Cuntz-Pimsner Algebras Associated with Substitution Tilings

by

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B.Sc., University of Victoria, 2007
M.Math., University of Waterloo, 2009

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ABSTRACT

A Cuntz-Pimsner algebra is a quotient of a generalized Toeplitz algebra. It is completely determined by a C^*-correspondence, which consists of a right Hilbert A-module, E, and a *-homomorphism from the C^*-algebra A into L(E), the adjointable operators on E. Some familiar examples of C^*-algebras which can be recognized as Cuntz-Pimsner algebras include the Cuntz algebras, Cuntz-Krieger algebras, and crossed products of a C^*-algebra by an action of the integers by automorphisms. In this dissertation, we construct a Cuntz-Pimsner Algebra associated to a dynamical system of a substitution tiling, which provides an alternate construction to the groupoid approach found in [3], and has the advantage of yielding a method for computing the K-Theory.

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Chapter 1

Introduction

Cuntz-Pimsner algebras have been used to construct a variety of familiar C^*-algebras, including the Cuntz algebras [2], Cuntz-Krieger algebras, and the crossed product $A \rtimes_\delta \mathbb{Z}$ of a C*-algebra, A, by an automorphism, δ. This paper concerns itself with constructing a Cuntz-Pimsner algebra which encodes the dynamics of a substitution tiling. The multiplication structure of the initial C^*-algebra A and the right and left actions of A on the Hilbert module strongly resemble matrix multiplication, and this resemblance will provide us with some helpful intuition when constructing these Cuntz-Pimsner algebras. In particular, our C^*-algebra will be a certain subalgebra of the subhomogeneous functions. Chapter 2 is devoted to providing some necessary background on Hilbert modules as is needed for the construction of a Cuntz-Pimsner algebra, and Chapter 3 gives a brief introduction to substitution tiling spaces. The reader is assumed to be familiar with basic C^*-algebra theory and the K-theory of C^*-algebras.

The crossed product $A \rtimes_\delta \mathbb{Z}$ is a well studied object. The substitution tiling that we construct using Cuntz-Pimsner algebras is implemented on a partial tiling, rather than a tiling of all of $\mathbb{R}^n$, and so the substitution does not implement an automorphism of the C^*-algebra. However, we show in Chapter 4.3, that $\mathcal{O}_E$ is isomorphic to $\overline{hC^*(R_S) \rtimes \mathbb{Z}h}$ for some positive $h \in C^*(R_S)$, where $C^*(R_S)$ is the stable Ruelle algebra associated with the substitution tiling.

One advantage of constructing a Cuntz-Pimsner algebra representation of the dynamics of a C^*-algebra associated with a substitution tiling is that it provides us with a method to compute the K-Theory of such objects, and this is the content of Chapter 5. In the case where our substitution is implemented on $\mathbb{R}$, it turns out that with the addition of some mild but subtle conditions, including that of forcing the border, we are able to characterize the K-groups of the dynamical systems constructed, completely in terms of the substitution matrix. We finish the chapter by outlining a method for calculating the K-groups in $\mathbb{R}^n$, $n \geq 2$. Unfortunately, the calculations even in the simplest of cases in $\mathbb{R}^2$, are very labour intensive, and in all practicality require the use of a computer to complete.

Chapter 2

Background

2.1 Hilbert Modules

A Cuntz-Pimsner algebra is a quotient of a C^*-subalgebra of adjointable operators on the Fock space generated from a Hilbert module E. Hence, we begin by introducing some basic Hilbert module theory. Since Hilbert modules generalize Hilbert spaces, we can often use our intuition gained from Hilbert spaces to guide us in our understanding of Hilbert modules, but we must be careful, as the similarities only go so far. For example, we are no longer guaranteed that, given a closed subspace, we can decompose the original space as a direct sum of that closed subspace and its orthogonal complement. This defect leads to others, including that bounded operators are no longer guaranteed to have a bounded adjoint. Much as in the development of a Hilbert space, we begin by defining the incomplete version of a Hilbert module, an inner product module.

Definition 2.1.1. *Let A be a C^*-algebra. A right inner product A-module is a complex linear space E which is equipped with a compatible right A-module structure:*

For all $\xi \in E$, *and* $a, b \in A$, *we have*

i) $\xi \cdot a = \xi a \in E$

ii) $\xi(a+b) = \xi a + \xi b$

iii) $\xi \cdot (ab) = (\xi \cdot a) \cdot b$

Furthermore there is a map $\langle \cdot, \cdot \rangle : E \times E \to A$ *which satisfies the following properties:*

For all $\xi, \zeta, \eta \in E$, $a \in A$ *and* $\alpha, \beta \in \mathbb{C}$, *we have*

i) $\langle \xi, \alpha\zeta + \beta\eta \rangle = \alpha\langle \xi, \zeta \rangle + \beta\langle \xi, \eta \rangle$

ii) $\langle \xi, \zeta \cdot a \rangle = \langle \xi, \zeta \rangle a$

iii) $\langle \xi, \zeta \rangle = \langle \zeta, \xi \rangle^*$

iv) $\langle \xi, \xi \rangle \geq 0$

v) $\langle \xi, \xi \rangle = 0$ *if and only if* $\xi = 0$.

Note that when we write $\langle \xi, \xi \rangle \geq 0$, we mean that the element $\langle \xi, \xi \rangle$ is a positive element in A. Recall that an element a in a C*-algebra is positive if $a = b^* b$ for some $b \in A$, or equivalently if it is self-adjoint and its spectrum is contained in the nonnegative reals. The first and third conditions imply that this inner product is conjugate linear in the first variable: For $\xi, \zeta, \eta \in E$, and $\alpha, \beta \in \mathbb{C}$, we have

$$\langle \alpha\xi + \beta\zeta, \eta \rangle = \langle \eta, \alpha\xi + \beta\zeta \rangle^* = (\alpha\langle \eta, \xi \rangle + \beta\langle \eta, \zeta \rangle)^* = \bar{\alpha}\langle \xi, \eta \rangle + \bar{\beta}\langle \xi, \zeta \rangle.$$

Using the second and third conditions, we get

$$\langle \xi \cdot a, \zeta \rangle = (\langle \zeta, \xi \cdot a \rangle)^* = (\langle \zeta, \xi \rangle a)^* = a^* \langle \xi, \zeta \rangle$$

which implies with *ii*) that

$$span\{\langle \xi, \zeta \rangle : \xi, \zeta \in E\}$$

is a two-side ideal in A.

We give a few simple examples.

Example 2.1.2. *The inner product $\mathbb{C}$-modules are the usual inner product spaces over $\mathbb{C}$ with the small exception that the inner product is conjugate linear in the first variable instead of the second.*

Example 2.1.3. *The C^*-algebra A is an inner product A-module in its own right, where the action of A on A is by right multiplication and the inner product is given by $\langle a, b \rangle = a^* b$. It is easy to verify that this claimed inner product satisfies the first four axioms, and the last follows from the C^* identity:*

$$\langle a, a \rangle = 0 \iff a^* a = 0 \iff \|a^* a\| = 0 \iff \|a\|^2 = 0 \iff a = 0$$

We might expect a Hilbert A-module to be a complete inner product A-module with respect a norm generated by the inner product and this is precisely the case. The norm is given by:

$$\|\xi\|_A = \|\langle \xi, \xi \rangle\|^{1/2}, \quad \xi \in E.$$

where the second norm is just that of the C^*-algebra A. Notice that in the previous example, the norm defined on A as a Hilbert module is the same as the norm on A

as a C^*-algebra. We do however need to prove that this does in fact define a norm and to that end, we first prove the Cauchy-Schwarz inequality for C*-valued inner products.

Lemma 2.1.4 (The Cauchy-Schwarz inequality)**.** *If E is an inner product A-module, and if $\xi, \zeta \in E$, then*

$$\langle \xi, \zeta \rangle^* \langle \xi, \zeta \rangle \leq \|\langle \xi, \xi \rangle\| \langle \zeta, \zeta \rangle$$

Note that the inequality is to be interpreted as $\|\langle \xi, \xi \rangle\| \langle \zeta, \zeta \rangle - \langle \xi, \zeta \rangle^* \langle \xi, \zeta \rangle$ is a positive element of A.

Proof. We first recall the following standard result about C*-algebras which can be found in [8]. If $a, b \in A$ are positive elements and $\rho(a) \leq \rho(b)$ for every state, ρ, on A, then $a \leq b$. Thus, it suffices to show that

$$\rho(\langle \xi, \zeta \rangle^* \langle \xi, \zeta \rangle) \leq \|\langle \xi, \xi \rangle\| \rho(\langle \zeta, \zeta \rangle)$$

for every state ρ on A, so fix ρ and note that the map $E \times E \to \mathbb{C}$ given by $(\mu, \eta) \to \rho(\langle \mu, \eta \rangle)$ is a positive sesquilinear form on E. We thus may apply the standard Cauchy-Schwarz inequality for complex-valued sesquilinear forms to get

$$|\rho(\langle \mu, \eta \rangle)| \leq \rho(\langle \mu, \mu \rangle)^{1/2} \rho(\langle \eta, \eta \rangle)^{1/2}.$$

Taking $\mu = \xi \langle \xi, \zeta \rangle$ and $\eta = \zeta$ in this inequality gives

$$\begin{aligned}\rho(\langle\xi,\zeta\rangle^*\langle\xi,\zeta\rangle) &= \rho(\langle\xi\langle\xi,\zeta\rangle,\zeta\rangle)\\ &\leq \rho(\langle\xi\langle\xi,\zeta\rangle,\xi\langle\xi,\zeta\rangle\rangle)^{1/2}\rho(\langle\zeta,\zeta\rangle)^{1/2} \qquad (2.1.1)\\ &= \rho(\langle\xi,\zeta\rangle^*\langle\xi,\xi\rangle\langle\xi,\zeta\rangle)^{1/2}\rho(\langle\zeta,\zeta\rangle)^{1/2}\end{aligned}$$

We would like to use the inequality $b^*cb \leq \|c\|b^*b$ for any $b \in A$ and any positive $c \in A$. To see that this inequality is valid, note that $b^*\|c\|b - b^*cb = b^*(\|c\| - c)b$ and since $\|c\| - c \geq 0$ for every positive c, we have $\|c\| - c = a^*a$ for some $a \in A^{\sim}$, the minimal unitization of A. Thus, $b^*(\|c\| - c)b = b^*a^*ab = (ab)^*ab \geq 0$.

Applying this result to inequality 2.1.1, we obtain, after squaring both sides,

$$\rho(\langle\xi,\zeta\rangle^*\langle\xi,\zeta\rangle)^2 \leq \|\langle\xi,\xi\rangle\|\rho(\langle\xi,\zeta\rangle^*\langle\xi,\zeta\rangle)\rho(\langle\zeta,\zeta\rangle)$$

and upon cancelling one of the $\rho(\langle\xi,\zeta\rangle^*\langle\xi,\zeta\rangle)$ terms from both sides, we have the desired result.

□

We are now in a position to prove that $\|\xi\|_A := \|\langle\xi,\xi\rangle\|^{1/2}$ defines a norm on our inner product A-module.

Corollary 2.1.5. *If E is an inner product A-module and $\xi \in E$, then*

$$\|\xi\|_A := \|\langle\xi,\xi\rangle\|^{1/2}$$

defines a norm on E such that $\|\xi \cdot a\|_A \leq \|\xi\|_A\|a\|$ for $a \in A$. The normed module is nondegenerate in the sense that for $\xi \in E$ and $a \in A$, elements of the form $\xi \cdot a$ span a dense subspace of E. In fact, span$\{\xi \cdot \langle\zeta,\eta\rangle : \xi,\zeta,\eta \in E\}$ is $\|\cdot\|_A$-dense in E.

Proof. Let $\lambda \in \mathbb{C}$ and $\xi \in E$. Then

$$\|\lambda\xi\|_A = \|\langle\lambda\xi, \lambda\xi\rangle\|^{1/2} = \|\bar{\lambda}\langle\xi, \xi\rangle\lambda\|^{1/2} = |\lambda|\|\langle\xi, \xi\rangle\|^{1/2}$$

Conditions iv) and v) on the inner product give $\|\xi\|_A \geq 0$ and $\|\xi\|_A = 0$ which is true if and only if $\xi = 0$. Lemma 2.1.4 implies that $\|\langle\xi, \zeta\rangle\|_A \leq \|\xi\|_A\|\zeta\|_A$ and hence

$$\begin{aligned}\|\xi + \zeta\|^2 &\leq \|\langle\xi, \xi\rangle\| + \|\langle\xi, \zeta\rangle\| + \|\langle\zeta, \xi\rangle\| + \|\langle\zeta, \zeta\rangle\| \\ &\leq \|\xi\|_A^2 + 2\|\xi\|_A\|\zeta\|_A + \|\zeta\|_A^2 \\ &= (\|\xi\| + \|\zeta\|)^2\end{aligned}$$

Thus $\|\cdot\|_A$ is a norm.

Next, we have

$$\|\xi \cdot a\|_A^2 = \|a^*\langle\xi, \xi\rangle a\| \leq \|a^*\|\|\xi\|_A^2\|a\| = \|a\|^2\|\xi\|_A^2$$

and so

$$\|\xi \cdot a\| \leq \|\xi\|_A\|a\| \tag{2.1.2}$$

as claimed.

To prove that $E \cdot \langle E, E\rangle$ is norm dense in E, first let B be the closed span of inner products $\langle\xi, \zeta\rangle$ with $\xi, \zeta \in E$. Recall that B is a closed ideal of A and so B contains an approximate identity $\{u_\lambda\}_{\lambda\in I}$ such that u_λ is a positive element and $\|u_\lambda\| \leq 1$ for each $\lambda \in I$. Then, since $u_\lambda^* = u_\lambda$ for each $\lambda \in I$, we have

$$\begin{aligned}\|\xi - \xi \cdot u_\lambda\|_A^2 &= \|\langle \xi, \xi\rangle - \langle \xi, \xi\rangle u_\lambda - u_\lambda\langle \xi, \xi\rangle + u_\lambda\langle \xi, \xi\rangle u_\lambda\| \\ &\leq \|\langle \xi, \xi\rangle - \langle \xi, \xi\rangle u_\lambda\| + \|u_\lambda\langle \xi, \xi\rangle - u_\lambda\langle \xi, \xi\rangle u_\lambda\| \\ &\leq \|\langle \xi, \xi\rangle - \langle \xi, \xi\rangle u_\lambda\| + \|\langle \xi, \xi\rangle - \langle \xi, \xi\rangle u_\lambda\|.\end{aligned}$$

Thus, given $\epsilon > 0$, there exists u_λ such that $\|\xi - \xi \cdot u_\lambda\| < \epsilon/2$. Since u_λ is in the ideal B, there exists $\{\xi_i\}_{i=1}^n$ and $\{\zeta_i\}_{i=1}^n$ in E such that $\|\sum_i^n \langle \xi_i, \zeta_i\rangle - u_\lambda\| < \epsilon/(2\|\xi\|_A)$. Then, using equation 2.1.2, we have

$$\begin{aligned}\|\xi - \xi \cdot \sum_i^n \langle \xi_i, \zeta_i\rangle\|_A &\leq \|\xi - \xi \cdot u_\lambda\|_A + \|\xi \cdot u_\lambda - \xi \cdot \sum_i^n \langle \xi_i, \zeta_i\rangle\|_A \\ &\leq \epsilon/2 + \|\xi\|_A\|u_\lambda - \sum_i^n \langle \xi_i, \zeta_i\rangle\| < \epsilon.\end{aligned}$$

□

We will see shortly that not only is $EA = \{\xi a \mid \xi \in E, a \in A\}$ dense in E, but in fact we have equality (by Lemma 2.1.10).

Definition 2.1.6. *A Hilbert A-module is an inner product A-module which is complete in the norm* $\|\cdot\|_A$*. It is said to be full if the ideal,*

$$I = span\{\langle \xi, \zeta\rangle : \xi, \zeta \in E\}$$

is dense in A.

It is said to be finitely generated if there exists a finite subset of elements $B \subset E$ *such that* $E = \{\xi a : \xi \in B,\ a \in A\}$ *and to be countably generated if there exists a countable subset of elements* $C \subset E$ *such that* $\{\xi a : \xi \in C,\ a \in A\}$ is dense in E.

From this point forward, we will drop the subscript A on the norm, unless clarification is required.

Example 2.1.7. *If A is a C^*-algebra, then $A \oplus A \oplus \cdots \oplus A$ is a Hilbert A-module, denoted A^n. For $\xi = \xi_1 \oplus \cdots \oplus \xi_n$ and $\eta = \eta_1 \oplus \cdots \oplus \eta_n$, the right action of A is given by*

$$\xi_1 \oplus \cdots \oplus \xi_n \cdot a = \xi_1 a \oplus \cdots \oplus \xi_n a$$

and the inner product is given by

$$\langle \xi, \eta \rangle = \sum_{i=1}^{n} \xi_i^* \eta_i.$$

It is easily verified that E is complete with respect to this inner product.

Example 2.1.8. *Let $\{E_n\}_{n=1}^{\infty}$ be a collection of Hilbert A-modules for a C^*-algebra, A. Then, we can construct the Hilbert A-module, $E = \bigoplus_{n=1}^{\infty} E_n$, which is defined to be the set of all sequences $(\xi_n)_{n=1}^{\infty}$ where $\xi_i \in E_i$ for each $i \in \mathbb{N}$, such that $\sum_{n=1}^{\infty} \langle \xi_n, \xi_n \rangle$ converges in A. The inner product is defined, for $\xi = (\xi_n)_{n=1}^{\infty}$ and $\zeta = (\zeta_n)_{n=1}^{\infty}$, by*

$$\langle \xi, \zeta \rangle = \sum_{n=1}^{\infty} \langle \xi_n, \zeta_n \rangle.$$

It is not clear at this point that the above sum converges, nor that this inner-product A-module is complete in the norm given by the inner product. To show that this sum converges, note that for any $N_1, N_2 \in \mathbb{N}$, with $N_1 < N_2$, we have by Lemma 2.1.4 that

$$\| \sum_{n=N_1}^{N_2} \langle \xi_n, \zeta_n \rangle \|^2 \leq \| \sum_{n=N_1}^{N_2} \langle \xi_n, \xi_n \rangle \| \| \sum_{n=N_1}^{N_2} \langle \zeta_n, \zeta_n \rangle \|.$$

Since the sums $\sum_{n=1}^{\infty} \langle \xi_n, \xi_n \rangle$ and $\sum_{n=1}^{\infty} \langle \zeta_n, \zeta_n \rangle$ converge to elements of A, if we let

$N_1 \to \infty$, the sums on the right hand side converge to zero. Thus, the sequence of partial sums of $\sum_{n=1}^{\infty}\langle \xi_n, \zeta_n\rangle$ is Cauchy and so we conclude that the sum on the left converges to an element in A.

To show that this inner-product A-module is complete is similar to showing that l_p is complete; we omit the details.

Example 2.1.9. *We define the Hilbert A-module,* $\mathcal{H}_A$*, by replacing each* E_i *in the previous example with a C*-algebra, A, with vectors,* $\xi = (a_n)_{n=1}^{\infty}$ *and* $\eta = (b_n)_{n=1}^{\infty}$*,* $a_n, b_n \in A$*, whose inner product is given by*

$$\langle \xi, \eta\rangle = \sum_{n=1}^{\infty} a_n^* b_n$$

This concludes our introduction to Hilbert modules. We complete the section with a very useful decomposition lemma, from which we can deduce the equality, $EA = E$.

Lemma 2.1.10. *Suppose that E is a Hilbert A-module,* $\xi \in E$ *and* $0 < \alpha < 1$*. Then there is an element* $\zeta \in E$ *such that* $\xi = \zeta|\xi|^{\alpha}$*, where* $|\xi| = \langle \xi, \xi\rangle^{\frac{1}{2}} \in A$*.*

Proof. First note that if $|\xi|$ is invertible, then we may take $\zeta = \xi \cdot |\xi|^{-\alpha}$, and we are done. Thus, we will assume $|\xi|$ is not invertible.

Consider the function for any $n \geq 1$, defined by

$$g_n(t) = \begin{cases} n^{\alpha} & 0 \leq t \leq \frac{1}{n} \\ t^{-\alpha} & t > \frac{1}{n} \end{cases}$$

which is an element in $C_0(\mathbb{R}^+ \cup 0)$, the continuous functions on the nonnegative reals which vanish at infinity. Then, $|\xi|$ is a positive element in A which we have assumed is not invertible, so that its spectrum includes the point 0. Thus, by the continuous functional calculus, $g_n(|\xi|) \in A^{\sim}$, where $A^{\sim}$ is the minimal unitization of A in the case that A is not unital, and is just A otherwise. To simplify notation, define $h_n = g_n(|\xi|) \in A^{\sim}$. Note that $E^{\sim}$, the inner product module where A has been replaced with $A^{\sim}$, is also a Hilbert module and when considered as vector spaces, $E^{\sim} = E$. We next want to show that $\{\xi h_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $E^{\sim}$. Fix $\epsilon > 0$ and pick N large enough such that $4N^{2(\alpha-1)} < \epsilon^2$ and $N < m < n$. Then,

$$\begin{aligned}\|\xi h_n(|\xi|) - \xi h_m(|\xi|)\|_A^2 &= \||\xi|^2(h_n(|\xi|) - h_m(|\xi|))^2\| \\ &= \sup_{t\in\sigma(|\xi|)} |t^2(h_n(t) - h_m(t))^2|\end{aligned}$$

where the second equality is due to the continuous functional calculus. Note that since $h_n(t) = h_m(t)$ for all $t > \frac{1}{N}$, we have

$$\sup_{t\in\sigma(|\xi|)} |t^2(h_n(t) - h_m(t))^2| = \sup_{0\leq t<\frac{1}{N}} |t^2(h_n(t) - h_m(t))^2|$$

At this point, there are two cases to check. First, if $0 \leq t \leq \frac{1}{n}$, then

$$\sup_{0\leq t<\frac{1}{N}} |t^2(h_n(t) - h_m(t))^2| \leq |\frac{1}{n^2}(n^\alpha - m^\alpha)^2| \leq |4n^{2(\alpha-1)}| < \epsilon^2$$

The second case is for $\frac{1}{n} < t \leq \frac{1}{m}$, in which case

$$\begin{aligned}\sup_{t<\frac{1}{N}} |t^2(h_n(t)-h_m(t))^2| &\leq \sup_{\frac{1}{n}<t\leq\frac{1}{m}} |t^2(t^{-\alpha}-m^{\alpha})^2| \\ &\leq \sup_{\frac{1}{n}<t\leq\frac{1}{m}} |(t^{1-\alpha}-tm^{\alpha})^2| \\ &\leq \sup_{\frac{1}{n}<t\leq\frac{1}{m}} |(m^{\alpha-1}-\frac{m^{\alpha}}{n})^2| \\ &\leq 4m^{2(\alpha-1)} < \epsilon^2\end{aligned}$$

and so $\xi h_n(|\xi|)$ is a Cauchy sequence as claimed. Our Hilbert module, $E^{\sim}$, is complete, and so this sequence converges to an element $\zeta \in E^{\sim}$. We claim that $\zeta|\xi|^{\alpha} = \xi$, or in other words, that $\xi h_n(|\xi|)|\xi|^{\alpha}$ converges to ξ. We can see that by the continuous functional calculus, $h_n(|\xi|)|\xi|^{\alpha} \in A$. Picking N such that $\frac{1}{N} < \epsilon$, we have that for all $n \geq N$

$$\begin{aligned}\|\xi h_n(|\xi|)|\xi|^{\alpha} - \xi\|_A = \||\xi|(h_n(|\xi|)|\xi|^{\alpha}-1)\| &= \sup_{t\in\sigma(|\xi|)} |t(h_n(t)t^{\alpha}-1)| \\ &= \sup_{0\leq t\leq\frac{1}{n}} |t(n^{\alpha}t^{\alpha}-1)| \leq \sup_{0\leq t\leq\frac{1}{n}} |t| \sup_{0\leq t\leq\frac{1}{n}} |n^{\alpha}t^{\alpha}-1| \\ &= \frac{1}{n}\cdot 1 < \epsilon.\end{aligned}$$

□

By taking $E = A$, the previous lemma implies that if $a \in A$, for any C*-algebra A, and $0 < \alpha < 1$, then $a = b(a^*a)^{\frac{\alpha}{2}}$ for some $b \in A$. This result looks a bit like polar decomposition, $a = u(a^*a)^{\frac{1}{2}}$ where u is a partial isometry, but we don't have this strong of a result in such a general setting. Recall that to apply the polar decomposition to an element a in a non-unital C*-algebra, A, we must move to the von Neumann algebra generated by a in order to find the partial isometry. It is nonetheless useful to be able to decompose a single element in a C*-algebra into a product of two.

2.2 The Space of Adjointable Operators

Just as in Hilbert space theory, we are more interested in the linear operators that act on the Hilbert modules than the actual space itself. For any Hilbert space H, every bounded linear operator on H has an adjoint which is also bounded. However, not every bounded linear operator on a Hilbert Module has a bounded adjoint. We make precise the definition of the adjoint of a bounded A-linear operator between Hilbert A-modules.

Definition 2.2.1. *Let E and F be Hilbert A-modules and $T : E \to F$ a bounded A-linear map. The adjoint of T, if it exists, is the bounded A-linear map $S : F \to E$ such that for all $\xi \in E$ and $\zeta \in F$*

$$\langle T\xi, \zeta \rangle = \langle \xi, S\zeta \rangle$$

In this case, we write $S = T^*$.

For an example of a bounded operator between Hilbert modules E and F which does not have a bounded adjoint, let X be a compact Hausdorff space, and $Y \subset X$ be a closed subset such that its complement is dense in X (for example, $Y = \{0\} \subset [0,1] = X$). Then let E and F be the Hilbert $C(X)$-modules given by $F = A = C(X)$ and $E = \{f \in A : f(x) = 0, \ \forall x \in Y\}$. Let $i : E \to F$ be the inclusion map, which is clearly bounded with norm 1. Suppose for a contradiction that i has an adjoint. Then, for $f \in E$ and the constant function 1 in F, $\overline{f} = \langle i(f), 1 \rangle = \langle f, i^*(1) \rangle = \overline{f} i^*(1)$.

Thus, $\overline{f} = \overline{f}i^*(1)$ for all $f \in E$, and so since $i^*(1)$ is continuous, $i^*(1)$ must be identically 1, but $1 \notin E$, a contradiction.

The lack of bounded adjoints is related to the fact that a closed submodule is not necessarily orthogonally complemented. In the above example, E is a closed, proper submodule of F, but the orthogonal complement of E in F is $\{0\}$.

Since we are interested in self adjoint algebras of operators, we restrict ourselves to considering only the adjointable operators (by adjointable, we mean those which have an adjoint) from a Hilbert module, E, to a Hilbert module, F, which we denote $L(E, F)$ and when $E = F$, we simply write $L(E)$ for $L(E, E)$. A simple application of the closed graph theorem shows that every adjointable operator between Hilbert modules is bounded. Furthermore, one can show that the adjointable operators on a Hilbert module under the operator norm form a C*-algebra.

For $\xi, \zeta \in E$, consider the operator $\xi \otimes \zeta^* \in L(E)$ defined on a vector $\eta \in E$ by

$$\xi \otimes \zeta^*(\eta) = \xi\langle\zeta, \eta\rangle$$

The adjoint of this operator is $\zeta \otimes \xi^*$. To see this, let $\eta_1, \eta_2 \in E$. Then we have

$$\begin{aligned}\langle\xi \otimes \zeta^*(\eta_1), \eta_2\rangle &= \langle\xi\langle\zeta, \eta_1\rangle, \eta_2\rangle = \langle\zeta, \eta_1\rangle^*\langle\xi, \eta_2\rangle = \langle\eta_1, \zeta\rangle\langle\xi, \eta_2\rangle \\ &= \langle\eta_1, \zeta\langle\xi, \eta_2\rangle\rangle = \langle\eta_1, \zeta \otimes \xi^*(\eta_2)\rangle.\end{aligned}$$

We denote by $K(E)$, the closure of the linear span of such operators in $L(E)$. In the case that E is a Hilbert space, then $K(E)$ is the usual compact operators. Even when E is not a Hilbert space, it is customary to call $K(E)$ the compact operators on E, even though many of them may not be compact in the usual sense. For example, if A is an infinite dimensional unital C*-algebra, then $1 \otimes 1^* \in K(A)$ is the identity operator on A which is certainly not compact.

Recall from example 2.1.8, that for any Hilbert A-modules, E and F, the direct sum of these modules is also a Hilbert A-module. Note then that $L(E) \oplus L(F) \subset L(E \oplus F)$, and so we have a copy of $L(E)$ and $L(F)$ in $L(E \oplus F)$ in the form of $L(E) \oplus 0$ and $0 \oplus L(F)$.

2.3 C*-Correspondences and the Interior Tensor Product

It will be useful for us to have not only a right action of our C*-algebra, A, on our Hilbert module, E, but also a left action. A *-homomorphism $\psi : A \to L(E)$ gives us a left action of A on E:

$$a \cdot \xi = \psi(a)\xi \quad a \in A, \ \ \xi \in E$$

Definition 2.3.1. *A C*-correspondence consists of a C*-algebra A, a Hilbert A-module, E, and a *-homomorphism, $\psi : A \to L(E)$ which gives a left action of A on E. We say that the C*-correspondence is*

i) *faithful* if ψ is injective

ii) *non-degenerate* if $\{\psi(a)\xi | a \in A, \xi \in E\}$ is dense in E

iii) *full* if $\{\langle \xi, \zeta \rangle | \xi, \zeta \in E\}$ is dense in A.

Example 2.3.2. *Let $m \geq n$ be positive integers, $E = M_{m,n}(\mathbb{C})$, $A = M_{n,n}(\mathbb{C})$, $\psi : M_{n,n}(\mathbb{C}) \to M_{m,m}(\mathbb{C})$ be the natural injection as matrices comprising only the top left $n \times n$ entries. The inner product of $M, N \in E$ is given by $\langle M, N \rangle_R = M^*N$, the expression on the right side being matrix multiplication where M^* is the conjugate transpose of the matrix, M. Then, E is a C*-correspondence which is necessarily* *degenerate unless* $m = n$*. If* $m = np$ *for some* $p \in \mathbb{N}$*, then we could define a unital *-homomorphism* $\psi : A \to L(E)$ *by* $\psi(a) = \bigoplus_{i=1}^{n} a$ *for* $a \in A$ *and this would be a non-degenerate C*-correspondence.*

We now wish to construct the interior tensor product of the Hilbert A-module E and the Hilbert B-module F and to do so, we first need a *-homomorphism $\psi : A \to L(F)$. Then, we can view F as a left A-module, by defining $a\zeta = \psi(a)\zeta$ for $a \in A$ and $\zeta \in F$. The algebraic tensor product of E and F over A, denoted $E \otimes_A F$, is defined to be the quotient of the vector space tensor product of E and F, denoted $E \otimes_{alg} F$, by the subspace generated by elements of the form

$$\xi a \otimes_A \zeta - \xi \otimes_A \psi(a)\zeta \quad a \in A, \xi \in E, \zeta \in F. \tag{2.3.1}$$

The right action of B, for $a \in A, \xi \in E, \zeta \in F$, is given by $(\xi \otimes_A \zeta)b = \xi \otimes_A (\zeta b)$.

Proposition 2.3.3. *With* A*,* B*,* E*,* F *and* ψ *as above,* $E \otimes_A F$ *is an inner product* B*-module with inner product given on simple tensors by*

$$\langle \xi_1 \otimes_A \zeta_1, \xi_2 \otimes_A \zeta_2 \rangle = \langle \zeta_1, \psi(\langle \xi_1, \xi_2 \rangle)\zeta_2 \rangle$$

Proof. First, we show that the given inner product defines a semi-inner product on $E \otimes_{alg} F$ and then we will show that $\{z \in E \otimes_{alg} F : \langle z, z \rangle = 0\}$ is precisely the subspace generated by elements of the form 2.3.1 to conclude that this semi-inner product actually defines an inner product on $E \otimes_A F$.

The above inner product formula extends by linearity to give a sesquilinear form on $E \otimes_{alg} F$, and so we just need to verify that $\langle z, z \rangle \geq 0$ for $z \in E \otimes_{alg} F$. We may assume that $z = \sum_{i=1}^{n} \xi_i \otimes_{alg} \zeta_i$ so that

$$\langle z, z\rangle = \sum_{i,j=1}^{n} \langle \zeta_i, \psi(\langle \xi_i, \xi_j\rangle)\zeta_j\rangle = \sum_{i=1}^{n} \langle \zeta_i, \psi^{(n)}(M)\zeta_i\rangle$$

where $\psi^{(n)}$ denotes the map from $M_n(A) \to M_n(L(E)) = L(E^n)$ which takes $X_{i,j}$ to $\psi(X_{i,j})$. Recall that the map ψ is said to be completely positive if $\psi^{(n)}$ is positive for all $n \in \mathbb{N}$. We need to use the fact that every *-homomorphism between C*-algebras is completely positive. Thus, if we show that the matrix, M, with (i,j)-entry given by $\langle \xi_i, \xi_j\rangle$ is positive, then, since $\psi^{(n)}(M)$ is positive, we can conclude that $\langle z, z\rangle \geq 0$. To see that $M \in M_n(A)$ is positive, first we identify $M_n(A)$ with $K(A^n)$. Then, note that for $a = (a_1, \cdots, a_n) \in A^n$,

$$\langle a, Ma\rangle = \sum_{i,j}^{n} \langle a_i, \langle \xi_i, \xi_j\rangle a_j\rangle = \sum_{i,j=1}^{n} a_i^* \langle \xi_i, \xi_j\rangle a_j = \langle \sum_{i=1}^{n} a_i \xi_i, \sum_{j=1}^{n} a_j \xi_j\rangle \geq 0.$$

For any Hilbert module, E, the operator $T \in L(E)$ is positive if and only if $\langle \xi, T\xi\rangle \geq 0$ for all $\xi \in E$ and so by taking $E = A^n$, we have that M is positive.

Let $z = \sum_{i=1}^{n} \xi_i a_i \otimes_A \zeta_i - \xi_i \otimes_A \psi(a_i)\zeta_i$. Then we have

$$\begin{aligned}
\langle z, z\rangle &= \langle \sum_{i=1}^{n} \xi_i a_i \otimes_A \zeta_i - \xi_i \otimes_A \psi(a_i)\zeta_i, \sum_{i=1}^{n} \xi_i a_i \otimes_A \zeta_i - \xi_i \otimes_A \psi(a_i)\zeta_i\rangle \\
&= \langle \sum_{i=1}^{n} \xi_i a_i \otimes_A \zeta_i, \sum_{i=1}^{n} \xi_i a_i \otimes_A \zeta_i\rangle - \langle \sum_{i=1}^{n} \xi_i a_i \otimes_A \zeta_i, \sum_{i=1}^{n} \xi_i \otimes_A \psi(a_i)\zeta_i\rangle \\
&- \langle \sum_{i=1}^{n} \xi_i \otimes_A \psi(a_i)\zeta_i, \sum_{i=1}^{n} \xi_i a_i \otimes_A \zeta_i\rangle + \langle \sum_{i=1}^{n} \xi_i \otimes_A \psi(a_i)\zeta_i, \sum_{i=1}^{n} \xi_i \otimes_A \psi(a_i)\zeta_i\rangle \\
&= \sum_{i,j=1}^{n} \langle \zeta_i, \psi(\langle \xi_i a_i, \xi_j a_j\rangle \zeta_j\rangle - \sum_{i,j=1}^{n} \langle \zeta_i, \psi(\langle \xi_i a_i, \xi_j\rangle \psi(a_j)\zeta_j\rangle \\
&- \sum_{i,j=1}^{n} \langle \psi(a_i)\zeta_i, \psi(\langle \xi_i, \xi_j a_j\rangle)\zeta_j\rangle + \sum_{i,j=1}^{n} \langle \psi(a_i)\zeta_i, \psi(\langle \xi_i, \xi_j\rangle)\psi(a_j)\zeta_j\rangle.
\end{aligned}$$

In the last term of this string of equalities, the first two sums are equal, and the last two sums are equal, since

$$\psi(\langle \xi_i a_i, \xi_j a_j\rangle) = \psi(\langle \xi_i a_i, \xi_j\rangle a_j) = \psi(\langle \xi_i a_i, \xi_j\rangle)\psi(a_j).$$

Thus, we conclude $\langle z, z\rangle = 0$.

Finally, we show that for any element $z \in E \otimes_{alg} F$ for which $\langle z, z\rangle = 0$, we have that z is of the form 2.3.1. Let $z = \sum_{i=1}^{n} \xi_i \otimes_{alg} \zeta_i$ such that $\langle z, z\rangle = 0$. Then, letting M be the matrix with i, j entry equal to $\langle \xi_i, \xi_j\rangle$, we have that

$$\langle z, z\rangle = \langle \zeta, \psi^{(n)}(M)\zeta\rangle$$

where $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in F^n$. Let $T = \psi^{(n)}(M)$. We have that $T \geq 0$, and so

$$0 = \langle \zeta, T\zeta\rangle = \langle T^{\frac{1}{2}}\zeta, T^{\frac{1}{2}}\zeta\rangle$$

and so $T^{\frac{1}{2}}\zeta = 0$, and similarly, $T^{\frac{1}{4}}\zeta = 0$.

We pause momentarily to note that we can view E^n as Hilbert A-module, using the inner product discussed previously. We an also view E^n as a Hilbert $M_n(A)$-module, by defining the right action of $M \in M_n(A)$ on $\xi \in E^n$ by matrix multiplication, with ξ viewed as a row vector. The inner product of $\xi, \zeta \in E^n$ is defined to be the matrix with i, j-entry equal $\langle \xi_i, \zeta_j\rangle$ where this second inner product is that of the Hilbert A-module E. While the norms induced by these inner products are different, they are equivalent, a fact that can be shown be using the equivalence of the norms of l_1 and l_∞ on $\mathbb{C}^n$. [7]

Let $\xi = (\xi_1, \cdots, \xi_n) \in E^n$ where we are viewing E^n to be a Hilbert $M_n(A)$-module, so, $\langle \xi, \xi\rangle = M$. By Lemma 2.1.10, there exists $\eta \in E^n$ such that $\xi = \eta M^{\frac{1}{4}}$. Then, $\psi^{(n)}(M^{\frac{1}{4}}) = T^{\frac{1}{4}}$ and letting $m_{i,j}$ be the matrix elements of $M^{\frac{1}{4}}$, $T^{\frac{1}{4}}$ has matrix elements $\psi(m_{i,j})$. Thus,

$$\xi_j = \sum_{i=1}^{n} \eta_i m_{i,j}, \qquad \text{and} \qquad \sum_{j=1}^{n} \psi(m_{i,j})\zeta_j = 0$$

and so

$$z = \sum_{i,j=1}^{n} (\eta_i m_{i,j} \otimes_{alg} \zeta_j - \eta_i \otimes_{alg} \psi(m_{i,j}))\zeta_j.$$

Thus, we have shown that our semi-inner product on $E \otimes_{alg} F$ defines an inner product on $E \otimes_A F$.

□

Definition 2.3.4. *The interior tensor product of the Hilbert modules, E and F, is defined to be the completion of the inner-product B-module $E \otimes_A F$ and is denoted $E \otimes_\psi F$ where $\psi : A \to L(F)$ is a *-homomorphism.*

We are now in a position to define and prove some results about an important class of operators. This operator (more precisely its image under a suitable extension and quotient) and its adjoint will be key in constructing our algebra $\mathcal{O}_E$.

Theorem 2.3.5. *For $\xi \in E$, define the operator $T_\xi \in L(F, E \otimes_\psi F)$ by $T_\xi(\eta) = \xi \otimes_\psi \eta$, for $\eta \in F$. We have the following:*

- *i) The adjoint, $T_\xi^* \in L(E \otimes_\psi F, F)$, for $\xi, \omega \in E$ and $\rho \in F$, is given by $T_\xi^*(\omega \otimes_\psi \rho) = \psi(\langle \xi, \omega \rangle)\rho$,*
- *ii) $T_\xi^* T_\eta = \psi(\langle \xi, \eta \rangle)$.*
- *iii) T_ξ is a bounded operator, and the norm is given by $\|T_\xi\| = \|\psi(\langle \xi, \xi \rangle)\|^{\frac{1}{2}}$,*

Proof. To see i), note that

$$\langle T_\xi \zeta, \omega \otimes_\psi \rho\rangle = \langle \xi \otimes_\psi \zeta, \omega \otimes_\psi \rho\rangle = \langle \zeta, \psi(\langle \xi, \omega\rangle)\rho\rangle.$$

so that $T_\xi^*(\omega \otimes_\psi \rho) = \langle \zeta, \psi(\langle \xi, \omega\rangle)\rho\rangle,$.

For ii), we have that $T_\zeta^* T_\xi(\eta) = T_\zeta^*(\xi \otimes_\psi \eta) = \psi(\langle \zeta, \xi\rangle)\eta$; that is,

$$T_\zeta^* T_\xi = \psi(\langle \zeta, \xi\rangle). \tag{2.3.2}$$

We'll use the results of i) and ii) to prove iii), starting with an application of the C^* identity: $\|T_\xi\|^2 = \|T_\xi^* T_\xi\| = \|\psi(\langle \xi, \xi\rangle)\|$. □

We also note that

$$\begin{aligned} T_\xi T_\zeta^*(\omega \otimes_\psi \rho) &= T_\xi(\psi(\langle \zeta, \omega\rangle)\rho) = \xi \otimes_\psi \psi(\langle \zeta, \omega\rangle)\rho \\ &= \xi \cdot \langle \zeta, \omega\rangle \otimes_\psi \rho = ((\xi \otimes \zeta^*) \otimes 1)(\omega \otimes_\psi \rho), \end{aligned}$$

where $\xi \otimes \zeta^* \in K(E)$ is the rank one operator given by $\xi \otimes \zeta^*(\rho) = \xi\langle \zeta, \rho\rangle$. The notation $\otimes 1$ denotes the canonical map $L(E) \to L(E \otimes_\psi F)$ which takes the operator T to $T \otimes 1$, where

$$T \otimes 1(\xi \otimes_\psi \zeta) = T(\xi) \otimes_\psi \zeta.$$

Lemma 2.3.6. *If $\psi : A \to L(E)$ is isometric, then so is $\otimes 1 : L(E) \to L(E \otimes_\psi F)$.*

Proof. Since $\otimes 1$ is a *-homomorphism, it suffices to show that the map is injective. To see this, let $T_1, T_2 \in L(E)$, $\eta \in E, \zeta \in F$. We have

$$
\begin{aligned}
& T_1 \otimes 1(\eta \otimes_\psi \zeta) = T_2 \otimes 1(\eta \otimes_\psi \zeta) \\
\Longrightarrow\ & T_1\eta \otimes_\psi \zeta = T_2\eta \otimes_\psi \zeta \\
\Longrightarrow\ & (T_1\eta - T_2\eta) \otimes_\psi \zeta = 0.
\end{aligned}
$$

Thus, $\|\langle \zeta, \psi(\langle\langle (T_1 - T_2)\eta, (T_1 - T_2)\eta\rangle\rangle)\zeta\rangle\| = 0$ for all $\eta \in E$, and $\zeta \in F$ and since $\psi(\langle\langle (T_1 - T_2)\eta, (T_1 - T_2)\eta\rangle\rangle)$ is positive, for this equality to hold for all $\zeta \in F$, $\psi(\langle\langle (T_1-T_2)\eta, (T_1-T_2)\eta\rangle\rangle) = 0$, for all $\eta \in E$. Thus, since ψ is injective, $T_1-T_2 = 0$. □

Consider the n-fold tensor product of the Hilbert A-module E with itself, $E \otimes_\psi \cdots \otimes_\psi E$, which we will denote by $E^{\otimes n}$, where $\psi : A \to L(E)$. We will drop the subscript ψ on $\otimes$ when the map ψ is clear as the notation easily becomes cluttered. Notice that we have that $E^{\otimes n}$ is a Hilbert A-module where the right A-module structure is obtained by $(\xi_1 \otimes \cdots \otimes \xi_n) \cdot a := \xi_1 \otimes \cdots \otimes (\xi_n \cdot a)$. We actually have a C*-correspondence, the left action defined by

$$\psi(a)(\xi_1 \otimes \cdots \otimes \xi_n) := \psi(a)(\xi_1) \otimes \cdots \otimes \xi_n. \tag{2.3.3}$$

where we have identified $\psi(a)$ with its image under the $n-1$ iterations of the map $\otimes 1$, to get $\psi(a) \otimes 1 \otimes \cdots \otimes 1$. We now have all of the pieces need to define our main object of interest.

2.4 The Cuntz-Pimsner Algebra

The Cuntz-Pimsner algebra $\mathcal{O}_E$ which we wish to construct is a quotient of an analogue of the Toeplitz algebra, $\mathcal{T}_E$, generated by the creation operators, $\mathcal{T}_\xi$, on the Fock space of a C*-correspondence E, which will we will define below.

Definition 2.4.1. *The Fock space of a C*-correspondence E over A is the Hilbert A-module defined as*

$$\mathcal{E}_+ = \bigoplus_{n=0}^{\infty} E^{\otimes n}$$

where by convention, $E^{\otimes 0} = A$. $\mathcal{E}_+$ is also a C-correspondence, where the left action is obtained by extending ψ using 2.3.3 to elements in $\mathcal{E}_+$, where $\psi(a)b = ab$ for $b \in E^{\otimes 0} = A$.*

Denote by $J(\mathcal{E}_+)$ the C*-algebra generated in $L(\mathcal{E}_+)$ by

$$\sum_{N=0}^{\infty} L(\bigoplus_{n=0}^{N} E^{\otimes n})$$

We now define an analogue in $L(\mathcal{E}_+)$ of $T_\xi \in L(E, E \otimes E)$. For each $\xi \in E$, let $\mathcal{T}_\xi \in L(\mathcal{E}_+)$ be the operator defined on the elementary tensor $\mu \in E^{\otimes n}$ by $\mathcal{T}_\xi(\mu) = \xi \otimes \mu \in E^{\otimes n+1}$ for all $n \in \mathbb{N}$. Extending this definition linearly to arbitrary elements in $\mathcal{E}_+$ defines an adjointable operator with adjoint analogous to that of T_ξ.

Lemma 2.4.2. *Given $m \in \mathbb{N}$, $\xi \in E$, and $\mu \in E^{\otimes m}$, the operator $\mathcal{T}_\xi$ defined by*

$$\mathcal{T}_\xi(\mu) = \xi \otimes \mu$$

is in $L(\mathcal{E}_+)$. Its adjoint, $\mathcal{T}_\xi^*$, is defined as follows. If $m = 0$ i.e. if $\mu \in A$, then

$$\mathcal{T}_\xi^*(\mu) = 0$$

and if $m \geq 1$, then

$$\mathcal{T}_\xi^*(\mu) = T_\xi^* \otimes 1_{m-1}(\mu)$$

Proof. Let $\eta_1 \in E$ and $\eta_2 \in E^{\otimes m}$ so that $\eta_1 \otimes \eta_2 \in E^{\otimes (m+1)}$. Then,

$$\langle \mathcal{T}_\xi(\mu), \eta_1 \otimes \eta_2 \rangle = \langle \xi \otimes \mu, \eta_1 \otimes \eta_2 \rangle = \langle \mu, \psi(\langle \xi, \eta_1 \rangle)\eta_2 \rangle$$

Thus, $\mathcal{T}_\xi^* = T_\xi^* \otimes 1_{m-1}$ for $m \geq 1$. We impose that $\mathcal{T}_\xi^*$ be zero when restricted to A, since $\mathcal{T}_\xi^*$ is not defined on A.

□

Let $M(\mathcal{E}_+)$ be the multiplier algebra of $J(\mathcal{E}_+)$, or explicitly, $M(\mathcal{E}_+) = \{T \in L(\mathcal{E}_+) \mid TJ(\mathcal{E}_+) \subset J(\mathcal{E}_+) \text{ and } J(\mathcal{E}_+)T \subset J(\mathcal{E}_+)\}$.

Lemma 2.4.3. *Let E be a C^*-correspondence and $\mathcal{T}_\xi$ be defined as before for $\xi \in E$. Then, $\mathcal{T}_\xi \in M(\mathcal{E}_+)$.*

Proof. Fix $J \in J(\mathcal{E}_+)$, so that there exists a positive integer N_1 such that for all $\mu \in E^{\otimes n}$ with $n \geq N_1$, $J\mu = 0$. Then, in this case, $\mathcal{T}_\xi J\mu = 0$. There also exists N_2 such that if $\mu \in E^{\otimes n}$ for $n < N_1$, then $J\mu = \sum_{n=0}^{N_2} \zeta_1 \otimes \cdots \otimes \zeta_n \in \bigoplus_{n=0}^{N_2} E^{\otimes n}$ so that

$$\mathcal{T}_\xi J\mu = \mathcal{T}_\xi \sum_{n=0}^{N_2} \zeta_1 \otimes \cdots \otimes \zeta_n = \sum_{n=0}^{N_2} \xi \otimes \zeta_1 \otimes \cdots \otimes \zeta_n \in \bigoplus_{n=0}^{N_2+1} E^{\otimes n}$$

Letting $N = max\{N_1, N_2+1\}$, $\mathcal{T}_\xi J \in L(\oplus_{n=0}^{N} E^{\otimes n}) \subset J(\mathcal{E}_+)$. A similar calculation shows that $J\mathcal{T}_\xi \in J(\mathcal{E}_+)$. □

Note that by definition, $J(\mathcal{E}_+)$ is an ideal in $M(\mathcal{E}_+)$, and so the quotient $M(\mathcal{E}_+)/J(\mathcal{E}_+)$ is well defined. We note at this point that $A \subset M(\mathcal{E}_+)$, since

$$A = span\overline{\{\mathcal{T}_\xi^* \mathcal{T}_\eta : \xi, \eta \in E\}} \subset M(\mathcal{E}_+).$$

Denote by S_ξ the class of the operator $\mathcal{T}_\xi$ in the quotient algebra $M(\mathcal{E}_+)/J(\mathcal{E}_+)$.

Definition 2.4.4. *Let E be a full and faithful C^*-correspondence over the C^*-algebra A with the left action of A given by the *-homomorphism $\psi : A \to L(E)$. The Cuntz-Pimsner algebra, $\mathcal{O}_E$, is the C^*-algebra generated in $M(\mathcal{E}_+)/J(\mathcal{E}_+)$ by all the operators*

S_ξ, with $\xi \in E$. The Toeplitz algebra $\mathcal{T}_E$ of E is the C^*-algebra generated in $L(\mathcal{E}_+)$ by the operators $\mathcal{T}_\xi$ with $\xi \in E$. Both $\mathcal{O}_E$ and $\mathcal{T}_E$ depend only on the isomorphism class of the C^*-correspondence E.

Recall that $J(\mathcal{E}_+)$ consists of operators of the form $T \in L(E^{\otimes 0} \oplus \cdots \oplus E^{\otimes n})$, which operate on the first $(n+1)$ summands of $\mathcal{E}_+$ and is zero on the rest of $\mathcal{E}_+$, i.e. on all $E^{\otimes k}$, $k > n$. Of course, $L(E^{\otimes n}) \subset L(E^{\otimes 0} \oplus \cdots \oplus E^{\otimes n})$ and is then contained in $J(\mathcal{E}_+)$ and becomes 0 in $M(\mathcal{E}_+)/J(\mathcal{E}_+)$. However, there is another inclusion of $L(E^{\otimes n})$ in $M(\mathcal{E}_+)/J(\mathcal{E}_+)$. For $T \in L(E^{\otimes n})$, we can define $T \otimes 1_{k-n}$ on $E^{\otimes k}$, for $k > n$. Then,

$$\tilde{T} = 0 \oplus \cdots \oplus 0 \oplus T \oplus (T \otimes 1_1) \oplus (T \otimes 1_2) \oplus \cdots$$

is an operator in $M(\mathcal{E}_+)$. Moding out by $J(\mathcal{E}_+)$ means the initial zero summands don't matter, and hence $L(E^{\otimes n}) \subset M(\mathcal{E}_+)/J(\mathcal{E}_+)$. Also observe that $A \subset L(E) \subset M(\mathcal{E}_+)/J(\mathcal{E}_+)$, and this is consistent with our earlier inclusion of A in $M(\mathcal{E}_+)$.

$$\tilde{T}\mu = T \otimes 1_{k-n}\mu = T(\mu_1 \otimes \cdots \otimes \mu_n) \otimes \mu_{n+1} \otimes \cdots \otimes \mu_k$$

and then identify T with the image of $\tilde{T}$ in $M(\mathcal{E}_+)/J(\mathcal{E}_+)$.

Proposition 2.4.5. *The elements of $\mathcal{O}_E$ satisfy the following relations:*

i) $S_\xi^* S_\zeta = \langle \xi, \zeta \rangle$ for every $\xi, \zeta \in E$, and so $A \subset \mathcal{O}_E$.

ii) $S_\zeta S_\xi^* = \zeta \otimes \xi^* \in K(E) \subset L(E)$ for every $\xi, \zeta \in E$.

iii) $S_\xi a = S_{\xi \cdot a}$, $a S_\xi = S_{\psi(a)\xi}$, for every $\xi \in E$ and $a \in A$.

iv) $R S_\xi = S_{R(\xi)}$, for every $\xi \in E$ and every $R \in L(E)$.

Proof. *i)*: Let $\xi, \zeta \in E$ and recall that the operator $T_\xi^* T_\zeta \in L(E)$ satisfies $T_\xi^* T_\zeta = \langle \xi, \zeta \rangle$. Now consider its analogue in $L(\mathcal{E}_+)$ acting on $\mu_1 \otimes \cdots \otimes \mu_n$, for $n \geq 0$

$$\mathcal{T}_\xi^* \mathcal{T}_\zeta(\mu_1 \otimes \cdots \otimes \mu_n) = \mathcal{T}_\xi^* \zeta \otimes \mu_1 \otimes \cdots \otimes \mu_n = \langle \xi, \zeta \rangle (\mu_1 \otimes \cdots \otimes \mu_n).$$

Since this equality holds for all $n \geq 0$ ($n \geq m$ for some $m \geq 0$ would do as well), it holds in the quotient too, and so $S_\xi^* S_\zeta = \langle \xi, \zeta \rangle$.

ii): To show equality in $\mathcal{O}_E$, it suffices to show equality of their corresponding representatives in $\mathcal{T}_E$ when restricted to $\bigoplus_{n=m}^\infty E^{\otimes n} \subset \mathcal{E}_+$ for some $m \in \mathbb{N}$. $S_\zeta S_\xi^*$ is the image under the quotient map of the operator $\mathcal{T}_\zeta \mathcal{T}_\xi^*$ which acts on an element $\mu = \mu_1 \otimes \cdots \otimes \mu_n \in E^{\otimes n}$ for $n \geq 1$ by

$$\mathcal{T}_\zeta \mathcal{T}_\xi^* (\mu_1 \otimes \cdots \otimes \mu_n) = \zeta \otimes \xi^*(\mu_1) \otimes \mu_2 \otimes \cdots \otimes \mu_n.$$

Now, $\zeta \otimes \xi^* \in \mathcal{O}_E$ as outlined just prior to this lemma, is the image of the operator in $\mathcal{T}_E$ that acts $\mu = \mu_1 \otimes \cdots \otimes \mu_n \in E^{\otimes n}$ for each $n \geq 1$ by

$$\zeta \otimes \xi^* \mu = \zeta \otimes \xi^*(\mu_1) \otimes \mu_2 \otimes \cdots \mu_n$$

Since both $S_\zeta S_\xi^*$ and $\zeta \otimes \xi^*$ are equal when restricted to elements in $E^{\otimes n}$ for $n \geq 1$, they are equal in the quotient $\mathcal{O}_E$.

iii): Any equalities which hold in $\mathcal{T}_E$ must also hold in $\mathcal{O}_E$. Thus, let $\mu = \mu_1 \otimes \cdots \otimes \mu_n \in E^{\otimes n}$, and note that

$$\mathcal{T}_\xi a(\mu) = \mathcal{T}_\xi(\psi(a)\mu) = \xi \otimes (\psi(a)\mu) = \xi a \otimes \mu = \mathcal{T}_{\xi a} \mu,$$

and

$$a\mathcal{T}_\xi(\mu) = a\xi \otimes \mu = (\psi(a)\xi) \otimes \mu = \mathcal{T}_{\psi(a)\xi}.$$

iv): Let $R \in L(E) \subset \mathcal{O}_E$, let $\mathcal{T}_\xi \in \mathcal{T}_E$ be the standard representative of S_ξ and let $\tilde{R}$ be the image in $\mathcal{T}_E$ of R. Then, for sufficiently large N we have that for all $\mu = \mu_1 \otimes \cdots \otimes \mu_n \in E^{\otimes n}$, $n \geq N$

$$\tilde{R}T_\xi\mu = \tilde{R}(\xi \otimes \mu) = R(\xi) \otimes \mu = T_{R(\xi)}\mu$$

and so $RS_\xi = S_{R\xi}$. □

Lastly, we present the universal property of the algebra $\mathcal{O}_E$, and a characterization of the kernel of the quotient map from $\mathcal{T}_E$ to $\mathcal{O}_E$, both proofs of which can be found in [9].

Theorem 2.4.6. *Let E be a full, faithful C^*-correspondence with $\psi : A \to L(E)$ and $\mathcal{O}_E$ the corresponding Cuntz-Pimsner algebra (Definition 2.4.4). Let B be any C^*-algebra and $\sigma : A \to B$ is any *-homomorphism with the property that there exist elements $t_\xi \in B$ satisfying*

1) $\alpha t_\xi + \beta t_\zeta = t_{\alpha\xi+\beta\zeta}$ for every $\xi, \zeta \in E$ and $\alpha, \beta \in \mathbb{C}$,

2) $t_\xi\sigma(a) = t_{\xi a}$ and $\sigma(a)t_\xi = t_{\psi(a)\xi}$ for every $\xi \in E$ and $a \in A$,

3) $t_\xi^* t_\zeta = \sigma(\langle\xi, \zeta\rangle)$ for every $\xi, \zeta \in E$,

4) $\sigma^{(1)}(\psi(a)) = \sigma(a)$ for every $a \in \psi^{-1}(K(E))$,

where $\sigma^{(1)} : K(E) \to B$ is given by $\sigma^{(1)}(\xi \otimes \eta^*) = t_\xi t_\eta^*$ and extended linearly and continuously.

Then, there exists a unique extension $\tilde{\sigma} : \mathcal{O}_E \to B$ of σ that maps S_ξ to t_ξ.

Theorem 2.4.7. *Let E be a full, faithful C^*-correspondence, with $\psi : A \to L(E)$. Let $I = \psi^{-1}(K(E))$ and $\mathcal{E}_{+,I} = \{\xi \in \mathcal{E} \ : \ \langle \xi, \xi \rangle \in I\}$. Then, $K(\mathcal{E}_{+,I}) \subset L(\mathcal{E}_{+,I})$ is precisely the kernel of the natural map $\mathcal{T}_E \to \mathcal{O}_E$. In other words, there is a short exact sequence*

$$0 \to K(\mathcal{E}_{+,I}) \to \mathcal{T}_E \to \mathcal{O}_E \to 0$$

Chapter 3

Tiling Spaces

The thesis concerns itself with a class of C*-dynamical systems associated to a substitution tiling. As such, we include a chapter outlining some basic terminology and facts.

Definition 3.0.1. *A* **tile** *is a subset of* $\mathbb{R}^d$ *that is homeomorphic to the closed unit ball in* $\mathbb{R}^d$ *and a* **tiling** *is a collection of tiles that cover* $\mathbb{R}^d$*, with pairwise disjoint interiors. A* **partial tiling** *is a collection of tiles that cover a subset of* $\mathbb{R}^d$*, with pairwise disjoint interiors. The* **support** *of a partial tiling,* P*, denoted* $supp(P)$*, is the union of all the tiles as a subset of* $\mathbb{R}^d$*.*

It will be useful for us to view a tiling T of $\mathbb{R}^d$ as a multivalued function from $\mathbb{R}^d$ into the tiles of T. That is, for $x \in \mathbb{R}^n$, $T(x) = \{t \in T \,:\, x \in t\}$ and similarly, for $U \subset \mathbb{R}^d$,

$$T(U) = \bigcup_{x \in U} \{t \in T \,:\, x \in t\}.$$

We can, in a similar way, consider a partial tiling P to be a map from the support of P into the tiles of P.

We will be interested in a specific class of tilings called substitution tilings. Let $p_1, \ldots, p_n$, be a finite set of tiles called prototiles. For $x \in \mathbb{R}^d$, we denote a translated tile by $p_i + x$, where x is a vector defining the translation. A substitution rule is a constant $\lambda > 1$ and, for each $i = 1, \cdots, n$, a partial tiling P_i of translates of $p_1, \cdots, p_n$ such that $supp(P_i) = \lambda supp(p_i)$. We define $\omega(p_i) = P_i$ and extend to translates of the prototiles by $\omega(p_i + x) = P_i + \lambda x, x \in \mathbb{R}^d$. We then extend ω in the obvious way to a partial tiling P, by applying ω to each tile in P. Note that we are able to define $\omega^k(P)$ for some partial tiling P by applying ω iteratively, since the image under ω of each partial tiling is another partial tiling. Let Ω be the set of all tilings, T, with the condition that any finite partial tiling $P \subset T$ is contained in $\omega^k(p_i + x)$ for some prototile p_i, $k \in \mathbb{N}$ and $x \in \mathbb{R}^d$. We call $T \in \Omega$ a substitution tiling.

Example 3.0.2. *An example of a substitution tiling in 2 dimensions is The Chair substitution, which is given by the sequence of images below, where we start with a prototile,* p_i, *in (1), inflate it to* $\lambda(p_i)$ *in (2), and re-tile it with prototiles in (3), resulting in* P_i. *The process is shown as it is applied to each prototile in* P_i *in (4) and (5):*

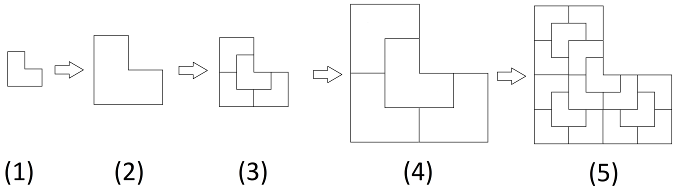

Figure 3.1: The Chair Substitution

Of particular interest in this thesis are one dimensional tilings, as working in higher dimensions rapidly increases the complexity of the calculations. Note first that in a one dimensional substitution, each tile corresponds to a closed interval of finite length. As a result, the information about our substitution is completely captured by an ordered substitution on letters: letting $a_1, \cdots, a_n$ be our n prototiles, defining for each i, $\omega(a_i) = a_{i_1} a_{i_2} \cdots a_{i_{k_i}}$ completely determines the substitution rule, which then allows us to compute the allowed lengths of the intervals, $supp(a_i)$, and the inflation constant from the substitution matrix. The substitution matrix is given by $\{a_{ij}\}_{i,j=1}^{n}$, where a_{ij} is a positive integer corresponding to the number of a_i prototiles whose translates appear in the substitution of a_j, $\omega(a_j)$. Note that the substitution matrix is a non-negative integer matrix and so with the added condition that the substitution matrix is primitive (to be defined below), we may apply the Perron-Frobenius Theorem to conclude that there is a largest positive eigenvalue, λ, which is the inflation constant of the substitution, and a corresponding nonnegative eigenvector of λ, $\{v_i\}_{i=1}^{n}$, where v_i gives the required (relative) length of the interval, *supp*(a_i).

Definition 3.0.3. *A matrix A is* **primitive** *if it is non-negative and its* m^{th} *power is positive for some natural number* m*.*

Primitivity of the substitution matrix M corresponds to imposing a mixing property on the one dimensional substitution ω: there is a k such that, for any letter a_i, $\omega^k(a_i)$ contains all letters $a_1, \cdots, a_n$.

Example 3.0.4. *Consider the substitution* ω *given by* $\omega(a) = aab$ *and* $\omega(b) = ab$*. We can visualize this with the following sequence:*

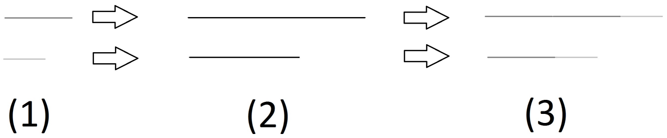

Figure 3.2: A 1-Dimensional Substitution

The first row of Figure 3.2 shows the tile a in (1), $\lambda(a)$ *in (2) and a retiled* $\lambda(a)$ *in (3) and the second row shows the tile b in (1),* $\lambda(b)$ *in (2) and a retiled* $\lambda(b)$ *in (3). The substitution matrix of this substitution is then*

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

which we can immediately see is primitive. We find the characteristic equation is $1 - 3\lambda + \lambda^2 = 0$*, and the Perron eigenvalue is* $\lambda = \frac{3+\sqrt{5}}{2} = \gamma^2$ *where* $\gamma = \frac{1+\sqrt{5}}{2}$*, the golden ratio and the corresponding Perron eigenvector* $v = \begin{pmatrix} \gamma \\ 1 \end{pmatrix}$

Thus, we conclude that the inflation constant is γ^2 *and the lengths of* a *and* b *respectively are* γ *and 1. We verify this result: the substitution increases the length of an* a *tile from* γ *to* $\gamma^2\gamma = (\gamma+1)\gamma = \gamma^2+\gamma = (\gamma+1)+\gamma = 2\gamma+1$*, and a* b *tile from 1 to* $\gamma^2 = \gamma + 1$*, where we have used the equality* $\gamma^2 = \gamma + 1$*. Thus,* a *is inflated to the length of 2* a*'s and 1* b*, and* b *is inflated to the length of an* a *and a* b*, as required. In particular, with* $a = [0, \gamma]$ *and* $b = [0, 1]$*, we have* $\omega(a) = \{a, a+\gamma, b+2\gamma\}$ *and* $\omega(b) = \{a, b+\gamma\}$*. Observe that moving the* a *and* b *to some other location has little effect; we just need to translate* $\omega(a)$ *(and* $\omega(b)$*) so that* $supp(\omega(a)) = \lambda supp(a)$ *(and* $supp(\omega(b)) = \lambda supp(b)$*).*

Chapter 4

A Cuntz-Pimsner Algebra Associated to a Substitution Tiling Space

4.1 A C^*-Algebra associated to a Partial Tiling

In this section we construct a C^*-algebra associated with a partial tiling obtained from a substitution rule. The dynamics from the substitution will be encoded using a Cuntz-Pimsner algebra constructed from an appropriate C^*-correspondence. For our construction to work, we need our initial partial tiling, denoted P_0, to contain all prototiles and all boundary intersections up to translation. This requirement is vague at the moment, but will be made more precise shortly. Although any choice of P_0 that satisfies these conditions will do, it will simplify some calculations if we choose one with as few tiles as possible. Given that ω is a substitution on the tiles in P_0, let $P_1 = \omega(P_0)$. We will also require that $P_0 \subset P_1$. Define P_k as the partial tiling given by k substitutions of P_0, so $P_k = \omega^k(P_0)$ and notice that the assumption that $P_0 \subset P_1$

implies that $P_j \subset P_k$ for all $j \leq k$. For the substitution $a \to aab$ and $b \to ab$, we can choose $P_0 = baab$ since it includes all letters, $\{a, b\}$ and all boundary intersections which will occur in subsequent substitutions, $\{aa, ab, ba\}$: $baab \to abaabaabab$. Notice that in this case, $P_0 \subset P_1$ as desired. In fact, P_0 appears as a subset of P_1 twice, which will not cause any problems, but will give us an (unimportant) option as to how we imbed the C^*-algebra associated with P_0 into the C^*-algebra associated with P_1 later.

Recall that we can consider a tiling T (or a partial tiling) as a multivalued function from $\mathbb{R}^d$ (or the support of the partial tiling) into the tiles of a tiling space T, so that for $x \in \mathbb{R}^d$, $T(x) = \{t \in T \ : \ x \in t\}$. Let $X_k = int(supp(P_k))$, the interior of the support of P_k. Define an equivalence relation, $R_k \subset X_k \times X_k$, by $(x, y) \in R_k$ if and only if $T(x) - x = T(y) - y$. When $(x, y) \in R_k$, we will also write $x \sim y$. This equivalence relation is saying that if we translate the tiles that contain x and y respectively so that x and y are on the origin, the tiles containing x should line up exactly with those containing y. Above, when we wrote that we need P_0 to contain all prototiles and boundary intersections, up to translation, we meant that we need P_0 to contain a member from every equivalence class that will occur in all subsequent substitutions. Generally, endowing an equivalence relation with a topology is a subtle matter, but here, we simply use the relative topology of $\mathbb{R}^{2d}$. This equivalence relation falls under a known class; with the relative topology of $\mathbb{R}^{2d}$, R_k is an étale equivalence relation (Theorem 4.1.2).

Definition 4.1.1. *An equivalence relation R on a locally compact metric space X is said to be étale if the canonical projections, $r, s : R \to X$, are local homeomorphisms; that is, for every $(x, y) \in R$, there exists an open neighbourhood U such that $r(U)$ is open and $r : U \to r(U)$ is a homeomorphism, and similarly for s.*

Theorem 4.1.2. *For each positive integer k, R_k is an étale equivalence relation.*

Proof. Fix $(x, y) \in R_k$, so that $T(x) - x = T(y) - y$. Recall that this means that if we shift the tiles $T(x)$ containing x and $T(y)$ containing y so that respectively x and y are at the origin, then $T(x) = T(y)$. Suppose $T(x) = \{t_1, \cdots, t_k\}$, so that x lies in each tile t_i for $i = 1, \cdots, k$, and the $\cup_{i=1}^{k} t_i$ covers an open ball $B_\epsilon(x)$ for some $\epsilon > 0$, where $B_\epsilon(x)$ is the open ball of radius ϵ about x. Since $(x, y) \in R_k$, we have $T(y) = \{t_1 - x + y, \cdots, t_k - x + y\}$, so that y lies in each $t_i - x + y$ and their union covers $B_\epsilon(y)$. If $x' \in B_\epsilon(x)$, then $T(x') = \{t_i \ : \ 1 \leq i \leq k, , x' \in t_i\}$, and $T(x' - x + y) = \{t_i - x + y \ : \ 1 \leq i \leq k, \ x' - x + y \in t_i - x + y\} = T(x') - x + y$. Thus, we have $(x', x' - x + y) \in R_k$ and hence $s(U) = B_\epsilon(x)$, $r(U) = B_\epsilon(y)$ where $U = \{(x', x' - x + y) \ : \ x' \in B_\epsilon(x)\}$. Next, we adjust ϵ to ensure that $r : U \to r(U)$ is injective. First, we note that $D = inf\{\|x - y\|, \ : \ x, y \in X, \ x \sim y, \ x \neq y\}$ is positive. In particular, it is not zero. Thus, choosing $\epsilon < D/2$ ensures that for each $x \in U$, the number of members in $[x]$ which are also in U is always 1. Then, $r(x) = r(y) \implies x \sim y \implies x = y$. That $r : U \to r(U)$ is surjective is given by definition, and easily seen to be continuous. □

Given an étale equivalence relation R_k, we can construct C^*-algebra in the following way. Recall that X_k is defined as $int(supp(P_k))$.

Definition 4.1.3. *Let $C^*(R_k)$ be the completion of the vector space $C_c(R_k)$ (the continuous compactly support functions on R_k) with*

$$f(x, y)^* = \overline{f(y, x)},$$

$$fg(x, y) = \sum_{z \in X_k \, : \, z \sim x} f(x, z) g(z, y)$$

and norm given by

$$\|f\|_I = sup\{\|\rho_x(f)\|_{[x]} \ : \ x \in X_k\}$$

with $f, g \in C_c(R_k)$, $(x, y) \in R_k$, *and* $[x]$ *denoting the equivalence class of* $x \in X_k$. *The norm* $\|\rho_x(f)\|_{[x]}$ *is the operator norm given by the representation of* f *on* $l_2([x])$, *that is*

$$\|\rho_x(f)\|_{[x]} = sup\{\|(f\xi)\| \,:\, \xi \in l_2([x]),\ \|\xi\| \leq 1\}$$

where

$$\|(f\xi)\| = (\sum_{z \in X_k \,:\, z \sim x} |f(y, z)\xi(z)|^2)^{\frac{1}{2}}.$$

It is intuitively helpful to notice that the multiplication of elements in $C^*(R_k)$ looks very similar to matrix multiplication. Note that it is not clear at this point that the norm defined above is bounded and this is the content of the next lemma.

Lemma 4.1.4. *The norm* $\|\cdot\|_I$ *defined in Definition 4.1.3 is bounded and is equivalent to the supremum norm on* $C_c(R_k)$.

Proof. The first thing to note, is that the number of members in the equivalence class of $x \in X_k$ has a maximum value which is less than or equal to the number of tiles in P_k. This is due to the fact that if $x \in t$ where t is a tile in P_k, then if $x \sim y$ for some $y \neq x$, then $y \in t'$, for some other tile $t' \in P_k$. It is possible that y is also in t, but y necessarily also belongs to another tile. Thus, for $f \in C_c(R_k)$, with $\xi \in l_2([x])$, $\|\xi\| \leq 1$ and letting N denote the number of tiles in P_k, we have that $(\sum_{z \in X_k \,:\, z \sim x} |f(y, z)\xi(z)|^2)^{\frac{1}{2}} \leq \sum_{z \in X_k \,:\, z \sim x} \|f\|_\infty \|\xi\| \leq N\|f\|_\infty$. This equality holds for all any $x \in X_k$ and any corresponding $\xi \in l_2([x])$, and so we have $\|f\|_I \leq N\|f\|_\infty$. It is also clear that for $\epsilon > 0$, $\|f\|_\infty \leq \|f\|_I + \epsilon$ which can be seen by choosing $(x_0, y_0) \in R_k$ such that $\|f\|_\infty \leq |f(x_0, y_0)| + \epsilon$. Then, define $\xi(x) = \delta_{x_0}$ where δ_{x_0} is defined to be one at x_0 and zero otherwise, and we have that $\|f\|_\infty \leq \|f(\xi)\| + \epsilon \leq \|f\|_I + \epsilon$. Since ϵ was arbitrary, the inequality follows. $\square$

Theorem 4.1.5. *As vector spaces,* $C^*(R_k) = C_0(R_k)$, *and the formulas for product and involution from Definition 4.1.3 also hold in* $C_0(R_k)$.

Proof. The proof is immediate, since $C_0(R_k)$ is the completion of $C_c(R_k)$ in the supremum norm, and we saw that the norm on $C_c(R_k)$ is equivalent to the supremum norm in Lemma 4.1.4. □

4.2 Encoding the Dynamics as a Cuntz-Pimsner Algebra

As outlined in Chapter 2.4, to construct a Cuntz-Pimsner Algebra which will encode the dynamics of our substitution, we first need a C^*-correspondence E over our C^*-algebra $A = C^*(R_0)$. The vectors of E are analogous to rectangular matrices (recall that $M_{m,n}(\mathbb{C})$ is naturally a right Hilbert $M_{n,n}(\mathbb{C})$ Module, with matrix multiplication as the right action), but the entries of the matrices are certain continuous functions. We make this precise in the following definition after a short proposition.

Proposition 4.2.1. *i) The equivalence relation $R_0 = R_1 \cap X_0 \times X_0$, and in particular, R_0 is an open subequivalence relation of R_1.*

ii) The inflated equivalence relation, λR_0 is an open subequivalence relation of R_1.

Proof. i): Fix $(x, y) \in R_0$. Since $X_0 \subset X_1$, (where $X_k = supp(P_k)$, $k >= 0$), $x \in X_1$ and $y \in X_1$. Since $x \sim y$ in R_0, $P_0(x) - x = P_0(y) - y$, viewing P_0 as a multivalued function from $\mathbb{R}^d$ into the tiles of P_0. Since the function P_0 is just the restriction of P_1 to P_0, $(x, y) \in R_1$ as well. We also have that $(x, y) \in X_0 \times X_0$. Thus, $R_0 \subset R_1 \cap X_0 \times X_0$.

Now let $(x, y) \in R_1 \cap X_0 \times X_0$. Since x and y are in $X_0 \times X_0$ and P_0 is just the restriction of P_1 to P_0, we have that $P_0(x) - x = P_0(y) - y$ and so $(x, y) \in R_0$.

ii): We first show $\lambda R_0 \subset R_1$. Let $(x, y) \in R_0$, so that $x, y \in X_0$ and $\lambda x, \lambda y \in X_1$. Since $(x, y) \in R_0$, $P_0(x) - x = P_0(y) - y$. But then, $\omega(P_0)(\lambda x) - \lambda x = \omega(P_0)(\lambda y) - \lambda y$

and so $P_1(\lambda x) - \lambda x = P_1(\lambda y) - \lambda y$ so that $(\lambda x, \lambda y) \in R_1$. $\square$

It will also be important to observe that $C^*(R_0)$ is subalgebra of $C^*(R_1)$ in two ways.

Proposition 4.2.2. *With the identifications $C^*(R_0) = C_0(R_0)$ and $C^*(R_1) = C_0(R_1)$ and the inclusion $C_0(R_0) \subset C_0(R_1)$ obtained by extending functions to be 0, $C^*(R_0)$ is a C^*-subalgebra of $C^*(R_1)$. It also appears as $\psi(C^*(R_0)) \subset C^*(R_1)$, an inflated version of $C^*(R_0)$ where*

$$\psi(a)(x,y) = \left\{ \begin{array}{cc} a(\lambda^{-1}x, \lambda^{-1}y), & (x,y) \in \lambda R_0 \\ 0, & \textit{otherwise} \end{array} \right\}.$$

Let α denote the former injection of $C^*(R_0)$ into $C^*(R_1)$.

Proof. $\square$

The results of both follow immediately from Lemma 4.2.1.

Definition 4.2.3. *Recall that by assumption, $P_0 \subset \omega(P_0) = P_1$. Thus, denoting as before the interior of the support of P_0 and P_1, as X_0 and X_1 respectively, we have $X_0 \subset X_1$, in a way that corresponds to how P_0 appears in P_1. Then, define $R_{1,0} = (X_1 \times X_0) \cap R_1$ and $E = C_0(R_{1,0})$ as a vector space.*

The right action of $A = C^*(R_0)$ for $\xi \in E$, $a \in A$ and $(x,y) \in R_{1,0}$ is given by

$$\xi \cdot a(x,y) = \sum_{z \in X_0 \,:\, z \sim y} \xi(x,z)a(z,x)$$

Notice that in a similar way, $C^*(R_1)$ can act on the left of E and so $C^*(R_1)$ is in a canonical way a subalgebra of $L(E)$. Thus, we define $\psi : A \to C^*(R_1) \subset L(E)$ for $(x,y) \in R_1$ by

$$\psi(a)(x,y) = \left\{ \begin{array}{cc} a(\lambda^{-1}x, \lambda^{-1}y), & (x,y) \in \lambda R_0 \\ 0, & otherwise \end{array} \right\}$$

where λ is the inflation constant of the substitution.

The A-valued inner product on E is given by

$$\langle \xi, \eta \rangle_A = \xi^* \eta, \tag{4.2.1}$$

where ξ^* is the conjugate transpose of ξ, and the product is the standard matrix type multiplication given by $\xi^*\eta(x,y) = \sum_{z:(x,z)\in R_{1,0}} \overline{\xi(z,x)}\eta(z,y)$. It is verified in the next lemma that this sesqui-linear form satisfies the axioms of an A-valued inner product.

Lemma 4.2.4. *The sesqui-linear form of equation 4.2.1 satisfies the axioms of an A-valued inner product.*

Proof. Most properties are tedious but easy to verify, so we just prove that $\xi^*\xi \geq 0$ and that $\xi^*\xi = 0$ if and only if $\xi = 0$. To see that $\xi^*\xi$ is a positive element in A, we show that the image of $\xi^*\xi$ under α in $C^*(R_1)$ is positive, where we are using the definition of α in Lemma 4.2.2. To see that $\alpha(\xi^*\xi)$ is positive in $C^*(R_1)$, we must find $b \in C^*(R_1)$ such that $b^*b = \alpha(\xi^*\xi)$. But note that as a vector space,$E = C_0(R_{1,0})$ is a subspace of $C_0(R_1)$, and so there is $b \in C_0(R_1)$ such that b is equal to ξ when restricted to $R_{1,0}$ and zero otherwise. Then, we have that $\alpha(\xi^*\xi) = b^*b$ as desired. □

Lemma 4.2.5. *The C^*-correspondence E over $C^*(R_0)$ as defined above is a* **full** *right Hilbert $C^*(R_0)$-module.*

Proof. First, for $a \in C^*(R_0)$, there exists $b, c \in C^*(R_0)$ such that $a = bc$ by Lemma 2.1.10. Next, $R_0 \subset R_{1,0}$, and so as vector spaces, $C_0(R_0)$ is a subspace of $C_0(R_{1,0})$ (here we consider $C_0(R_0)$ to be its embedded image in $C_0(R_{1,0})$) by extending the functions to be zero off of $R_0 \subset R_{1,0}$. Let $\xi, \eta \in E$ be supported only on $R_0 \subset R_{1,0}$, so that when their domains are restricted to R_0, $\xi^* = b$ and $\eta = c$. Then, $\langle \xi, \eta \rangle = \xi^* \eta = bc = a$.

□

Let $B = C^*(R_1)$. We, in fact, have constructed an $B - A$ equivalence bimodule once we add the extra B-valued inner product given by $\langle \xi, \eta \rangle_B = \xi \eta^*$, where $\xi\eta^*(x, y) = \sum_{z \in P_0} \xi(x, z)\overline{\eta(y, z)}$. Note that for $\xi, \eta, \mu \in E$ we have $\xi \langle \eta, \mu \rangle_A = \xi \eta^* \mu = \langle \xi, \eta \rangle_B \, \mu$, and so we just need to verify that the B-valued inner product is dense in B. The value in this identification is that we can conclude that A and B are Morita equivalent [7], and so the K-groups of A coincide with those of B, which will be useful for us later.

Lemma 4.2.6. *The Hilbert module, E, as defined above is a* **full** *left Hilbert* $C^*(R_1)$*-module.*

Proof. It will suffice to find $\{\xi_i\}_{i=1}^N \subset E$ such that $\sum_{i=1}^N \langle \xi_i, \xi_i \rangle_B$ is strictly positive on the diagonal of R_1, since then given $a \in C^*(R_1)$, and $\epsilon > 0$, we can find n such that $\|(\sum_{i=1}^N \langle a^n \xi_i, \xi_i \rangle_B)^{\frac{1}{n}} - a\| < \epsilon$. Since R_1 is an étale equivalence relation, by Lemma 4.1.2, we know that around each point $(x, y) \in R_1$ there is an open neighbourhood $U(x, y)$, which can and will be chosen to be the image of an open ball in the relative topology, which is homeomorphic to its image under the two projections r and s onto X_1, the support in $\mathbb{R}^d$ of P_1. Moreover, we can choose these balls so that the infimum of the length of the radii is positive. Denote this infimum length by r_0. Since R_1 is pre-compact, and we are covering by balls of radii greater than r_0, we can find a finite sub cover, $\cup_{i=1}^N U(x_i, x_i)$.

Next, we will find an element $\xi \in R_{1,0}$ such that $\langle \xi_i, \xi_i \rangle_B$ is supported only on $U(x_i, x_i)$, where it is positive. First note that for $(x_i, x_i) \in R_1$, there exists $(x_i, z_i) \in$

$R_{1,0}$, since we have assumed that X_0 contains an element from every equivalence class of R_k, $k \geq 0$. Moreover, if $x_i \in X_0 \subset X_1$ is equivalent to $z_i \in X_1$, then by the same reasoning as in Lemma 4.1.2, there exists open balls $B_\epsilon(x_i)$ and $B_\epsilon(z_i)$ such that for $x_i' \in B_\epsilon(x_i)$, $x_i' \sim x_i' + z_i - x_i \in B_\epsilon(z_i)$, where $\epsilon > 0$ is chosen as in Lemma 4.1.2. Thus, $B_\epsilon(x_i, z_i) \cap R_{1,0}$ is an open neighbourhood of $(x_i, z_i) \in R_{1,0}$. Define ξ_i so that it is zero except on $B_\epsilon(x_i, z_i) \cap R_1$ where it is positive. Then, $\langle \xi_i, \xi_i \rangle_B > 0$ on $U(x_i, x_i)$. Thus, we can find $\{\xi_i\}_{i=1}^N \subset E$ such that $\langle \xi_i, \xi_i \rangle_B > 0$ on $U(x_i, x_i)$ for each i, so that $\sum_{i=1}^N \langle \xi_i, \xi_i \rangle_B$ is a strictly positive element.

□

The C^*-correspondence, E, then completely determines the Cuntz-Pimsner algebra $\mathcal{O}_E$, which is constructed as outlined in Chapter 2. In the next section, we show that $\mathcal{O}_E$ is isomorphic to a full corner of a crossed product C^*-algebra.

4.3 Encoding the Dynamics as a Crossed Product by $\mathbb{Z}$

In this section, we show that the Cuntz-Pimsner system associated to a substitution tiling that we constructed in the previous section can also be constructed as a groupoid C^*-algebra or more specifically, as $\overline{hC^*(Q_S) \rtimes \mathbb{Z}h}$, where $C^*(Q_S)$ is a groupoid C^*-algebra and h is a certain positive element in $C^*(Q_S) \subset C^*(Q_S) \rtimes \mathbb{Z}$, both of which we will define below. The element h is essentially playing the role of a projection, and restricting us to a corner of $C^*(Q_S) \rtimes \mathbb{Z}$. We begin by constructing $C^*(Q_S)$.

Let T be a substitution tiling of $\mathbb{R}^d$. For each integer $k \geq 0$, define an equivalence relation, $Q_k \subset \mathbb{R}^d \times \mathbb{R}^d$ given by $x \sim_k y \iff T(\omega^k(x)) - \lambda^k x = T(\omega^k(y)) - \lambda^k y$. Note that $Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_k \subset \cdots$. Note here that Q_k is given the relative topology of $\mathbb{R}^d \times \mathbb{R}^d$ and that it is also an étale equivalence relation, the proof of which is essentially the same as the proof of Theorem 4.1.2. Consider the vector space $C_c(Q_k)$ of continuous compactly supported functions on Q_k. Define an involution $f(x,y)^* = \overline{f(y,x)}$ where $\overline{f(x,y)}$ is the complex conjugate of $f(x,y)$. Define a multiplication on $C_c(Q_k)$ by $fg(x,y) = \sum_{z:z\sim_k x} f(x,z)g(z,y)$, where we note that this sum is finite since these functions are compactly supported, so that each function is zero on all but finitely many of the points (x,z) such that $x \sim_k z$. We define a C^*-norm on $f \in C_c(Q_k)$ by

$$\|f\| = sup\{\|\rho_x(f)\|_{[x]} \ : \ x \in R^d\}$$

where the norm $\|\rho_x(f)\|_{[x]}$ is the operator norm given by the representation of f on $B(l_2([x]))$, that is

$$\|\rho_x(f)\|_{[x]} = sup\{\|(f\xi)\| \ : \ \xi \in B(l_2([x]))\}$$

where

$$\|(f\xi)\| = (\sum_{z\in\mathbb{R}^d \, : \, z\sim_k x} |f(y,z)\xi(z)|^2)^{\frac{1}{2}}.$$

Let $C^*(Q_k)$ denote the C^*-algebra obtained by completing $C_c(Q_k)$ in the norm above. Note that since $Q_k \subset Q_{k+1}$ is an open subset, $C_0(Q_k) \subset C_0(Q_{k+1})$ is a $*$-subalgebra. In particular, $C^*(Q_k)$ is a C^*-subalgebra of $C^*(Q_{k+1})$ and so we can define a direct limit C^*-algebra:

$$C^*(Q_S) = lim_{k\to\infty} C^*(Q_k)$$

where the connecting maps are just the natural inclusions. Observe that much like how $C^*(R_0)$ is a sub algebra of $C^*(R_1)$ in two ways, $\overline{hC^*(Q_0)h}$ is a sub algebra of $\overline{hC^*(Q_1)h}$ in two ways, and so there was some choice in how the connecting maps were defined: first, $Q_0 \subset Q_1$, and so $\overline{hC^*(Q_0)h}$ is a sub algebra of $\overline{hC^*(Q_1)h}$ under this imbedding. This is the one used in the construction of the direct limit. Secondly, since P_0 is contained in its image under the substitution, a "shrunk down" version of $\overline{hC^*(Q_0)h}$ also appears as a sub algebra of $\overline{hC^*(Q_1)h}$ as $\overline{\alpha(h)C^*(Q_1)\alpha(h)}$.

Consider the map $\alpha : C^*(Q_S) \to C^*(Q_S)$ given by $\alpha(f)(x, y) = f(\lambda x, \lambda y)$. This map is an automorphism of $C^*(Q_S)$ and so we can construct the crossed product $C^*(Q_S) \rtimes_\alpha \mathbb{Z}$. Recall from the previous section the partial tiling P_0, which contains all prototiles and all possible local configurations up to translation. Let $h \in C^*(Q_0)$ be such that $h(x, y) = 0$ if $x \neq y$, $h(x, x) > 0$ for all $x \in P_0$, and $h(x, x) = 0$ for all $x \notin P_0$. This is like a diagonal projection, except it's a bump function on the diagonal. We claim $\overline{hC^*(Q_S) \rtimes_\alpha \mathbb{Z}h} \cong \mathcal{O}_E$, where $\mathcal{O}_E$ is the Cuntz-Pimsner algebra defined in the previous section. In order to prove this assertion, we will use the universal property of $\mathcal{O}_E$. To do so, we first need to define a homomorphism $\sigma : A \to \overline{hC^*(Q_S) \rtimes_\alpha \mathbb{Z}h}$. First, note that $\overline{hC^*(Q_0)h}$ is a sub-algebra of $\overline{hC^*(Q_S) \rtimes_\alpha \mathbb{Z}h}$, and it is easy to see that $hC_c(Q_0)h = C_0(R_0)$, and hence $A = C^*(R_0) \cong \overline{hC^*(Q_0)h}$.

Note that $\overline{hC^*(Q_S) \rtimes_\alpha \mathbb{Z}h}$ is generated by elements of the form $h\xi uh$, where $\xi \in C^*(Q_S)$ and u is the unitary which implements the automorphism α: $u\xi u^* = \alpha(\xi)$. It is helpful to notice that

$$h\xi uh = h\xi\alpha(h)u$$

so that we can really think of $h\xi\alpha(h)$ for $\xi \in C^*(Q_1)$ as being a function $\tilde{\xi}$ which is supported on a rectangular subset (which looks like $R_{1,0}$) of Q_0. In fact, the collection of vectors of the form of $\tilde{\xi}$ is isomorphic as a vector space to $C_0(R_{1,0})$. Thus, it is clear that $h\xi uh$ is a good candidate for t_ξ. We next apply the universal property of $\mathcal{O}_E$ to deduce the existence of a homomorphism from $\mathcal{O}_E$ to $\overline{hC^*(Q_S) \rtimes_\alpha \mathbb{Z}h}$.

Proposition 4.3.1. *Let* $B = \overline{hC^*(Q_S) \rtimes_\alpha \mathbb{Z}h}$, *and* $\sigma : A \to \overline{hC^*(Q_S) \rtimes_\alpha \mathbb{Z}h}$ *be as* *above. Then, for $\xi, \eta \in E$, we have $t_\xi, t_\eta \in B$ with $t_\xi = \xi u$ and $t_\eta = \eta u$ such that*

1) *$\alpha t_\xi + \beta t_\zeta = t_{\alpha\xi+\beta\zeta}$ for every $\xi, \zeta \in E$ and $\alpha, \beta \in \mathbb{C}$*

2) *$t_\xi \sigma(a) = t_{\xi a}$ and $\sigma(a) t_\xi = t_{\psi(a)\xi}$ for every $\xi \in E$ and $a \in A$*

3) *$t_\xi^* t_\zeta = \sigma(\langle \xi, \zeta \rangle)$ for every $\xi, \zeta \in E$*

4) *$\sigma^{(1)}(\psi(a)) = \sigma(a)$ for every $a \in \psi^{-1}(K(E))$*

and so, by the universal property of $\mathcal{O}_E$, there exists an extension $\tilde{\sigma} : \mathcal{O}_E \to B$ which maps S_ξ to t_ξ.

Proof. First, we observe that $R_{1,0} \subset Q_1$ is open and so extending functions in $C_0(R_{1,0})$ to be zero in $(R_{1,0} \cap Q_1)^c$ means that $E = C_0(R_{1,0}) \subset C^*(Q_1) \subset C^*(Q_S)$ and in particular $E = \overline{hC^*(Q_1)\alpha(h)}$.

1) Immediate.

2) We have that

$$t_\xi \sigma(a) = \xi u \alpha^{-1}(a) = \xi a u = t_{\xi a}$$

and

$$\sigma(a) t_\xi = \alpha^{-1}(a) t_\xi = \psi(a) \xi u = t_{\psi(a)\xi}.$$

3)

$$t_\xi^* t_\eta = (\xi u)^* \eta u = u^* \xi^* \eta u = \alpha^{-1}(\xi^* \eta) = \sigma(\langle \xi, \eta \rangle)$$

4) First, $\psi(a) \in C^*(R_1) \cong K(E)$ so $\psi(a)$ can be approximated by finite sums of the form $\sum_i \xi_i \eta_i^*$. Thus, it suffices to prove it for such elements, and then the result extends by continuity.

$$\sigma^{(1)}(\psi(a)) = \sum_i t_{\xi_i} t^*_{\eta_i} = \sum_i \xi_i u(\eta_i u)^* = \sum_i \xi_i u u^* \eta_i^* = \sum_i \xi_i \eta_i^*$$

$$= \psi(a) = \alpha^{-1}(a) = \sigma(a)$$

□

Now that we have a homomorphism $\tilde{\sigma} : \mathcal{O}_E \to B$, we want to show that it is actually an isomorphism.

Theorem 4.3.2. *The homomorphism $\tilde{\sigma} : \mathcal{O}_E \to B$ is an isomorphism.*

Proof. We first show that $\tilde{\sigma}$ is isometric. By Proposition 4.4 from [5], the injectivity of $\tilde{\sigma}$ is equivalent to that of $\tilde{\sigma}|_C$, where $C \subset \mathcal{O}_E$, is the subalgebra given by the closed span of elements of the form $S_{\xi_k} S_{\xi_{k-1}} \cdots S_{\xi_1} S^*_{\eta_1} S^*_{\eta_2} \cdots S^*_{\eta_k}$, where k is not fixed. Let $a = \sum_{i=1} S_{\xi_{k_i}} S_{\xi_{k_i-1}} \cdots S_{\xi_1} S^*_{\eta_1} S^*_{\eta_2} \cdots S^*_{\eta_{k_i}}$, where the sum is finite. It suffices to prove injectivity for such elements, as the result then follows by continuity. Let C_1 denote the closed span of elements of the form $S_\xi S^*_\eta$, and let C_k denote the closed span of elements of the form $S_{\xi_k} S_{\xi_{k-1}} \cdots S_{\xi_1} S^*_{\eta_1} S^*_{\eta_2} \cdots S^*_{\eta_k}$, where in this case, k is fixed. As vector spaces, $C_k \cong C_0(R_k)$, and recall that $R_k \subset R_{k+1}$ for all $k \geq 1$, and so $C_0(R_k) \subset C_0(R_{k+1})$, where the inclusion is given by extending the functions to be zero on $R_{k+1} \backslash R_k$. In particular, we can always view a as an element in $C_0(R_k)$ for some sufficiently large k, and $\tilde{\sigma}$ is then the identity map into the vector space $C_0(R_k) \subset B$. Thus, $\tilde{\sigma}|_C$ is injective, and so too is $\tilde{\sigma}$.

Next, we show that the range of $\tilde{\sigma}$ is dense in B. Fix $b \in B$, so that b can be approximated arbitrarily well by a finite sum $b' = \sum_{i=-m}^{n} h q_i u^i h$ where $u^{-i} = u^{*i}$, and $q_i \in C^*(Q_S)$. It suffices to find an element in $\mathcal{O}_E$ which is mapped to hqu^ih, for $q \in C^*(Q_S)$ and $i \geq 0$, since then mapping an appropriate linear combination of such elements and their involutions will produce b'. Since $q \in C^*(Q_S)$, there exists $k \geq 0$ such that $q \in C^*(Q_k)$. Note that $hC^*(Q_k)h$ is generated by elements of the form $t_{\xi_1} \cdots t_{\xi_k} t^*_{\eta_1} \cdots t^*_{\eta_k}$, with ξ_i, η_i of the form $ha\alpha(h)$ for $a \in C^*(Q_1)$ and so it follows that $hC^*(Q_k)u^i h$ is generated by elements of the form $t_{\xi_1} \cdots t_{\xi_{k+i}} t^*_{\eta_1} \cdots t^*_{\eta_k}$, so it suffices to find an element in $\mathcal{O}_E$ that is mapped to $t_{\xi_1} \cdots t_{\xi_{k+i}} t^*_{\eta_1} \cdots t^*_{\eta_k}$, which is easy, since

$$\tilde{\sigma}(S_{\xi_1} \cdots S_{\xi_{k+i}} S^*_{\eta_1} \cdots S^*_{\eta_k}) = t_{\xi_1} \cdots t_{\xi_{k+i}} t^*_{\eta_1} \cdots t^*_{\eta_k}.$$

□

In the case that Ω described above contains no periodic tilings, it, along with $\omega : \Omega \to \Omega$ is an example of a Smale space. Our groupoid Q_S can be identified with the groupoid of stable equivalence, restricted to the unstable set of T as follows. In a general Smale space (X, ϕ), two points x, y in X are stably equivalent (unstably equivalent) if $d(\phi^n(x), \phi^n(y))$ tends to zero as n tends to infinity (negative infinity). In Ω, a tiling T' is unstably equivalent to T if and only if $T' = T - x$, for some vector $x \in \mathbb{R}^d$. So the map $x \in \mathbb{R}^d \to T - x$ is a bijection from $\mathbb{R}^d$ to the unstable class of T, $\Omega^u(T)$. The latter is given a natural topology and this bijection is a homeomorphism. The stable equivalence class of a tiling T' is those T'' such that $\omega^k(T') = \omega^k(T'')$ on $B_\epsilon(0)$ for some $k \geq 0$, $\epsilon > 0$. For $T' = T - x$, $T'' = T - y$, this is simply our equivalence relation Q_k above and $Q_S = \cup_k Q_k$.

Chapter 5

Computing the K-Theory of these C^*-Algebras

A key result in [9] which will be of great use to us in this section is the six term cyclic exact sequence of Figure 5.1 which connects the K-groups of $\mathcal{O}_E$ to the K-groups of other more easily understood C^*-algebras.

$$\begin{tikzcd}
K_0(K(\mathcal{E}_{I,+})) \arrow[r, "{[1-\otimes E]_0}"] & K_0(\mathcal{T}_E) \arrow[r] & K_0(\mathcal{O}_E) \arrow[d] \\
K_1(\mathcal{O}_E) \arrow[u] & K_1(\mathcal{T}_E) \arrow[l] & K_1(K(\mathcal{E}_{I,+})) \arrow[l, "{[1-\otimes E]_1}"]
\end{tikzcd}$$

Figure 5.1: Pimsner's Six Term Exact Sequence

As we will see later, $K_0(K(\mathcal{E}_{I,+}))$ and $K_0(\mathcal{T}_E)$ are both isomorphic to $K_0(A)$ and so a good description of the K-theory of A and the connecting maps in the exact sequence of Figure 5.1 will be key in our computation of the K-theory of $\mathcal{O}_E$.

5.1 The 1-Dimensional Case

In this section, we present the method for computing the K-Theory of the Cuntz-Pimsner Algebra associated with a 1-dimensional substitution tiling. The K-theory of a C^*-algebra is often revealed by understanding the ideal structure. We take advantage of the cyclic six term exact sequence of the K-groups obtained from the short exact sequences generated by an ideal I of a C^*-algebra A:

$$0 \to I \to A \to A/I \to 0$$

which gives

$$\begin{tikzcd}
K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[d] \\
K_1(A/I) \arrow[u] & K_1(A) \arrow[l] & K_1(I) \arrow[l]
\end{tikzcd}$$

Figure 5.2: The Six Term Exact Sequence of K-Groups

In many cases, some of these K-groups are easily determined while others are not, and this exact sequence, along with knowledge of the connecting maps, may allow us to deduce certain K-groups which are hard to compute directly.

As before, let ω be a substitution on n letters, $a_1, \cdots, a_n$, and assume that the substitution matrix of ω is primitive (Definition 3.0.3). In order to describe the procedure, we need to first introduce some notation. Note that in the definition of our equivalence relation, R_k, as applied to a 1-dimensional tiling, points in the intersection of two tiles (intervals) had their own equivalence classes depending on the ordered pair of intersecting tiles. Let V_k denote the set of all such points in X_k, which we will call vertices in analogy with the terminology of graphs (where we recall that X_k denotes the interior of the support of the partial tiling P_k). We will also call a point $(v, w) \in R_K$ a vertex if both v and w are vertices in X_k. Note that the pair of the left and right tiles define the equivalence class of each vertex in R_k, and we will use the letter m (or m_k if it is not clear) to denote the number of equivalence classes of vertices in R_k. In the 1-dimensional case, the set X_k is an open interval of a certain length (the sum of the lengths of the tiles in P_k). Note that $X_k \setminus V_k$ then consists of a set of disjoint open intervals, corresponding to the interiors of the supports of the tiles in the partial tiling. Again, in analogy with the terminology of graphs, we call the open intervals edges, and denote the set of all edges in $X_k \setminus V_k$ by L_k (L for lines; unfortunately, the obvious choice of E is already taken by our C^*-correspondence). Note that the equivalence relation given by R_k carries over to $L_k \times L_k$, simply by $R_K \cap (L_k \times L_k)$, and is again an étale equivalence relation.

Lemma 5.1.1. *If we identify $C^*(R_k)$ with $C_0(R_k)$ as linear spaces as in Theorem 4.1.5, then the set of functions which are zero at the vertices of R_k is an ideal in $C^*(R_k)$, which we will denote I_k. Furthermore, I_k has the following description:*

$$I_k = \oplus_{i=1}^{n} M_{k_i} \otimes C_0(int(supp(a_i)))$$

where n is the number of letters in the substitution, and M_{k_i} is the complex $k_i \times k_i$ matrices where k_i is given by the number of edges in L_k which are the support of the letter a_i.

Proof. That I_k is closed as a *-algebra is straight forward, so we will only verify the absorbing property. Let $a \in C^*(R_k)$ and $b \in I_k$. Let $(v, w) \in R_k$ be a vertex. Then, $ab(v, w) = \sum_{z \sim w} a(v, z)b(z, w) = 0$ since (z, w) is a vertex in R_k for all $z \sim w$. The proof for ba is similar. This is a general fact; ideals in $C_r^*(G)$ come from G-invariant open subsets of the unit space for G, a principal groupoid, [10].

To show the second assertion, first note L_k is a disjoint union of intervals; more specifically, $L_k \cong \dot{\cup}_{i=1}^{n}\{1, \cdots, k_i\} \times int(a_i)$. Applying to L_k the equivalence relation that was used on X_k to obtain R_k, we get $\cup_{i=1}^{n}\{1, \cdots, k_i\}^2 \times int(a_i)$.that the edges (which are open and disjoint) inherit an equivalence relation from that on the points, where we can say two edges are equivalent if each point of each edge has an equivalent point in the other. This amounts to defining any two edges corresponding to the same letter to be equivalent. The obstruction to this view in R_k, was that the endpoints of two equivalent edges may not have belonged to the same equivalence class. Using this new view of the equivalence relation, we can see the claimed form I_k.

□

The advantage of this characterization of I_k is that the K-theory of this type of object is easily computed. In particular, we will use the six-term exact sequence obtained from $0 \to I_0 \to A \to A/I_0 \to 0$ to compute the K-theory of A. Thus, in preparation, we next examine the structure of $C^*(R_k)/I_k$, and in particular $A/I_0 = C^*(R_0)/I_0$.

Lemma 5.1.2. *The quotient $C^*(R_k)/I_k \cong \oplus_{i=1}^{m} M_{j_i}$, where m is the number is equivalence classes of vertices in X_k, M_{j_i} denotes the $j_i \times j_i$ matrices over $\mathbb{C}$, and j_i is the number of members in the i^{th} equivalence class for $i = 1, \cdots, m$.*

Proof. Let $a \in C^*(R_k)$, so that $a + I_k$ is an equivalence class in $C^*(R_k)/I_k$. Let $\{[v_i]\}_{i=1}^{m}$ denote the m equivalence classes of vertices in X_k, and for each i, let $\{v_{i1}, v_{i2}, \cdots, v_{ij_i}\}$ be the j_i members in the equivalence class of v_i. The isomorphism maps a to

$$\begin{pmatrix} a(v_{11},v_{11}) & a(v_{11},v_{12}) & \cdots & a(v_{11},v_{1j_1}) \\ a(v_{12},v_{11}) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a(v_{1j_1},v_{11}) & \cdots & \cdots & a(v_{1j_1},v_{1j_1}) \end{pmatrix} \oplus \cdots$$

$$\oplus \begin{pmatrix} a(v_{m1},v_{m1}) & a(v_{m1},v_{m2}) & \cdots & a(v_{m1},v_{mj_m}) \\ a(v_{m2},v_{m1}) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a(v_{mj_m},v_{m1}) & \cdots & \cdots & a(v_{mj_m},v_{mj_m}) \end{pmatrix}$$

It is a straight forward, though tedious, procedure to verify that this map is a *-homomorphism. Injectivity is immediate, since an element mapped to the zero element is zero on all of the vertices, and as a result is a function in I_k. Surjectivity follows by defining a continuous function which takes on the required values at the vertices, which can be done since the minimum distance between any pair of vertices exists and is positive.

□

We recall again the fact that ideals in $C_r^*(G)$ come from G-invariant open subsets of the unit space for G, a principal groupoid, [10]. Consider now the short exact sequence

$$0 \to I_k \to C^*(R_k) \to C^*(R_k)/I_k \to 0$$

which induces the 6-term cyclic exact sequence of Figure 5.3.

Of particular interest in the 1-dimensional case is when $k = 0$: due to the characterization of I_0 and A/I_0 given above, we can immediately determine their K-

$$\begin{tikzcd}
K_0(I_k) \arrow[r] & K_0(C^*(R_k)) \arrow[r] & K_0(C^*(R_k)/I_k) \arrow[d] \\
K_1(C^*(R_k)/I_k) \arrow[u] & K_1(C^*(R_k)) \arrow[l] & K_1(I_k) \arrow[l]
\end{tikzcd}$$

Figure 5.3: The Six Term Exact Sequence of K-Groups of $C^*(R_k)$

groups. We use the facts that $K_0(M_j) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$, $K_1(M_j) \cong K_1(\mathbb{C}) \cong 0$, $K_0(M_j \otimes C_0(0,1)) \cong K_0(C_0(0,1)) \cong 0$, and $K_1(M_j \otimes C_1(0,1)) \cong K_1(C_0(0,1)) \cong \mathbb{Z}$. As a result, $K_*(I_0)$ and $K_*(C^*(R_0)/I_0)$ depend only on the number of letters in the substitution, and the number of equivalence classes of vertices. Thus, if the substitution has n letters and m equivalence classes of vertices, then $K_0(I_0) = 0$, $K_1(I_0) \cong \mathbb{Z}^n$, $K_0(A/I_0) \cong \mathbb{Z}^m$, and $K_1(A/I_0) = 0$. These results gives us the exact sequence

$$K_0(I_0) \to K_0(A) \to K_0(A/I_0) \to K_1(I_0) \to K_1(A) \to K_1(A/I_0) \tag{5.1.1}$$

or after replacing isomorphic groups

$$0 \to K_0(A) \to \mathbb{Z}^m \to \mathbb{Z}^n \to K_1(A) \to 0 \tag{5.1.2}$$

In particular, the map $K_0(A/I_0) \to K_1(I_0)$ is the exponential map, δ_0 and so $K_0(A) \cong ker(\delta_0)$, which is computable since we know $K_0(A/I_0)$ and $K_1(I_0)$. Similarly, we have $K_1(A) \cong coker(\delta_0)$, which is also computable. We record these results as part of the next lemma.

Lemma 5.1.3. *Let A be as before, where we denote the n letters of the substitution by $a_1, \cdots, a_n$ and the number of equivalence classes of vertices by m. Then, $K_0(I_0) \cong 0$, $K_1(I_0) \cong \mathbb{Z}^n$, $K_0(A/I_0) \cong \mathbb{Z}^m$, and $K_1(A/I_0) = 0$. Furthermore, the generators of*

$K_1(I_0)$ correspond to the equivalence classes of the unitaries in I_0, $u_1, \cdots, u_n$, where u_i is a function which is 0 off the diagonal of R_0, 1 on the diagonal, except on a segment corresponding to an a_i letter, where it wraps around the complex unit circle once in the positive direction.

Proof. It is well known fact that a generator of $K_1(C_0(0,1))$ is $u(t) = e^{2\pi i t}$ and we saw above that $K_1(I_0) \cong K_1(\oplus_{i=1}^n M_{k_i} \otimes C_0(int(supp(a_i)))) \cong K_1(\oplus_{i=1}^n C_0(int(supp(a_i)))) \cong \oplus_{i=1}^n K_1(C_0(int(supp(a_i))))$. □

Theorem 5.1.4. *For $A = C^*(R_k)$, $K_1(A) \cong \mathbb{Z}$.*

Proof. We know that $K_1(I_0) \cong \mathbb{Z}^n$ where n is the number of letters in the substitution. As in Lemma 5.1.3, we let $u_1, \cdots, u_n$ in I_0 denote the n generators of $K_1(I_0)$ and denote the map from $K_1(I_0)$ to $K_1(A)$ in equation 5.1.1 by i_*. Due to the exactness, i_* is surjective. Moreover, i_* is the induced map between the K_1 groups given by the inclusion map of I_0 into A. Thus, $i_*([u_i])$ has a representative in A given by a function which is one on the diagonal of R_0 except on the support of the a_i letter where it wraps once around the complex unit circle in the positive direction. The point here though, is that there is no longer the restriction of the vertices on the diagonal taking the value 1, so that $i_*([u_i]) \sim i_*([u_j])$ for all i, j. Thus, $K_1(A)$ is generated by a single element, and so $K_1(A) \cong \mathbb{Z}$.

□

Example 5.1.5. *Consider the substitution from before, $a \to aab$, $b \to ab$. We took $P_0 = baab$, and so we can see that we have two letters, a and b, and three vertices at the intersection from left to right of b and a, a and a, and a and b. Each of the three vertices is an equivalence class, and so $A/I_0 \cong \mathbb{C}^3$ and $I_0 \cong M_2 \otimes C_0(int(a)) \oplus M_2 \otimes C_0(int(b))$. We then know that $K_0(A/I_0) \cong \mathbb{Z}^3$ and $K_1(I_0) \cong \mathbb{Z}^2$. We also know that $K_1(A) \cong \mathbb{Z}$ from Theorem 5.1.4. Then using the exact sequence 5.1.1, we can* *conclude that* $K_0(A) \cong \mathbb{Z}^2$. *Note: we could have also compute* $K_0(A)$ *by calculating the kernel of the exponential map.*

Our goal is to compute the K-groups of $\mathcal{O}_E$. We follow Pimsner's approach from [9]. Using the results we covered in Section 2.4, in particular Theorem 2.4.7, we have that $K_*(K(\mathcal{E}_{+,I})) \cong K_*(I)$ and $K_*(\mathcal{T}_E) \cong K_*(A)$ and that the connecting map between these two is given by $[1 - \otimes E]_*$, where 1 is the identity map, and I is the ideal in A given by $I = \psi^{-1}(K(E))$ (not to be confused with I_k). In our case, $A = I$, since $\psi(A) \subset C^*(R_1) \cong K(E)$. We outline our procedure for computing the map $[\otimes E]_*$ in the general case, and then we will explicitly compute it in the case of Example 3.0.4.

Recall that I_B denotes the ideal in $B = C^*(R_1)$ consisting of those functions which are zero on the vertices of R_1, $\psi : A \to B$, the inflation map, and λ, the inflation constant. Let J be the ideal in A given by functions $\xi \in A$ such that $\xi(x, y) = 0$ if $\lambda(x)$ (or $\lambda(y)$) is a vertex of R_1, or in other words, $J = \psi^{-1}(I_1)$. Consider the commutative diagram of Figure 5.4 (we will define the six vertical maps later):

$$\begin{tikzcd}
0 \arrow[r] & K_0(A) \arrow[r, "q_*"] \arrow[d, "G_*"] & K_0(A/I_0) \arrow[r, "\delta_0"] \arrow[d, "G_{q*}"] & K_1(I_0) \arrow[r, "i_*"] \arrow[d] & K_1(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r] \arrow[d, "\phi_*"] & K_0(A/J) \arrow[r] \arrow[d, "\phi_{q*}"] & K_1(J) \arrow[r] \arrow[d] & K_1(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(B) \arrow[r] \arrow[d, "\alpha_*^{-1}"] & K_0(B/I_B) \arrow[r] \arrow[d, "\alpha_{q*}^{-1}"] & K_1(I_B) \arrow[r] \arrow[d] & K_1(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r, "q_*"] & K_0(A/I_0) \arrow[r, "\delta_0"] & K_1(I_0) \arrow[r, "i_*"] & K_1(A) \arrow[r] & 0
\end{tikzcd}$$

Figure 5.4: Commutative Diagram for Computing the Connecting Map $[1 - \otimes E]_*$

Each row comes from the six term cyclic exact sequence of Figure 5.3. Our goal is to compute the far left and right vertical maps from top to bottom, as we will show later that these correspond to the map $[\otimes E]_0$ and $[\otimes E]_1$ of 5.1. Denote from left to right, the vertical maps from the first row to the last row as $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4. The general idea is that we are able to compute Γ_2, Γ_3, q_*, δ_0 and i_*, which then allows us to deduce Γ_1 and Γ_4 from the commutativity of the diagram.

We wish to define the map G_*, which is induced by a homotopy $g : X_0 \times [0,1] \to X_0$ (recall X_0 is the support of P_0). Suppose $P_0 = a_1 \cdots a_n$, so that $P_1 = \omega(a_1) \cdots \omega(a_n)$. We denote $\omega(a_i) = b_{i1} \cdots b_{in}$. Define $X_{a_i} = supp(a_i)$, and $X_{b_i} = supp(\phi^{-1}(b_{i1})) \subset X_{a_i}$. In other words, X_{b_i} is the subset of X_{a_i} which is mapped to the first letter in the substitution of a_i by ω. The homotopy that we wish to define will have the property that $g(x, 0) = x$ for all $x \in X_0$ and $g(X_{b_i}, 1) = X_{a_i}$. The existence of such a homotopy is clear from Figure 5.5, but is also easy to define explicitly in specific cases:

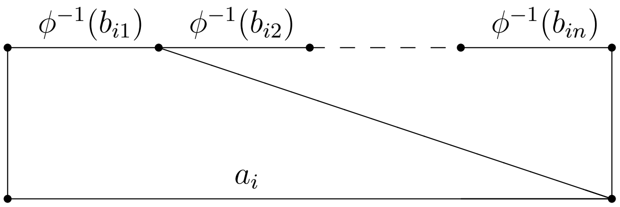

Figure 5.5: A Visual Representation of the Homotopy, g

where $\phi^{-1}(b_{i1}), \cdots, \phi^{-1}(b_{in})$ correspond to the inverse image under ϕ of the substituted letters of a_i. Note in the homotopy, $\phi^{-1}(b_{i2}), \cdots, \phi^{-1}(b_{in})$ are smoothly contracted down to the end point of a_i and $\phi^{-1}(b_{i1})$ is smoothy inflated until it is the same length as a_i and is thusly mapped onto a_i. This process is extended to the image of any partial tiling under a substitution by simply applying the procedure to the image of each letter.

The next step is to extend the homotopy to the equivalence relation on the partial tiling. For the equivalence relation on P_0 given by $R_0 \subset P_0 \times P_0$, the homotopy on P_0 induces in a natural way a homotopy on $P \times P$, which then induces a homotopy on $R \subset P \times P$ by restriction: for $(x, y) \in R_0$, $(g(t, x), g(t, y)) \in R_0$ for all $t \in [0, 1]$.

Recall that a homotopy between spaces X and Y, $g : X \to Y$ induces a homotopy $\tilde{g} : C_0(Y) \to C_0(X)$ by $\tilde{g}(f) = f \circ g$. Thus, as desired, we have obtained a path G_t from $A = C^*(R_0)$ to A, where $G_t(f)(x, y) = f(g(t, x), g(t, y))$. We claim, and prove below, that when G_1 is restricted to the ideal $I_0 \subset A$, the image, $G_1(I_0)$, is a subset of $J \subset A$, so that G_1 also induces a well defined $*$-homomorphism from A/I_0 into A/J, which we denote G_q. Thus, the desired map G_{q*} is simply the induced map of G_q on the K-groups; i.e. $G_{q*}([p] - [q]) = [G_q(p)] - [G_q(q)]$ where $p, q \in A/I_0$ are representative projections for the element $[p] - [q] \in K_0(A/I_0)$.

Lemma 5.1.6. *$G_1(I_0) \subset J$.*

Proof. By definition, the image of the vertices of R_0 under $g(1, \dot{})$ contains the pre images of the vertices of R_1, and so a function which is zero on the vertices of R_0 will be mapped by G_1 to a function which is zero on the pre images of the vertices of R_1. □

The next map we need to compute is ϕ_*. First, recall the inflation map $\phi : A \to B$, given by $\phi(a(x, y)) = a(\lambda^{-1}x, \lambda^{-1}y)$ where λ is the inflation constant of the substitution. Next, note that $\phi(J) \subset I_1$, since by definition $\phi(j)$ vanishes on the vertices of B for any $j \in J$. Thus, ϕ extends to a well defined map from $\phi_q : A/J \to B/I_1$ and then ϕ_{q*} is defined to be the induced map on the K-groups.

Lastly, we need to define α_*^{-1}, in order to define its induced quotient map, α_{q*}^{-1}. One might hope that it is simply the induced map on the K-groups of α^{-1}, but while α_* is invertible, α is not. The easiest way to define α_*^{-1} is as the map implementing the Morita Equivalence between B and A. Recall that to show two C^*-algebras are Morita Equivalence, we need a C^*-correspondence between them. In the case of B and A, E is exactly that C^*-correspondence.

Since A and B are Morita equivalent, their K-groups are isomorphic, and the isomorphism is given by tensoring the C^*-correspondence between them. We can view the elements in $K_0(B)$ as projective right Hilbert B modules, each of which is given by pB^n, where p is a projection in $L(B^n)$ for some $n \in \mathbb{Z}^+$. Then, the isomorphism between $K_0(B)$ and $K_0(A)$ is given by

$$[pB^n] \to [pB^n \otimes E] \in K_0(A).$$

Lemma 5.1.7. *With* $B = C^*(R_1)$, $A = C^*(R_0)$ *and* p *a projection in* $L(B^n)$ *for some* $n \in \mathbb{Z}^+$, $pA^n \otimes_A E \cong \phi_n(p)B^n \otimes_B E$ *as right Hilbert* A*-modules, where* $\otimes_B$ *is balanced by the identity map.*

Proof. First note that we have the equality, $\phi_n(p)(b_k)_{k=1}^n b \otimes_B \xi = \phi_n(p)(b_k)_{k=1}^n \otimes_B b\xi$. Let $\pi : pA^n \otimes_A E \to \phi_n(p)B^n \otimes_B E$ denote the homomorphism defined by $\pi(p(a_k)_{k=1}^n \otimes_A \xi) = \phi_n(p)((\phi(a_k))_{k=1}^n) \otimes_B \xi$. It suffices to show that π is isometric and that the range of π is dense. Let $\sum_{i=1}^m p((a_k^{(i)})_k) \otimes \xi_i \in pA^n \otimes_A E$. Then,

$$\begin{aligned}
\|\sum_{i=1}^m p((a_k^{(i)})_k) \otimes \xi_i\|_A^2 &= \|\sum_{i,j=1}^m \langle \xi_i, \phi(\langle p((a_k^{(i)})_k), p((a_k^{(j)})_k)\rangle)\xi_j\rangle\| \\
&= \|\langle \sum_{i=1}^m \phi_n(p)(\phi(a_k^{(i)})_k) \otimes_B \xi_i, \sum_{i=1}^m \phi_n(p)(\phi(a_k^{(i)})_k) \otimes_B \xi_i\rangle\| \\
&= \|\sum_{i=1}^m \phi_n(p)\phi((a_k^{(i)})_k) \otimes_B \xi_i\|_A
\end{aligned}$$

Next, we show that the range of π is dense. Since π is a module homomorphism, it suffices to prove it for an elementary tensor $\phi_n(p)(b_k^{(i)})_k \otimes \xi \in \phi_n(p)B^n \otimes_B E$, where $(b_k^{(i)})_k \in B^n$ is zero in all entries except the i^{th} where it takes the value $b \in B$. Let $(e_m)_m \subset A$ be the approximate unit given by the n^{th} roots of a function on R_0 which is strictly positive and less than or equal to 1 on the diagonal of R_1, and 0 otherwise.

Note that $\phi(e_m)_m \subset B$ is an approximate unit for B. Given $\epsilon > 0$, we can pick m so that

$$\|\phi_n(p)(\phi(e_m)b_k^{(i)})_k \otimes \xi - \phi_n(p)(b_k^{(i)})_k \otimes \xi\| < \epsilon.$$

Now, $\phi_n(p)(\phi(e_m)b_k^{(i)})_k \otimes \xi = \phi_n(p)(\phi(e_{m,k}^{(i)})_k) \otimes b\xi$ where $(e_{m,k}^{(i)})_k$ is zero in each entry except the i^{th} where it takes the value e_m and $\phi_n(p)(\phi(e_{m,k}^{(i)})_k) \otimes b\xi = \pi(p(e_{m,k}^{(i)})_k) \otimes b\xi)$. Thus the range of π is dense in $\phi_n(p)B^n \otimes_B E$.

□

The map Γ_2 consists of $\alpha_{q*}^{-1} \circ \phi_{q*} \circ G_{q*}$. The corresponding maps of Γ_1 are denoted respectively as G_*, ϕ_* and α_*^{-1}. G_* is induced by G_1 which is homotopic to the identity, and so by homotopy invariance, G_* is the identity on the K-groups of A. Thus, the composition of all three of these maps takes $[pA^n]$ to $[\phi^{(n)}(p)B^n \otimes E] = [pA^n \otimes E]$, where the equality is due to Lemma 5.1.7 and so we see that in Pimsner's sequence we have that Γ_1 is $[\otimes E]_0$ and similarly, Γ_4 is $[\otimes E]_1$.

In the 1-dimensional case, with the added condition that the substitution of each letter starts with the same letter, a condition which we will explore more in the next section, we can in general compute Γ_4, and we find that it is always the identity map. To see this, first let u_i denote the unitary corresponding to the letter a_i as in Lemma 5.1.3 where we have defined the letters in the substitution to be $a_1, \cdots, a_n$ as usual. We compute directly $\Gamma_3([u_i]_1)$, one map at a time. First, $G_*([u_i]_1) = [v_{i1}]_1$, where v_{i1} is zero off the diagonal, and 1 on the diagonal except on the preimage of the first letter of the substitution of a_i, where it wraps once around the unit circle in the positive direction. Since the substitution of each letter begins with the same letter, which we may assume is a_1, we find that $\alpha_*^{-1}(\phi_*(G_*([u_i]_1))) = [u_1]_1$.

We also have that $i_* = [1, 1, \cdots, 1]$, since as we saw before, the image of each generator of $K_1(I_0)$ is homotopic in $K_1(A)$. Thus, by the commutativity of the diagram, we must have that $i_*\Gamma_3 = \Gamma_4 i_*$, which implies that $\Gamma_4 = 1$.

Consider the six term cyclic exact sequence of Figure 5.6 given by Pimsner in [9]:

$$\begin{tikzcd}
K_0(K(\mathcal{E}_{I,+})) \arrow[r, "{[1-\otimes E]_0}"] & K_0(\mathcal{T}_E) \arrow[r] & K_0(\mathcal{O}_E) \arrow[d] \\
K_1(\mathcal{O}_E) \arrow[u] & K_1(\mathcal{T}_E) \arrow[l] & K_1(K(\mathcal{E}_{I,+})) \arrow[l, "{[1-\otimes E]_1}"']
\end{tikzcd}$$

Figure 5.6: Pimsner's Six term Exact Sequence

Since presently we have that $I = \phi^{-1}(B) = A$, and using the results in [9] that $K_*(K(\mathcal{E}_{I,+})) \cong K_*(I)$ and $K_*(\mathcal{T}_E) \cong K_*(A)$, we find that in our case, $K_*(K(\mathcal{E}_{I,+})) \cong K_*(A)$, $K_*(\mathcal{T}_E) \cong K_*(A)$. Combining these results with the fact that $K_1(A) \cong \mathbb{Z}$, we obtain the new exact sequence:

$$\begin{tikzcd}
K_0(A) \arrow[r, "1-\Gamma_1"] & K_0(A) \arrow[r] & K_0(\mathcal{O}_E) \arrow[d] \\
K_1(\mathcal{O}_E) \arrow[u] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "1-\Gamma_4"']
\end{tikzcd}$$

Figure 5.7: Pimsner's Six Term Exact Sequence Applied to $\mathcal{O}_E$

In the next example, we go through in detail the procedure for computing the K groups of the Cuntz-Pimsner algebra of the prototype example from above.

Example 5.1.8. *Recall the 1 dimensional substitution given by* $a \to aab$ *and* $b \to ab$. *We saw above that* $K_0(A) \cong \mathbb{Z}^2$, $K_0(A/I) \cong \mathbb{Z}^3$, $K_1(I) \cong \mathbb{Z}^2$, *and* $K_1(A) \cong \mathbb{Z}$. *We claim that the exponential map is given by the matrix*

$$\delta_0 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Let $p_1, p_2, p_3 \in A/I$ *be the three projections which generate* $K_0(A/I)$*, where each corresponds to a function which is one on the diagonal vertex, which we'll denote* v_1*,* v_2 *and* v_3*, corresponding to the intersection of the edges* b, a*,* a, a *and* a, b *respectively, and zero elsewhere. Recall to compute the exponential map, we first seek preimages* A *of* p_1*,* p_2 *and* p_3 *which we denote respectively,* a_1*,* a_2*, and* a_3*. For* $i = 1, 2, 3$*, we need* a_i *to take the value one at the vertex* v_i*, and zero on the other vertices; in other words,* a_i *is a bump function on the diagonal of* R_0 *around the vertex* v_i*, and zero elsewhere. Since* a_i *has support only on the diagonal, we can really think of it as being a function of one variable,* $a_i(t)$*, where* t *takes values in an open interval of appropriate length. The exponential map is then given as* $\delta_0([p_i]) = exp(2\pi\, i\, a_i)$ *and in particular* $exp(2\pi\, i\, a_i)$ *is zero off the diagonal, and can be expressed as* $exp(2\pi\, i\, a_i(t))$ *on the diagonal. Notice that* $exp(2\pi\, i\, a_1(t))$ *wraps once around the complex unit circle in the positive direction on the* b *edge, and once around the complex unit circle in the negative direction on the* a *edge, which corresponds to the unitaries* u_1^* *and* u_2*, which in the basis* $\{u_1, u_2\}$ *is given by*

$$\delta_0([p_1]) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Similarly,

$$\delta_0([p_2]) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\delta_0([p_3]) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We want to use this information to figure out the quotient map $q_* : K_0(A) \to K_0(A/I)$*. Using* $ran(q_*) = ker(\delta_0) = span_{\mathbb{Z}}\{(0,1,0),(1,0,1)\}$*, and choosing a basis for* $K_0(A)$ *to be* $\{q_*^{-1}(0,1,0),\, q_*^{-1}(1,0,1)\}$*, with respect to these bases, we can write*

$$q_* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We next define the vertical map $G_* : K_0(A/I) \to K_0(A/J)$*. To do so, we first define the homotopy* $g : P_0 \times [0,1] \to P_0$*. Since our substitution is given by* $a \to aab$ *and* $a \to ab$*,*

To define the homotopy on each b *segment, we identify the subsegments of* b *which map to* a *and* b*, and simply denote these as* a *and* b *respectively, where we note that* $\phi^{-1}\omega(b) = \phi^{-1}(ab)$ *has the same length as* b*. Then, our homotopy* g *stretches* a *while shrinking* b *to the right endpoint of* b *so that* $ab \to b$*, as is illustrated in Figure 5.8:*

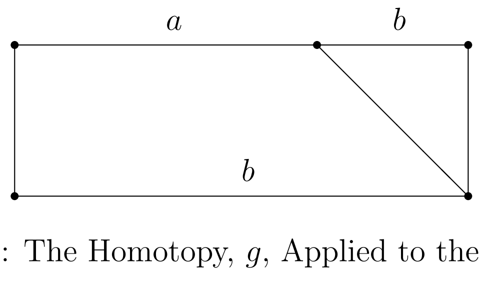

Figure 5.8: The Homotopy, g, Applied to the Substitution $b \to ab$

Note that the ab *segment is just a relabelling of* b*, and that this is really a homotopy from* b *to* b*, or more precisely, from the support of* b *to the support of* b*.*

For a*, we write it as an* aab *segment, it stretches the first* a *while shrinking the* ab *segment to the right endpoint of* a*, so that* $aab \to a$*, as is illustrated in Figure 5.21:*

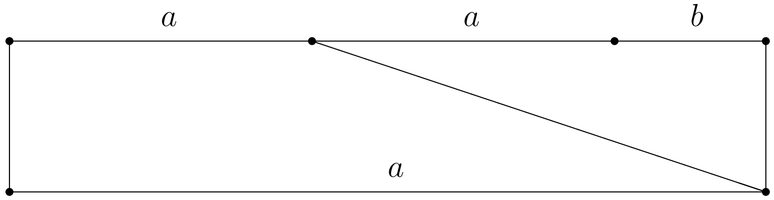

Figure 5.9: The Homotopy, g, Applied to the Substitution $a \to aab$

We then extend this process to all of P_0 *as is illustrated in figure 5.10:*

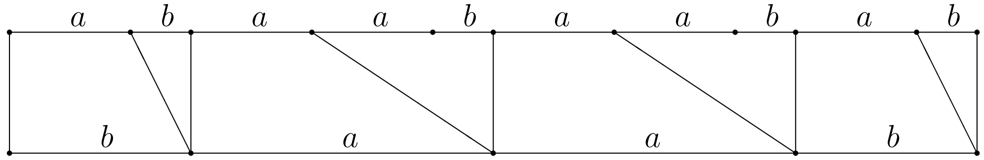

Figure 5.10: The Homotopy, g, Applied to the Substitution $a \to aab$ and $b \to ab$ on P_0

Lastly, we extend it to $R_0 = P_0 \times P_0$*. This homotopy induces a homotopy* $G : A \times [0,1] \to A$*, where* $G(A,0) = A$ *and* $G(A,1)$ *is a proper subset of* A *with the property that* $K_*(A) \cong K_*(G(A,1))$*, since homotopic maps induce the same map on K-theory. Recall that* $G(I,1) \subset J$*, so that* G *induces a well defined map* G_q *from* A/I *into* A/J*.*

Let us pause to consider J *and* A/J*.* $J \cong \oplus_{i=1}^{5} M_2 \otimes C(c_i)$*, where* $\{c_i\}_{i=1}^{5}$ *refers to the 5* R_0*-equivalence classes of edges, 3* a*'s and 2* b*'s, each with 2 members. For* A/J*, there are 6* R_0*-equivalence classes of vertices where the elements in* J *vanish, 3 with 2 members and 3 with 1 member, and so we find that* $A/J \cong \oplus_{i=1}^{3} M_2(\mathbb{C}) \bigoplus \oplus_{j=1}^{3} \mathbb{C}$*. In particular,* $K_0(A/J) \cong \mathbb{Z}^6$*.*

Now we compute G_{q*} *on the basis elements of* $K_0(A/I) \cong \mathbb{Z}^3$. *We'll describe the procedure for* $(0,1,0)$, *which involves choosing and element in* $s \in A$ *which maps to* p_2 *under the quotient map* q, *applying* G *to* s *and then examining its image under the quotient map from* A *to* A/J, *and determining its class in* $K_0(A/J)$.

Let U *be a small neighbourhood around* $(v_2, v_2) \in R_0$ *and let* $s \in A$ *be an element such that* $s > 0$ *on* U, $s(v_1, v_1) = 1$ *and* $s = 0$ *otherwise. Then,*

$$G(s(v_i, v_i), 1) = \begin{cases} 1 & : i = 3, 4, 5 \\ 0 & : i = 1, 2, 6 \end{cases}$$

and so its image in A/J *is given by a function which is* 1 *at* v_3, v_4, v_5 *and zero otherwise. This is a projection, and its class in* $K_0(A/J)$ *is* $(0,0,1,1,1,0)$. *The others are obtained similarly and we get:*

$$G_*(1,0,0) = (1,1,0,0,0,0)$$

$$G_*(0,1,0) = (0,0,1,1,1,0)$$

$$G_*(0,0,1) = (0,0,1,1,0,1)$$

Recall that $\phi : A \to B$ *is given by inflating an element in* A *into one in* B: *for* $\xi \in A$, $\phi(\xi(x,y)) = \xi(\lambda^{-1}x, \lambda^{-1}y)$. *Next note that* $\phi(J) \subset I_B$, *and so the induced map* $\phi : A/J \to B/I_B$ *is well defined. The effect of this map is to introduce more equivalences. Note that* $B/I_B \cong M_4(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_2(\mathbb{C})$ *since* $w_1 \sim w_4 \sim w_7 \sim w_9$, $w_2 \sim w_5 \sim w_8$ *and* $w_3 \sim w_6$ *and in particular,* $K_0(B/I_B) \cong \mathbb{Z}^3$. *Our goal is to calculate* $\phi_*(G_*(1,0,0))$, $\phi_*(G_*(0,1,0))$, *and* $\phi_*(G_*(0,0,1))$. *Again, we'll show the details for* $\phi_*(G_*(0,1,0))$. *We use the same function* $s \in A$ *as above. Recall that*

$$g(s(v_i, v_i), 1) = \begin{cases} 1 & : i = 3, 4, 5 \\ 0 & : i = 1, 2, 6 \end{cases}$$

Then, $\phi(g(s, 1))$ will be a function which is 1 on the diagonal of R_1 from (w_3, w_3) to (w_5, w_5) and zero on the other vertices. The image of $\phi(g(s, 1) \in B$ under the quotient map to B/I_B is a function given by which is 1 at each of the vertices (w_i, w_i), $i = 3, 4, 5$, and since these three vertices are respectively in each equivalence class of B/I_B, it's image in $K_0(B/I_B)$ is $(1, 1, 1)$. The others are obtained similarly and we get:

$$\phi_*(G_*(1, 0, 0)) = (1, 0, 1)$$

$$\phi_*(G_*(0, 1, 0)) = (1, 1, 1)$$

$$\phi_*(G_*(0, 0, 1)) = (1, 1, 1)$$

Lastly, consider the map $\alpha : A \to B$, which injects A as a subalgebra of B without inflation: recall that $R_0 \subset R_1$, and so an image of A is contained in B by restricting the elements of B to R_0. Also note that

$$K_0(B/I_B) \cong K_0(M_4(\mathbb{C})) \oplus K_0(M_3(\mathbb{C})) \oplus K_0(M_2(\mathbb{C}))$$

and

$$K_0(A/I) \cong K_0(\mathbb{C}) \oplus K_0(\mathbb{C}) \oplus K_0(\mathbb{C})$$

so that α_* is simply the identity map, and so α_*^{-1} is as well.

Thus, as a matrix with respect to the basis of $K_0(A/I)$ used in the preceding discussion, given by p_1, p_2, and p_3, we find that

$$\Gamma_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then, by commutativity of the diagram, we have that the matrix $\Gamma_1 : K_0(A) \to K_0(A)$ *satisfies*

$$q_*\Gamma_1 = \Gamma_2 q_* = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

with

$$q_* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

from which we conclude that

$$\Gamma_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

A similar procedure gives

$$\Gamma_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

.

The exactness of row 1 of the diagram implies that

$$i_* = \begin{pmatrix} 1 & 1 \end{pmatrix},$$

and then the commutativity of the diagram gives $\Gamma_4 = 1$.

Thus, with $K_0(A) \cong \mathbb{Z}^2$, $K_1(A) \cong \mathbb{Z}$, $\Gamma_4 - id = 0$ *and* $\Gamma_1 - id$ *an isomorphism,* *Pimsner's 6-term exact sequence gives Figure 5.11:*

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r, "\Gamma_1 - id"] & \mathbb{Z}^2 \arrow[r, "0"] & K_0(\mathcal{O}_E) \arrow[d, "\cong"] \\
K_1(\mathcal{O}_E) \arrow[u, "0"] & \mathbb{Z} \arrow[l, "\cong"'] & \mathbb{Z} \arrow[l, "0"']
\end{tikzcd}$$

Figure 5.11: The Six Term Exact Sequence of the Substitution $a \to aab$, $b \to ab$

so that $K_0(\mathcal{O}_E) \cong \mathbb{Z}$ *and* $K_1(\mathcal{O}_E) \cong \mathbb{Z}$.

Example 5.1.9. *The next example to investigate is the substitution* $a \to a^n$, *where* a^n *denotes* n *consecutive* a*'s. It suffices to take* $P_0 = aa$, *since* a *is the only letter that appears, so the only vertex equivalence class is formed by the intersection of two* a *edges. As always, we use the notation* $A = C^*(R_0)$, *and* I_0 *is the ideal in* A *corresponding to elements in* A *which vanish on the vertices of* A. *In this case,* $K_0(I_0) \cong 0$, $K_1(I_0) \cong \mathbb{Z}$, $K_0(A/I_0) \cong \mathbb{Z}$ *and* $K_1(A/I_0) \cong 0$. *Thus, we have the exact sequence*

$$0 \longrightarrow K_0(A) \xrightarrow{q_*} \mathbb{Z} \xrightarrow{\delta_0} \mathbb{Z} \xrightarrow{i_*} K_1(A) \longrightarrow 0$$

We first compute the exponential map δ_0. *By Lemma 5.1.2,* $A/I_0 \cong \mathbb{C}$, *since there is only* 1 *vertex which is formed by the intersection of the two* a *edges. Let* $[p] \in K_0(A/I_0)$, *where* $p \in A/I_0$ *is the projection which is* 1 *on the vertex (on the diagonal) of* R_0 *and zero otherwise. Since all non-zero projections in* A/I_0 *are equivalent in* $K_0(A/I_0)$, $[p]$ *is a generator for* $K_0(A/I_0)$ *and its image will completely define the exponential map. As in the previous example, when the vertex is formed by the intersection of two of the same letters, it is mapped to zero by the exponential map. Thus,* $\delta_0 = 0$ *in this case and as a result,* $K_0(A) \cong \mathbb{Z}$ *and* $K_1(A) \cong \mathbb{Z}$. *Moreover,* q_*

and i_* *are isomorphisms, and so by choosing appropriate bases, we may assume each is the identity map.*

We next compute G_{q*}*,* ϕ_{q*} *and* α_{q*}^{-1}*, so that we can use the commutativity of Diagram 5.4 to determine* Γ_1*. Note that* $K_0(A/J) \cong M_2(\mathbb{C}) \oplus \cdots \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$ *where there are* $n-1$ *occurrences of* $M_2(\mathbb{C})$ *in the direct sum. This is due to the fact that in* X_0*, there are* n *equivalence classes of points where functions in* J *vanish,* $n-1$ *of which occur in the interior of an a edge, so that each class contains two elements (one on each edge). The last class consists of only the vertex (where the two a edges meet).*

Let p *be the projection in* A/I_0 *given by the function which is 1 on the diagonal vertex of* R_0 *and zero otherwise. Since the class of this projection is a generator for* $K_0(A/I)$*, we can use it to define* G_{q*} *by examining its image. It is straight forward to verify that*

$$G_{q*}([p]) = [p_1 \oplus p_1 \oplus \cdots \oplus p_1 \oplus 1] \tag{5.1.3}$$

where $p_1 \in M_2(\mathbb{C})$ *is the projection with a 1 in the top left entry and zeros elsewhere, where we have used the identification that* $A/J \cong M_2(\mathbb{C}) \oplus \cdots \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$*.*

Applying ϕ_{q*} *has the effect of making the support points of each projection in the direct sum of equation 5.1.3 equivalent. Thus, the image of this projection under* ϕ_{q*} *is*

$$\left[\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}\right]_0$$

where I_n *is an* $n \times n$ *identity matrix, and the zero's correspond to appropriately sized zero matrices. Using the identification* $K_0(B/I_1) \cong \mathbb{Z}$*, we can see the* K *class of this element is* $n \in \mathbb{Z}$*.*

Lastly, again we note that α_{q*} *is the identity map on the K-groups, and so* α_*^{-1} *is as well. Thus, putting the maps together, we find that* $\Gamma_2 : \mathbb{Z} \to \mathbb{Z}$ *is defined by* $\Gamma_2(1) = n$*. Then, since* q_* *is the identity, we find that* $\Gamma_1 : \mathbb{Z} \to \mathbb{Z}$ *is defined by* $\Gamma_1(1) = n$ *as well.*

It is straight forward to verify that Γ_3 *is the identity map, and since* i_* *is the identity map, we can conclude that* Γ_4 *is as well by the commutativity of the diagram. Inputting these identifications into Pimsner's 6-term exact sequence gives*

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "(n-1)"] & \mathbb{Z} \arrow[r, "\pi_1"] & K_0(\mathcal{O}_E) \arrow[d, "\pi_2"] \\
K_1(\mathcal{O}_E) \arrow[u, "0"] & \mathbb{Z} \arrow[l, "\cong"'] & \mathbb{Z} \arrow[l, "0"']
\end{tikzcd}$$

Figure 5.12: The Six Term Exact Sequence of the Substitution $a \to a^n$

where we have used the fact that $(n-1)$ *(viewed as a* 1×1 *matrix, i.e. multiplication by* $n-1$*) is injective, so that the preceding map must be 0, which then allows us to conclude that the map from* $\mathbb{Z}$ *to* $K_1(\mathcal{O}_E)$ *must be an isomorphism. Finally, we determine* π_1 *and* π_2*. The range of* $(n-1)$ *is* $(n-1)\mathbb{Z}$*, and so this must be the kernel* π_1*. Thus, we have a copy of* $\mathbb{Z}_{n-1}$ *in* $K_0(\mathcal{O}_E)$*, and also a copy of* $\mathbb{Z}$*, since* π_2 *must be onto. Thus, we conclude that* $K_1(\mathcal{O}_E) \cong \mathbb{Z}$ *and* $K_0(\mathcal{O}_E) \cong \mathbb{Z} \oplus \mathbb{Z}_{n-1}$*.*

5.2 Forcing the Border

Forcing the border is an extra condition which we can impose on a substitution tiling, which, with a mild additional condition, will give us a simple formula for computing the K-groups of our Cuntz-Pimsner algebra in the case where our substitution tiling is on $\mathbb{R}$. Although our results will be for tilings on $\mathbb{R}$, we give the general definition of forcing the border for a substitution on $\mathbb{R}^n$.

Definition 5.2.1. *Let P_0 be a partial tiling of $\mathbb{R}^n$ made up of a subset of the prototiles $p_1, \cdots, p_k$. Let ω be a substitution as before. Assume that all subsequent substitutions of P_0 consist only of translations of the prototiles $p_1, \cdots, p_k$. A substitution tiling is said to* **force the border** *if there exists a positive integer m such that the neighbouring tiles of the supertile $\omega^m(p_i + x)$ are the same for any translate of p_i by x in P_0 (since P_0 may contain many translated copies of p_i for fixed i).*

Example 5.2.2. *Suppose in a 1-dimensional substitution tiling on the letters $\{a_i\}_{i=1}^n$ that there exists a positive integer m such that for all $k \geq m$ and $1 \leq i \leq n$, $\omega^k(a_i)$ starts with a_j and ends with a_l. This substitution will force the border since for any i, a_i will have some letters a_s and a_t on its left and right respectively, so that $\omega^k(a_i)$ will have $\omega^k(a_s)$, which must end with a_l, on its left and $\omega^k(a_t)$, which must start with a_j on its right.*

Returning to the setting in $\mathbb{R}$, let W be a finite set whose elements we denote by letters. Let W^* be the set of all words on W, so $W^* = \bigcup_{n=1}^{\infty} W^n$ where W^n is the set of all words consisting of n letters of W. The substitution $\omega : W \to W^*$ extends to a map which we will also denote $\omega : W^* \to W^*$, where $\omega(a_1 a_2 \cdots a_n) = \omega(a_1)\omega(a_2) \cdots \omega(a_n)$. For an element in $\mathbb{Z}W$ which is all finite $\mathbb{Z}$-linear combinations of words in W^*, just extend ω linearly, so that now we view omega as a map $\omega : \mathbb{Z}W^* \to \mathbb{Z}W^*$. Let $\tilde{W}^2 = \{a_i a_{i+1} : \omega^k(a) = \cdots a_i a_{i+1} \cdots$, for some $a \in W$ and $k \geq 1\}$. In other words, $\tilde{W}^2$ consists of all pairs of letters of W that eventually appear after substituting some letter of W enough times. Define $\lambda : \mathbb{Z}(W^* \setminus W) \to \mathbb{Z}W^*$ by $\lambda(a_1 \cdots a_n) = a_1 \cdots a_{n-1}$ and extend linearly. Next define $\omega_2 : \mathbb{Z}W^2 \to \mathbb{Z}W^*$ by $\omega_2(ab) = \omega(a)b_1$ where $\omega(b) = b_1 b_2 \cdots b_n$. Again, for a general element in $\mathbb{Z}W^2$, just extend ω_2 linearly. We begin with a simple lemma which will be useful later.

Lemma 5.2.3. *The following diagram commutes:*

$$\begin{tikzcd}
\mathbb{Z}W^2 \arrow[r, "\lambda"] \arrow[d, "\omega_2"] & \mathbb{Z}W \arrow[d, "\omega"'] \\
\mathbb{Z}W^* \arrow[r, "\lambda"] & \mathbb{Z}W^*
\end{tikzcd}$$

Figure 5.13: Commutative Diagram of ω and ω_2

Proof. Since all maps are linear, it suffices to prove it for an element of the form $ab \in \mathbb{Z}W*$. Let $\omega(b) = b_1 \cdots b_m$. Then, we have

$$\begin{aligned}
\omega(\lambda(ab)) &= \omega(a) \\
&= \lambda(\omega(a)b_1) \\
&= \lambda(\omega_2(ab))
\end{aligned}$$

□

Lemma 5.2.4. *Define $\beta : W^2 \to W^2$ by $\beta(ab) = a_n b_1$ where $\omega(a) = a_1 \cdots a_n$ and $\omega(b) = b_1 \cdots b_m$. There exists $k \geq 1$ such that $\beta^{k+1}(W^2) = \beta^k(W^2)$. In other words, β is a permutation on $\beta^k(W^2)$ so that there exists $l \geq 1$ such that $\beta^{k+l}(ab) = \beta^k(ab)$ for all $ab \in W^2$.*

Proof. First note that W^2 is finite and $\beta^k(W^2)$ must be nonempty for all $k \geq 0$. Thus, it will suffice to show that $\beta^{k+1}(W^2) \subset \beta^k(W^2)$, since then we have a sequence consisting of finite, nonempty, decreasing sets so that they eventually must all become equal. Once $\beta^{k+1}(W^2) = \beta^k(W^2)$, β must be acting as a permutation on $\beta^k(W^2)$.

Thus, we prove that $\beta^{k+1}(W^2) \subset \beta^k(W^2)$. It is clear that $\beta(W^2) \subset W^2$, since W^2 contains all possible pairs. Then $\beta^{k+1}(W^2) = \beta^k(\beta(W^2)) \subset \beta^k(W^2)$. □

Lemma 5.2.5. *Suppose ω forces its border, so that for some positive integer, k, the neighbouring tiles of the super tile $\omega^k(p_i)$ are the same for any $p_i \in P_0$ and choose k large enough to also satisfy Lemma 5.2.5. If $ab_1, ab_2 \in \beta^k(W^2)$, then $b_1 = b_2$.*

Proof. Since ω forces its border after k substitutions, $\omega^{k+j}(a)$ will always have the same letters on either side of it, regardless of which letters surrounded a at the start for any $j \geq 0$. Thus, $\beta^{k+j}(ab) = \beta^{k+j}(ac)$ for all $ab, ac \in W^2$ and positive integers j. By the previous lemma, we also know that there exists a positive integer l such that $\beta^{k+nl}(ab) = \beta^k(ab)$, for all $ab \in W^2$ and any positive integer n. Thus, $ab_1 = \beta^{kl}(ab_1) = \beta^{kl}(ab_2) = ab_2$, where the middle equality follows since $kl \geq k$ and the other equalities follow since kl is a multiple of l, and $ab_1, ab_2 \in \beta^k(W^2)$, so β^{kl} acts as the identity. Lastly, since $ab = cd$ only holds if $a = c$ and $b = d$, so $ab_1 = ab_2$ gives $b_1 = b_2$. □

Recall $\omega_2 : \mathbb{Z}W^2 \to \mathbb{Z}W^*$ which acts by

$$\omega_2(ab) = a_1 \cdots a_n b_1, \quad \text{where } \omega(a) = a_1 \cdots a_n,\ \omega(b) = b_1 \cdots b_m.$$

Let H and G be the subgroups of $\mathbb{Z}W^*$ given by $H = span_{\mathbb{Z}}\{\omega_2^k(ab) \ :\ ab \in W^2,\ k \in \mathbb{N}\}$ and $G = span_{\mathbb{Z}}\{\omega^k(a) \ :\ a \in W,\ k \in \mathbb{N}\}$.

Define $\sigma_1 : \mathbb{Z}W^* \to \mathbb{Z}W$ by $\sigma_1(a_1 \cdots a_n) = a_1 + \cdots + a_n$ and $\sigma_2 : \mathbb{Z}(W^* \backslash W) \to \mathbb{Z}W^2$ by $\sigma_2(a_1 \cdots a_n) = a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n$ and extend both linearly.

Lemma 5.2.6. *If ω forces its border, then $\tilde{\lambda} : H \to G$ is bijective, where $\tilde{\lambda}$ is defined on $a_1 \cdots a_n \in H$ by $\tilde{\lambda}(a_1 \cdots a_n) = a_1 \cdots a_{n-1}$, and extended linearly for general elements.*

Proof. Suppose $\tilde{\lambda}(h_1) = \tilde{\lambda}(h_2)$ for some $h_1, h_2 \in H$. Then,

$$h_1 = \sum_i n_i \omega^k(a_i) b_i, \qquad h_2 = \sum_i m_i \omega^k(c_i) d_i$$

where n_i, m_i are integers. Then, $\tilde{\lambda}(h_1) = \sum_i n_i \omega^k(a_i) = \sum_i m_i \omega^k(c_i) = \tilde{\lambda}(h_2)$. For these two sums to be equal, the elements on either side must pair off (where without adjusting notation, we assume that each sum has been simplified to consist of as few terms as possible). Without loss of generality, assume that $n_i \omega^k(a_i) = m_i \omega^k(c_i)$ by relabelling if necessary. We must have $n_i = m_i$, but also, since ω forces the border at level k, $b_i = d_i$ for all i. Thus, $h_1 = h_2$, and so $\tilde{\lambda}$ is injective.

For surjectivity, fix $g \in G$, so that $g = \sum_i \omega^k(a_i)$. Then, $\sum_i \omega_2^k(a_i b_i) = \sum_i \omega^k(a_i) b_{i1} \in H$ and

$$\tilde{\lambda}(\sum_i \omega_2^k(a_i b_i)) = \sum_i \tilde{\lambda}(\omega^k(a_i) b_{i1}) = \sum_i \omega^k(a_i)$$

□

Lemma 5.2.7. *Denote ω by ω_1 for this lemma. The mappings σ_i and ω_i satisfy $\sigma_i \omega_i^k = \sigma_i^k \omega_i^k = (\sigma_i \omega_i)^k$, $i = 1, 2$.*

Proof. It is clear from the definitions that σ_i is an idempotent for $i = 1, 2$, so $\sigma_i^k = \sigma_i$. Notice that if $\sigma_i \omega_i^2 = (\sigma_i \omega_i)^2$, then

$$\begin{aligned}
(\sigma_i \omega_i)^k &= (\sigma_i \omega_i)^{k-2} \sigma_i \omega_i \sigma_i \omega_i \\
&= (\sigma_i \omega_i)^{k-2} \sigma_i \omega_i^2 \\
&= (\sigma_i \omega_i)^{k-3} (\sigma_i \omega_i \sigma_i \omega_i) \omega_i \\
&= (\sigma_i \omega_i)^{k-3} (\sigma_i \omega_i^2) \omega_i = (\sigma_i \omega_i)^{k-3} \sigma_i \omega_i^3 \\
&= \cdots = \sigma_i \omega_i^k
\end{aligned}$$

Thus, it suffices to prove that $\sigma_i\omega_i^2 = (\sigma_i\omega_i)^2$ in each case, $i = 1, 2$.

Case: i=1 We will show that $\sigma_1\omega_1^2(a) = (\sigma_1\omega_1)^2(a)$ for $a \in W$, and the result extends $\mathbb{Z}$-linearly to elements in $\mathbb{Z}W$.

$$\begin{aligned}
\sigma_1\omega_1^2(a) &= \sigma_1\omega_1(a_1 \cdots a_n) \\
&= \sigma_1(a_1^{(1)}a_1^{(2)} \cdots a_1^{(n_1)}a_2^{(1)} \cdots a_2^{(n_2)} \cdots a_n^{(1)} \cdots a_n^{(n_n)}) \\
&= a_1^{(1)} + a_1^{(2)} + \cdots + a_1^{(n_1)} + a_2^{(1)} + \cdots + a_2^{(n_2)} + \cdots + a_n^{(1)} + \cdots + a_n^{(n_n)} \\
&= \sigma_1(a_1^{(1)}a_1^{(2)} \cdots a_1^{(n_1)} + a_2^{(1)} \cdots a_2^{(n_2)} + \cdots + a_n^{(1)} \cdots a_n^{(n_n)}) \\
&= \sigma_1\omega_1(a_1 + a_2 + \cdots + a_n) \\
&= \sigma_1\omega_1\sigma_1(a_1a_2 \cdots a_n) \\
&= \sigma_1\omega_1\sigma_1\omega_1(a) \\
&= (\sigma_1\omega_1)^2(a)
\end{aligned}$$

Case: i=2 Again we'll show that $\sigma_2\omega_2^2(ab) = (\sigma_2\omega_2)^2(ab)$, where ab is a single word in W^2 and the result again extends $\mathbb{Z}$-linearly to elements in $\mathbb{Z}W^2$. Let $\omega_2(ab) = a_1a_2 \cdots a_nb_1$ where $\omega(a) = a_1 \cdots a_n$ and $\omega(b) = b_1 \cdots b_m$. Let $\omega_2(a_j) = a_j^{(1)}a_j^{(2)} \cdots a_j^{(n_j)}$ for $j = 1, \cdots, n$ and $\omega(b_1) = b_1^{(1)}b_1^{(2)} \cdots b_1^{(m_2)}$. Then,

$$\begin{aligned}
\sigma_2\omega_2^2(ab) &= \sigma_2\omega_2(a_1a_2 \cdots a_nb_1) \\
&= \sigma_2(a_1^{(1)}a_1^{(2)} \cdots a_1^{(n_1-1)}a_1^{(n_1)}a_2^{(1)}a_2^{(2)} \cdots a_2^{(n_2-1)}a_2^{(n_2)} \cdots a_n^{(n_n)}b_1^{(1)}) \\
&= a_1^{(1)}a_1^{(2)} + \cdots + a_n^{(n_n-1)}a_n^{(n_n)} + a_n^{(n_n)}b_1^{(1)}
\end{aligned}$$

and

$$\begin{aligned}(\sigma_2\omega_2)^2(ab) &= \sigma_2\omega_2(a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_nb_1)\\ &= \sigma_2(a_1^{(1)}a_1^{(2)}\cdots a_1^{(n_1-1)}a_1^{(n_1)}a_2^{(1)} + \cdots + a_n^{(1)}a_n^{(2)}\cdots a_n^{(n_n)}b_1^{(1)})\\ &= a_1^{(1)}a_1^{(2)} + \cdots + a_n^{(n_n-1)}a_n^{(n_n)} + a_n^{(n_n)}b_1^{(1)}\end{aligned}$$

Thus, $(\sigma_2\omega_2)^2 = \sigma_2^2\omega_2^2 = \sigma_2\omega_2^2$.

□

Lemma 5.2.8. *Let* $\alpha_2^k = (\sigma_2\omega_2)^k = \sigma_2\omega_2^k$. *The following diagram commutes:*

$$\begin{tikzcd}
\mathbb{Z}W^2 \arrow[r, "\lambda"] \arrow[d, "\omega_2^k"] \arrow[dd, bend right, "\alpha_2^k"'] & \mathbb{Z}W \arrow[d, "\omega^k"'] \arrow[dd, bend left, "\alpha_1^k"] \\
H \arrow[r, "\tilde{\lambda}"] \arrow[d, "\sigma_2"] & G \arrow[d, "\sigma_1"'] \\
\mathbb{Z}W^2 \arrow[r, "\lambda"] & \mathbb{Z}W
\end{tikzcd}$$

Figure 5.14: Commutative Diagram of α_1^k and α_2^k

Proof. All the maps are linear, so it will suffice to prove it for generators. Fix $ab \in W^2$ and let b_1 be the first letter of $\omega^k(b)$. Then

$$\omega^k\lambda(ab) = \omega^k(a) = \lambda(\omega^k(a)b_1) = \lambda(\omega_2^k(ab))$$

Fix $\omega^k(a)b_1 = \omega_2^k(ab) \in H$. Then

$$\begin{aligned}\sigma_1\tilde{\lambda}\omega_2^k(ab) &= \sigma_1\tilde{\lambda}(\omega^k(a)b_1) = \sigma_1\omega^k(a) = \sigma_1(a_1a_2\cdots a_n) = a_1 + a_2 + \cdots + a_n\\ &= \lambda(a_1a_2 + a_2a_3 + \cdots + a_nb_1) = \lambda\sigma_2(a_1a_2\cdots a_nb_1)\\ &= \lambda\sigma_2\omega_2^k(ab)\end{aligned}$$

□

Our goal is to show that λ restricts to an isomorphism between $ker(1-\alpha_2)$ and $ker(1-\alpha_1)$ and also restricts to an isomorphism between $coker(1-\alpha_2)$ and $coker(1-\alpha_1)$. We present this in a sequence of lemmas.

Lemma 5.2.9. *The map $\lambda : ker(1-\alpha_2) \to ker(1-\alpha_1)$ is injective.*

Proof. Suppose $a \in ker(1-\alpha_2)$ and $\lambda(a) = 0$. Then $\tilde{\lambda}\omega_2^k(a) = \omega^k\lambda(a) = 0$. But then $\omega_2(a) = 0$ since $\tilde{\lambda}$ is an isomorphism. Then $\sigma_2\omega_2^k(a) = \sigma_2^k\omega_2^k(a) = \alpha_2^k(a) = 0$, but $\alpha_2(a) = a$ so $a = \alpha_2^k(a) = 0$.

□

Next we will prove surjectivity of this restriction of λ, which itself will be broken down into several lemmas. We first present a general result about groups which will be used repeatedly.

Lemma 5.2.10. *Let C be a group, $\beta : C \to C$ an endomorphism, and $i : \beta^k(C) \to C$ be the imbedding of $\beta^k(C)$ into C. The diagram of Figure 5.15 commutes and $(co)ker(1-\beta)|_{\beta^k(C)} = (co)ker(1-\beta)$.*

Proof. Commutativity is immediate. We first check that the kernels are equal. We have that $ker(1-\beta)|_{\beta^k(C)} \subset ker(1-\beta)$, since restricting a map to a subset of its domain can only reduce the kernel. Let $c \in ker(1-\beta)$. Then $\beta(c) = c$, so

$$\begin{tikzcd}
\beta^k(C) \arrow[r, "i"] \arrow[d, "1-\beta"'] & C \arrow[d, "1-\beta"] \\
\beta^k(C) \arrow[r, "i"] & C
\end{tikzcd}$$

Figure 5.15: Commutative Diagram of $1-\beta$ and i

$\beta^n(c) = c$ for all $n \geq 1$. Thus, $c = \beta^k(c) \in \beta^k(C)$ and $(1-\beta)c = 0$. Thus, $ker(1-\beta) \subset ker(1-\beta)|_{\beta^k(C)}$.

Next we check that the cokernels are equal. The isomorphism is given explicitly by the induced quotient map of $\beta^k : C \to \beta^k(C)$, denoted $\beta^k_q : C/(1-\beta)C \to \beta^k(C)/(1-\beta)\beta^k(C)$. We first note that this map is well defined since for $c+(1-\beta)C = 0+(1-\beta)C$ means that $c \in (1-\beta)C$, and so $\beta^k(c) \in \beta^k(1-\beta)C = (1-\beta)\beta^k(C)$. Thus, $\beta^k_q(c+(1-\beta)C) = \beta^k(c)+(1-\beta)\beta^k(C) = 0+(1-\beta)\beta^k(C)$. To verify that β^k_q is surjective, fix $y \in \beta^k(C)/(1-\beta)\beta^k(C)$, so $y = y_0+(1-\beta)\beta^k(C)$ for some $y_0 \in \beta^k(C)$. Then, $y_0 = \beta^k(x_0)$ for some $x_0 \in C$ and so $\beta^k_q(x_0+(1-\beta)C) = \beta^k(x_0)+(1-\beta)\beta^k(C) = y_0+(1-\beta)\beta^k(C)$.

Lastly, we show that β^k_q is injective. First notice that $c+(1-\beta)C = \beta(c)+(1-\beta)C = \beta^n(c)+(1-\beta)C$ for all $c \in C$ and all $n \geq 1$. Fix $x \in C/(1-\beta)C$, so $x = c+(1-\beta)C$ for some $c \in C$, and suppose that $\beta^k_q(x) = 0$, so that $\beta^k(c) \in (1-\beta)\beta^k(C)$. Since $(1-\beta)\beta^k(C) \subset (1-\beta)C$, $\beta^k(c) \in (1-\beta)C$. Thus, $c+(1-\beta)C = \beta^k(c)+(1-\beta)C = 0+(1-\beta)C$ and so β^k_q is injective. □

Lemma 5.2.11. *The diagram Figure 5.16 commutes and* $(co)ker(1-\alpha_i)|_{\alpha_i^k(\mathbb{Z}W^i)} = (co)ker(1-\alpha_i)$. *Thus when proving our results, it suffices to work with this diagram.*

Proof. The commutivity follows immediately from 5.2.8, and the other result follows from Lemma 5.2.10.

$$\begin{tikzcd}
\alpha_2^k(\mathbb{Z}W^2) \arrow[r, "\lambda"] \arrow[d, "\omega_2"] \arrow[dd, bend right=50, "\alpha_2"'] & \alpha_1^k(\mathbb{Z}W) \arrow[d, "\omega"'] \arrow[dd, bend left=50, "\alpha_1"] \\
H \arrow[r, "\tilde{\lambda}"] \arrow[d, "\sigma_2"] & G \arrow[d, "\sigma_1"'] \\
\alpha_2^k(\mathbb{Z}W^2) \arrow[r, "\lambda"] & \alpha_1^k(\mathbb{Z}W)
\end{tikzcd}$$

Figure 5.16: Commutative Diagram of α_1 and α_2

□

Lemma 5.2.12. *The map* $\lambda : ker(1-\alpha_2) \to ker(1-\alpha_1)$ *is surjective.*

Proof. Let $b \in ker(1-\alpha_1)$. Since $\omega^k(b) \in G$, and $\tilde{\lambda}$ is an isomorphism between H and G, there exists $h \in H$ such that $\tilde{\lambda}(h) = \omega^k(b)$. ω_2^k is surjective by definition, so there exists $a \in \mathbb{Z}W^2$ such that $\tilde{\lambda}\omega_2^k(a) = \omega^k(b)$, and then by commutivity, $\omega^k\lambda(a) = \omega^k(b)$.

At this point we have $a \in \mathbb{Z}W^2$ with $\omega^k\lambda(a) = \omega^k(b)$. First, we need to adjust a so that it is in $ker(1-\alpha_2)$, but still has the same properties that we've shown a to have. Note that $\lambda\alpha_2^k(a) = \alpha_1^k\lambda(a) = \alpha_1^k(b) = b$, and by Lemma 5.2.11, $\lambda\alpha_2^{k+n}(a) = \alpha_1^{k+n}\lambda(a) = b$ for all $n \geq 1$.

Then,

$$\tilde{\lambda}\omega_2\alpha_2^k(a) = \omega\lambda\alpha_2^k(a) = \omega\alpha_1\lambda\alpha_2^k(a) = \omega\lambda\alpha_2^{k+1}(a) = \tilde{\lambda}\omega_2\alpha_2^{k+1}(a)$$

Thus, $\omega_2\alpha_2^k(a) = \omega_2\alpha_2^{k+1}(a)$ which gives $\alpha_2^{k+1}(a) = \alpha_2^{k+2}(a)$ and so $\alpha_2^{k+1}(a) \in ker(1-\alpha_2)$ and $\lambda\alpha_2^{k+1}(a) = b$.

Note that by the commutivity of the diagram elements in $ker(1-\alpha_2)$ are mapped into elements in $ker(1-\alpha_1)$. In particular, $\lambda\alpha_2^{k+1}(a) \in ker(1-\alpha_1)$.

We have that $\omega^k\lambda\alpha_2^{k+1}(a) = \omega^k(b)$ and so if ω^k is injective when restricted to $ker(1-\alpha_1)$, then $\lambda\alpha_2^{k+1}(a) = b$. Suppose $\omega^k(b) = 0$ for $b \in ker(1-\alpha_1)$. Then $0 = \sigma_1^k\omega^k(b) = \alpha_1^k(b) = b$. Thus, ω^k is injective on $ker(1-\alpha_1)$ and so $\lambda\alpha_2^{k+1}(a) = b$.

$\square$

Thus, we have established that λ is an isomorphism between $ker(1-\alpha_2)$ and $ker(1-\alpha_1)$.

Next, we prove that λ is an isomorphism between $coker(1-\alpha_2)$ and $coker(1-\alpha_1)$. We begin with a general lemma about maps between quotient groups.

Lemma 5.2.13. *Let S_1 and S_2 be groups with normal subgroups T_1 and T_2 respectively. Suppose $\pi : S_1 \to S_2$, and that $\pi(T_1) \subset T_2$. Let π_q be the induced map between quotients: $\pi_q : S_1/T_1 \to S_2/T_2$. Then, π_q is well defined and*

i) if $\pi : S_1 \to S_2$ is surjective, then $\pi_q : S_1/T_1 \to S_2/T_2$ is surjective,

ii) if $\pi : S_1 \to S_2$ is injective, and $\pi(T_1) = T_2$, then $\pi_q : S_1/T_1 \to S_2/T_2$ is injective

Proof. First, we verify that π_q is well defined. Suppose $s \in S_1$ and $s + T_1 = 0 + T_1$, so that $s \in T_1$. Then, $\pi(s) \in T_2$ by assumption, so $\pi_q(s + T_1) = \pi(s) + T_2 = 0 + T_2$. Thus, π_q is well defined.

i): Fix $s_2 + T_2 \in S_2/T_2$. Since π is surjective, there exists $s_1 \in S_1$ such that $\pi(s_1) = s_2$. Then, $\pi_q(s_1 + T_1) = \pi(s_1) + T_2 = s_2 + T_2$.

ii): Fix $s_1 + T_1 \in ker(\pi_q)$. Then, $0 + T_2 = \pi_q(s_1 + T_1) = \pi(s_1) + T_2$, so $\pi(s_1) \in T_2$. Since $\pi(T_1) = T_2$, there exists $t_1 \in T_1$ such that $\pi(t_1) = \pi(s_1)$, but π is injective so $t_1 = s_1$. Thus, $s_1 + T_1 = 0 + T_1$.

$\square$

Lemma 5.2.14. *Let N_i denote the (normal) subgroup $(1-\alpha_i)\mathbb{Z}W^i$ of $\mathbb{Z}W^i$ for $i = 1, 2$. The following diagram commutes and the induced maps between the quotients are all well-defined. Also, $\tilde{\lambda}_q : H/\omega_2^k(N_2) \to G/\omega^k(N_1)$ is an isomorphism.*

$$\begin{tikzcd}
\mathbb{Z}W^2/N_2 \arrow[r, "\lambda_q"] \arrow[d, "\omega^k_{q2}"] \arrow[dd, bend right, "\alpha^k_{q2}"'] & \mathbb{Z}W/N_1 \arrow[d, "\omega^k_{q1}"'] \arrow[dd, bend left, "\alpha^k_{q1}"] \\
H/\omega_2^k(N_2) \arrow[r, "\tilde{\lambda}_q"] \arrow[d, "\sigma_{q2}"] & G/\omega^k(N_1) \arrow[d, "\sigma_{q1}"'] \\
\mathbb{Z}W^2/N_2 \arrow[r, "\lambda_q"] & \mathbb{Z}W/N_1
\end{tikzcd}$$

Figure 5.17: Commutative Diagram of α^k_{q1} and α^k_{q2}

Proof. We immediately see that the maps ω^k_{q2} and ω^k_q are well-defined since in each case the equivalence class of 0 is mapped into the equivalence class of 0. The same is true for σ_{q2} and σ_{q1} since

$$\sigma_2\omega_2^k(N_2) = \alpha_2^k(N_2) = \alpha_2^k(1-\alpha_2)\mathbb{Z}W^2 = (1-\alpha_2)\alpha_2^k\mathbb{Z}W^2 \subset N_2$$

and

$$\sigma_1\omega^k(N_1) = \alpha_1^k(N_1) = \alpha_1^k(1-\alpha_1)\mathbb{Z}W = (1-\alpha_1)\alpha_1^k\mathbb{Z}W \subset N_1.$$

To check that $\tilde{\lambda}_q : \mathbb{Z}W^2/N_2 \to \mathbb{Z}W/N_1$ is well defined, we verify that $\tilde{\lambda}\omega_2^k(N_2) \subset$

$\omega_k(N_1)$. Using Lemma 5.2.8 and Lemma 5.2.11 to replace $\mathbb{Z}W^2$ and $\mathbb{Z}W$ respectively with $\alpha_2^k(\mathbb{Z}W^2)$ and $\alpha_1^k(\mathbb{Z}W)$, we have that

$$\tilde{\lambda}\omega_2^k(1-\alpha_2)\alpha_2^k\mathbb{Z}W^2 = \omega^k(1-\alpha_1)\alpha_1^k\lambda\mathbb{Z}W^2 \subset \omega^k(N_1).$$

Recall $\tilde{\lambda} : H \to G$ is an isomorphism. If we show that $\tilde{\lambda}$ as a map from $\omega_2^k(N_2)$ to $\omega^k(N_1)$ is an isomorphism, then Lemma 5.2.13 implies that $\tilde{\lambda_q}$ is also an isomorphism. Since $\tilde{\lambda} : H \to G$ is an isomorphism, it will suffice to show that $\tilde{\lambda}$ maps $\omega_2^k(N_2)$ onto $\omega^k(N_1)$, since the restriction of an injective map is automatically injective. Again using Lemma 5.2.11, we can replace $\mathbb{Z}W^2$ and $\mathbb{Z}W$ respectively with $\alpha_2^k(\mathbb{Z}W^2)$ and $\alpha_1^k(\mathbb{Z}W)$. First we verify that $\tilde{\lambda}$ maps $\omega_2^k(N_2)$ into $\omega^k(N_1)$. Fix $a \in \mathbb{Z}W^2$. Then,

$$\tilde{\lambda}\omega_2^k(1-\alpha_2)\alpha_2^k(a) = \omega^k\lambda(1-\alpha_2)\alpha_2^k(a) = \omega^k(1-\alpha_1)\alpha_1^k\lambda(a) \in \omega^k(N_1)$$

Next we check that $\tilde{\lambda}$ maps $\omega_2^k(N_2)$ onto $\omega^k(N_1)$. Fix $b \in \omega^k(N_1)$, so $b = \omega^k(1-\alpha_1)\alpha_1^k(b_1)$, for some $b_1 \in \mathbb{Z}W$. Since $\lambda : \alpha_2^k(\mathbb{Z}W^2) \to \alpha_1^k(\mathbb{Z}W)$ is onto, there exists $a \in \alpha_2^k(\mathbb{Z}W^2)$ such that $\lambda(a) = b_1$. Then,

$$b = \omega^k(1-\alpha_1)\alpha_1^k(\lambda(a)) = \omega^k(\lambda(1-\alpha_2)\alpha_2^k(a) = \tilde{\lambda}\omega_2^k(1-\alpha_2)\alpha_2^k(a)$$

and so $\tilde{\lambda}_q$ is an isomorphism as claimed.

□

Lastly, we show that λ_q is in fact an isomorphism, so that $coker(1-\alpha_2) \cong coker(1-\alpha_1)$.

Lemma 5.2.15. *The map $\lambda_q : coker(1-\alpha_2) \to coker(1-\alpha_1)$ is an isomorphism.*

Proof. We start by showing λ_q is injective. Let $[a] \in coker(1-\alpha_2)$, and suppose that $\lambda_q([a]) = 0$ or equivalently, that $\lambda(a) \in ran(1 - \alpha_2)$ for some $a \in \mathbb{Z}W^2$ in the equivalence class of $[a]$. Then, $\tilde{\lambda}_q \tilde{\omega}^k_{2q}([a]) = \omega^k_{q1}\lambda_q([a]) = 0$ which implies that $\tilde{\omega}_{q2}([a]) = 0$ since $\tilde{\lambda}_q$ is an isomorphism. Then,

$$\begin{aligned}\alpha^k_{q2}([a]) = 0 &\implies (1 - \alpha^k_{q2})([a]) = [a] \implies [a] \in ran(1 - \alpha^k_{q2}) \implies a \in ran(1 - \alpha^k_2)\\ &\implies [a] = 0\end{aligned}$$

and so λ_q is injective.

For surjectivity, start by fixing $[b] \in coker(1 - \alpha_1)$. Then, since $\tilde{\lambda}_q$ is an isomorphism, there exists an $[h] \in H/\omega^k_2(N_2)$ such that $\tilde{\lambda}_q([h]) = \omega^k_{q1}([b])$, and then since ω^k_{q2} is surjective, there exists $[a] \in \mathbb{Z}W^2/N_2$ such that $\tilde{\lambda}_q\omega^k_{q2}([a]) = \omega^k_{q1}([b]) = \omega^k_{q1}\lambda_q([a])$. Thus, if ω^k_{q1} is injective, then $\lambda_q([a]) = [b]$. Suppose $\omega^k_{q1}([x]) = 0$. Then,

$$\begin{aligned}\alpha^k_{q1}([x]) = 0 &\implies (1 - \alpha^k_{q1})([x]) = [x] \implies x \in ran(1 - \alpha^k_1)\\ &\implies [x] = 0.\end{aligned}$$

Thus, ω^k_{q1} is injective and so λ_q is surjective as claimed.

□

Recall from Section 5.1 that $K_0(\mathcal{O}_E) \cong coker(id - \Gamma_1) \oplus \mathbb{Z}$ and $K_1(\mathcal{O}_E) \cong ker(id - \Gamma_1) \oplus \mathbb{Z}$ and note that in the above, $\alpha_2 = \Gamma_2$, (Γ_2 is the second vertical map in diagram 5.4, a truncated version of which is provided below in Diagram 5.18) and so we need to show that $ker(1 - \Gamma_2) = ker(1 - \Gamma_1)$, in order to conclude that we can replace Γ_1 with α_1 (note that the matrix form of α_1 is the substitution matrix). Initially we had hoped that when the substitution forces the border, that $(co)ker(id - \Gamma_1) \cong (co)ker(id - \Gamma_2)$

so that the K-groups would be completely determined by the substitution matrix, but as will be shown in Example 5.2.19, it isn't quite enough. If, however, we add the condition that the substitution of each letter, starts with the same letter, then we do get that $K_0(\mathcal{O}_E) \cong coker(id - \alpha_1) \oplus \mathbb{Z}$ and $K_1(\mathcal{O}_E) \cong ker(id - \alpha_1) \oplus \mathbb{Z}$. This extra requirement may seem very strong, but in practice, most substitutions which force the border already satisfy this condition. We finish by proving that adding this extra condition is sufficient to determine the K-groups by the substitution matrix, and presenting a few examples which show that this extra condition, along with that of forcing the border are both necessary.

To prove that the addition of this new condition is sufficient, we first recall a truncated version of the commutative diagram of Figure 5.4:

$$\begin{tikzcd}
0 \arrow[r] & K_0(A) \arrow[r, "q_*"] \arrow[d, "\Gamma_1"] & K_0(A/I_0) \arrow[r, "\delta_0"] \arrow[d, "\Gamma_2"] & K_1(I_0) \arrow[r, "i_*"] \arrow[d, "\Gamma_3"] & K_1(A) \arrow[r] \arrow[d, "\Gamma_4"] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r, "q_*"] & K_0(A/I_0) \arrow[r, "\delta_0"] & K_1(I_0) \arrow[r, "i_*"] & K_1(A) \arrow[r] & 0
\end{tikzcd}$$

Figure 5.18: Truncated Commutative Diagram

Observe that we may replace Γ_i with $1 - \Gamma_i$ for each $i = 1, 2, 3, 4$ giving the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_0(A) \arrow[r, "q_*"] \arrow[d, "1 - \Gamma_1"] & K_0(A/I_0) \arrow[r, "\delta_0"] \arrow[d, "1 - \Gamma_2"] & K_1(I_0) \arrow[r, "i_*"] \arrow[d, "1 - \Gamma_3"] & K_1(A) \arrow[r] \arrow[d, "1 - \Gamma_4"] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r, "q_*"] & K_0(A/I_0) \arrow[r, "\delta_0"] & K_1(I_0) \arrow[r, "i_*"] & K_1(A) \arrow[r] & 0
\end{tikzcd}$$

Figure 5.19: Truncated Commutative Diagram with $1 - \Gamma_i$

The first thing to observe, is that with the added condition that the substitution of each letter starts with the same letter, the map Γ_3 is now computed as

$$\Gamma_3 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

which in particular means that $\Gamma_3\delta_0 = 0$. This result will be used repeatedly and so we record it as a lemma.

Lemma 5.2.16. *If a 1-dimensional substitution tiling has the property that it forces the border, and the substitution of each letter begins with the same letter, then the composition* $\Gamma_3\delta_0$ *is the zero mapping, as is the composition* $\delta_0\Gamma_2 = 0$.

Proof. The range of δ_0 is the $\mathbb{Z}$-span of vectors which have one entry equal to one, one entry equal to negative one and the rest zero, all of which are in the kernel of Γ_3. That $\delta_0\Gamma_2 = 0$ follows from the commutativity of Diagram 5.18. □

We are now ready to begin proving the main result of this section. We will first prove that with the added condition on the substitution that $ker(1-\Gamma_1) \cong ker(1-\Gamma_2)$, and second that $coker(1-\Gamma_1) \cong coker(1-\Gamma_2)$. Then, since we have already shown the equivalences $ker(1-\alpha_1) \cong ker(1-\Gamma_2)$ and $coker(1-\alpha_1) \cong coker(1-\Gamma_2)$, we will have the desired result.

Theorem 5.2.17. *Let* α_1 *be the matrix representation of a 1-dimensional substitution tiling* ω *on* n *letters,* $a_1, \cdots, a_n$ *that forces the border and with the property that for each letter, the substitution begins with* a_i*, for a fixed* i*. Then, the* K*-groups of the corresponding Cuntz-Pimsner algebra are given by*

$$K_0(\mathcal{O}_E) = coker(1-\alpha_1) \oplus \mathbb{Z} \quad \text{and} \quad K_1(\mathcal{O}_E) = ker(1-\alpha_1) \oplus \mathbb{Z}$$

Proof. We begin by proving the result for $K_1(\mathcal{O}_E)$ since the method is more straight forward, and gives insight into the strategy for $K_0(\mathcal{O}_E)$. Our method will be to show that $q_*(ker(1-\Gamma_1)) \cong ker(1-\Gamma_2)$, which is equivalent to showing $ker(1-\Gamma_1) \cong ker(1-\Gamma_2)$ since q_* is injective.

1) $q_*(ker(1-\Gamma_1)) \subset ker(1-\Gamma_2)$:

   Let $x \in ker(1-\Gamma_1)$. Then, we have that $0 = q_*(1-\Gamma_1)(x) = (1-\Gamma_2)q_*(x)$ by the commutativity of Diagram 5.19 and so $q_*(x) \in ker(1-\Gamma_2)$.

2) $ker(1-\Gamma_2) \subset q_*(ker(1-\Gamma_1))$:

   Let $x \in ker(1-\Gamma_2)$, so that $x = \Gamma_2(x)$. We have that $\delta_0\Gamma_2(x) = 0$ by Lemma 5.2.16. But then, $\Gamma_2(x) \in ker(\delta_0)$ which implies that $\Gamma_2(x) \in ran(q_*)$, by the exactness of the rows. Thus, there exists $y \in K_0(A)$ such that $q_*(y) = \Gamma_2(x) = x$.

Before we prove the analogous containments for the cokernels, we first want to consider the following diagram: where the subscript "q's" are indicating that these are the induced maps on the quotients. We first note that these maps are all well defined by showing the zero equivalence class is mapped into the zero equivalence class for a vertical and a horizontal map, and the rest are analogous: $q_*(1-\Gamma_1)(K_0(A)) = (1-\Gamma_2)q_*(K_0(A)) \subset (1-\Gamma_2)(K_0(A/I_0))$; and $\Gamma_1(1-\Gamma_1)K_0(A) = (1-\Gamma_1)\Gamma_1 K_0(A) \subset (1-\Gamma_1)K_0(A)$. Our strategy will be to show that q_{q*} is an isomorphism. To do so, we will show that q_{q*} is injective, that $ranq_{q*} = ker\delta_{q0}$, and lastly that q_{q*} is surjective.

$$\begin{tikzcd}
0 \arrow[r] & \frac{K_0(A)}{(1-\Gamma_1)K_0(A)} \arrow[r, "q_{q*}"] \arrow[d, "\Gamma_{q1}"] & \frac{K_0(A/I_0)}{(1-\Gamma_2)K_0(A/I_0)} \arrow[r, "\delta_{q0}"] \arrow[d, "\Gamma_{q2}"] & \frac{K_1(I_0)}{(1-\Gamma_3)K_1(I_0)} \arrow[r, "i_{q*}"] \arrow[d, "\Gamma_{q3}"] & \frac{K_1(A)}{(1-\Gamma_4)K_1(A)} \arrow[r] \arrow[d, "\Gamma_{q4}"] & 0 \\
0 \arrow[r] & \frac{K_0(A)}{(1-\Gamma_1)K_0(A)} \arrow[r, "q_{q*}"] & \frac{K_0(A/I_0)}{(1-\Gamma_2)K_0(A/I_0)} \arrow[r, "\delta_{q0}"] & \frac{K_1(I_0)}{(1-\Gamma_3)K_1(I_0)} \arrow[r, "i_{q*}"] & \frac{K_1(A)}{(1-\Gamma_4)K_1(A)} \arrow[r] & 0
\end{tikzcd}$$

Figure 5.20: Truncated Commutative Diagram of Quotients

First, we'll show that q_{q*} is injective. Suppose $q_{q*}[(x)] = [q_*(x)] = [0]$ where $q_*(x)$ is a member of the equivalence class. Then, $q_*(x) \in (1-\Gamma_2)K_0(A/I_0)$, or analogously, that $q_*(x) = a - \Gamma_2(a)$ for some $a \in K_0(A/I_0)$. Then,

$$0 = \delta q_*(x) = \delta(a - \Gamma_2(a)) = \delta(a) - \delta\Gamma_2(a) = \delta(a)$$

where the first equality follows from the exactness of the rows of Diagram 5.18. The last equality follows since $\delta\Gamma_2(a) = 0$, by Lemma 5.2.16. Thus, $\delta(a) = 0$, so that again by the exactness of the rows of Diagram 5.18, there exists $y \in K_0(A)$ such that $q_*(y) = a$. Thus, $q_*(x) = q_*(y) - \Gamma_2 q_*(y) = q_*(y) - q_*\Gamma_1(y) = q_*(y - \Gamma_1(y))$. Thus, since q_* is injective, $x = y - \Gamma_1(y)$, and so $[x] = [0]$.

Next, we show that $ran(q_{q*}) \subset ker(\delta_{q*})$. Fix $[x] \in ran(q_{q*})$, so that $[x] = [q_*(y)]$ for some $y \in K_0(A)$. Then, $\delta_{q*}([x]) = [\delta_* q_*(y)] = [0]$.

To see that $ker(\delta_{q*}) \subset ran(q_{q*})$, let $[x] \in ker(\delta_{q*})$, so that $\delta_{q*}[x] = [\delta_*(x)] = [0]$. Then, notice that $x + (1 - \Gamma_2)(-x) = \Gamma_2(x)$ is in the equivalence class of $[x]$, and $\delta(\Gamma_2(x)) = 0$ by Lemma 5.2.16. Thus, there exists $y \in K_0(A)$ such that $q_*(y) = \Gamma_2(x)$, and so $q_{q*}([y]) = [q_*(y)] = [\Gamma_2(x)] = [x]$.

Thus, the last thing to prove is that q_{q*} is surjective. Suppose $[x] \in coker(1-\Gamma_2)$, so that $[x] = x + (1 - \Gamma_2)K_0(A/I_0)$ for some $x \in K_0(A/I_0)$. Then, by Lemma 5.2.16, $\Gamma_3\delta = 0$, so $(1 - \Gamma_3)\delta(x) = \delta(x)$ which means that $\delta_{q0}([x]) \in [0]$. Thus, since $ker(\delta_{q*}) \subset ran(q_{q*})$, there exists $[a] \in coker(1 - \Gamma_1)$ such that $q_{q*}([a]) = [x]$.

□

We now present two substitutions, one which forces the border, one which does not, both of which have the same substitution matrix, but which have different K-groups.

Example 5.2.18. *Consider the substitutions* ω *which takes* $0 \to 0101$ *and* $1 \to 0011$

and ω_2 *which takes* $0 \to 0110$ *and* $1 \to 1001$ *(the second is the square of the Thue-Morse substitution). The first one forces the border, the second one does not, and although the substitution matrices are the same, we will show that the* K_0 *groups are different from which we conclude that the addition of forcing the border was a required condition. We use the same method as in section 5.1 to compute the two* Γ_1 *maps corresponding to* ω_1 *and* ω_2*. First, we'll compute* Γ_1 *for* ω_1*. We need to first define a suitable* P_0 *which needs to contain the pairs* 01, 10, 00 *and* 11*. Thus, let* $P_0 = 00110$*, so that* $P_1 = 01010101001100110101$*. It becomes quickly evident that these computations become long even with relatively simple substitutions.*

Define as before, $A = C^*(R_0)$*,* $B = C^*(R_1)$*, and the ideals* $I \subset A$ *and* $I_B \subset B$ *which consist of functions on* R_0 *and* R_1 *respectively which vanish at the vertices. Again we take advance of the six term exact sequences given by a* C^**-algebra and a closed ideal, as illustrated in the following diagram:*

$$\begin{tikzcd}
0 \arrow[r] & K_0(A) \arrow[r, "q_*"] \arrow[d] & K_0(A/I) \arrow[r, "\delta_0"] \arrow[d, "i_*"] & K_1(I) \arrow[r, "i_*"] \arrow[d] & K_1(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r] \arrow[d] & K_0(A/J) \arrow[r] \arrow[d, "G_*"] & K_1(J) \arrow[r] \arrow[d] & K_1(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(B) \arrow[r] \arrow[d] & K_0(B/I_B) \arrow[r] \arrow[d, "\gamma_*"] & K_1(I_B) \arrow[r] \arrow[d] & K_1(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[r] & K_1(I) \arrow[r] & K_1(A) \arrow[r] & 0
\end{tikzcd}$$

Figure 5.21: Commutative Diagram of the Thue-Morse Example

Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ *denote the 4 vertical maps, from the top row to the bottom row, listed from left to right. We want to compute* Γ_1 *and* Γ_4*, but to do so, we'll compute* Γ_2 *and* Γ_3 *and use the commutativity of the diagram.*

We begin by computing the exponential map $\delta_0 : K_0(A/I) \to K_1(I)$*. First note that* $A/I \cong \mathbb{C}^4$ *since there are 4 1-element equivalence classes of distinct points in* A/I*. Thus,* $K_0(A/I) \cong \mathbb{Z}^4$*. We use the same argument used in section 5.1 of lifting a projection in* $p \in A/I$ *to a self adjoint element in* $s \in A$*, and then* $\delta_0(p) = e^{is}$*. , the projections at the vertex of* 00 *and* 11 *will be in the kernel of the* δ_0*. The projections at the vertices* 01 *and* 10 *will be mapped to* $u + v^*$ *and* $v + u^*$ *respectively, where* u *and* v *are unitaries which rotate once around the unit circle in the positive direction on the* 0 *and* 1 *intervals respectively. Let* $\{e_i\}_{i=1}^4$ *be the standard basis for* $\mathbb{Z}^4 \cong K_0(A/I)$ *where* e_i *corresponds to the* i^{th} *vertex from left to right in* R_0 *(note that in* R_0*, there are no off diagonal vertices). Let* $\{f_1, f_2\}$ *be the standard basis for* $\mathbb{Z}^2 \cong K_1(I)$ *where* f_1 *and* f_2 *correspond to a unitary which wraps once in the position direction around the complex circle on the* 0 *interval and the* 1 *interval respectively. The exponential map has a matrix representation with respect to this basis:*

$$\delta_0 = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

The kernel of this map is isomorphic to $\mathbb{Z}^3$*, from which we conclude that* $K_0(A) \cong \mathbb{Z}^3$*. Define a basis for* $K_0(A)$ *by* $\{q_*^{-1}(e_1), q_*^{-1}(e_3), q_*^{-1}(e_2 + e_4)\}$*. Then, we have a matrix representation*

$$q_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We next calculate a matrix for the vertical map Γ_2*. Using the same homotopy procedure, we see that a projection which is 1 at the vertex* 00 *is mapped to a projection* *in* A/J *which is 1 at the first four vertices, and zero elsewhere. Thus, in* $K_0(A/I) \cong \mathbb{Z}^4$*, this corresponds to* $(0, 2, 0, 2)$.

Thus, $\Gamma_2(1, 0, 0, 0) = (0, 2, 0, 2)$. *Similarly,*

$$\Gamma_2(0,1,0,0) = (0,2,0,2) \quad \Gamma_2(0,0,1,0) = (1,1,1,1) \quad \Gamma_2(0,0,0,1) = (1,1,1,1)$$

In matrix form,

$$\Gamma_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

Now we impose the commutativity of the diagram, giving that $\Gamma_2 q_* = q_* \Gamma_1$.

Thus,

$$\Gamma_2 q_* = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

and so

$$\Gamma_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

Now,

$$\Gamma_1 - I = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

has $ker(\Gamma_1 - I) = \{0\}$ *and* $coker(\Gamma_1 - I) = \{0\}$ *and so* $K_0(\mathcal{O}_E) \cong \mathbb{Z}$ *and* $K_1(\mathcal{O}_E) \cong \mathbb{Z}$.

The substitution matrix is given by

$$\Omega = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

so that

$$\Omega - I = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

which also has $ker(\Omega - I) = \{0\}$ *and* $coker(\Omega - I) = \{0\}$. *Thus, the two methods for computing the K-groups agree in this case.*

A similar calculation for the other substitution, ω_2 *which does not force the border gives*

$$\Gamma_1 - I = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

but $ker(\Gamma_1 - I) \cong \mathbb{Z}$ *and so* $K_0(\mathcal{O}_E) \cong \mathbb{Z}^2$. *Thus, despite having the same substitution matrix, the* K*-theory of these two dynamical systems is different.*

Next, we provide an example which shows the necessity of the condition that the substitution of each letters begins with the same letter by giving an example of a substitution which forces the border, but does not start with the same letter in each substitution, and showing in this case that $ker(id - \Gamma_1)$ and $ker(id - \Gamma_2)$ are not isomorphic, so that the K-groups cannot be obtained by substitution matrix.

Example 5.2.19. *The substitution, ω, we are examining is given by*

$$a \to abc$$

$$b \to bab$$

$$c \to abc$$

We take $P_0 = abcba$, since this has all letters and pairs of letters that occur in subsequent substitutions. Also, $P_1 = abcbababcbababc$ and P_0 appears inside as is required. We first show that this substitution forces its border. Note that P_1 consists of an alternation of the blocks abc and bab, which also results in every other letter being b. This means that P_2 will also consist of alternations of the blocks abc and bab, again, with every other letter being b, as will P_k, for $k \geq 1$. Thus, we can see the partial tilings $\omega(a)$ and $\omega(c)$ will always be sandwiched between two b tiles, and $\omega(b)$ will always have a c to its left and an a to its right, and so the substitution forces its border as claimed.

Then, R_0 and R_1 are the usual equivalence relations on P_0 and P_1 respectvely, $A = C^*(R_0)$, I is the usual ideal in A of functions vanishing at the vertices of R_0, J is the ideal in A consisting of functions which vanish at the points in R_0 which the substitution maps to vertices of R_1. We take $B = C^*(R_1)$ and I_B is the ideal in B consisting of functions which vanish at the vertices of R_1.

Consider the commutative diagram of Figure 5.22. Let Γ_1, Γ_2, Γ_3, Γ_4, be the maps, respectively from left to right obtained in each case by composing the three vertical maps from top to bottom. We can compute and represent Γ_2 as a matrix according to the basis of $K_0(A/I)$ given in order by the projections at the vertices of ab, bc, cb, and ba. We note that

$$\begin{tikzcd}
0 \arrow[r] & K_0(A) \arrow[r, "q_*"] \arrow[d] & K_0(A/I) \arrow[r, "\delta_0"] \arrow[d] & K_1(I) \arrow[r, "i_*"] \arrow[d] & K_1(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r] \arrow[d] & K_0(A/J) \arrow[r] \arrow[d] & K_1(J) \arrow[r] \arrow[d] & K_1(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(B) \arrow[r] \arrow[d] & K_0(B/I_B) \arrow[r] \arrow[d] & K_1(I_B) \arrow[r] \arrow[d] & K_1(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[r] & K_1(I) \arrow[r] & K_1(A) \arrow[r] & 0
\end{tikzcd}$$

Figure 5.22: Commutative Diagram for Forcing the Border Example

$$\Gamma_2(ab) = ab + bc + ca$$

$$\Gamma_2(bc) = ab + 2ba$$

$$\Gamma_2(cb) = ab + bc + ca$$

$$\Gamma_2(ba) = ab + 2ba$$

Then,

$$\Gamma_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

Next, we compute the exponential map δ_0 again as a matrix according to the same basis for $K_0(A/I)$ and the basis of $K_1(I)$ given by three unitaries, u, v, w, which correspond to functions which are 0 off the diagonal of R_0, and wind around the unit *circle once on the* (a,a)*,* (b,b)*, and* (c,c) *intervals on the diagonal of* R_0 *respectively and are 1 otherwise. Then, with respect to these bases, the matrix representation is*

$$\delta_0 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

Then, we have that $K_0(A) \cong ker(\delta_0) \cong \mathbb{Z}^2$*. Notice that* $ker(\delta_0)$ *is generated by* $(1,0,0,1)$ *and* $(0,1,1,0)$*. Thus, we can generate* $K_0(A)$ *by the pre images of these vectors under* q_**, which allows us, according to this basis of* $K_0(A)$*, and the same basis of* $K_0(A/I)$ *as before, to compute* q_**.*

$$q_* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then, by the commutativity of the diagram, we have that $\Gamma_2 \circ q_* = q_* \circ \Gamma_1$ *from which we conclude that*

$$\Gamma_1 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

Now, $ker(\Gamma_1 - id)$ *is trivial, but*

$$\Gamma_2 - id = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

has the vector $(0,1,1,-2)$ *in its kernel, so the two kernels are not isomorphic.*

5.3 The General n-Dimensional Case

In this section, we outline the procedure for computing the K-groups of the C^*-algebra system constructed above applied to a general n-dimensional substitution tiling. The strategy is based on understanding the ideal structure of the C^*-algebra $A = C^*(R_0)$ associated with the (partial) tiling space. In the one dimensional case, there was only one ideal which was of importance in A, which consisted of functions which vanish on the vertices. Let's consider now the 2-dimensional case and assume our prototiles $a_1, \cdots, a_n$ are closed polygons for simplicity. The most obvious ideal to examine first is that consisting of functions which vanish on the boundary of the tiles, which we denote I_1. We have the short exact sequence:

$$0 \to I_1 \to A \to A/I_1 \to 0$$

which gives

$$\begin{tikzcd}
K_0(I_1) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I_1) \arrow[d] \\
K_1(A/I_1) \arrow[u] & K_1(A) \arrow[l] & K_1(I_1) \arrow[l]
\end{tikzcd}$$

Figure 5.23: Six Term Exact Sequence from Ideal Structure

Since I_1 is easily described as $\oplus_{i=1}^{n} M_{j_i} \otimes C_0(int(a_i))$ where a_i is one of the polygonal prototiles, we find $K_0(I_1) \cong \mathbb{Z}^n$ and $K_1(I_1) \cong 0$. However, the K-groups of A and A/I_1 are not obvious. However, A/I_1 can be view as functions whose domain is the boundary of the tiles, with the multiplication structure inherited from A. There is another obvious ideal now in A/I_1; that which consists of functions in A/I_1 which vanish as the vertices of A/I_1, where the vertices consist of points which lie in 3 or more tiles. Denote this ideal in A/I_1 by I_2. First, note that the K-groups of I_2 are easily computable, since

$$I_2 \cong \oplus_{j=1}^{m} M_{k_j}(C_0(int(e_j)))$$

where the e_j are the various edges of the polygons a_j, m is the number of equivalence classes of edges and k_j is the number of members in the equivalence class of e_j. Also, the K-groups of $(A/I_1)/I_2$ are easily computable, since

$$(A/I_1)/I_2 \cong \oplus_{i=1}^{l} M_{j_i} v_i$$

where l is the number of equivalence classes of vertices, $v_1, \cdots, v_l$ and j_i is the number of members in the equivalence class of v_i. Thus, from the short exact sequence:

$$0 \to I_2 \to A/I_1 \to (A/I_1)/I_2$$

we have

$$\begin{tikzcd}
K_0(I_2) \arrow[r] & K_0(A/I_1) \arrow[r] & K_0((A/I_1)/I_2) \arrow[d] \\
K_1((A/I_1)/I_2) \arrow[u] & K_1(A/I_1) \arrow[l] & K_1(I_2) \arrow[l]
\end{tikzcd}$$

Figure 5.24: Six Term Exact Sequence from Ideal Structure - Second Level

Thus, we are now able to compute the K-groups of A/I_1, which in turn allows us to use Diagram 5.23 compute the K-groups of A.

In $\mathbb{R}^n$, there is a finite increasing sequence of ideals, in an analogous way to those in $\mathbb{R}^2$, $I_1 \subset I_2 \subset \cdots \subset I_n$, where I_1 consists of functions which vanish on the boundary of the tiles. We then view the boundary of the tiles as its own tiling space, so that $I_2 \subset A/I_1$ consists of functions which vanish on the boundary of these new tiles. We iterate this process until we reach a six term exact sequence where we know 4 of the six terms, from which we deduce the other two. Then, we iterate the procedure back, finally computing the K-groups of A.

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