing fewer branch vertices than D, a contradiction. ■

Corollary 3.7. *Let T be a tainted shadow tree with a minimally stained γ-set D. If $v_{c_1} \in D$ ($v_{c_k} \in D$, respectively), then $d(v_{c_1}, v_0) \equiv 1 \pmod 3$ ($d(v_{c_k}, v_d) \equiv 1 \pmod 3$, respectively).*

Proof. Suppose $v_{c_1} \in D$ and let w be the first vertex of P in D. By Lemma 3.6, $d(v_{c_1}, w) \equiv 0 \pmod 3$. Since w dominates v_0, $w \in \{v_0, v_1\}$. However, if $w = v_0$, then $D' = (D - \{w\}) \cup \{v_1\}$ is a γ-set of T that does not satisfy Lemma 3.6. Hence $w = v_1$ and $d(v_0, v_{c_1}) \equiv 1 \pmod 3$. Similarly, $d(v_{c_k}, v_d) \equiv 1 \pmod 3$ if $v_{c_k} \in D$. ■

Let $i \in \{1, 2, ..., k\}$. If D is a natural γ-set of T, then $D \cap (V(B_i - v_{c_i})) = \{u_{i,j} : j \equiv 1 \pmod 3\}$. If $v_{c_i} \in D$ and

$$D' = (D - \{u_{i,j} : j \equiv 1 \pmod 3\}) \cup \{u_{i,j} : j \equiv 0 \pmod 3\} \cup \{u_{i,b_i}\},$$

then D' is a γ-set of T and we call D' the *i-conversion* of D. Similarly, for $i' \neq i$, if $\{v_{c_i}, v_{c_{i'}}\} \subseteq D$ and

$$D'' = (D' - \{u_{i',j} : j \equiv 1 \pmod 3\}) \cup \{u_{i',j} : j \equiv 0 \pmod 3\} \cup \{u_{i',b_{i'}}\},$$

then D'' is also a γ-set of T and we call D'' the *$\{i, i'\}$-conversion* of D. The main theorem of this section follows.

Theorem 3.8. *Let T be a tainted 1-cap shadow tree whose only free edges are x leading and y trailing free edges. Then T is one of the following trees:*

(i) *a spider $S(r, r+x, r+y)$, where $r \equiv x \equiv y \equiv 2 \pmod 3$,*

(ii) *a tree with exactly two branch vertices and overlap sequence $-x, 1, -y$ such that $x \equiv y \equiv 2 \pmod 3$,*

(iii) *a tree with k branch vertices, $k \geq 2$, and overlap sequence $-x, 0, 0, ..., 0, 2, -y$ such that $x \equiv 1 \pmod 3$ and $y \equiv 2 \pmod 3$, or its reverse.*

Proof. Suppose the statement of Theorem 3.8 is not true. Amongst all tainted 1-cap shadow trees without internal free edges that do not satisfy (i), (ii) or (iii), let T be a smallest one. By Corollary 1.8 we may assume that $x, y \in \{0, 1, 2\}$ and thus that T is radial. Let D be a minimally stained natural γ-set of T and let $v_{c_\alpha} \in D$. Define the vertices z and z' as in Lemma 3.6. If $z = v_d$ and $z' = v_0$, then T has exactly one branch vertex and it follows from Corollary 3.7 that (i) holds, so assume without loss of generality that $z \neq v_d$. We consider two cases, depending on the choice of z.

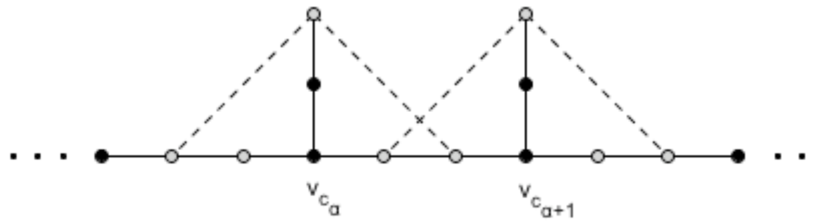
Case 1 $z = v_{c_{\alpha+1}}$. Then $z \in D$ and by Lemma 3.6 $d(v_{c_\alpha}, v_{c_{\alpha+1}}) \equiv 0 \pmod 3$. Define the vertex z'' for $v_{c_{\alpha+1}}$ similar to the vertex z for v_{c_α}.

Recall that the branches B_α and $B_{\alpha+1}$ have lengths b_α and $b_{\alpha+1}$. Also, $d(v_{c_\alpha}, v_{c_{\alpha+1}}) = b_\alpha + b_{\alpha+1} - h_{\alpha+1}$. Now $b_\alpha \equiv b_{\alpha+1} \equiv 2 \pmod 3$ and $d(v_{c_\alpha}, v_{c_{\alpha+1}}) \equiv 0 \pmod 3$, hence $h_{\alpha+1} \equiv 1 \pmod 3$. Let X be the $\{\alpha, \alpha+1\}$-conversion of D. Then $\{u_{\alpha,b_\alpha}, u_{\alpha+1,b_{\alpha+1}}\} \subseteq X$ and for $i \in \{\alpha, \alpha+1\}$, $\text{PN}(u_{i,b_i}, X) = \{u_{i,b_i}\}$. See Figure 3.8.

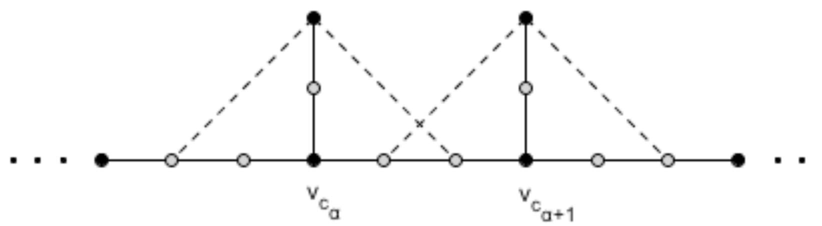
Let $T' = T - \{u_{\alpha,b_\alpha}, u_{\alpha+1,b_{\alpha+1}}\}$ and let Δ'_α and $\Delta'_{\alpha+1}$ be the triangles of T' corresponding to the triangles Δ_α and $\Delta_{\alpha+1}$ of T. Let $h'_{\alpha+1}$ be the overlap of Δ'_α and $\Delta'_{\alpha+1}$. Since T has no internal free edges and $h_{\alpha+1} \geq 1$, $h'_{\alpha+1} \geq -1$. If $\Delta_{\alpha-1}$ exists, let h'_α be the overlap of $\Delta_{\alpha-1}$ and Δ'_α, otherwise let $|h'_\alpha|$ be the number of leading free edges of T'. Similarly, if $\Delta_{\alpha+2}$ exists, let $h'_{\alpha+2}$ be the overlap of $\Delta'_{\alpha+1}$ and $\Delta_{\alpha+2}$, otherwise let $|h'_{\alpha+2}|$ be the number of trailing free edges of T'.

Since $\text{PN}(u_{i,b_i}, X) = \{u_{i,b_i}\}$ for $i \in \{\alpha, \alpha+1\}$, $\gamma(T') \leq \gamma(T) - 2$ and therefore

$$\gamma_b(T') \leq \gamma(T') \leq \gamma(T) - 2 = \gamma_b(T) - 2 = \text{rad}(T) - 2 = \text{rad}(T') - 2. \quad (3.4)$$



(a) Here $\{v_{c_\alpha}, v_{c_{\alpha+1}}\} \subseteq D$.



(b) The $\{\alpha, \alpha+1\}$-conversion of D.

Figure 3.8: Case 1 in the proof of Theorem 3.8

Let m be the cardinality of a maximum split-set of T'. By Corollary 1.4, $\gamma_b(T') = \text{rad}(T') - \lceil \frac{m}{2} \rceil$, hence by (3.4), $m \geq 3$. Since $h'_{\alpha+1} \geq -1$, the only possible free edges of T' are x leading free edges, y trailing free edges, possibly an edge to the left of Δ'_α, possibly an edge to the right of $\Delta'_{\alpha+1}$, and possibly an edge between Δ'_α and $\Delta'_{\alpha+1}$. Since none of the x leading or y trailing free edges of T is a split-edge, we deduce that $m = 3$, $h'_{\alpha+1} = -1$ and

$$h'_\alpha = \begin{cases} -1 & \text{if } \Delta_{\alpha-1} \text{ exists} \\ -x-1 & \text{otherwise} \end{cases}$$
$$h'_{\alpha+2} = \begin{cases} -1 & \text{if } \Delta_{\alpha+2} \text{ exists} \\ -y-1 & \text{otherwise} \end{cases}.$$

Suppose $\Delta_{\alpha+2}$ exists. Then $h'_{\alpha+2} = -1$ and therefore $h_{\alpha+2} = 0$. This in turn implies that $d(v_{c_{\alpha+1}}, v_{c_{\alpha+2}}) \equiv 1 \pmod 3$ and $d(v_{c_{\alpha+1}}, u_{\alpha+2,1}) \equiv 2 \pmod 3$. Now if $v_{c_{\alpha+2}} \in D$, then $z'' = v_{c_{\alpha+2}}$, otherwise $z'' = u_{\alpha+2,1} \in D$ (since D is a natural γ-set). But by Lemma 3.6, $d(v_{c_{\alpha+1}}, z'') \equiv 0 \pmod 3$, a contradiction. We deduce that $\Delta_{\alpha+2}$ does not exist. Therefore $\alpha + 1 = k$. By Corollary 3.7, $d(v_{c_{\alpha+1}}, v_d) \equiv 1 \pmod 3$, that is, $y \equiv 2 \pmod 3$ and so $y = 2$. Similarly, $\alpha = 1$ (hence $\alpha + 1 = k = 2$) and $x = 2$. Finally, $h_{\alpha+1} = h'_{\alpha+1} + 2 = 1$. Therefore (ii) holds, contrary to the choice of T.

By symmetry, (ii) holds if $z' = v_{c_{\alpha-1}}$. We therefore assume henceforth that $z' \neq v_{c_{\alpha-1}}$.

Case 2 $z = u_{\alpha+1,1}$. Then $z \in D$ and by Lemma 3.6, $d(v_{c_\alpha}, z) \equiv 0 \pmod 3$ and $d(v_{c_\alpha}, v_{c_{\alpha+1}}) \equiv 2 \pmod 3$. Therefore $h_{\alpha+1} \equiv 2 \pmod 3$. If $z' = u_{\alpha-1,i}$, then similarly $d(v_{c_{\alpha-1}}, v_{c_\alpha}) \equiv 2 \pmod 3$ and $h_{\alpha-1} \equiv 2 \pmod 3$. Then $T' = T - \{u_{\alpha,b_\alpha}\}$ has no internal free edges, hence is radial, so that $\gamma_b(T') = \gamma_b(T)$. But if X is the α-conversion of D, then $X - \{u_{\alpha,b_\alpha}\}$ is a dominating set of T'. This means that

$$\gamma_b(T') \leq \gamma(T') < \gamma(T) = \gamma_b(T) = \gamma_b(T'),$$

which is impossible. Therefore $z' = v_0$ and $\alpha = 1$. By Corollary 3.7, $d(v_0, v_{c_1}) \equiv 1 \pmod 3$; hence $x \equiv 2 \pmod 3$ and so $x = 2$.

Suppose $h_2 \geq 5$. Since T has no nested triangles it follows that the branch B_1 at v_{c_1} has length at least 5. Let $T'' = T - \{u_{1,b_1}, u_{1,b_1-1}, u_{1,b_1-2}, u_{1,b_1-3}\}$. Then T'' has exactly $x + 4 = 6$ leading free edges, $y \in \{0, 1, 2\}$ trailing free edges, and no internal free edges since $h_2 \geq 5$.

If, as above, X is the 1-conversion of D, then $\{u_{1,b_1}, u_{1,b_1-2}\} \subseteq X$ and $X - \{u_{1,b_1}, u_{1,b_1-2}\}$ dominates T''. Therefore $\gamma(T'') = \gamma(T) - 2$. Now $\gamma_b(T'') \leq \gamma_b(T'') \leq \gamma(T'') = \gamma(T) - 2 = \text{rad}(T) - 2 = \text{rad}(T'') - 2$. Let m be the cardinality of a maximum split-set M of T''. By Corollary 1.4, $\gamma_b(T'') = \text{rad}(T'') - \left\lceil \frac{m}{2} \right\rceil$, hence $m \geq 3$. But it is impossible to find a split-set of cardinality three amongst six consecutive free edges (see Figure 3.9), and none of the trailing free edges is a split-edge. Hence $h_2 = 2$.

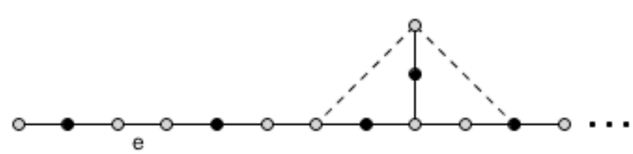


Figure 3.9: There is no way to get a split-set with three edges from six consecutive edges. The first possible edge that could be in the split-set is e, but then the two edges following e could not be in the set.

Let H be the subtree of T obtained by deleting all the vertices of B_1 except v_{c_1}. Then $D - \{u_{1,1}, u_{1,4}, ..., u_{1,b_1-1}\}$ dominates H and $\gamma(H) = \gamma(T) - m_1 - 1$.

Since $\gamma_b(H) \leq \gamma(H)$, H has a maximum split-set M of cardinality $m \geq 1$. By Corollary 1.4, $\gamma_b(H) = \text{rad}(H) - \lceil \frac{m}{2} \rceil = \text{rad}(T) - \lceil \frac{m}{2} \rceil$. Therefore $m \geq 2m_1 + 1$. The rightmost vertex of Δ_1 is v_{2b_1+2} and, since $h_2 = 2$, the leftmost vertex of Δ_2 is v_{2b_1}. The leading free edges of H are the edges on the path $R = v_0, v_1, ..., v_{2b_1}$, and H has no internal free edges. Hence M consists of edges of R. Note that $2b_1 = 6m_1 + 4$. Since each component of $H - M$ has even positive diameter, the set $M = \{v_2v_3, v_5v_6, ..., v_{6m_1+2}v_{6m_1+3}\}$ of cardinality $m = 2m_1 + 1$ is the unique maximum split-set of H.

Let T_d be the component of $H - M$ that contains v_d; it has even diameter, exactly one leading free edge, and at least one branch vertex. By Theorem 2.2, T_d is a 1-cap tree. If T_d is a clear tree, it satisfies Theorem 3.5(a) or (b), and if it is a tainted tree, then, by the choice of T, it satisfies (i), (ii) or (iii). But since T_d has exactly one leading free edge it does not satisfy (i) or (ii). We examine the other possibilities.

- If T_d satisfies Theorem 3.5(a), then $y = 1$ since $\text{diam}(T_d)$ is even and T_d has one leading free edge. Hence T satisfies (the reverse of) (iii).
- If T_d satisfies Theorem 3.5(b), then T_d has an odd number of leading free edges, an odd number of trailing free edges, and one odd overlap, so that $\text{diam}(T_d)$ is odd, contrary to M being a split-set.
- If T_d satisfies (iii), then it has one leading free edge, two trailing free edges, and only even overlaps, so that $\text{diam}(T_d)$ is odd, again a contradiction.

This completes the proof of Theorem 3.8. ■

3.5 Summary

Define the classes $\mathcal{T}_1 - \mathcal{T}_6$ of shadow trees as follows. For $\bar{b} = (b_1, b_2, ..., b_k)$, where $b_i \equiv 2 \pmod 3$, $i = 1, 2, ..., k$, and $\overline{h} = (-x, h_2, ..., h_k, -y)$, let

$$\mathcal{T}_1 = \{T(\bar{b}, \overline{h}) : x \equiv 1 \pmod 3 \text{ and } h_i = 0 \text{ for } i = 2, ..., k\}$$

$$\mathcal{T}_2 = \{T(\bar{b}, \overline{h}) : x \equiv y \equiv 1 \pmod 3,\ h_i = 1 \text{ for exactly one } i, \text{ and } h_j = 0 \text{ if } j \neq i\}$$

$$\mathcal{T}_3 = \{T(\bar{b}, \overline{h}) : x \equiv y \equiv 1 \pmod 3,\ h_i = 3 \text{ for exactly one } i, \text{ and } h_j = 0 \text{ if } j \neq i\}$$

$$\mathcal{T}_4 = \{T(\bar{b}, \overline{h}) : k = 1 \text{ and } x \equiv y \equiv 2 \pmod 3\}$$

$$\mathcal{T}_5 = \{T(\bar{b}, \overline{h}) : k = 2,\ h_2 = 1 \text{ and } x \equiv y \equiv 2 \pmod 3\}$$

$$\mathcal{T}_6 = \{T(\bar{b}, \overline{h}) : x \equiv 1 \pmod 3,\ y \equiv 2 \pmod 3,\ h_i = 0 \text{ for } i = 2, ..., k-1, \text{ and } h_k = 2\}.$$

Note that the definitions of $\mathcal{T}_6$ and some instances of $\mathcal{T}_1$ (the cases $y \equiv 0$ or $2 \pmod 3$) are not symmetrical with respect to x and y; however, we also consider a tree to be in one of these classes if we can reverse its diametrical path P to fit the criteria. We summarize the results of Chapter 3 in the following theorem.

Theorem 3.9. *Let T be a shadow tree without internal free edges whose branches all have length congruent to $2 \pmod 3$. Then T is a 1-cap tree if and only if $T \in \bigcup_{i=1}^{6} \mathcal{T}_i$.*

Chapter 4

Manipulating 1-cap Trees

In this chapter we discuss joining 1-cap shadow trees to form new 1-cap shadow trees with internal free edges. We then determine 1-cap supertrees of shadow trees with the same broadcast and domination numbers.

4.1 The Classes $\mathcal{T}_1 - \mathcal{T}_6$

We begin by examining the classes $\mathcal{T}_1 - \mathcal{T}_6$ defined in Section 3.5 in more detail. If $T \in \mathcal{T}_i$ for some i, where T has x leading and y trailing free edges, we also write, cryptically, $T = xT_iy$. Of course, if $xT_iy \in T_i$, then $yT_ix \in T_i$; the only difference is that in our representation yT_ix is a tree of the form xT_iy with its diametrical path reversed. A tree of the form $2T_11$ is shown in Figure 4.1. Note that xT_iy denotes a member of an infinite class of trees, not just a single specific tree, and when we write $T = xT_iy$ we mean that T is any shadow tree $T(\overline{b}, \overline{h})$ where $\overline{b}$ and $\overline{h}$ satisfy the definition of T_i.

We further partition $\mathcal{T}_1$ into the three subclasses $\mathcal{T}_{1,i}$, $i = 0, 1, 2$, where

$$\mathcal{T}_{1,i} = \{xT_1y : x \equiv 1 \pmod 3 \text{ and } y \equiv i \pmod 3,\ i = 0, 1, 2\}.$$

Let $\mathcal{T} = \bigcup_{i=1}^{6} \mathcal{T}_i$. If $T, T' \in \mathcal{T}$, say $T = kT_i\ell$ with diametrical path $P = v_0, v_1, ..., v_d$, and $T' = k'T_j\ell'$ with diametrical path $P' = v'_0, v'_1, ..., v'_{d'}$, , we denote the tree obtained by joining v_d to v'_0 by $T + T' = kT_i\ell + k'T_j\ell'$, and say that $T + T'$ is the *sum* of T and T'. The tree T in Figure 4.1 can be written as $T = 2T_11 + 1T_11$ if we consider uv to be the joining edge, or as $T = 2T_42 + 0T_11$ if we consider vw to be the joining edge. If $T = T_1 + T_2 + \cdots + T_k$, where T_i is of the form xT_iy for each i, we also write

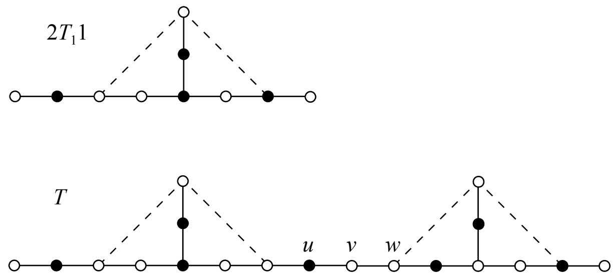


Figure 4.1: $T = 2T_1 1 + 1T_1 1 = 2T_4 2 + 0T_1 1$

$T = k(xT_i y)$.

It follows from Theorem 1.7 that the broadcast and domination numbers of trees in $\mathcal{T}$ with a large number of leading or trailing free edges can easily be determined in terms of the respective parameters of trees with fewer leading and trailing free edges.

Observation 4.1. *Suppose $T = kT_i\ell$, where $k = 3m + k'$ and $\ell = 3n + \ell'$, $k', \ell' \in \{0, 1, 2\}$ and let $T' = k'T_i\ell'$. Then $\gamma_b(T) = \gamma(T) = \gamma(T') + m + n$.*

For each $i = 1, 2, ..., 6$, we define the *core* of $\mathcal{T}_i$ by

$$\operatorname{Cor} \mathcal{T}_i = \{xT_iy \in \mathcal{T}_i : x, y \in \{0, 1, 2\}\}.$$

Then $\operatorname{Cor} \mathcal{T} = \bigcup_{i=1}^{6} \operatorname{Cor} \mathcal{T}_i$. Similarly, for $x'T_iy' \in \mathcal{T}_i$, let $\operatorname{Cor} x'T_iy' = xT_iy$, where $x \equiv x' \pmod 3$, $y \equiv y' \pmod 3$ and $x, y \in \{0, 1, 2\}$. The next observation follows in a similar way as Observation 4.1.

Observation 4.2. *For any $T, T' \in \mathcal{T}$, $T + T'$ is a 1-cap tree if and only if $\operatorname{Cor} T + \operatorname{Cor} T'$ is a 1-cap tree.*

Observation 4.3. *If T_1 and T_2 are subtrees of the tree T such that every vertex of T is contained in T_1 or T_2, then $\gamma_b(T) \leq \gamma_b(T_1) + \gamma_b(T_2)$.*

4.2 Joining 1-Cap Trees

It may seem intuitive that the sum of any two 1-cap trees is another 1-cap tree, but this is not always the case. For example, as illustrated in Figure 4.2, no tree of the form $1T_2 1 + 1T_2 1$ is a 1-cap tree. Note that $\operatorname{diam}(1T_2 1)$ is odd.

The tree T in Figure 4.3 is the sum of the 1-cap trees T' and T'', both of which have even diameter and are radial, but T is not 1-cap.

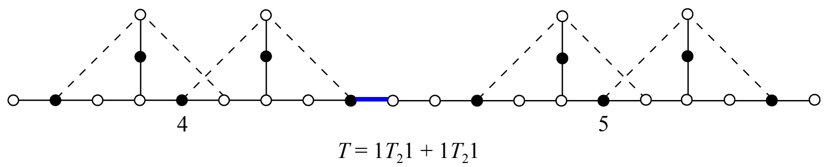


Figure 4.2: The tree $T = 1T_2 1 + 1T_2 1$ is not a 1-cap tree

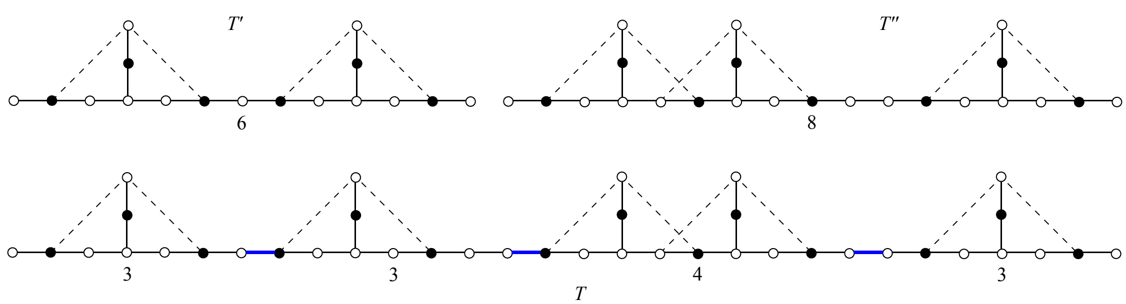


Figure 4.3: T' and T'' are 1-cap trees but T is not

Theorem 4.4. *Let $F_1, F_2, ..., F_k \in \mathcal{T}$ and $T = F_1 + F_2 + \cdots + F_k$. For $i = 1, ..., k$, let $F_i' = \operatorname{Cor} F_i$ and $T' = F_1' + F_2' + \cdots + F_k'$. Let e_i be the edge joining the diametrical path of F_i' to the diametrical path of F_{i+1}', $i = 1, ..., k-1$. If $\{e_1, e_2, ..., e_{k-1}\}$ contains a maximum split-set of T', then T is 1-cap.*

Proof. By Observation 4.2 we may assume that $F_i \in \operatorname{Cor} \mathcal{T}$ for each i, i.e., $F_i' = F_i$. Then each F_i is radial. Let M be a maximum split-set of T that is contained in $\{e_1, e_2, ..., e_{k-1}\}$. Let $H_1, H_2, ..., H_r$ be the components of $T - M$; each H_i is radial and has even diameter. Then $\gamma_b(T) = \sum_{i=1}^{r} \gamma_b(H_i)$. If $H_i = F_j$ for some i and j, then H_i is a 1-cap tree. Suppose there exists $H \in \{H_1, H_2, ..., H_r\}$ such that $H \neq F_i$ for any $i = 1, ..., k$. Since $M \subseteq \{e_1, e_2, ..., e_{k-1}\}$, there exist $s, t \in \{1, 2, ..., k\}$, $s < t$, such that $H = F_s + \cdots + F_t$. Then $\operatorname{diam}(H) = \sum_{i=s}^{t} \operatorname{diam}(F_i) + (t - s)$. Hence

$$\begin{aligned}\sum_{i=s}^{t} \operatorname{diam}(F_i) + (t-s) = \operatorname{diam}(H) = 2\gamma_b(H) &\leq 2\sum_{i=s}^{t} \gamma_b(F_i) \\ &= 2\sum_{i=s}^{t} \operatorname{rad}(F_i) = 2\sum_{i=s}^{t} \left\lceil \frac{\operatorname{diam}(F_i)}{2} \right\rceil. \end{aligned} \tag{4.1}$$

Now, there are $t-s+1$ terms in $\sum_{i=s}^{t} \left\lceil \frac{\operatorname{diam}(F_i)}{2} \right\rceil$. But $\operatorname{diam}(H)$ is even, hence $\operatorname{diam}(F_i)$ is even for at least one $i \in \{s, ..., t\}$. Therefore

$$2\sum_{i=s}^{t} \left\lceil \frac{\operatorname{diam}(F_i)}{2} \right\rceil \leq \sum_{i=s}^{t} \operatorname{diam}(F_i) + (t-s).$$

Thus equality holds throughout (4.1) and we have that $\gamma_b(H) = \sum_{i=s}^{t} \gamma_b(F_i)$, so that

$$\gamma_b(H) \leq \gamma(H) \leq \sum_{i=s}^{t} \gamma(F_i) = \sum_{i=s}^{t} \gamma_b(F_i) = \gamma_b(H),$$

that is, $\gamma(H) = \gamma_b(H)$ and H is a 1-cap tree. Therefore each H_i, $i = 1, ..., r$, is a 1-cap tree. The result now follows from Theorem 2.2. ■

Corollary 4.5. *If $F_1, F_2, ..., F_k \in \mathcal{T}_{1,1}$ then $F_1 + F_2 + \cdots + F_k$ is 1-cap. If $F_1, F_2 \in \mathcal{T}_{1,1} \cup \mathcal{T}_4$ then $F_1 + F_2$ is 1-cap.*

Proof. Suppose that $F_i \in \mathcal{T}_{1,1}$ for all $i \in \{1, 2, \cdots, k\}$. Then $\operatorname{diam}(\operatorname{Cor} F_i)$ is even for each i and if e_i is the edge joining the diametrical path of $\operatorname{Cor}(F_i)$ to the diametrical path of $\mathrm{Cor}(F_{i+1})$, $i = 1, ..., k-1$, then $\{e_1, e_2, ..., e_{k-1}\}$ is a maximum split-set of $F_1 + F_2 + \cdots F_k$, and the result follows from Theorem 4.4.

If $F_1, F_2 \in \mathcal{T}_{1,1} \cup \mathcal{T}_4$ then $\mathrm{diam}(\mathrm{Cor}\, F_i)$ is even for $i = 1, 2$ and e_1, the edge joining the diametrical path of $\mathrm{Cor}(F_1)$ to the diametrical path of $\mathrm{Cor}(F_2)$, is a maximum split-set. The result follows from Theorem 4.4. ■

Corollary 4.6. *If* $F_1, F_2, ..., F_k \in \mathcal{T}$ *and* $\mathrm{Cor}\, F_1 + \mathrm{Cor}\, F_2 + \cdots + \mathrm{Cor}\, F_k$ *is radial, then* $F_1 + F_2 + \cdots + F_k$ *is* 1*-cap.*

Proof. If $\mathrm{Cor}\, F_1 + \mathrm{Cor}\, F_2 + \cdots + \mathrm{Cor}\, F_k$ is radial, then its maximum split-set is empty. Thus the condition of Theorem 4.4 holds vacuously and $F_1 + F_2 + \cdots + F_k$ is a 1-cap tree. ■

We now determine exactly when the sum of two trees in $\mathcal{T}$ is a 1-cap tree. Again, the reader is reminded to investigate the website
`http://www.math.uvic.ca/~slunney/GraphGUI.html`.

Theorem 4.7. *If* $F_1, F_2 \in \mathcal{T}$*, then* $F_1 + F_2$ *is a* 1*-cap tree if and only if one of the following conditions (or its reverse) holds.*

(*i*) $F_i \in \mathcal{T}_{1,1} \cup \mathcal{T}_4$ *for* $i = 1, 2$.

(*ii*) $F_1 \in \mathcal{T}_{1,1}$ *and* $F_2 \in \mathcal{T} - (\mathcal{T}_{1,1} \cup \mathcal{T}_4)$.

(*iii*) $F_1 \in \mathcal{T}_4$ *and* $\mathrm{Cor}\, F_2 = 0T_11$.

(*iv*) $\mathrm{Cor}\, F_1 = 1T_10$ *or* $1T_12$ *while* $\mathrm{Cor}\, F_2 = 0T_11$ *or* $2T_11$.

(*v*) $\mathrm{Cor}\, F_1 = 1T_12$ *while* $\mathrm{Cor}\, F_2 = 1T_12$ *and has exactly one branch vertex (or the reverse).*

Proof. (*i*) If $F_1, F_2 \in \mathcal{T}_{1,1} \cup \mathcal{T}_4$, then $\mathrm{diam}(\mathrm{Cor}\, F_i)$ is even for $i = 1, 2$ and the result follows from Corollary 4.5.
(*ii*) Assume that $F_1 \in \mathcal{T}_{1,1}$ and $F_2 \in \mathcal{T} - (\mathcal{T}_{1,1} \cup \mathcal{T}_4)$. Then $\mathrm{diam}(\mathrm{Cor}\, F_1)$ is even and $\mathrm{diam}(\mathrm{Cor}\, F_2)$ is odd, so that $\mathrm{diam}(\mathrm{Cor}\, F_1 + \mathrm{Cor}\, F_2)$ is even. If M is a nonempty split-set of $\mathrm{Cor}\, F_1 + \mathrm{Cor}\, F_2$, then $|M| \geq 2$. But no leading or trailing free edge is a split-edge (there are at most two leading and at most two trailing free edges). Further, $\mathrm{Cor}\, F_1$ has only one trailing free edge while $\mathrm{Cor}\, F_2$ either has at most one leading free edge, or two leading free edges and the last free edge (the one farthest from F_1) is

followed by a part of F_2 with odd diameter. In the former case $\operatorname{Cor} F_1 + \operatorname{Cor} F_2$ has only three internal free edges and thus no split-set consisting of internal free edges. In the latter case $\operatorname{Cor} F_1 + \operatorname{Cor} F_2$ has four internal free edges, but the fourth one is not a split-edge. In either case $M = \varnothing$ and the result follows from Corollary 4.6.
(iii) Assume that $F_1 \in \mathcal{T}_4$ and $F_2 \in \mathcal{T} - (\mathcal{T}_{1,1} \cup \mathcal{T}_4)$. Then $\operatorname{diam}(\operatorname{Cor} F_1)$ is even and $\operatorname{diam}(\operatorname{Cor} F_2)$ is odd. If $\operatorname{Cor} F_2$ has two leading free edges, then $\operatorname{Cor} F_1 + \operatorname{Cor} F_2$ has five internal free edges, thus has a split-set of cardinality two. If $\operatorname{Cor} F_2$ has one leading free edge, then $\operatorname{Cor} F_1 + \operatorname{Cor} F_2$ has four internal free edges. Since $\operatorname{Cor} F_2$ has odd diameter, its leading free edge is followed by a part of $\operatorname{Cor} F_2$ of even diameter, hence $\operatorname{Cor} F_1 + \operatorname{Cor} F_2$ has a split-set of cardinality two. In either case it is easy to verify that $\gamma_b(F_1 + F_2) = \gamma_b(F_1) + \gamma_b(F_2) - 1$ while $\gamma(F_1 + F_2) = \gamma(F_1) + \gamma(F_2) = \gamma_b(F_1) + \gamma_b(F_2)$. Hence $F_1 + F_2$ is not a 1-cap tree. The only remaining case is when F_2 is of the form $0T_11$, in which case $\operatorname{Cor} F_1 + \operatorname{Cor} F_2$ is radial. By Corollary 4.6, $F_1 + F_2$ is a 1-cap tree.

Parts *(i)* – *(iii)* of the statement of Theorem 4.4 deal with the cases where at least one of $\operatorname{diam}(\operatorname{Cor} F_1)$ and $\operatorname{diam}(\operatorname{Cor} F_2)$ is even. Hence only the case where $\operatorname{diam}(\operatorname{Cor} F_i)$ is odd for $i = 1, 2$ remains.
(iv) and *(v)* By Observation 4.2 we may assume that $F_i \in \operatorname{Cor} \mathcal{T}$ for each i, that is, $F_i = \operatorname{Cor} F_i$. If $\operatorname{diam}(F_i)$ is odd for $i = 1, 2$, then $\operatorname{diam}(F_1 + F_2)$ is odd. Hence if $F_1 + F_2$ is not radial then it has a maximum split-set of odd cardinality. Since $\operatorname{diam}(F_1)$ and $\operatorname{diam}(F_2)$ are odd, F_1 and F_2 are one of the following types of trees: $1T_10$, $1T_12$, $1T_21$, $1T_31$, $2T_52$, $1T_62$ or their reverses. If e is the edge joining F_1 to F_2, then $\{e\}$ is not a split-set of $F_1 + F_2$. If $F_1 + F_2$ is radial, then it is a 1-cap tree by Corollary 4.6.

- Suppose $F_1 = 1T_10$. If F_2 has one leading free edge f, then f is followed by a part of F_2 of even diameter, hence $\{f\}$ is a split-set. If F_2 has 0 leading free edges, then $F_1 + F_2$ has no internal split-edge and hence is radial. Note that $1T_10 + 2T_11 = 1T_11 + 1T_11$, which is a 1-cap tree. It can be verified that $1T_10 + F_2$ is a 1-cap tree if and only if F_2 is of the form $0T_11$ or $2T_11$, as shown in Table 4.1.

- Suppose $F_1 = 0T_11$. Then the trailing free edge of F_1 is always a split-edge, and it can be shown that no 1-cap tree is formed.

- Suppose $F_1 = 1T_12$. If $F_2 = 0T_11$, then $F_1 + F_2 = 1T_11 + 1T_11$, which is a 1-cap tree. If $F_2 = 2T_11$, then $F_1 + F_2 = 1T_11 + 3T_11$, which is a 1-cap tree because

+	$1T_10$	$0T_11$	$1T_12$	$2T_11$	$1T_21$	$1T_31$	$2T_52$	$1T_62$	$2T_61$
$1T_10$	no	**yes**	no	**yes**	no	no	no	no	no
$0T_11$	no	no	no	no	no	no	no	no	no
$1T_12$	no	**yes**	$\star$	**yes**	no	no	no	no	no
$2T_11$	no	no	no	$\star$	no	no	no	no	no
$1T_21$	no	no	no	no	no	no	no	no	no
$1T_31$	no	no	no	no	no	no	no	no	no
$2T_52$	no	no	no	no	no	no	no	no	no
$1T_62$	no	no	no	no	no	no	no	no	no
$2T_61$	no	no	no	no	no	no	no	no	no

Table 4.1: Is the sum of two 1-cap trees whose cores have odd diameters also 1-cap?

$1T_11 + 0T_11$ is a 1-cap tree. Now suppose F_2 is of the form $1T_12$. If F_2 has at least two branches, then $F_1 + F_2$ is not a 1-cap tree. However, if F_2 has exactly one branch vertex, then $F_1+F_2 = 1T_12+2T_42$, which is 1-cap, and thus F_1+F_2 is 1-cap too. This is indicated by a $\star$ in Table 4.1.

- By symmetry, $2T_11+2T_11$ is a 1-cap tree if and only if the first tree has exactly one branch vertex.

It can be verified that these are the only 1-cap trees of this nature. ■

The tree T in Figure 4.3 can be written as the sum $1T_11 + 0T_11 + 1T_21 + 1T_11$, joined by the edges ab, uv and pq. However, $\{ab, vw, pq\}$ is the unique maximum split-set of T and $\{ab, vw, pq\} \not\subseteq \{ab, uv, pq\}$. We suspect that a slightly modified version of the converse of Theorem 4.4 is true:

Conjecture 4.1. *Let $F_1, F_2, ..., F_k \in \mathcal{T}$ and $T = F_1 + F_2 + \cdots + F_k$. For $i = 1, ..., k$, let $F_i' = \operatorname{Cor} F_i$. Define the trees H_i, $i = 1, 2, ..., k$, as follows.*

- *If there exists $i = 1, 2, ..., k-1$ such that $F_i' = 1T_10$ and $F_{i+1}' = 2T_11$, or $F_i' = 1T_12$ and $F_{i+1}' = 0T_11$, define $H_i = H_{i+1} = 1T_11$.*
- *If there exists $i = 1, 2, ..., k-1$ such that $F_i' = F_{i+1}' = 1T_12$ and F_{i+1}' has exactly one branch vertex, define $H_i = 1T_11$ and $H_{i+1} = 2T_42$.*
- *If there exists $i = 1, 2, ..., k-1$ such that $F_i' = F_{i+1}' = 2T_11$ and F_i' has exactly one branch vertex, define $H_i = 2T_42$ and $H_{i+1} = 1T_11$.*

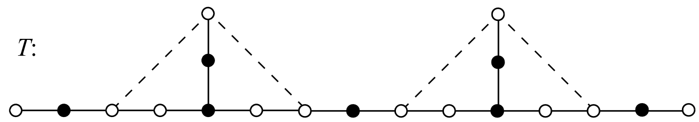


Figure 4.4: A 1-cap tree with internal free edges that is not the sum of trees in $\mathcal{T}$

- *Otherwise let* $H_i = F_i'$.

Let $T' = H_1 + H_2 + \cdots + H_k$ *and let* e_i *be the edge of* T' *joining the diametrical path of* H_i *to the diametrical path of* H_{i+1}*,* $i = 1, ..., k-1$. *Then* T *is* 1*-cap if and only if* $\{e_1, e_2, ..., e_{k-1}\}$ *contains a maximum split-set of* T'.

Conjecture 4.1, if true, does not give all 1-cap trees with internal free edges. The radial 1-cap tree T in Figure 4.4 cannot be written as the sum of two trees in $\mathcal{T}$. Note that $T = T(\overline{b}, \overline{h})$, where $\overline{b} = (2, 2)$ and $\overline{h} = (-2, -2, -2)$. The pattern does not generalize: $T(\overline{b}, \overline{h})$, where $\overline{b} = (2, 2, 2)$ and $\overline{h} = (-2, -2, -2, -2)$ is not 1-cap (and also not radial).

4.3 Resuscitating 1-Cap Trees from 1-Cap Shadow Trees

Cockayne et. al. [3] determined necessary and sufficient conditions for a subtree T of T' to have equal domination numbers. Let $W_1, ..., W_t$ be the nontrivial components of $T - E(T')$. For each $i \in \{1, ..., t\}$, let u_i be the unique vertex of $V(T') \cap V(W_i)$. The vertex u_i is called the *hinge* of W_i and we say that W_i *is hinged at* u_i. Let U_1 (respectively U_2) be the set of hinges of subtrees W_i that are stars hinged at a central vertex (respectively at a leaf that is not also a central vertex). Note that $U_1 \cap U_2 = \emptyset$.

Proposition 4.8. [3] *Let* T' *be a subtree of the tree* T. *Then* $\gamma(T) = \gamma(T')$ *if and only if*

(*i*) *each subtree* W_i *is either a star hinged at its centre or a star hinged at a leaf, and*

(*ii*) T' *has a* γ*-set* D *with* $U_1 \subseteq D$ *and* $U_2 \subseteq \{v \in D : \mathrm{PN}(v, D) = \{v\}\}$.

Suppose that S is a shadow tree and we want to determine which trees T have S as a shadow tree (i.e. $S = S_T$), and $\gamma(T) = \gamma(S)$ (we already know that $\gamma_b(T) = \gamma_b(S)$ from Theorem 1.5). Note that by convention leaves are not included in dominating sets.

If D is any γ-set for S and $v \in D$, then we can add any number of leaves to v, as v will dominate any adjacent leaves. Furthermore, if S has P_5 as a subtree where the first, third, and fifth vertex on P_5 (say w_1, w_3, and w_5) are in D, then we can add a vertex u to w_3 and any number of leaves to u. Now u will take the place of w_3 in D', the new dominating set of equal cardinality and the other two vertices on P_5 are already dominated by w_1 and w_5.

Chapter 5

Conclusion

5.1 Summary

In Chapter 3 we characterized 1-cap shadow trees with no internal free edges and with branch lengths congruent to 2 (mod 3). Then in Chapter 4 we discuss when 1-cap trees of this class can be joined or manipulated to create other 1-cap trees.

5.2 Future Work

We close by briefly mentioning a number of open problems on 1-cap trees. The first of these is Conjecture 4.1, part of which has been proved in Theorem 4.4.

Conjecture 4.1 *Let* $F_1, F_2, ..., F_k \in T$ *and* $T = F_1 + F_2 + \cdots + F_k$*. For* $i = 1, ..., k$*, let* $F'_i = \operatorname{Cor} F_i$*. Define the trees* H_i*,* $i = 1, 2, ..., k$*, as follows.*

- *If there exists* $i = 1, 2, ..., k-1$ *such that* $F'_i = 1T_10$ *and* $F'_{i+1} = 2T_11$*, or* $F'_i = 1T_12$ *and* $F'_{i+1} = 0T_11$*, define* $H_i = H_{i+1} = 1T_11$*.*

- *If there exists* $i = 1, 2, ..., k-1$ *such that* $F'_i = F'_{i+1} = 1T_12$ *and* F'_{i+1} *has exactly one branch vertex, define* $H_i = 1T_11$ *and* $H_{i+1} = 2T_42$*.*

- *If there exists* $i = 1, 2, ..., k-1$ *such that* $F'_i = F'_{i+1} = 2T_11$ *and* F'_i *has exactly one branch vertex, define* $H_i = 2T_42$ *and* $H_{i+1} = 1T_11$*.*

- *Otherwise let* $H_i = F_i$*.*

Let $T' = H_1 + H_2 + \cdots + H_k$ and let e_i be the edge of T' joining the diametrical path of H_i to the diametrical path of H_{i+1}, $i = 1, ..., k-1$. Then T is 1-cap if and only if $\{e_1, e_2, ..., e_{k-1}\}$ contains a maximum split-set of T'.

As also mentioned in Section 4.2, Conjecture 4.1, if true, does not give all 1-cap trees with internal free edges.

Question 5.1. *Is the radial* 1*-cap tree in* Figure 4.4 *the only* 1*-cap shadow tree with branches of length congruent to* $2 \pmod 3$ *that cannot be written as the sum of* 1*-cap trees in the class* $\mathcal{T}$*?*

It is unlikely that this is the case.

Problem 5.1. *Determine all* 1*-cap shadow trees with branches of length congruent to* $2 \pmod 3$ *that contain internal free edges.*

Since the class of 1-cap trees with branches of length congruent to $1 \pmod 3$ is completely characterized in [12], only the trees with branches of length congruent to $0 \pmod 3$ remain to be considered in the study of trees, all of whose branches have the same length (modulo 3).

Problem 5.2. *Determine all* 1*-cap trees with branches of length congruent to* $0 \pmod 3$.

Finally, the case where the shadow trees have branches of arbitrary length remains open.

Problem 5.3. *Characterize the class of all* 1*-cap trees.*

Bibliography

[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*. Fourth Edition, Chapman & Hall, Boca Raton, 2005.

[2] E.J. Cockayne, S. Goodman and S.T. Hedetniemi, A linear-time algorithm for the domination number of a tree, *Inform. Proc. Lett.*, **4** (1975), 41-44.

[3] E.J. Cockayne, S. Herke and C.M. Mynhardt, Broadcasts and Domination in Trees, *Discrete Math.* (2009), doi:10.1016/j.disc.2009.12.012.

[4] J. Dabney, A linear-time algorithm for broadcast domination in a tree, *Master's thesis, Clemson University*, 2007.

[5] J. Dabney, B.C. Dean and S.T. Hedetniemi, A linear-time algorithm for broadcast domination in a tree, *Networks*, **53(2)** (2009), 160-169.

[6] J. Dunbar, D. Erwin, T. Haynes, S.M. Hedetniemi and S.T. Hedetniemi, Broadcasts in graphs, *Discrete Applid Math.* **154** (2006), 59-75.

[7] J. Dunbar, S.M. Hedetniemi and S.T. Hedetniemi, Broadcasts in trees, Manuscript, 2003.

[8] D. Erwin, Dominating broadcasts in graphs, *Bulletin of the ICA* **42** (2004), 89-105.

[9] P. Heggernes and D. Lokshtanov, Optimal broadcast domination in polynomial time, *Discrete Math.* **36** (2006), 3267-3280.

[10] S. Herke, Dominating broadcasts in graphs, *Master's dissertation, University of Victoria*, 2009.

[11] S. Herke and C.M. Mynhardt, Radial Trees, *Discrete Math.* **309** (2009), 5950-5962.

[12] C.M. Mynhardt and J. Wodlinger, Broadcasts and Domination in Trees, (2010), submitted.

[13] C.M. Mynhardt and J. Wodlinger, Uniquely Radial Trees, (2010), submitted.

[14] S.M. Seager, Dominating broadcasts of caterpillars, *Ars Combin.* **88** (2008), 307-319.

Appendix A

Algorithms

Algorithm A.1 implements the procedure for finding a maximum split-set of a shadow tree outlined in [10]. Note that the free edges of T are required as part of the input. Algorithm A.2 uses Algorithm A.1 and Corollary 1.4 to compute the broadcast number for a shadow tree.

Algorithm A.1. *SplitSet*(T)
Input: Shadow tree T with diametrical path $P = v_0, v_1, ..., v_{diam}$ with determined free edges.
Output: A maximum split-set M for T where M is an array of integers corresponding to the indices of the first vertex incident with the edges in the split-set.

$M = boolean[diam]$
$a = -1$
$temp = 0$
$temp2 = 0$
$lastE = -1$
$disFromLast = 0$
$disFromEnd = 0$
if *no leading free edges* ***then***
  $temp = 0$
end if
for $i = temp \rightarrow numFreeEdges - 1$ ***do***
  if *a=-1* ***then***
    $temp2 = lastE$

```
      else
        temp2 = a
      end if
      disFromLast = freeEdges.vertex(i) − temp2
      if disFromLast > 1 AND disFromLast is ODD then
        disFromLast = diam − indexOfFreeEdge(i)
        if disFromEnd > 1 AND disFromLast is ODD then
          lastE = indexOfFreeEdge(i)
          if a > −1 then
            M.add(vertex(a))
          end if
          M.add(vertex(lastE))
          a = −1
        end if
      else
        a = freeEdges.vertex(i)
      end if
    end for
    return M
```

Algorithm A.2.
Input: Shadow tree S with diametrical path $P = v_0, v_1, ..., v_{diam}$
Output: $\gamma_b(S)$

$$\textbf{\textit{return}}\ \operatorname{rad}(T) - \left(\left\lceil \frac{|SplitSet(T)|}{2} \right\rceil\right)$$

Algorithm A.3 finds a dominating set (and therefore the domination number) for a shadow tree T. The algorithm works by dominating all branches so that every third vertex along the branch is in the dominating set, starting with the support vertex for the leaf of the branch (u_{i,b_i-1}). After the branches have been dominated the algorithm breaks the diametrical path into undominated paths and again picks every third vertex to be in the dominating set.

Algorithm A.3.
Input: Shadow tree T with diametrical path $P = v_0, v_1, ..., v_{diam}$ with $k \geq 1$ branches, overlap sequence h, and branch length sequence b; $h[i]$ is the i^{th} overlap and $b[i]$ is

the length of branch i.

Output: A γ-set D for T.

for $i = 0 \rightarrow k - 1$ ***do***
  for $j = 0 \rightarrow \left(\left\lceil \frac{b[i]}{3} \right\rceil - 1\right)$ ***do***
    $D.add(vertex(i, b[i] - 1 - 3j)$
  end for
end for
for $i = 1 \rightarrow -1 * h[0] + b[0]; i+ = 3$ ***do***
  if *!inD(i)* ***then***
    $D.add(vertex(i))$
  end if
end for
for $i = 0 \rightarrow k - 2$ ***do***
  $miniDiam = b[i+1] - b[i] - 1$
  $j = 0$
  if $inD(b[i])$ ***then***
    $j = 3$
  else if $inD(b[i] - 1)$ OR $inD(vertex(i, 1))$ ***then***
    $j = 2$
  else if $inD(b[i] - 2)$ ***then***
    $j = 1$
  end if
  while $j \leq (miniDiam + 1)$ ***do***
    if $!inD(b[i] + j$ ***then***
      $D.add(b[i] + j)$
    end if
    $j+ = 3$
  end while
end for
$j = 0$
if $inD(b[k-1])$ ***then***
  $j = 3$
else if $inD(b[k-1] - 1)$ OR $inD(vertex(k-1, 1))$ ***then***
  $j = 2$

else if $inD(b[k-1]-2)$ ***then***
  $j = 1$
end if
$miniDiam = -1 * h[k] - b[k-1] - 1$
while $j \leq (miniDiam + 1)$ ***do***
  if $!inD(b[k-1] + j$ ***then***
    $D.add(b[i] + j)$
  end if
  $j+ = 3$
end while
return D

Algorithm A.4 takes an arbitrary tree T with diametrical path $P = v_0, v_1, ..., v_{diam}$ and outputs the shadow tree $S_{T,P}$. It works by finding all branch vertices, which are just those vertices v_i along P with $\deg(v_i) \geq 3$. The branch lengths are then calculated by finding the eccentricity of v_i in the subtree T_{v_i}, where T_{v_i} is the component that v_i is in of the forest $T - (P - v_i)$. The overlaps h_i are calculated using the formula $h_i = b_i + b_{i+1} - (c_{i+1} - c_i)$, where c_i is the index of the i^{th} branch vertex along P. Note that Algorithms A.5 and A.6 are called by Algorithm A.4.

Algorithm A.4.
Input: A tree T with diametrical path $P = v_0, v_1, ..., v_{diam}$
Output: The triple(k, b, h) for the shadow tree $S = S_{T,P}$ for T with respect to P where S has k branches, branch length sequence b, and overlap sequence h.

$k = 0$
$br[] = new\ int[diam]$
for $i = 0 \rightarrow diam - 1$ ***do***
  if $degree(v_i) > 2$ ***then***
    $br[k++] = v_i$
  end if
end for
if $k = 0$ ***then***
  return $null$
end if
$b[] = new\ int[k]$

$h[] = new\ int[k+1]$
for $i = 1 \rightarrow k-1$ ***do***
$b[i] = brLen(br[i])$
end for
$h[0] = b[0] - br[0]$
for $i = 1 \rightarrow k-1$ ***do***
$h[i] = b[i-1] + b[i] - (br[i] - br[i-1])$
end for
$h[k] = br[k-1] + b[k-1] - diambr[0]$
return (k, b, h)

Algorithm A.5. *brLen(vertex v)*
Input: A vertex v_i on a diametrical path $P = v_0, v_1, ..., v_{diam}$ of a tree T where degree$(v_i) > 2$
Output: The length of the branch for which v_i is the branch vertex on the shadow tree $S_{T,P}$.

$max = 0$
$temp = 0$
for all *vertices* $w \in (N[v] \cap \overline{P})$ ***do***
$temp = ecc(w, v)$
if $temp > max$ ***then***
$max = temp$
end if
end for
return max

Algorithm A.6. *ecc(vertex u, vertex v)*
Input: Two adjacent vertices $u, v \in V(T)$ where T is a tree
Output: The length of the longest path (u, w) which does not go through v where w is any other vertex in $V(T)$.

$max = 0$
$temp = 0$
if *degree*(u)*=1* ***then***
return *1*

end if
for all *vertices* $x \in (N[u] \cap \overline{P})$ ***do***
  $temp = ecc(x, u)$
  if $x! = v$ ***then***
    $temp = ecc(x, u)$
    if $temp > max$ ***then***
      $max = temp + 1$
    end if
  end if
end for
return max