MANAGEMENT OF A FOREST STAND: DETERMINATION OF OPTIMAL THINNING RATE, FERTILIZATION RATE AND ROTATION AGE IN THE PRESENCE OF RISK OF CATASTROPHIC DESTRUCTION

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ABSTRACT

A continuous time model for determining the optimal thinning, fertilization and rotation age for a forest, and optimal expenditure and rotation age for a cocoa plantation under the assumption of catastrophic destruction are studied. The hazard is assumed to be age dependent in the first instance. The analysis is then carried over to age and volume, and age and thinning rate dependent hazards.

In each case the resulting problem is transformed appropriately to enable the use of the Pontryagin Maximum Principle of Optimal Control Theory in its solution. Both single and ongoing rotations of the stand are considered and the optimal policies can be determined using numerical iterations.

It is shown that the singular path in the thinning model for the age dependent hazard satisfies an implicit equation. In the cases of age and volume, age and thinning rate dependent hazards, the singular paths satisfy ordinary differential equations which can be solved for given boundary conditions. Similarly optimal fertilization under age, age and volume dependent hazards are shown to obey ordinary differential equations which again could be solved for given boundary conditions. In the cocoa plantation problem, an implicit equation for the optimal accumulated expenditure is derived; from this one can obtain the optimal expenditure for the treatment of the plantation.

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DEDICATION

In memory of Afanjiwole and Etro̲pe

Chapter 1

INTRODUCTION

The problem of determining an optimal thinning policy and rotation age jointly for a stand of trees has been considered by Clark and De Pree (1979) using the methods of Optimal Control Theory, and by Kilkki and Vaisanen (1969) using Dynamic Programming. However, the above authors did not consider these policies under the more realistic situation in which the possibility of catastrophic destruction of the trees through fires, insect infestation etc. is present. Burt (1965), Reed (1984) and Martell (1980) considered the effect of such catastrophic destruction on the optimal rotation age of a stand while Reed and Errico (1985), (1986) studied the effects of forest fires on the long-run stand yield and rotation age, and on optimal harvest scheduling for a forest comprising many stands.

In this thesis, we consider the model used by Clark and De Pree (1979) under the assumption that the stand of trees is subject to random catastrophic destruction. In Chapter 2 the notations are defined and the optimization model is derived for the case in which the probability of destruction is assumed to depend solely on the age of the stand. In Chapter 3, this optimization model is appropriately transformed to allow the use of the Pontryagin Maximum Principle of Optimal Control

Theory to find the optimal thinning policy and rotation age. The problem of optimal fertilization and rotation age for a stand is considered in Chapter 5 as a further application of the techniques developed in Chapters 2 and 3. Also in Chapter 5 an application concerning the optimal rotation age and the optimal expenditure rate on cultivation etc. for a vulnerable cocoa plantation is discussed.

In the above mentioned chapters, the occurrence of stand destruction is modelled by a hazard function which depends only on the age of the stand. However, the hazard (probability of destruction) could be assumed to depend on other factors such as the volume of trees, or the rate of thinning at the current time. In Chapter 6 the hazard rate is assumed to depend in turn on both age (t) and the volume of trees $x(t)$; and on age (t) and the rate of thinning $u(t)$. Also in Chapter 6, the assumption of age and volume dependent hazard is applied to the fertilization problem developed as an application in Chapter 5. Again necessary conditions for optimality are determined using the Pontryagin Maximum Principle. Numerical schemes for determining optimal policies are suggested for all these cases and some numerical examples are given.

Chapter 2

NOTATION AND MODEL

Consider a commercial stand of trees growing according to the dynamic equation

$$2.1 \qquad \frac{dx}{dt} = W(t,x) \quad , \qquad\qquad x(0)=x_0$$

where $x(t)$ is the volume of timber at time t and x_0 is the known volume at the planting time 0. An example of this growth rate function (see Clark and De Pree (1979), and Kilkki and Vaisanen (1969)) is given by

$$W(t,x) = f(x)g(t)$$

where $f(x)$ is positive and concave, and $g(t)$ is positive and decreasing. If the stand is being thinned at the rate $u(t)$ ($u(t)$ is assumed to satisfy $0 \le u(t) \le u_{max}$ where u_{max} is some maximum possible thinning rate), then the dynamics can be described as

$$2.2 \qquad \frac{dx}{dt} = W(t,x) - u(t) \; , \qquad\qquad t>0$$

Now, suppose that the stand is subject to the possibility of catastrophic destruction through fire, wind damage, insect infestation, etc., and that such catastrophes result in total destruction of the stand of trees. Suppose also that reforestation is carried out whenever the stand is clear-cut at the rotation age, T , or whenever the trees are destroyed through a catastrophe before the optimal rotation age. Denote the associated costs by C_1 and C_2 respectively.

Let Z be a random variable denoting the time at which the stand is destroyed (either through catastrophic destruction or clear-cut harvesting), then the present value (P.V) of net revenue over a single cycle discounted to the start of the period is random and is given by (see Reed (1987))

$$\text{P.V} = \begin{cases} \int_0^Z e^{-\delta t} R(t)u(t)dt - C_2 e^{-\delta Z}, & Z<T \\ \int_0^T e^{-\delta t} R(t)u(t)dt + (q(T)x(T)-C_1)e^{-\delta T}, & Z=T \end{cases} \tag{2.3}$$

where u(t) is the rate of thinning ,

R(t) is the unit net revenue from thinning trees of age t,

q(T) is the unit net revenue from clear-cutting at age T and

δ is the instantaneous discount rate.

Note that we would expect $q(t) \equiv R(t) \equiv 0$ for t sufficiently small, reflecting the fact that below a certain age trees have no commercial value. Also since thinning is more expensive than clear-cut harvesting we expect $q(t) > R(t)$ whenever $R(t) > 0$.

Let h(t) denote the hazard function, which specifies the instantaneous probability rate of catastrophic stand destruction at age t given that the stand has survived until age t. Before stating some results related to the hazard function we shall assume that the hazard function, h(t), is nonnegative, continuous and nonincreasing, over meaningful ages. The hazard function we have in mind for fire risk is one which is almost constant during the tender ages of the stand but which decreases as the stand matures and the crowns close keeping the undergrowth moist. When the foliage is completely developed the hazard then remains almost constant thereafter.

Symbolically h(t) can be written as

2.4 $$h(t) = \lim_{\Delta t \to 0} \left\{ \frac{P(t < Z \leq t + \Delta t \mid Z > t)}{\Delta t} \right\}$$

which uniquely determines the survivor function (see e.g Kalbfliesch 1985)

2.5 $$S(t) = \begin{cases} \exp\{-\int_0^t h(u)du\} & t < T \\ 0 & t \geq T \end{cases}$$

(see appendix A), specifying the probability that the stand of trees is not destroyed by age t . The probability distribution for the random variable Z, has a cumulative distribution function (c.d.f) given by (see Kalbfliesch 1985)

$$F(z) = P(Z \le z) = \begin{cases} 1-S(z) & \text{if} \quad z < T \\ 1 & \text{if} \quad z \ge T \end{cases} \quad . \tag{2.6}$$

Thus the random variable Z possesses a probability density function on (0,T) given by

$$f(z) = h(z)\exp\{-\textstyle\int_0^z h(t)dt\} = h(z)S(z) \tag{2.7}$$

but at the point T, the distribution is not continuous there being an atom of probability of size

$$\exp\{-\textstyle\int_0^T h(u)du\} = S(T^-) \quad ,$$

say, at this point. The statistical expectation of the present value, M, (2.3) can be written, from (2.6),

as (see Reed 1987)

$$\begin{aligned} M = \int_0^T\{\int_0^Z & e^{-\delta t}R(t)u(t)dt \\ & -C_2e^{-\delta z}\}f(z)dz \\ & +[\int_0^T e^{-\delta t}R(t)u(t)dt+(q(T)x(T)-C_1)e^{-\delta T}]S(T^-) \end{aligned} \tag{2.8}$$

By changing the order of integration in the double integral and evaluating it; carrying out the single integral on the second line of (2.8) by parts and simplifying (see Reed 1987) the resulting expression (see appendix A for the details), (2.8) reduces to

$$2.9 \qquad M = \int_0^T [R(t)u(t) + \delta C_2] e^{-\delta t} S(t)dt + (q(T)x(T) + C_2 - C_1)e^{-\delta T} S(T^-) - C_2$$

The setting is now appropriate for a consideration of the ongoing rotations problem. We denote the sequence of successive independent and identically distributed cycle lengths by the sequence of random variables $\{Z_1, Z_2, Z_3, \ldots\}$ with each Z_n having the distribution function given in (2.6). Then the expected present value of net reward associated with the infinite number of rotations of the stand can be expressed as (see Reed 1987)

$$2.10 \qquad J = \sum_{n=1}^{\infty} E[\exp\{-\delta(Z_1 + \ldots + Z_{n-1})\}]M$$

Recalling that the Z_n are independent random variables, we can further express J as

2.11 $$J = \sum_{n=1}^{\infty} \{ \prod_{j=1}^{n-1} E[\exp\{-\delta Z_j\}]M\}$$

$$= \sum_{n=1}^{\infty} \{E(\exp[-\delta Z])\}^{n-1}M$$

$$= M/\{1-E[e^{-\delta Z}]\} .$$

Also

2.12 $$E(e^{-\delta Z}) = \int_0^{\infty} e^{-\delta z} dF(z)$$

$$= \int_0^T e^{-\delta z} f(z)dz + e^{-\delta T} S(T^-) .$$

On evaluating the integral in (2.12) by parts and simplifying further one obtains

2.13 $$E(e^{-\delta Z}) = 1 - \int_0^T \delta e^{-\delta z} S(z)dz .$$

Substituting (2.9) and (2.13) in (2.11) one obtains an explicit expression for the expected present net value from an infinite number of rotations of the stand:

2.14 $$J = \frac{\{\int_0^T [R(t)u(t) + \delta C_2] e^{-\delta t} S(t)dt + [q(T)x(T) + C_2 - C_1] e^{-\delta T} S(T^-) - C_2\}}{\int_0^T \delta e^{-\delta t} S(t)dt}$$

At this point, the problem of optimally managing the stand of trees over many rotations and assuming the possibility of catastrophic destruction reduces to maximizing J as given in (2.14) over thinning rate u(t) and rotation age T subject to the dynamic equation (2.2), the survivor function constraint (2.5) and the nonnegativity constraint $u(t) \geq 0$ for all t in the interval [0,T]. In the next chapter we consider a reformulation of this optimization problem to enable the application of the Pontryagin Maximum Principle of Optimal Control Theory.

Chapter 3

OPTIMAL POLICIES WITH AGE DEPENDENT HAZARD

In order to be able to use the techniques of the Maximum Principle (see eg Kamien and Schwartz 1981) to determine the optimal thinning policy and rotation age for the stand of trees, we need to reformulate the optimization problem as stated in Chapter 2. Now, (2.11) can be expressed (see Reed 1987) as :

3.1 $$J = M + E(e^{-\delta Z})J ,$$

where J, M, and Z are as defined previously in chapter 2.

Assume that, for any fixed nonnegative L the expression

3.2 $$M + E(e^{-\delta Z})L$$

can be maximized over the controls u(t) and T and subject to the equations (2.2) and (2.5). This solves the forest management problem for a single rotation assuming a fixed "site-value" L for a newly replanted site. In the case of the ongoing-rotations problem, the L that is used in determining optimal u(t) and T is given by the value of J that solves (3.1) which can be obtained iteratively using the Secant Method (see eg. Atkinson 1978). Thus if the maximization of (3.2) could be achieved <u>for any L</u>, then the ongoing-rotations problem could be solved by finding the value of L that solves (3.1). This can be achieved numerically.

The major task involved in the determination of the optimal policies is therefore that of maximizing (3.2) over u(t) and T subject to the dynamic equation (2.2) and the survivor constraint (2.5). Using the expression in (2.13) rewrite (3.2) as

$$3.3 \qquad N = M + [1 - \int_0^T \delta e^{-\delta z} S(z)dz]L$$

Then substituting the expression for M, (2.9) in (3.3) and carrying out some simplification we obtain the following expression (see Reed 1987) ;

$$3.4 \qquad N = \int_0^T [R(t)u(t) - \delta(L - C_2)]e^{-\delta t}S(t)dt$$
$$+ [q(T)x(T) + C_2 - C_1]e^{-\delta T}S(T^-) + L - C_2$$

Thus, we consider the problem of maximizing N (3.4) over the the controls u(t) and T subject to the dynamic equation (2.2) and the survivor constraint (2.5) for some fixed L. This latter constraint can be expressed as a dynamic equation thereby allowing the problem to be formulated as a familiar optimal-control problem.

Let

$$3.5 \qquad y(t) = -\ln S(t)$$

From equation (3.5) and the survivor function (2.5) it is easy to see that on [0,T)

3.6 $$y(t) = \int_0^t h(u)du \ ,$$

3.7 $$\frac{dy}{dt} = h(t) \ ,$$

and

3.8 $$S(t) = e^{-y(t)}$$

With the above transformations the objective functional is written as

3.9 $$N = \int_0^T [R(t)u(t) - \delta(L - C_2)]e^{-\delta t - y(t)}dt + [q(T)x(T) + C_2 - C_1]e^{-\delta T - y(T)} + L - C_2$$

and the optimization problem becomes: maximize N (3.9) over the controls u(t) and T subject to the dynamic equations (2.2) and (3.7) and the harvest-rate constraint $u_{max} \geq u(t) \geq 0$. The variables x and y are the state variables. Note that we have a dynamic optimization problem involving a terminal payoff and a free terminal time. Problems of this sort are readily handled using the Pontryagin Maximum Principle (see e.g Kamien and Schwartz 1981).

We are now set to apply the Maximum Principle (see Appendix B for detailed statement of the Maximum Principle). To start with, we define the <u>Hamiltonian function</u> H as

3.10 $$H = [R(t)u(t) - \delta(L - C_2)]e^{-\delta t-y(t)} + \lambda_1[W(t,x) - u(t)] + \lambda_2 h(t)$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the adjoint (multiplier) variables satisfying the adjoint equations[1]

3.11 $$\frac{d\lambda_1}{dt} = - H_x = - \lambda_1 W_x$$

3.12 $$\frac{d\lambda_2}{dt} = - H_y = [R(t)u(t) - \delta(L-C_2)]e^{-\delta t-y(t)}$$

and the transversality conditions

3.13 $$\lambda_1(T) = G_x = q(T)e^{-\delta T-y(T)}$$

3.14 $$\lambda_2(T) = G_y = - [q(T)x(T) + C_2 - C_1]e^{-\delta T-y(T)}$$

We note that $G(T,x(T),y(T))$ is the terminal payoff function which is defined by

3.15 $$G(T,x,y) = [q(T)x(T)+C_2-C_1]e^{-\delta T-y(T)} + L - C_2 \quad .$$

If the rotation period T were fixed it would follow from the Maximum Principle (see Appendix B), that if there were an optimal solution $u^*(t)$ which was piecewise continuous on the time interval $[0,T]$ and $x^*(t)$ was the corresponding optimal path for $x(t)$, then it is necessary that there exist multiplier functions such that the

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[1] Here and elsewhere letter subscripts denote partial derivatives.

differential equations (2.2), (3.7), (3.11) and (3.12) hold, together with the boundary conditions $x(0)=x_0$, $y(0)=0$, (3.13) and (3.14). In addition $u^*(t)$ would maximize $H(t,x^*,y,u,\lambda_1,\lambda_2)$ over the set of admissible controls U for every t in the interval [0,T] (see Kamien and Schwartz 1981) and $H_u=0$.

The problem involving a free-terminal time could be solved by first assuming the parameter T is fixed and determining the optimal thinning policy and subsequently obtaining the maximum of the objective functional (2.14) for such fixed T. This process would then be repeated for various values of T to obtain the optimal rotation period T^* and the corresponding optimal thinning policy (see e.g Reed 1987 where this technique is used to determine optimal protection policies, and Clark and De Pree 1979 where it is used to determine optimal thinning). It is however, our wish to use a different approach which requires a third transversality condition to be satisfied at the optimal rotation age, (see Kamien and Schwartz 1981) namely

$$3.16 \qquad H(T,x(T),y(T),u(T),\lambda_1(T),\lambda_2(T)) + G_T = 0$$

Using the fact that $u(T) = 0$, (since optimally there will be no thinning at the rotation age) and the transversality conditions (3.13) and (3.14), and carrying out some simplifications we arrive at the following explicit equation for the third transversality condition :

$$3.17 \qquad q(T)W(T,x) - [q(T)x(T) + C_2 - C_1]h(T) - \delta[q(T)x(T) + L - C_1] = 0 \quad .$$

The Hamiltonian function (3.10) is linear in the control u(t) and the associated switching function (see e.g Clark 1976) is

$$\sigma(t) = R(t)e^{-\delta t - y(t)} - \lambda_1(t) \quad . \tag{3.18}$$

Optimal thinning policy therefore involves the use of

$$u^*(t) = \begin{cases} 0 \ , & \text{if } \sigma(t) < 0 \\ u_{max} \ , & \text{if } \sigma(t) > 0 \end{cases} \tag{3.19}$$

which is a bang-bang control policy. This bang-bang policy would however, not apply if the switching function is zero. In this case we have what is called the <u>singular control</u>.

Suppose that singular control holds on some time interval, then

$$\sigma(t) = R(t)e^{-\delta t - y(t)} - \lambda_1(t) \equiv 0 \tag{3.20}$$

on the interval of singular control; differentiating (3.20) with respect to time, t, we have the condition

$$[e^{-\delta t - y(t)}][R'(t) - \{\delta + h(t)\}R(t)] - \lambda_1'(t) \equiv 0 \tag{3.21}$$

Using the adjoint equation (3.11), the condition for singular control (3.20) and the condition (3.21) we can show that the equation for the resulting singular path is

$$3.22 \qquad W_x = [\delta + h(t) - \frac{R'(t)}{R(t)}]$$

Clark and De Pree (1979) following Kilkki and Vaisanen (1969) used a specific form for the biological growth function W(t,x). They assumed that W(t,x)=g(t)f(x). Substituting this in (3.22) gives the equation for the singular path

$$f'(x) = \frac{1}{g(t)}[\delta + h(t) - \frac{R'(t)}{R(t)}] \quad .$$

which is the same as the singular control path obtained by Clark and De Pree (1979), with the only difference being the term, h(t) (hazard function) added to the discount rate.

We therefore have the following result (see Crabbé (1982)):

The singular path for optimal thinning, if it exists, for a stand of trees growing under the risk of catastrophic destruction with time dependent hazard rate is the equivalent of the risk-free deterministic counterpart with the discount rate adjusted upward by addition of the hazard rate function. The consequence is that, optimal volume levels will be lower when there is the risk of catastrophic destruction than when there is no risk; furthermore the thinning rate will be higher.

We now give (following Clark and De Pree (1979) and Clark (1976)) a capital theoretic interpretation to singular path equation (3.22). Let

$$V(t) = x(t)R(t)S(t) \quad .$$

Since S(t) represents the probability that the stand survives until time t , and x(t)R(t) represents the revenue that could be earned at time t through thinning the stand to a zero level at that time, using an infinite impulse of thinning, we see that V(t), denotes the expected instantaneous cash value of the stand at time t. This can be taken as a measure of the capital value of the stand. The productivity of this capital is

$$\begin{aligned}\frac{dV}{dt} &= x'(t)R(t)S(t) + x(t)R'(t)S(t) + x(t)R(t)S'(t)\\ &= W(t,x)R(t)S(t) + x(t)R'(t)S(t) + x(t)R(t)S'(t)\\ &= I(t,x), \quad \text{say.}\end{aligned}$$

The marginal productivity of the forest stand capital will therefore be

$$\begin{aligned}\frac{dI}{dV} &= \frac{dI}{dx} \div \frac{dV}{dx}\\ &= W_x + \frac{R'(t)}{R(t)} - h(t)\end{aligned}$$

Setting this to δ, we obtain the singular path equation (3.22). Therefore on the singular path (3.22) the capital theory rule (see e.g Clark 1976) that the discount (or interest) rate is equal to the marginal productivity of the forest capital, is satisfied.

Chapter 4

NUMERICAL EXAMPLE

In this section we consider a numerical scheme for solving the optimal thinning problem described in the previous chapters. Denote by $\tilde{x}(t)$ the singular path ($\tilde{x}(t)$ is defined implicitly by 3.22). Assuming that the biological growth process is such that it is possible to stay on the singular path, then as described in Clark and De Pree (1979) the optimal policy will involve a period $[0,t_1)$ when no thinning takes place, followed by a period $[t_1,t_2)$ during which the singular path is tracked. Finally there will be a period $[t_2,T)$ leading up to the optimal clear-cut harvest time T during which no thinning takes place. A degenerate case will be when $t_1=t_2$; in this case it would never be optimal to practice thinning. Note that if it is not feasible to stay on the singular path, (because it is either growing at a rate faster than that of natural biological growth, or declining faster than is possible with the maximum rate of thinning, u_{max}) then a problem of "blocked intervals" (see e.g Clark 1976) arises. Numerical solution will be much more difficult in this case although it will still involve bang-bang-singular control. In the sequel we shall ignore this possibility and assume that the optimal policy is of the type described above.

The following is a method for determining numerically the ages t_1, t_2 and T.

Solve the differential equation (2.1) with initial condition $x(0)=x_0$ until the condition for singular control (3.22) holds using for example the Merson form of the Runge-Kutta method (see Anonymous (1984)). This provides the age of first-thinning t_1.

Next we consider the problem of determining t_2, the time for final thinning, and T the clear-cut harvest age jointly. Clearly $t_2<T$. This follows from the assumption $q(t)>R(t)$ since this implies greater rewards from clear-cutting than thinning, and thence that a policy of continuous thinning until time of clear-cutting must be suboptimal. The following numerical scheme can be used for the joint determination of t_2 and T. Firstly choose candidate values $\hat{t}_2$ and $\hat{T}$ for t_2 and T. Then solve the system

$$4.1 \qquad \begin{aligned} \frac{dx}{dt} &= W(t,x) \\ \frac{d\lambda_1}{dt} &= -\lambda_1 W_x \end{aligned}$$

on $[\hat{t}_2,\hat{T}]$ as an initial value problem using values $x(\hat{t}_2)$ computed from the singular path equation (3.22) and $\lambda_1(\hat{t}_2)$ computed from the condition for singular control (3.20) (NB: the adjoint variables are continuous being the solutions to ordinary differential equations). The solution of the above system (4.1) can be achieved through the Merson form of the Runge-Kutta method (see Anonymous 1984). This gives values $x(\hat{T})$ and $\lambda_1(\hat{T})$. We require that these values satisfy the transversality conditions (3.13) and (3.17). Note that we can ignore the adjoint equation for λ_2 and the corresponding transversality condition since λ_1 and x do not depend upon λ_2. Using an iterative root-finding algorithm (eg. binary search followed by a combination of the methods of linear interpolation, extrapolation and bisection, see Anonymous 1984) we seek values $\hat{t}_2$ and $\hat{T}$ to satisfy (3.13) and (3.17).

For the "once-and-for-all" forest with a known "site-value", L, the above scheme is sufficient for the determination of t_1, t_2, and T^*; in this case $C_1=C_2=0$ since there will be no reforestation costs incurred. However, in the "ongoing" forest case we need to determine the t_1, t_2 and T^* corresponding to $L=J^*$, the value which solves (3.1). The determination of J^* is achieved iteratively using for example the Secant Method (see eg. Atkinson 1978). In determining J^* we fix L and use the above schemes to find t_1, t_2, T and the thinning rate u(t). We then use these corresponding optimal policies for the fixed L to calculate the value of M (2.9) and check whether this value together with the fixed value of L solve equation (3.1). This process is then repeated until such an L is found using the Secant iterations mentioned above. The evaluation of the statistical expectation in (3.1) is straightforward when t_1 and t_2 are known - it can be achieved through evaluating the integral in (2.13) using a quadrature method such as that described by Patterson (see Patterson 1968). In the following examples the numerical computations in the process of solution were accomplished through the use of NAG (Anonymous 1984) Fortran Library routines using an IBM 3083 processor. It is obvious how the scheme can be modified to provide a scheme for the case in which h(t) is identically equal to zero (the deterministic model).

In our numerical example, we shall use the growth rate function used by Kilkki and Vaisanen (1969) and also by Clark and De Pree (1979):

$$4.2 \qquad \frac{dx}{dt} = W(t,x) = 11.38t^{-1.23}xe^{-0.003x} \qquad x(50)=129$$

where x is the volume of usable timber per hectare of Scotch pine at age t years. We shall also use the unit revenue function estimated by Clark and De Pree (1979)

$$4.3 \qquad R(t) = 0.337t + 5.06 \qquad (\mathrm{Fmk}/m^3)$$

for $t \geq 50$. We assume a net revenue ratio $(q(t)/R(t))$ Of 1.01. The forms of hazard function considered are $h_0(t) \equiv 0.013$ (see Murphy (1982)), $h_1(t)=0.0167\exp(-t^2/5000)$, $h_2(t)=0.0367\exp(-t^2/1800)$ (see Reed 1987) and discount rates varying from 0.01 through to 0.08.

In table 1 we present some results on switch-on time, t_1, switch-off time, t_2 and the cutting age T for the single rotation of a stand. Note that in these results the constants C_1 and C_2 were set to zero, and also L was set to zero although it could have been set to some positive value.

For discount rates of more than 0.04 with a constant hazard of 0.013, optimal management for the single rotation problem involves clearcutting at age 50 with no thinning. This optimal policy applies to the case with hazard given by $h_1(t)$ and discount rates of more than 0.05. It is clear from table 1 that the higher the hazard the lower the switch-on, switch-off and cutting ages. Higher discount rates have similar effect on these ages.

Table 1: Optimal switch-on age, t_1, switch-off age, t_2, and cutting age, T^* for the single rotation of a stand.

$$h_1(t)=0.0167\exp(-t^2/5000)$$

		DISCOUNT RATE							
HAZARD		.01	.02	.03	.04	.05	.06	.07	.08
	t_1	63	57	53	50	--	--	--	--
0.013	t_2	95	75	62	54	--	--	--	--
	T^*	100	79	65	57	50	50	50	50
	t_1	68	60	55	51	--	--	--	--
$h_1(t)$	t_2	160	99	72	58	--	--	--	--
	T^*	171	105	76	61	52	50	50	50

We next present some graphs for optimal and singular paths using a discount rate of 0.02, a constant hazard of 0.013 and the time-dependent hazard given by $h_1(t)$ in figures 1 and 2. The full curves represent optimal paths and the dashed curves represent singular paths. The effect of lowering of the singular path when there is the risk of fire become apparent when these singular paths are compared to those obtained in Clark and De Pree (1979). This lowering effect is more pronounced with increasing hazard. Similar graphs can be drawn using other discount rates and hazards.

In both figures 1 and 2, the optimal volume levels decrease with increasing age. Also optimal volumes decrease with increasing discount rates. However the values of the parameters may be such that the former conclusion will not hold. This is the situation in the case with discount rate of .01 and hazard given by $h_2(t)=0.0267\exp(-t^2/1800)$ (see table 3 in appendix). Note that in this case the singular path increases over the age interval (50,70). This constitutes an upward branch of the singular path and the question arises as to whether one can stay on this branch or not. To answer this question we solve the differential equation for volume growth (4.2) with the intial condition $x(50)=321.3$ forward using Runge-Kutta-Merson method and determine whether this path lies above the singular path or not. If so it is still possible to thin the forest and stay on the singular path but if not the singular path will not be feasible. It turns out that the path obtained fron solving (4.2) is above the

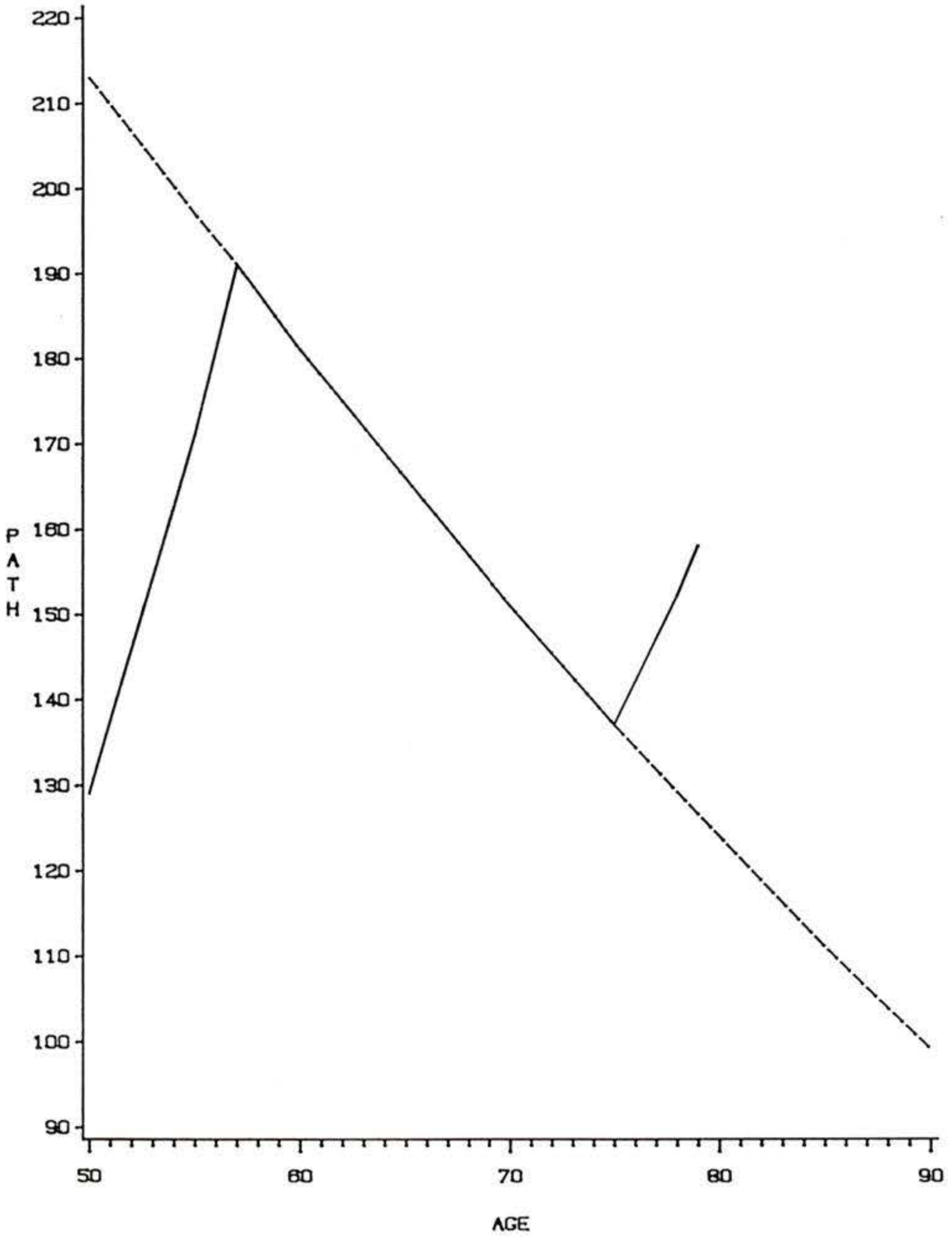

Figure 1: Optimal and singular paths with constant hazard of 0.013, discount rate of 0.02 for the single rotation problem

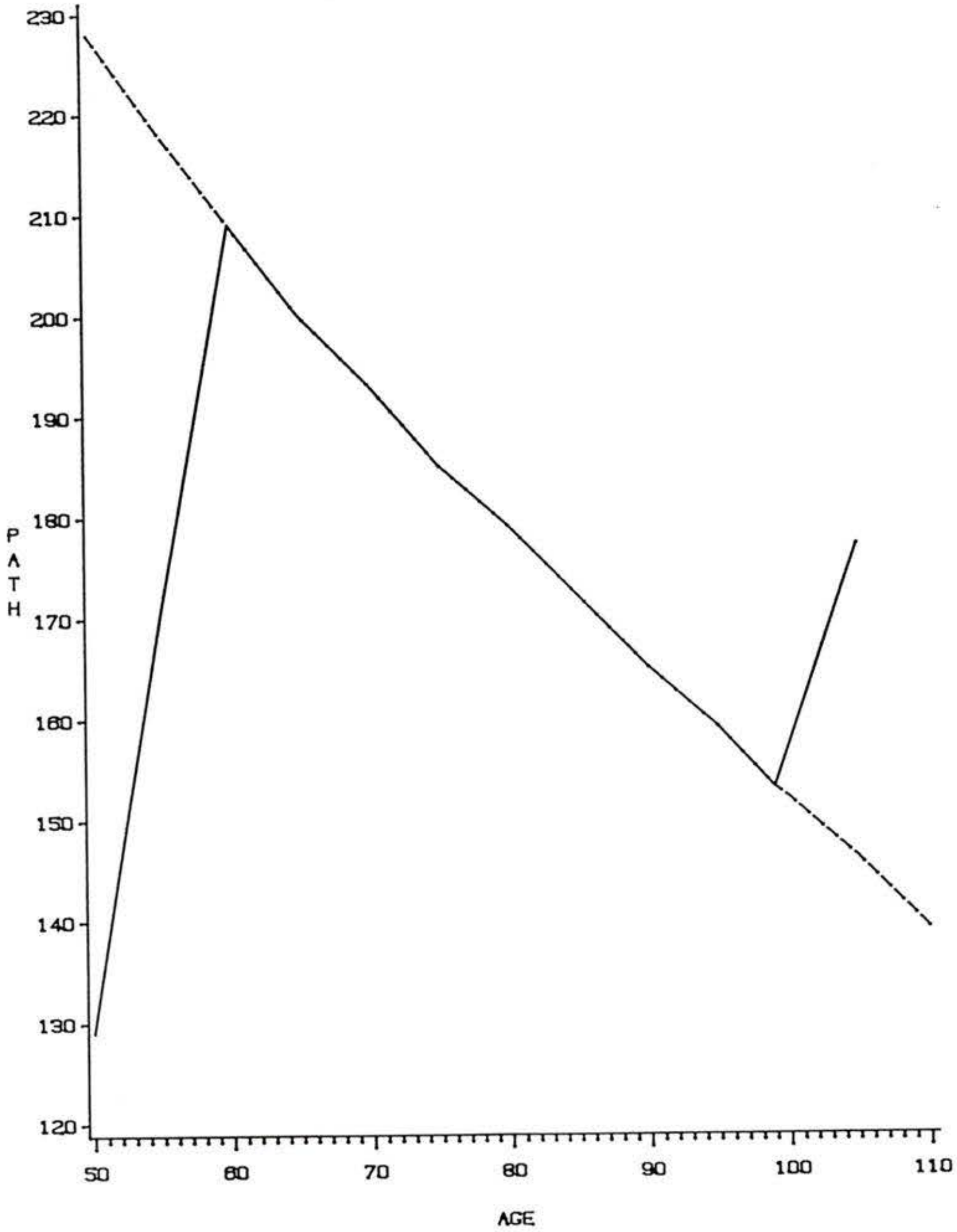

Figure 2: Optimal and singular paths with $h(t)=0.0167\exp(-t^2/5000)$, discount rate $=0.02$ for the single rotation problem

singular path. Thus the singular path is feasible. Comparing these results with those obtained by Clark and De Pree 1979, one can see that the presence of hazard has the the effect of increasing optimal thinning rates and these increase as the hazard rates increase. We remark that the singular paths are independent of reforestation costs and the "site-value". We note that the switch-on, switch-off and cutting ages are 57, 75 and 79 respectively in figure 1. The respective ages in figure 2 are 60, 99 and 105. Taking figure 1 as an example, optimal management involves no thinning during the age interval (0,57). Then there is a period [57,75) of thinning followed by no thinning with subsequent clearcutting at age 79.

For a numerical example for the "ongoing" rotations problem we shall set C_1=800 FMK, C_2=900 FMK, and the clearcutting to thinning revenue ratio (q/R) to 1.01. We shall also assume a constant hazard of 0.013 and discount rate 0.02. The results are displayed in figure 3. The optimal "land-expectation" value corresponding to the above chosen values is 887 FMK and this gives switch-on, switch-off and rotation ages of 57, 72 and 76 respectively. Comparing this with the results for a single rotation one can see that in the case of ongoing rotations the switch-off and clear-cut ages are lower although the switch-on age is unchanged.

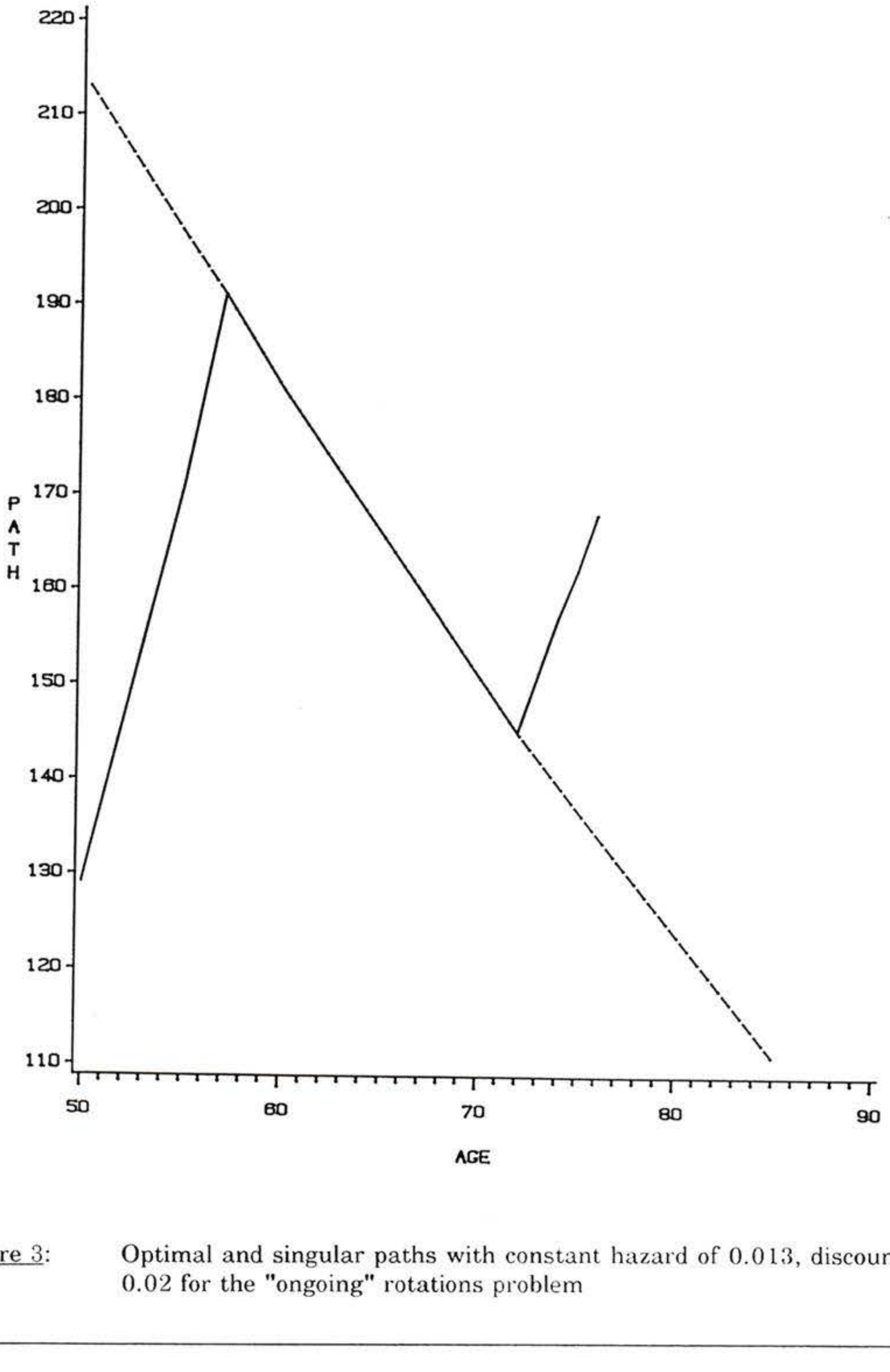

Figure 3: Optimal and singular paths with constant hazard of 0.013, discount rate of 0.02 for the "ongoing" rotations problem

Chapter 5

FURTHER APPLICATIONS

In this Chapter we consider two further applications of the techniques used in Chapter 3. The first application involves the determination of optimal fertilization and rotation age for a forest (for the effects of fertilization on growth see eg. Barclay and Brix 1985, 1986). The second application considers the problem of optimal rotation and treatment of a cocoa plantation. Both discussions will assume the possibility of catastrophic destruction through fire, wind damage, insect infestation etc., with the hazard rate being age-dependent (i.e $h=h(t)$). The assumptions on the age-dependent hazard rate are the same as those given in Chapter 2. We shall ignore thinning in the first application. If thinning were introduced it would lead to a control problem with two control variables.

Consider now a forest stand and let $Q(t)$ be the rate of fertilizer application ($ per unit time) at age t and let p denote the fixed unit price of timber. Suppose that the result of fertilization is a growth curve of the form (see Clark 1976)

$$5.1 \qquad \frac{dx}{dt} = W(t,x) + U(t,Q) \qquad x(0)=x_0$$

where we assume that $U(t,Q)\geq 0$, $U_Q>0$ and $U_{QQ}<0$. The assumption on the second derivative reflects diminishing returns for fertilizer application. Note that the volume at age t, depends on previous fertilization "history". Note also that in contrast to the thinning model (2.2) fertilization has no immediate reward (but has cost), whereas thinning provides immediate revenue.

Using the techniques of chapter 2 one can show that an explicit expression for the net revenue, J, to be earned over infinite rotations of the stand will be

$$J = \frac{\left[\int_0^T[\delta C_2-Q(t)]e^{-\delta t}S(t)dt + [px(T)+C_2-C_1]e^{-\delta T}S(T) - C_2\right]}{\int_0^T \delta e^{-\delta t}S(t)dt} \tag{5.2}$$

To manage the stand optimally we need to determine a fertilization rate $Q(t)$ $(0\leq t\leq T)$ and a rotation age T such that (5.2) is maximized, subject to the dynamic equation (5.1), the survivor constraint (2.5) and the constraint $0\leq Q(t)\leq Q_{max}$ where Q_{max} is the maximum possible fertilization rate.

It is not difficult, in the light of the previous discussions (in particular chapter 3), to see that the above optimization problem can be reduced to

Maximize

$$N = \int_0^T [\delta(C_2-L)-Q(t)]e^{-\delta t-y(t)}dt + [px(T)+C_2-C_1]e^{-\delta T-y(T)} + L - C_2 \tag{5.3}$$

over the controls Q(t) and T, subject to the dynamic equations (5.1) and (3.7), and the constraint $Q_{max} \geq Q(t) \geq 0$ for t in the interval [0,T].

Now let λ_1, λ_2 be the adjoint variables, then (see Appendix B for detailed statement of the Maximum Principle) the Hamiltonian function is defined as

$$H = [\delta(C_2-L)-Q(t)]e^{-\delta t-y(t)} + \lambda_1[W(t,x) + U(t,Q)] + \lambda_2 h(t) \tag{5.4}$$

and the terminal payoff function is

$$G = [px(T) + C_2 - C_1]e^{-\delta T-y(T)} + L - C_2 \tag{5.5}$$

Note that now the Hamiltonian is non-linear in the control Q (unless U is linear in Q), so bang-bang control will not be optimal. The necessary conditions for optimal fertilization are that the adjoint variables λ_1 and λ_2, the state variables x and y, and the control Q(t) satisfy jointly the dynamic equations (5.1) and (3.7), and the adjoint equations (3.11) and (5.6) below,

$$\frac{d\lambda_2}{dt} = [\delta(C_2-L)-Q(t)]e^{-\delta t-y(t)} \tag{5.6}$$

with the terminal transversality conditions

5.7 $$\lambda_1(T) = pe^{-\delta T-y(T)} \quad ,$$

5.8 $$\lambda_2(T) = -\ [px(T)+C_2-C_1]e^{-\delta T-y(T)} \quad ,$$

and

5.9 $$p[W(T,x(T)) + U(T,0)] - \delta[px(T) + L - C_1] - [px(T) + C_2 - C_1]h(T) = 0 \ .$$

Note that, the last condition is the free terminal time condition which should be satisfied if we are considering the problem as a free terminal time problem. If in addition Q(t) maximizes the Hamiltonian function H for every t in the interval [0,T], then it provides the optimal fertilization schedule.

For optimal fertilization set the partial derivative of H with respect to Q(t) to zero i.e.,

5.10 $$-\ e^{-\delta t-y(t)} + \lambda_1 U_Q = 0$$

which implies that

5.11 $$\lambda_1 = \frac{e^{-\delta t-y(t)}}{U_Q}$$

Differentiate both sides of (5.11) with respect to t, and make use of (5.11), and the adjoint equation (3.11) to obtain the following differential equation

$$5.12 \qquad \frac{dQ}{dt} = \frac{[W_x - \delta - h(t)]U_Q - U_{tQ}}{U_{QQ}}$$

(see appendix A for details), which when solved jointly with the dynamic equation (5.1) using given boundary conditions will give the optimal fertilization pattern (provided it is positive) and the optimal volume levels. If the solution has $Q(t)\leq 0$ for some t, then clearly it cannot be optimal to fertilize over the whole life of the stand.

For a special case of this model suppose that U(t,Q) is linear in Q i.e

$$U(t,Q) = l(t)Q$$

say. It then follows that the optimal fertilization pattern involves bang-bang/singular control. Precisely

$$Q^*(t) = \{ \begin{array}{ll} 0 & \text{if } \sigma(t)<0 \\ Q_{max} & \text{if } \sigma(t)>0 \end{array}$$

where $\sigma(t)$ is the switching function of the resulting Hamiltonian and is given by

$$\sigma(t) = \lambda_1(t)l(t) - e^{-\delta t-y(t)}$$

We recall that singular control applies when the switching function is zero. It can then be shown that on the singular path the following equation holds;

5.13 $$W_x = \delta + h(t) - l'(t)/l(t)$$

An interpretation similar to the one given on page 16 applies here when this equation is compared with its deterministic counterpart. The numerical scheme of Chapter 4 can be used to obtain optimal policies for this problem. In the case when it is optimal to refrain from fertilizing at some ages the problem becomes more complicated. Because of limitations of time and computing resources no numerical examples are given here.

We now move to a second application; that of determining the optimal replanting age and treatment regime for a plantation of cocoa trees, or similar fruit producing trees subject to catastrophic destruction through fires and disease. Let C be the cost of replanting the trees either because of declining fruit production or because the plantation is destroyed through fires and let p be the unit price of cocoa. Let Z be the random variable denoting the time at which the cocoa trees are destroyed by fire or disease, or when the trees are cut for replantation, and assume that cut cocoa trees are of no commercial value. We also assume a known fruit production rate v(t) (tons per year, say). Then, considering ongoing rotations of the trees and using methods discussed earlier one can deduce that the expected present value of revenue to be earned from infinite number of rotations will be

$$5.14 \qquad J = \frac{\int_0^T \{pv(t)+\delta C\}e^{-\delta t}S(t)dt - C}{\int_0^T \delta e^{-\delta t}S(t)dt}$$

For optimal replanting age set the first derivative of J with respect to T to zero and simplify to obtain the following equation

$$5.15 \qquad \frac{pv(T) + \delta C}{\int_0^T \{pv(t)+\delta C\}e^{-\delta t}S(t)dt - C} = \frac{\delta}{\int_0^T \delta e^{-\delta t}S(t)dt}$$

A simplification of (5.15) leads to

$$5.16 \qquad \frac{pv(T)}{\int_0^T pv(t)e^{-\delta t}S(t)dt - C} = \frac{1}{\int_0^T e^{-\delta t}S(t)dt}$$

Therefore, for optimal replanting age we need to find a T which solves (5.16). When we let $h(t)\equiv 0$ i.e. $S(t)\equiv 1$, the result is the equation to be satisfied by the optimal replanting age in the corresponding deterministic model (see Clark (1976)).

Now rewrite (5.15) as

5.17 $$pv(T^{*}) = \delta(LEV-C)$$

where LEV (Land expectation value or site value) is the value of J in (5.14) evaluated at the optimal rotation age T^{*} (T^{*} is the value of T that solves (5.16)). In a perfect market LEV would represent the price of a newly planted site. Multiply both sides of (5.17) by a small time interval Δ to obtain

5.18 $$pv(T^{*})\Delta = \delta(LEV-C)\Delta$$

This is a variant of the familiar Faustmann formula (see e.g Clark 1976, Reed 1984) and has the following economic interpretation:

<u>At the optimal replanting age the expected revenue earned in time Δ through not replanting equals the revenue that would be earned in the same period through replanting and selling the site, or in other words equals the opportunity cost of retaining the trees.</u>

Now suppose that, the plantation is being managed with expenditure on treatment at the rate u(t) ($ per unit time) at age t. This treatment cost might result from weed control, fertilization, etc., but does not include cost of protecting the plantation against destruction by fires. Assume that the production rate v(t) (tons per year, say) depends on the accumulated expenditure (see eg. Reed 1988 b) in treatment according to

5.19 $$v(t) = D(w,t)$$

where

5.20 $$w(t) = \int_0^t u(s)ds ,$$

is the accumulated treatment expenditure and D is assumed to be nondecreasing, positive and concave. It follows that

5.21 $$\begin{aligned} v'(t) &= D_w w'(t) + D_t \\ &= D_w u + D_t \end{aligned}$$

since from (5.20)

5.22 $$w'(t) = u(t)$$

The expected revenue to be earned over infinite rotations net of costs is

5.23 $$J = \frac{\int_0^T \{pv(t)+\delta C-u(t)\}e^{-\delta t}S(t)dt - C}{\int_0^T \delta e^{-\delta t}S(t)dt}$$

Optimal management therefore involves finding u(t) ($0\leq t\leq T$) and T such that (5.23) is maximized subject to the survivor constraint (2.5), the dynamic equations (5.21), (5.22) and the condition $u_{max}\geq u(t)\geq 0$. Thus we have three state variables v, w, y (however y falls out) and one control variable u. We can then translate the optimization problem into the following optimal control problem:

Maximize

5.24 $$N = \int_0^T\{pv(t)+\delta(C-L)-u(t)\}e^{-\delta t-y(t)}dt + L - C$$

over the controls u(t) and T, subject to the dynamic equations

(5.21), (5.22) and (3.7) for a fixed L.

The Hamiltonian function is (see Appendix B for detailed statement of the Maximum Principle)

5.25 $$H = \{pv(t)+\delta(C-L)-u(t)\}e^{-\delta t-y(t)} + \lambda_1[D_w u(t) + D_t] + \lambda_2 u + \lambda_3 h(t)$$

where $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$ are the multiplier variables. This Hamiltonian is linear in u, so one gets bang-bang-singular expenditure with switching function

5.26 $$\sigma(t) = -e^{-\delta t-y(t)} + \lambda_1 D_w + \lambda_2 ,$$

and so optimal expenditure involves

5.27 $$u^*(t) = \begin{cases} 0 & \text{if } \sigma(t)<0 \\ u_{max} & \text{if } \sigma(t)>0 \end{cases}$$

The adjoint equations to be satisfied are

5.28 $$\lambda_1'(t) = -pe^{-\delta t-y(t)}$$

5.29 $$\lambda_2'(t) = -\lambda_1(D_{ww}u + D_{tw})$$

5.30 $$\lambda_3'(t) = \{pv(t)+\delta(C-L)-u(t)\}e^{-\delta t-y(t)}$$

Since the terminal payoff function is a constant, the terminal transversality conditions are

5.31 $$\lambda_1(T) = 0$$

5.32 $$\lambda_2(T) = 0$$

5.33 $$\lambda_3(T) = 0$$

and the free terminal time condition is

5.34 $$pv(T) + \delta(C-L) - u(T) = 0$$

Using the fact that the switching function is identically equal to zero on the singular path, and the adjoint equations (5.28), (5.29) we can show that the following equation holds on the singular path

5.35 $D_W = [\delta + h(t)]/p$

The above equation (5.35) can be solved implicitly to determine $w(t)$ on the singular path; hence treatment expenditure, $u(t)=w'(t)$ can be determined on the singular path. The schemes suggested in chapter 4 can be used (in a modified form) to determine the switch-on and switch-off times for expenditure, and the rotation age.

Equation (5.35) can be rewritten as

$$pD_W = \delta + h(t)$$

and the following interpretation can be given:

The marginal productivity of expenditure (or investment) on the singular path is equal to the discount rate adjusted by the hazard function.

The above interpretation stems from the fact that $v(t)=D(w(t),t)$ where $v(t)$ is the rate of fruit production and therefore we can think of D_W as marginal productivity of investment with pD_W being the value of this productivity.

Chapter 6

OPTIMAL POLICIES WITH AGE AND VOLUME, AGE AND THINNING RATE DEPENDENT HAZARDS

It is possible that the hazard rate does not depend only on the age of the stand but also on the volume of trees at the time. A second possibility is that the hazard rate depends on both the age and the rate of thinning. We first discuss optimal forest thinning and rotation age under the former assumption and then look at the application of the techniques to the fertilization model. Finally optimal forest thinning and rotation age are discussed under the latter assumption.

Let the hazard rate, h, depend on age, t, and the volume of trees x(t). In general h would be a function of both t and x. However we will confine our attention to a simple separable form:

$$6.1 \qquad h = a(t)\phi(x)$$

where $\phi(x)$ is assumed to be a continuous, nonnegative and decreasing function. This represents the phenomenon of a lowering of the probability of fire occurrence as the density of the forest increases. By definition the survivor function is

$$6.2 \qquad S(t) = \exp\{-\int_0^t a(u)\phi(x(u))du\} \qquad ;$$

Note that this depends on the "history" of the forest size up until the current age t.

The probability density function of the time of destruction on [0,T) is

$$6.3 \qquad \begin{aligned} f(z) &= a(z)\phi(x(z))S(z) \\ &= a(z)\phi(x(z))\exp\{-\int_0^z a(t)\phi(x(t))dt\} \end{aligned}$$

with c.d.f given by (2.6) . Also the expressions for M given in (2.9) and N in (3.4) remain unchanged under the present assumption about the form of the hazard rate function. The optimization problem is

Maximize N (3.4)

over the controls u(t) and T subject to the dynamic equation (2.2), and the survivor constraint (6.2) for a fixed L .

Again, making the substitution (3.5) in the above, the expression for N reduces to that given in (3.9). Therefore, our problem simplifies to

Maximize N (3.9)

over the controls u(t) and T, subject to the dynamic equations (2.2) and

6.4 $$\frac{dy}{dt} = a(t)\phi(x) \quad .$$

The only difference to the optimization problem of Chapter 3 lies in the dynamic equation (6.4) for the survival probability. The corresponding Hamiltonian function is (see Appendix B for detailed statement of the Maximum Principle)

6.5 $$H = \{R(t)u(t) - \delta[L-C_2]\}e^{-\delta t-y(t)} + \lambda_1[W(t,x)-u(t)] + \lambda_2 a(t)\phi(x) \;,$$

and the adjoint equation for λ_1 is

6.6 $$\frac{d\lambda_1}{dt} = - \lambda_1 W_x - \lambda_2 a(t)\phi'(x) \; .$$

The adjoint equation for λ_2 is the same as the one given in equation (3.12). Also the terminal transversality conditions (3.13) and (3.14) still hold.

Once again the Hamiltonian, H (6.5), is linear in the control variable u(t) and therefore the bang-bang optimal policy (3.19) with the switching function given by (3.18) is still optimal. However, because of the difference in the adjoint equation for λ_1 the singular path equation is different.

Using the condition for singular control (3.20), condition (3.21) and the adjoint equation (6.6), we can show that on the singular path

$$6.7 \qquad \{R'(t) - [\delta+a(t)\phi(x) - W_x]R(t)\}e^{-\delta t-y(t)} + \lambda_2 a(t)\phi'(x) = 0 \quad .$$

Note how the equation (6.7) easily reduces to the singular path condition (3.22) for the age-dependent hazard when we let $\phi(x)\equiv 1$. Now rewrite (6.7) as

$$6.8 \qquad W_x = \delta+\phi(x)a(t) - \frac{R'(t)}{R(t)} - \frac{\lambda_2}{R(t)}\phi'(x)a(t)e^{\delta t+y(t)}$$

Comparing this with equation (3.22) one can see that the effect on the singular path of a density-dependent hazard is given by the last term in (6.8). Since this is positive in sign and W_x is decreasing, it can be seen that when the hazard decreases with increased forest volume, the optimal thinning rates are lower than in the case of a hazard independent of density. Again comparing this equation, (6.8), with the equation (27) of Clark and De Pree (1979) for the no hazard case it can be seen that the presence of a volume-dependent hazard has the effect of introducing two new terms on the right hand side of the singular path equation. The first term, $a(t)\phi(x)$, would be present even for a volume-independent hazard (c.f (3.22)) and has a similar effect to an increase in the discount rate; i.e it results in a <u>faster</u> rate of thinning. The second additional term (the last term on the right hand side of (6.8)) depends on how the hazard changes with volume (via $\phi'(x)$). It is negative in sign (since $\lambda_2<0$ and $\phi'(x)<0$) and so has a similar effect as a decrease in the discount rate; i.e it results in a slower rate of thinning. The overall effect of the presence of a volume-dependent hazard is ambiguous, depending on the relative magnitudes of these two extra terms. A similar situation arises in fishery modelling where Reed (1988 a) shows that the presence of a density-dependent hazard can result in either an increase or decrease in the optimal biomass level.

Now, make λ_2 the subject of the equation (6.7) to obtain

$$6.9 \qquad \lambda_2 = \frac{\{[\delta+a(t)\phi(x) - W_x]R(t) - R'(t)\}e^{-\delta t-y(t)}}{a(t)\phi'(x)}$$

Note that there is a minor change in the third transversality condition (3.17). In this case, it is given by

$$6.10 \qquad q(T)W(T,x(T)) - [q(T)x(T)+C_2-C_1]a(T)\phi(x(T)) - \delta[q(T)x(T)+L-C_1] = 0 \quad .$$

To motivate the discussion of a numerical scheme for the above problem we rewrite (6.9) as

$$6.11 \qquad \lambda_2(t) = A(x,t)e^{-\delta t-y(t)}$$

where

$$6.12 \qquad A(t,x) = \frac{\{[\delta+a(t)\phi(x) - W_x]R(t) - R'(t)\}}{a(t)\phi'(x)}$$

Then we can show that the singular path $\tilde{x}(t)$ satisfies

$$6.13 \qquad x'(t) = \frac{R(t)W(t,x) - A_t + \{\delta + a(t)\phi(x)\}A - \delta(L-C_2)}{A_x + R(t)}$$

The above differential equation can be solved given an initial (or final) condition $\tilde{x}(0)$ to obtain the singular path $\tilde{x}(t)$. We now give a scheme for determining t_1 (the switch-on time), t_2 (the switch-off time) and T (the cutting age).

Starting with some likely values of $\tilde{x}(0)$, t_2 and T say $\hat{x}(0)$, $\hat{t}_2$ and $\hat{T}$ respectively, solve the singular path equation (6.13) above using the initial condition $\hat{x}(0)$ to obtain $\tilde{x}(t)$ on $[0,\hat{T}]$. In obtaining t_1 (the switch-on time) we solve (2.1) until $x(t)-\tilde{x}(t)=0$. With $\tilde{x}(t)$ known over $[t_1,\hat{t}_2]$ y(t) can be determined over the same time interval since (6.1) holds. On $[t_1,\hat{t}_2]$ λ_1 is given by (3.18) and λ_2 by (6.9). Therefore x, y, λ_1 and λ_2 are known at $\hat{t}_2$. We can then solve the system of equations (2.1), (6.4), (6.6) and (6.14) below

$$6.14 \qquad \lambda_2'(t) = [-\delta(L-C_2)]e^{-\delta t-y(t)}$$

as an initial value problem on $[\hat{t}_2,\hat{T}]$ towards $\hat{T}$ and evaluate $x(\hat{T})$, $\lambda_1(\hat{T})$, and $\lambda_2(\hat{T})$. We require that the three conditions (6.10), (3.13) and (3.14) hold simultaneously. Thus we have three equations (6.10), (3.13) and (3.14) involving three unknowns $\tilde{x}(0)$, t_2 and T which (in principle) can be solved numerically.

Rewrite the singular path equation as

$$6.15 \qquad x'(t) = \frac{W(t,x) - K(x,t)/R(t)}{1 + A_x/R(t)}$$

where

$$K(x,t) = A_t + [\delta + a(t)\phi(x)]A - \delta(L-C_2)$$

Depending on the sign of A_x and K we may or may not have a feasible singular path. In particular if $K>0$ and $A_x>0$ then $x'(t)<W(x,t)$ and singular path will be feasible.

Once the singular path has been obtained it then becomes easy to calculate the corresponding optimal thinning rates by the equation which we derive as follows; differentiate both sides of (6.11) with respect to time, t, substitute in the expressions for $x'(t)$ from (2.2) and for $\lambda_2'(t)$ from (3.12) and make u(t) the subject of the resulting equation to obtain

$$6.16 \qquad u(t) = \frac{\delta[L-C_2]+A_t+WA_x-A[\delta+a(t)\phi(x)]}{R(t) + A_x}$$

We now give a numerical example for the above problem. To this end we assume that

$$W(t,x) = 0.062x\exp(-0.003x)$$

$$\phi(x) = 1/x$$

$$a(t) \equiv 0.013 .$$

We set C_1, C_2 and L t0 zero FMK, and the unit revenue from thinning is as given in (4.3). The clear-cutting to thinning revenue ratio (q/R) is assumed to be 1.2 and the discount rate is 0.02. We remark that the above simple functions were chosen to make the integration of the singular path possible. For more realistic functions and parameter values the singular path equation appears to have a singularity which makes numerical solution of the singular path impossible. In such cases no numerical example can be given for the above problem. The optimal volume at age 50 was found to be 276, and the initial thinning, final thinning and clear-cut ages were respectively 67, 98, 119. The above results are illustrated in figure 4 and the FORTRAN program will be available on request. Note that the above results are for the single rotation problem. The numerical scheme for the ongoing rotations problem suggested in Chapter 4 is applicable to this problem. However, due to constraints on time and computing funds we do not give a numerical example for the ongoing rotations problem.

We next discuss how the age and volume dependent hazard techniques presented above apply to the fertilization problem introduced in Chapter 5. Since most of the results derived earlier on are applicable to this problem, we shall state the optimization model to be considered here without explicitly working through the preliminaries, trusting that the

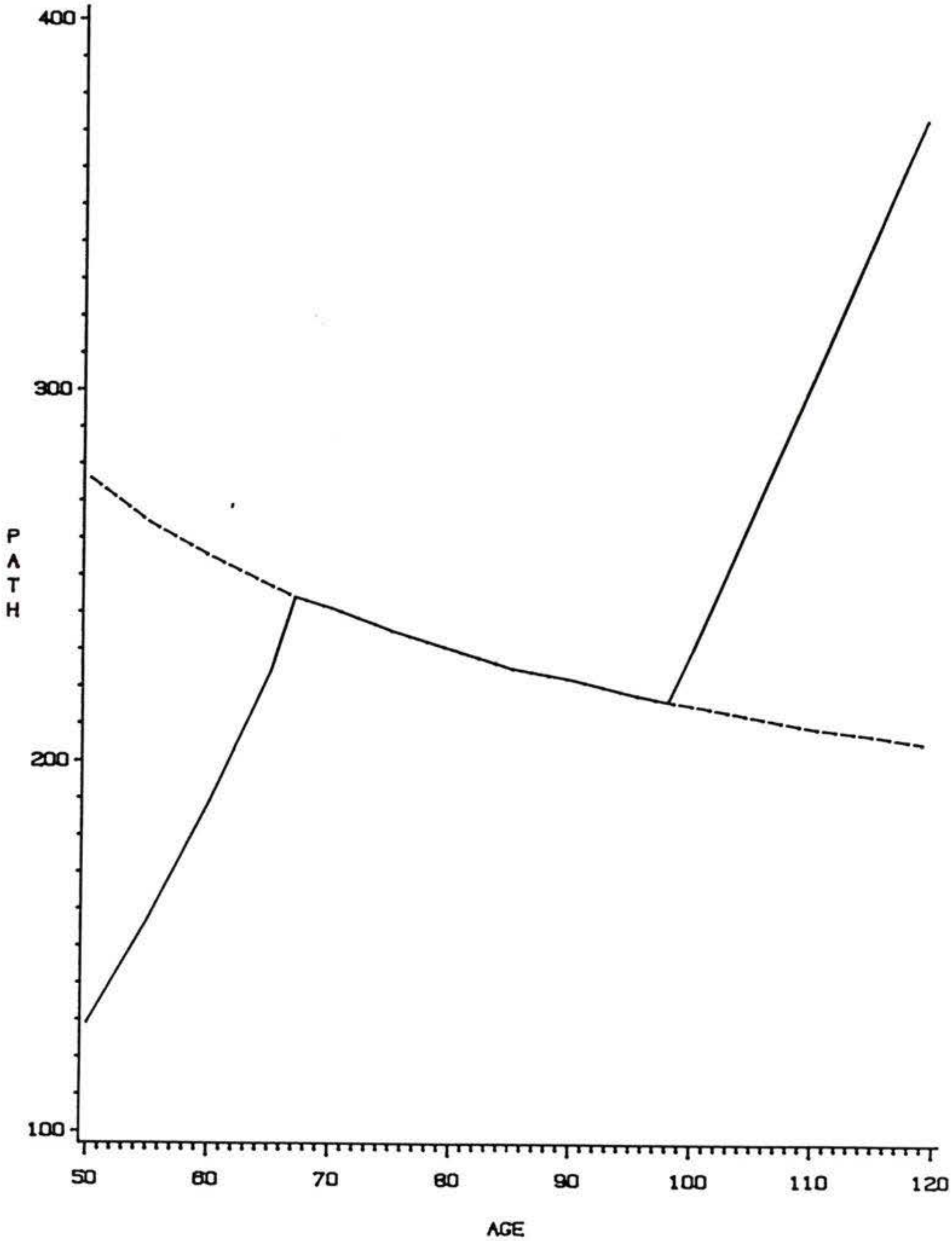

Figure 4: Optimal and singular paths with hazard given by h(x)=0.013/x, discount rate of 0.02 for the single rotation problem

discussion will still be meaningful. So, in the optimal fertilization problem under the present assumption about the hazard function we need to

Maximize N (5.3)

over the controls Q(t) and T, subject to the dynamic equations (5.1), (6.4) and the constraint $Q_{max} \geq Q(t) \geq 0$.

Let λ_1 and λ_2 be the adjoint variables and define the Hamiltonian function by (see Appendix B)

$$6.17 \qquad H = \{\delta(C_2-L)-Q(t)\}e^{-\delta t-y(t)} + \lambda_1[W(t,x)+U(t,Q)] + \lambda_2 a(t)\phi(x)$$

In comparison to the previous thinning problem, this one will not be linear unless U is linear in Q. We will consider the problem as a nonlinear and as usual it is necessary that the multiplier equations (6.6), (5.6), and the terminal transversality conditions (5.8), (5.7) be satisfied by the adjoint variables, Q(t) and y(t). Again if the initial condition y(0)=0 holds then Q(t), maximizes H (the Hamiltonian function) for every t in the age interval [0,T]. The free terminal time condition is as given below (equation (6.18));

$$6.18 \qquad -\delta[px(T)+L-C_1] + p[W(T,x(T))+U(T,0)] - [px(T) + C_2 - C_1]a(T)\phi(x(T)) = 0$$

The conditions (5.10) and (5.11) for optimal fertilization are also true for this consideration. Therefore differentiating both sides of (5.11) with respect to t and using (5.11), we obtain after further simplification the following differential equation for optimal fertilization;

$$6.19 \qquad Q'(t) = \frac{\{\lambda_2 a(t)\phi'(x)\}\{U_Q\}^2 + \{[W_x - \delta - a(t)\phi(x)]U_Q - U_{tQ}\}e^{-\delta t - y(t)}}{\{U_{QQ}\}e^{-\delta t - y(t)}}$$

(see appendix A for the details). This differential equation can then be solved jointly with the dynamic equation (5.1) and the adjoint equation for λ_2 in (5.6), for given boundary conditions to obtain optimal fertilization policy and the corresponding optimal volume levels. The adjoint variable in (6.19) could have been eliminated but the result is a long and complicated second order differential equation which will require two boundary conditions for its solution.

The equation (6.19) is nonautonomous and consequently the solution path could cross the t-axis more than once over the interval [0,T]. We shall assume that the the solution is positive over [0,T].

Let x(0) be given and guess $\tilde{x}(0)$, Q(0), T by say $\hat{x}(0)$, $\hat{Q}(0)$, $\hat{T}$ respectively. Then $\lambda_1(0)$ and $\lambda_2(0)$ can be determined from (5.11) and (6.9) respectively and let $y(0)=0$. Solve the differential equations

(6.19), (5.1), (5.6), (6.4) and (6.6) as an initial value problem towards $\hat{T}$ and calculate $x(\hat{T})$, $\lambda_1(\hat{T})$, $\lambda_2(\hat{T})$ and and $y(\hat{T})$. We require that the conditions (6.18), (5.8) and (5.7) hold simultaneously. Otherwise we revise the values of $\tilde{x}(0)$, $Q(0)$, T and repeat the above process. This procedure is continued until the conditions are met giving the optimal fertilization, volume and cutting age.

In both problems the scheme for the corresponding optimal policies for the "ongoing" forest is similar to the one outlined in chapter 4.

Finally we turn to the problem of finding optimal policies with age and thinning rate dependent hazard. We therefore suppose that the hazard rate , h, is of the form

6.20 $$h = a(t)b(u(t))$$

where $u(t)$ is the thinning rate at time t and $b(u)$ is an increasing, differentiable and nonnegative function. The reason for increasing $b(u)$ is due to fact that more thinning results in more dried up branches and leaves on the site and this is believed to increase the probability of fire occurring. Again, defining Z as the random variable denoting the time of stand destruction through fire or clearcutting, and using previous results, the respective survivor function, and p.d.f for Z are given as below;

6.21 $$S(z) = \exp\{-\int_0^z a(t)b(u(t))dt\}$$

6.22 $$f(z) = a(z)b(u(z))S(z)$$

Again S(z) depends on the history of thinnings up to time z. The c.d.f is given by (2.6) and the expected P.V of a single cycle (net of costs) discounted to the start of the cycle by (2.8). It is therefore obvious that the optimization problem becomes:

Maximize N (3.4)

over the controls u(t) and T subject to the dynamic equation (2.2) and the survivor constraint (6.21).

Now use the usual transformation $y = -\ln S(t)$ to simplify the problem to

Maximize N (3.9)

over the controls u(t) and T subject to dynamic equations (2.2) and

$$\frac{dy}{dt} = a(t)b(u(t)) \tag{6.23}$$

Even though equations (2.9), (3.4) and (3.9) remain unchanged for this problem we should remember that the hazard rate functions are different.

Introducing λ_1 and λ_2 as the multiplier variables, the Hamiltonian function is given by (see Appendix B)

$$\begin{aligned} H = {} & [R(t)u(t) - \delta(L-C_2)]e^{-\delta t-y(t)} \\ & +\lambda_1[W(t,x)-u(t)] + \lambda_2 a(t)b(u) \end{aligned} \tag{6.24}$$

The adjoint equations are given by equations (3.11) and (3.12). Again, there is no change in the form of the terminal payoff function G, and consequently the terminal transversality conditions (3.13) and (3.14) still apply here and will not be stated explicitly. Since we have been considering our problem as a free terminal time problem we need the third transversality condition which is given in its simplest form as

$$q(T)W(T,x(T)) - \delta[q(T)x(T) + L - C_1] - [q(T)x(T) + C_2 - C_1]a(T)b(0) = 0 \tag{6.25}$$

The equation (6.25) above was derived using the transversality condition (3.16) and the fact that $u(T)=0$.

The Hamiltonian function will be linear or nonlinear in the control according as $b(u)$ is linear or nonlinear in the thinning rate $u(t)$. For now, we shall assume that $b(u)$ is nonlinear and consider the problem as a nonlinear one.

For the optimal thinning rate, set the partial derivative of H with respect to the control variable $u(t)$ to zero ie.,

$$R(t)e^{-\delta t-y(t)} - \lambda_1 + \lambda_2 a(t)b'(u) = 0 \tag{6.26}$$

Now solving for λ_2 in equation (6.26) we have

$$\lambda_2 = \frac{\lambda_1 - R(t)e^{-\delta t-y(t)}}{a(t)b'(u)} \tag{6.27}$$

Differentiate both sides of equation (6.27) with respect to time, t. λ_2 can then be eliminated by substituting the expression for $\lambda_2'(t)$ from equation (3.12). This shows that except for the time intervals over which $u(t)=0$, the optimal thinning rate satisfies the following first order differential equation which could be solved jointly with the dynamic equation (2.2) and multiplier equation (3.11) using given boundary conditions to obtain optimal thinning rates and volume levels:

$$\frac{du}{dt} = \frac{[R(t)u(t)-\delta(L-C_2)][a(t)b'(u)]^2e^{-\delta t-y(t)} +[-\lambda_1 W_x+\{R(t)[\delta+a(t)b(u)]-R'(t)\}e^{-\delta t-y(t)}]a(t)b'(u) +[\lambda_1-R(t)e^{-\delta t-y(t)}]a'(t)b'(u)}{-[\lambda_1-R(t)e^{-\delta t-y(t)}]a(t)b''(u)} \tag{6.28}$$

(find detailed calculations in appendix A). Unfortunately, it is not obvious from the above equation, how the age and thinning rate dependent hazard optimal volume levels compare with those discussed previously. We note that this equation can be solved together with the adjoint equation for λ_1 (3.11) and the dynamic equation (2.2) for given initial conditions. A numerical scheme is not suggested to this problem because of the inherent complexity.

Chapter 7

SUMMARY

The methods of Optimal Control Theory are used to show that the optimal thinning rate under the assumption of the risk of catastrophic destruction of a forest stand with age dependent hazard is bang-bang-singular. The singular path is governed by an implicit equation which could have been obtained from the deterministic counterpart by adjusting the discount rate factor by the hazard function as asserted in Crabbé (1982) for similar problems. The singular path was shown to satisfy the capital theory rule that the discount rate is equal to the marginal productivity of capital.

In the case of the age and volume dependent hazard problem, a similar equation was obtained but it could not be solved independently since it involved one of the adjoint variables. However it was possible to derive a first order ordinary differential equation for the singular path. Once again optimal thinning is bang-bang-singular. It was shown whether the singular path is feasible. Crabbé's assertion does not apply here nor in the age and thinning rate dependent hazard problem. The assumption of age and volume dependent hazard may result in slower or faster thinning rates.

When the hazard is age and thinning rate dependent with the hazard being nonlinear in the thinning-rate, the bang-bang-singular policy is no longer optimal. No implicit equation could be obtained for the optimal path but a first order differential equation was obtained for optimal thinning rate. This equation can be solved jointly with the dynamic equations for biological growth and thinning, and one of the adjoint variables with given boundary conditions to give the optimal volumes and thinning rates with age.

Numerical schemes were suggested for all of the problems except the last and numerical examples were given for the age dependent, and age and volume dependent hazard problems. In these numerical examples we realize that the presence of risk results in more thinning activity and shorter rotation ages. Numerical examples were not given for all the problems considered in this thesis. This was due to constraints on computing funds and time.

The techniques developed in the solution of the above problems can also be used to solve the problem of determining the optimal fertilization and rotation age for a forest stand under the assumption of age, age and volume dependent hazards. Also in these problems the optimal policy was not bang-bang-singular but the optimal fertilization rates satisfy ordinary differential equations. Thus optimal paths could be obtained by solving these equations jointly with others.

Finally an application concerning the optimal treatment expenditure rate and rotation age for cocoa (or other similar fruit producing) plantation was considered and it was shown that the optimal expenditure rate involved a bang-bang-singular policy. In this problem the rate of fruit production was assumed to depend on the accumulated past expenditure to date. It was shown that the optimal accumulated expenditure satisfies an implicit equation on the singular path from which one can obtain optimal expenditure for tree treatment. On this singular path the marginal productivity of expenditure is equal to the the discount rate adjusted by the hazard function.

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APPENDIX A

DETAILED DERIVATIONS

The following are the details involved in the derivation of certain equations or expressions used in the main text of the thesis. The equation number is indicated in the heading.

<u>Derivation (2.5)</u> (see eg. Kalbfleisch 1985)

Let the hazard function h(t) be defined by

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t<Z\leq t+\Delta t \mid Z>t)}{\Delta t}$$

where Z is the random variable denoting the time of stand destruction. Then the probability that the stand survives past time t is given as

$$S(t) = \exp\{-\int_0^t h(r)dr\}$$

<u>Proof:</u>

$$h(t) = \lim_{\Delta t \to 0} \left\{ \frac{P(t<Z\leq t+\Delta t \mid Z>t)}{\Delta t} \right\}$$

$$= \lim_{\Delta t \to 0} \frac{P(t<Z\leq t+\Delta t)}{P(Z>t)\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{S(t)\Delta t}$$

where F is the c.d.f of Z.

$$h(t) = \lim_{\Delta t \to 0} \frac{S(t) - S(t+\Delta t)}{S(t)\Delta t}$$

$$\text{A.1} \qquad h(t) = -\frac{S'(t)}{S(t)}$$

Rewrite (A.1) as $h(r) = -S'(r)/S(r)$ and integrate both sides over the interval from 0 to t. Use the condition $S(0)=1$ to get the required result.

Derivation (2.7) (see eg. Kalbfleisch 1985)

Show that the p.d.f of Z (Z has the survivor function given in Derivation (2.5)) is

$$f(t) = S(t)h(t)$$

Proof:

Let $F(t)$ be the c.d.f of Z, then

$$F(t) = P(Z \leq t) = 1-S(t)$$

But from theory the p.d.f f of Z is given as

$$f(t) = F'(t) = -S'(t) \quad . \tag{A.2}$$

Use the expression for $-S'(t)$ in (A.1) to obtain the result.

Derivation (2.9)

We show that equation (2.9) is a simplification of (2.8).

Proof:

Removing the first curly brackets in equation (2.8) and pulling out the integral from the second curly brackets we have

$$\begin{aligned}\Pi = \int_0^T \int_0^z e^{-\delta t} R(t)u(t)f(z)dtdz \\ - \int_0^T C_2 e^{-\delta z} f(z)dz \\ + \{\int_0^T e^{-\delta t} R(t)u(t)dt\} S(T^-) + \{q(T)x(T)-C_1\}\, e^{-\delta T} S(T^-)\end{aligned}$$

Now change the order of integration in the double integral to obtain

$$\Pi = \int_0^T \{\int_t^T f(z)dz\} e^{-\delta t} R(t)u(t)dt$$

$$- \int_0^T C_2 e^{-\delta z} f(z)dz$$

$$+ \{\int_0^T e^{-\delta t} R(t)u(t)dt\} S(T^-) + \{q(T)x(T)-C_1\}\ e^{-\delta T} S(T^-)$$

Recalling that $-S'(t)=f(t)$ (see (A.2)) and evaluating the inner integral of the double integral we rewrite the above equation as

$$\Pi = \int_0^T \{-S(T^-)+S(t)\} e^{-\delta t} R(t)u(t)dt$$

$$- \int_0^T C_2 e^{-\delta z} f(z)dz$$

$$+ \{\int_0^T e^{-\delta t} R(t)u(t)dt\} S(T^-) + \{q(T)x(T)-C_1\}\ e^{-\delta T} S(T^-)$$

Cancelling out we arrive at the following expression

$$\Pi = \int_0^T e^{-\delta t} R(t)u(t)S(t)dt$$

$$- \int_0^T C_2 e^{-\delta z} f(z)dz$$

$$+ \{q(T)x(T)-C_1\}\ e^{-\delta T} S(T^-)$$

Evaluate the second integral by parts

$$\Pi = \int_0^T e^{-\delta t} R(t)u(t)S(t)dt$$

$$- C_2\{[-e^{-\delta z}S(z)]\Big|_0^T - \int_0^T \delta e^{-\delta z}S(z)dz\}$$

$$+ \{q(T)x(T)-C_1\}\ e^{-\delta T}S(T^-)$$

Combine the two integrals and simplify the result of the integration by parts to obtain

$$\Pi = \int_0^T e^{-\delta t}\{R(t)u(t)+\delta C_2\}S(t)dt$$

$$+ C_2\{e^{-\delta T}S(T)-1\} + \{q(T)x(T)-C_1\}e^{-\delta T}S(T^-)$$

and a further simplification gives

$$\Pi = \int_0^T e^{-\delta t}\{R(t)u(t)+\delta C_2\}S(t)dt$$

$$+ \{q(T)x(T)+C_2-C_1\}e^{-\delta T}S(T^-) - C_2$$

which is expression (2.9).

Derivation (5.12)

We show the detailed derivation of equation (5.12).

Proof:

Differentiate both sides of (5.11)

$$\lambda_1'(t) = \frac{-[\delta+y'(t)]e^{-\delta t-y(t)}U_Q - e^{-\delta t-y(t)}\frac{d}{dt}\{U_Q\}}{[U_Q]^2}$$

$$= -\frac{e^{-\delta t-y(t)}W_x}{U_Q}$$

by the adjoint equation (3.11) and equation (5.11). Cancelling out factors and cross multiplying we have

$$\text{A.3} \qquad [\delta+y'(t)]U_Q + \frac{d}{dt}\{U_Q\} = W_x U_Q$$

Carry out the differentiation operation in the above equation and factorize the other expressions to

$$U_{tQ} + U_{QQ}Q'(t) = [W_x - \delta - h(t)]U_Q$$

Expressing Q'(t) in terms of the other terms gives the equation (5.12).

Derivation (6.19)

We show the detailed derivation of equation (6.19).

Proof:

Since the condition (5.11) also applies to this problem we use the first result of Derivation (5.12) to obtain

$$\lambda_1'(t) = \frac{-[\delta+y'(t)]e^{-\delta t-y(t)}U_Q - e^{-\delta t-y(t)}\frac{d}{dt}\{U_Q\}}{[U_Q]^2}$$

$$= \quad - \frac{e^{-\delta t-y(t)}W_x}{U_Q} - \lambda_2 a(t)\phi'(x)$$

by the adjoint equation (6.6) and the equation (5.11). Now perform the total derivative

$$\frac{[\delta+y'(t)]e^{-\delta t-y(t)}U_Q + e^{-\delta t-y(t)}\{U_{tQ} + U_{QQ}Q'(t)\}}{[U_Q]^2}$$

$$= \quad \frac{e^{-\delta t-y(t)}W_x}{U_Q} + \lambda_2 a(t)\phi'(x)$$

Now solve for Q'(t) to obtain (6.19).

Derivation (6.28)

Show that (6.28) derives from (6.27)

Proof:

Differentiate both sides of (6.27)

$$\lambda_2'(t) = \frac{[\lambda_1'(t)-\{R'(t)-R(t)(\delta+y'(t))\}e^{-\delta t-y(t)}]a(t)b'(u) -[\lambda_1-R(t)e^{-\delta t-y(t)}]\{a'(t)b'(u)+a(t)b''(u)u'(t)\}}{[a(t)b'(u)]^2}$$

$$= [R(t)u(t) - \delta(L-C_2)]e^{-\delta t-y(t)}$$

by equation (3.12). Cross multiplying we have

$$[\lambda_1'(t)-\{R'(t)-R(t)(\delta+y'(t))\}e^{-\delta t-y(t)}]a(t)b'(u) - [\lambda_1 - R(t)e^{-\delta t-y(t)}]\{a'(t)b'(u)+a(t)b''(u)u'(t)\}$$

$$= [R(t)u(t) - \delta(L-C_2)]e^{-\delta t-y(t)}[a(t)b'(u)]^2$$

Now make u'(t) the subject of the above equation to obtain

$$u'(t) = \frac{[\lambda_1'(t)-\{R'(t)-R(t)(\delta+y'(t))\}e^{-\delta t-y(t)}]a(t)b'(u) - [\lambda_1 - R(t)e^{-\delta t-y(t)}]\{a'(t)b'(u)\} - [R(t)u(t) - \delta(L-C_2)]e^{-\delta t-y(t)}[a(t)b'(u)]^2}{[\lambda_1-R(t)e^{-\delta t-y(t)}]a(t)b''(u)}$$

which is the required result (note that $y'(t)=h(t)b(u)$, see (6.6), and the expression for $\lambda_1'(t)$ in (3.11)).

APPENDIX B

THE PONTRYAGIN MAXIMUM PRINCIPLE

We now state the Pontryagin Maximum Principle in the form used in this thesis. For detailed discussion and heuristic proof of the theorem see eg. Kamien and Schwartz (1981). The optimization problems solved in this thesis can be stated in a general form as: Select a terminal time T and a control u(t), in the control set $0 \leq u(t) \leq u_{max}$, over the time interval $0 \leq t \leq T$ to

maximize

$$\int_0^T D(t,x(t),y(t),u(t))dt \quad + \quad G(T,x(T),y(T))$$

subject to the dynamic equations

B.1 $$x'(t) = A(t,x,u) \qquad x(0)=x_0$$

B.2 $$y'(t) = h(t,x,u) \qquad y(0)=0 \ ,$$

The state variables are x, and y. The functions G, D are respectively called terminal payoff (or "scrap-value"), revenue functions and are known continuously differentiable functions of the arguments, none of which is a derivative. To state the Maximum Principle we define the Hamiltonian function, H, as

$$H = D(t,x,y,u) + \lambda_1 A(t,x,u) + \lambda_2 h(t,x,u)$$

where λ_1 and λ_2 are additional unknown differentiable functions called multiplier functions of t on $0 \leq t \leq T$. The Pontryagin Maximum Principle states that if there is an optimal solution $u^*(t)$ which is piecewise continuous on the interval $[0,T]$ and $x^*(t)$ is the corresponding optimal trajectory for $x(t)$, then it is necessary that there exist differentiable multiplier functions $\lambda_1(t)$, $\lambda_2(t)$ such that the differential equations (B.1), (B.2), and (B.3), (B.4) below hold.

B.3 $$\lambda_1'(t) = -H_x$$

B.4 $$\lambda_2'(t) = -H_y$$

together with the following boundary (or "transversality") conditions for λ_1 and λ_2

B.5 $$\lambda_1(T) = G_x(T,x(T),y(T))$$

B.6 $$\lambda_2(T) = G_y(T,x(T),y(T))$$

B.7 $$H(T,x(T),y(T),u(T),\lambda_1(T),\lambda_2(T)) + G_T(T,x(T),y(T)) = 0$$

In addition $u^*(t)$ maximizes $H(t,x^*,y,u,\lambda_1,\lambda_2)$ for every t in the interval [0,T]. The previous statement means that if $0<u^*<u_{max}$ then $H_u=0$ at u^*, if $u^*=0$ then $H_u\leq 0$ at u^*, and if $u^*=u_{max}$ then $H_u\geq 0$ at u^*. If the terminal time T is fixed (and is not at the choice of the controller) then the free-terminal time condition (B7) is not required.

APPENDIX C

THE ROUTINE TO SOLVE FOR THE SINGULAR PATH

```
C THIS PROGRAM COMPUTES THE SINGULAR PATHS FOR VARIOUS
C DISCOUNT RATE AND AGE DEPENDENT HAZARD FUNCTIONS.
      DOUBLE PRECISION VOL(8,16), DELTA(8)
      DOUBLE PRECISION A, B, C, EPS, ETA, H, A2, B2, X
      DOUBLE PRECISION FX
      INTEGER          IFAIL, I, J, NT(16)
      EXTERNAL         FX
      COMMON           A, B, C, DELTA, NT, I, J

C INITIALIZATION OF CONSTANTS IN SINGULAR
C PATH EQUATION (3.22)
      A     = 11.38D0
      B     = 1.23D0
      C     = 0.003D0

C INITIALIZATION OF ARRAY NT TO CONTAIN AGES AT WHICH
C SINGULAR PATH VALUES ARE TO BE COMPUTED
      DO 16 L=1,16
      NT(L)=50+(L-1)*5
   16 CONTINUE

C INITIALIZATION OF ARGUMENTS FOR THE SUBROUTINE C05AGF
C (SEE ANONYMOUS 1984) WHICH SOLVES THE SINGULAR
C PATH EQUATION
      H     = 2.0D0
      EPS   = 1.0D-5
      ETA   = 0.0D0
      IFAIL= 0

C FOR A FIXED DISCOUNT RATE PROGRAM SEGMENT SOLVES THE
C CORRESPONDING SINGULAR PATH WITH AGE VARYING FROM 50 TO
C 125 DISCOUNT RATE (ARRAY DELTA) VARIES FROM 0.01 TO 0.08
      DO 12 L=1,8
      J     = L
      DELTA(J) = 0.01D0*J

      DO 10 M=1,16
```

```
      I     = M
      X     = 300.0D0

      CALL C05AGF(X, H, EPS, ETA, FX, A2, B2, IFAIL)

      VOL(L,M) = X
   10 CONTINUE
   12 CONTINUE

C WRITING OF RESULTS
      WRITE(8,6)
    6 FORMAT('1',25X,'DISCOUNT RATE')
      WRITE(8,8) (DELTA(I),I=1,8)
    8 FORMAT('0','AGE(YEARS)',4X,F3.2,7(5X,F3.2))

      DO 4 I=1,16
      WRITE(8,2) NT(I),(VOL(M,I),M=1,8)
    2 FORMAT('0',3X,I3,4X,8(3X,F5.1))
    4 CONTINUE

      STOP
      END

C SUBPOGRAM EVALUATES VALUE OF FUNCTION REPRESENTING
C SINGULAR PATH
      DOUBLE PRECISION FUNCTION FX(XX)
      DOUBLE PRECISION   RST1, RST2, RST3, HT, DEXP
      DOUBLE PRECISION   A, B, C, DELTA(8), XX
      INTEGER            NT(16), I, J
      COMMON             A, B, C, DELTA, NT, I, J

      HT  = 0.013D0
C     HT  = 0.0267D0*DEXP(-DFLOAT(NT(I))**2/1800)
      RST1= (1-C*XX)*DEXP(-C*XX)
      RST2= DELTA(J)-(0.337D0/(0.337D0*NT(I)+5.06D0))+HT
      RST3= (NT(I)**B)/A
      FX  = RST1-RST2*RST3
      RETURN
      END
```

APPENDIX D

THE ROUTINE TO CHECK FOR BLOCKED INTERVALS

```
C PROGRAM COMPUTES SINGULAR PATH AND GROWTH (WITHOUT
C THINNIG) OVER AGE INTERVAL [50,70] FOR  DISCOUNT
C RATE OF 0.01 AND HAZARD FUNCTION OF
C 0.0267EXP(-T^2/1800)
      DOUBLE PRECISION A2, B2, X, VOL(30), DELTA
      DOUBLE PRECISION A, B, C, EPS, ETA, H
      DOUBLE PRECISION HG, TEND, TENDG, TOLG
      DOUBLE PRECISION TG, WG(1,7), YG, TN(30)
      DOUBLE PRECISION FX
      INTEGER          IFAIL, I
      INTEGER          IFAILG, IRG, JG, NG, IG
      EXTERNAL         FX, FCNG, OUTG
      COMMON           A,B,C,DELTA,TN,TENDG,HG,I,IG

C INITIALIZATION OF CONSTANTS IN THE SINGULAR PATH, AND
C NATURAL GROWTH EQUATIONS (2.1) AND (3.22)
      A    = 11.38D0
      B    = 1.23D0
      C    = 0.003D0

C INITIALIZATION OF ARRAY NT WITH AGES AT SINGULAR PATH
C AND NATURAL GROWTH VALUES ARE TO BE COMPUTED
      DO 16 L=1,21
      TN(L)=50.0D0+(L-1)
   16 CONTINUE

C INITIALIZATION OF ARGUMENTS FOR
C FOR C05AGF (SEE ANONYMOUS 1984)
C TO SOLVE SINGULAR PATH FOR FIXED DISCOUNT RATE (DELTA)
      H    = 2.0D0
      EPS  = 1.0D-5
      ETA  = 0.0D0
      IFAIL= 0
      DELTA= 0.01D0

      DO 10 M=1,21
      I    = M
```

```
      X     = 300.0D0

      CALL C05AGF(X, H, EPS, ETA, FX, A2, B2, IFAIL)

      VOL(M)= X
   10 CONTINUE

C INITIALIZATION OF ARGUMENTS FOR SUBROUTINE D02BBF (SEE
C ANONYMOUS 1984) NATURAL GROWTH EQUATION (2.1) WITH
C INITIAL CONDITION 129 AT AGE 50
      NG    = 1
      IRG   = 0
      TOLG  = 1.0D-5
      TG    = TN(1)
      TENDG = TN(1)+20.0D0
      TEND  = TN(1)+20.0D0
      YG    = VOL(1)
      HG    = (TENDG-TG)/20
      WRITE(6,*) 'DEBUG : TN(1), TG, TENDG, HG',
     *                    TN(1), TG, TENDG, HG
      IG    = 19
      IFAILG= 1

      CALL D02BBF(TG, TEND, NG, YG, TOLG, IRG,
     *            FCNG, OUTG, WG, IFAILG)

      WRITE(8,99997) IFAILG
      IF(TOLG.LT.0) WRITE(8,99995)
      WRITE(8,6)
    6 FORMAT('1',25X,'DISCOUNT RATE')
      WRITE(8,8) DELTA
    8 FORMAT('0','AGE(YEARS)',4X,F3.2)

C PRINTING SINGULAR PATH
      DO 4 I=1,21
      WRITE(8,2) TN(I), VOL(I)
    2 FORMAT('0',3X,F5.1,7X,F5.1)
    4 CONTINUE

      STOP
99995 FORMAT(24H RANGE TOO SHORT FOR TOL)
99997 FORMAT(8H IFAIL =, I1)
      END

C SUBPROGRAM EVALUATES VALUE OF FUNCTION (3.22) FOR
C SOLVING THE SINGULAR PATH
      DOUBLE PRECISION FUNCTION FX(XX)
      DOUBLE PRECISION A, B, C, DELTA, XX
      DOUBLE PRECISION RST1, RST2, RST3, HT, DEXP
      DOUBLE PRECISION TENDG, HG, TN(30)
```

```
      INTEGER            I, IG
      COMMON             A,B,C,DELTA,TN,TENDG,HG,I,IG

      HT   = 0.0267D0*DEXP(-(TN(I)**2)/1800)

      RST1 = (1-C*XX)*DEXP(-C*XX)
      RST2 = DELTA-(0.337D0/(0.337D0*TN(I)+5.06D0))+HT
      RST3 = (TN(I)**B)/A
      FX   = RST1-RST2*RST3
      RETURN
      END

C COMPUTES THE GROWTH RATE FUNCTION (4.2)
      SUBROUTINE FCNG(T, Y, F)
      DOUBLE PRECISION T,Y,F, DEXP
      F    = 11.38D0*(T**(-1.23D0))*Y*DEXP(-0.003D0*Y)
      RETURN
      END

C PRINTS VOLUME FROM NATURAL GROWTH AT SPECIFIED AGES
      SUBROUTINE OUTG(T,Y)
      DOUBLE PRECISION DFLOAT, A, B, C, DELTA
      DOUBLE PRECISION T, Y, HG, TENDG, TN(30)
      INTEGER            I, IG
      COMMON             A,B,C,DELTA,TN,TENDG,HG,I,IG
      WRITE(8,99999) T,Y
      WRITE(6,*) 'DEBUG : TENDG', TENDG
      T    = TENDG-IG*HG
      WRITE(6,*) 'DEBUG : T', T
      IG   = IG - 1
      RETURN
99999 FORMAT(1H , F7.2,5X,F7.2)
      END
```

APPENDIX E

THE ROUTINE TO SOLVE FOR SWITCH-ON AGE

```
C PROGRAM COMPUTES INITIAL THINNING AGE FOR VARIOUS
C DISCOUNT RATES AND AGE DEPENDENT HAZARD FUNCTION
      DOUBLE PRECISION HMAX, TOL, TI, TEND, W(1,7), XI
      DOUBLE PRECISION DEL(8), TR, XR
      DOUBLE PRECISION G
      INTEGER          I, IFAIL, IRELAB, J, N, NOUT, M
      EXTERNAL         FCN, G
      COMMON           DEL, M
      DATA NOUT /6/

      WRITE(NOUT,99999)
C INITIALIZATION OF ARGUMENT FOR SUBROUTINE D02BHF (SEE
C ANONYMOUS 1984) FOR SOLVING EQUATION (2.1) UNTIL SINGULAR
C PATH HOLDS TO DETERMINE INITIAL THINNING AGE. VARIABLES
C TI AND XI ARE INITIAL CONDITIONS FOR THE EQUATION AND
C ARRAY DEL CONTAINS DISCOUNT RATES.
      IRELAB=1
      HMAX  = 0.01D0
      TI    = 50.0D0
      XI    = 129.0D0
      N     = 1
      TEND  = 110.0D0
      WRITE(6,*) 'DEBUG : HT=.013 ; TI, XI =', TI, XI

      DO 30 M=1,8
      DEL(M)= 0.01D0*M
      TOL   = 1.0D-5
      TR    = TI
      XR    = XI
      IFAIL = 0

      CALL D02BHF(TR, TEND, N, XR, TOL,
     *            IRELAB, HMAX, FCN, G, W, IFAIL)

      WRITE(NOUT,99998) IFAIL
      IF(TOL.LT.0) WRITE(NOUT,99996)
      IF(IFAIL.EQ.0) WRITE(NOUT,99997) X,Y
```

```
C  20 CONTINUE

C PRINTING OF RESULTS
      WRITE(6,*) 'DEBUG : D.R, T1 =', DEL(M), TR
C     IF(M.LT.3) GO TO 30
C     XEND=40.0D0
   30 CONTINUE

C     WRITE(NOUT,99992)

      STOP
99999 FORMAT(4(1X/),39H DETERMINATION OF INITIAL
     *       THINNING TIME/1X)
99998 FORMAT(17H IN D02BHF IFAIL=, I3)
99997 FORMAT(16H ROOT OF Y(1) AT,F7.4/12H SOLUTION IS,E13.5)
99996 FORMAT(40H OVER ONE-THIRD STEPS CONTROLLED BY HMAX)
99995 FORMAT(22H CALCULATION WITH TOL=, E8.1)
C9994 FORMAT(' ',7X,F3.2,9X,F4.1)
C9993 FORMAT(' ',4X,2F10.5)
C9992 FORMAT('1',2X,'DISCOUNT RATE',4X,'T1')
      END


C SUBPROGRAM COMPUTES NATURAL GROWTH RATE (4.2)
      SUBROUTINE       FCN(T, X, F)
      DOUBLE PRECISION     T, X, F
      DOUBLE PRECISION     DEXP

      F      = 11.38D0*(T**(-1.23D0))*X*DEXP(-0.003D0*X)

      RETURN
      END


C CHECKS WHETHER AGE AND VOLUME FROM NATURAL GROWTH SATISFY
C SINGULAR PATH EQUATION (3.22)
      DOUBLE PRECISION FUNCTION G(T, X)
      DOUBLE PRECISION              T, X
      DOUBLE PRECISION              RST1, R, DEL(8), HT, R1
      DOUBLE PRECISION              DEXP
      INTEGER                       M
      COMMON                        DEL, M

      HT     = 0.013D0

      RST1   = (1-0.003D0*X)*DEXP(-0.003D0*X)
      R      = (T**1.23D0)/11.38D0
      R1     = (DEL(M)-(0.337D0/(0.337D0*T+5.06D0))+HT)

      G      = RST1-R*R1
```

```
      RETURN
      END
```

APPENDIX F

THE ROUTINE TO SOLVE FOR SWITCH-OFF AND ROTATION AGE FOR THE SINGLE ROTATION PROBLEM

```
C PROGRAM COMPUTES FINAL THINNING TIME AND ROTATION AGE
C FOR THE SINGLE ROTATION PROBLEM GIVEN DISCOUNT RATE
C AND AGE DEPENDENT HAZARD
      DOUBLE PRECISION T2, A, B, H, EPS, ETA
      DOUBLE PRECISION T, Y(2), DEL, C1, C2, RO, SV, TT1
      DOUBLE PRECISION F
      INTEGER          IFAIL
      EXTERNAL         F
      COMMON           T, Y, DEL, C1, C2, RO, SV, TT1
      DATA             NOUT /6/

C INITIALIZATION OF CONSTANTS IN EQUATIONS TO BE SOLVED
C DEL (DISCOUNT RATE), T2 (GUESSED INITIAL THINNING AGE),
C RO (REVENUE RATIO q/R), C1 AND C2 ARE REFORESTATION
C COST, SV IS THE SITE VALUE.
      DEL  = 0.08D0
      T2   = 35.00D0
      RO   = 1.02D0
      C1   = 0.0D0
      C2   = 0.0D0
      SV   = 0.0D0
      WRITE(NOUT,99999)
      WRITE(6,*) 'DEBUG : H1; DEL, T2, RO =', DEL, T2, RO
      WRITE(6,*) 'DEBUG : C1, C2, L =', C1, C2, SV

C INITIALIZATION OF PARAMETERS FOR C05AGF (SEE ANONYMOUS
C 1984) TO SOLVE THE TRANSVERSALITY CONDITION (3.13)
      H    = 0.2D0
      EPS  = 1.0D-5
      ETA  = 0.0D0
      IFAIL=0
C     WRITE(6,*) 'DEBUG : DEL , RO , & T2 =', DEL, RO, T2

      CALL C05AGF(T2, H, EPS, ETA, F, A, B, IFAIL)

      WRITE(NOUT,99998) IFAIL
```

```
C PRINTING OF RESULTS
      WRITE(NOUT,99997) T2, T
      STOP
99999 FORMAT(4(1X/),48H DETERMINATION OF FINAL
     *       THINNING & ROTATION TIME)
99998 FORMAT(17H IN C05AGF IFAIL=, I3)
99997 FORMAT(10H T2   =    ,F13.5,10X, 7H T  =  ,F13.5 )
      END


C EVALUATES THE THE CONDITION (3.13) AFTER T HAS BEEN FOUND
C TO SATISFY CONDITION (3.17)
      DOUBLE PRECISION FUNCTION F(TT)
      DOUBLE PRECISION            TT
      DOUBLE PRECISION X, EPS1, ETA1
      DOUBLE PRECISION FX, G, S15AEF, DSQRT, DEXP
      DOUBLE PRECISION TS, TEND, Y1(2), TOL, HMAX, W(2,20)
      DOUBLE PRECISION T, Y(2), DEL, C1, C2
      DOUBLE PRECISION RO, SV, YTF, YT, XYT, TT1
      INTEGER          NFMAX, IFAIL1
      INTEGER          N, IRELAB, IFAIL2, IFAIL3
      EXTERNAL         FX, G, FCN
      COMMON           T, Y, DEL, C1, C2, RO, SV, TT1

C PASSING TIME ARGUMENT TO SUBPROGRAM FX
      TT1   = TT
C INITIALIZATION OF PARAMETERS FOR C05AJF
C (SEE ANONYMOUS 1984) WHICH SOLVES FOR
C INITIAL VOLUME CONDITION FOR SYSTEM (4.1)
      X     = 200.8D0

      EPS1  = 1.0D-5
      ETA1  = 0.0D0
      NFMAX = 200
      IFAIL1= 0

      CALL C05AJF(X, EPS1, ETA1, FX, NFMAX, IFAIL1)

C INITIALIZATION OF PARAMETERS FOR D02BHF
C (SEE ANONYMOUS 1984) WHICH SOLVES
C SYSTEM (4.1) UNTIL CONDITION (3.17) HOLDS
      TS    = TT
      TEND  = 200.0D0
      N     = 2
      Y1(1) = X

C     IFAIL3= 1
C     XYT   = TT/DSQRT(5000.0D0)
C     YT    = S15AEF(XYT,IFAIL3)
C     YTF   = 1.0465173D0*YT
```

```
      YTF   = 0.013D0*TT
      Y1(2) = (0.337D0*TT+5.06D0)*DEXP(-DEL*TT-YTF)

      TOL   = 1.0D-5
      IRELAB= 1
      HMAX=0.0D0
      IFAIL2=0

      CALL D02BHF(TS, TEND, N, Y1, TOL, IRELAB,
     *            HMAX, FCN, G, W, IFAIL2)


C     IFAIL3= 1
C     XYT   = TS/DSQRT(5000.0D0)
C     YT    = S15AEF(XYT,IFAIL3)
C     YTF   = 1.0465173D0*YT
      YTF   = 0.013D0*TS
C EVALUATING CONDITION (3.13)
      F     = Y1(2)-RO*(0.337D0*TS+5.06D0)*DEXP(-DEL*TS-YTF)

      Y(1)  = Y1(1)
      Y(2)  = Y1(2)
      T     = TS
      WRITE(6,*) 'DEBUG : T2, T =', TT, TS
      RETURN
      END

C EVALUATES RIGHT HAND SIDES OF SYSTEM (4.1)
      SUBROUTINE   FCN(TTT, Y1,     FS)
      DOUBLE PRECISION TTT, Y1(2), FS(2)
      DOUBLE PRECISION FZ1, FZ2, R1, R2
      DOUBLE PRECISION DEXP

      R1    = 11.38D0*(TTT**(-1.23D0))
      R2    = DEXP(-0.003D0*Y1(1))
      FS(1) = R1*Y1(1)*R2
      FZ1   = -Y1(2)*R1
      FZ2   = R2*(1-0.003D0*Y1(1))
      FS(2) = FZ1*FZ2
      RETURN
      END

C EVALUATES (3.17) FOR GIVEN AGE
      DOUBLE PRECISION FUNCTION G(TTT, Y1)
      DOUBLE PRECISION            TTT, Y1(2) ,REV
      DOUBLE PRECISION T, Y(2), DEL, C1, C2, RO, SV, TT1
      DOUBLE PRECISION R1, R2, R3, R4, HTG
      DOUBLE PRECISION DEXP
      COMMON           T, Y, DEL, C1, C2, RO, SV, TT1
```

```
      REV   = RO*(0.337D0*TTT+5.06D0)
      HTG   = 0.013D0
C     HTG   = 0.0167D0*DEXP(-TTT**2/5000)
C     WRITE(6,*) 'DEBUG : HTG= ', HTG
      R1    = REV*11.38D0
      R2    = (TTT**(-1.23D0))*Y1(1)*DEXP(-0.003D0*Y1(1))
      R3    = (REV*Y1(1)+C2-C1)*HTG
      R4    = (REV*Y1(1)-C1+SV)*DEL
      G=R1*R2-R3-R4
      RETURN
      END

C EVALUATES SINGULAR PATH FUNCTION (3.22)
      DOUBLE PRECISION FUNCTION FX(XX)
      DOUBLE PRECISION             XX
      DOUBLE PRECISION T, Y(2), DEL, C1, C2, RO, SV, TT1
      DOUBLE PRECISION RST1, RST2, RST3, HT
      DOUBLE PRECISION DEXP
      COMMON           T, Y, DEL, C1, C2, RO, SV, TT1

C     HT    = 0.0167D0*DEXP(-TT1**2/5000)
      HT    = 0.013D0
C     WRITE(6,*) 'DEBUG : HT= ',HT
      RST1  = (1 - 0.003D0*XX)*DEXP(-0.003D0*XX)
      RST2  = DEL -(0.337D0/(0.337D0*TT1 + 5.06D0)) + HT
      RST3  = (TT1**1.23D0)/11.38D0
      FX    = RST1 - RST2*RST3
      RETURN
      END
```

APPENDIX G

THE ROUTINE TO SOLVE FOR SWITCH-OFF AND ROTATION AGE FOR THE "ONGOING" ROTATIONS PROBLEM

```
C PROGRAM COMPUTES FINAL THINNING TIME AND ROTATION AGE
C FOR THE "ONGOING" ROTATIONS PROBLEM FOR GIVEN DISCOUNT
C RATE AND AGE DEPENDENT HAZARD FUNCTION
      DOUBLE PRECISION T, Y(2), DEL, C1, C2, RO
      DOUBLE PRECISION SV, T1, T2, TT1, TR
      DOUBLE PRECISION FO, SCALE, TOL, XJG, C(26)
      DOUBLE PRECISION FG, DSQRT, X02AAF
      INTEGER          IFAIL, IND, IR
      EXTERNAL         FG
      COMMON           T,Y,DEL,C1,C2,RO,SV,T1,T2,TT1,TR

C INITIALIZATION OF PARAMETERS FOR C05AXF
C (SEE ANONYMOUS 1984) TO SOLVE (3.1).
C RO, DEL, C1, C2 ARE AS DEFINED PREVIOUSLY.
C TR AND TI ARE AGE VARIABLES USED IN PASSING ARGUMENTS
C TO SUBPROGRAMS
      XJG   = 300.0D0
      DEL   = 0.01D0
      T1    = 62.79D0
      TR    = 78.92D0
      RO    = 1.01D0
      C1    = 800.0D0
      C2    = 900.0D0
      WRITE(6,*) 'DEBUG : DEL, RO, T1 =', DEL, RO, T1
      WRITE(6,*) 'DEBUG : C1, C2, J =', C1, C2, XJG
      WRITE(6,*) '**'

      SCALE = XJG*DSQRT(X02AAF( 0.0D0))
      IR    = 0
      IND   = 1
      IFAIL = 0
      TOL   = 1.0D-5

   20 CALL C05AXF(XJG, FO, TOL, IR, SCALE, C, IND, IFAIL)

      IF(IND.EQ.0) GO TO 40
```

```
      FO    = FG(XJG)
      WRITE(6,*) '***'
      GO TO 20
   40 WRITE(6,*) 'DEBUG : TOL =', TOL
      IF(IFAIL.GT.0) GO TO 60
C PRINTING OF RESULTS
      WRITE(6,*) 'DEBUG : J, T1, T2, T =', XJG, T1, T2, T
   60 WRITE(6,*) 'DEBUG : IFAIL OF MAIN =', IFAIL
      STOP
      END

C EVALUATES (3.1) BY FIRST DETERMINING FINAL THINNING AND
C CUTTING AGE WITH INITIAL THINNING AGE GIVEN IN MAIN AS TI.
      DOUBLE PRECISION FUNCTION FG(RJ)
      DOUBLE PRECISION            RJ
      DOUBLE PRECISION T2, EPS, ETA, T3, A1, B1, H
      DOUBLE PRECISION YTF, XYT, YT, R1
      DOUBLE PRECISION T, Y(2), DEL, C1, C2
      DOUBLE PRECISION RO, SV, T1, TT1, TR
      DOUBLE PRECISION D01AHF, DSQRT, S15AEF, F, FUN, DEXP
      DOUBLE PRECISION A, B, ANS, RELERR
      INTEGER          IFAIL, N, NLIMIT
      EXTERNAL         F, FUN
      COMMON           T,Y,DEL,C1,C2,RO,SV,T1,T2,TT1,TR
      DATA             NOUT /6/
C PASSING ARGUMENT
      SV    = RJ
C     WRITE(6,*) 'DEBUG : L = ', SV
C INITIALIZATION OF PARAMETERS FOR C05AGF (SEE ANONYMOUS
C 1984) TO SOLVE FOR FINAL THINNING AGE
      T3    = TR
      H     = 0.2D0
      EPS   = 1.0D-5
      ETA   = 0.0D0
      IFAIL = 0

      CALL C05AGF(T3, H, EPS, ETA, F, A1, B1, IFAIL)

      T2    = T3
C     WRITE(NOUT,99998) IFAIL
      WRITE(6,*) 'DEBUG : T1, T2, T =', T1, T2, T

      YTF   = 0.013D0*T

C     XYT   = T/DSQRT(5000.0D0)
C     IFAIL = 0
C     YT    = S15AEF(XYT,IFAIL)
C     YTF   = 1.0465173D0*YT
C EVALUATION OF (3.1) STARTS
      R1    = (RO*Y(1)*(0.337D0*T+5.06D0) + C2-C1)
```

```
     *         *DEXP(-DEL*T-YTF) -C2
C     WRITE(6,*) 'DEBUG : R1  =', R1

      IF (T1.LT.T2) GO TO 80
      ANS   = 0.0D0
      GO TO 100
C INITIALIZATION OF PARAMETERS FOR D01AHF (SEE ANONYMOUS
C 1984) TO EVALUATE INTEGRAL IN (3.1)
   80 A     = T1
      B     = T2
      NLIMIT= 0
      EPS   = 1.0D-5
      IFAIL = 0

      ANS   = D01AHF(A, B, EPS, N, RELERR, FUN,
     *               NLIMIT, IFAIL)

C     WRITE(6,99996) IFAIL
  100 FG    = R1+ANS
      WRITE(6,*) 'DEBUG : ANS, R1, FG L =', ANS, R1, FG, SV
      RETURN

99998 FORMAT(24H IN C05AJF OF FG IFAIL =, I3)
99996 FORMAT(24H IN D01AHF OF FG IFAIL =, I3)
      END

C EVALUATES THE VALUE OF THE INTEGRAND IN (3.1)
C THIS ALSO INVOLVES FINDING THINNING RATES FOR GIVEN AGE
      DOUBLE PRECISION FUNCTION FUN(TT)
      DOUBLE PRECISION              TT
      DOUBLE PRECISION T, Y(2), DEL, C1, C2
      DOUBLE PRECISION RO, SV, T1, T2, TT1, R5
      DOUBLE PRECISION YTF, R1, PI, RT, XYT, S1, S2, TR
      DOUBLE PRECISION S15AEF, X01AAF, U, DSQRT, DEXP
      INTEGER          IFAIL
      EXTERNAL         U
      COMMON           T,Y,DEL,C1,C2,RO,SV,T1,T2,TT1,TR

      RT    = 0.337D0*TT+5.06D0

      YTF   = 0.013D0*TT

C     PI    = X01AAF(PI)
C     S1    = 0.0167D0*DSQRT(5000.0D0*PI)/2
C     WRITE(6,*) 'DEBUG : S1 =', S1
C     IFAIL = 1
C     XYT   = TT/DSQRT(5000.0D0)
C     S2    = S15AEF(XYT,IFAIL)
C     YTF   = S1*S2
C U IS FUNCTION SUBPROGRAM FOR SOLVING THINNING RATE
```

```
      R5    = U(TT)
C     WRITE(6,*) 'DEBUG : UT, YTF, TT =', R5, YTF, TT
      R1    = DEXP(-DEL*TT-YTF)
C     WRITE(6,*) 'DEBUG : RT, R1, R5 =', RT, R1, R5
C     WRITE(6,*) 'DEBUG : SV, C2, DEL =', SV, C2, DEL
      FUN   = (RT*R5 - DEL*(SV-C2))*R1
C     WRITE(6,*) 'DEBUG : FUN =', FUN
      RETURN
      END

C SUBPROGRAM FOR SOLVING THINNING RATE
      DOUBLE PRECISION FUNCTION U(TT)
      DOUBLE PRECISION            TT
      DOUBLE PRECISION T, Y(2), DEL, C1, C2
      DOUBLE PRECISION RO, SV, T1, T2, TT1, TR
      DOUBLE PRECISION U4, EPS4, ETA4, H, A, B
      DOUBLE PRECISION FX4
      INTEGER          IFAIL
      EXTERNAL         FX4
      COMMON           T,Y,DEL,C1,C2,RO,SV,T1,T2,TT1,TR
C PASSSING ARGUMENT TO SUBPROGRAM FX
      TT1   = TT
C INITIALIZATION OF PARAMETERS FOR C05AGF (SEE ANONYMOUS
C 1984) TO SOLVE FOR THINNING RATE AT GIVEN AGE
      U4    = 10.0D0
      H     = 0.1D0
      EPS4  = 1.0D-5
      ETA4  = 0.0D0
      IFAIL = 0

      CALL C05AGF(U4, H, EPS4, ETA4, FX4, A, B, IFAIL)

      U     = U4
      RETURN
      END

C EVALUATES FUNCTION FOR OPTIMAL THINNING
      DOUBLE PRECISION FUNCTION FX4(UU)
      DOUBLE PRECISION              UU
      DOUBLE PRECISION RS1, RS2, RS3, RS4, HT, HPT, RS12
      DOUBLE PRECISION S1, S2, S3, S4
      DOUBLE PRECISION DEXP, DSQRT, X02AAF, FX
      DOUBLE PRECISION X, EPS1, ETA1
      DOUBLE PRECISION T, Y(2), DEL, C1, C2
      DOUBLE PRECISION RO, SV, T1, T2, TT1, TR
      INTEGER          IFAIL, IND, IR, NFMAX
      EXTERNAL         FX
      COMMON           T,Y,DEL,C1,C2,RO,SV,T1,T2,TT1,TR

C INITIALIZATION OF PARAMETERS FOR C05AJF (SEE ANONYMOUS
```

```
C 1984) FOR SOLVING SINGULAR PATH
      EPS1  = 1.0D-5
      ETA1  = 0.0D0
      X     = 100.0D0
      NFMAX = 200
      IFAIL = 0

      CALL C05AJF(X, EPS1, ETA1, FX, NFMAX, IFAIL)

      HT    = 0.013D0
      HPT   = 0.0D0
C     HT    = 0.0167D0*DEXP(-TT1**2/5000)
C     HPT   = 2*TT1*HT/5000

      RS1   = 11.38D0*(TT1**(-1.23D0))*X
      RS12  = DEXP(-0.003D0*X)
      RS2   = RS12*(2.0D0-0.003D0*X)
      S1    = 1.23D0*(TT1**0.23D0)
      S2    = DEL-(0.337D0/(0.337D0*TT1+5.06D0))+HT
      RS3   = S1*S2
      S3    = (TT1**1.23D0)
      S4    = 0.113569D0/((0.337D0*TT1+5.06D0)**2)+HPT
      RS4   = S3*S4
      FX4   = 0.003D0*(RS12*RS1-UU)*RS2+(RS3+RS4)/11.38D0
      RETURN
C9998 FORMAT(7H TOL = , F10.6)
C9997 FORMAT(9H ROOT IS , F14.6)
99996 FORMAT(37H ERROR EXIT, IN C05AXF OF FX4 IFAIL =, I2)
      END

C EVALUATES (3.13) BY FIRST DETERMINING T SATISFYING (3.17)
      DOUBLE PRECISION FUNCTION F(TT)
      DOUBLE PRECISION           TT
      DOUBLE PRECISION X, EPS1, ETA1
      DOUBLE PRECISION FX, G, S15AEF, DSQRT, DEXP
      DOUBLE PRECISION TS, TEND, Y1(2), TOL
      DOUBLE PRECISION HMAX, W(2,20), T1, T2, TR
      DOUBLE PRECISION T, Y(2), DEL, C1, C2
      DOUBLE PRECISION RO, SV, YTF, YT, XYT, TT1
      INTEGER          NFMAX, IFAIL1
      INTEGER          N, IRELAB, IFAIL2, IFAIL3
      EXTERNAL         FX, G, FCN
      COMMON           T,Y,DEL,C1,C2,RO,SV,T1,T2,TT1,TR
C PASSING ARGUMENT TO FX
      TT1   = TT
C INITIALIZATION OF PARAMETERS FOR C05AJF (SEE ANONYMOUS
C 1984) TO SOLVE INITIAL VOLUME CONDITION FOR SYSTEM (4.1)
      X     = 200.8D0

      EPS1  = 1.0D-5
```

```
      ETA1  = 0.0D0
      NFMAX = 200
      IFAIL1= 0

      CALL C05AJF(X, EPS1, ETA1, FX, NFMAX, IFAIL1)

C INITIALIZATION OF PARAMETERS FOR D02BHF (SEE ANONYMOUS
C 1984) TO SOLVE SYSTEM (4.1) UNTIL CONDITION (3.17) HOLDS
      TS    = TT
      TEND  = 250.0D0
      N     = 2
      Y1(1) = X

C     IFAIL3= 1
C     XYT   = TT/DSQRT(5000.0D0)
C     YT    = S15AEF(XYT,IFAIL3)
C     YTF   = 1.0465173D0*YT
      YTF   = 0.013D0*TT
      Y1(2) = (0.337D0*TT+5.06D0)*DEXP(-DEL*TT-YTF)

      TOL   = 1.0D-5
      IRELAB= 1
      HMAX=0.0D0
      IFAIL2=0

      CALL D02BHF(TS, TEND, N, Y1, TOL, IRELAB, HMAX,
     *              FCN, G, W, IFAIL2)

C     IFAIL3= 1
C     XYT   = TS/DSQRT(5000.0D0)
C     YT    = S15AEF(XYT,IFAIL3)
C     YTF   = 1.0465173D0*YT
      YTF   = 0.013D0*TS
C CALCULATION OF CONDITION (3.13)
      F     = Y1(2)-RO*(0.337D0*TS+5.06D0)*DEXP(-DEL*TS-YTF)

      Y(1)  = Y1(1)
      Y(2)  = Y1(2)
      T     = TS
      RETURN
      END

C EVALUATES RIGHT HAND SIDES OF SYSTEM (4.1)
      SUBROUTINE   FCN(TTT, Y1,    FS)
      DOUBLE PRECISION TTT, Y1(2), FS(2)
      DOUBLE PRECISION FZ1, FZ2, R1, R2
      DOUBLE PRECISION DEXP

      R1    = 11.38D0*(TTT**(-1.23D0))
      R2    = DEXP(-0.003D0*Y1(1))
```

```
      FS(1) = R1*Y1(1)*R2
      FZ1   = -Y1(2)*R1
      FZ2   = R2*(1-0.003D0*Y1(1))
      FS(2) = FZ1*FZ2
      RETURN
      END

C EVALUATES THE CONDITION (3.17)
      DOUBLE PRECISION FUNCTION G(TTT, Y1)
      DOUBLE PRECISION            TTT, Y1(2) ,REV
      DOUBLE PRECISION T, Y(2), DEL, C1, C2
      DOUBLE PRECISION RO, SV, T1, T2, TT1, TR
      DOUBLE PRECISION R1, R2, R3, R4, HTG
      DOUBLE PRECISION DEXP
      COMMON           T,Y,DEL,C1,C2,RO,SV,T1,T2,TT1,TR

      REV   = RO*(0.337D0*TTT+5.06D0)
      HTG   = 0.013D0
C     HTG   = 0.0167D0*DEXP(-TTT**2/5000)
C     WRITE(6,*) 'DEBUG : HTG= ', HTG
      R1    = REV*11.38D0
      R2    = (TTT**(-1.23D0))*Y1(1)*DEXP(-0.003D0*Y1(1))
      R3    = (REV*Y1(1)+C2-C1)*HTG
      R4    = (REV*Y1(1)-C1+SV)*DEL
      G     = R1*R2 - R3 - R4
      RETURN
      END

C EVALUATES SINGULAR PATH FUNCTION (3.22)
      DOUBLE PRECISION FUNCTION FX(XX)
      DOUBLE PRECISION             XX
      DOUBLE PRECISION T, Y(2), DEL, C1, C2
      DOUBLE PRECISION RO, SV, T1, T2, TT1, TR
      DOUBLE PRECISION RST1, RST2, RST3, HT
      DOUBLE PRECISION DEXP
      COMMON           T,Y,DEL,C1,C2,RO,SV,T1,T2,TT1,TR

C     HT    = 0.0167D0*DEXP(-TT1**2/5000)
      HT    = 0.013D0

C     WRITE(6,*) 'DEBUG : HT= ',HT
      RST1  = (1 - 0.003D0*XX)*DEXP(-0.003D0*XX)
      RST2  = DEL -(0.337D0/(0.337D0*TT1 + 5.06D0)) + HT
      RST3  = (TT1**1.23D0)/11.38D0
      FX    = RST1 - RST2*RST3
      RETURN
      END
```

APPENDIX H

SINGULAR PATHS FOR A CONSTANT AND AGE DEPENDENT HAZARDS

Table 2: Singular path $x^{*}(t)$ (m^3/hectare) for a constant hazard of 0.013

AGE	DISCOUNT RATE							
(YEARS)	.01	.02	.03	.04	.05	.06	.07	.08
50	271	213	169	132	101	74	50	29
55	257	197	151	113	82	54	29	7
60	243	181	134	95	63	35	10	-
65	229	166	118	79	46	17	-	-
70	216	151	102	62	29	-	-	-
75	204	137	88	47	13	-	-	-
80	191	124	73	33	-	-	-	-
85	179	111	60	19	-	-	-	-
90	168	99	47	5	-	-	-	-
95	157	87	35	-	-	-	-	-
100	146	76	23	-	-	-	-	-
105	136	65	12	-	-	-	-	-
110	126	54	.4	-	-	-	-	-
115	116	44	-	-	-	-	-	-
120	107	34	-	-	-	-	-	-
125	97	24	-	-	-	-	-	-

Table 3: Singular path $x^*(t)$ (m^3/hectare) for $h(t)=0.0167\exp(-t^2/5000)$.

AGE	DISCOUNT RATE							
(YEARS)	.01	.02	.03	.04	.05	.06	.07	.08
50	292	228	180	142	110	82	57	35
55	287	218	167	127	93	64	39	16
60	283	209	155	113	78	48	22	–
65	280	200	144	100	64	33	6	–
70	277	193	134	88	51	19	–	–
75	275	185	124	77	38	6	–	–
80	273	179	115	66	27	–	–	–
85	272	172	106	56	15	–	–	–
90	270	165	97	46	4	–	–	–
95	268	159	89	36	–	–	–	–
100	266	152	80	26	–	–	–	–
105	264	146	72	17	–	–	–	–
110	261	139	64	8	–	–	–	–
115	257	132	55	–	–	–	–	–
120	254	125	47	–	–	–	–	–
125	249	118	39	–	–	–	–	–

Table 4: Singular path $x^*(t)$ (m^3/hectare) for $h(t)=0.0267\exp(-t^2/1800)$.

AGE	DISCOUNT RATE							
(YEARS)	.01	.02	.03	.04	.05	.06	.07	.08
50	321	248	196	154	120	91	65	42
55	326	244	187	143	107	76	49	25
60	330	239	178	131	93	61	33	9
65	333	233	168	120	80	47	18	-
70	333	226	158	107	67	33	3	-
75	332	218	148	96	54	19	-	-
80	329	210	137	84	41	6	-	-
85	325	201	126	72	28	-	-	-
90	318	191	115	60	16	-	-	-
95	311	182	105	48	4	-	-	-
100	304	172	94	37	-	-	-	-
105	296	162	83	26	-	-	-	-
110	288	153	73	15	-	-	-	-
115	279	143	63	5	-	-	-	-
120	271	134	53	-	-	-	-	-
125	263	125	44	-	-	-	-	-

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