Introduction to AF Algebras and their Dimension Groups

by

Tao Cao
B.S., Wuhan University , 1986

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

ACCEPTED
CULTY OF GRADUATE STUDIES

DEAN

E

We accept this thesis as conforming
to the required standard

[redacted]

Dr. J. Phillips, Supervisor (Department of Mathematics and Statistics)

[redacted]

Dr. C. R. Miers, Department member (Department of Mathematics and Statistics)

[redacted]

Dr. W. W. Wadge, Outside Member (Department of Computer Science)

[redacted]

Dr. A. Lau, External Examiner (University of Alberta)

University of Victoria

Supervisor: Dr. J. Phillips

Abstract

AF algebras arised in the study of the algebraic classification of C-*algebras. The structure of an AF algebra can be described by a convenient notation called a Bratteli diagram. AF algebras are classified by certain discrete, ordered abelian groups which are just the inductive limits of inductive systems of groups of the form $\mathbb{Z}^r$. These groups are called dimension groups and they are just K_0 groups in K-theory.

Examiners:

[redacted]

Dr. J. Phillips, Supervisor (Department of Mathematics and Statistics)

[redacted]

Dr. C. R. Miers, Department member (Department of Mathematics and Statistics)

[redacted]

Dr. W. W. Wadge, Outside Member (Department of Computer Science)

[redacted]

Dr. A. Lau, External Examiner (University of Alberta)

Contents

Acknowledgements

At this point in the thesis, I would like to thank my supervisor, Dr. J. Phillips of the Department of Mathematics and Statistics, University of Victoria, for his continuous encouragement, infinite patience, and helpful advice during the past two years and for his kind assistance in the preparation of this manuscript. I would also like to thank Dr. C. R. Miers and Dr. I. Putnam of the Department of Mathematics and Statistics, University of Victoria, for numerous discussions and useful suggestions throughout the writing of this thesis. Finally, I would like to express my gratitude to my parents for their moral support and encouragement throughout my studies.

To My Parents

Chapter 1

Preliminaries and Notations

Here follows a list of basic results and properties of C*-algebra which will be used without reference throughout the thesis. Everything can be found in [22] or [1]. In addition to this we only need a few fundamentals about inductive limits, ordered groups and Grothendieck groups. In any case, throughout the thesis, we try to give the appropriate references for the particular results.

1. Conventions:

   We use the calligraphic symbols for sets of numbers $\mathcal{N} \subseteq \mathcal{Z} \subseteq \mathcal{Q} \subseteq \mathcal{R} \subseteq \mathcal{C}$ and let $\mathcal{N}^r \subseteq \mathcal{Z}^r \subseteq \cdots$ indicate r-tuples $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$ of such numbers. We let $(\mathcal{Z}^r)^+$ denote the set of $\alpha \in \mathcal{Z}^r$ with $\alpha_i \geq 0$ for all i, and we let $(\mathcal{Q}^r)^+, (\mathcal{R}^r)^+$ be the corresponding subsets of $\mathcal{Q}^r$ and $\mathcal{R}^r$.

2. Definition of a C*-algebra:

Definition 1.1 *A* **C*-algebra** *is a Banach algebra A with an involution $a \to a^*$ satisfying $\|a^*a\| = \|a\|^2$ for all $a \in A$.*

Let H be an infinite dimensional Hilbert space, then $B(H)$, the algebra of bounded operators on H, is a C*-algebra, and it can be shown that any C*-algebra is isomorphic to a norm-closed *-invariant subalgebra of $B(H)$ ([6] : 2.6.2). Denote by $\mathcal{K}(H)$ the algebra of compact operators on H. Then $\mathcal{K}(H)$ is also a C*-algebra. Usually we will only consider separable Hilbert spaces.

3. Unitizing:

For a C*-algebra A, we can construct a unique unital C*-algebra (i.e., it has a multiplicative identity), denoted A^1, such that A is an ideal in A^1 and A^1/A is isomorphic to $\mathcal{C}$. Elements of A^1 consist of pairs (a, λ) with $a \in A$ and $\lambda \in \mathcal{C}$. The multiplication on A^1 is defined by setting

$$(a, \lambda)(b, \gamma) = (ab + \lambda b + \gamma a, \lambda\gamma).$$

Then the unit in A^1 is the element $(0, 1)$. Letting $(a, \lambda)^* = (a^*, \overline{\lambda})$, we can make A^1 a *-algebra. Moreover we can represent A^1 as bounded operators on A by left multiplication:

$$\begin{aligned} \pi :\ & A^1 \longrightarrow B(A) \\ \text{via} :\ & \pi(a + \lambda)b = ab + \lambda b. \end{aligned}$$

$B(A)$ has a norm $\|\cdot\|_{\mathrm{op}}$ induced by the norm on A. We define a norm on A^1 by setting

$$\|a+\lambda\| = \|\pi(a+\lambda)\|_{\mathrm{op}}.$$

This is the unique norm on A^1 such that the embedding map of A into A^1 is isometric and A^1 is a C*-algebra([24]).

If A itself is unital, $A^1 \cong A \oplus C$ via the map $(a,\lambda) \to (a+\lambda 1_A) \oplus \lambda$. In any case $A^1/A \cong C$. The quotient map is denoted by ϕ_0, that is, $\phi_0 : A^1 \to C$ defined by $\phi_0((a,\lambda)) = \lambda$. By $\tilde{A}$ we refer to A when A is unital, and to A^1 when it is not.

4. Exact Sequences:

A sequence of C*-algebras and *-homomorphisms

$$F \xrightarrow{\phi} G \xrightarrow{\psi} H$$

is **exact** at G if $\mathrm{im}\phi = \ker\psi$, that is, if $\psi \circ \phi$ is the zero map (giving $\mathrm{im}\phi \subseteq \ker\psi$), and if every $g \in G$ such that $\psi(g) = 0$ has the form $g = \phi(f)$ for some $f \in F$ (giving $\mathrm{im}\phi \supseteq \ker\psi$).

In the exact sequence

$$0 \longrightarrow F \xrightarrow{\phi} G \xrightarrow{\psi} H \longrightarrow 0$$

(with zero morphisms at the ends), the map ϕ is injective (since $\ker\phi = 0$), and the map ψ is surjective (since $\mathrm{im}\psi = \ker 0 = H$). Such exact sequences are called **short exact** sequences.

The above short exact sequence is **split** exact if there exists a map $\lambda : H \to G$ such that $\psi \circ \lambda = \mathrm{id}$. And in this case, $G \cong F \oplus H$ as a vector space, but not generally as algebras.

In particular, there is a unique unital map $\mathcal{C} \xrightarrow{j} A^1$, defined by $j(\lambda) = (0, \lambda)$, that makes the sequence $0 \longrightarrow A \xrightarrow{i} A^1 \underset{j}{\overset{\phi_0}{\rightleftarrows}} \mathcal{C} \longrightarrow 0$ split exact.

5. Matrices with entries in C*-algebra:

   For any C*-algebra $A, M_n(A)$ will denote the algebra of $n \times n$ matrices over A.

   $$M_n(A) \cong A \otimes M_n(\mathcal{C})$$

   $$via: \quad [a_{ij}] \longrightarrow \textstyle\sum_{i,j} a_{ij} \otimes e_{ij},$$

   where e_{ij} is the matrix in which the entry in i, j position is 1 and all others are 0. $M_n(A)$ is a C*-algebra in an appropriate norm. Let (H, ϕ) be the universal representation of A, so the *-homomorphism $\phi : M_n(A) \longrightarrow M_n(B(H))$ is injective. The norm is defined by $\|a\| = \|\phi(a)\|$ for $a \in M_n(A)$, where $M_n(B(H))$ is identified with $B(H^n)$.

6. Here follows some notation and concepts which will be assumed throughout the thesis.

   For an algebra A, we say that $a \in A$ is **self-adjoint** if $a = a^*$, **idempotent** if $a = a^2$, and a **projection** if $a = a^* = a^2$. We say that $v \in A$ is a **partial isometry** if $v^*v = e$ and $vv^* = f$ are projections in A. If A is unital, we say that $u \in A$ is **unitary** if $u^*u = uu^* = 1_A$, and we let $U(A)$ denote the group of all unitaries in A.

Each $u \in U(A)$ defines an automorphism

$$\mathrm{Ad}\ u : A \longrightarrow A$$

by $\mathrm{Ad}\ u(a) = uau^*$. An automorphism ϕ of A is **inner** if $\phi = \mathrm{Ad}\ u$ for some $u \in U(A)$.

Chapter 2

Introduction

Classification in one of the major challenges facing C*-algebras, both in theory and applications. The problem of classifying C*-algebras which are inductive limits of sequences of finite dimensional C*-algebras has been considered for years. In [15], Glimm classified the C*-algebras (UHF algebras) which are inductive limits of:

$$M_{p_1} \xrightarrow{\phi_{12}} M_{p_2} \xrightarrow{\phi_{23}} \cdots,$$

where p_k divides p_{k+1} and $\phi_{k,k+1}$ is a unital morphism from M_{p_k} to $M_{p_{k+1}}$ defined by $x \to x \otimes I_p$ with $p = \frac{p_{k+1}}{p_k}$. Glimm showed that the isomorphism class depends only on the prime factors of the dimensions of the finite dimensional factors M_{p_k}'s.

In [7], Dixmier removed the assumption that the morphisms be unital. He classified the C*-algebras (matroid C*-algebras), which are the inductive limits of sequences:

$$M_{p_1} \xrightarrow{\psi_{12}} M_{p_2} \xrightarrow{\psi_{23}} \cdots,$$

where ψ's are arbitrary morphisms.

A C*-algebra A is said to be **approximately finite (AF)** if A is the norm-closure of the inductive limit of an ascending sequence of finite dimensional C*-algebras.

Gilmm and Dixmier gave the classifications for the special AF algebras when the finite dimensional C*-subalgebras are restricted to be simple. In [4], Bratteli gave a complete description of AF algebras. He showed that if a C*-algebra has a dense subalgebra which is the union of an increasing sequence of finite dimensional C*-subalgebras then this dense subalgebra is unique up to isomorphism as an involutive algebra, and determines the isomorphism type of the C*-algebra.

In [14], Elliott proved that the (stable) classification of AF algebras can be reduced via K-theory ([11]) to the classification of certain discrete, ordered abelian groups called dimension groups. In particular, it was shown that the ordered groups that arise are just those that can be written as inductive limits of direct systems of groups of the form $\mathbb{Z}^r$, where the latter is ordered by the simplicial positive cone

$$(\mathbb{Z}^r)^+ = \{(\alpha_, \ldots, \alpha_r) : \alpha_i \in \mathbb{Z}^+\}.$$

Using K-theory, it was shown that the dimension group is just the K_0 group of the algebra.

The purpose of this thesis is to present a survey of the work of Glimm, Dixmier, Bratteli, Elliott and Effros.

We give a brief outline of the thesis. In chapter 3, the description of inductive limits and some results for finite dimensional C*-algebras, as preparation for AF algebras, are discussed. Basic structures of AF algebras and the embedding maps (canonical homomorphisms) of subalgebras of given AF algebras are presented in detail. The major tool for analyzing AF algebras, the Bratteli diagram, is introduced, and the graphical representations which easily reveal the properties of given AF algebras are defined.

In chapter 4, we give an alternative characterization of AF algebras (Theorem 4.2), and a sufficient and necessary condition for isomorphism of two AF algebras (Theorem 4.9).

In chapter 5, we give an important observation. It is shown that AF algebras are classified by their dimension groups together with the ranges of their dimension functions (Theorem 5.11, Theorem 5.15).

In Chapter 6, we introduce C*-algebraic K_0 -theory. Using it, we point out that the dimension group is actually just the K_0 group of the algebra. Thus many results in the preceding chapters can be stated in term of K-theory.

To end this thesis, we discuss an example as a detailed application.

Chapter 3

Introduction to AF Algebras

3.1 Directed Systems and Inductive Limits

The material of this section is preparatory for the definition of an AF algebra. In the first part our objects are general sets. And then, since our interests are in groups and algebras, we will have to pay more attention to the binary operations of the set-theoretic results we have already obtained.

3.1.1 General Cases

Definition 3.1 *A relation* $m \leq n$ *in a set* I *is called a* **quasi-order** *if it is reflexive and transitive.*

Definition 3.2 *A* **directed set** I *is a quasi-ordered set such that for each pair* $l, m \in I$*, there exists* $n \in I$ *for which* $l \leq n$, $m \leq n$*. A directed set* I' *is a* **subset** *of* I $(I' \subset I)$ *if* $n \in I'$ *implies* $n \in I$ *and* $m \leq n$ *in* I' *implies* $m \leq n$ *in* I*.*

Definition 3.3 *A* **direct system of sets $\{A_n\}$ over a directed set I** *is a function which attaches to each $n \in I$ a set A_n, and, to each pair $m, n \in I$ with $m \leq n$, a morphism ϕ_{mn}: $A_m \to A_n$, such that, $\forall m \in I$,*

$$\phi_{mm} = \text{identity},$$

and for $l \leq m \leq n$ in I,

$$\phi_{ln} = \phi_{mn} \circ \phi_{lm}.$$

In particular, if the index set is just $\mathcal{N}$, we can also write the direct system $\{A_n\}_{n \in \mathcal{N}}$ as:

$$A_1 \xrightarrow{\phi_{12}} A_2 \xrightarrow{\phi_{23}} A_3 \xrightarrow{\phi_{34}} \cdots$$

and we have $\phi_{mn} : A_m \to A_n$ with

$$\phi_{mn} = \phi_{n-1,n} \circ \cdots \circ \phi_{m,m+1}$$

for $m \leq n$.

Remark 3.4 *Each set A determines a* **constant system** $\{A_n\}_{n \in I}$ *over an arbitrary directed set I, where $A_n = A$ and $\phi_{mn} = id$ for all $m, n \in I$.* **A map $\theta = \{\theta_n\}$ of direct systems** $\{A_n\}_{n \in I}$, $\{B_n\}_{n \in I}$, *which are over the same index set I, is a sequence of maps*

$$\theta_n : A_n \longrightarrow B_n$$

for which the diagrams

$$\begin{tikzcd}
A_m \arrow[r, "\theta_m"] \arrow[d] & B_m \arrow[d] \\
A_n \arrow[r, "\theta_n"] & B_n
\end{tikzcd}$$

are commutative for all $m, n \in I$ with $m \leq n$. We shall write

$$\theta : \{A_n\} \to \{B_n\}.$$

Definition 3.5 *An* **inductive limit** *(or* **direct limit***) of a direct system of sets $\{A_n\}$ is a pair (A_∞, ϕ), consisting of a set A_∞ and a map $\phi = \{\phi_{n\infty}\}$ from $\{A_n\}$ to the constant system $\{A_\infty\}$ such that for any map $\theta = \{\theta_n\}$ from $\{A_n\}$ to a constant system $\{H\}$ there exists a unique map λ such that the diagrams*

$$\begin{tikzcd}
A_n \arrow[rr, "\theta_n"] \arrow[dr, "\phi_{n\infty}"'] & & H \\
& A_\infty \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

commute for all n.

We shall also denote A_∞ by $\varinjlim A_n$, which makes sense by the following uniqueness result.

Remark 3.6 *Inductive limits exist and are essentially unique. An inductive limit may always be constructed by the following process.*

Given a direct system $\{A_n\}_{n \in I}$, we define an equivalence relation "$\equiv$" on

$$\bigcup_n (\{n\} \times A_n),$$

the "disjoint union" of $\{A_n\}$. We say (l,a), $(m,b) \in \bigcup_n(\{n\} \times A_n)$, **are equivalent**, denoted by $(l,a) \equiv (m,b)$, if and only if there exists $n \in I$ with $l \leq n$, $m \leq n$ and $\phi_{ln}(a) = \phi_{mn}(b)$. We define the equivalence classes for each (l,a),

$$[(l,a)] := \{(m,b) \in \bigcup_n(\{n\} \times A_n) : \ (m,b) \equiv (l,a)\}.$$

We shall use $(\bigcup_n(\{n\} \times A_n))/\!\equiv$ to denote $\{[(l,a)] : \ (l,a) \in \bigcup_n(\{n\} \times A_n)\}$, the set of the equivalence classes. Finally we define

$$\begin{aligned} \phi_{l\infty} : \quad A_l \ &\longrightarrow (\bigcup_n(\{n\} \times A_n))/\!\equiv \\ \text{by} : \quad a \ &\longrightarrow [(l,a)], \ a \in A_l. \end{aligned}$$

Then $(\bigcup_n(\{n\} \times A_n))/\!\equiv$ is the inductive limit of $\{A_n\}$.

To verify the universal property in Definition 3.5, we just need to define λ: $A_\infty \longrightarrow H$ by $\lambda([(n,a)]) = \theta_n(a)$. To see that λ is well-defined, assume $[(l,a)] = [(m,b)]$, then $(l,a) \equiv (m,b)$, there exists $n \in I$ with $\phi_{ln}(a) = \phi_{mn}(b)$. By the commutative diagrams

$$\begin{tikzcd} A_l \arrow[r, "\theta_l"] \arrow[d, "\phi_{ln}"'] & H \arrow[d, "\text{id}"'] \\ A_n \arrow[r, "\theta_n"] & H \end{tikzcd}$$

and

$$\begin{tikzcd} A_m \arrow[r, "\theta_m"] \arrow[d, "\phi_{mn}"'] & H \arrow[d, "\text{id}"'] \\ A_n \arrow[r, "\theta_n"] & H \end{tikzcd}$$

$\theta_l(a) = \theta_n \circ \phi_{ln}(a) = \theta_n \circ \phi_{mn}(b) = \theta_m(b)$. i.e., $\lambda([(l,a)]) = \lambda([(m,b)])$. Thus λ is well-defined.

Remark 3.7 *Given a map $\theta : \{A_n\} \to \{B_n\}$, then the composition*

$$\{A_n\} \xrightarrow{\theta} \{B_n\} \xrightarrow{\psi} \varinjlim B_n$$

determines a map

$$\lambda_\theta : \varinjlim A_n \longrightarrow \varinjlim B_n$$

such that the diagram

$$\begin{tikzcd}
\{A_n\} \arrow[r, "\theta"] \arrow[d] & \{B_n\} \arrow[d] \\
\varinjlim A_n \arrow[r, "\lambda_\theta"] & \varinjlim B_n
\end{tikzcd}$$

Thus inductive limit is a functor from direct systems to sets.

3.1.2 About Special Cases

In the special cases when A_n's are (abelian) groups, algebras or other particular objects instead of general sets, more details have to be considered when we define inductive limits of the system $\{A_n\}$.

Definition 3.8 *By a* **direct system of (abelian) groups** $\{G_n\}$ *we shall mean a sequence of (abelian) groups G_n together with homomorphisms*

$$\phi_{mn} : G_m \longrightarrow G_n, \qquad m \leq n$$

such that

$$\phi_{ln} = \phi_{mn} \circ \phi_{lm}, \qquad l \leq m \leq n$$

and $\forall m \in I$,

$$\phi_{mm} = \text{identity}.$$

The definition of a direct system of algebras $\{A_n\}$ is similar to the above.

A homomorphism of two direct systems of (abelian) groups (or, algebras) can be defined as in Remark 3.4.

Definition 3.9 *The* **inductive limit** *(or* **direct limit***) of a direct system of (abelian) groups* $\{G_n\}$ *is a pair* (G_∞, ϕ)*, consisting of the (abelian) group* G_∞ *and a homomorphism* $\phi = \{\phi_{n\infty}\}$*:* $\{G_n\} \to \{G_\infty\}$ *such that for any homomorphism* $\theta = (\theta_n)$ *from* $\{G_n\}$ *to a constant system of (abelian) groups* $\{H\}$ *there exists a unique homomorphism* λ *such that the diagrams*

$$\begin{tikzcd}
G_n \arrow[rr, "\theta_n"] \arrow[dr, "\phi_{n\infty}"'] & & H \\
& G_\infty \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

commute for all n.

We shall also denote G_∞ by $\varinjlim G_n$.

Given a direct system of (abelian) groups $\{G_n\}$. The inductive limit of $\{G_n\}$, $\varinjlim G_n$ or G_∞ (in set-theoretic sense, cf Defintion 3.5) has a natural (abelian) group structure. The binary operation on it is defined as the following way:

For $a, b \in G_\infty$, there exist $a_m \in G_m$ and $b_n \in G_n$, such that $\phi_{m\infty}(a_m) = a$, $\phi_{n\infty}(b_n) = b$. By directed property, for m, n, we have n' with $m \leq n'$, $n \leq n'$, Now, $\phi_{n'\infty}(\phi_{nn'}(b_n)) = b$. Changing notations we may assume $m \leq n$ (still using n to denote n', b_n to denote $\phi_{nn'}(b_n)$). Let $a'_m = \phi_{mn}(a_m) \in G_n$, and define

$$a + b := \phi_{n\infty}(a'_m + b_n).$$

Then G_∞ becomes a(n) (abelian) group.

The "+" defined above is well-defined. In fact, if we assume that there are other $a_k \in G_k$, $b_l \in G_l$ with $\phi_{k\infty}(a_k) = a$, $\phi_{l\infty}(b_l) = b$. By directed property, for m, n, k, l, we have k', l' such that $k \leq k'$, $m \leq k'$, $n \leq l'$, $k' \leq l'$, $l \leq l'$. Changing notations (still let k, l denote k', l' and still let a_k, b_l denote $\phi_{kk'}(a_k)$, $\phi_{ll'}(b_l)$) we may assume $m \leq k$, $n \leq l$ and $k \leq l$.

Since

$$\begin{tikzcd}
G_m \arrow[rr, "\phi_{mk}"] \arrow[dr, "\phi_{m\infty}"'] & & G_k \arrow[dl, "\phi_{k\infty}"] \\
& G_\infty &
\end{tikzcd}$$

is commutative and since $\phi_{m\infty}(a_m) = \phi_{k\infty}(a_k)$ we have (change k bigger enough if necessary)

$$a_k = \phi_{mk}(a_m).$$

Similarly we have $\phi_{nl}(b_n) = b_l$. So,

$$\begin{aligned}
\phi_{n\infty}(a'_m + b_n) &= \phi_{n\infty} \circ \phi_{mn}(a_m) + \phi_{n\infty}(b_n) \\
&= \phi_{m\infty}(a_m) + \phi_{l\infty} \circ \phi_{nl}(b_n) \\
&= \phi_{l\infty} \circ \phi_{kl} \circ \phi_{mk}(a_m) + \phi_{l\infty}(b_l) \\
&= \phi_{l\infty} \circ \phi_{kl}(a_k) + \phi_{l\infty}(b_l) \\
&= \phi_{l\infty}(a'_k + b_l)
\end{aligned}$$

where

$$a'_k = \phi_{kl}(a_k).$$

So, the binary operation "+" in G_∞ is well-defined.

For direct systems of algebras, the binary operations "addition" and "multiplication" in inductive limits can be defined as above. It follows that the inductive limits are algebras.

However, the existence of inductive limits for groups (and algebras) can be proved directly, in a different way. Given a direct system $\{G_n\}_{n\in I}$ with ϕ, let $\sum\limits_{n\in I}^{\oplus} G_n$ be the direct sum of $\{G_n\}_{n\in I}$, we may define a map $\mathcal{I}_n$ for each n:

$$\mathcal{I}_n : G_n \longrightarrow \sum_{n\in I}^{\oplus} G_n$$

$$\text{via: } (\mathcal{I}_n(a))_m = \left\{ \begin{array}{lll} a & \text{if} & m = n \\ 0 & \text{if} & m \neq n \end{array} \right\} \text{ for } a \in G_n. \tag{3.1}$$

Let Q be the subgroup of $\sum\limits_{n\in I}^{\oplus} G_n$ generated by all the following forms:

$$\mathcal{I}_n(\phi_{mn}(a)) - \mathcal{I}_m(a), \tag{3.2}$$

where $m \leq n$, and $a \in G_m$. Then the inductive limit, $\varinjlim G_n$ or G_∞, of $\{G_n\}$ is the factor group

$$(\sum_{n\in I}^{\oplus} G_n)/Q.$$

It remains to show that $(\sum\limits_{n\in I}^{\oplus} G_n)/Q$ satisfies the universal property. Given a homomorphism $\theta = \{\theta_n\}$: $\{G_n\} \longrightarrow \{H\}$, where H is a(n) (abelian) group, by the definition of the direct sum (Chapter 4, [16]), there exists a unique homomorphism γ such that

$$\begin{tikzcd}
G_n \arrow[rr, "\theta_n"] \arrow[dr, "\mathcal{I}_n"'] & & \mathrm{H} \\
& \sum\limits_{n\in I}^{\oplus} G_n \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

commute for all n.

Let $\pi : \sum\limits_{n\in I}^{\oplus} G_n \longrightarrow (\sum\limits_{n\in I}^{\oplus} G_n)/Q$ be the quotient map. Define $\phi_{n\infty} = \pi \circ \mathcal{I}_n$ and define

$$\lambda : \ (\sum_{n\in I}^{\oplus} G_n)/Q \longrightarrow H$$

$$\text{by}: \ \lambda(a+Q) = \gamma(a), \ a \in \sum_{n\in I}^{\oplus} G.$$

i.e., $\gamma = \pi \circ \lambda$. To show λ is well-defined, it is sufficient to show $Q \subseteq \ker \gamma$. Since each element in Q is a finite sum of the forms 3.2 or the inverses of the forms (such as $-\mathcal{I}_n(\phi_{mn}(a)) + \mathcal{I}_m(a))$, we just need to show $\gamma(\mathcal{I}_n(\phi_{mn}(a)) - \mathcal{I}_m(a)) = 0$ for any pair m, n with $m \leq n$ and $a \in G_m$. In fact,

$$\begin{aligned} & \gamma(\mathcal{I}_n(\phi_{mn}(a)) - \mathcal{I}_m(a)) \\ = \; & (\gamma \circ \mathcal{I}_n)(\phi_{mn}(a)) - (\gamma \circ \mathcal{I}_m)(a) \\ = \; & \theta_n(\phi_{mn}(a)) - \theta_m(a) \end{aligned}$$

which equals to 0 by the commutative diagram

$$\begin{tikzcd} G_m \arrow[r, "\theta_m"] \arrow[d, "\phi_{mn}"'] & H \arrow[d, "\mathrm{id}"'] \\ G_n \arrow[r, "\theta_n"] & H. \end{tikzcd}$$

Thus $\theta_n = \mathcal{I}_n \circ \gamma = \mathcal{I}_n \circ \pi\lambda = \phi_{n\infty} \circ \lambda$. Therefore the diagrams

$$\begin{tikzcd} G_n \arrow[rr, "\theta_n"] \arrow[dr, "\phi_{n\infty}"'] & & \mathrm{H} \\ & G_\infty \arrow[ur, "\lambda"'] & \end{tikzcd}$$

commute for all n.

From the structure of G_∞, we have a very useful criterion for inductive limits:

Proposition 3.10 *(G_∞, $\{\phi_{n\infty}\}$) is the inductive limit of direct system of (abelian) groups $\{G_n\}_{n \in I}$ if and only if:*

1. *For any $a \in G_\infty$, there exists $n \in I$, and $a_n \in G_n$, such that $\phi_{n\infty}(a_n) = a$.*

2. For $a \in G_m$ with $\phi_{m\infty}(a) = 0$, there exists $n \in G_n$ with $m \leq n$ such that $\phi_{mn}(a) = 0$.

Let $\{A_n\}$ be a directed system of algebras. If each A_n is C*-algebra and each ϕ_{mn} $(m \leq n)$ is *- homomorphism, A_∞ is a *-algebra. Noting the ϕ_{mn} are norm-decreasing, for $a \in A_\infty$, we set

$$\|a\| := \lim_{n\to\infty} \|\phi_{mn}(a_m)\|$$

where $\phi_{m\infty}(a_m) = a$. Then $\|\cdot\|$ is a norm on A_∞ by the Proposition 3.10. The Banach space completion of A_∞ with this norm becomes a C*-algebra by extending the addition, multiplication and involution to operations of the same type on the norm-closure of A_∞, which we still call inductive limit of the C*-algebras $\{A_n\}$ and denote by A_∞ or $\varinjlim A_n$ if no ambiguity can result.

3.2 AF Algebras

A C*-algebra A is called **approximately finite dimensional** or an AF algebra if there is a sequence of finite dimensional *-subalgebras $A_1 \subseteq A_2 \subseteq \cdots$ with $A = \overline{(\cup A_n)}$, the norm closure of $\cup A_n$.

If we define $\phi_{mn} : A_m \to A_n$ as the embedding map when $m \leq n$, then $\{A_n\}_{n\in\mathcal{N}}$ is a direct system and A is (isomorphic to) the (closure of the) inductive limit of this direct system. We call such a sequence $\{A_n\}$ an **approximating system.**

If A is a unital AF algebra with $A = \overline{(\cup A_n)}$, then $A_n + \mathcal{C}1$ is a finite dimensional *-subalgebra and $A_n \subseteq A_n + \mathcal{C}1 \subseteq A_{n+1} + \mathcal{C}1$. We may therefore assume that each A_n contains the unit 1_A of A. In this case, we say $\{A_n\}$ is a **unital approximating system.** In all that follows, that "A is an AF algebra with approximating system $\{A_n\}_{n\in\mathcal{N}}$" or "$A = \overline{\cup A_n}$", we shall mean that $\{A_n\}$ is an increasing sequence of finite dimensional C*-subalgebras of A such that $A = \overline{\cup A_n}$. If A is unital, we assume each A_n contains the identity of A unless otherwise stated.

We now give some examples of AF algebras.

Example 3.11 *([15]) A UHF algebra is an AF algebra. A C*-algebra A is called* **uniformly hyperfinite (UHF) of type** $\{p_n\}$ *if there is a sequence of factors* $\{F_n\}_{n\in\mathcal{N}}$ *in A with following properties:*

1. *F_n contains the unit of A and is of type I_{p_n}* [1]*,*
2. $F_{n-1} \subset F_n$,
3. $p_n \to \infty$ *as* $n \to \infty$,
4. $A = \overline{\cup F_n}$.

Let A_n be F_n, then A is an AF algebra.

Example 3.12 *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $\mathcal{K}$ be the set of compact operators on $\mathcal{H}$. Since there exists a countable orthonormal set*

$$\{\xi_1, \xi_2, \cdots\}$$

---

[1] A factor of type I_n, $n < \infty$, is nothing but the $n \times n$ matrix algebra M_n over C.

such that

$$\mathcal{H} = \overline{span\{\xi_1, \xi_2, \cdots\}}.$$

We can define P_n *to be the projection on the closed set*

$$\overline{span\{\xi_1, \xi_2, \cdots, \xi_n\}}.$$

Let $A_n = P_n \mathcal{K} P_n \cong M_n(\mathcal{C})$ *, then* $\mathcal{K} = \overline{(\cup A_n)}$*, so* $\mathcal{K}$ *is an AF algebra which is not unital.*

Before considering homomorphisms between any two AF algebras, we had better study homomorphisms of finite dimensional C*-algebras first.

3.3 Finite Dimensional C*-Algebras

In this section we shall study some basic propositions of finite dimensional C*-algebras which are very important to AF algebras. Most of the details in this section can be found in [28].

Given a (nonzero) C*-algebra A, let A_1 denote its closed unit ball, it is well known that A_1 admits an extreme point if and only if A is unital (Page 48, [28]). Now assume that A is a finite dimensional, fixed throughout the remainder of this section. Then A_1 is compact, therefore it has an extreme point by Krein-Milman Theorem. Hence A is always unital. Here we shall prove that every ideal I of A is of the form $I = Ae_I$ where e_I is the identity of I and e_I is a central projection of A. Moreover A can be decomposed into a direct sum and each summand is an ideal of A and is isomorphic to some matrix algebra.

Lemma 3.13 *Every ideal I of A is of the form $I = Ae$ for some central projections $e \in A$.*

Proof : Let e be the identity of I (then e is a projection). For any $x \in A$ we have $xe \in I, x^*e \in I$, hence

$$exe = e(xe) = xe, \text{and}, ex^*e = e(x^*e) = x^*e.$$

So

$$ex = (x^*e)^* = (ex^*e)^* = exe = xe.$$

Thus e commutes with every $x \in A$. ∎

The center $C(A)$ of A, is an abelian C*-subalgebra of A, it is finite dimensional since A is finite dimensional. Hence the spectrum[2] $\widehat{C}$ of $C(A)$ is a finite set, say $\{\omega_1, \omega_2, \ldots, \omega_r\}$. Thus $C(A)$ is isometrically *-isomorphic to the set of continuous functions $\mathbf{C}(\widehat{C})$ on $\widehat{C}$ via the Gelfand transform $\Gamma : C(A) \longrightarrow \mathbf{C}(\widehat{C})$ defined by $(\Gamma a)(\omega_j) = \omega_j(a)$ for $a \in C(A)$. For each k, $1 \leq k \leq r$, let $e^{(k)} \in \mathbf{C}(\widehat{C})$ be such that $e^{(k)}(\omega_j) = \delta_{kj}$. Then for $1 \leq j \leq r$,

$$(e^{(k)})^2(\omega_j) = e^{(k)}(\omega_j)e^{(k)}(\omega_j) = e^{(k)}(\omega_j) = \overline{e^{(k)}(\omega_j)},$$

---

[2] The spectrum of an abelian C*-algebra A is the set of all nonzero complex homomorphisms of A.

and

$$(\sum_{k=1}^{r} e^{(k)})(\omega_j) = \omega_j.$$

Then $e^{(k)}$'s (or $\Gamma^{-1}e^{(k)}$) are orthogonal projections which are in $C(A)$ by functional calculus and are called **eigenprojections**, and $\sum_{k=1}^{r} e^{(k)} = 1_A$. Hence we have the spectral formula for $C(A)$:

$$C(A) = \mathcal{C}e^{(1)} \oplus \cdots \oplus \mathcal{C}e^{(r)},$$

and A is decomposed into the direct sum

$$A = \sum_{k=1}^{r\oplus} {}^{\oplus} Ae^{(k)}.$$

Each $Ae^{(k)}$ is a factor, that is, $C(Ae^{(k)})$ consists of the scalar multiples of the identity $e^{(k)}$. In fact, if $ae^{(k)} \in C(Ae^{(k)})$, then $\forall b \in A$, $(ae^{(k)})(be^{(k)}) = (be^{(k)})(ae^{(k)})$. Since $e^{(k)} \in C(A)$, we have $(ae^{(k)})b = b(ae^{(k)})$. So, $ae^{(k)} \in C(A)$, Thus $ae^{(k)} \in C(A)e^{(k)}$ and $C(Ae^{(k)}) \subseteq C(A)e^{(k)} = \mathcal{C}e^{(k)}$. The other containing way is obvious. Hence by the lemma, $Ae^{(k)}$ is simple.

Theorem 3.14 *A can be decomposed into the direct sum*

$$A = \sum_{k=1}^{r} {}^{\oplus} F_k.$$

where each F_k is isomorphic to the algebra $M_{p_k}(\mathcal{C})$ of $p_k \times p_k$ matrices. The sequence of $\{p_1, \ldots, p_r\}$ of positive integers is uniquely determined by A up to permutation.

Proof : Without loss of generality we assume that A is simple. Let B be a maximal abelian self-adjoint subalgebra of A which exists by Zorn's lemma. The spectrum of B, $\widehat{B}$, is finite since B is finite dimensional. Let $\{e_i\}_{i=1}^n$ be the eigenprojections on $\widehat{B}$. Then $\{e_i\}$ are orthogonal and $\sum_{i=1}^n e_i = 1$, and

$$B \cong \mathcal{C}e_1 \oplus \mathcal{C}e_2 \oplus \cdots \oplus \mathcal{C}e_n. \tag{3.3}$$

It follows

$$(e_iAe_i)e_j = e_j(e_iAe_i) = \delta_{ij}(e_iAe_j),$$

and e_iAe_i, $1 \leq i \leq n$, commutes with every e_j, $1 \leq j \leq n$. So e_iAe_i commutes with B since $\{e_j\}$ generate B. Thus $e_iAe_i \subset B$ since B is maximal abelian. Thus, $e_iAe_i \subseteq e_iBe_i = \mathcal{C}e_i$.

For any fixed i, j, we shall show that there is a partial isometry e_{ij} from e_i to e_j. First we notice that $e_iAe_j \neq \{0\}$. Since if $e_iAe_j = \{0\}$, we have $\{0\} = A\{0\}A = A(e_iAe_j)A$. But $Ae_iA(\neq \{0\})$ is an ideal of A, so $Ae_iA = A$ since A is simple. Thus $\{0\} = A(e_iAe_j)A = A^2 = A$; that is a contradiction. Thus there exists $x_0 \in A$ with $e_ix_0e_j \neq 0$. Let $x = e_ix_0e_j$. We have then $x = e_ixe_j$ and

$$x^*x = e_jx^*xe_j = \lambda e_j \text{ and } xx^* = e_ixx^*e_i = \mu e_i$$

for some λ and $\mu > 0$. But

$$\lambda = \|x^*x\| = \|x\|^2 = \|xx^*\| = \mu,$$

so we have

$$x^*x = \lambda e_j \text{ and } xx^* = \lambda e_j.$$

Therefore we have $u = \lambda^{-1/2}x \in A$ with

$$u^*u = e_j \text{ and } uu^* = e_i,$$

Now for each $i, 1 \leq i \leq n$, let $u_i \in A$ such that $u_i^*u_i = e_1$ and $u_iu_i^* = e_i$. Put $e_{ij} = u_iu_j^*,\ 1 \leq i,j \leq n$. We then have

$$\begin{cases} e_{ij}^* = e_{ji}, \\ \sum_{i=1}^n e_{ii} = 1, \\ e_{ij}e_{kl} = \delta_{jk}e_{il}. \end{cases}$$

We claim that $e_iAe_j = Ce_{ij}$ for $i,j = 1,2,\ldots,n$. In fact, if $x \in e_iAe_j$, then $xe_{ji} \in (e_iAe_j)e_{ji} \subseteq e_iAe_i$, so that $xe_{ji} = \lambda e_i$ for some $\lambda \in C$; Hence we get

$$x = xe_j = xe_{ji}e_{ij} = \lambda e_ie_{ij} = \lambda e_{ij}.$$

For each $x \in A$, let $\lambda_{ij}(x)$ be the scalar such that $e_ixe_j = \lambda_{ij}(x)e_{ij}$. It follows then that

$$x = \sum_{i,j=1}^{n} e_ixe_j = \sum_{i,j=1}^{n} \lambda_{ij}(x)e_{ij}.$$

Define

$$h: \quad A \longrightarrow M_n(C),$$

$$\text{via:} \quad x \longrightarrow (\lambda_{ij}(x)).$$

Tedious computing shows that h is a *-homomorphism. Moreover, h is injective since if $\lambda_{ij}(x) = \lambda_{ij}(y)$ for all i, j, we have

$$\begin{aligned} x &= \sum e_i x e_j = \sum \lambda_{ij}(x) e_{ij} \\ &= \sum \lambda_{ij}(y) e_{ij} = \sum e_i y e_j \\ &= y. \end{aligned}$$

Given $[\lambda_{ij}] \in M_n(C)$, then we have $x = \sum \lambda_{ij} e_{ij} \in A$ and $e_i x e_j = \lambda_{ij} e_{ij}$ for all $1 \leq i, j \leq n$. Thus h is surjective, therefore h is a *-isomorphism.

∎

Let $e^{(k)}$ denote the maximal projection in F_k, then the $e^{(k)}$, $k = 1, \cdots, r$ are the minimal projections of $C(A)$, the center of A, and we have

$$1_A = \sum_{k=1}^{r} e^{(k)}.$$

We let $\{e_{ij}^{(k)}\}_{i,j=1}^{p_k}$ denote a set of matrix units for F_k. We will say that

$$\{e_{i,j}^{(k)} \in F_k : i, j = 1, \ldots, p_k; k = i, \ldots, r\}$$

is a set of **matrix units for** A if the $e_{i,j}^{(k)}$'s span A linearly and satisfy the following conditions:

$$e_{ij}^{(k)} e_{st}^{(q)} = \delta_{kq} \delta_{js} e_{it}^{(k)} \qquad (3.4)$$

$$(e_{ij}^{(k)})^* = e_{ji}^{(k)}. \qquad (3.5)$$

If the $e_{ij}^{(k)}$'s satisfy (3.4) and (3.5) without necessarily spanning A, we shall call them **matrix units in** A.

Since $e^{(k)}$ is the identity of F_k for each k, we have

$$e^{(k)} = \sum_{i=1}^{p_k} e_{ii}^{(k)}. \tag{3.6}$$

We give some notation for matrix algebras here. Let M_n denote $M_n(\mathcal{C})$, and let $M_{\vec{p}}$, where $\vec{p} = (p_1, \ldots, p_r) \in \mathcal{Z}^r$, denote:

$$\sum_{k=1}^{r}{}^{\oplus} M_{p_k}(\mathcal{C}).$$

3.4 Canonical Homomorphisms

In this section we shall describe the homomorphisms of finite-dimensional C*-algebras and AF algebras.

If A_1 and A_2 are finite dimensional C*-algebras with

$$A_1 = \sum_{j=1}^{s}{}^{\oplus} M_{q_j},\ A_2 = \sum_{i=1}^{r}{}^{\oplus} M_{p_i},$$

or shortly, $A_1 = M_{\vec{q}}$, $A_2 = M_{\vec{p}}$, where $\vec{q} = (q_1, \ldots, q_s)$, $\vec{p} = (p_1, \ldots, p_r)$, and $\phi : A_1 \to A_2$ is a *-homomorphism, then ϕ can be described by an $r \times s$ matrix $[k_{ij}]$ of nonnegative integers. Let $\phi_i : M_{\vec{q}} \longrightarrow M_{p_i}$ be the i'th coordinate function, i.e., $\forall a \in A_1$,

$$\begin{aligned} & [0_{(p_1+\cdots+p_{i-1})} \oplus \phi_i(a) \oplus 0_{(p_{i+1}+\cdots+p_r)}] \\ = \ & [0_{(p_1+\cdots+p_{i-1})} \oplus I_{p_i} \oplus 0_{(p_{i+1}+\cdots+p_r)}]\phi(a)[0_{(p_1+\cdots+p_{i-1})} \oplus I_{p_i} \oplus 0_{(p_{i+1}+\cdots+p_r)}] \end{aligned}$$

then $\phi(a) = \phi_1(a) \oplus \cdots \oplus \phi_r(a)$. Regard ϕ_i as a representation[3] of $A_1 = M_{\vec{q}}$, it breaks up as a direct sum of irreducible representations. There is exactly one

---

[3] A representation of a C*-algebra A is a pair (π, H), where H is a Hilbert space and $\pi : A \to B(H)$ is a *-homomorphism. π is irreducible if the only projections in $B(H)$ that commute with every $\pi(a)$, $a \in A$, are 0 and I.

irreducible representation of A_1 for each summand M_{q_j}. k_{ij} is the multiplicity of this representation in ϕ_i, called the multiplicity of partial embedding of M_{q_i} into M_{p_i} ([2]).

Now we can give the definition of canonical homomorphisms. Let us begin from simple algebras first.

Given $p, q \in \mathcal{N}$ and $k \in \mathcal{Z}^+$ such that $kq \leq p$, we define a homomorphism

$$\phi : M_q \longrightarrow M_p$$

$$\text{by} \qquad a \longrightarrow \overbrace{a \oplus a \oplus \cdots \oplus a}^{k} \oplus 0_h.$$

where $kq + h = p$.

More generally, given $\vec{q} = (q_1, q_2, \ldots, q_s) \in \mathcal{N}^s$, $\vec{p} = (p_1, p_2, \ldots, p_r) \in \mathcal{N}^r$ and an $r \times s$ matrix $\phi = [k_{ij}]$, $k_{ij} \in \mathcal{Z}^+$ such that :

$$\phi(\vec{q}) \leq \vec{p}, \qquad \phi(\vec{q}) + \vec{h} = \vec{p}:$$

$$\begin{bmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & & \vdots \\ k_{r1} & \cdots & k_{rs} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_s \end{bmatrix} + \begin{bmatrix} h_1 \\ \vdots \\ h_r \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_r \end{bmatrix}.$$

We define a *-homomorphism

$$\phi : M_{\vec{q}} \longrightarrow M_{\vec{p}}$$

by

$$\begin{aligned} a_1 \oplus a_2 \oplus \cdots \oplus a_s \longrightarrow \ & (\overbrace{a_1 \oplus \cdots \oplus a_1}^{k_{11}} \oplus \overbrace{a_2 \oplus \cdots \oplus a_2}^{k_{12}} \oplus \cdots \oplus 0_{h_1}) \\ & \oplus (\overbrace{a_1 \oplus \cdots \oplus a_1}^{k_{21}} \oplus \overbrace{a_2 \oplus \cdots \oplus a_2}^{k_{22}} \oplus \cdots \oplus 0_{h_2}) \oplus \cdots . \end{aligned}$$

We say that ϕ defined above is a **canonical homomorphism.**

The following two definitions are introduced for comparing pairs of algebras (or any other algebraic objects) with their morphisms.

Any two homomorphisms of algebras $A_1 \xrightarrow{\phi} A_2$, $B_1 \xrightarrow{\psi} B_2$ are **isomorphic** if there exist two isomorphisms $\gamma : A_1 \longrightarrow B_1$, $\delta : A_2 \longrightarrow B_2$ for which

$$\begin{tikzcd}
A_1 \arrow[r, "\phi"] \arrow[d, "\gamma"'] & A_2 \arrow[d, "\delta"'] \\
B_1 \arrow[r, "\psi"] & B_2
\end{tikzcd}$$

commutes.

Any two homomorphisms of unital algebras $\phi, \psi : A_1 \longrightarrow A_2$ are **inner equivalent** if there exist inner automorphisms $\gamma : A_1 \longrightarrow A_1$, $\delta : A_2 \longrightarrow A_2$ for which

$$\begin{tikzcd}
A_1 \arrow[r, "\phi"] \arrow[d, "\gamma"'] & A_2 \arrow[d, "\delta"'] \\
A_1 \arrow[r, "\psi"] & A_2
\end{tikzcd}$$

commutes.

By using matrix units, O. Bratteli([4]) proved:

Theorem 3.15 *Let $A_1 \cong M_{\vec{q}}$, $A_2 \cong M_{\vec{p}}$ be two finite dimensional C^*-algebras with the same unit, and suppose that $A_1 \subseteq A_2$. Then there exists a unique canonical homomorphism $\phi = [k_{ij}]$ such that the the embedding map $A_1 \hookrightarrow A_2$ is isomorphic to $M_{\vec{q}} \xrightarrow{\phi} M_{\vec{p}}$.*

The above theorem can be regarded as a corollary of the next theorem.

Theorem 3.16 *(Alfred Hales[9]) Given $\vec{q} \in \mathcal{N}^s$, $\vec{p} \in \mathcal{N}^r$ any homomorphism*

$$\phi : M_{\vec{q}} \longrightarrow M_{\vec{p}}$$

is inner equivalent to a unique canonical homomorphism.

Proof : Having showed that

$$\phi(a) = \phi_1(a) \oplus \cdots \oplus \phi_r(a) \qquad \text{where} \qquad \phi_k : M_{\vec{q}} \to M_{p_k},$$

we may restrict our attention to the case in which $r = 1$, i.e.,

$$\phi : M_{\vec{q}} \longrightarrow M_p, \qquad p \in \mathcal{N}.$$

for some $p \in \mathcal{N}$. Letting

$$\{e_{ij}^{(k)} : k = 1, \ldots, s;\ i, j = 1, \ldots, q_k\}$$

denote the matrix units for $M_{\vec{q}}$, the projections

$$\{e^{(k)} : e^{(k)} = \sum_{i=1}^{q_k} e_{ii};\ k = 1, \ldots, s\}$$

are orthogonal, and thus

$$\{\phi(e^{(k)}) : k = 1, \ldots, s\}$$

are orthogonal. Regarding M_p as linear transformations on C^p, we let $\{x_l^k : l = 1, \ldots, l_k\}$ be an orthonormal basis in $\phi(e_{11}^{(k)})C^p$. Then

$$\{x_{jl}^k : x_{jl}^k = \phi(e_{j1}^{(k)})x_l^k;\ j = 1, \ldots, q_k;\ l = 1, \ldots, l_k\}$$

is an orthonormal basis for $\phi(e^{(k)})\mathcal{C}^p$ and we complete $\{x^k_{jl}\}$ to an orthonormal basis for $\mathcal{C}^p$ by introducing additional vectors $x_1, \ldots, x_h$. If

$$a = \sum_{i,j,k} a^k_{ij} e^{(k)}_{ij} \in M_{\vec{q}},$$

then

$$\begin{aligned}
\phi(a)x^{k_0}_{j_0 l} &= \textstyle\sum_{i,j,k} a^k_{ij}\phi(e^{(k)}_{ij})\phi(e^{(k)}_{j_0 1})x^{k_0}_l \\
&= \textstyle\sum_i a^{k_0}_{ij_0}\phi(e^{(k_0)}_{i1})x^{k_0}_l \\
&= \textstyle\sum_i a^{k_0}_{ij_0} x^{k_0}_{il}, \\
\phi(a)x_j &= 0,
\end{aligned}$$

and using the ordered basis

$$x^1_{11}, x^1_{21}, \ldots, x^1_{q(1)1}, x^1_{12}, \ldots, x^1_{1l_1}, \ldots, x^1_{q(1)l_1}, x^2_{11}, \ldots, x_1, \ldots, x_h$$

$\phi(a)$ has the matrix

$$\overbrace{[a^1_{ij}] \oplus [a^1_{ij}] \oplus \cdots}^{l_1} \oplus \overbrace{[a^2_{ij}] \oplus [a^2_{ij}] \oplus \cdots}^{l_2} \cdots 0_h.$$

Letting $\{f_i : i = 1, \ldots, p\}$ be the canonical basis for $\mathcal{C}^p$ and v be the unitary with $vx^1_{11} = f_1$, $vx^1_{21} = f_2, \ldots$, it follows that $v\phi(a)v^*$ has the above form with respect to the f_i.

For the uniqueness, suppose that $\phi = [k_{ij}]$ and $\psi = [k'_{ij}]$ define inner equivalent homomorphisms:

$$M_{\vec{q}} \longrightarrow M_{\vec{p}}$$

i.e., there exist inner automorphisms γ=Ad u, δ=Ad v:

$$M_{\vec{q}} \xrightarrow{\gamma} M_{\vec{q}},$$

$$M_{\vec{p}} \xrightarrow{\delta} M_{\vec{p}}$$

(for some u, v are unitaries in $M_{\vec{q}}$, $M_{\vec{p}}$ respectively.) such that:

$$\begin{tikzcd}
M_{\vec{q}} \arrow[r, "\phi"] \arrow[d, "\gamma"'] & M_{\vec{p}} \arrow[d, "\delta"'] \\
M_{\vec{q}} \arrow[r, "\psi"] & M_{\vec{p}}
\end{tikzcd}$$

is commutative. i.e., $\psi \circ \gamma = \delta \circ \phi$.

First, we show that there is a unitary w in $M_{\vec{p}}$ with $\psi =$ Ad $w \circ \phi$.

Letting I, I' be the identity matrices of $M_{\vec{q}}$, $M_{\vec{p}}$ respectively, and letting $e = \psi(I)$, $u' = \psi(u) + (I' - e)$, we have

$$\begin{aligned}
u'(u')^* &= [\psi(u) + (I' - e)][\psi(u) + (I' - e)]^* \\
&= e + \psi(u)(I' - e) + (I' - e)\psi(u)^* + (I' - e) \\
&= I' + \psi(u) + \psi(u^*) - \psi(I)\psi(u^*) + \psi(u)\psi(I) \\
&= I'.
\end{aligned}$$

Similarly $(u')^*u' = I'$, thus u' is unitary in $M_{\vec{p}}$. It is fairly simple to verify $u'\psi(a)(u')^* = \psi(uau^*)$ for all a in $M_{\vec{q}}$. letting δ'=Ad u', we have $\delta' \circ \psi = \psi \circ \gamma$. So the diagram

$$\begin{tikzcd}
M_{\vec{q}} \arrow[r, "\psi"] \arrow[d, "\gamma"'] & M_{\vec{p}} \arrow[d, "\delta'"'] \\
M_{\vec{q}} \arrow[r, "\psi"] & M_{\vec{p}}
\end{tikzcd}$$

is commutative.

It follows that

$$\psi = (\delta')^{-1} \circ (\psi \circ \gamma) = (\delta')^{-1} \circ (\delta \circ \phi) = ((\delta')^{-1} \circ \delta) \circ \phi = \theta \circ \phi.$$

where $\theta = (\delta')^{-1} \circ \delta$ is inner since δ', δ are inner. Thus $\psi = \text{Ad } w \circ \phi$ for some unitary w in $M_{\vec{p}}$.

Now we show $\psi = \phi$. Let $w = w_1 \oplus \cdots \oplus w_r$,

$$\begin{aligned} w\phi(a)w^* &= w_1(\overbrace{a_1 \oplus \cdots \oplus a_1}^{k_{11}} \oplus a_2 \oplus \cdots \oplus 0_{h_1})w_1^* \oplus w_2(\cdots)w_2^* \oplus \cdots \\ &= (\overbrace{a_1 \oplus \cdots \oplus a_1}^{k'_{11}} \oplus a_2 \oplus \cdots \oplus 0'_{h_1}) \oplus \cdots . \end{aligned}$$

Thus we assume $r = 1 (M_{\vec{p}} = M_p)$, and write $w = w_1,\ l_j = k_{1j},\ l'_j = k'_{1j},\ h = h_1,\ h' = h'_1$. Let $a_1 = I_{q_1},\ a_2 = a_3 = \cdots = 0$,

$$w(\overbrace{I_{q_1} \oplus \cdots \oplus I_{q_1}}^{l_1} \oplus 0 \oplus 0 \cdots)w^* = (\overbrace{I_{q_1} \oplus \cdots \oplus I_{q_1}}^{l'_1} \oplus 0 \oplus 0 \cdots),$$

and since inner automorphisms preserve trace, $l_1 = l'_1$. Similarly, $l_j = l'_j$, and it follows

$$h = p - \sum l_i q_i = p - \sum l'_i q_i = h'.$$

And the result follows. ∎

3.5 Bratteli Diagrams

Let A be an AF algebra with $A = \overline{\cup A_n}$, then by preceding section each

$$A_n \;\cong\; \sum_{k=1}^{r_n}{}^{\oplus} M_{p_{nk}}, \tag{3.7}$$

The structure of A is completely determined by the series of central summands of the A_n and the embeddings of A_n into A_{n+1}.

Here we introduce a kind of convenient graph called a **Bratteli Diagram** for describing the structure of A: draw the diagram with one row for each n, write down $p_{n1}, \ldots, p_{nr_n}$ in each n'th row for each summand $M_{p_{nk}}$ of A_n, draw k_{ij} segments between $p_{nj}, p_{n+1,i}$, where $[k_{ij}]$ is the canonical homomorphism of these two matrix algebras.

We give some examples of AF algebra A with its Bratteli Diagrams:

Example 3.17 *A is Finite dimensional C^*-algebra.*

We can assume $A = \sum_{k=1}^{r}{}^{\oplus} M_{p_k}$ *and* $A = \overline{(\cup A_n)}$ *where each* $A_n = A$. *So* A *is an AF algebra and its Bratteli diagram is*

$$\begin{array}{cccc}
p_1 & p_2 & \cdots & p_r \\
| & | & & | \\
p_1 & p_2 & \cdots & p_r \\
| & | & & | \\
p_1 & p_2 & \cdots & p_r \\
\vdots & \vdots & & \vdots
\end{array}$$

Example 3.18 *A is a UHF algebra.*

The number of lines between p_n and p_{n+1} is $\frac{p_{n+1}}{p_n}$.

Example 3.19 *This example is closely related to the algebra we mentioned before. Let $\mathcal{H}$, $\mathcal{K}$ and P_n have the same meaning as in Example 3.12. Let $A_n = P_n\mathcal{K}P_n + \mathcal{C}I, A = \mathcal{K} + \mathcal{C}I$, where I is identity map on $\mathcal{H}$. So*

$$\begin{aligned} A_n &= P_n\mathcal{K}P_n + \mathcal{C}I \\ &= P_n\mathcal{K}P_n \oplus \mathcal{C}(I - P_n) \\ &\cong B(\mathcal{H}_n) \oplus \mathcal{C}I_\infty \\ &\cong M_n(\mathcal{C}) \oplus M_1(\mathcal{C}), \end{aligned}$$

where I_∞ is the identity $\aleph_0 \times \aleph_0$ matrix. Thus for $a \in A_n$,

$$a = P_naP_n + (I - P_n)a(I - P_n)$$

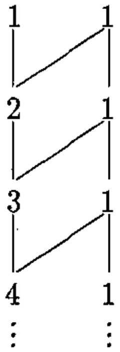
$$\cong \left|\begin{array}{c|c} P_naP_n & \bigcirc \\ \hline \bigcirc & \begin{matrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \ddots \end{matrix} \end{array}\right|$$

$$= \left|\begin{array}{c|c|c} P_naP_n & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \bigcirc \\ \hline 0 \cdots 0 & \lambda & \\ \hline \bigcirc & & \begin{matrix} \lambda & & \\ & \ddots & \\ & & \ddots \end{matrix} \end{array}\right|$$

thus $a \in A_{n+1}$. *Moreover,*

$$\begin{aligned} A &= \mathcal{K} + \mathcal{C}I \\ &= \overline{(\cup P_n \mathcal{K} P_n)} + \mathcal{C}I \\ &= \overline{(\cup P_n \mathcal{K} P_n + \mathcal{C}I)} \\ &= \overline{(\cup (P_n \mathcal{K} P_n + \mathcal{C}I))} \\ &= \overline{(\cup A_n)}. \end{aligned}$$

Therefore A *is an AF algebra, and the Bratteli diagram is*



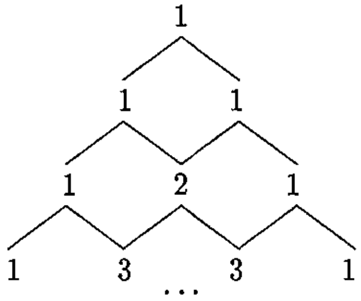
Example 3.20 A *is the inductive limit of:*

$$M_1 \xrightarrow{\phi_{12}} M_1 \oplus M_1 \xrightarrow{\phi_{23}} M_1 \oplus M_2 \oplus M_1 \xrightarrow{\phi_{34}} M_1 \oplus M_3 \oplus M_3 \oplus M_1 \longrightarrow \cdots$$

where

$$\phi_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \phi_{23} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \ \phi_{34} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ \ldots.$$

Thus the embedding maps are indicated by the following Bratteli diagram as *follows:*



Reference: [4], [7], [9], [13], [28].

Chapter 4

Isomorphisms of AF Algebras

Most results in this chapter are proved by O. Bratteli in [4]. Theorem 4.7 is one of the key results in this thesis. For the sake of completeness, here we present an outline of a part of Bratteli's work in [4].

4.1 Another Definition for AF Algebras

Lemma 4.1 *Let A be a C^*-algebra on a Hilbert space H, let $\varepsilon > 0$ and let n be a positive integer. Then there exists a $\delta = \delta(\varepsilon, n) > 0$ such that if*

1. $\{e_{ij}^{(k)} : i, j = 1, 2, \ldots, n_k;\ k = 1, 2, \ldots, m\}$ *is a family of matrix units for a finite dimensional C^*-algebra on H with unit I_H such that $\sum_{k=1}^{m} n_k = n$,*
2. *there exists $x_{ij}^{(k)} \in A$ such that $\|x_{ij}^{(k)} - e_{ij}^{(k)}\| < \delta$,*

   then there exists a family $\{f_{ij}^{(k)}\}$ of matrix units in A such that

$$\|f_{ij}^{(k)} - e_{ij}^{(k)}\| < \varepsilon.$$

Proof : By Lemma A.1 we suppose without loss of generality that $x_{ii}^{(k)}$ is a projection. For $i \neq j$ and $1 \leq k, q \leq m$,

$$\begin{aligned} \|x_{ii}^{(k)} x_{jj}^{(q)}\| &= \|x_{ii}^{(k)} x_{jj}^{(q)} - e_{ii}^{(k)} x_{jj}^{(q)} + e_{ii}^{(k)} x_{jj}^{(q)} - e_{ii}^{(k)} e_{jj}^{(q)} + e_{ii}^{(k)} e_{jj}^{(q)}\| \\ &\leq \|x_{ii}^{(k)} x_{jj}^{(q)} - e_{ii}^{(k)} x_{jj}^{(q)}\| + \|e_{ii}^{(k)} x_{jj}^{(q)} - e_{ii}^{(k)} e_{jj}^{(q)}\| \\ &\leq \|x_{ii}^{(k)} - e_{ii}^{(k)}\| + \|x_{jj}^{(q)} - e_{jj}^{(q)}\| < 2\delta. \end{aligned}$$

By Lemma A.2, we may assume that $\{x_{ii}^{(k)} : i = 1, \ldots, n_k, k = 1, \ldots, m\}$ is an orthogonal family of projections and by Lemma A.4 that for $i \neq j$, $x_{ij}^{(k)}$ is a partial isometry from $x_{jj}^{(k)}$ to $x_{ii}^{(k)}$. Now, for each k, $\sum_i x_{ii}^{(k)}$ is a projection and defining $e^{(k)} = \sum_i e_{ii}^{(k)}$,

$$\|e^{(k)} - \sum_i x_{ii}^{(k)}\| \leq \sum_i \|e_{ii}^{(k)} - x_{ii}^{(k)}\|$$

which for $\delta < \frac{1}{n}$ is less than 1. Let $f_{ij}^{(k)} = (x_{1i}^{(k)})^* x_{1j}^{(k)}$. If $\delta < \min(\frac{1}{n}, \frac{\varepsilon}{2})$ then $\{f_{ij}^{(k)}\}$ is a family of matrix units and

$$\begin{aligned} \|f_{ij}^{(k)} - e_{ij}^{(k)}\| &\leq \|(x_{1i}^{(k)})^* x_{1j}^{(k)} - (x_{1i}^{(k)})^* e_{1j}^{(k)}\| + \|(x_{1i}^{(k)})^* e_{1j}^{(k)} - e_{ij}^{(k)}\| \\ &= \|(x_{1i}^{(k)})^* x_{1j}^{(k)} - (x_{1i}^{(k)})^* e_{1j}^{(k)}\| + \|(x_{1i}^{(k)})^* e_{1j}^{(k)} - e_{i1}^{(k)} e_{1j}^{(k)}\| \\ &\leq \|x_{1j}^{(k)} - e_{1j}^{(k)}\| + \|(x_{1i}^{(k)})^* - (e_{1i}^{(k)})^*\| \\ &< 2\delta \\ &< \varepsilon \end{aligned}$$

This proof is complete. ∎

Theorem 4.2 *Suppose that A is a separable C^*-algebra with unit 1_A, then A is an AF algebra if and only if given $x_1, x_2, \ldots, x_n \in A$ and $\varepsilon > 0$, there exist a finite dimensional C^*-subalgebra B of A and elements $y_1, y_2, \ldots, y_n \in B$ such that $\|x_i - y_i\| < \varepsilon,\ i = 1, 2, \ldots, n$.*

Furthermore, if we are initially also given a finite dimensional subalgebra B_0, we may choose $B \supseteq B_0$.

Proof : The necessity follows from the definition of AF algebras clearly.

For sufficiency:

Let $\{d_i\}_{i \in \mathcal{N}}$ be a dense sequence in

$$\{a \in A : \|a\| < \frac{1}{2}\}.$$

Without loss of generality, suppose the algebras B, B_0 contain 1_A. We shall construct an increasing sequence $\{A_n\}$ of finite dimensional subalgebras of A such that for all n there exists $b_k \in A_n,\ k = 1, \ldots, n$, such that $\|b_k - d_k\| < 2^{-n},\ k = 1, \ldots, n$.

Since $\|d_1\| < \frac{1}{2}$, A_1 may be chosen arbitrarily. Suppose that A_n has been constructed and has the required properties. Assume for each n,

$$A_n \cong \sum_{k=1}^{r_n}{}^{\oplus} M_{p_{nk}}.$$

Let $\{e_{ij}^{(nk)}\}$ be matrix units for A_n. Define

$$\varepsilon = \frac{1}{2^{n+1}}(1 + 4\sum_{k=1}^{r_n} p_{nk}^2)^{-1}$$

By using hypothesis of the theorem and Lemma 4.1, there exist a finite dimensional subalgebra A' of A and a set of matrix units $\{f_{ij}^{(k)}\}$ in A' such that

$$\|f_{ij}^{(k)} - e_{ij}^{(nk)}\| < \delta,$$

where δ is the $\delta(\varepsilon, (nk))$ of Lemma A.3, and such that there exists $b'_k \in A'$, $k = 1, \ldots, n+1$, such that

$$\|b'_k - d_k\| < \varepsilon.$$

By Lemma A.3, there exists a single partial isometry $w \in A$ such that $wf_{11}^{(k)}$ is a partial isometry between $f_{11}^{(k)}$ and $e_{11}^{(nk)}$, and $\|e_{11}^{(nk)} - wf_{11}^{(k)}\| < \varepsilon$, for all $k = 1, \ldots, r_n$. Let

$$a = \sum_{k=1}^{r_n} \sum_{i=1}^{p_{nk}} e_{i1}^{(nk)} w f_{i1}^{(k)}.$$

Then $a \in A$ and a is unitary with $af_{ij}^{(k)}a^* = e_{ij}^{(nk)}$. Define $A_{n+1} = aA'a^*$. Then A_{n+1} is a finite dimensional subalgebra of A isomorphic with A', and $A_n \subseteq A_{n+1}$. Let $b_k = ab'_k a^* \in A_{n+1}$. Then

$$\begin{aligned}
\|b_k - d_k\| &\leq \|d_k - b'_k\| + \|b'_k - b_k\| \\
&< \varepsilon + \|b'_k - ab'_k a^*\| \\
&= \varepsilon + \|\sum_{kqst}(f_{ss}^{(k)} b'_k f_{tt}^{(q)} - e_{s1}^{(nk)} w f_{1s}^{(k)} b'_k f_{t1}^{(q)} w^* e_{1t}^{(nq)}\| \\
&\leq \varepsilon + (\sum_{k=1}^{n_n} p_{nk})^2 \sup_{kqst} \|f_{ss}^{(k)} b'_k f_{tt}^{(q)} - e_{s1}^{(nk)} w f_{1s}^{(k)} b'_k f_{t1}^{(q)} w^* e_{1t}^{(nq)}\|.
\end{aligned}$$

Now

$$
\begin{aligned}
&\|f_{ss}^{(k)}b_k'f_{tt}^{(q)} - e_{s1}^{(nk)}wf_{1s}^{(k)}b_k'f_{t1}^{(q)}w^*e_{1t}^{(nq)}\| \\
&\le \|f_{ss}^{(k)}b_k'f_{tt}^{(q)} - f_{ss}^{(k)}b_k'f_{t1}^{(q)}w^*e_{1t}^{(nq)}\| + \|(f_{ss}^{(k)} - e_{s1}^{(nk)}wf_{1s}^{(k)})b_k'f_{t1}^{(q)}w^*e_{1t}^{(nq)}\| \\
&\le \|f_{tt}^{(q)} - f_{t1}^{(q)}w^*e_{1t}^{(nq)}\| + \|f_{ss}^{(k)} - e_{s1}^{(nk)}wf_{1s}^{(k)}\| \\
&\le \|f_{1t}^{(q)} - w^*e_{1t}^{(nq)}\| + \|f_{s1}^{(k)} - e_{s1}^{(nk)}w\| \\
&\le \|f_{1t}^{(q)} - e_{1t}^{(nq)}\| + \|e_{t1}^{(nq)} - e_{t1}^{(nq)}w\| + \|f_{s1}^{(k)} - e_{s1}^{(nk)}\| + \|e_{s1}^{(nk)} - e_{s1}^{(nk)}w\| \\
&< 4\varepsilon.
\end{aligned}
$$

Hence

$$\|b_k - d_k\| < \varepsilon + 4(\sum_{k=1}^{n_k} p_{nk})^2\varepsilon = \frac{1}{2^{n+1}}.$$

By induction, a sequence $\{A_n\}$ with the required properties exists. Then $\{d_i\} \in \overline{\cup A_n}$, so $\overline{\cup A_n} = A$.

If we are initially also given a finite dimensional subalgebra B_0, we can construct a finite dimensional subalgebra $\tilde{B}$ of A from B_0 and B such that this subalgebra contains B_0 and B.

Let $\{u_{ij}^{(k)}\}$ be a family of matrix units for B_0. By the hopothesis, we may assume that there exist $x_{ij}^{(k)} \in B$ such that

$$\|x_{ij}^{(k)} - u_{ij}^{(k)}\| < \delta$$

where the δ is the δ in Lemma 4.1. By Lemma 4.1 there exists a family $\{v_{ij}^{(k)}\}$ of matrix units in B such that

$$\|u_{ij}^{(k)} - v_{ij}^{(k)}\| < \varepsilon.$$

Similarly as above, there exists a unitary u in A such that

$$\text{ad } u(u_{ij}^{(k)}) = v_{ij}^{(k)}.$$

Thus ad $u(B_0)$ is in B. Let $\tilde{B} = \text{ad } u^*(B)$, then $B_0 \subseteq \tilde{B}$ and $\tilde{B}$ has the desired property (we now have to assume in B that $\|u^* x_i u - y_i\| < \varepsilon$ instead of $\|x_i - y_i\| < \varepsilon$). ∎

4.2 Isomorphisms of AF Algebras

Lemma 4.3 *Let A be a unital AF algebra with $A = \overline{(\cup A_n)}$, and let B be a finite dimensional subalgebra of A. Then given $\varepsilon > 0$, there exists $u \in U(A)$ and $n \in \mathcal{N}$ such that*

$$\|u - 1_A\| < \varepsilon \textit{ and } Ad\, u(B) \subseteq A_n.$$

Proof : This is essentialy the proof of the previous theorem. Without loss of generality we assume $1_A \in B$. Let $\{f_{ij}^{(k)}\}_{k=1}^{m}$ be matrix units for B, and suppose $1 \leq i, j \leq N$ for all $f_{ij}^{(k)}$. Let $\varepsilon_1 = \frac{\varepsilon}{3mN}$ and let $\delta = \delta(\varepsilon_1, m)$ of Lemma A.3.

By Lemma 4.1 there exist an A_n and a family $\{e_{ij}^{(k)}\}_{k=1}^{m}$ of matrix units in A_n such that

$$\|f_{ij}^{(k)} - e_{ij}^{(k)}\| < \delta$$

By Lemma A.3, there exists a partial isometry $w \in A$ such that $ww^* = e_{11}^{(k)}$, $w^*w = f_{11}^{(k)}$ and such that

$$\|e_{11}^{(k)} - e_{11}^{(k)} w\| < \varepsilon_1.$$

Define

$$u = \sum_k \sum_i e_{i1}^{(k)} w f_{1i}^{(k)}.$$

Then u is unitary, and

$$e_{ij}^{(k)} = u f_{ij}^{(k)} u^*$$

Thus,

$$A_n \supseteq \text{Ad } u(B).$$

Furthermore,

$$\begin{aligned}
\|e_{ii}^{(k)} - e_{ii}^{(k)} u\| &= \|e_{ii}^{(k)} - e_{ii}^{(k)} e_{i1}^{(k)} w f_{1i}^{(k)}\| \\
&\leq \|e_{1i}^{(k)} - e_{11}^{(k)} w f_{1i}^{(k)}\| \\
&\leq \|e_{1i}^{(k)} - f_{1i}^{(k)}\| + \|f_{1i}^{(k)} - e_{11}^{(k)} w f_{1i}^{(k)}\| \\
&\leq \|e_{1i}^{(k)} - f_{1i}^{(k)}\| + \|f_{11}^{(k)} - e_{11}^{(k)} w f_{11}^{(k)}\| \\
&\leq \|e_{1i}^{(k)} - f_{1i}^{(k)}\| + \|f_{11}^{(k)} - e_{11}^{(k)}\| + \|e_{11}^{(k)} - e_{11}^{(k)} w\| \\
&< 2\delta + \varepsilon_1 < 3\varepsilon_1.
\end{aligned}$$

Thus

$$\begin{aligned}
\|1_A - u\| &= \|\textstyle\sum_k \sum_i (e_{ii}^{(k)} - e_{ii}^{(k)} u)\| \\
&\leq \textstyle\sum_k \sum_i \|e_{ii}^{(k)} - e_{ii}^{(k)} u\| \\
&< m \times N \times 3\varepsilon_1 \\
&= \varepsilon.
\end{aligned}$$

∎

Lemma 4.4 *Let A be unital C^*-algebra, let B_1, B_2 be two finite dimensional *-subalgebras of A containing 1_A. Let $\phi : B_1 \to B_2$ be a *-isomorphism such that $\|\phi - I|_{B_1}\| < 1$, where I is the identity map of A. Then ϕ is an inner automorphism of A restricted to B_1, i.e., $\exists u \in U(A)$ such that*

$$\phi(x) = Ad\, u(x), \quad x \in B_1.$$

Proof : Suppose that $\{e_{ij}^{(k)}\}$ is a set of matrix units for B_1. Let $f_{ij}^{(k)} = \phi(e_{ij}^{(k)})$. Then $\{f_{ij}^{(k)}\}$ is a set of matrix units for B_2. Since

$$\|f_{11}^{(k)} - e_{11}^{(k)}\| = \|(\phi - I)e_{11}^{(k)}\| < \|e_{11}^{(k)}\| = 1$$

for all k, by Lemma A.3, there exists a partial isometry $w \in A$ such that $f_{11}^{(k)} w e_{11}^{(k)}$ is a partial isometry from $e_{11}^{(k)}$ to $f_{11}^{(k)}$ for all k. Let $u = \sum_k \sum_i f_{i1}^{(k)} w e_{1i}^{(k)}$. Then u is unitary and $u e_{ij}^{(k)} u^* = f_{ij}^{(k)} = \phi(e_{ij}^{(k)})$. So $\phi = \text{Ad}\, u$. ∎

Lemma 4.5 *Let $A = \overline{(\cup A_n)}$ and let B be a finite dimensional *-subalgebra of A such that $A_1 \subseteq B$. Then there exists $n \in \mathcal{N}$ and $u \in U(A)$ such that*

(1) $Ad\, u(B) \subseteq A_n$

(2) $Ad\, u(x) = x \quad$ *for* $x \in A_1$

Proof : By Lemma 4.3, there exists $v \in U(A)$ and $n \in \mathcal{N}$ such that

$$\|v - 1_A\| < \frac{1}{3} \text{ and } \text{Ad}\, v(B) \subseteq A_n.$$

Let $A_1' = \text{Ad } v(A_1) \subsetneq A_n$ and define an isomorphism

$$\phi : \ A_1 \longrightarrow A_1'$$

$$\text{via}: \ \phi(x) = \text{Ad } v(x).$$

Then, for $x \in A_1$,

$$\begin{aligned}\|\phi(x) - x\| &= \|vxv^* - x\| \\ &\leq \|vxv^* - xv^*\| + \|xv^* - x\| \\ &\leq 2\|x\| \cdot \|v - 1_A\| \\ &< \tfrac{2}{3}\|x\|;\end{aligned}$$

Thus $\|\phi - I|_{A_1}\| \leq \frac{2}{3} < 1$. By Lemma 4.4, there exists $w \in U(A_n)$ such that $\phi(x) = \text{Ad } w(x), \ x \in A_1$. Let $u = w^*v$, then

$$\text{Ad } u(B) = uBu^* = w^*vBv^*w \subsetneq w^*A_nw = A_n,$$

since $w \in A_n$. For $x \in A_1$,

$$uxu^* = w^*vxv^*w = w^*\phi(x)w = \phi^{-1}(\phi(x)) = x.$$

Thus $\text{Ad } u(x) = x$ for all $x \in A_1$. ∎

Definition 4.6 *An* **equivalence** *between approximating systems* $\{A_n\}, \{B_n\}$ *for a* C^**-algebra* A *is a sequence of unitaries* $u_n \in \tilde{A}$ *(cf. 3, Chapter 1) to-*

gether with subsequences $\{A_{n(k)}\}, \{B_{m(k)}\}$ *such that for all* k*:*

(1) $Ad\, u_k(A_{n(k)}) \subseteq B_{m(k)}$

(2) $Ad\, u_l(A_{n(l)}) \supseteq B_{m(k)}$ *for large enough* l

(3) *The diagrams*

$$\begin{tikzcd} A_{n(k)} \arrow[r, "Ad\, u_k"] \arrow[d] & B_{m(k)} \arrow[d] \\ A_{n(k+1)} \arrow[r, "Ad\, u_{k+1}"] & B_{m(k+1)} \end{tikzcd}$$

are commutative.

Theorem 4.7 *Any two approximating systems for an AF algebra are equivalent.*

Proof : Let $A = \overline{(\cup A_n)} = \overline{(\cup B_n)}$, we construct $\{u_n\}$ by induction. By using unitizing A, we may assume A is unital. Choose $n(1) = 1$. By Lemma 4.3, there exist $m(1)$ and u_1 such that

$$A_{n(1)} \xrightarrow{\text{Ad } u_1} B_{m(1)}. \tag{4.1}$$

Suppose now $u_1, v_1, \ldots, u_n$ have been constructed such that

$$\begin{tikzcd}
A_{n(k-1)} \arrow[r, "\mathrm{Ad}\ u_{k-1}"] \arrow[d] & B_{m(k-1)} \arrow[d] \arrow[dl, "\mathrm{Ad}\ v_{k-1}"'] \\
A_{n(k)} \arrow[r, "\mathrm{Ad}\ u_k"] & B_{n(k)}
\end{tikzcd}$$

commutes.

Let $B' = \mathrm{Ad}\ u_k^*(B_{m(k)})$. Then B' is a finite dimensional *-subalgebra of A and since $\mathrm{Ad}\ u_k(A_{n(k)}) \subseteq B_{m(k)}$, we have $A_{n(k)} \subseteq B'$. By Lemma 4.5 there exist $v \in U(A)$ and positive integer $n(k+1) > n(k)$ such that $\mathrm{Ad}\ v(B') \subseteq A_{n(k+1)}$ and $\mathrm{Ad}\ v(x) = x, x \in A_{n(k)}$. Let $v_k = vu_k^*$. Then

$$\begin{aligned}
\mathrm{Ad}\ v_k(B_{m(k)}) &= v_k B_{m(k)} v_k^* = u_k v^* B_{m(k)} v u_k = u_k B' u_k^* \\
&\subseteq A_{n(k+1)}
\end{aligned}$$

And if $x \in A_{n(k)}$, then

$$\begin{aligned}
\mathrm{Ad}\ v_k(\mathrm{Ad}\ u_k(x)) &= v_k(u_k x u_k^*) v_k^* \\
&= v u_k^* u_k x u_k^* u_k v^* \\
&= v x v = x
\end{aligned}$$

u_{n+1} is then constructed in an analogous fashion by "rotating" $v_n^* A_{n(k+1)} v_n$ into an algebra $B_{m(k+1)}$ by $\mathrm{Ad}\ u$ such that $B_{m(k)}$ is kept fixed, and define $u_{n+1} = uv_n^*$. Hence we have the commutative diagram:

$$\begin{tikzcd}
A_{n(k-1)} \arrow[r, "\mathrm{Ad}\ u_{k-1}"] \arrow[d] & B_{m(k-1)} \arrow[d] \arrow[dl, "\mathrm{Ad}\ v_{k-1}"'] \\
A_{n(k)} \arrow[r, "\mathrm{Ad}\ u_{k}"] \arrow[d] & B_{m(k)} \arrow[d] \arrow[dl, "\mathrm{Ad}\ v_{k}"'] \\
A_{n(k+1)} \arrow[r, "\mathrm{Ad}\ u_{k+1}"] & B_{m(k+1)}.
\end{tikzcd}$$

By induction we obtain the following commutative diagram:

$$\begin{tikzcd}
A_{n(1)} \arrow[r, "\mathrm{Ad}\ u_{1}"] \arrow[d] & B_{m(1)} \arrow[d] \arrow[dl, "\mathrm{Ad}\ v_{1}"'] \\
A_{n(2)} \arrow[r, "\mathrm{Ad}\ u_{2}"] \arrow[d] & B_{m(2)} \arrow[d] \arrow[dl, "\mathrm{Ad}\ v_{2}"'] \\
A_{n(3)} \arrow[r, "\mathrm{Ad}\ u_{3}"] & B_{m(3)} \\
& \vdots &
\end{tikzcd}$$

Furthermore

$$\begin{aligned} B_{m(k)} &\subseteq B_{m(k+1)} \\ &= \text{Ad } u_{k+1}(\text{Ad } v_k B_{m(k)}) \\ &\subseteq \text{Ad } u_{k+1}(A_{m(k+1)}) \qquad \forall k. \end{aligned}$$

And the proof is complete. ∎

Corollary 4.8 *Let* $A = \overline{\cup A_n} = \overline{\cup B_n}$. *Then there exists an automorphism* α *of* A *such that given* $n \in \mathcal{N}$, *there exists* $m \in \mathcal{N}$ *such that* $\alpha(B_n) \subseteq A_m, A_n \subseteq \alpha(B_m)$.

Proof : In the proof of Theorem 4.7, we may define a morphism $\alpha : \cup_k A_{m(k)} \to \cup_k B_{m(k)}$ by $\alpha|_{A_{m(k)}} = \text{Ad } u_k$, since $\text{Ad } u_{k+1}|_{A_{n(k)}} = \text{Ad } u_k$, α is well-defined.

α is surjective because if $y \in B_{m_k}$ we have $y = \text{Ad } u_{k+1}(\text{Ad } v_k(y)) = \alpha(\text{Ad } v_k(y))$ and $\text{Ad } v_k(y) \in A_{m(k+1)}$.

Moreover since $\alpha|_{A_{m(k)}}$ is injective and hence isometric, α is an isometric *-isomorphism of $\cup A_{n(k)}$ onto $\cup B_{m(k)}$. Since these two sets are dense in A, α may be extended to an automorphism of A.

Given $n \in \mathcal{N}$, there exists k such that $n \leq m(k)$ and $n \leq n(k)$. Thus

$$\alpha(A_n) \subseteq \alpha(A_{n(k)}) \subseteq B_{m(k)}$$

and

$$\begin{aligned} B_n \subseteq B_{m(k)} &= \text{Ad } u_{k+1}(\text{Ad } v_k(B_{m(k)})) \\ &\subseteq \text{Ad } u_{k+1}(A_{n(k+1)}) \\ &= \alpha(A_{n(k+1)}) \end{aligned}$$

And choosing $m \geq n(k+1)$ and $m \geq m(k)$ we complete the proof. ∎

Using preceding theorems, Bratteli ([4]) showed the following theorem:

Theorem 4.9 *Let unital AF algebras* $A = \overline{\cup A_n}$, $B = \overline{\cup B_n}$ *respectively. Then* A *is isomorphic to* B *if and only if there exists a subsequence* $\{A_{n(k)}\}$ *and each* $A_{n(k)}$ *contains a finite dimensional *-subalgebra* B'_k *containing 1 and*

1. $\{B'_n\}$ *increasing, and there exists an isomorphism* $\phi : \cup_n B_n \to \cup_n B'_n$ *such that* $\phi(B_n) = B'_n$ *for all* n.
2. $\forall n \in \mathcal{N}$, $\exists k \in \mathcal{N}$ *such that* $A_n \subseteq B'_k$.

Proof : For sufficiency.

$\phi|_{B_n} : B_n \to B'_n$ is an isometry. Thus ϕ is an isometry. By 2 we have $\cup A_n \subseteq \cup B'_n$. So $\cup A_n = \cup B'_n$ since $B'_k \subseteq A_{n(k)}$. Hence ϕ extends continuously to an isomorphism from $B = \overline{(\cup B'_n)}$ to $A = \overline{(\cup A_n)}$.

For necessity.

Let $\psi : B \to A$ be a *-isomorphism. Since ψ is an isometry, $\cup\psi(B_n)$ is a dense subset of A, thus $A = \overline{(\cup\psi(B_n))}$. By Corollary 4.8, there exits an automorphism φ of A, and an increasing sequence $\{n(k)\}_k$ of positive integers such that $\varphi(\psi(B_k)) \subseteq A_n(k)$, $k = 1, 2, \ldots$, and such that for all n there exists k such that $A_n \subseteq \varphi(\psi(B_k))$. Let $B'_k = \varphi \circ \psi(B_k)$ and $\phi = \varphi \circ \psi|_{\cup_n B_n}$. Then items 1 and 2 of the theorem are fulfilled.

∎

Reference: [4], [11].

Chapter 5

Dimension Groups

The algebraic classification of C*-algebras has proved to be a difficult problem. To date, even for the case of separable C*-algebras, such a detailed and general classification appears to be hopelessly out of reach. If one restricts attention to AF algebras however, then quite a satisfactory theory is available. In Elliott's paper([14]), it was proved that the (stable) classification of the *AF* algebras can be reduced to the classification of certain discrete, ordered abelian groups, called dimension groups.

In this chapter we shall consider the classification of some more general algebras, locally semisimple algebras. The similar results for AF algebras will come out consequently.

In addition to dimension groups, the important concepts in this chapter are orders of groups and dimension functions. The main results of this chapter are Theorem 5.9, Theorem 5.11, and Theorem 5.15. At the end of this chapter we give a sufficient and necessary condition for a an abelian group to be dimension group without proof which belongs to Effros, Handelman, and Shen ([10]).

Since our interest is ultimately in AF algebras, we shall assume the index set for a directed system is $\mathcal{N}$ unless otherwise stated.

5.1 Preliminaries

Definition 5.1 *By an* **ordered group**, *we mean an abelian group G together with a subset G^+, called the* **positive cone** *or* **ordering**, *such that*

(1) $G^+ + G^+ \subseteq G^+$,

(2) $G^+ - G^+ = G$,

(3) $G^+ \cap (-G^+) = \{0\}$,

(4) *if $g \in G$ and $ng \in G^+$ for some $n \in \mathcal{N}$ then $g \in G^+$.*

Definition 5.2 *By a* **simplicial ordering** *on group $\mathcal{Z}^r$ we shall mean the usual ordering (see Chapter 1)*

$$(\mathcal{Z}^r)^+ = \{(a_1, a_2, \ldots, a_r) : a_i \in \mathcal{Z}^+, i = 1, 2, \ldots, r\}.$$

The group $\mathcal{Z}^r$ with this order is called a **simplicial group**.

Definition 5.3 *A homomorphism ϕ from an ordered group G into an ordered group H is* **positive**, *and write $\phi \geq 0$, if $\phi(G^+) \subseteq H^+$, i.e., if $a \leq b$ in G implies $\phi(a) \leq \phi(b)$ in H. ϕ is an* **order isomorphism** *if ϕ is a group isomorphism with $\phi, \phi^{-1} \geq 0$.*

Given a direct system of ordered groups, positive maps:

$$G_1 \xrightarrow{\phi_{12}} G_2 \xrightarrow{\phi_{23}} \cdots,$$

the set-theoretic inductive limit

$$G_\infty = \varinjlim G_n$$

is an ordered group under the usual limit addition operation, and the positive cone

$$G_\infty^+ = \bigcup_n \phi_{n\infty}(G_n^+).$$

Thus $g \in G_\infty$ is said to be positive if there exists an index n and $g_n \in G_n^+$ such that $\phi_{n\infty}(g_n) = g$.

Definition 5.4 *An element $u \in G^+$ is called an* **order unit** *if $\forall g \in G^+, \exists n > 0$ with $0 \leq g \leq nu$.*

Definition 5.5 *An ordered group G is a* **dimension group** *if it is isomorphic to the inductive limit of a direct system of simplicially ordered groups together with positive homomorphism:*

$$\mathcal{Z}^{r_1} \xrightarrow{\phi_{12}} \mathcal{Z}^{r_2} \xrightarrow{\phi_{23}} \cdots.$$

Definition 5.6 *A* **scale** *of a dimension group G is a subset D of G^+ which is generating, hereditary, and directed, i.e.,*

(1) *For each $g \in G^+, \exists d_1, \ldots, d_r \in D$ with $g = d_1 + \cdots + d_r$.*

(2) *If $0 \leq g \leq d$ and $d \in D$, then $g \in D$.*

(3) *Given $d_1, d_2 \in D, \exists d_3 \in D$ with $d_1 \leq d_3$ and $d_2 \leq d_3$.*

A **scaled dimension group** *is a dimension group* G *with a distinguished scale.*

Given a dimension group G with an order unit u, then we have a scale $D(u)$ which is a singly generated (by u) hereditary subset of G^+. That is $D(u) = \{g \in G^+ : g \leq u\}$. We shall call $D(u)$ a **unital scale** and with this scale we shall call G a **unital dimension group.**

Another special case is $D = G^+$, we shall call G^+ the **stable scale** of G.

5.2 Dimension Groups and Locally Semisimple Algebras

Letting A be an algebra, we define equivalence classes of idempotents in A. Recall that $e \in A$ is idempotent if $e^2 = e$. Let E_A denote the set of idempotents in A. Given $e, f \in E_A$, we say e is (algebraically) **equivalent** to f, denoted by $e \sim f$, if there exist $x, y \in A$ such that $xy = e$ and $yx = f$.

It is easily verified that "$\sim$ " is reflexive and symmetric.

Suppose that $e, f, g \in E_A$ and $e \sim f$, $f \sim g$. By the definition, we may assume $xy = e$, $yx = f$ and $uv = f$, $vu = g$ for some $x, y, u, v \in A$. Then

$$e = e^2 = (xy)(xy) = xfy = xuvy = (xu)(vy)$$

$$g = g^2 = (vu)(vu) = vfu = vyxu = (vy)(xu)$$

that is,

$$e \sim g.$$

Thus "$\sim$" is an equivalence relation on the set E_A of idempotents in A.

Moreover, if $e_1 \sim e_2$, $f_1 \sim f_2$ and e_i is orthogonal to $f_i (e_i f_i = f_i e_i = 0)$, then $e_1 + f_1 \sim e_2 + f_2$, the additive operation which we shall call **addition** induces a partially defined binary relation in the set of equivalence classes by setting $[e] + [f] = [e' + f']$, where $e \sim e'$, $f \sim f'$ and $e'f' = 0$. This set with such partially defined addition is called the **local semigroup** of A and we denote it by $\Delta(A)$.

The **dimension function** dim is the canonical map from the set E_A of idempotents to the equivalence classes $\Delta(A)$:

$$\begin{aligned} \dim_A : &\quad E_A \longrightarrow \Delta(A) \\ \text{via} : &\quad e \longrightarrow [e] \end{aligned}$$

An algebra A is **semisimple** if A is isomorphic to some direct sum of matrix algebras, i.e.,

$$A \cong \sum_k {}^{\oplus} M_{p_k}(C).$$

A is called **locally semisimple** if A is isomorphic to the inductive limits of a direct system of semisimple finite dimensional algebras.

Lemma 5.7 *Let A be locally semisimple, and let B and C be isomorphic finite dimensional subalgebras of A. Suppose that $\phi : B \to C$ is an isomorphism such that for all idempotents e in B, $dim_A e = dim_A \phi(e)$ in $\Delta(A)$. Then ϕ extends to an automorphism of A.*

Proof : Letting A^1, B^1, C^1 be the unitizations of A, B, C respectively, we may identify A, B, C with their images in A^1, B^1, C^1 respectively. The corresponding map $\phi^1 : B \oplus \mathcal{C} \longrightarrow C \oplus \mathcal{C}$ is given by $\phi^1((b,\lambda)) = (\phi(b),\lambda)$. Observe there are only two kinds of idempotents in A^1, namely (e,0) and (e,1). If $(e,1) \in E_{A^1}$, then by $(e,1)(e,1) = (e,1)$ we have $e^2 + 2e = e$. It follows that $(-e)(-e) = -e$, therefore $(-e,0)$ is in E_{A^1} and is orthogonal to $(0,1)$. It is well known that for any two idempotents in a locally semisimple algebra, e is algebraically equivalent to f if and only if e is homotopically equivalent to f, i.e., there exist $\{e_t\}_{t\in[0,1]}$, such that $t \longrightarrow e_t$ is continuous and $e_0 = e$, $e_1 = f$. Thus if idempotents $e \sim f$ in A, then $(e,0) \sim (f,0)$ and $(-e,1) \sim (-f,0)$ in A^1 since they are homotopiclly equivalent respectively (via $\{(e_t,0)\}$ and $\{(-e_t,1)\}$). Thus

$$\Delta(A^1) = \{[(e,0) : e \in E_A\} \bigcup \{[(-e,1)] : e \in E_A\},$$

where the union is disjoint union. We may Identify E_A and $\Delta(A)$ with subsets of E_{A^1} and $\Delta(A^1)$, for $(e,0) \in E_{A^1}$ we have

$$\dim_{A^1}\phi^1((e,0)) = \dim_{A^1}((e,0)),$$

and,

$$\dim_{A^1}\phi^1((e,1)) = \dim_{A^1}((e,1)),$$

since $\phi(e) = e$ in A.

Thus we may initially assume that A is unital, $1_A \in B$ and $1_A \in C$, and ϕ preserves the identity.

Let $\{e_{ij}^{(k)}\}$ be matrix units in B. Since $e_{11}^{(k)} \sim \phi(e_{11}^{(k)})$, there exist x_0^k and y_0^k in A such that

$$x_0^k y_0^k = e_{11}^{(k)}$$

$$y_0^k x_0^k = \phi(e_{11}^{(k)})$$

Let

$$x = \sum_{k,i} e_{i1}^{(k)} x_0^k \phi(e_{1i}^{(k)})$$

$$y = \sum_{k,i} \phi(e_{i1}^{(k)}) y_0^k e_{1i}^{(k)}.$$

Then

$$xy = \sum_{k,i} e_{ii}^{(k)}$$

$$yx = \phi(\sum_{k,i} e_{ii}^{(k)})$$

Thus $xy = yx = 1_A$. Moreover we have:

$$\begin{aligned} ye_{st}^{(q)}x &= (\sum_{k,i}\phi(e_{i1}^{(k)})y_0^k e_{1i}^{(k)})e_{st}^{(q)}(\sum_{k,i} e_{i1}^{(k)} x_0^k \phi(e_{1i}^{(k)})) \\ &= \phi(e_{s1}^{(q)})y_0^q e_{1t}^{(q)}(\sum_{k,i} e_{i1}^{(k)} x_0^k \phi(e_{1i}^{(k)})) \\ &= \phi(e_{s1}^{(q)})y_0^q e_{1t}^{(q)} e_{t1}^{(q)} x_0^q \phi(e_{1t}^{(q)}) \\ &= \phi(e_{s1}^{(q)})y_0^q e_{11}^{(q)} x_0^q \phi(e_{1t}^{(q)}) \\ &= \phi(e_{st}^{(q)}) \end{aligned}$$

and thus for any $b \in B, ybx = \phi(b)$.

Define

$$\psi : \quad A \longrightarrow A$$

$$\text{via:} \quad \psi(a) = yax$$

Then $\psi|_B = \phi$. ∎

Lemma 5.8 *Suppose that A and B are algebras. Let B_1 be semisimple finite dimensional subalgebra of B. Suppose that $dim_B(E_{B_1})$, as a local semigroup, is isomorphic to a sub local semigroup of $\Delta(A)$. Then there exists a subalgebra A_1 of A and an isomorphism $\phi : A_1 \to B_1$ such that*

$$dim_B \circ \phi = dim_A|_{A_1} .$$

i.e., the diagram

$$\begin{tikzcd} E_{B_1} \arrow[d, "dim_B"'] & E_{A_1} \arrow[l, "\phi"'] \arrow[d, "dim_A"] \\ dim_B(E_{B_1}) \arrow[r, "\theta"] & \Delta(A) \end{tikzcd}$$

commutes.

Proof : Identify $\dim_B(E_{B_1})$ with a sub local semigroup of $\Delta(A)$. Let $\{f^{(1)}, \ldots, f^{(r)}\}$ be a maximal set of orthogonal minimal idempotents in B_1. Then there exist orthogonal minimal idempotents $e^{(1)}, \ldots, e^{(r)}$ in A such that

$$\dim_A e^{(i)} = \dim_B(f^{(i)}). \tag{5.1}$$

If the sum of certain of the $f^{(i)}$'s, say $\sum_{i=1}^{m} f^{(p_i)}$, is the unit of a minimal two-sided ideal F of B_1, then the corresponding $e^{(p_i)}$'s are equivalent, and therefore are contained in a subalgebra D of A which is isomorphic to F. The sum of such subalgebras D, say $D_1, \ldots, D_m$, where each D_i is isomorphic to each minimal two-sided ideal, say F_i, of B, is direct sum.

Thus

$$B_1 = \sum_{i=1}^{m} {}^{\oplus} F_i \text{ and } A_1 = \sum_{i=1}^{m} {}^{\oplus} D_i$$

and together with

$$\phi_i : D_i \longrightarrow F_i$$

defined via

$$\phi_i(e^{(j)}) = f^{(j)} \qquad (e^{(j)} \in D_i).$$

Let

$$\phi = \phi_1 \oplus \cdots \oplus \phi_m$$

By (5.1):

$$\dim_A|_{A_1} = \dim_B \circ \phi.$$

∎

Theorem 5.9 *Let A and B be locally semisimple algebras and $\phi : \Delta(A) \to \Delta(B)$ be isomorphism. Then A and B are isomorphic.*

Proof : Suppose that A, B are inductive limits of $A_1 \subseteq A_2 \subseteq \cdots$, and $B_1 \subseteq B_2 \subseteq \cdots$ respectively. We shall construct sequences

$$A_{m(1)} \subseteq A'_{m(1)} \subseteq A_{m(2)} \subseteq A'_{m(2)} \subseteq \cdots$$

$$B_{n(1)} \subseteq B_{n(2)} \subseteq \cdots$$

with isomorphisms $\psi_i : A'_{m(i)} \to B_{n(i)}$, $i = 1, 2, \ldots$, such that ψ_{i+1} extends ψ_i. The $\psi_i' s$ then extends to $\psi : A \to B$ uniquely as required.

Let $m(1) = 1$. Since $\phi(\dim_A(E_{A_{m(1)}})) \subseteq \Delta(B)$, by lemma 5.8, there exists a subalgebra $B'_{m(1)}$ of B and isomorphism $\phi_1 : A_{m(1)} \to B'_{m(1)}$, such that

$$\dim_B \circ \phi_1 = \phi \circ \dim_A|_{A_1}$$

Choose $n(1)$ such that

$$B'_{m(1)} \subseteq B_{n(1)}.$$

By lemma 5.8 again, there exists a subalgebra $A'_{m(1)}$ of A with an isomorphism $\psi_1 : A'_{m(1)} \to B_{n(1)}$ such that

$$\dim_B \circ \psi_1 = \phi \circ \dim_A|_{A'_{m(1)}} .$$

$$\begin{tikzcd}
A_{m(1)} \arrow[dr, "\phi_1"'] & & & & A'_{m(1)} \arrow[dl, "\psi_1"] \\
& B'_{m(1)} & \subseteq & B_{n(1)} &
\end{tikzcd}$$

Since $\psi_1^{-1} \circ \phi_1$ is a monomorphism from $A_{m(1)}$ into $A'_{m(1)}$ and

$$\begin{aligned} \phi \circ \dim_A|_{A_{m(1)}} &= \dim_B \circ \phi_1 \\ &= (\phi \circ \dim_A|_{A'_{m(1)}} \circ \psi_1^{-1}) \circ \phi_1 \\ &= \phi \circ \dim_A|_{A'_{m(1)}} \circ (\psi_1^{-1} \circ \phi_1). \end{aligned}$$

By lemma 5.7, $\psi_1^{-1} \circ \phi_1 = \rho_1|_{A_{m(1)}}$ for some inner automorphism ρ_1 of A (with the property $\dim_A e = \dim_A \rho_1(e)$ for all $e \in E_A$). We may suppose that $\psi_1^{-1} \circ \phi_1$ is the identity on $A_{m(1)}$ (otherwise replacing $A'_{m(1)}$ by $\rho_1^{-1} A'_{m(1)}$ and ψ_1 by $\psi_1 \rho_1$). That is, $A_{m(1)} \subseteq A'_{m(1)}$ and ψ_1 extends ϕ_1.

Next, choose $m(2) > m(1)$ such that $A'_{m(1)} \subseteq A_{m(2)}$.

Since $\phi(\dim_A(E_{A_{m(2)}})) \subseteq \Delta(B)$, by lemma 5.8, there exists a subalgebra $B'_{m(2)}$ of B with an isomorphism $\phi_2 : A_{m(2)} \to B'_{m(2)}$ such that

$$\dim_B \circ \phi_2 = \phi \circ \dim_A|_{A_{m(2)}}$$

By lemma 5.7, suppose that $\phi_2 \circ \psi_1^{-1}$ extends to σ, an automorphism of B. Replacing ϕ_2 by $\sigma^{-1} \circ \phi_2$, we may suppose that $B_{n(1)} \subseteq B'_{m(2)}$ and ϕ_2 extends ψ_1.

$$\begin{tikzcd}
A_{m(1)} \arrow[dr, "\phi_1"'] & \subseteq & A'_{m(1)} \subseteq A_{m(2)} \arrow[dl, "\psi_1"] \arrow[dr, "\phi_2"'] & & A'_{m(2)} \subseteq A_{m(3)} \arrow[dl, "\psi_2"] & \\
& B'_{m(1)} \subseteq B_{n(1)} & & B'_{m(2)} \subseteq B_{n(2)} & & \cdots
\end{tikzcd}$$

Similarly, we have $B_{n(2)}$, $A'_{m(2)}$, $A_{m(3)}$, and an isomorphism $\psi_2 : A'_{m(2)} \rightarrow B_{n(2)}$ where ψ_2 extends ϕ_2 and also extends ψ_1. Thus the proof is completed by induction. ∎

Remark 5.10 *Let A be a locally semisimple algebra, that is, A is (isomorphic to) the inductive limit of*

$$A_1 \longrightarrow A_2 \longrightarrow \cdots, \tag{5.2}$$

and each A_i is semisimple finite dimensional,

$$A_i \ \cong\ \sum_{k=1}^{r_i}{}^{\oplus} M_{P_{ik}}. \tag{5.3}$$

By Theorem 3.16, each homomorphism ϕ_{ij} can be characterized up to inner equivalence by a canonical homomorphism which can be expressed as a matrix. We still denote this matrix by ϕ_{ij} which also determines a homomorphism $\phi_{ij} : \mathcal{Z}^{r_i} \rightarrow \mathcal{Z}^{r_j}$ by ordinary matrix multiplication. Then 5.2 indues a direct system

$$\sum_{k=1}^{r_1}{}^{\oplus} M_{P_{1k}} \xrightarrow{\phi_{12}} \sum_{k=1}^{r_2}{}^{\oplus} M_{P_{2k}} \xrightarrow{\phi_{23}} \cdots, \tag{5.4}$$

and an associated direct system of simplicial groups:

$$\mathcal{Z}^{r_1} \xrightarrow{\phi_{12}} \mathcal{Z}^{r_2} \xrightarrow{\phi_{23}} \cdots. \tag{5.5}$$

Thus a dimension group arises which is the inductive limit of the above system.

We will give a different approach to dimension groups using K-theory in the next chapter.

Theorem 5.11 *Let A be a locally semisimple algebra. Then $\Delta(A)$, the set of equivalence classes with the partial binary operation "addition", is isomorphic to a scale of a dimension group.*

Proof : If A is finite dimensional, the result is clear. To see this, assume $e \sim f$ in $A \cong \sum_{k=1}^{r}{}^{\oplus} M_{P_k}$, then $e \cong e_1 \oplus \cdots \oplus e_r$ and $f \cong f_1 \oplus \cdots \oplus f_r$. By linear algebra theory, we have $e_j \sim f_j$ for $j = 1, \ldots, r$, and therefore

$$(\text{rank } e_1, \ldots, \text{rank } e_r) = (\text{rank } f_1, \ldots, \text{rank } f_r).$$

On the other hand, if $(\text{rank } e_1, \ldots, \text{rank } e_r) = (\text{rank } f_1, \ldots, \text{rank } f_r)$ we have $e \sim f$. So we can well define an embedding map $\Delta(A) \longrightarrow (\mathcal{Z}^r)^+$ by mapping $[e]$ to $(\text{rank } e_1, \ldots, \text{rank } e_r)$ which is a monomorphism of local semigroups.

In the general case, assume that A is the inductive limit of :

$$A_1 \xrightarrow{\phi_{12}} A_2 \xrightarrow{\phi_{23}} \cdots,$$

where each A_i is semisimple finite dimensional and each ϕ_{ij} is a homomorphism. By restriction, E_A is the inductive limit of:

$$E_{A_1} \longrightarrow E_{A_2} \longrightarrow \cdots.$$

This follows from the fact that an element e in A is an idempotent if it is the image of $e_i \in A_i$ with $e_i^2 = e_i$ for some i.

Thus $\Delta(A)$ is the inductive limit of:

$$\Delta(A_1) \xrightarrow{(\phi_{12})_*} \Delta(A_2) \xrightarrow{(\phi_{23})_*} \cdots.$$

This follows from the fact that if $e, f \in E_A$ with $e \sim f$ then $\exists e_i \sim f_i$ in E_{A_i} for some i such that

$$e = \phi_{i\infty}(e_i), \quad f = \phi_{i\infty}(f_i).$$

But for each i there exists an embedding

$$\Delta(A_i) \longrightarrow \mathbb{Z}^{r_i}$$

of $\Delta(A_i)$ as a scale of simplicial group of $\mathbb{Z}^{r_i}$, which we proved above. Hence the system:

$$\begin{tikzcd}
\Delta(A_1) \arrow[r] \arrow[d] & \Delta(A_2) \arrow[r] \arrow[d] & \cdots \\
\mathbb{Z}^{r_1} \arrow[r] & \mathbb{Z}^{r_2} \arrow[r] & \cdots
\end{tikzcd}$$

showes $\Delta(A)$ can be embeded in the inductive limit of:

$$\mathbb{Z}^{r_1} \longrightarrow \mathbb{Z}^{r_2} \longrightarrow \cdots.$$

It is now straightfoward to check that $\Delta(A)$ is a generating, hereditary and directed subset of the positive cone $(\bigcup_i \phi_{i\infty}(\mathbb{Z}^{r_i})^+)$ of the above inductive limit.

∎

Lemma 5.12 *Let $\phi : G_1 \to G_2$ be a morphism of two simplicial groups and let u_1 and u_2 be two order units of G_1, G_2 respectively. Let $D(u_1)$ and $D(u_2)$ denote the corresponding unital scales. Suppose $\phi(D(u_1)) \subseteq D(u_2)$. Then there exists a morphism $\psi : A_1 \to A_2$ of locally semisimple finite dimensional algebras such that the induced $\psi : \Delta(A_1) \to \Delta(A_2)$ is isomorphic to the restriction $\phi : D(u_1) \to D(u_2)$.*

We give an example instead of a proof. Following this example one can prove the general case easily.

Example 5.13 *Assume $G_1 = G_2 = \mathcal{Z} \oplus \mathcal{Z}, u_1 = (3,2), u_2 = (4,3)$. Then*

$$\begin{aligned} D(u_1) &= \{(x_1, x_2) : x_1 = 0,1,2,3; x_2 = 0,1,2\} \\ D(u_2) &= \{(y_1, y_2) : y_1 = 0,1,2,3,4; y_2 = 0,1,2,3\} \end{aligned}$$

Let

$$\phi = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \text{ with } k_{ij} \geq 0$$

and

$$\forall X \in D(u_1), \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} X = Y \in D(u_2).$$

In particular,

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Let $A_1 = M_3 \oplus M_2$ *and* $A_2 = M_4 \oplus M_3$,

and let $\psi = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ *be the canonical homomorphism from* A_1 *to* A_2.

For any two idempotents $e,\ f \in A_1$, *We have* $e = e_1 \oplus e_2$ *and* $f = f_1 \oplus f_2$ *with* $e_1,\ f_1 \in M_3$ *and* $e_2,\ f_2 \in M_2$. *Then* $e \sim f$ *if and only if* $e_1 \sim f_1$ *and* $e_2 \sim f_2$ *if and only if* $(\mathit{rank}\ e_1, \mathit{rank}\ e_2) = (\mathit{rank}\ f_1, \mathit{rank}\ f_2)$.

Similarly, we have analogous results for A_2.

Therefore, $\Delta(A_1) \cong D(u_1)$ *and* $\Delta(A_2) \cong D(u_2)$. *To show* $\Delta(A_1) \xrightarrow{\psi} \Delta(A_2)$ *is isomorphic to* $D(u_1) \xrightarrow{\phi} D(u_2)$, *we must show that the diagram*

$$\begin{tikzcd}
\Delta(A_1) \arrow[r, "\psi"] \arrow[d, "\cong"'] & \Delta(A_2) \arrow[d, "\cong"'] \\
D(u_1) \arrow[r, "\phi"] & D(u_2)
\end{tikzcd}$$

commutes.

Briefly, let $[e] = [e_1 \oplus e_2] \in \Delta(A_1)$. *Then*

$$\begin{aligned}\psi([e]) = \quad & [(\overbrace{e_1 \oplus \cdots \oplus e_1}^{k_{11}} \oplus \overbrace{e_2 \oplus \cdots \oplus e_2}^{k_{12}} \oplus 0_{h_1}) \\ & \oplus(\overbrace{e_1 \oplus \cdots \oplus e_1}^{k_{21}} \oplus \overbrace{e_2 \oplus \cdots \oplus e_2}^{k_{22}} \oplus 0_{h_2})].\end{aligned}$$

The image of $\psi([e])$ *in* $D(u_2)$ *is*

$$(k_{11}\mathit{rank}\ e_1 + k_{12}\mathit{rank}\ e_2, k_{21}\mathit{rank}\ e_1 + k_{22}\mathit{rank}\ e_2)$$

which just is $\phi(\mathit{rank}\ e_1, \mathit{rank}\ e_2)$. *Therefore the above diagram commutes.*

Lemma 5.14 *Let A_1 and A_2 be semisimple finite dimensional algebras. Let $\phi : \Delta(A_1) \to \Delta(A_2)$ be a morphism. Then ϕ extends to a morphism $A_1 \to A_2$.*

This is seen in the same way as preceding lemma.

Theorem 5.15 *Let G be a scaled dimension group with scale D. Then there exists a locally semisimple algebra A such that $D \cong \Delta(A)$.*

Proof : Suppose that G is the inductive limit of the system:

$$\mathcal{Z}^{r_1} \xrightarrow{\phi_{12}} \mathcal{Z}^{r_2} \xrightarrow{\phi_{23}} \cdots,$$

where the morphisms are positive.

Since D is hereditary and (upward) directed, we may choose order units u_i in $\mathcal{Z}^{r_i}$ for each i such that the unital scale $D(u_{i+1})(\subseteq (\mathcal{Z}^{r_{i+1}})^+)$ contains the image of $D(u_i)$ and such that $\cup\phi_{i\infty}(D(u_i)) = D$.

For details, let $D\backslash\{0\} = \{d_1, d_2, \ldots\}$, by (3) of Definition 5.6, we have $c_1 \in D$ with $d_1 \leq c_1$ and for c_1, d_2, we can choose $c_2 \in D$ with $c_1 \leq c_2$ and $d_2 \leq c_2$. By induction we may select $c_1, c_2, \ldots \in D$ with $d_i \leq c_i$ and $c_1 \leq c_2 \leq \cdots$. We define a sequence $i(k) \in \mathcal{N}$, $v_k, u_k \in (\mathcal{Z}^{r_{i(k)}})^+$ as follows. Since $d_1 \leq c_1$ there exist $i(1)$ and $v_1, u_1 \in \mathcal{Z}^{r_{i(1)}}$, $0 \leq v_1 \leq u_1$, with $v_1 \to d_1$, $u_1 \to c_1$ ($\phi_{i(1)\infty}(v_1) = d_1$ and so on). Since $d_2 \leq c_2$ and $c_1 \leq c_2$, there exist $i(2) \geq i(1)$ and $v_2, u_2 \in \mathcal{Z}^{r_{i(2)}}$, $0 \leq v_2 \leq u_2$, $u_1' \leq u_2 (u_1' = \phi_{i(1),i(2)}(u_1))$, with $v_2 \to d_2$, $u_2 \to c_2$.

By induction, assume we have $u_k, v_k \in \mathcal{Z}^{r_{i(k)}}$ with $0 \leq v_k \leq u_k$, $u'_{k-1} \leq u_k$ and $v_k \rightarrow d_k$, $u_k \rightarrow c_k$. Since $d_{k+1} \leq c_{k+1}$ and $c_k \leq c_{k+1}$, there exist $i(k+1) \geq i(k)$ and $u_{k+1}, v_{k+1} \in \mathcal{Z}^{r_{i(k+1)}}$ with $0 \leq v_{k+1} \leq u_{k+1}$, $u'_k \leq u_{k+1}$(u'_k is image of u_k in $\mathcal{Z}^{r_{i(k+1)}}$). Changing notations, we may assume $u_i, v_i \in \mathcal{Z}^{r_i}$.

Thus we have a unital scale $D(u_i)$ of $\mathcal{Z}^{r_i}$ and $\phi_{ij}(D(u_i)) \subseteq D(u_j)(i \leq j)$, and $\cup\phi_{n\infty}(D(u_i)) = D$ since $D \setminus \{0\} = \{d_1, d_2, \ldots\}$ are contained in $\cup\phi_{n\infty}(D(u_i))$.

By Lemma 5.12, there exist morphisms

$$A_1 \longrightarrow A_2,\ A'_2 \longrightarrow A_3,\ A'_3 \longrightarrow A_4,\ \ldots$$

of semisimple finite dimensional algebras such that induced morphisms

$$\Delta(A_1) \longrightarrow \Delta(A_2),\ \Delta(A'_2) \longrightarrow \Delta(A_3),\ \Delta(A'_3) \longrightarrow \Delta(A_4), \ldots$$

are isomorphic to

$$D(u_1) \longrightarrow D(u_2),\ D(u_2) \longrightarrow D(u_3),\ D(u_3) \longrightarrow D(u_4), \ldots,$$

the above isomorphisms in particular determines:

$$\Delta(A_2) \longrightarrow \Delta(A'_2),\ \Delta(A_3) \longrightarrow \Delta(A'_3), \ldots.$$

By Lemma 5.14, we have morphisms

$$A_2 \longrightarrow A'_2,\ A_3 \longrightarrow A'_3,\ A_4 \longrightarrow A'_4, \ldots.$$

Thus we have a long sequence

$$A_1 \longrightarrow A_2 \longrightarrow A_2' \longrightarrow A_3 \longrightarrow A_3' \longrightarrow A_4 \longrightarrow \cdots,$$

inducing the given morphisms

$$\Delta(A_1) \longrightarrow \Delta(A_2) \longrightarrow \Delta(A_2') \longrightarrow \Delta(A_3) \longrightarrow \Delta(A_3') \longrightarrow \Delta(A_4) \cdots$$

The subsequence

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow \cdots,$$

induces a system:

$$\Delta(A_1) \longrightarrow \Delta(A_2) \longrightarrow \Delta(A_3) \longrightarrow \Delta(A_4) \longrightarrow \cdots,$$

that is isomorphic to

$$D(u_1) \longrightarrow D(u_2) \longrightarrow D(u_3) \longrightarrow D(u_4) \longrightarrow \cdots$$

of which the inductive limit is just D.

Letting $A = \varinjlim A_n$ we have that $\Delta(A)$ is isomorphic to D. This is required.

∎

Remark 5.16 *An ordered group G is* **unperforated** *if $nx \geq 0$ for $x \in G$ and some $n \in \mathcal{N}$ implies $x \in G^+$. G has the* **Riesz Interpolation Property** *if $x_1, x_2, y_1, y_2 \in G$ with $x_1, x_2 \leq y_1, y_2$, then there is a $z \in G$ with $x_1, x_2 \leq z \leq y_1, y_2$.*

In [10], it is proved:

Theorem 5.17 *An ordered group is a dimension group if and only if it is countable, unperforated, and has the Riesz Interpolation Property.*

Reference: [12], [14], [27].

Chapter 6

Basic K_0-Theory of C*-Algebras

Informally, K-theory can be described as the linear algebra of very large matrices. In C*-algebraic K-theory, one considers matrix algebras over a C*-algebra. The basic concepts of this theory are functors $K_0(A)$, $K_1(A)$ from the category of C*-algebras A to the category of abelian groups which reflect some of properties of A.

In the first section of this chapter, we present the definition and basic properties of Grothendieck groups. In the second we prove some elementary results concerning $K_0(A)$. In the third section we show how AF-algebras can be classified.

6.1 Grothendieck Group

Definition 6.1 *Let S be an abelian semigroup with identity 0, the* **Grothendieck group** *of S is a group $G(S)$ together with a homomorphism $\theta : S \to G$ such* *that*

1. *$\theta(S)$ generates G,*
2. *any homomorphism $S \xrightarrow{\psi} H$ of S into a group H extends uniquely to a homomorphism $G(S) \xrightarrow{\lambda} H$ such that the diagram*

$$\begin{tikzcd}
S \arrow[rr, "\psi"] \arrow[dr, "\theta"'] & & H \\
& G(S) \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

commutes.

We shall show that Grothendieck group $G(S)$ consists of equivalence classes of formal difference, $x - u$, of elements $x, u \in S$.

Definition 6.2 *Given x, u and y, v in an abelian semigroup S, two formal difference $x - u, y - v$ are said to be* **equivalent**, *(denoted by $x - u \sim y - v$), if there exists $r \in S$ such that*

$$x + v + r = y + u + r$$

This relation is obviously symmetric and reflexive. Furthermore, if

$$x - u \sim y - v \text{ and } y - v \sim z - w$$

then we have

$$x + v + r = y + u + r$$

$$y + w + s = z + v + s$$

for some $r, s \in S$. It follows that:

$$x + w + (v + r + s) = y + w + u + s = z + u + (v + r + s)$$

and hence $x - u \sim z - w$. We conclude that "$\sim$" is transitive and, hence, is an equivalent relation.

Let $G(S)$ be the set of equivalence classes of formal differences $x - u$, then $G(S)$ inherits an abelian operation from S (via:$(x - u) + (y - v) = (x + y) - (u + v)$) under which it becomes a group.

In fact, all $x - x$, $x \in S$, belong to a single equivalence class in $G(S)$, and since

$$(y - v) + (x - x) = (y + x) - (v + x) \sim y - v$$

$x - x$ is an identity for $G(S)$.

For any $x - u$, the element $u - x$ is its inverse in $G(S)$. Therefore, $G(S)$ is a group.

For $x \in S$, $x - 0 \in G(S)$, thus we can define a map θ:

$$\theta : S \longrightarrow G(S)$$

since $(x - 0) + (y - 0) = (x + y) - 0$, θ is a homomorphism.

The following property of $G(S)$ which we construct above shows that $G(S)$ admits the universal property of Definition 6.1, thus $G(S)$ is just the Grothendieck group of S.

Proposition 6.3 *If $\psi : S \to H$ is any homomorphism of S into a group H,* *then there is a unique (up to isomorphism) homomorphism* $\lambda : G(S) \to H$ *such that* $\psi = \lambda \circ \theta$, *i.e., the diagram*

$$\begin{tikzcd}
S \arrow[rr, "\psi"] \arrow[dr, "\theta"'] & & H \\
& G(S) \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

commutes.

Proof : Given $\psi : S \to H$ and $x - u \in G(S)$ define

$$\begin{aligned} \lambda :\ & G(S) \longrightarrow H \\ by :\ & (x-u) \longrightarrow \psi(x) + (-\psi(u)). \end{aligned}$$

If $x - u \sim y - v$, then $x + v + r = y + u + r$ for some $r \in S$. Thus $\psi(x+v+r) = \psi(y+u+r)$. We have then $\psi(x) - \psi(u) = \psi(y) - \psi(u)$. That is $\lambda(x-u) = \lambda(y-v)$. λ is well defined and the desired univeral property follows.

∎

6.2 K_0 Functor

6.2.1 Equivalence Relations

Given unital C*-algebra A, let $M_n(A)$ denote the C*-algebra of all $n \times n$ matrices over A.

Definition 6.4 *If e and f are idempotents in $M_n(A)$, we say e and f are* **Murray - von Neumann equivalent** *(denoted by $e \simeq f$) if $\exists v, w \in M_n(A)$ so that $vw = e, wv = f$.*

It is well-known that for any idempotent e in a C*-algebra A (not necessarily unital) there exists a projection $e' \in A$ which is (Murray-von Neumann) equivalent to e ([24]).

For the **unital** C*-algebra, let $P_n(A)$ be the set of projections in $M_n(A)$. We have natural injections and injective homomorphisms:

$$P_1(A) \longrightarrow \cdots \longrightarrow P_n(A) \longrightarrow P_{n+1}(A) \longrightarrow \cdots$$

$$M_1(A) \longrightarrow \cdots \longrightarrow M_n(A) \longrightarrow M_{n+1}(A) \longrightarrow \cdots$$

defined by

$$e \longrightarrow e \oplus 0$$

$$a \longrightarrow a \oplus 0$$

respectively.

Let $P(A) = \bigcup_{n \in \mathcal{N}} P_n(A)$, $M_\infty(A) = \bigcup_{n \in \mathcal{N}} M_n(A)$. If $a \in M_n(A), b \in M_m(A)$, we define

$$a \oplus b = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_{n+m}(A) \subseteq M_\infty(A)$$

Definition 6.5 *For two projections e and f in $M_\infty(A)$, e is* **stably equivalent** *to f (written $e \sim f$) if $e \oplus I_n$ is Murray-von Neumann equivalent to*

$f \oplus I_n$ for some $n \in \mathcal{N}$. That is, if there exist v and w in $M_\infty(A)$ such that $vw = e \oplus I_n$ and $wv = f \oplus I_n$.

Considering

$$P_1(A) \longrightarrow P_2(A) \longrightarrow \cdots$$

if $e \sim f$, then $\exists v, w \in M_\infty(A)$ such that

$$vw = e \oplus I_k \quad wv = f \oplus I_k \qquad \text{for some } k.$$

then

$$\begin{aligned} (v \oplus 0)(w \oplus 0) &= (e \oplus I_k) \oplus 0, \\ (w \oplus 0)(v \oplus 0) &= (f \oplus I_k) \oplus 0. \end{aligned}$$

By linear algebra, there exists an $n \times n$ elementary matrix e' (n large enough) such that $e'e' = I_{n\times n}$ and $e'((e \oplus I_k) \oplus 0)e' = (e \oplus 0) \oplus I_k$. Identifying e' with it image in $M_\infty(A)$ and letting $v' = e'(v \oplus 0)e'$, $w' = e'(w \oplus 0)e'$ we have

$$(e \oplus 0) \sim (f \oplus 0).$$

Thus, the injective maps preserve equivalence classes. However, the maps need not be injective module equivalence. That is, $P_n(A)/\sim \longrightarrow P(A_{n+1})/\sim$ need not be injective.

Proposition 6.6 *Let A be a C^*-algebra and let e, f be two projections in $M_\infty(A)$, then $e \simeq f$ if and only if there exists a unitary $u \in M_\infty(A)$ such that $Ad\ u(e) = f$.*

Proof : Sufficiency. Let $v = ue$, $w = u^*f$. Then

$$vw = ueu^*f = ff = f$$

$$wv = u^*fue = u^*(ueu^*)ue = e.$$

Thus

$$e \simeq f.$$

Necessity. Assume $vw = e$ and $wv = f$ for some $v,\ w \in M_k(A)$. Let $x = evf$, $y = fwe$. Then

$$xy = e, yx = f$$

and

$$exf = xf = ex = x,\ fye = fy = ye = y.$$

Now $x^*x = fx^*xf \in fAf$, and similarly $yy^* \in fAf$ (a C*-algebra with unit f).

Moreover:

$$(x^*x)(yy^*) = x^*ey^* = x^*y^* = f^* = f,$$

$$(yy^*)(x^*x) = y(xy)^*x = yex = yx = f.$$

So that x^*x has inverse yy^* in fAf.

Let $|y^*| = (yy^*)^{\frac{1}{2}}$ (by functional calculus) which is the inverse in fAf of $|x| = (x^*x)^{\frac{1}{2}}$. Let $u_1 = x|y^*|$. Then

$$\begin{aligned} u_1^*u_1 &= |y^*|^*x^*x|y^*| \\ &= |y^*|^*|x|^2|y^*| \\ &= (|y^*|^*|x|)(|x||y^*|) \\ &= f^*f \\ &= f \end{aligned}$$

and similarly,

$$u_1u_1^* = e.$$

Let $u_2 = eu_1f$ and then $u_2^* = fu_1^*e$, we have :

$$u_2u_2^* = e, \; u_2^*u_2 = f$$

and

$$eu_2f = u_2, \; fu_2^*e = u_2^*.$$

Let

$$u = \begin{bmatrix} u_2^* & 1-e \\ 1-f & u_2 \end{bmatrix} \in M_{2k}(A)$$

then

$$u^* = \begin{bmatrix} u_2 & 1-f \\ 1-e & u_2^* \end{bmatrix}$$

and $u(e \oplus 0)u^* = f \oplus 0$. ∎

Corollary 6.7 *Let A be a C^*-algebra, let $e, f \in P(A)$. Then $e \sim f$ if and only if there exists a unitary $u \in M_\infty(A)$ such that $Ad\, u(e \oplus I_n) = f \oplus I_n$ for some n.*

Definition 6.8 *Let A be a unital C^*-algebra, and let e, $f \in P_n(A)$. e and f are* **unitarily equivalent** *if there is a unitary $u \in M_n(A)$ with $Ad\, u(e) = f$.*

According to the remark after Definition 6.4 and to Corollary 6.7 we shall restrict our attention to projections instead of general idempotents, to unitaries instead of general invertible elements sine both two approaches give the same results.

6.2.2 The Definition of K_0

Let $[e]$ denote the stable equivalence class of the projection e in $M_\infty(A)$. We now consider the set of equivalence classes of projections in $M_\infty(A)$ subject to addition defined as

$$\begin{aligned} [e] + [f] &= [\mathrm{diag}(e, f)] &&= [e \oplus f] \\ &= [e' \oplus f'] && \text{if } e' \sim e, f' \sim f. \end{aligned}$$

Now, letting $P(A)/\!\sim := \{[e] | \, e \in P(A)\}$, we have an abelian semigroup under addition defined above. From this semigroup we then construct the Grothendieck group of $P(A)/\!\sim$. This gives us the abelian group consisting of all formal difference $[e] - [f]$ of equivalence classes of projections in $P(A)$. We denote this group by $K_0(A)$. Hence we have:

Definition 6.9 *For a unital C^*-algebra A, $K_0(A)$ is the Grothendieck group of the semigroup of stable equivalence classes of projections in $M_\infty(A)$.*

Thus, elements of $K_0(A)$ for A unital can be viewed as formal differences $[e]-[f]$ for e and f in $P(A)$.

Given a unital homomorphism $\phi : A \to B$ from a unital C*-algebra A into a unital C*-algebra B, it induces homomorphisms $\phi_n : M_n(A) \to M_n(B)$ by $[a_{ij}] \to [\phi(a_{ij})]$, which in turn restricts to a map

$$\phi : P(A) \longrightarrow P(B)$$

Since ϕ preserves equivalence and direct sums, it defines a homomorphism $[\phi] : P(A)/\!\sim \longrightarrow P(B)/\!\sim$ and thus defines a group homomorphism

$$\phi_* = K_0([\phi]) : K_0(A) \longrightarrow K_0(B)$$

satisfying

$$\phi_*(\dim e) = \dim \phi(e).$$

Thus K_0 may be regarded as a functor from unital C*-algebras to abelian groups.

In the case where A is not necessarily unital, let A^1 denote A with a unit adjointed and consider the unital extension

$$0 \longrightarrow A \xrightarrow{i} A^1 \xrightarrow{\phi_0} C \longrightarrow 0$$

where

$$\begin{aligned} i \quad &: A \longrightarrow A^1 \quad \text{via: } i(a) = (a,0) \\ \phi_0 \quad &: A^1 \longrightarrow \mathcal{C} \quad \text{via: } \phi_0((a,\lambda)) = \lambda. \end{aligned}$$

Now ϕ_0 indues a map

$$(\phi_0)_\infty : M_\infty(A^1) \longrightarrow M_\infty(\mathcal{C}).$$

thus ϕ_0 induces

$$\begin{aligned} (\phi_0)_* \quad &: K_0(A^1) \longrightarrow K_0(\mathcal{C}), \\ \text{by} \quad &: (\phi_0)_*([e] - [f]) = [(\phi_0)_\infty(e)] - [(\phi_0)_\infty(f)]. \end{aligned}$$

Let $\tilde{K}_0(A) := \ker(\phi_0)_*$, we now define the K_0 group for A non-unital.

Definition 6.10 *For A not necessarily unital*

$$K_0(A) := \tilde{K}_0(A).$$

we observe below that when A is unital, the tow definitions of $K_0(A)$ are naturally isomorphic. The natural homomorphism $j : \mathcal{C} \to A^1$ defined by $j(\lambda) = (0, \lambda)$ induces $K_0(\mathcal{C}) \xrightarrow{j_*} K_0(A^1)$.

And similarly we have

$$K_0(A^1) \xrightarrow{(\phi_0)_*} K_0(\mathcal{C})$$

And since $\mathcal{C} \xrightarrow{j} A^1 \xrightarrow{\phi_0} \mathcal{C}$ is the identity. So

$$K_0(\mathcal{C}) \xrightarrow{j_*} K_0(A^1) \xrightarrow{(\phi_0)_*} K_0(\mathcal{C})$$

is the identity map, hence

$$K_0(A^1) \cong \tilde{K_0}(A) \oplus K_0(\mathcal{C}).$$

And so, we have for A unital

$$\begin{aligned}\tilde{K_0}(A) &= \ker(\phi_0)_* \\ &= \ker((\phi_0)_* : K_0(A \oplus \mathcal{C}) \longrightarrow K_0(\mathcal{C})) \\ &\cong \ker(K_0(A) \oplus K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{C})) \\ &= K_0(A), \quad \text{in the unital sense.}\end{aligned}$$

This $K_0(A)$ defined by $\tilde{K_0}(A)$ is called reduced K_0. However, because of $\tilde{K_0}(A) = K_0(A)$ for unital A, we continue to use simple $K_0(A)$ where it is understood to which definition we refer, based on the algebra A. No confusion should arise.

Example 6.11 *$K_0(\mathcal{C}) = \mathcal{Z}$ and $K_0(M_n(\mathcal{C})) = \mathcal{Z}$.*

Let $A = M_n(\mathcal{C})$, hence we have the natural isomorphism

$$M_\infty(A) = M_\infty(M_n(\mathcal{C})) \cong M_\infty(\mathcal{C}).$$

Then if $e, f \in P(A)$ with $e \sim f$, then $e \oplus I_n$ is Murray-von Neumann equivalent to $f \oplus I_n$ for some n. Hence by linear algebra, we have

$$\begin{aligned}\mathit{rank}\ (e \oplus I_n) &= \mathit{rank}\ e + n \\ \mathit{rank}\ (f \oplus I_n) &= \mathit{rank}\ f + n\end{aligned}$$

and since rank $(e \oplus I_n) =$ *rank* $f \oplus I_n$,

$$rank\ e = rank\ f.$$

Also, if rank $e =$ *rank* f, *then we can construct matrices* v *and* w *such that* $vw = e$ *and* $wv = f$ *so that* $e \sim f$.

Hence projections in $M_\infty(A)$ *are equivalent if and only if they have the same rank. Thus we may well define a homomorphism*

$$\begin{aligned} dim: &\ \ K_0(M_n(\mathcal{C})) \longrightarrow \mathcal{Z} \\ by &\ \ [e] - [f] = rank\ (e) - rank\ (f). \end{aligned}$$

The map dim is injective, since if dim$([e]-[f]) = 0$ *then rank* $(e) =$ *rank* (f) *and* $e \sim f$, *i.e.,* $[e]-[f] = 0$. *dim is obviously surjective since* $1 =$ *rank*$([1_1])$. *Thus we have that*

$$K_0(M_n(\mathcal{C})) \cong \mathcal{Z},$$

and

$$K_0(\mathcal{C}) \cong \mathcal{Z}.$$

Similarly we have

$$K_0(\sum_{k=1}^{r}{}^{\oplus} M_{p_k}(\mathcal{C})) \cong \mathcal{Z}^r.$$

Motivated by this example, we shall define a dimension function for a general C*-algebra A and we could regard $K_0(A)$ as a "generalized dimension" group.

6.2.3 The Dimension Function

Let A be a unital C*-algebra, for any projection $e \in P_n(A)$, we have

$$e \oplus 0_\infty \in P(A)$$

so

$$\theta([e \oplus 0_\infty]) = [e \oplus 0_\infty] - [0] \in K_0(A).$$

where θ is the natural map from $P(A)/\sim$ into $K_0(A)$ as in Definition 6.1.

We define the dimension function

$$\begin{aligned} \dim :\ & \textstyle\bigcup_n P_n(A) \longrightarrow K_0(A) \\ \text{via} :\ & e \longrightarrow \theta([e \oplus 0_\infty]) \end{aligned}$$

If A is not unital, considering the short exact sequence:

$$0 \longrightarrow A \xrightarrow{i} A^1 \xrightarrow{\phi_0} C \longrightarrow 0,$$

it reduces:

$$0_\infty \longrightarrow M_\infty(A) \xrightarrow{i_\infty} M_\infty(A^1) \xrightarrow{(\phi_0)_\infty} M_\infty(C) \longrightarrow 0_\infty,$$

and

$$0 \longrightarrow K_0(A) \xrightarrow{i_*} K_0(A^1) \xrightarrow{(\phi_0)_*} K_0(C) \longrightarrow 0,$$

where $K_0(A) = \tilde{K}_0(A)$. For $e = [e_{ij}] \in P_n(A)$,

$$i_n(e) = [(e_{ij}, 0)] \in P_n(A),$$

then $(\phi_0)_n \circ i_n(e) = 0$(thus dim $P(A) \subseteq K_0(A)$).

It follows then that $\theta(i_\infty[e\oplus 0_\infty])$ which is in $K_0(A^1)$, is actually in $K_0(A)$. We may define the dimension function dim the same as for unital algebras:

$$\begin{aligned} \dim: &\quad \cup_n P_n(A) \longrightarrow K_0(A) \\ \text{by}: &\quad e \longrightarrow \theta(i_\infty[e\oplus 0_\infty]). \end{aligned}$$

6.2.4 K_0 as a Functor

Given arbitrary C*-algebras A, B and a homomorphism $\phi : A \to B$ we have a unital homomorphism

$$\begin{aligned} \phi^1: &\quad A^1 \longrightarrow B^1 \\ \text{by}: &\quad (a,\lambda) \longrightarrow (\phi(a),\lambda). \end{aligned}$$

Since the diagram

$$\begin{tikzcd} A^1 \arrow[r, "\phi_0"] \arrow[d, "\phi^1"'] & C \arrow[d, "\text{id}"] \\ B^1 \arrow[r, "\phi_0"] & C. \end{tikzcd}$$

is commutative, we have

$$\begin{tikzcd} K_0(A^1) \arrow[r] \arrow[d, "\phi^1_*"'] & \mathbb{Z} \arrow[d, "\text{id}"] \\ K_0(B^1) \arrow[r] & \mathbb{Z}. \end{tikzcd}$$

Hence ϕ^1_* restricts to a homomorphism

$$\phi_* : K_0(A) \longrightarrow K_0(B).$$

Thus K_0 may be regarded as a functor from (not necessarily unital) C*-algebras to abelian groups.

6.2.5 The Order Structure of $K_0(A)$

Let A be a C*-algebra, define

$$K_0(A)^+ := \dim(P(A))$$

and let

$$\begin{aligned}\Delta(A) = &\ \dim(P_1(A)) \\ = &\ \{\dim e : e \in P_1(A)\} \subseteq K_0(A).\end{aligned}$$

Generally speaking, $K_0(A)^+$ does not satisfy the Definition 5.1 (cf. [2], 6.3). For instance, if $A = \mathbf{C}_0(R^2)$, then $K_0(A) = \mathcal{Z}$ and $K_0(A)^+ = \{0\}$. Furthermore, an ordered group should be torsion free but $K_0(A)$ could have torsion[1]. But, we still say that a homomorphism $\phi : K_0(A_1) \to K_0(A_2)$ is positive if $\phi(K_0(A_1)^+) \subseteq K_0(A_2)^+$, where A_1, A_2 are two C*-algebras. And we still call $\Delta(A)$ a scale of $K_0(A)$.

However in particular cases, $K_0(A)^+$ can be a positive cone on $K_0(A)$. Here is an Example:

---

[1] For example, for the high projective space $\mathcal{R}P^3 = S^3/\sim$, where

$$S^3 = \{(x, y, z, t) \in \mathcal{R}^4 : x^2 + y^2 + z^2 + t^2 = 1\}$$

and $\vec{x} \sim \vec{y}$ iff $\vec{x} = \pm\vec{y}$, $K_0(\mathrm{C}(\mathcal{R}P^3)) \cong \mathcal{Z} \oplus \mathcal{Z}_2$ ([24]).

Example 6.12 *The $K_0(\mathcal{C})^+$, $K_0(M_n(\mathcal{C}))^+$ are ordinary ordering on $\mathcal{Z}$.*

$$\begin{aligned} K_0(M_n(\mathcal{C}))^+ &= dim\{e : e \in P(A)\} \\ &= \{\theta([e]) : e \in P(A)\} \\ &\cong \{rank([e]) : e \in P(A)\} \\ &= \mathcal{Z}^+. \end{aligned}$$

In particular,

$$K_0(\mathcal{C})^+ \cong \mathcal{Z}^+.$$

∎

Similarly, we have:

$$K_0(\sum_{k=1}^{r}{}^{\oplus} M_{p_k}(\mathcal{C}))^+ \cong (\mathcal{Z}^r)^+.$$

Remark 6.13 *If A is an AF algebra with $A = \overline{\bigcup A_n}$, then $\Delta(A)$ is isomorphic to $\Delta(\bigcup A_n)$. In fact, given $\varepsilon > 0$ and a projection in A, since A is the norm closure of $\bigcup A_n$ and by Lemma A.1, there exists a projection f in $\bigcup A_n$ such that*

$$\|e - f\| < \varepsilon.$$

If $e_1 \sim e_2$, there exists $u \in M_k(A)$ for some k with $ad\, u(e_1) = e_2$ (in $M_k(A)$). Choose f_1, f_2 in $\bigcup A_n$ with $\|f_1 - e_1\| < \varepsilon$, $\|f_2 - e_2\| < \varepsilon$. We may assume that $ad\, u(f_1)$ is orthogonal to f_2 (otherwise we can choose a $2k \times 2k$ unitary *matrix* w *such that* $w(uf_1u^* \oplus 0_k)w^* = 0_k \oplus uf_1u^*$*). Now, if* $\varepsilon < \frac{1}{2}$

$$\begin{aligned} \|ad\ u(f_1) - f_2\| = & \ \|uf_1u^* - f_2\| \\ = & \ \|uf_1u^* - ue_1u^* + ue_1u^* - f_2\| \\ = & \ \|uf_1u^* - ue_1u^* + e_2 - f_2\| \\ = & \ \|uf_1u^* - ue_1u^*\| + \|e_2 - f_2\| \\ < & \ 2\varepsilon < 1. \end{aligned}$$

Thus ad $(f_1) = f_2$*. On the other hand, if* f_1 *is unitarily equivalent to* f_2 *in* $M_k(\bigcup A_n)$*, then they are unitarily equivalent in* $M_k(A)$*. Therefore* $\Delta(A) \cong \Delta(\bigcup A_n)$*.*

6.3 An Application to AF Algebras

Our task in this section is to show that the "dimension group" which arose in the preceding chapter is actually just the K_0 group. Using K-theory, dimension groups for AF algebras may be easily computed. The main result of this section is that any two AF algebras A and B are isomorphic if and only if there is an isomorphism from $K_0(A)$ onto $K_0(B)$ carrying $\Delta(A)$ onto $\Delta(B)$.

Let A be an AF algebra with $A = \overline{(\cup A_n)}$. Then the system

$$A_1 \xrightarrow{\phi_{12}} A_2 \xrightarrow{\phi_{23}} \cdots$$

determines

$$K_0(A_1) \xrightarrow{\sigma_{12}} K_0(A_2) \xrightarrow{\sigma_{23}} \cdots,$$

where $\sigma_{ij} = (\phi_{ij})_*$. And the $A_n \xrightarrow{\phi_{n\infty}} A$ determines $K_0(A_n) \xrightarrow{(\phi_{n\infty})_*} K_0(A)$ for each n. Thus we have a unique homomorphism λ:

$$\begin{tikzcd}
K_0(A_n) \arrow[rr, "(\phi_{n\infty})_*"] \arrow[dr, "\sigma_{n\infty}"'] & & K_0(A) \\
& \varinjlim K_0(A_n). \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

We shall prove that λ is an isomorphism.

Lemma 6.14 *If A is a unital AF algebra with unital approximating system $\{A_n\}$, then given projections e, $f \in A_n$ for some n and a unitary $u \in A$ with $Ad\, u(e) = f$, there exist $A_m \supseteq A_n$ and a unitary $w \in A_m$ with $Ad\, w(e) = f$.*

Proof : For the e, f and u in the condition, we choose $A_m \supseteq A_n$ and a unitary $v \in A_m$ with $\|u - v\| < \frac{1}{2}$. Then,

$$\begin{aligned}
\|\mathrm{Ad}\ v(e) - f\| &= \|\mathrm{Ad}\ v(e) - \mathrm{Ad}\ u(e)\| \\
&= \|vev^* - ueu^*\| \\
&\leq \|vev^* - veu^* + veu^* - ueu^*\| \\
&\leq \|ve\|\|v^* - u^*\| + \|v - u\|\|eu^*\| \\
&< 1,
\end{aligned}$$

and since Ad $v(e)$ and f are both projections in A_m, by Lemma A.3 there is a unitary $v' \in A_m$ with

$$\mathrm{Ad}\ (v'v)(e) = \mathrm{Ad}\ v'(\mathrm{Ad}\ v(e)) = f.$$

Let $w = v'v$ and we complete the proof. ∎

Lemma 6.15 *If A is a unital AF algebra such that e, f and g are projections in A with $e \perp g$, $f \perp g$, and $e+g$ unitarily equivalent to $f+g$, then e is unitarily equivalent to f.*

Proof : Since $C^*(e, g, 1)$, $C^*(f, g, 1)$ are finite dimensional, by Lemma 4.3, we may choose an integer n and unitaries u, $v \in A$ such that

$$\text{Ad } u(e+g) \in A_n, \quad \text{Ad } v(f+g) \in A_n.$$

Since Ad $u(e)$+Ad $u(g)$ and Ad $v(f)$+Ad $v(g)$ are unitarily equivalent in A_n, by Lemma 6.14 they are unitarily equivalent in A_m for some $m \geq n$. But unitary equivalence is determined by the rank in a matrix algebra, it is clear that Ad $u(e)$ and Ad $v(f)$ are unitarily equivalent in A_m, and thus e and f are unitarily equivalent in A. ∎

Theorem 6.16 *If A is an AF algebra with approximating system $\{A_n\}$, the map*

$$\lambda : \varinjlim K_0(A_n) \longrightarrow K_0(A)$$

is an isomorphism.

Proof : First, assume that A is unital.

Given $g \in P(A)$, then since dim is onto, $g = \dim_A e$ with some $e \in P_k(A)$ for some k. Since $M_k(A)$ is the inductive limit of $\{M_k(A_n)\}$, by

Lemma 6.14, there exist a unitary $u \in A$ and integer n such that:

$$\text{Ad } u(e) \in M_k(A_n).$$

Thus e is stably equivalent to a projection (still denoted by e) in $M_k(A_n)$, i.e., $\dim_{A_n} e \in K_0(A_n)$.

We have

$$\dim_A e = (\phi_{n\infty})_* \circ \dim_{A_n} e,$$

thus

$$g = \lambda \circ \sigma_{n\infty}(\dim_{A_n} e),$$

hence λ is surjection, since $P(A)$ generates $K_0(A)$,

Given an element $g \in \varinjlim K_0(A_n)$ with $\lambda(g) = 0$. There exists A_n and $g_n \in K_0(A_n)$ with $g = \sigma_{n\infty}(g_n)$. Then

$$(\phi_{n\infty})_*(g_n) = \lambda \circ \sigma_{n\infty}(g_n) = \lambda(g) = 0.$$

Let $g_n = [e] - [f] \in K_0(A_n)$, it follows that $[e] - [f]$ is stably equivalent to $0 = [0] - [0]$ in $K_0(A)$, i.e., for suitable k, $e \oplus 0_\infty \oplus I_k$, $f \oplus 0_\infty \oplus I_k$ are Murry-von Neumann equivalent and hence unitarily equivalent.

Since $\overline{M_\infty(A)}$ is an AF algebra, by Lemma 6.15 we have that $e \oplus 0_\infty$, $f \oplus 0_\infty$ are unitarily equivalent. By Lemma 6.14 $e \oplus 0_l$ $f \oplus 0_l$ are unitarily equivalent via some unitary matrix over A_m for some $m \geq n$. Thus it follows that $[e] - [f]$ is stably equivalent to $[0] - [0]$ in $K_0(A_m)$, i.e., $(\phi_{nm})_*(g_n) = 0$ in $K_0(A_m)$ and $g = 0$. Therefore λ is injective.

If A is not unital, we consider the following diagrams:

From

$$\begin{tikzcd}
0 \arrow[r] & K_0(A_n) \arrow[r] \arrow[d] & K_0(A_n^1) \arrow[r] \arrow[d] & \mathcal{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & K_0(A_{n+1}) \arrow[r] & K_0(A_{n+1}^1) \arrow[r] & \mathcal{Z} \arrow[r] & 0
\end{tikzcd}$$

we have

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim K_0(A_n) \arrow[r] & \varinjlim K_0(A_n^1) \arrow[r] & \mathcal{Z} \arrow[r] & 0
\end{tikzcd}$$

is exact. Thus we have:

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim K_0(A_n) \arrow[r] & \varinjlim K_0(A_n^1) \arrow[r] \arrow[d, "\lambda"] & \mathcal{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r] & K_0(A^1) \arrow[r] & \mathcal{Z} \arrow[r] & 0
\end{tikzcd}$$

and we just proved that the λ is isomorphism. Since the right-hand square commutes, we restrict λ to an isomorphism from $\varinjlim K_0(A_n)$ onto $K_0(A)$. This is required. ∎

Remark 6.17 *For AF algebra $A = \overline{(\cup A_n)}$, by Theorem 6.16, every class in $K_0(A)$ is the image of a class in $K_0(A_n)$ for some n. On the other hand, we have dim $P(A) \subseteq K_0(A)$. Moreover $K_0(A)$ is generated by the classes of elements in $P(A)$. Since for each A_n, $K_0(A_n)$ is isomorphic to a simplicial* *group, thus by Remark 5.10, Remark 6.13 and the proceding theorem, it follows that $K_0(A)$ is the dimension group for A with a scale dim $P(A)$, which arose in the preceding chapter.*

Corollary 6.18 *If AF algebra A is the (C^*-algebra) inductive limit of the direct system (5.4), then $K_0(A)$ is (isomorphic to) to scaled dimension group of (5.5).*

Corollary 6.19 *Let A and B be two AF algebras, then the following are equivalent:*

(1) *A is isomorphic to B;*

(2) *Any two dense AF subalgebras of A and B respectively are isomorphic;*

(3) *$K_0(A)$ is isomorphic to $K_0(B)$ as scaled dimension groups.*

Proof : By Lemma 5.7, Remark 6.13, Theorem 6.16, Remark 6.17, the results are automatic. ∎

Reference: [2], [9], [11], [24], [28].

Chapter 7

An Example

We have seen that AF algebras are classified by their scaled dimension groups. The classification problems include the computations of the dimension groups and the determination of the scales. Usually the dimension groups of AF algebras are given as inductive limits, and their algebraic constructions can be read from the Bratteli diagrams, but the intrinsic definition of the dimension groups of given AF algebras are always attractive. Here we shall compute the dimension group of the algebra in Example 3.20.

The algebra A in Example 3.20 is the inductive limit of

$$\{\sum_{k=0}^{n}{}^{\oplus} M_{c(n,k)}\},$$

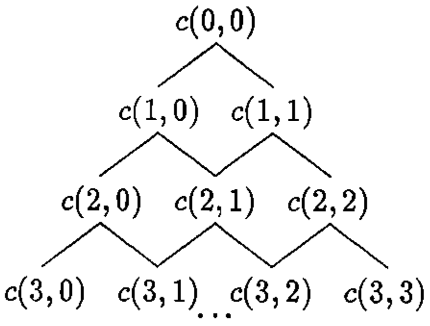
where $c(n,k) = \frac{n!}{k!(n-k)!}$ are the binomial coefficients. Letting

$$A_n = \sum_{k=0}^{n}{}^{\oplus} M_{c(n,k)},$$

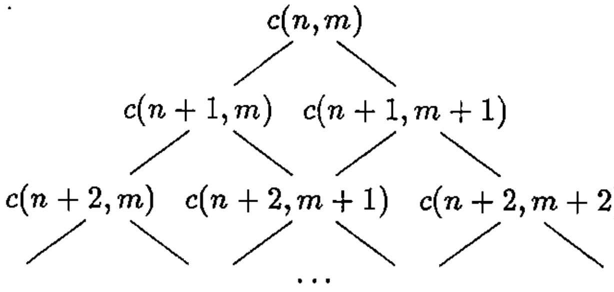
we have

$$A = \overline{(\cup A_n)}.$$

The Bratteli diagram is



By [4] (3.3, 5.5), the ideal $J_{c(n,m)}$ of A ($J_{c(n,m)} \neq \{0\}$) is represented by "pyramids" on the diagram, starting from $c(n,m)$:



Thus

$$J_{c(n,m)} = \overline{\bigcup_{k=0}^{+\infty}\Big(\sum_{j=m}^{m+k}{}^{\oplus} M_{c(n+k,j)}\Big)}.$$

The center of $A_n = \sum_{k=0}^{n}{}^{\oplus} M_{c(n,k)}$ is $C(A_n) = \sum_{k=0}^{n}{}^{\oplus} \lambda_k I_{c(n,k)}$, where $\lambda_k \in \mathcal{C}$ and $I_{c(n,k)}$ is the $c(n,k) \times c(n,k)$ identity matrix.

From the Bratteli diagram, $\sum_{k=0}^{n}{}^{\oplus} \lambda_k I_{c(n,k)}$ is embedded into A_{n+1} as

$$\lambda_0 I_{c(n+1,0)} \oplus \Big[\sum_{k=0}^{n}{}^{\oplus}(\lambda_k I_{c(n+1,k)} \oplus \lambda_{k+1} I_{c(n+1,k+1)})\Big] \oplus \lambda_n I_{c(n+1,n+1)}.$$

Thus it is in $C(A_{n+1})$ if and only if $\lambda_0 = \lambda_1 = \cdots = \lambda_n$.

We claim that the center $C(A)$ of A is the scalar multiples (thus $C(A)$ is the inductive limit of $\{\sum_{k=0}^{n}{}^{\oplus}\lambda I_{c(n,k)} : \lambda \in \mathcal{C}\}_{n=1}^{+\infty}$).

$C(A)$ is a commutative C*-algebra, then by the Gelfand Transformation, it is isometrically *-isomorphic to $\mathbf{C}(\widehat{C(A)})$ the set of all continuous functions on some Hausdorff space (the maximal ideal space). To prove the claim, it is sufficient to show that $\widehat{C(A)}$ is just one-point set. Assume $x_1, x_2 \in \widehat{C(A)}$ with $x_1 \neq x_2$, then there exist two disjoint open sets U_1, U_2 containing x_1, x_2 respectively. Let

$$I_1 = \{f : f \text{ continuous and } f(x) = 0 \text{ for } x \notin U_1\}$$

$$I_2 = \{f : f \text{ continuous and } f(x) = 0 \text{ for } x \notin U_2\}.$$

Then I_1 and I_2 are two ideals of $\mathbf{C}(\widehat{C(A)})$ with $I_1 I_2 = \{0\}$.

Identifying I_1, I_2 to be ideals of $C(A)$, and letting:

$$J_1 = \overline{\text{span}(I_1 \cdot A)},$$

$$J_2 = \overline{\text{span}(I_2 \cdot A)},$$

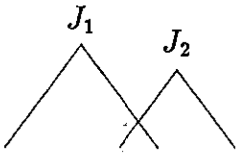
we have two ideals in A. We observe that $J_1 \cdot J_2 = \{0\}$:

For any $a \in J_1,\ b \in J_2$, we may assume

$$a \leftarrow \sum_{i=1}^{n_k} \lambda_i^k a_i^k, \quad b \leftarrow \sum_{j=1}^{m_k} \mu_j^k b_j^k,$$

where $\lambda_i \in I_1,\ a_i \in A,\ i = 1, \ldots, n_k;\ \mu_j \in I_2,\ b_j \in A,\ j = 1, \ldots, m_k,\ k = 1, 2, \ldots$.

Then since $\lambda_i^s \mu_j^t = 0$, we have $ab = 0$ since the λ's and μ's are in $C(A)$. Thus $J_1 \cdot J_2 = \{0\}$. But from the Bratteli diagram:



$J_1 \cdot J_2$ can not be $\{0\}$. The contradiction proves the claim.

Next we show that the dimension group of A is isomorphic to $P[t]$, the algebra of polynomials in t with integer coefficients with usual addition and the positive cone $\{f : f(t) > 0,\ \forall t \in [0,1]\}$. In what follows, we shall denote by $P_n[t]$ all those polynomials with degree less than n.

By Example 6.12 and Corollary 6.18, we have that the dimension group of A is the inductive limit of

$$\mathcal{Z} \xrightarrow{\phi_{12}} \mathcal{Z} \oplus \mathcal{Z} \xrightarrow{\phi_{23}} \mathcal{Z} \oplus \mathcal{Z} \oplus \mathcal{Z} \xrightarrow{\phi_{34}} \cdots.$$

The ϕ's are the same matrices as in Example 3.20.

Given $n \in \mathcal{N}$, we define a map ψ_n,

$$\begin{aligned} \psi_n :&\quad \mathcal{Z}^n \longrightarrow P_{n-1}[t] \\ by :&\quad (z_1, \ldots, z_n) \longrightarrow \sum_{k=0}^{n-1} z_{k+1} t^k (1-t)^{n-1-k}. \end{aligned}$$

For $m = 1, 2, \ldots, n$, by binomial theorem and assuming that $c(n,k) = 0$ if $k < 0$ or $k > n$ we have:

$$\begin{aligned} 1 &= [t + (1-t)]^{(n-1)-m} \\ &= \textstyle\sum_{k=0}^{n-1} c(n-1-m, k-m) t^{k-m} (1-t)^{[(n-1)-m]-(k-m)}, \end{aligned}$$

then we have

$$t^m = \sum_{k=0}^{n-1} c(n-1-m, k-m) t^k (1-t)^{n-1-k}$$

and similarly we have

$$(1-t)^m = \sum_{k=0}^{n-1} c(n-1-m,k)t^k(1-t)^{n-1-k}.$$

Denoting $t^k(1-t)^{n-k}$ by $T(n,k)$, $n \in \mathcal{N}$, and $k = 0,1,\ldots,n$, it follows that $\{T(n,k) : n \in \mathcal{N}, \text{ and } k = 0,1,\ldots,n\}$ generates $P[t]$. For fixed $n \in \mathcal{N}$, $T(n,k)$'s are linearly independent. In fact, if there exist integers $a_0,\ldots,a_n$ such that for any $t \in \mathcal{R}$:

$$a_0(1-t)^n + a_1t(1-t)^{n-1} + \cdots + a_nt^n = 0,$$

we have $a_0 = 0$ since it is the coefficient item if we expend the left hand side of above equation (or let $t = 1$). Thus, we have

$$a_1t(1-t)^{n-1} + a_2t^2(1-t)^{n-2} + \cdots + a_nt^n = 0,$$

therefore

$$a_1(1-t)^n + a_2t(1-t)^{n-2} + \cdots + a_nt^{n-1} = 0,$$

thus $a_1 = 0$. By induction, we have all $a_i = 0$, $1 \le i \le n$. Therefore it becomes obvious that ψ_n defined above is an isomorphism.

Moreover, using

$$T(n,k) = T(n+1,k) + T(n+1,k+1),$$

we embed $P_n[t] = \{\sum_{k=0}^{n} \lambda_k t^k(1-t)^{n-k}\}$ into $P_{n+1}[t]$. Then $P[t]$ is the inductive limit of $\{P_n[t]\}$. The tedious computation shows that the direct system $\{P_n[t]\}$ is isomorphic to $\{\mathcal{Z}^n\}$ under the map $\psi = \{\psi_n\}$.

We define $f > 0$ if and only if there exists $n \in \mathcal{N}$ such that the coefficients z_{k+1} of the expansion

$$f = \sum_{k=0}^{n-1} z_{k+1} t^k (1-t)^{n-k}$$

are nonegative (if and only if every z_k in $(z_1, \ldots, z_n)$ is nonegative.)

However, if $f(t) > 0$ for $t \in [0,1]$, we have $f > 0$. In fact, given

$$f = \sum_{m=0}^{p} a_m t^m$$

we have a sequence of $\{f_n\}$ convergent to f uniformly on $t \in [0,1]$, where

$$f_n = \sum_{m=1}^{p} a_m t \frac{t - \frac{1}{n}}{1 - \frac{1}{n}} \cdots \frac{t - \frac{m-1}{n}}{1 - \frac{m-1}{n}}, \quad n \geq p.$$

For n large enough, $f_n(t) > 0 \ \forall t \in [0,1]$. Then

$$\begin{aligned} f &= \textstyle\sum_{m=0}^{p} a_m t^m \\ &= \textstyle\sum_{m=0}^{p} a_m \sum_{k=0}^{n-1} c(n-1-m, k-m) t^k (1-t)^{n-1-k} \\ &= \textstyle\sum_{k=0}^{n-1} [\sum_{m=0}^{p} a_m c(n-1-m, k-m)] t^k (1-t)^{n-1-k}, \end{aligned}$$

and,

$$\begin{aligned} & \textstyle\sum_{m=0}^{p} a_m c(n-1-m, k-m) \\ = \; & c(n-1, k) \textstyle\sum_{m=0}^{p} a_m \frac{k(k-1)\cdots(k-m+1)}{(n-1)(n-2)\cdots(n-m)} \\ = \; & c(n-1, k) f_n(\frac{k}{n}) \frac{n}{n-m} \\ > \; & 0, \end{aligned}$$

i.e., $f > 0$.

If $f > 0$ then $f(t) > 0$ for $t \in [0,1]$, then we conclude that $P[t]$ is (isomorphic to) the dimension group of A with the positive cone $\{f : f(t) > 0 \ \forall t \in [0,1]\}$.

Reference: [4], [25].

Appendix A

Some Analytic Results About Concrete C*-Algebras

We shall need the following lemmas which are proved in [15].

Lemma A.1 *Let* $\varepsilon > 0$. *There is a* $\delta = \delta(\varepsilon) > 0$ *such that if* A *is a* C^*-*algebra acting on a Hilbert space* H, *if* e *is a projection on* H *and if there is an* $a \in A$ *with* $\|e - a\| < \delta$ *then there is a projection* $f \in A$ *with* $\|e - f\| < \varepsilon$.

Lemma A.2 *Given* $\varepsilon > 0$ *and* $n \in \mathcal{N}$, *there exists a* $\delta = \delta(\varepsilon, n) > 0$ *such that if* A *is* C^*-*algebra with* $\{e_i\}_{i=1}^{n}$ *a family of projections in* A *for which* $\|e_i e_j\| < \delta$ *for* $i \neq j$, *then there is an orthogonal family* $\{e_i'\}_{i=1}^{n}$ *of projections in* A, *with* $\|e_i' - e_i\| < \varepsilon$.

Lemma A.3 *If* $\{e_i\}_{i=1}^{n}$ *and* $\{f_i\}_{i=1}^{n}$ *are each orthogonal families of projections in a* C^*-*algebra* A, *and if* $\|e_i - f_i\| < 1$, *then there is a partial isometry* $w \in A$ *such that* $e_i w f_i$ *is a partial isometry from* f_i *to* e_i. *If* $\varepsilon > 0$ *there is* *a* $\delta = \delta(\varepsilon, n) > 0$ *such that if* $\|e_i - f_i\| < \delta$ *then* w *can be chosen so that* $\|e_i - e_i w f_i\| < \varepsilon$. *If* $\sum_i e_i = 1_A$ *then* w *can be chosen so that* $\|1_A - w\| < \varepsilon$.

Lemma A.4 *Given* $\varepsilon > 0$,*there is a* $\delta = \delta(\varepsilon) > 0$ *with the following property: let* A *be a* C^**-algebra acting on a Hilbert space* H. *Let* e_1 *and* e_2 *(respectively* f_1 *and* f_2*) be orthogonal projections in* A *(respectively operators on* H*). Suppose that* $\|e_i - f_i\| < \delta, i = 1, 2$ *and suppose that there is a partial isometry* v *from* f_1 *to* f_2 *and an* $a \in A$ *such that* $\|v - a\| < \delta$. *Then there is a partial isometry* $w \in A$ *from* e_1 *to* e_2 *and* $\|v - w\| < \varepsilon$.

Bibliography

[1] Arveson, W., *An Invitation to C^*-Algebras*, Graduate Texts in Mathematics, Vol 39, Spring-Verlag, New York, Heidelberg, Berlin, 1976.

[2] Blackadar, B., *K-Theory for Operator Algebras*, Mathematical Science Research Institute. Publications, Vol.5, Springer-Verlag, 1986.

[3] Bonsall, F. F. and Duncun, J., *Complete Normed Algebras.* Spring-Verlag, Berlin, 1973.

[4] Bratteli, O., *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. 171 (1972), 195-234.

[5] Conway, J. B., *A Course in Functional Analysis*, 2ed edition, Graduate Texts in Mathematics, Vol 96, Spring-Verlag, New York, Heidelberg, Berlin, 1990.

[6] Dixmier, J., *Les C^*-Algébras et Leurs Représentation*, Gauthier-Villars, Paris, 1964; (English Translation:*C^*-Algebras*, North-Holland, Amsterdam, 1977).

[7] Dixmier, J., *On some C^*-algebras considered by Glimm*, J. Functional Anal. 1(1967), 182-203.

[8] Douglas, R. G., *Banach Algebra Techniques in Operator Theory.* Academic Press, London, New York, 1972.

[9] Effros, E., *Dimensions and C^*-Algebras*, CBMS Reginal Conf. Ser. in Math., no.46, Amer. Math. Soc., Providence, 1981.

[10] Effros, E., Handelman, D. and Shen, C.-L., *Dimension groups and their affine representations*, Amer. J. Math. 102(1980), 385-407.

[11] Effros, E. and Rosenberg, J., *C^*-Algebras with approximately inner flip*, Pac. J. Math. 77(1978), 417-443.

[12] Effros, E. and Shen, C.-L., *Approximately finite C^*-algebras and continued fractions*, Indiana Univ. Math. J. 29(1980), 191-204.

[13] Eilenberg, S. and Steenrod, N., *Foundations of Algebraic Topology*, Princeton Univ. Press, Princeton, 1952.

[14] Elliott, G., *On the classification of inductive limits of sequences of semisimple finite dimensional algebras*, J. Algebra 38(1976), 29-44.

[15] Glimm, J., *On a certain class of operator algebras*, Trans. Amer. Math. Soc. 95(1960), 318-340.

[16] Grove, L. C., *Algebra*, Academic Press, New York, London, Paris, Toronto, 1983.

[17] Hungerford, T. W., *Algebra*, Holt, Rinehart and Winston, Inc., New York, Chicago, San Francisco, 1974.

[18] Kadison, R. V. and Ringrose, J. R., *Fundamentals of the Theory of Operator Algebras, Vol.I*, Academic Press, London, New York, 1983.

[19] Kadison, R. V. and Ringrose, J. R., *Fundamentals of the Theory of Operator Algebras, Vol.II*, Academic Press, Academic Press, London, New York, 1983.

[20] Murphy, G. J., *C^*-Algebras and Operator Theory*, Academic Press, Boston, San Diego, New York, London, Toronto, 1990.

[21] Nishihara, A. R., *On the proofs of the Bott periodicity theorem*, M. Sc. Thesis, 1990.

[22] Pedersen, G., *C^*-Algebras and Their Automorphism Groups.* Academic Press, London, New York, San Francisco, 1979.

[23] Phillips, J., *K-Theory relative to a semifinite factor*, Indiana University Math. J 39(1990), 339-354.

[24] Phillips, J., *Notes on C^*-algebras and K-Theory*(unpublished).

[25] Renault, J., *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Math., vol.793, Springer-Verlag, New York, Heidelberg, Berlin, 1980.

[26] Sakai, S., *C^*-Algebras and W^*-Algebras*, Springer-Verlag, New York, Heideberg, Berlin, 1971.

[27] Shen, C.-L., *On the classification of the ordered groups associated with the approximately finite dimensional C^*-algebras*, Duke Math. J. 46(1979), 613-633.

[28] Takesaki, M., *Theory of Operator Algebras I*, Springer-Verlag, New York, Heidelberg, Berlin, 1979.

[29] Taylor, J., *Banach algebras and topology*, Algebras in Analysis, (ed. Williamson, J. H.), Academic Press, 1975.

[30] Wegge-Olsen, N. E., *Introduction to operator algebra K-Theory and generalized index theory*, M. Sc. Thesis, 1989.

VITA

Surname: Cao

Given Names: Tao

Place of Birth: Wuhan, P. R. China

Date of Birth: Nov. 6th, 1964

Educational Institutions Attended:

University of Victioria	1990 to 1992
Wuhan University	1982 to 1986

Degree Awarded:

B.S.	Wuhan University	1986

Honours and Awards:

University of Victoria Fellowship	1990-1992
People's Prize	1982-1986

PARTIAL COPYRIGHT LICENSE

Title of Thesis: **Introduction to AF Algebras and their Dimension Groups**

Author [redacted]

TAO CAO
(Name in Block Letter)

September 30, 1992
(Date)