University of Victoria

Proper Rainbow Saturation in Graphs

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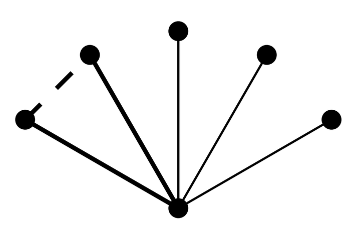
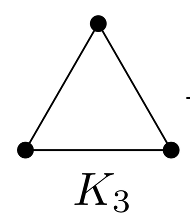
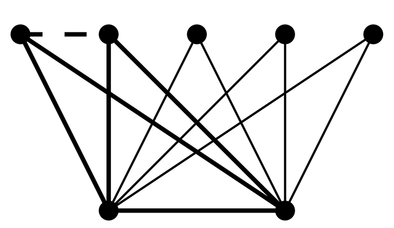
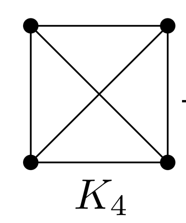
Graph Saturation

Let G and H be graphs. Say G is H-*saturated* if:

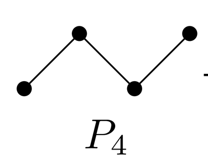
1. G contains no copy of H;
2. Adding any edge to G creates a copy of H.

Examples:

 is  -saturated.

 is  -saturated.

The *saturation number* $\mathrm{sat}(n, H)$ is the minimum number of edges in an H-saturated graph with n vertices.

Your Turn: Which of the following graphs are  -saturated?

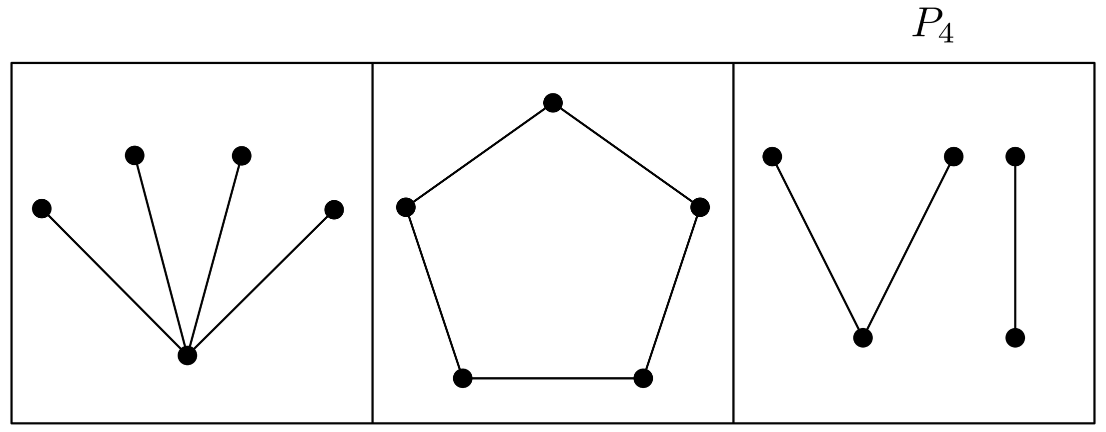
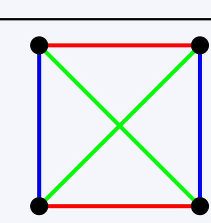
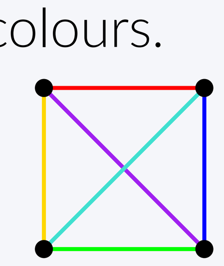


Figure 1

Proper Rainbow Saturation

An edge-colouring is *proper* if any two edges that share a vertex receive different colours.





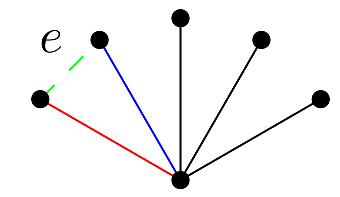
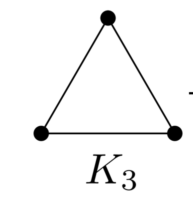
An edge-colouring is *rainbow* if every edge receives a different colour.

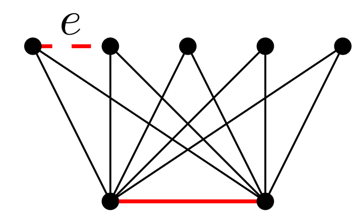
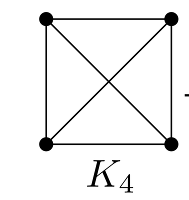
Let G and H be graphs. Say G is *properly rainbow H-saturated* if:

1. There exists a proper edge-colouring of G that contains no rainbow copy of H;
2. For any non-edge $e \in E(\overline{G})$, every proper edge-colouring of $G + e$ contains a rainbow copy of H.

The *proper rainbow saturation number* $\mathrm{sat}^*(n, H)$ is the minimum number of edges in a properly rainbow H-saturated graph with n vertices.

Examples

 is properly rainbow  -saturated.

 is *not* properly rainbow  -saturated.

Original Results:

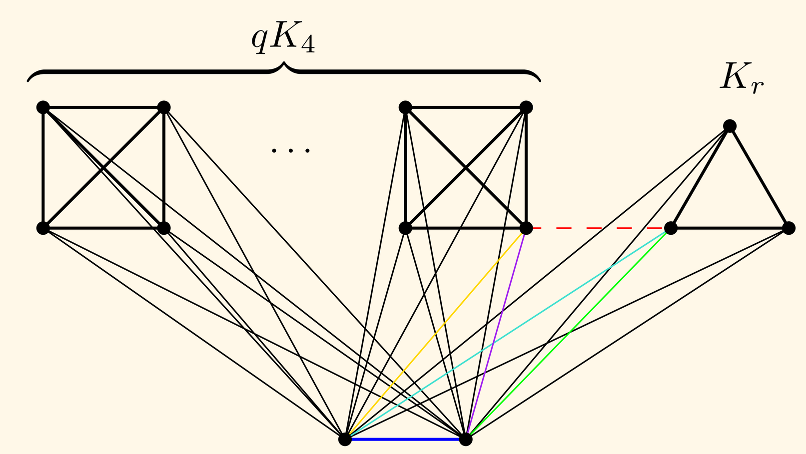
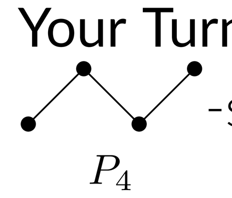


Figure 2

Theorem 1. *The graph in Figure 2 is properly rainbow K_4-saturated.*

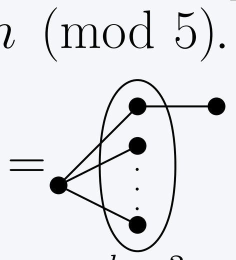
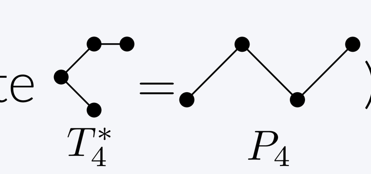
Corollary 2. *For all $n \geq 7$, $\mathrm{sat}^*(n, K_4) \leq \frac{7n}{2} - 6$.*

Your Turn: Which of the graphs in Figure 1 are properly rainbow  -saturated?

Trees

Prior Results:

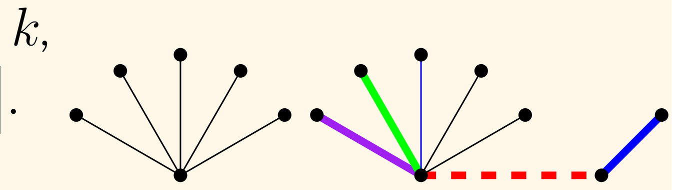
Theorem 3 ([2]). *For $n \geq 16$, $\lfloor \frac{4n}{5} \rfloor \leq \mathrm{sat}^*(n, P_4) \leq \frac{4n}{5} + \frac{14c}{5}$, where $0 \leq c \leq 4$ and $c \equiv -n \pmod 5$.*

For any $k \geq 4$, let $T_k^* =$  (note ).

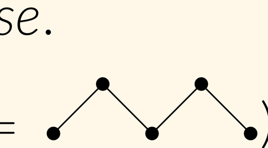
Theorem 4 ([4]). *For $k \geq 5$ and $n \geq k + 2$, $\mathrm{sat}(n, T_k^*) = n - \lfloor \frac{n+k-2}{k} \rfloor$.*

Original Results:

Theorem 5. *For $k \geq 4$ and $n \geq k$, $\mathrm{sat}^*(n, T_k^*) = \mathrm{sat}(n, T_{k+1}^*) = n - \lfloor \frac{n+k-2}{k} \rfloor$.*

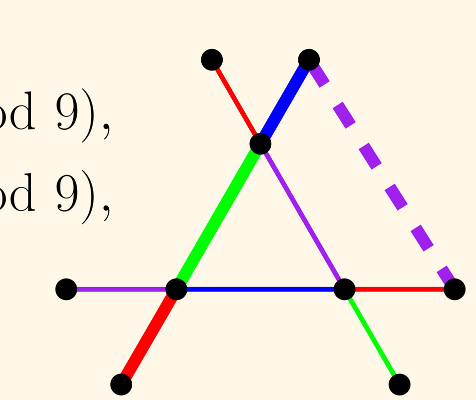


Corollary 6. $\mathrm{sat}^*(n, P_4) = \begin{cases} \lceil \frac{4n}{5} \rceil, & n \equiv 1 \pmod 5, \\ \lfloor \frac{4n}{5} \rfloor, & \text{otherwise.} \end{cases}$

Let P_k be the *path* on k vertices (e.g. $P_5 =$ ).

Theorem 7. *For $n \geq 9$,*

$$n-1 \leq \mathrm{sat}^*(n, P_5) \leq \begin{cases} n-1, & n \equiv 1 \text{ or } 2 \pmod 9, \\ n, & n \equiv 0 \text{ or } 3 \pmod 9, \\ n+1, & \text{otherwise.} \end{cases}$$



General Bounds

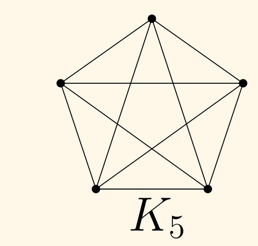
Prior Result:

Theorem 8 ([2]). *If H contains no induced even cycle, then there exists a constant C such that for all n, $\mathrm{sat}^*(n, H) \leq Cn$.*

Original Results:

Proposition 9. *For any graph H, there exists a constant C such that for all n, $\mathrm{sat}^*(n, H) \leq Cn$.*

Let K_k be the *complete graph* on k vertices (all edges between vertices are present).



Theorem 10. *There exist constants C_1, C_2 such that for all $k \geq 4$ and $n \geq C_1 k^3 / \ln k$,*

$$(k-2)(n-k+2) + \binom{k-2}{2} \leq \mathrm{sat}^*(n, K_k) \leq \left(\frac{1}{4} + \frac{C_2}{\ln k}\right) k^3 n.$$

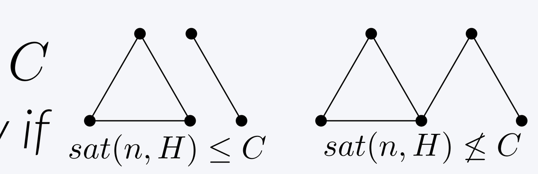
Theorem 11. *For all $k \geq 6$ and $n \geq k$,*

$$n - \left\lfloor \frac{n-1}{k} \right\rfloor - 1 \leq \mathrm{sat}^*(n, P_k) \leq \left(\frac{k}{2} - 1\right) n.$$

Graphs with an Isolated Edge

Prior Result:

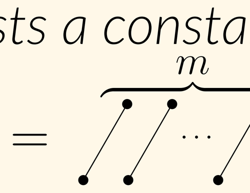
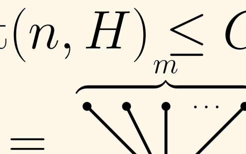
Theorem 12 ([8]). *There exists a constant C such that $\mathrm{sat}(n, H) \leq C$ for all n if and only if H has an isolated edge.*

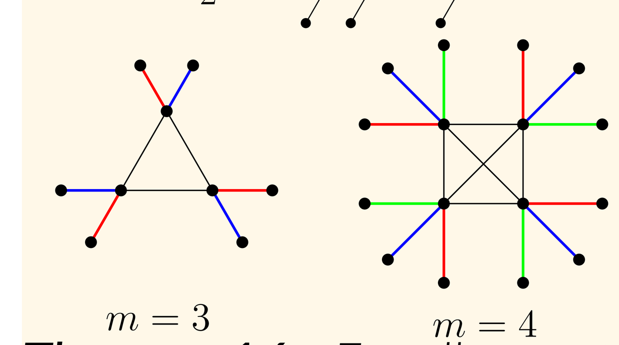


Original Results:

Theorem 13. *If H does not have an isolated edge, then there exists $C > 0$ such that $\mathrm{sat}^*(n, H) \geq Cn$ for all $n > 1$.*

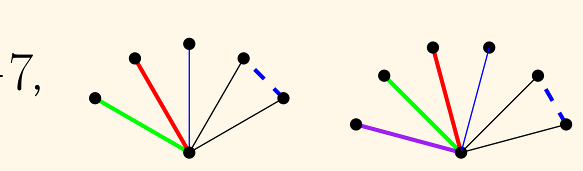
Question 14. *Is it the case that for any graph H with an isolated edge, there exists a constant C such that $\mathrm{sat}(n, H) \leq C$ for all n?*

Let $mK_2 =$  and $K_{1,m} \cup K_2 =$ .



Theorem 15. *For $m \geq 3$ and $n \geq m^2$, we have $\mathrm{sat}^*(n, mK_2) \leq \frac{3m(m-1)}{2}$.*

Theorem 16. *For all $m \geq 1$ and $n \geq m+7$, we have $\mathrm{sat}^*(n, K_{1,m} \cup K_2) = m + 3$.*



References

[1] Noga Alon, Tao Jiang, Zevi Miller, and Dan Pritikin. Properly colored subgraphs and rainbow subgraphs in edge-colorings with local constraints. *Random Structures Algorithms*, 23(4):409–433, 2003.

[2] Neal Bushaw, Daniel Johnston, and Puck Rombach. Rainbow saturation. *Graphs Combin.*, 38(5):Paper No. 166, 12, 2022.

[3] P. Erdős, A. Hajnal, and J. W. Moon. A problem in graph theory. *Amer. Math. Monthly*, 71:1107–1110, 1964.

[4] Jill Faudree, Ralph J. Faudree, Ronald J. Gould, and Michael S. Jacobson. Saturation numbers for trees. *Electron. J. Combin.*, 16(1):Research Paper 91, 19, 2009.

[5] Ralph J. Faudree and Ronald J. Gould. Saturation numbers for nearly complete graphs. *Graphs Combin.*, 29(3):429–448, 2013.

[6] Ralph J. Faudree, Ronald J. Gould, and Michael S. Jacobson. Weak saturation numbers for sparse graphs. *Discuss. Math. Graph Theory*, 33(4):677–693, 2013.

[7] Robert E. Jamison, Tao Jiang, and Alan C. H. Ling. Constrained Ramsey numbers of graphs. *J. Graph Theory*, 42(1):1–16, 2003.

[8] L. Kászonyi and Zs. Tuza. Saturated graphs with minimal number of edges. *J. Graph Theory*, 10(2):203–210, 1986.

[9] L. Lovász. Flats in matroids and geometric graphs. In *Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977)*, pages pp 45–86. Academic Press, London, 1977.