Non 3-Choosable Bipartite Graphs and The Fano Plane

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DMS-854-IR May 2003

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Abstract

It is known that the smallest complete bipartite graph which is not 3-choosable has 14 vertices. We show that the extremal configuration is unique.

1 Introduction

A *list colouring* of a graph G is a proper vertex colouring of G in which every vertex v is assigned a colour from a set (or list) $L(v)$ of colours. The graph G is k*-choosable*, or k*-list colourable*, if the condition that every list has size at least k is sufficient to guarantee the existence of a list colouring of G. The *choice number*, or *list chromatic number*, of G is the smallest positive integer k such that G is k-choosable. A recent survey on list colourings is [5].

Both Erdös, Rubin and Taylor [2] and Vizing [4] observed that the choice number of a complete bipartite graph can be arbitrarily large. Erdos, Rubin and Taylor demonstrated that the choice number of $K_{7,7}$ is at least four. Their example consisted of assigning the seven lines in the Fano plane to the seven vertices in each set in the bipartition. It is proved in [3] that the choice number of any bipartite graph with at most thirteen vertices is less than or equal to three.

In this report we show that the fourteen vertex example of Erdös, Rubin and Taylor is unique up to renaming the colours. The proof is a detailed case analysis. The cases to be considered are determined in Lemma 2.8. Our initial goal was to undertake a combinatorial computing project and examine all possibilities, up to isomorphism. In order to be able to limit the size of the search, results were established regarding the structure of the lists in an example where a list colouring does not exist. Eventually there were enough such results for a proof.

For a discussion of the many other contexts in which the Fano configuration arises, see [1].

2 Preparatory Results

We assume throughout this article that $K_{a,c}$ is a complete bipartite graph, where $a \leq c$, in which lists of size three are assigned to the vertices. We use $\mathbf{A} = A_1, A_2, \ldots, A_a$ to denote the collection of lists assigned to the vertices in one set of the bipartition, and $\mathbf{C} = C_1, C_2, \ldots, C_c$ to denote the collection of lists assigned to the vertices in other set of the bipartition. Define $N = |A_1 \cup A_2 \cup \cdots \cup A_a \cup C_1 \cup C_2 \cup \cdots \cup C_c|$. For integers $a \leq b$ we use $[a, b]$ to denote the set $\{a, a+1, \ldots, b\}$. We will use the following results from [3].

Theorem 2.1 ([3]) *Every bipartite graph on at most thirteen vertices is 3-choosable.*

Let $\mathbf{X} = X_1, X_2, \ldots, X_n$ be a collection of sets. A *transversal* of $\mathbf{X}$ is a set $T \subseteq X_1 \cup X_2 \cup \cdots \cup X_n$ such that $T \cap X_i \neq \emptyset$ for $i = 1, 2, \ldots, n$.

Lemma 2.2 ([3]) *Suppose the complete bipartite graph $K_{a,c}$ has the collections $\mathbf{A}$ and $\mathbf{C}$ as the lists assigned to vertices in the sets in the bipartition. The following are equivalent:*

1. *$K_{a,c}$ is not list colourable from the lists in $\mathbf{A}$ and $\mathbf{C}$.*
2. *Every transversal of $\mathbf{A}$ has a subset that is a list in $\mathbf{C}$.*
3. *Every transversal of $\mathbf{C}$ has a subset that is a list in $\mathbf{A}$.*
4. *Each subset of $[1, N]$ is either disjoint from some list in $\mathbf{A}$ or has a subset that is a list in $\mathbf{C}$.*

Lemma 2.3 ([3]) *If $K_{a,c}$ is not 3-choosable, then $N \geq 5$.*

Theorem 2.4 ([3]) *If $K_{a,c}$ is not 3-choosable, then $a \cdot \binom{N-3}{l} + c \cdot \binom{N-3}{l-3} \geq \binom{N}{l}$ for all $0 \leq l \leq N$ where $\binom{N-k}{l} = 0$ for $l > N-3$ and $\binom{N-k}{l-k} = 0$ for $l < k$.*

Corollary 2.5 ([3]) *If $K_{a,c}$ is not 3-choosable, then $a + c \geq \frac{2\binom{N}{l}}{\binom{N-3}{l}+\binom{N-3}{l-3}}$ for all $0 \leq l \leq N$ where $\binom{N-k}{l} = 0$ for $l > N-3$ and $\binom{N-k}{l-k} = 0$ for $l < k$.*

We will make repeated, often implicit, use of the following fact.

Proposition 2.6 ([3]) *Suppose G is a bipartite graph on fourteen vertices and there is an assignment of lists for which there is no list colouring of G. Then every colour appears in lists on both sides of the bipartition, and no list is assigned to two vertices on the same side of the bipartition.*

We will say that the pair (x, y) is in the set X when $\{x, y\} \subseteq X$.

Lemma 2.7 ([3]) *Let G be a bipartite graph in which lists have been assigned to the vertices of G. Let $N = \left|\bigcup_{v\in V} L(v)\right|$. Suppose there is no list colouring of G and some pair of colours does not appear together in a list. Then there is a collection of lists $L'(v), v \in V$, such that $|L'(v)| = |L(v)|$ for every $v \in V$, and $\left|\bigcup_{v\in V} L(v)\right| = N - 1$.*

Lemma 2.8 *Suppose $a + c = 14$. If there is a collection of lists of size three for which there is no list colouring of $K_{a,c}$, then there is such a collection in one of the following situations:*

1. *$N = 7$ and $a = 7$*
2. *$N = 8$ and $a =$ 5, 6 or 7.*
3. *$N = 9$ and $a =$ 5, 6 or 7.*

Proof. Suppose $K_{a,c}$ is not 3-choosable. The fourteen lists of size three together contain 42 not necessarily distinct pairs. Since $\binom{10}{2} = 45$, by Lemma 2.7 it may be assumed that $N \leq 9$.

By Lemma 2.3, $N \geq 5$. By Proposition 2.6 no list is assigned to two vertices on the same side of the bipartition. However, the inequality from Corollary 2.5 is satisfied for neither the pair $N = 5$ and $l = 2$, nor the pair $N = 6$ and $l = 3$. Therefore, $N \geq 7$. Further, using $N = 7$ and $l = 3$ in the formula of Theorem 2.4 we obtain $a \geq 7$. Using $N = 8$ and $l = 3$, we obtain $a \geq 5$, and using $N = 9$ and $l = 4$, we obtain $a \geq 5$. Finally, $a \leq c$ and $a + c = 14$ implies that $a \leq 7$ for any value of N. The result follows. □

Lemma 2.9 *Suppose $a + c = 14$. If any element is in more than $a - 3$ lists of $\mathbf{A}$ or $c - 3$ lists of $\mathbf{C}$, then there is a list colouring of $K_{a,c}$.*

Proof. Suppose $K_{a,c}$ is not 3-choosable, and that $\mathbf{A}$ and $\mathbf{C}$ are collections of lists from which there is no list colouring of $K_{a,c}$. By Lemma 2.8, we have $5 \leq a \leq 7$ and $7 \leq c \leq 9$.

Now suppose some element is in $a - 2$ lists of $\mathbf{A}$. Without loss of generality, assume $1 \in A_1 \cap A_2 \cap \cdots \cap A_{a-2}$. Then for any $x \in A_{a-1}$ and $y \in A_a$, $\{1, x, y\}$ is a transversal of $\mathbf{A}$. By Lemma 2.2, every transversal of $\mathbf{A}$ has a subset that is a list in $\mathbf{C}$. Hence, for every choice of x and y the elements in $\{1, x, y\}$ are all distinct and $\{1, x, y\}$ is a list in $\mathbf{C}$. Hence, $c = 9$. However, all nine lists in $\mathbf{C}$ contain 1. Therefore, $\{1\}$ is a transversal of $\mathbf{C}$, and, by Lemma 2.2, $K_{a,c}$ can be properly coloured, a contradiction. Thus no element is in more than $a-3$ lists of $\mathbf{A}$.

Suppose some element is in $c-2$ lists of $\mathbf{C}$. It can be similarly shown that $\mathbf{C}$ has nine transversals of size three. Since $a \leq 7$, this contradicts the fact that every transversal of $\mathbf{C}$ has a subset that is a list in $\mathbf{A}$. The result follows. □

Lemma 2.10 *Suppose $7 \leq N \leq 9$. If some element is in more than three lists of $\mathbf{A}$ (or $\mathbf{C}$), then there is a list colouring of $K_{7,7}$.*

Proof: Suppose $\mathbf{A}$ and $\mathbf{C}$ are collections of lists from which there is no list colouring of $K_{7,7}$. By Lemma 2.2, every transversal of $\mathbf{A}$ (respectively $\mathbf{C}$) has size at least three, and every transversal of $\mathbf{A}$ (respectively $\mathbf{C}$) of size three is a list in $\mathbf{C}$ (respectively $\mathbf{A}$).

Now suppose that some element is in four lists of $\mathbf{A}$. Without loss of generality, assume $1 \in A_i$ for all $i \in [1,4]$. By Lemma 2.9, $1 \notin A_5 \cup A_6 \cup A_7$. Hence, $|A_5 \cup A_6 \cup A_7| \leq 8$, and $|A_5 \cap A_6| + |A_5 \cap A_7| + |A_6 \cap A_7| \geq 1$. Also note that $A_5 \cap A_6 \cap A_7 = \emptyset$, since $\mathbf{A}$ has no transversal of size two.

We claim that $|A_i \cap A_j| \leq 1$ for any $i, j \in [5,7]$ where $i \neq j$. To see this, assume that $|A_5 \cap A_6| = 2$. Then, for any $x \in A_5 \cap A_6$ and any $y \in A_7$, $\{1, x, y\}$ is a transversal of $\mathbf{A}$. Since $\mathbf{A}$ has no transversal of size two, there are six distinct transversals of this type. Since each of these six transversals is a list in $\mathbf{C}$, there are six lists in $\mathbf{C}$ that contain 1. Lemma 2.9 then asserts that a list colouring exists, a contradiction. Hence, $1 \leq |A_5 \cap A_6| + |A_5 \cap A_7| + |A_6 \cap A_7| \leq 3$.

Case 1: Suppose $|A_5 \cap A_6| + |A_6 \cap A_7| + |A_5 \cap A_7| = 3$. Then any two of A_5, A_6 and A_7 have exactly one element in common. For any distinct $i, j, k \in \{5, 6, 7\}$, the set $\{1, x, y\}$ where $x \in A_i \cap A_j$ and $y \in A_k - (A_i \cap A_j)$ is a transversal of $\mathbf{A}$. Each such transversal is a list in $\mathbf{C}$. Hence, at least six lists in $\mathbf{C}$ contain 1. Lemma 2.9 then asserts that a list colouring exists, a contradiction.

Case 2: Suppose $|A_5 \cap A_6| + |A_6 \cap A_7| + |A_5 \cap A_7| = 2$. Without loss of generality, assume that $|A_5 \cap A_6| = 1$ and $|A_5 \cap A_7| = 1$. Since A_5, A_6 and A_7 have no common element, we may assume that $A_5 = \{2, 3, 4\}$, $A_6 = \{2, 5, 6\}$ and $A_7 = \{3, 7, 8\}$. Then $\{1, 2, 3\}$, $\{1, 2, 7\}$, $\{1, 2, 8\}$, $\{1, 3, 5\}$ and $\{1, 3, 6\}$ are all transversals of $\mathbf{A}$. This gives five lists of $\mathbf{C}$, each containing 1. Lemma 2.9 then asserts that a list colouring exists, a contradiction.

Case 3: Suppose $|A_5 \cap A_6| + |A_6 \cap A_7| + |A_5 \cap A_7| = 1$. Without loss of generality, we may assume $A_5 = \{2, 3, 4\}$, $A_6 = \{2, 5, 6\}$ and $A_7 = \{7, 8, 9\}$. Then $\{1, 2, 7\}$, $\{1, 2, 8\}$ and $\{1, 2, 9\}$ are transversals of $\mathbf{A}$ and therefore lists in $\mathbf{C}$. Suppose these are C_1, C_2 and C_3, respectively.

For any $x \in A_5 - \{2\}$, $y \in A_6 - \{2\}$ and $z \in A_7$, the set $\{1, x, y, z\}$ is a transversal of $\mathbf{A}$. Let S be the set of these twelve distinct transversals. Note that none of C_1, C_2 or C_3 is a subset of any set in S, and no list in $\mathbf{C}$ is a subset of more than three sets in S. Furthermore, a list C_i is a subset of three sets in S only if it contains 1. Since every set in S has a subset that is a list in $\mathbf{C}$, each of C_4, C_5, C_6 and C_7 must be a subset of exactly three sets in S. Therefore, $1 \in C_i$ for all $i \in [4,7]$. However, this means that $\{1\}$ is a transversal of $\mathbf{C}$ which is a contradiction.

Hence, if some element appears in either three lists of $\mathbf{A}$ or three lists of $\mathbf{C}$, then G can be properly coloured. $\square$

3 Nine Colours

Lemma 3.1 *Suppose $N = 9$. T If there is no list colouring of $K_{5,9}$ then no pair appears in more than one list in $\mathbf{A}$.*

Proof: Suppose that $\mathbf{A}$ and $\mathbf{C}$ are families of lists for which there is no list colouring of $K_{5,9}$. Then, by Lemma 2.2, every transversal of $\mathbf{A}$ has size at least three, and every transversal of $\mathbf{A}$ of size three is a list in $\mathbf{C}$.

By Lemma 2.9, no element appears in more than two lists of $\mathbf{A}$. Since every element appears in at least one list of $\mathbf{A}$, there are six elements in exactly two lists each and three elements in exactly one list each. Say every element in $[1,6]$ is in exactly two lists of $\mathbf{A}$.

Suppose some pair, say $(1,2)$, is in two lists, say A_1 and A_2. Then there are two elements of $[3,6]$ each of which is in neither A_1 nor A_2. Assume $3 \in A_3 \cap A_4$ and 4 is in two of A_3, A_4 and A_5.

Suppose $4 \in A_3 \cap A_4$. Then for each $x \in A_1 \cap A_2$ and $y \in A_3 \cap A_4$, and $z \in A_5$, $\{x, y, z\}$ is a transversal of $\mathbf{A}$. There are twelve such transversals. This contradicts the fact that every transversal of size three is a list in $\mathbf{C}$.

Suppose $4 \in A_3 \cap A_5$. Then $\{1,3,4\}$ and $\{2,3,4\}$ are transversals of $\mathbf{A}$. It is also the case that for each $x \in A_5 - \{4\}$, $\{1,3,x\}$ and $\{2,3,x\}$ are transversals of $\mathbf{A}$. And, for each $y \in A_4 - \{3\}$, both $\{1,4,y\}$ and $\{2,4,y\}$ are transversals of $\mathbf{A}$. This gives a total of ten transversals of $\mathbf{A}$ of size three. This contradicts the fact that every transversal of $\mathbf{A}$ of size three is a list in $\mathbf{C}$. $\square$

Lemma 3.2 *Suppose $N = 9$. If there is no list colouring of $K_{5,9}$, then no element appears in more than four lists of $\mathbf{C}$.*

Proof: Suppose that $\mathbf{A}$ and $\mathbf{C}$ are families of lists for which there is no list colouring of $K_{5,9}$. Then, by Lemma 2.2, every transversal of $\mathbf{C}$ has size at least three, and every transversal of $\mathbf{C}$ of size three is a list in $\mathbf{A}$.

By Lemma 2.9, no element is in more than six lists in $\mathbf{C}$ and no element is in more than two lists of $\mathbf{A}$.

Suppose some element is in exactly six lists of $\mathbf{C}$. Say $x \in C_1 \cap C_2 \cap \cdots \cap C_6$, but $x \notin C_7 \cup C_8 \cup C_9$. Then $|C_7 \cup C_8 \cup C_9| \leq 8$ and two of C_7, C_8 and C_9 contain a common element. Say $y \in C_7 \cap C_8$. Then for every $z \in C_9$, $\{x, y, z\}$ is a transversal of $\mathbf{C}$. Since $\mathbf{C}$ has no transversal of size two, there are three transversals of $\mathbf{C}$ of size three. Hence, there are 3 lists in $\mathbf{A}$ containing x. This is a contradiction to Lemma 2.9.

Suppose some element is in exactly five lists of **C**. Say $x \in C_1 \cap C_2 \cap \cdots \cap C_5$, but $x \notin C_6 \cup C_7 \cup C_8 \cup C_9$. Then $|C_6 \cup C_7 \cup C_8 \cup C_9| \leq 8$ and at least two of C_6, C_7, C_8 and C_9 have an element in common. Suppose $y \in C_6 \cap C_7$. If $y \in C_8 \cup C_9$, then there are three transversals of **C** containing y. Hence, there are three lists in **A** containing y. This is a contradiction to Lemma 2.9. Hence, we may assume that $y \notin C_8 \cup C_9$. Suppose $C_8 \cap C_9 \neq \emptyset$. Then $\{x, y, z\}$ is a transversal of **C** for each $z \in C_8 \cap C_9$. Since **A** has at most one list containing the pair (x, y), it must be the case that $|C_8 \cap C_9| \leq 1$.

Suppose $|C_8 \cap C_9| = 1$. Say $r \in C_8 \cap C_9$, $\{x, y, r\}$ is a transversal of **C** and therefore a list in **A**. For each $z \in C_8 - C_9$ and $w \in C_9 - C_8$, the set $\{x, y, z, w\}$ is a transversal of **C**. There are four such transversals. Since $\{x, y, r\}$ is the only list in **A** containing the pair (x, y), no list in **A** is a subset of more than one of these transversals. There is one list in **A** containing x but not y, and one list containing y but not x. Each such list is a subset of at most one of these transversals. Hence, at least two of these transversals have no subset in **A**.

Suppose $C_8 \cap C_9 = \emptyset$. For each $z \in C_8$ and $w \in C_9$, $\{x, y, z, w\}$ is a transversal of **C**. There are nine such transversals. At most one list in **A** contains the pair (x, y). This list is a subset of at most three of the nine transversals. A list containing exactly one of x or y is a subset of at most one of these transversals. Hence, at most four of these transversals have a subset in **A**, so there is a transversal of **C** that has no subset in **A**, contrary to Lemma 2.2. □

Lemma 3.3 *If $N = 9$, then there is a list colouring of $K_{5,9}$.*

Proof: Suppose there is no list colouring of $K_{5,9}$. By Lemma 2.9 no element is in more than two lists of **A**. By Proposition 2.6, every element is in at least one list of **A**. Then we may assume that every element in $[1, 6]$ appears in exactly two lists of **A**. Suppose $1 \in A_1 \cap A_2$. Then there is some element from $[2, 6]$ is not in $A_1 \cup A_2$. Say $2 \in A_3 \cap A_4$.

Suppose some element from $[3, 6]$ is in the list A_5. Since 3 is in two lists, we may assume that $3 \in A_5 \cap A_1$. Then $\{1, 2, 3\}$ is a transversal of **A**, as is $\{1, 2, x\}$ for each $x \in A_5 - \{3\}$ and $\{2, 3, y\}$ for each $y \in A_2 - \{1\}$. This gives five transversals of **A** of size three, each containing 2. By Lemma 2.2 each of these transversals is a list in **C**. This contradicts Lemma 3.2.

Hence, $A_5 = \{7, 8, 9\}$. By Lemma 3.1 every pair appears in at most one list of **A**. So, without loss of generality, we can assume that $A_1 = \{1, 3, 4\}$ and $A_2 = \{1, 5, 6\}$. Without loss of generality, assume $3 \in A_3$. Similarly, $4 \in A_3 \cup A_4$. Since the pair $(3, 4)$ is in at most one list of **A**, $4 \in A_4$. By a similar argument, we may assume, without loss of generality, that $5 \in A_3$ and $6 \in A_4$. Therefore, for each $x \in \{7, 8, 9\}$, the sets $\{1, 2, x\}$, $\{3, 6, x\}$, $\{4, 5, x\}$ are all transversals of **A**. By Lemma 2.2, each of these transversals is a list in **C**. It then follows that $\{1, 3, 5\}$ is a transversal in **C**. However, this transversal is not a list in **A** which is a contradiction to Lemma 2.2. Hence, G can not be properly coloured. □

Lemma 3.4 *Suppose $N = 9$. If there is no list colouring of $K_{6,8}$, then no element is in more than four lists of* **C**.

Proof: Suppose that **A** and **C** are collections of lists from which there is no list colouring of $K_{6,8}$. Then, by Lemma 2.2, every transversal of **C** has size at least three, and every transversal of **C** of size three is a list in **A**.

Suppose some element is in five lists of **C**. Without loss of generality, assume 1 is in lists C_1 through C_5. By Lemma 2.9, 1 is in none of C_6, C_7 or C_8. Hence, $C_6 \cup C_7 \cup C_8 \subseteq [2, 9]$, and some pair of C_6, C_7 and C_8 share a common element. Say $2 \in C_6 \cap C_7$. Then for any $x \in C_8$, $\{1, 2, x\}$ is a transversal of **C** and thus a list in **A** . Since **C** has no transversal of size two, this gives three lists of **A** containing the pair $(1, 2)$. Say these are lists A_1, A_2 and A_3. By Lemma 2.9, neither 1 nor 2 is in any of A_4, A_5 or A_6. Since $N = 9$, it must be the case that some pair of A_4, A_5 and A_6 have a common element. Say $y \in A_4 \cap A_5$. Hence, for all $z \in A_6$, $\{1, y, z\}$ and $\{2, y, z\}$ are transversals of **A** and therefore lists in **C**. Since A has no transversal of size two, this gives six lists in **C** each containing y. But Lemma 2.9 now implies that a list colouring exists, a contradiction. □

Lemma 3.5 *Suppose $N = 9$ and each element appears in exactly two lists of* **A**. *If some element appears in more than three lists of* **C** *then there is a list colouring of $K_{6,8}$.*

Proof: Suppose that every element appears in exactly two lists of **A**, but there is no list colouring of $K_{6,8}$. Then, by Lemma 2.2, every transversal of **A** (respectively **C**) has a subset that is a list in **C** (respectively **A**).

Suppose that some element is in four lists of **C**. Without loss of generality, assume $1 \in C_1 \cap C_2 \cap C_3 \cap C_4$. By Lemma 3.4, 1 is not in any other list of **C**. Since **C** has no transversal of size two, $C_5 \cap C_6 \cap C_7 \cap C_8 = \emptyset$. Furthermore, for any distinct $i, j, k \in [5, 8]$, $C_i \cap C_j \cap C_k = \emptyset$. Otherwise, $\{C_5, C_6, C_7, C_8\}$ would have three transversals of size two, resulting in three transversals of **C** of size three all containing 1. Since every element is in exactly two lists of **A**, this contradicts the fact that every transversal of **C** of size three is a list in **A**.

Since $C_5 \cup C_6 \cup C_7 \cup C_8 \subseteq [2, 9]$, some pair of C_5 through C_8 must have a common element. Without loss of generality, assume $2 \in C_5 \cap C_6$. Therefore, $2 \notin C_7 \cup C_8$.

Case 1: Suppose $|C_7 \cap C_8| = 2$. Then for any $x \in C_7 \cap C_8$, $\{1, 2, x\}$ is a transversal of **C**, and, therefore, a list in **A**. Let A_1 and A_2 be the lists resulting from these transversals. Now, $\{1, 2, y, z\}$ is a transversal of **C** where $y \in C_7 - C_8$ and $z \in C_8 - C_7$. Since neither A_1 nor A_2 is a subset of $\{1, 2, y, z\}$ and no other list in **A** contains either 1 or 2, then $\{1, 2, y, z\}$ has no subset in **A**, a contradiction.

Case 2: Suppose $|C_7 \cap C_8| = 1$. Then for $x \in C_7 \cap C_8$, $\{1, 2, x\}$ is a transversal of **C**. Say this is list A_1. There are also four transversals of the form $\{1, 2, y, z\}$ where $y \in C_7 - \{x\}$ and $z \in C_8 - \{x\}$. Note that A_1 is not a subset of any of these four transversals. Furthermore, no list is a subset of more than two of these transversals, and a list is a subset of two of them only if it contains the pair $(1, 2)$. Since each element appears in exactly two lists of $\mathbf{A}$, there is a single value of $i \in [2, 6]$ such that $1 \in A_i$. Similarly, there is a single value of $j \in [2, 6]$ such that $2 \in A_j$. If $i = j$, then A_i is a subset of at most two of the four transversals. If $i \neq j$, then each of A_i and A_j is a subset of at most one of the transversals. In either case, there are two transversals of $\mathbf{C}$ with no subset in $\mathbf{A}$, a contradiction.

Case 3: Suppose $C_7 \cap C_8 = \emptyset$. For any $y \in C_7$ and $z \in C_8$, $\{1, 2, y, z\}$ is a transversal of $\mathbf{C}$. Let S be the set of these nine transversals. A list in $\mathbf{A}$ is a subset of zero, one or three elements of S. It is a subset of three of them only if it contains $(1, 2)$ and it is a subset of one of them only if it contains 1 or 2 but not both. Since 1 and 2 each appear in exactly two lists of $\mathbf{A}$, at most six elements of S have a subset that is a list in $\mathbf{A}$. Therefore, at least three elements of S have no subset which is a list in $\mathbf{A}$, so there is a list colouring of $K_{6,8}$. □

Lemma 3.6 *If $N = 9$, then for any collection of lists $\mathbf{A}$ and $\mathbf{C}$, there is a list colouring of $K_{6,8}$*

Proof: Assume that $\mathbf{A}$ and $\mathbf{C}$ are collections of lists for which there is no list colouring of $K_{6,8}$. Then, by Lemma 2.2, every transversal of $\mathbf{A}$ (respectively $\mathbf{C}$) has a subset that is a list in $\mathbf{C}$ (respectively $\mathbf{A}$). Since $N = 9$, either some element appears in three lists of $\mathbf{A}$, or every element appears in exactly two lists of $\mathbf{A}$.

Case 1: Suppose there is an element that appears in three lists of $\mathbf{A}$. Without loss of generality, assume $1 \in A_1 \cap A_2 \cap A_3$. By Lemma 2.9, $A_4 \cup A_5 \cup A_6 \subseteq [2, 9]$. Then two of A_4, A_5 and A_6 have an element in common. Without loss of generality, assume $2 \in A_4 \cap A_5$. If $2 \in A_6$, then $\mathbf{A}$ has a transversal of size two, a contradiction. Hence, we may assume that $A_6 = \{3, 4, 5\}$. Then $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 2, 5\}$ are transversals of $\mathbf{A}$. We may assume these are lists C_1, C_2 and C_3, respectively.

Suppose $3 \in A_4$. Then for any $x \in A_5 - \{2\}$, $\{1, 3, x\}$ is a transversal of $\mathbf{A}$. Since there are no transversals of size two, $3 \notin A_5$, so there are two such transversals. If both of these lists are in $\mathbf{C}$, there are five lists in $\mathbf{C}$ containing 1, contrary to Lemma 3.4. Hence, $3 \notin A_4$. It can be similarly shown that $4, 5 \notin A_4$ and $3, 4, 5 \notin A_5$.

Suppose there is some $x \in (A_4 \cap A_5) - \{2\}$. Then $\{1, x, 3\}$, $\{1, x, 4\}$ and $\{1, x, 5\}$ are transversals of A and therefore lists in $\mathbf{C}$. These three lists together with C_1, C_2 and C_3 give six lists in $\mathbf{C}$ all containing 1, contrary to Lemma 3.4. Hence, $A_4 \cap A_5 = \{2\}$.

Now, for any $x \in A_4 - \{2\}$ and $y \in A_5 - \{2\}$, $\{1, 3, x, y\}$ is a transversal of $\mathbf{A}$. Let S be the set of these four distinct transversals. Note that none of C_1, C_2 or C_3 is a subset of any set in S.

If none of C_4 through C_8 contains the pair $(1,3)$, then for each choice of x and y, either $\{1,x,y\}$ or $\{3,x,y\}$ is a list in $\mathbf{C}$. This means C_4 through C_8 contain two lists with 1 or three lists with 3. Thus there are five lists of $\mathbf{C}$ containing 1 or five lists of $\mathbf{C}$ containing 3, contrary to Lemma 3.4. Therefore, at least one of C_4 through C_8 contains the pair $(1,3)$. Without loss of generality, assume $(1,3)$ is in C_4.

Now, let S' be set of elements of S that have no subset in $\{C_1,C_2,C_3,C_4\}$. Since C_4 is a subset of at most two sets in S, $|S'| \geq 2$. By Lemma 3.4, none of C_5 through C_8 contains 1. Since each set in S' has a subset that is a list in $C' = \{C_5,C_6,C_7,C_8\}$, at least two lists in C' contain 3. Say $3 \in C_5 \cap C_6$. Hence, $2 \in C_1 \cap C_2 \cap C_3$ and $3 \in C_4 \cap C_5 \cap C_6$. Hence, for any $x \in C_7$ and $y \in C_8$, $\{2,3,x,y\}$ is a transversal of $\mathbf{C}$. Let T be the set of these transversals. Since $1 \notin C_7 \cup C_8$, none of A_1, A_2 or A_3 is a subset of any transversal of the form $\{2,3,x,y\}$. Since the pair $(2,3)$ is in no list of $\mathbf{A}$, there is no choice of x and y such that $x=y$. Hence, $|T| = 9$, and a list of $\mathbf{A}$ is a subset of at most one set in T. Hence, for each choice of x and y, either $\{2,x,y\}$ or $\{3,x,y\}$ is a list in $\mathbf{A}$. This contradicts the fact that $\mathbf{A}$ contains six lists.

Case 2: Suppose every element appears in exactly two lists of $\mathbf{A}$. Without loss of generality, assume $1 \in A_1 \cap A_2$. Obviously, some element from $[2,9]$ is in neither A_1 nor A_2. Suppose $2 \notin A_1 \cup A_2$. Without loss of generality, assume $2 \in A_3 \cap A_4$. Then for any $x \in A_5$ and $y \in A_6$, $\{1,2,x,y\}$ is a transversal of $\mathbf{A}$. Let S be the set of these transversals. Let $\mathbf{C}_{1,2}$ be the number of lists in $\mathbf{C}$ containing the pair $(1,2)$, and assume $\mathbf{C}_{1,2} \leq 2$.

Suppose $|A_5 \cap A_6| = 2$. Then, without loss of generality, $A_5 = \{3,4,5\}$ and $A_6 = \{3,4,6\}$. Then $\{1,2,3\}$ and $\{1,2,4\}$ are lists in $\mathbf{C}$. Since $\{1,2,5,6\}$ is a transversal of $\mathbf{A}$ and $C_{1,2} \leq 2$, without loss of generality $\{1,5,6\}$ is a list in $\mathbf{C}$. If $|A_3 \cap A_4| \geq 2$ then, for any $x \in (A_3 \cap A_4) - \{2\}$, both $\{1,x,3\}$ and $\{1,x,4\}$ belong to $\mathbf{C}$, contrary to Lemma 3.5. Thus $|A_3 \cap A_4| = 1$. Let $A_3 = \{2,w,x\}$ and $A_4 = \{2,y,z\}$. Then $\{1,3,w,y\}, \{1,3,w,z\}, \{1,3,x,y\}$ and $\{1,3,x,z\}$ are all transversals of $\mathbf{A}$. Each of these has a subset that is a list in $\mathbf{C}$. Since 1 already belongs to three lists in C, it can not belong to any of these subsets. Thus, 3 belongs to all four subsets, contrary to Lemma 3.5.

Suppose $|A_5 \cap A_6| = 1$. The situation where $|A_1 \cap A_2| = 2$ or $|A_3 \cap A_4| = 2$, is covered by the argument in the previous paragraph, so assume $|A_1 \cap A_2| = |A_3 \cap A_4| = 1$. Let $A_1 = \{1,u_1,u_2\}, A_2 = \{1,v_1,v_2\}, A_3 = \{2,w_1,w_2\}, A_4 = \{2,x_1,x_2\}, A_5 = \{3,y_1,y_2\}$ and $A_5 = \{3,z_1,z_2\}$. Then, for $i,j \in \{1,2\}$ we have $u_i \neq v_j$, $w_i \neq x_j$, and $y_i \neq z_j$. The set $\{1,2,3\}$ is a transversal of $\mathbf{A}$, hence it is a list in $\mathbf{C}$. Let $S = \{\{1,2,y_i,z_j\}, \{1,3,w_i,x_j\}, \{2,3,u_i,v_j\} : 1 \leq i,j \leq 2\}$. Then $|S| = 12$. Each element of S is a transversal of $\mathbf{A}$, so each has a subset that is a list in $\mathbf{C}$. It follows from the structure of the lists in $\mathbf{A}$ that no list in $\mathbf{C}$ is a subset of more than two elements of S. Suppose that $C_{1,2} = 2$, that is, a second list in $\mathbf{C}$ contains the pair $(1,2)$. This list is a subset of at most two elements of S, so there are at least five other lists in $\mathbf{C}$ that are subsets of elements of S. Each of these five must contain a 1, a 2 or a 3. By Lemma 3.5, no element is in more than 3 lists of $\mathbf{C}$. Since both 1 and 2 are already in two lists of $\mathbf{C}$, and 3 is in one list of $\mathbf{C}$, at most four further lists of $\mathbf{C}$ can contain a 1, a 2, or a 3. Therefore $C_{1,2} = 1$. There are at least six lists in $\mathbf{C}$ that are subsets of elements of S, and each of these contains a 1, a 2, or a 3. Since $\{1,2,3\}$ is an element of $\mathbf{C}$, the set $\{1,2,3\}$ has non-empty intersection with at least seven of the eight lists in $\mathbf{C}$. Therefore either $\{1,2,3\}$ or $\{1,2,3,p\}$ is a transversal of $\mathbf{C}$, for some $p \in [4,9]$. No such transversal has a subset that is a list in $\mathbf{A}$, a contradiction.

Suppose $|A_5 \cap A_6| = 0$. Then $\mathbf{A}$ has nine transversals of the form $\{1,2,x,y\}$, where $x \in A_5$ and $y \in A_6$. A list in $\mathbf{C}$ is a subset of zero, one or three of these. It is a subset of three of them only if it contains the pair $(1,2)$, and it is a subset of one of them only if it contains 1 or 2 but not both. Since $C_{1,2} \leq 2$, at most eight of these nine transversals have a subset that is a list in $\mathbf{C}$. Therefore, some transversal of $\mathbf{A}$ has no subset that is a list in $\mathbf{C}$ contrary to Lemma 2.2. Therefore, $C_{1,2} = 3$.

Without loss of generality, assume $C_1 = \{1,2,3\}$, $C_2 = \{1,2,4\}$ and $C_3 = \{1,2,5\}$. By Lemma 3.5, $1,2 \notin C_4 \cup C_5 \cup \cdots \cup C_8$. Therefore, each pair from $\{1,2\} \times [6,9]$ must appear in some list in $\mathbf{A}$. Hence, $A_1 \cup A_2 \subseteq \{1\} \cup [6,9]$ and $A_3 \cup A_4 \subseteq \{2\} \cup [6,9]$. Since each element is in exactly two lists of $\mathbf{A}$, this means $A_5 \cup A_6 \subseteq \{3,4,5\}$. This implies $A_5 = A_6$ which is a contradiction. Hence, there is a list colouring of $K_{6,8}$. □

Lemma 3.7 *Suppose $N = 9$. If any pair appears in more than two lists of $\mathbf{A}$ (or $\mathbf{C}$), then there is a list colouring of $K_{7,7}$.*

Proof: Suppose some pair, say $(1,2)$, is in each of A_1, A_2 and A_3. Since these three lists are all distinct, assume that $A_1 = \{1,2,3\}$, $A_2 = \{1,2,4\}$ and $A_3 = \{1,2,5\}$. By Lemma 2.10, we may assume no element appears in more than three lists of $\mathbf{A}$. Then $A_i \subseteq [3,9]$ for $i \in [4,7]$. Furthermore, by Lemma 2.2, every transversal of $\mathbf{A}$ of size three is a list in $\mathbf{C}$.

Let $Z = [1,5] \times [6,9]$. None of the lists A_1, A_2 and A_3 contains a pair from Z, and every other list in $\mathbf{A}$ or $\mathbf{C}$ contains either no pairs or exactly two pairs from Z. Therefore, $\mathbf{C}$ contains at most fourteen pairs from Z. By Lemma 2.7, we may assume that every pair in Z is in some list of $\mathbf{A}$ or $\mathbf{C}$. Since Z contains twenty pairs, $\mathbf{A}$ contains at least six pairs from Z. Without loss of generality, we may assume that A_4, A_5 and A_6 each contain two pairs from Z. Since each of these lists is a subset of $[3,9]$, $A_i \cap [3,5] \neq \emptyset$ for $i \in [4,6]$.

Case 1: Suppose none of 3, 4 or 5 is in two of the lists A_4, A_5, A_6 and A_7. Then $|A_i \cap [3,5]| = 1$ for $i \in [4,6]$ and $A_7 \subseteq [6,9]$. Without loss of generality, assume $A_7 = \{7,8,9\}$. Then exactly six pairs from Z are in $\mathbf{A}$ and fourteen pairs are in $\mathbf{C}$. Therefore, no pair from Z is in more than one list of $\mathbf{A}$ or $\mathbf{C}$.

Since A_4, A_5 and A_6 each contain two elements from $[6,9]$, either some element of $\{7,8,9\}$ is in two of the lists A_4, A_5 and A_6, or $6 \in A_i$ for every $i \in [4,6]$.

Suppose some element in $\{7, 8, 9\}$ is in two of A_5, A_6 and A_7. Without loss of generality, assume that $7 \in A_5 \cap A_6$. Then for any $x \in A_6 \cap \{3, 4, 5\}$ the sets $\{1, 7, x\}$ and $\{2, 7, x\}$ are transversals of $\mathbf{A}$ and, therefore, lists in $\mathbf{C}$. This contradicts the fact that each pair from Z is in exactly one list.

Hence, 6 is in all three lists, and $\{1, 6, 7\}$ and $\{1, 6, 8\}$ are both transversals of $\mathbf{A}$. Hence, these are lists in $\mathbf{C}$ which contradicts the fact that each pair from Z is in exactly one list. Therefore, there is a list colouring of $K_{7,7}$.

Case 2: Suppose one of 3, 4 or 5 is in two of the lists A_4, A_5, A_6 and A_7. Without loss of generality, assume that $3 \in A_4 \cap A_5$. Since 3 is in at most three lists in $\mathbf{A}$, $3 \notin A_6 \cup A_7$.

Suppose $x \in A_6 \cap A_7$. Then $\{1, 3, x\}$ and $\{2, 3, x\}$ are both transversals of $\mathbf{A}$, and, therefore, lists in $\mathbf{C}$. Say they are C_1 and C_2, respectively. If there exists $w \in (A_6 \cap A_7) - \{x\}$ then $\{1, 3, w\}$ and $\{2, 3, w\}$ are transversals of $\mathbf{A}$, and therefore lists in $\mathbf{C}$. Thus, four lists in $\mathbf{C}$ contain the element 3, contradicting Lemma 3.5. Therefore, $A_6 \cap A_7 = \{x\}$. Now, for any $y \in A_6 - \{x\}$ and $z \in A_7 - \{x\}$, the sets $\{1, 3, y, z\}$ and $\{2, 3, y, z\}$ are transversals of $\mathbf{A}$. This gives eight distinct transversals of $\mathbf{A}$. Now, for any $i \in [1, 7]$, if $3 \notin C_i$ then C_i is a subset of at most one of these transversals, otherwise it is a subset of at most three. Since C_1 and C_2 are subsets of none of these transversals and, by Lemma 2.10, at most one other list in $\mathbf{C}$ contains 3, some transversal does not have a subset in $\mathbf{C}$. Hence, by Lemma 2.2, a list colouring exists.

Suppose A_6 and A_7 are disjoint. Then there are nine transversals of $\mathbf{A}$ of the form $\{1, 3, x, y\}$ and nine of the form $\{2, 3, x, y\}$ where $x \in A_6$ and $y \in A_7$. However, at least four lists of $\mathbf{C}$ do not contain 3 and each one is a subset of at most one transversal. Since no list in $\mathbf{C}$ is a subset of more than three of these transversals, at least five transversals do not have a subset in $\mathbf{C}$. Hence, by Lemma 2.2, a list colouring exists. □

Corollary 3.8 *If the elements x and y each appear in three lists of $\mathbf{A}$ (respectively $\mathbf{C}$), but the pair (x, y) is in no list of $\mathbf{A}$ (respectively $\mathbf{C}$), then there is a list colouring of $K_{7,7}$.*

Proof: Without loss of generality, assume $1 \in A_i$ for $i = 1, 2, 3$ and $2 \in A_j$ for $j = 4, 5, 6$. Then for each $x \in A_7$, $\{1, 2, x\}$ is a transversal of $\mathbf{A}$. By Lemma 3.7, the pair $(1, 2)$ is in at most two lists of $\mathbf{C}$. Therefore, one of these transversals is not in $\mathbf{C}$ and, by Lemma 2.2, a list colouring exists. □

Lemma 3.9 *Suppose $N = 9$. If there is no list colouring of $K_{7,7}$, then for any collection of lists $\mathbf{A}$ and $\mathbf{C}$ there is a list A' in $\mathbf{A}$ such that*

1. $|A' \cap C_i| \leq 1$ *for all* $i \in [1, 7]$
2. *A' is disjoint from exactly three lists in $\mathbf{A}$.*

Proof: By Lemma 2.7, we may assume that every pair of elements appears in some list of **A** or **C**. Since each list contains three pairs, the lists in **C** together contain at most twenty-one of these pairs. Hence, the lists of **A** contain fifteen pairs that are not found in **C**. This means some list in **A** contains three pairs not found in **C**. Let A_1 be such a list. Hence, $|A_1 \cap C_i| \leq 1$ for all $i \in [1, 7]$.

Without loss of generality, assume that $A_1 = \{1, 2, 3\}$.

Case 1: Suppose that A_1 is disjoint from at most two lists in **A**. Without loss of generality assume that $A_1 \cap A_i \neq \emptyset$ for all $i \in [2, 5]$. Then for any $x \in A_6$ and $y \in A_7$, the set $\{1, 2, 3, x, y\}$ is a transversal of **A**. By Lemma 2.2, we may assume that each such transversal has a subset that is a list in **C**. Then for every $x \in A_6$ and $y \in A_7$, there is a list C_i such that $|C_i \cap \{1, 2, 3, x, y\}| = 3$. Since $|C_i \cap \{1, 2, 3\}| \leq 1$ for all $i \in [1, 7]$, A_1, A_6 and A_7 are pairwise disjoint. This means there are nine distinct transversals of the form $\{1, 2, 3, x, y\}$. Since no list in **C** is a subset of more than one of these transversals then at least two transversals of **A** have no subset in **C** which is contradiction. Hence, at least three lists in **A** are disjoint from A_1.

Case 2: Suppose that four lists of **A** are disjoint from A_1. These four lists along with A_1 do not contain any pair from the set $X = [1, 3] \times [4, 9]$. The remaining two lists in **A**, say A_2 and A_3, each contain at most two pairs from X. By Lemma 2.7 we may assume that every pair from X is in some list of **A** or **C**. Since the lists of **C** collectively contain at most fourteen of the pairs from X, then each of A_2 and A_3 contain exactly two pairs and C_i contains exactly two pairs for each $i \in [1, 7]$. Furthermore, no pair from X is in two lists of **A** or **C**.

For each $i \in [1, 3]$, define X_i to be the set $\{i\} \times [4, 9]$. Since each list in **C** contains a pair from X, $|C_i \cap A_1| \geq 1$ for all $i \in [1, 7]$. Hence $|C_i \cap A_1| = 1$ for all $i \in [1, 7]$. Therefore, each list in **C** contains either no pairs or exactly two pairs from X_1. Hence, the lists of **C** collectively contain an even number of pairs from X_1. Since X_1 contains six pairs, the lists A_2 and A_3 together contain an even number of pairs from X_1. Similar statements can be made regarding X_2 and X_3.

The collection **A** contains exactly four pairs from X. Suppose **A** contains four pairs from X_1, say, and none from X_2 and X_3. Then, **C** contains two pairs from X_1 and six from each of X_2 and X_3. Without loss of generality, $1 \in C_1, 2 \in C_2 \cap C_3 \cap C_4, 3 \in C_5 \cap C_6 \cap C_7$, and $2, 3 \notin C_1$. Thus, there exist two transversals of **C** of the form $\{x, 2, 3\}$, where $x \in C_1 - \{1\}$. But only one list in **A** contains the pair $(2, 3)$, so this is impossible. Thus, **A** must contain exactly two pairs from each of two of the sets X_1, X_2 and X_3. Without loss of generality, assume **A** contains no pair from X_1, and contains two pairs from each of X_2 and X_3.

If each of A_2 and A_3 were to contain one pair from each of X_2 and X_3, then the pair $(2, 3)$ would be in both A_2 and A_3. Since this pair is also in A_1, by Lemma 3.7 a list colouring would exist. Hence, we may assume that A_2 contains two pairs from X_2 and A_3 contains two pairs from X_3.

This means **C** contains six pairs from X_1, and four pairs from each of X_2 and X_3. Since $|C_i \cap \{1,2,3\}| = 1$ we may assume that $1 \in C_1 \cap C_2 \cap C_3$, $2 \in C_4 \cap C_5$ and $3 \in C_6 \cap C_7$. Since no two lists in **C** contain the same pair from X, $C_6 \cap C_7 = \{3\}$. Hence, there are four distinct transversals of **C** of the form $\{1,2,x,y\}$ where $x \in C_6 - \{3\}$ and $y \in C_7 - \{3\}$. By Lemma 2.2, we may assume that each of these transversals has a subset that is a list in **A**. However, A_1 is not a subset of any such transversal. Neither are any of the four lists in **A** disjoint from A_1, nor is A_3. This implies A_2 is a subset of all four transversals which is impossible.

Hence, A_1 is disjoint from exactly three lists in **A**. □

Theorem 3.10 *Suppose $N = 9$. Then for any collection of lists* **A** *and* **C**, *$K_{7,7}$ can be properly coloured.*

Proof: By Lemma 2.7, we assume that every pair is in some list of $\mathbf{A} \cup \mathbf{C}$, and, by Lemma 2.2, that every transversal of **A** (respectively **C**) has a subset that is a list in **C** (respectively **A**). By Lemma 3.9, we can assume that $A_1 = \{1,2,3\}$, $A_i \cap \{1,2,3\} \neq \emptyset$ for all $i \in [2,4]$, $A_i \cap \{1,2,3\} = \emptyset$ for all $i \in [5,7]$ and $|C_i \cap \{1,2,3\}| \leq 1$ for all $i \in [1,7]$. This means that **A** contains at least four and at most six pairs from the set $X = [1,3] \times [4,9]$ and **C** contains at least twelve and at most fourteen pairs from X. Hence, at least six lists in **C** contain an element from $\{1,2,3\}$.

Define X_1, X_2 and X_3 as in the previous proof.

Case 1: Suppose $C_i \cap \{1,2,3\} \neq \emptyset$ for all $i \in [1,7]$. By Lemma 2.10 and Corollary 3.8, we may assume that $1 \in C_1 \cap C_2 \cap C_3$, $2 \in C_4 \cap C_5$ and $3 \in C_6 \cap C_7$. Hence, **C** contains at most four pairs from each of X_2 and X_3. Hence, A contains at least two pairs from each of X_2 and X_3.

Suppose there is some $x > 3$ such that $x \in C_6 \cap C_7$. Then $\{1,2,x\}$ is a transversal of **C** and therefore a list in **A**. Say, $A_2 = \{1,2,x\}$. It is also follows that **C** contains exactly three pairs from X_3. So, **A** must contain three pairs from X_3. Therefore, we may assume that $3 \in A_3 \cap A_4$. Now, by Proposition 2.6, there is a second transversal of **C** of the form $\{1,2,y,z\}$ where $y \in C_6 - \{3,x\}$, $z \in C_7 - \{3,x\}$, and $y \neq z$. However, no list in **A** is a subset of this transversal which is a contradiction. Therefore, $C_6 \cap C_7 = \{3\}$. Similarly, $C_4 \cap C_5 = \{2\}$.

For any $x \in C_6 - \{3\}$ and $y \in C_7 - \{3\}$, the set $\{1,2,x,y\}$ is a transversal of **C**. There are four such transversals. For any $a \in C_4 - \{2\}$ and $b \in C_5 - \{2\}$ the set $\{1,3,a,b\}$ is also a transversal. There are four transversals of this type. Any list in **A** is a subset of at most two of these eight transversals, while the lists A_1, A_5, A_6 and A_7 are subsets of none of these. Hence, at most six of these transversals have a subset that is a list in **A**, a contradiction.

Case 2: Suppose exactly six lists in **C** contain an element from $\{1,2,3\}$ and some element in $\{1,2,3\}$ is in only one list of **C**. By Lemma 2.10 and Corollary 3.8, we may assume that $1 \in C_1 \cap C_2 \cap C_3$, $2 \in C_4 \cap C_5$, $3 \in C_6$ and $C_7 \cap \{1,2,3\} =$

$\emptyset$. Therefore, at most four pairs from X_2 and at most two pairs from X_3 are in $\mathbf{C}$. Therefore, $\mathbf{A}$ contains two pairs from X_2 and four pairs from X_3. Since A_2, A_3 and A_4 are the only lists of $\mathbf{A}$ containing any pair from X_3, 3 must be in at least two of these sets. In fact, 3 appears in exactly two of these sets since $3 \in A_1$ and no element appears in $\mathbf{A}$ four times. Assume $3 \in A_3 \cap A_4$. Then $A_3 \cup A_4 \subseteq [3,9]$. It follows that $2 \in A_2$ and $A_2 \subseteq \{2\} \cup [4,9]$.

Now, for any $x \in C_6 - \{3\}$ and $y \in C_7$, the set $\{1,2,x,y\}$ is a transversal of $\mathbf{C}$. The only list in $\mathbf{A}$ that could be a subset of any such transversal is A_2. Hence, there is no choice of x and y such that $x = y$. Therefore, there are four transversals of this type and A_2 is a subset of at most two of them. This implies that there are at least two transversals of $\mathbf{C}$ with no subset that is a list in $\mathbf{A}$, a contradiction.

Case 3: Exactly six lists in $\mathbf{C}$ contain an element from $\{1,2,3\}$ and every element in $\{1,2,3\}$ is in exactly two lists of $\mathbf{C}$. Without loss of generality, assume $1 \in C_1 \cap C_2$, $2 \in C_3 \cap C_4$, $3 \in C_5 \cap C_6$ and $C_7 \cap \{1,2,3\} = \emptyset$. Since $\mathbf{C}$ contains at least twelve pairs from X, $\mathbf{C}$ contains exactly four pairs from each of X_1, X_2 and X_3. Since $\mathbf{A}$ contains at most six pairs from X, then $\mathbf{A}$ contains two pairs from each of X_1, X_2 and X_3. Therefore, $\{1,2,3\} \subseteq A_2 \cup A_3 \cup A_4$.

Suppose that the pair $(1,2)$ appears in some list in $\mathbf{A}$ besides A_1. Without loss of generality assume that $1,2 \in A_2$. Since there is only one pair from each X_1 and X_2 in A_2, $1,2 \in A_3 \cup A_4$. Since the pair $(1,2)$ is in both A_1 and A_2, by Lemma 3.7, we may assume that $1 \in A_3$ and $2 \in A_4$. Since $A_i \subseteq [4,9]$ for $i = 5,6,7$ then for some $i \neq j$, $i,j \in [5,7]$, $A_i \cap A_j \neq \emptyset$. Assume that $4 \in A_5 \cap A_6$. Then for any $y \in A_7$, the set $\{1,2,4,y\}$ is a transversal of $\mathbf{A}$. Since no list in $\mathbf{C}$ contains the pair $(1,2)$ and every transversal of $\mathbf{A}$ of size three is a list in $\mathbf{C}$, $4 \notin A_7$. Hence, there are three transversals of $\mathbf{A}$ of the form $\{1,2,4,y\}$. Since no list in $\mathbf{C}$ contains both 1 and 2 then for any $y \in A_7$ there is a list $\{a,4,y\}$ in $\mathbf{C}$ for some $a \in \{1,2\}$. Hence, either two lists of $\mathbf{A}$ contain the pair $(1,4)$ or two lists contain the pair $(2,4)$. This contradicts the fact that no pair from X appears in more than one list. Hence, the pair $(1,2)$ is in no list of $\mathbf{A}$ besides A_1.

We may similarly assume that each of the pairs $(1,3)$ and $(2,3)$ appears in no list of $\mathbf{A}$ besides A_1. Hence, $|A_i \cap \{1,2,3\}| = 1$ for all $i = 2,3,4$. Since $\{1,2,3\} \subseteq A_2 \cup A_3 \cup A_4$, we may assume that $1 \in A_2$, $2 \in A_3$ and $3 \in A_4$.

Recall that $C_7 \subseteq [4,9]$. Without loss of generality, assume $C_7 = \{7,8,9\}$. Now , for any $a \in \{1,2,3\}$, at most two pairs from $\{a\} \times [7,9]$ are in $\mathbf{A}$. Hence, $\mathbf{C}$ contains at least one such pair. This means $C_7 \cap (C_{2i-1} \cup C_{2i}) \neq \emptyset$ for all of $i = 1,2$ and 3. Assume $C_7 \cap C_i \neq \emptyset$ for $i = 1,3,5$.

Let $C_2 = \{1,y,z\}$. Then for any $a \in C_1 \cap C_7$ the sets $\{2,3,a,y\}$ and $\{2,3,a,z\}$ are transversals of $\mathbf{C}$. Since the only list in $\mathbf{A}$ containing both 2 and 3 is $\{1,2,3\}$ there is no choice of a such that $a = y$ or $a = z$. Since A_3 and A_4 are the only lists besides $\{1,2,3\}$ containing 2 and 3 respectively, assume $A_3 = \{2,a,y\}$ and $A_4 = \{3,a,z\}$.

Let $C_4 = \{2, w, x\}$. Then for any $b \in C_3 \cap C_7$, $\{1, 3, b, w\}$ and $\{1, 3, b, x\}$ are transversals of $\mathbf{C}$. As before, we can conclude that $A_2 = \{1, b, w\}$ and $A_4 = \{3, b, x\}$. From the two representations of A_4, we have $\{a, z\} = \{b, x\}$. Since no pair in X is in both A_3 and C_4 then $\{w, x\} \cap \{a, y\} = \emptyset$. Hence, $b = a$. However, this means that A_2 and C_1 both contain the pair $(1, a)$ which contradicts the fact that no pair from X is in more than one list. □

4 Eight Colours

Lemma 4.1 *Suppose $a + c = 14$ and $N = 8$. If there is no list colouring of $K_{a,c}$. then one of the following holds.*

1. *For any $i \neq j$, $|A_i \cap A_j| \leq 1$.*
2. *For any $i \neq j$, $|C_i \cap C_j| \leq 1$.*
3. *For any i and j, $|A_i \cap C_j| \geq 1$*

Proof: Let T be the set of all 4-subsets of $[1, 8]$ and let X be any set in T. If X is a transversal of $\mathbf{A}$ then, by Lemma 2.2, $C_i \subset X$ for some $i \in [1, 7]$. If X is not a transversal, $A_i \cap X = \emptyset$ for some $i \in [1, 7]$. Hence, every set in T is either disjoint from some list in $\mathbf{A}$ or has a subset that is a list in $\mathbf{C}$.

Each list in $\mathbf{A}$ is disjoint from exactly five sets in T and each list in $\mathbf{C}$ is a subset of exactly five sets in T. Since T contains seventy sets and there are fourteen lists in $\mathbf{A} \cup \mathbf{C}$, then each set in T is either disjoint from exactly one list in $\mathbf{A}$ or has exactly one subset in $\mathbf{C}$, but not both. The result follows. □

Lemma 4.2 *Suppose $a + c = 14$ and $N = 8$. If some element is in more than $a - 4$ lists of $\mathbf{A}$ (respectively $\mathbf{C}$) then there is a list colouring of $K_{a,c}$.*

Proof: By Lemma 2.9, if some element is in more than $a - 3$ lists of $\mathbf{A}$, then $K_{a,c}$ can be properly coloured.

Suppose some element is in $a-3$ lists of $\mathbf{A}$. Without loss of generality, assume $1 \in A_1 \cap A_2 \cap \cdots \cap A_{a-3}$. Then $A_{a-2} \cup A_{a-1} \cup A_a \subseteq [2, 8]$. Therefore, some pair of A_{a-2}, A_{a-1} and A_a contain a common element. Assume $2 \in A_{a-2} \cap A_{a-1}$. Then for each $x \in A_a$ the set $\{1, 2, x\}$ is a transversal of $\mathbf{A}$. It follows that either one of these transversals is not a list in $\mathbf{C}$ or there are three lists in $\mathbf{C}$, each containing the pair $(1, 2)$. Hence, by Lemma 2.2 and Lemma 4.1,a list colouring exists. □

Theorem 4.3 *Suppose $a + c = 14$, and $N = 8$. Then for any collection of lists $\mathbf{A}$ and $\mathbf{C}$, there is a list colouring of $K_{a,c}$.*

Proof: Suppose $a = 5$. Since $N = 8$, some element appears in at least two lists of $\mathbf{A}$. Hence, there is an element in more than $a - 4$ elements of $\mathbf{A}$ and, by Lemma 4.2, a list colouring exists.

Suppose $a = 6$. Since $N = 8$, some element appears in three lists of $\mathbf{A}$. Therefore, by Lemma 4.2, a list colouring exists.

Suppose $a = 7$, and $\mathbf{A}$ and $\mathbf{C}$ are collections of lists for which $K_{7,7}$ has no list colouring. Then, by Lemma 4.2, no element is in more than three lists of $\mathbf{A}$. Hence, at least five elements are in exactly three lists of $\mathbf{A}$. It follows that for some A_i, every element in A_i appears in exactly three lists of $\mathbf{A}$. Assume that $A_1 = \{1, 2, 3\}$ and each element in A_1 is in three lists of $\mathbf{A}$. Furthermore, by Lemma 4.1, A_1 is the only list in $\mathbf{A}$ containing any of the pairs $(1, 2)$, $(1, 3)$ and $(2, 3)$. Hence, we may assume without loss of generality that $A_2 = \{1, 4, 5\}$, $A_3 = \{1, 6, 7\}$, $2 \in A_4 \cap A_5$, $3 \in A_6 \cap A_7$.

The sets $\{1, 2, 3\}$, $\{2, 3, 4, 6\}$, $\{2, 3, 4, 7\}$, $\{2, 3, 5, 6\}$ and $\{2, 3, 5, 7\}$ are all transversals of $\mathbf{A}$. Therefore, by Lemma 2.2, they have subsets that are lists in $\mathbf{C}$. Hence, we may assume that $C_1 = \{1, 2, 3\}$. Since the pair $(2, 3)$ is in at most one list of $\mathbf{C}$, either $\{2, 4, 6\}$ or $\{3, 4, 6\}$ is a list in $\mathbf{C}$. Without loss of generality, assume $C_2 = \{2, 4, 6\}$. By Lemma 4.1, it follows that $\{3, 4, 7\}$, $\{3, 5, 6\}$ and $\{2, 5, 7\}$ are lists of $\mathbf{C}$. We may assume these are lists C_3 through C_5, respectively.

By Lemma 4.1, $A_4 \cap C_3$ and $A_4 \cap C_4$ are both non-empty. Since 3 is not in A_4, $A_4 \cap \{4, 7\} \neq \emptyset$ and $A_4 \cap \{5, 6\} \neq \emptyset$. It can be similarly shown that $A_5 \cap \{4, 7\} \neq \emptyset$ and $A_5 \cap \{5, 6\} \neq \emptyset$. Hence, $A_4, A_5 \subseteq \{2, 4, 5, 6, 7\}$. Since A_4 and A_5 are the only lists in $\mathbf{A}$ containing 2, the pair $(2, 8)$ is in no list in $\mathbf{A}$. However, the pair $(2, 8)$ is in none of C_1, C_2 or C_5. Since these are the only lists in $\mathbf{C}$ containing 2, the pair $(2, 8)$ is not in any list of $\mathbf{C}$. By Lemma 2.7, this implies a list colouring exists, a contradiction. $\square$

5 Seven Colours

Lemma 5.1 *Suppose $N = 7$. Either every element appears in exactly three lists of $\mathbf{A}$ (respectively $\mathbf{C}$) or there is a list colouring of $K_{7,7}$.*

Proof: By Lemma 2.10 no element is in more than three lists in $\mathbf{A}$. Since $|\mathbf{A}| = 7$, each element is in exactly three lists in $\mathbf{A}$. $\square$

Lemma 5.2 *Suppose $N = 7$ and $\mathbf{A}$ and $\mathbf{C}$ are collections of lists for which $K_{7,7}$ is not list colourable. Then every pair of elements appears in exactly one list of $\mathbf{A}$ (respectively $\mathbf{C}$).*

Proof: By Lemma 2.10, no element is in more than three lists of $\mathbf{A}$ or three lists of $\mathbf{C}$. Since $a = c = 7$, each element must appear in exactly three lists in each of $\mathbf{A}$ and $\mathbf{C}$.

Now suppose the pair $(1, 2)$ is in no list of $\mathbf{A}$. We may assume, without loss of generality, that $1 \in A_1 \cap A_2 \cap A_3$, $2 \in A_4 \cap A_5 \cap A_6$ and $A_7 = \{3, 4, 5\}$. Then the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 2, 5\}$ are transversals of $\mathbf{A}$, and, by Lemma 2.2, lists in $\mathbf{C}$.

Since 6 is in exactly three lists of **A**, it must either be in at least two of A_1, A_2 and A_3 or at least two of A_4, A_5 and A_6. Without loss of generality, assume $6 \in A_1 \cap A_2$. Consider any $x \in A_3 - \{1,7\}$. Then $x \in [3,6]$. If $x = 6$ then the set $\{2,3,6\}$ is a transversal of **A**, otherwise the set $\{2,6,x\}$ is a transversal of **A**. In either case, the transversal is not a list in **C** since at most three lists of **A** contain 2. This contradicts Lemma 2.2.

Therefore, every pair of elements appears in some list in **A**. Since there are 21 such pairs and each of the seven lists in **A** contain exactly three, each pair appears in exactly one list in **A**. □

Lemma 5.3 *Suppose* $N = 7$ *and* **A** *and* **C** *are collections of lists for which* $K_{7,7}$ *is not list colourable. Then* $\mathbf{A} = \mathbf{C}$.

Proof: Without loss of generality, assume $A_1 = \{1,2,3\}$. Then, by Lemma 5.2, none of $(1,2)$, $(2,3)$ or $(1,3)$ is in A_i for any $i \in [2,7]$. By Lemma 2.10, each of 1, 2 and 3 is in exactly three lists. We may therefore assume, without loss of generality, that $1 \in A_2 \cap A_3$, $2 \in A_4 \cap A_5$ and $3 \in A_6 \cap A_7$. Hence, the set $\{1,2,3\}$ is a transversal of **A**. By Lemma 2.2, this set must be a list in **C**. Therefore, every list in **A** is also in **C** and it follows that $\mathbf{A} = \mathbf{C}$. □

Theorem 5.4 *Suppose* $a + c = 14$. *Then, for any assignment of lists of size three to its vertices, there is a list colouring of* $K_{a,c}$, *unless* $a = c = 7$ *and the collections of lists* **A** *and* **C** *are the following, up to isomorphism:*

$$\mathbf{A} = \mathbf{C} = \{1,2,3\}, \{1,4,7\}, \{1,5,6\}, \{2,4,6\}, \{2,5,7\}, \{3,4,5\}, \{3,6,7\}$$

Proof: It has been shown that unless $N = 7$ and $a = c = 7$ there is a list colouring of $K_{a,c}$. Hence, assume that $N = a = c = 7$. In **A** we are requiring seven 3-subsets of $[1,7]$ such that every pair appears in exactly one subset. This is a Steiner Triple System on seven points, and it is well known that there is only one of these up to isomorphism. □

Corollary 5.5 *Let* G *be a bipartite graph on at most fourteen vertices. Then* G *is 3-choosable unless* $G = K_{7,7}$.

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