

***SOME FAMILIES OF
GENERALIZED ELLIPTIC-TYPE INTEGRALS***

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DMS-663-IR

May 1994

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Abstract

The object of the present paper is first to point out a hitherto unnoticed relationship between two seemingly different families of generalized elliptic-type integrals which were studied recently. This relationship and some other simple techniques are then employed in order to derive many properties (including, for example, analytic continuations and asymptotic behaviors) of these families. Several interesting special cases, involving simpler elliptic-type integrals, are also considered.

1991 *Mathematics Subject Classification*. Primary 33E05; Secondary 33C75, 33C05.

Keywords. Elliptic integrals, analytic continuation, asymptotic expansion, radiation physics, Gamma function, modified Bessel function, Laplace transform, Abelian theorem, Gauss hypergeometric function, quadratic transformation, Chebyshev polynomials, Euler's transformation.

1. Introduction

Motivated by their importance or potential for applications in certain problems in radiation physics, several recent works were devoted exclusively to the study of various interesting generalizations of the complete elliptic integrals $K(\kappa)$ and $E(\kappa)$ defined by

$$K(\kappa) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2 \theta}} \quad (\kappa^2 < 1) \quad (1)$$

and

$$E(\kappa) := \int_0^{\pi/2} \sqrt{1 - \kappa^2 \sin^2 \theta} \, d\theta \quad (\kappa^2 < 1), \quad (2)$$

respectively. For example, Epstein and Hubbell [4] (and, subsequently, Weiss [15]) studied the following family of elliptic-type integrals:

$$\Omega_j(k) := \int_0^\pi (1 - k^2 \cos \theta)^{-j - \frac{1}{2}} d\theta \quad (3)$$

$(j = 0, 1, 2, \dots; 0 \leq k < 1),$

which were further extended by Kalla and Al-Saqabi [9] to allow the parameter j to take on *complex* values. By comparing the definitions (1) and (2) with the Epstein-Hubbell definition (3), we readily obtain the relationships:

$$\Omega_0(k) = \frac{\kappa\sqrt{2}}{k} K(\kappa) \quad \text{and} \quad \Omega_1(k) = \frac{\kappa\sqrt{2}}{k(1 - k^2)} E(\kappa) \quad (4)$$

$$\left(\kappa^2 = \frac{2k^2}{1 + k^2} \right).$$

Kalla [7], on the other hand, considered a further generalization of $\Omega_\mu(k)$ ($\mu \in \mathbb{C}$) in the form:

$$S_\mu(k, \lambda) := \int_0^\pi \frac{\sin^{2\lambda} \theta}{(1 - k^2 \cos \theta)^{\mu + \frac{1}{2}}} d\theta \quad (5)$$

$(0 \leq k < 1; \Re(\lambda) > -\frac{1}{2}; \mu \in \mathbb{C}),$

so that, obviously,

$$S_\mu(k, 0) = \Omega_\mu(k) \quad (0 \leq k < 1; \mu \in \mathbb{C}), \quad (6)$$

the *additional* constraint:

$$\Re(\mu) > -\frac{1}{2},$$

imposed upon the definitions of $\Omega_\mu(k)$, $S_\mu(k, \lambda)$, *et cetera*, by many earlier authors (*cf.*, *e.g.*, [1], [7], [8], [9], and [10]) being *unnecessary*.

Another seemingly independent generalization of $\Omega_\mu(k)$ ($\mu \in \mathbb{C}$) may be recalled here in the following (slightly modified) form (*cf.* Mohamed-Murid [11] and Bromberg [2]):

$$\begin{aligned} H(\nu, m, \kappa) &:= \int_0^{2\pi} \frac{\cos m\theta}{(1 - \kappa \cos \theta)^\nu} d\theta \\ &= 2 \int_0^\pi \frac{\cos m\theta}{(1 - \kappa \cos \theta)^\nu} d\theta \end{aligned} \quad (7)$$

$$(\nu \in \mathbb{R}; m = 0, \pm 1, \pm 2, \dots; -1 < \kappa < 1).$$

Clearly, we have

$$\Omega_\mu(k) = \frac{1}{2} H\left(\mu + \frac{1}{2}, 0, k^2\right). \quad (8)$$

In the present paper we first prove the hitherto unnoticed relationship:

$$H(\nu, m, k^2) = \frac{(\nu)_m k^{2m}}{2^{m-1} \left(\frac{1}{2}\right)_m} S_{\nu+m-\frac{1}{2}}(k, m) \quad (9)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; 0 \leq k < 1),$$

where, and in what follows, $(\nu)_m := \Gamma(\nu + m)/\Gamma(\nu)$, in terms of the familiar Gamma functions. Since

$$H(\nu, -m, \kappa) = H(\nu, m, \kappa) \quad (10)$$

$$(\nu \in \mathbb{R}; m = 0, \pm 1, \pm 2, \dots; -1 < \kappa < 1),$$

which is an *immediate* consequence of the definition (7), the relationship (9) exhibits the fact that the elliptic-type integral $H(\nu, m, \kappa)$, studied by Mohamed-Murid [11] and Bromberg [2], is actually a *special* case of the elliptic-type integral $S_\mu(k, \lambda)$ studied earlier by Kalla [7]. We also make use of the relationship (9), and some other simple techniques, in order to derive various properties (including, for example, analytic continuations and asymptotic behaviors) of these families of elliptic-type integrals. For systematic and detailed studies of several *additional* families of generalized elliptic-type integrals, one may refer to the works by (among other authors) Al-Saqabi [1], Kalla *et al.* [10], and Siddiqi [13].

2. The Relationship (9) and Its Applications

We begin by recalling that the Chebyshev polynomials of the first kind are standardized by putting (see, for details, Erdélyi *et al.* [5, Vol. II, pp. 184-185])

$$T_n(\cos \theta) = \cos n\theta, \quad (11)$$

where

$$T_n(x) = {}_2F_1 \left(-n, n; \frac{1}{2}; \frac{1-x}{2} \right) \quad (n = 0, 1, 2, \dots), \quad (12)$$

in terms of the Gauss hypergeometric function. These polynomials are represented explicitly by the finite series:

$$T_n(x) = \frac{n}{2} \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r (n-r-1)!}{r! (n-2r)!} (2x)^{n-2r} \quad (n = 1, 2, 3, \dots) \quad (13)$$

and the Rodrigues formula:

$$T_n(x) = \frac{(-1)^n}{2^n \left(\frac{1}{2}\right)_n} (1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n} \left\{ (1-x^2)^{n-\frac{1}{2}} \right\} \quad (14)$$

$$(n = 0, 1, 2, \dots).$$

In view of (11) and (14), we find from (7) with $\cos \theta = x$ (and $\kappa = k^2$) that

$$H(\nu, m, k^2) = \frac{(-1)^m}{2^{m-1} \left(\frac{1}{2}\right)_m} \int_{-1}^1 (1-k^2x)^{-\nu} \frac{d^m}{dx^m} \left\{ (1-x^2)^{m-\frac{1}{2}} \right\} dx,$$

which, upon m times integration by parts, yields

$$H(\nu, m, k^2) = \frac{(\nu)_m k^{2m}}{2^{m-1} \left(\frac{1}{2}\right)_m} \int_{-1}^1 (1-k^2x)^{-\nu-m} (1-x^2)^{m-\frac{1}{2}} dx \quad (15)$$

$$(\nu \in \mathbb{R}; \quad m = 0, 1, 2, \dots; \quad 0 \leq k < 1).$$

Finally, upon setting $x = \cos \theta$ in (15), we obtain

$$H(\nu, m, k^2) = \frac{(\nu)_m k^{2m}}{2^{m-1} \left(\frac{1}{2}\right)_m} \int_0^\pi \frac{\sin^{2m} \theta}{(1-k^2 \cos \theta)^{\nu+m}} d\theta \quad (16)$$

$$(\nu \in \mathbb{R}; \quad m = 0, 1, 2, \dots; \quad 0 \leq k < 1),$$

and the relationship (9) would follow at once when we interpret the integral in (16) by means of the definition (5). Since [cf. Definition (5)]

$$S_\mu(\sqrt{k}, \lambda) = S_\mu(-\sqrt{k}, \lambda) \quad (17)$$

and, in particular,

$$H(\nu, m, \kappa) = (-1)^m H(\nu, m, -\kappa), \quad (18)$$

which can easily be verified from the definition (7), there is no loss of generality in assuming that $0 \leq k < 1$ (instead of $-1 < \kappa < 1$).

An explicit power-series expansion of the generalized elliptic-type integral $S_\mu(k, \lambda)$ was given by Kalla [7]. We recall Kalla's result [7, p. 217, Equation (4)] in the following (corrected) form:

$$S_\mu(k, \lambda) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2}) \Gamma(\mu + 2n + \frac{1}{2})}{(2n)! \Gamma(\lambda + n + 1)} k^{4n} \quad (19)$$

$$(0 \leq k < 1; \Re(\lambda) > -\frac{1}{2}; \mu \in \mathbb{C}),$$

which, in terms of the Gauss hypergeometric function, becomes

$$S_\mu(k, \lambda) = \frac{\sqrt{\pi} \Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)} {}_2F_1\left(\frac{1}{2}\mu + \frac{1}{4}, \frac{1}{2}\mu + \frac{3}{4}; \lambda + 1; k^4\right) \quad (20)$$

$$(0 \leq k < 1; \Re(\lambda) > -\frac{1}{2}; \mu \in \mathbb{C}).$$

Applying the relationship (9), (19) and (20) readily yield the following power-series and Gauss hypergeometric representations for $H(\nu, m, \kappa)$:

$$H(\nu, m, \kappa) = \frac{\sqrt{\pi}}{2^{m-1} \Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2}) \Gamma(\nu + m + 2n)}{(2n)! (m+n)!} \kappa^{m+2n} \quad (21)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; -1 < \kappa < 1)$$

or, equivalently,

$$H(\nu, m, \kappa) = \frac{\pi}{2^{m-1} \Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(\nu + m + 2n)}{n! (m+n)!} \frac{\kappa^{m+2n}}{2^{2n}} \quad (22)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; -1 < \kappa < 1),$$

and

$$H(\nu, m, \kappa) = \frac{\pi(\nu)_m \kappa^m}{2^{m-1} m!} {}_2F_1\left(\frac{\nu+m}{2}, \frac{\nu+m+1}{2}; m+1; \kappa^2\right) \quad (23)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; -1 < \kappa < 1).$$

In precisely the same manner, we are led to the following hypergeometric representations for $H(\nu, m, \kappa)$:

$$H(\nu, m, \kappa) = \frac{\pi(\nu)_m \kappa^m}{2^{m-1} m!} (1+\kappa)^{-\nu-m} \cdot {}_2F_1\left(\nu+m, m+\frac{1}{2}; 2m+1; \frac{2\kappa}{1+\kappa}\right) \quad (24)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; -\frac{1}{3} < \kappa < 1);$$

$$H(\nu, m, \kappa) = \frac{\pi(\nu)_m \kappa^m}{2^{m-1} m!} (1-\kappa)^{-\nu-m} \cdot {}_2F_1\left(\nu+m, m+\frac{1}{2}; 2m+1; \frac{2\kappa}{\kappa-1}\right) \quad (25)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; \kappa \in (-\infty, -1) \cup (\frac{1}{3}, \infty) \setminus \{1\});$$

$$H(\nu, m, \kappa) = \frac{\pi(\nu)_m \kappa^m}{2^{m-1} m!} (1+\kappa)^{\frac{1}{2}-\nu} (1-\kappa)^{-m-\frac{1}{2}} \cdot {}_2F_1\left(m-\nu+1, m+\frac{1}{2}; 2m+1; \frac{2\kappa}{\kappa-1}\right) \quad (26)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; \kappa \in (-\infty, -1) \cup (\frac{1}{3}, \infty) \setminus \{1\});$$

$$H(\nu, m, \kappa) = \frac{\pi(\nu)_m \kappa^m}{2^{m-1} m!} (1+\kappa)^{-m-\frac{1}{2}} (1-\kappa)^{\frac{1}{2}-\nu} \cdot {}_2F_1\left(m-\nu+1, m+\frac{1}{2}; 2m+1; \frac{2\kappa}{1+\kappa}\right) \quad (27)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; -\frac{1}{3} < \kappa < 1);$$

$$H(\nu, m, \kappa) = \frac{\pi(\nu)_m \kappa^m}{2^{m-1} m!} (1-\kappa^2)^{\frac{1}{2}-\nu} \cdot {}_2F_1\left(\frac{m-\nu+1}{2}, \frac{m-\nu}{2}+1; m+1; \kappa^2\right) \quad (28)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; -1 < \kappa < 1).$$

The hypergeometric representations (24) to (28), and many more, can indeed be deduced also from (23) by appealing to one or the other linear and quadratic transformations for the Gauss hypergeometric function (see, for example, Erdélyi *et al.* [5, Vol. I, Chapter 2]). In particular, the hypergeometric transformations (24) and (25) would follow from (23) when we apply the quadratic transformation [5, Vol. I, p. 112, Equation 2.11(17)]:

$$\begin{aligned} {}_2F_1\left(a, a + \frac{1}{2}; c; z^2\right) &= (1+z)^{-2a} {}_2F_1\left(2a, c - \frac{1}{2}; 2c - 1; \frac{2z}{1+z}\right) \\ &(|\arg(1+z)| \leq \pi - \epsilon \quad (0 < \epsilon < \pi)) \end{aligned} \quad (29)$$

with

$$a = \frac{\nu + m}{2}, \quad c = m + 1, \quad \text{and} \quad z = \pm \kappa.$$

Furthermore, in view of Euler's transformation [5, Vol. I, p. 64, Equation 2.1.4(23)]:

$$\begin{aligned} {}_2F_1(a, b; c; z) &= (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z) \\ &(|\arg(1-z)| \leq \pi - \epsilon \quad (0 < \epsilon < \pi)), \end{aligned} \quad (30)$$

(23) would immediately yield the hypergeometric representation (28) which appears erroneously in Bromberg [2, p. 33, Equation (38)].

If we apply (30) to the general hypergeometric representation (20), we obtain

$$\begin{aligned} S_\mu(k, \lambda) &= \frac{\sqrt{\pi} \Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda + 1)} (1 - k^4)^{\lambda - \mu} \\ &\cdot {}_2F_1\left(\lambda - \frac{1}{2}\mu + \frac{1}{4}, \lambda - \frac{1}{2}\mu + \frac{3}{4}; \lambda + 1; k^4\right), \end{aligned} \quad (31)$$

which, in view of the relationship [5, Vol. I, p. 128, Entry 3.2(24)]:

$$\begin{aligned} P_\nu^\mu(z) &= \frac{2^\mu (z^2 - 1)^{-\frac{1}{2}\mu} z^{\mu + \nu}}{\Gamma(1 - \mu)} \\ &\cdot {}_2F_1\left(-\frac{\mu + \nu}{2}, \frac{1 - \mu - \nu}{2}; 1 - \mu; 1 - \frac{1}{z^2}\right) \quad (\Re(z) > 0), \end{aligned} \quad (32)$$

yields the following known representation (*cf.* Kalla [7, p. 218, Equation (6)]):

$$\begin{aligned} S_\mu(k, \lambda) &= \frac{2^\lambda \sqrt{\pi} \Gamma\left(\lambda + \frac{1}{2}\right)}{k^{2\lambda} (\sqrt{1 - k^4})^{\mu - \lambda + \frac{1}{2}}} P_{\mu - \lambda - \frac{1}{2}}^{-\lambda} \left(\frac{1}{\sqrt{1 - k^4}}\right) \\ &(0 \leq k < 1; \Re(\lambda) > -\frac{1}{2}; \mu \in \mathbb{C}) \end{aligned} \quad (33)$$

in terms of the Legendre function (*cf.*, *e.g.*, Erdélyi *et al.* [5, Vol. I, Chapter 3]). By appealing to the relationship (9), (33) immediately leads us to the representation:

$$H(\nu, m, \kappa) = \frac{2\pi(\nu)_m}{(1 - \kappa^2)^{\frac{1}{2}\nu}} P_{\nu-1}^{-m} \left(\frac{1}{\sqrt{1 - \kappa^2}} \right) \quad (34)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; -1 < \kappa < 1),$$

which would follow *directly* from the hypergeometric representations (28) and (32).

Many recurrence relations (that is, contiguous function relations) for the Gauss hypergeometric function are recorded by (for example) Rainville [12, pp. 71-72, Exercises 21, 22, and 23]. Some of these known results can be applied to the hypergeometric representations (23) to (28) with a view to obtaining various recurrence relations for $H(\nu, m, \kappa)$. For instance, if we apply the familiar result [12, p. 71, Exercise 21(8)] to the hypergeometric representation (23), we shall obtain the following corrected version of a known recurrence relation (*cf.* [2, p. 33, Equation (46)]):

$$(\nu - 1)(2\nu - 1)H(\nu, m, \kappa) = (\nu - m - 1)(\nu + m - 1)H(\nu - 1, m, \kappa) \quad (35)$$

$$+ \nu(\nu - 1)(1 - \kappa^2)H(\nu + 1, m, \kappa).$$

Rewriting the definition (5) in the forms:

$$S_\mu(k, \lambda) = \int_0^\pi \frac{\sin^{2\lambda-1} \theta \sin \theta}{(1 - k^2 \cos \theta)^{\mu+\frac{1}{2}}} d\theta$$

and

$$S_\mu(k, \lambda) = \int_0^\pi \frac{\sin^{2\lambda} \theta}{(1 - k^2 \cos \theta)^{\mu+\frac{3}{2}}} d\theta - k^2 \int_0^\pi \frac{\sin^{2\lambda} \theta \cos \theta}{(1 - k^2 \cos \theta)^{\mu+\frac{3}{2}}} d\theta,$$

and integrating by parts, it is easily seen that (*cf.* Kalla [7, Section 4])

$$(2\mu - 1)k^4 S_\mu(k, \lambda) = 2(2\lambda - 1) [S_{\mu-1}(k, \lambda - 1) - S_{\mu-2}(k, \lambda - 1)]; \quad (36)$$

$$(2\mu - 1)(2\mu - 3)k^4 S_\mu(k, \lambda) = 4(2\lambda - 1)(2\lambda - 3) S_{\mu-2}(k, \lambda - 2) \quad (37)$$

$$- 4(2\lambda - 1)(2\lambda - 2) S_{\mu-2}(k, \lambda - 1);$$

$$2(2\lambda + 1) S_\mu(k, \lambda) = 2(2\lambda + 1) S_{\mu+1}(k, \lambda) - (2\mu + 3)k^4 S_{\mu+2}(k, \lambda + 1); \quad (38)$$

$$(2\mu - 4\lambda + 1) S_\mu(k, \lambda) = (2\mu + 1) S_{\mu+1}(k, \lambda) - 2(2\lambda - 1) S_\mu(k, \lambda - 1). \quad (39)$$

Making use of the relationship (9) in each of these recursion formulas, we can obtain the corresponding results for $H(\nu, m, \kappa)$. Thus, for example, (36) yields

$$(\nu - 1)\kappa H(\nu, m, \kappa) = (\nu - 1) H(\nu, m - 1, \kappa) - (\nu + m - 2) H(\nu - 1, m - 1, \kappa), \quad (40)$$

which can be proven *directly* from the definition (7) by writing

$$\cos m\theta = \cos(m - 1)\theta \cos \theta - \sin(m - 1)\theta \sin \theta.$$

If, on the other hand, we rewrite the definition (7) in the form:

$$\begin{aligned} H(\nu, m, \kappa) &= 2 \int_0^\pi \frac{(1 - \kappa \cos \theta) \cos m\theta}{(1 - \kappa \cos \theta)^{\nu+1}} d\theta \\ &= 2 \int_0^\pi \frac{\cos m\theta - \frac{1}{2} \kappa [\cos(m - 1)\theta + \cos(m + 1)\theta]}{(1 - \kappa \cos \theta)^{\nu+1}} d\theta, \end{aligned}$$

we at once obtain the following *four-term* recurrence relation for $H(\nu, m, \kappa)$:

$$\begin{aligned} H(\nu + 1, m, \kappa) - H(\nu, m, \kappa) \\ = \frac{1}{2} \kappa [H(\nu + 1, m - 1, \kappa) + H(\nu + 1, m + 1, \kappa)]. \end{aligned} \quad (41)$$

Finally, we integrate (7) by parts to get

$$\begin{aligned} H(\nu, m, \kappa) &= \frac{2\nu\kappa}{m} \int_0^\pi \frac{\sin m\theta \sin \theta}{(1 - \kappa \cos \theta)^{\nu+1}} d\theta \\ &= \frac{\nu\kappa}{m} \int_0^\pi \frac{\cos(m - 1)\theta - \cos(m + 1)\theta}{(1 - \kappa \cos \theta)^{\nu+1}} d\theta, \end{aligned}$$

and this immediately leads us to the following recurrence relation:

$$2m H(\nu, m, \kappa) = \nu\kappa [H(\nu + 1, m - 1, \kappa) - H(\nu + 1, m + 1, \kappa)], \quad (42)$$

which was derived by Bromberg [2, p. 33, Equation 49)] by making use of (23) and a known contiguous function relation for the Gauss hypergeometric function.

3. Further Properties and Asymptotic Expansions

From the definition (7) it is easily seen that

$$H(0, m, \kappa) = H(\nu, m, 0) = H(-1, m, \kappa) = 0. \quad (43)$$

Furthermore, since [6, p. 367, Entry 3.613(1)]

$$\int_0^\pi \frac{\cos nx}{1 + a \cos x} dx = \frac{\pi}{\sqrt{1 - a^2}} \left(\frac{\sqrt{1 - a^2} - 1}{a} \right)^n \quad (a^2 < 1),$$

we also have

$$H(1, m, \kappa) = \frac{2\pi}{\sqrt{1 - \kappa^2}} \left(\frac{1 - \sqrt{1 - \kappa^2}}{\kappa} \right)^m \quad (\kappa^2 < 1), \quad (44)$$

which follows readily from (23) in view of a familiar reduction formula for the Gauss hypergeometric function [5, Vol. I, p. 101, Entry 2.8(6)]. As a matter of fact, the hypergeometric representation (23) immediately yields

$$H(m + 1, m, \kappa) = \frac{\pi(2m)! \kappa^m}{2^{m-1}(m!)^2} (1 - \kappa^2)^{-m - \frac{1}{2}}. \quad (45)$$

Next we apply the known identity [5, Vol. I, p. 101, Entry 2.8(11)]:

$$\cos az = {}_2F_1\left(\frac{1}{2}a, -\frac{1}{2}a; \frac{1}{2}; \sin^2 z\right),$$

and we find from (7) that

$$H(\nu, m, \kappa) = 2\pi F_3\left(\frac{1}{2}m, \frac{1}{2}\nu, -\frac{1}{2}m, \frac{1}{2}\nu + \frac{1}{2}; 1; 1, \kappa^2\right) \quad (46)$$

$$(\nu \in \mathbb{R}; m = 0, 1, 2, \dots; -1 < \kappa < 1),$$

where F_3 denotes an Appell function of the third kind, defined by [5, Vol. I, p. 224, Equation 5.7.1(8)]

$$F_3(a, a', b, b'; c; x, y) = \sum_{m, n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}$$

$$(\max\{|x|, |y|\} < 1).$$

Analytic continuations of such generalized elliptic-type integrals as $S_\mu(k, \lambda)$ and $H(\nu, m, k^2)$, and their asymptotic behaviors in the neighborhood of $k^2 = 1$, can be obtained by suitably considering those of their Gauss hypergeometric representations given in the preceding section. For example, since [5, Vol. I, p. 108, Equation 2.10(2)]

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1(a, 1-c+a; 1-b+a; z^{-1}) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1(b, 1-c+b; 1-a+b; z^{-1}) \end{aligned} \quad (47)$$

$$(a-b \neq 0, \pm 1, \pm 2, \dots; c \neq 0, -1, -2, \dots; |\arg(-z)| \leq \pi - \epsilon \ (0 < \epsilon < \pi)),$$

we thus find from the hypergeometric representation (25) that

$$\begin{aligned} H(\nu, m, \kappa) &= \frac{\sqrt{2\pi} \Gamma(\nu - \frac{1}{2})}{\Gamma(\nu)} \kappa^{-\frac{1}{2}} (1 - \kappa)^{\frac{1}{2} - \nu} \\ &\cdot {}_2F_1\left(\frac{1}{2} + m, \frac{1}{2} - m; \frac{3}{2} - \nu; -\frac{1 - \kappa}{2\kappa}\right) \\ &+ \frac{2^{1-\nu} \sqrt{\pi} (\nu)_m \Gamma(\frac{1}{2} - \nu)}{\Gamma(1 - \nu + m)} \kappa^{-\nu} \\ &\cdot {}_2F_1\left(\nu + m, \nu - m; \frac{1}{2} + \nu; -\frac{1 - \kappa}{2\kappa}\right) \end{aligned} \quad (48)$$

$$(\nu - \frac{1}{2} \neq 0, \pm 1, \pm 2, \dots; m = 0, 1, 2, \dots; \kappa > \frac{1}{3}),$$

where the exceptional cases (when $\nu - \frac{1}{2}$ is an integer) can be handled by means of another known result [5, Vol. I, p. 109, Equation 2.10(7)]. More generally, if we apply (47) to the hypergeometric representation:

$$\begin{aligned} S_\mu(k, \lambda) &= \frac{\sqrt{\pi} \Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)} (1 + k^2)^{\lambda - \mu} (1 - k^2)^{-\lambda - \frac{1}{2}} \\ &\cdot {}_2F_1\left(2\lambda - \mu + \frac{1}{2}, \lambda + \frac{1}{2}; 2\lambda + 1; \frac{2k^2}{k^2 - 1}\right), \end{aligned} \quad (49)$$

which follows from (31) in view of the quadratic transformation (29) with

$$a = \lambda - \frac{1}{2}\mu + \frac{1}{4}, \quad c = \lambda + 1, \quad \text{and} \quad z = -k^2,$$

or to the *equivalent* representation:

$$S_\mu(k, \lambda) = \frac{\sqrt{\pi} \Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)} (1 - k^2)^{-\mu - \frac{1}{2}} {}_2F_1\left(\lambda + \frac{1}{2}, \mu + \frac{1}{2}; 2\lambda + 1; \frac{2k^2}{k^2 - 1}\right), \quad (50)$$

which follows from (49) in view of the Euler transformation (30), we shall obtain

$$\begin{aligned} S_\mu(k, \lambda) &= \frac{2^{\lambda - \frac{1}{2}} \Gamma(\lambda + \frac{1}{2}) \Gamma(\mu - \lambda)}{\Gamma(\mu + \frac{1}{2})} k^{-2\lambda - 1} (1 - k^2)^{\lambda - \mu} \\ &\quad \cdot {}_2F_1\left(\frac{1}{2} + \lambda, \frac{1}{2} - \lambda; 1 + \lambda - \mu; -\frac{1 - k^2}{2k^2}\right) \\ &+ \frac{2^{2\lambda - \mu - \frac{1}{2}} \Gamma(\lambda + \frac{1}{2}) \Gamma(\lambda - \mu)}{\Gamma(2\lambda - \mu + \frac{1}{2})} k^{-2\mu - 1} \\ &\quad \cdot {}_2F_1\left(\frac{1}{2} + \mu, \frac{1}{2} - 2\lambda + \mu; 1 - \lambda + \mu; -\frac{1 - k^2}{2k^2}\right) \end{aligned} \quad (51)$$

$$(\mu - \lambda \neq 0, \pm 1, \pm 2, \dots; \lambda + \frac{1}{2} \neq 0, -1, -2, \dots; k^2 > \frac{1}{3}),$$

which may be compared with (for example) an earlier observation by Al-Saqabi [1, p. 335, Equation (39)].

A *direct* approach to the asymptotic behavior problems of the aforementioned types, which is based upon an Abelian theorem for the Laplace transforms (*cf.*, *e.g.*, Doetsch [3, p. 231, Theorem 34.1]), was made *seemingly erroneously* by Weiss [15], Kalla [7], Kalla *et al.* ([8], [10]), and Siddiqi [13]. We aim at presenting here a simple way to circumvent a certain problem which seems to have been overlooked by each of these earlier authors. We choose to illustrate this approach by considering the generalized elliptic-type integral $S_\mu(k, \lambda)$ defined by (5). Indeed, by appealing to the familiar Γ -function formula:

$$\frac{\Gamma(\rho)}{s^\rho} = \int_0^\infty t^{\rho-1} e^{-st} dt \quad (52)$$

$$(\Re(\rho) > 0; \Re(s) > 0)$$

with

$$\rho = \mu + \frac{1}{2} \quad \text{and} \quad s = 1 - k^2 \cos \theta,$$

we find from (5) that

$$S_\mu(k, \lambda) = \frac{1}{\Gamma(\mu + \frac{1}{2})} \int_0^\pi \sin^{2\lambda} \theta \left(\int_0^\infty t^{\mu - \frac{1}{2}} e^{-(1 - k^2 \cos \theta)t} dt \right) d\theta \quad (53)$$

$$(0 \leq k < 1; \Re(\lambda) > -\frac{1}{2}; \Re(\mu) > -\frac{1}{2}),$$

Upon inversion of the order of integration in (53), which can be justified under the conditions already stated, we have

$$S_\mu(k, \lambda) = \frac{1}{\Gamma(\mu + \frac{1}{2})} \int_0^\infty t^{\mu - \frac{1}{2}} e^{-t} \left(\int_0^\pi e^{k^2 t \cos \theta} \sin^{2\lambda} \theta d\theta \right) dt \quad (54)$$

$$(0 \leq k < 1; \Re(\lambda) > -\frac{1}{2}; \Re(\mu) > -\frac{1}{2}).$$

Now use the integral representation [14, p. 79, Equation 3.71(9)]:

$$I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^\pi e^{\pm z \cos \theta} \sin^{2\nu} \theta d\theta \quad (\Re(\nu) > -\frac{1}{2}) \quad (55)$$

for the modified Bessel function $I_\nu(z)$, and (54) with t replaced by t/k^2 would yield (cf. [7, p. 217, Equation (5)])

$$S_\mu(k, \lambda) = \frac{2^\lambda \sqrt{\pi} \Gamma(\lambda + \frac{1}{2})}{k^{2\mu+1} \Gamma(\mu + \frac{1}{2})} \int_0^\infty e^{-t(1-k^2)/k^2} \left\{ t^{\mu-\lambda-\frac{1}{2}} e^{-t} I_\lambda(t) \right\} dt \quad (56)$$

$$(0 \leq k < 1, \Re(\lambda) > -\frac{1}{2}; \Re(\mu) > -\frac{1}{2}),$$

which expresses $S_\mu(k, \lambda)$ as the Laplace transform of the function

$$f(t) := t^{\mu-\lambda-\frac{1}{2}} e^{-t} I_\lambda(t), \quad (57)$$

and with the parameter

$$s := \frac{1 - k^2}{k^2} \rightarrow 0 \quad \text{as} \quad k^2 \rightarrow 1.$$

In order to apply the aforementioned Abelian theorem [3, p. 231, Theorem 34.1], we notice from (57) and the known asymptotic expansion [14, p. 203, Equation 7.23(2)]:

$$I_\nu(z) \sim \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu - n + \frac{1}{2})} \frac{(2z)^{-n}}{n!} \quad (|z| \rightarrow \infty)$$

that

$$f(t) \sim \frac{t^{\mu-\lambda-1}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda + n + \frac{1}{2})}{\Gamma(\lambda - n + \frac{1}{2})} \frac{(2t)^{-n}}{n!} \quad (t \rightarrow \infty). \quad (58)$$

If we choose to substitute this asymptotic expansion for $f(t)$ on the right-hand side of (56), and perform term-by-term integration, the resulting integrals would converge under the conditions:

$$\Re(\mu - \lambda - n) > 0 \quad (n = 0, 1, 2, \dots),$$

which are obviously impossible to satisfy as long as n is *unbounded* above.

A simple way to overcome the above difficulty in properly applying the aforementioned Abelian theorem for the Laplace transforms is to consider only the first few terms of the asymptotic expansion (58). Thus, making use of the Γ -function formula (52) *correctly*, we finally obtain the asymptotic expansion [cf. Equation (51)]:

$$S_{\mu}(k, \lambda) \sim \frac{2^{\lambda-\frac{1}{2}} \Gamma(\lambda + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} k^{-2\lambda-1} (1 - k^2)^{\lambda-\mu} \sum_{n=0}^N \frac{\Gamma(\lambda + n + \frac{1}{2}) \Gamma(\mu - \lambda - n)}{n! \Gamma(\lambda - n + \frac{1}{2})} \left(-\frac{1 - k^2}{2k^2}\right)^n \quad (59)$$

$$(\Re(\mu - \lambda - n) > 0 \quad (n = 0, 1, 2, \dots, N); \quad k^2 \rightarrow 1),$$

which, in view of the relationship (9), yields the asymptotic expansion [cf. Equation (48)]:

$$H(\nu, m, \kappa) \sim \frac{\sqrt{2\pi}}{\Gamma(\nu)} k^{-\frac{1}{2}} (1 - \kappa)^{\frac{1}{2}-\nu} \sum_{n=0}^N \frac{\Gamma(m + n + \frac{1}{2}) \Gamma(\nu - n - \frac{1}{2})}{n! \Gamma(m - n + \frac{1}{2})} \left(-\frac{1 - \kappa}{2\kappa}\right)^n \quad (60)$$

$$(m = 0, 1, 2, \dots; \quad \Re(\nu - n - \frac{1}{2}) > 0 \quad (n = 0, 1, 2, \dots, N); \quad \kappa \rightarrow 1).$$

Both (59) and (60) are derivable also from the general results (51) and (48), respectively.

Acknowledgements

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

References

- [1] B.N. Al-Saqabi, A generalization of elliptic-type integrals, *Hadronic J.* **10**(1987), 331-337.
- [2] S. Bromberg, Resultados sobre una integral elíptica generalizada, *Rev. Técn. Fac. Ingr. Univ. Zulia* **15**(1992), no. 1, 29-35.
- [3] G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation* (Translated from the German by W. Nader), Springer-Verlag, New York, Heidelberg, and Berlin, 1974.
- [4] L.F. Epstein and J.H. Hubbell, Evaluation of a generalized elliptic-type integral, *J. Res. Nat. Bur. Standards B: Math. and Math. Phys.* **67B**(1963), 1-17.
- [5] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vols. **I** and **II**, McGraw-Hill, New York, Toronto, and London, 1953.
- [6] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Corrected and Enlarged Edition prepared by A. Jeffrey and Incorporating the Fourth Edition prepared by Yu. V. Geronimus and M. Yu. Tseytlin), Academic Press, New York, London, Toronto, and Tokyo, 1980.
- [7] S.L. Kalla, Results on generalized elliptic-type integrals, in *Mathematical Structures - Computational Mathematics - Mathematical Modelling*, Vol. **2** (B. Sendov, Editor), Bulgarian Academy of Sciences, Sofia, 1984, 216-219.
- [8] S.L. Kalla and B. Al-Saqabi, On a generalized elliptic-type integral, *Rev. Brasileira Fís.* **16**(1986), 145-156.
- [9] S. Kalla and B. Al-Saqabi, Some results for a class of generalized elliptic-type integrals, *Resultate Math.* **20**(1991), 507-516.
- [10] S.L. Kalla, S. Conde, and J.H. Hubbell, Some results on generalized elliptic-type integrals, *Applicable Anal.* **22**(1986), 273-287.
- [11] A.H. Mohamed-Murid, *Numerical Computation of Some Complete Elliptic Integrals of the First and Second Kinds*, M.S. Thesis, Iowa State University of Science and Technology, Ames, Iowa, 1986.

- [12] E.D. Rainville, *Special Functions*, Macmillan, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.
- [13] R.N. Siddiqi, On a class of generalized elliptic-type integrals, *Rev. Brasileira Fis.* **19**(1989), 137-147.
- [14] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Second Edition, Cambridge University Press, Cambridge, London, and New York, 1944.
- [15] G.H. Weiss. A note on a generalized elliptic integral, *J. Res. Nat. Bur. Standards B: Math. and Math. Phys.* **68B**(1964), 1-2.