Abstract

Supervisor: Professor F. I. Cooperstock

We develop new variational techniques, acting on classes of Lagrangians with the same functional dependence but arbitrary functional form, for the derivation of general, strongly conserved quantities, supplementing the usual procedure for deriving weak conservation laws via Noether's theorem. Using these new techniques we generate and generalize virtually all energy-momentum complexes currently known. In the process we discover and understand the reason for the difficulties associated with energy-momentum complexes in general relativity.

We study a Palatini variation of a novel Lagrangian due to Nissani. We find that Nissani's principal claim, that his Lagrangian specifies Riemannian geometry in the presence of a generalized matter tensor, is not in fact justifiable, and prove that his Lagrangian is not unique.

We speculate on the possibility of deriving a general-relativistic analog of Maxwell's current equation, a matter current equation, yielding an entirely new approach to the idea of energy-momentum in general relativity. We develop the $SL(2,C) \times U(1)$ spinor formalism naturally combining the gravitational and electromagnetic potentials in a single object—the spinor connection. Variably charged matter is rigourously introduced, through the use of spin densities, in the unified potential theories we develop.

We generate both the Einstein-Maxwell equations and new equations. The latter generalize both the Maxwell equation and the Einstein equation which includes a new "gravitational stress-energy tensor". This new tensor exactly mimicks the electromagnetic stress-energy tensor with Riemann tensor contractions replacing Maxwell tensor contractions. We briefly consider the introduction of matter. A Lagrangian generalizing the two spinor Dirac equations has no gravitational currents and the electromagnetic currents must be on the light cone. A Lagrangian generalizing the Pauli equations has both gravitational and electromagnetic currents. The equations of both Lagrangians demonstrate beautifully how the divergence of

the total stress-energy tensor vanishes in this formalism. In the theory of the generalized Einstein-Maxwell and Pauli equations we succeed in deriving an equation describing a generalized matter-charge current density.

Examiners:

Supervisor Dr. F. I. Cooperstock

Dr. A. Watton

Dr. G. G. Miller

Dr. C. S. Wu

Dr. A. Das

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Chapter 1

Introduction

In this dissertation we present new work on variational principles and gauge theories in general relativity. It basically consists of two parts: the development of variational techniques on classes of Lagrangians and the development of $SL(2,C)\times U(1)$ gauge theories of the combined gravitational-electromagnetic field, loosely bridged by a chapter in which we investigate a class of Lagrangians based on that of Nissani [23]. However, the underlying motivation for the work reflects a common theme; a search for a "good" description of energy-momentum in general relativity.

Conventional work in general relativity (and other classical theories) describes energy-momentum as a conserved quantity. This approach has been fairly well developed in terms of *weak* conservation laws (dependent on the field equations), generally derived via Noether's theorem. In the words of J.C. duPlessis [8]:

> "Conservation laws are mostly associated with the invariance properties of problems in the calculus of variations and a general procedure for obtaining such conserved quantities were (sic) laid down in 1918 by E. Noether. However, it is in fact the case that many so-called conservation laws exist independently of the particular variational principle employed to describe the physical situation. There appears to be no general formalism to accomodate these laws."

It is just such a general formalism for *strong* conservation laws (independent of the field equations) which is developed in the first part of this dissertation.

It is worth noting that, while we are here concerned primarily with general relativity, the techniques we develop are quite general and have a much wider applicability. We present canonical procedures for the manipulation of whole classes of Lagrangians that share the same transformation law and functional dependence, but are otherwise arbitrary in functional form, and for the derivation therefrom of generalized conserved quantities. These techniques are applicable to any type of Lagrangian or argument with a known transformation law.

Einstein's theory of general relativity provides an ideal example for the demonstration of these new procedures. When derived via a Hilbert variation the theory is of second order in the derivatives of the metric and, hence, considerably more complex than other theories. But general relativity presents other problems.

Along with the principle of equivalence, one of the cornerstones of general relativity is the idea of covariance. Just as quantum mechanics deems that good quantities be observable, general relativity requires them to be covariant[1]. Thus it is disconcerting that, while a covariant momentum vector does exist for a point particle, general energy-momentum complexes are *not* covariant—in sharp contrast to the stress-energy tensor which describes the non-gravitational energy-momentum density.

Another problem with energy-momentum complexes in general relativity is that they are valid only near infinity in asymptotically flat spacetimes (and usually only in asymptotically Cartesian coordinates). It has been argued that, in view of the (strong form of the) equivalence principle, a transformation to freely falling coordinates will eliminate the gravitational field and, hence, gravitational energy-momentum must be inherently unlocalizable. This argument is fallacious. "No Γ's means no 'gravitational field'..." (Misner, Thorne and Wheeler [21]) is simply wrong. The absence of a 'gravitational field', ie. curvature, is determined by the vanishing of the Riemann tensor. Synge [30] long ago suggested the retirement of the equivalence principle and Ohanion [24] has shown that, in the presence of tidal

[1] With the notable exception of the connection which, of course, occupies a special position in the theory.

effects (ie. any non-homogeneous gravitational field), the (strong form of the) equivalence principle fails even for arbitrarily small volumes—it isn't even locally true[2]. On the other hand it can be argued that localization is necessary (Rindler [29]). Briefly, in view of mass-energy equivalence we expect all energy—including gravitational energy—to gravitate, and thus its location should be significant in a theory of gravity. Peters [27] has shown that the location of the gravitational energy density can affect the predicted perihelion precession in a nonlinear extension of Newtonian gravity and that in general relativity this effect also depends on the trace of the gravitation stress. In any case both Weinberg's energy-momentum complex [32] and Penrose's quasi-local mass [26] are claimed to be local quantities; though neither is derived via a conservation law.

Finally we bring up a problem implied by the use of the plural in the previous paragraphs. In general relativity there are no fewer than three energy-momentum complexes in common use (Einstein [9], Landau and Lifshitz [17], Møller [22]) and an infinite number are known (Goldberg [14], Komar [15]). None has proven wholly satisfactory. This multiplicity is related to the non-covariant nature of these complexes and the freedom inherent in conservation laws. With an arbitrary choice for a transformation law *any* divergenceless quantity may be added, ad hoc, to a particular energy-momentum complex in order to generate another.

Thus we are led to the following objectives in our search for a good energy-momentum complex: it should be covariant and its derivation should be as unambiguous as possible so as to lead to a unique quantity. As a further bonus, we may also wish to demand symmetry so that it defines a conserved angular momentum complex.

Our attempt to attain these objectives will centre upon a powerful but little known resource; a set of invariance relations derived from the transformation laws of the Lagrangian and its arguments [6,20]. As we shall see, these invariance

[2]There is little doubt that a transformation to freely falling coordinates would *minimize* a local energy density, but this is hardly surprising. For example, the same is true for a particle's kinetic energy in special relativity.

relations are intimately connected to the variational process and may be used to "integrate"[3] conserved quantities in a very natural fashion. However, the functional derivatives of the Lagrangian, in terms of which the invariance relations are written, are not generally the covariant quantities we wish to work with in general relativity. Hence we follow duPlessis [8] and introduce tensor quantities concomitant to these functional derivatives whereupon we rewrite the invariance relations in tensor form[4].

We will make the derivation as unambiguous as possible by sticking as closely as we may to the actual quantities involved in the variation. We will convert the invariance relations to covariant form by substituting tensor concomitants, changing partial derivatives to covariant derivatives and simplifying only via exact cancellations and previously converted invariance relations. In the "integration" of the conserved quantity we will attempt to eliminate only those expressions which will *"integrate"* to zero, generally simplifying as above. As we shall see, the invariance relations provide a natural direction to the path we take.

Following these guidelines will provide a compact derivation of a number of new and well-known momentum complexes and generalizations thereof. The general expression from which particular complexes are generated will have several advantages. In particular, it will be mathematically simpler in that most of its properties may be deduced by inspection. While we will fail to attain our goals of covariance and locality, the reasons for the failure will become apparent, lying in the choice of the fundamental quantities on which the theory is based and the choice of the invariance group with which we generate the conserved quantities.

Another possible approach to conserved quantities is via currents. The Maxwell equations (and their generalizations, the Yang-Mills equations) appear

[3]Here and in the following we will loosely use the term "integrate" to represent the phrase "take the antidivergence."

[4]DuPlessis used the concomitant invariance relations to define ad hoc conserved quantities, but apparently knew nothing of their connection to the variational principle and that his ad hoc quantities appear in actual conservation laws.

in two forms

$$F^{ab}{}_{;b} = j^a \tag{1.1}$$

$$\varepsilon^{abcd} F_{cd;b} = 0 \tag{1.2}$$

where (1.1) clearly implies that $\sqrt{-g}\, j^a$ is conserved. As there is a natural general relativistic analog of (1.2) in the Bianchi identities, one is led to speculate about the possibility of a momentum density equation corresponding to (1.1). Unfortunately, general relativity, as usually formulated, exhibits no such correspondence. The closest analog is found via the Palatini variational principle, in which the metric and connection are assumed to be independent. The equation resulting from the variation of the connection then establishes the relation between them. But it is clear that this equation might be altered if we add to the Lagrangian further terms in the connection. In order to investigate this possibility we consider the Palatini variation of a class of Lagrangians based on that of Nissani [23].

Nissani presents a novel Lagrangian which, he claims, generates the Einstein equations and specifies Riemannian geometry (connection equals Christoffel symbol) in the presence of a generalized matter tensor. However we will show that, in fact, this latter claim is unjustified. Riemannian geometry must be assumed in order for the additional terms in the Euler-Lagrange equation to vanish properly. In addition we also present a new Lagrangian which possesses properties like Nissani's, demonstrating that Nissani's Lagrangian is not unique.

The use of alternate Lagrangians like that of Nissani will indeed affect the resulting equations in a Palatini variation, but, as yet, we will have seen no indication how this might lead us to some sort of energy-momentum current. However, part of the motivation for Nissani's work was to investigate the classical analog of Carmeli's [3,4,5] $SL(2,C)$ gauge theory Lagrangian for general relativity. If we also recall that our failure with regard to energy-momentum complexes was related to our choice of fundamental quantities and invariance group, it becomes natural to consider gauge theoretic formulations of general relativity based on alternate fundamental quantities.

Most gauge theories of general relativity are based on the group $SL(2,C)$. However we will choose the group $SL(2,C) \times U(1)$, with the section and spinor connection as fundamental quantities, because the resulting spinor formalism combines the gravitational and electromagnetic potentials in a single object—the spinor connection. Not only does this greatly facilitate comparison between the two fields, but, in a quantum mechanical sense at least, this constitutes a unification of gravity and electromagnetism. It is natural to consider the significance, if any, of this fact.

In general relativity, the usual unification criteria deal with fields and seem to go back to Einstein [10] who suggested two possible points of view, the first stronger and preferable, which we paraphrase here:

> (1) That the field appear as a unified covariant entity—ie. not separable, under the transformation group(s), into covariant parts—as per the Maxwell tensor.
> (2) That both the field equations and the Lagrangian be unified entities—ie. not separable into invariant parts—as per Maxwell's equations and the usual Maxwell Lagrangian.

Our field quantity will fail to conform to condition (1) and neither our field equations nor Lagrangians will satisfy condition (2). However, it is interesting that in electromagnetic theory, the paradigm of unification and the model behind Einstein's reasoning, both conditions follow from the unification (under the Lorentz and the $U(1)$ gauge transformations) of the electromagnetic potential. Of perhaps more importance is the fact that, since Einstein's day, the Aharanov-Bohm experiment has altered our ideas about the reality of the electromagnetic potential (see Feynman, et al [11]) and gauge potentials in general.

At issue is not which adjective may or may not be used in describing a theory but how much content may be squeezed into a theory based on a certain formalism. In this case the $SL(2,C) \times U(1)$ spinor formalism has enabled the definition of a unified gravitational-electromagnetic potential. If this unification has any significance we would expect the corresponding Euler-Lagrange equation to be an equation in the gravitational-electromagnetic field (unified or not) which, in the presence of matter, becomes a relation for a matter-charge current density. That is we expect a theory which if not of Yang-Mills type is at least a close relative.

We will present several Lagrangians that result in the Einstein-Maxwell equations and we find that the current equation is of the desired form but, as we should expect, only the charge current is nonzero. However it is possible to devise Lagrangians with which the gravitational analog of the electromagnetic current is also nonzero. The interpretation of the resulting system of equations is unclear since the Einstein equation has been inevitably "damaged" by the appearance of a new gravitational stress-energy tensor. But the form of the new ${}_{grav}T_{ab}$ is of some interest—proportional to a contraction of the Bel-Robinson tensor, it is exactly analogous to the electromagnetic stress-energy tensor with Riemann tensor contractions replacing those of the Maxwell tensor.

Finally, in an attempt to clarify the significance of these new equations, we briefly consider the introduction of matter; and in a spinor formulation of the theory it is natural to investigate spinor type matter (although we will not go so far as to attempt quantization). For a Lagrangian generating the two-spinor Dirac equations we find that there can be no gravitational currents and that electromagnetic currents must be on the light cone; that is particles must be massless. For a Lagrangian generating the Pauli equations (generalizations of the Klein-Gordon equations) we find that both gravitational and electromagnetic currents are allowed. The properties of this system favor a general relativistic gauge theory with the "damaged" form of the Einstein equation. Thus we ultimately arrive at a theory containing a fully generalized matter-charge current density, if only at the cost of "damaging" Einstein's equation.

Chapter 2

Invariance Properties

2.1 Introduction

The value of the variational principle in field theory lies in the generation of the Euler-Lagrange equations and the derivation of associated conserved quantities for a particular system or theory. While the techniques developed herein may be used to elucidate certain properties of the Euler-Lagrange equations, their primary advantage is in the improvement and generalization of the procedures for deriving conserved quantities. Central to these techniques will be the application of the invariance relations we derive in Section 2.3 of this chapter. However, as first derived, these relations will not be in covariant form. Thus in Section 2.4 we introduce the necessary tensor concomitants (due to duPleissus [8]) and, in Section 2.5, rewrite the invariance relations in terms of them.

In this and the following two chapters we will be considering a scalar density Lagrangian for both the gravitational and electromagnetic field. We include the electromagnetic part in the Lagrangian and derive its contribution to the conserved quantities, despite the fact that the electromagnetic energy-momentum density can be (and usually is) obtained directly from the Einstein-Maxwell equations. Generally, the electromagnetic energy-momentum density is derived as a conserved quantity only within the framework of classical electromagnetism. However, the derivation involves manipulations to ensure symmetry. We perform the operation here, in the presence of gravity, because the symmetrization process may also be applied to the gravitational part of the combined energy-momentum complex and provides insight into this and intermediate complexes. In doing so we are led (for the first time, to the best of the author's knowledge) to a derivation of the Landau and Lifshitz pseudotensor via a variational principle [6].

2.2 Definitions and Transformation Laws

In this and the following chapter we consider the general Hilbert variation of a scalar action

$$S = \int L\, d^4x \tag{2.1}$$

where the Lagrangian L is a scalar density with the functional dependence

$$L = L(g_{ab}, g_{ab,c}, g_{ab,cd}, \phi_a, \phi_{a,b}) \tag{2.2}$$

but as yet unknown functional form, ϕ_a is a vector and g_{ab} is a symmetric tensor. As is usual in a Hilbert variation we assume that our manifold is Riemannian but, in the interest of greater generality and, especially, to clarify and control the introduction and elimination of zeros in the upcoming "integration" of a general conserved quantity, we will not identify g_{ab} with the metric until we actually consider the specific case of gravitational and electromagnetic fields.

We introduce the following notation for the functional derivatives of L

$$\begin{aligned} \Lambda^{ab} &= \frac{\partial L}{\partial g_{ab}}, \qquad \Lambda^{abc} = \frac{\partial L}{\partial g_{ab,c}}, \qquad \Lambda^{abcd} = \frac{\partial L}{\partial g_{ab,cd}} \\ \Phi^{a} &= \frac{\partial L}{\partial \phi_a}, \qquad \Phi^{ab} = \frac{\partial L}{\partial \phi_{a,b}} \end{aligned} \tag{2.3}$$

which obey the symmetry relations

$$\Lambda^{ab} = \Lambda^{ba} \tag{2.4}$$

$$\Lambda^{abc} = \Lambda^{bac} \tag{2.5}$$

$$\Lambda^{abcd} = \Lambda^{bacd} = \Lambda^{abdc}. \tag{2.6}$$

Under a coordinate transformation $\tilde{x}^i = \tilde{x}^i(x^a)$ with the definitions

$$C = \det(C^a{}_i)$$
$$C^a{}_i = \frac{\partial x^a}{\partial \tilde{x}^i}, \qquad C^a{}_{ij} = \frac{\partial^2 x^a}{\partial \tilde{x}^i\,\partial \tilde{x}^j}, \qquad C^a{}_{ijk} = \frac{\partial^3 x^a}{\partial \tilde{x}^i\,\partial \tilde{x}^j\,\partial \tilde{x}^k} \tag{2.7}$$

the transformation laws of L and its arguments are

$$\tilde{L} = CL \tag{2.8}$$

$$\tilde{g}_{ij} = C^a{}_i C^b{}_j g_{ab} \tag{2.9}$$

$$\tilde{g}_{ij,k} = C^a{}_i C^b{}_j C^c{}_k g_{ab,c} + (C^a{}_{ik} C^b{}_j + C^a{}_i C^b{}_{jk}) g_{ab} \tag{2.10}$$

$$\begin{aligned}\tilde{g}_{ij,k\ell} = {} & C^a{}_i C^b{}_j C^c{}_k C^d{}_\ell g_{ab,cd} \\ & +(C^a{}_{i\ell} C^b{}_j C^c{}_k + C^a{}_i C^b{}_{j\ell} C^c{}_k + C^a{}_i C^b{}_j C^c{}_{k\ell} + C^a{}_{ik} C^b{}_j C^c{}_\ell + C^a{}_i C^b{}_{jk} C^c{}_\ell) g_{ab,c} \\ & +(C^a{}_{ik\ell} C^b{}_j + C^a{}_{ik} C^b{}_{j\ell} + C^a{}_{i\ell} C^b{}_{jk} + C^a{}_i C^b{}_{jk\ell}) g_{ab}\end{aligned} \tag{2.11}$$

$$\tilde{\phi}_i = C^a{}_i \phi_a \tag{2.12}$$

$$\tilde{\phi}_{i,j} = C^a{}_i C^b{}_j \phi_{a,b} + C^a{}_{ij} \phi_a. \tag{2.13}$$

In the next section we use these transformation laws to derive transformation laws for, and invariance relations in, the functional derivatives of L. As we will see, the quantities (2.3) are not all tensorial. Thus we find it convenient to introduce appropriate tensor concomitants in Section 2.4 and rewrite the invariance relations in terms of them in Section 2.5. While neither the invariance relations nor the tensor concomitants are new (see, for instance, [8,20]), they appear to have found little application in the literature. We will find that, using them, we may generate a surprising amount of general information, without reference to the exact functional form of any specific Lagrangian, which may be directly applied to the "integration" of a general conserved quantity in Chapter 3.

2.3 Invariance Relations

We now consider the transformation laws of $\tilde{L}(\tilde{g}_{ij}, \tilde{g}_{ij,k}, \tilde{g}_{ij,k\ell}, \tilde{\phi}_i, \tilde{\phi}_{i,j})$ and its arguments. Differentiating these relations with respect to the arguments of L will yield transformation laws for the functional derivatives of L. For example, differentiating (2.8) with respect to g_{ab} we find

$$\frac{\partial \tilde{L}}{\partial \tilde{g}_{ij}} \frac{\partial \tilde{g}_{ij}}{\partial g_{ab}} + \frac{\partial \tilde{L}}{\partial \tilde{g}_{ij,k}} \frac{\partial \tilde{g}_{ij,k}}{\partial g_{ab}} + \frac{\partial \tilde{L}}{\partial \tilde{g}_{ij,k\ell}} \frac{\partial \tilde{g}_{ij,k\ell}}{\partial g_{ab}} = C \frac{\partial L}{\partial g_{ab}} \tag{2.14}$$

or, introducing the notation (2.3) and noting (2.9–2.11),

$$\tilde{\Lambda}^{ij} C^a{}_i C^b{}_j + \tilde{\Lambda}^{ijk} (C^a{}_i C^b{}_j)_{,k} + \tilde{\Lambda}^{ijk\ell} (C^a{}_i C^b{}_j)_{,k\ell} = C \Lambda^{ab}. \tag{2.15}$$

Transformation laws for the rest of the quantities (2.3) are derived similarly. They are

$$\tilde{\Lambda}^{ijk} C^a{}_i C^b{}_j C^c{}_k + \tilde{\Lambda}^{ijk\ell} [(C^a{}_i C^b{}_j C^c{}_k)_{,\ell} + (C^a{}_i C^b{}_j)_{,k} C^c{}_\ell] = C \Lambda^{abc} \tag{2.16}$$

$$\tilde{\Lambda}^{ijk\ell} C^a{}_i C^b{}_j C^c{}_k C^d{}_\ell = C \Lambda^{abcd} \tag{2.17}$$

$$\tilde{\Phi}^i C^a{}_i + \tilde{\Phi}^{ij} C^a{}_{ij} = C \Phi^a \tag{2.18}$$

$$\tilde{\Phi}^{ij} C^a{}_i C^b{}_j = C \Phi^{ab}. \tag{2.19}$$

We see that $\Lambda^{abcd}$and Φ^{ab} are tensor densities, while it appears that Λ^{ab}, Λ^{abc} and Φ^a are not (if Φ^{ab} is antisymmetric, Φ^a will be a vector density—see the discussion in Section 3.4).

In order to derive the invariance relations we differentiate the transformed Lagrangian $\tilde{L}(\tilde{g}_{ij}, \tilde{g}_{ij,k}, \tilde{g}_{ij,k\ell}, \tilde{\phi}_i, \tilde{\phi}_{i,j})$ with respect to $C^a{}_i$ and its derivatives after which we consider the special case of the identity transformation. For example, differentiating (2.8) with respect to $C^p{}_{qrs}$ we have

$$\frac{\partial \tilde{L}}{\partial \tilde{g}_{ij,k\ell}} \frac{\partial \tilde{g}_{ij,k\ell}}{\partial C^p{}_{qrs}} = 0 \tag{2.20}$$

or, introducing the compact notation (2.3), noting (2.11), and taking care to consider all possible permutations of the symmetric covariant indices

$$\tfrac{1}{6}\Lambda^{ijk\ell}[\delta^a_p(\delta^q_i\delta^r_k\delta^s_\ell+\delta^q_k\delta^r_\ell\delta^s_i+\delta^q_\ell\delta^r_i\delta^s_k+\delta^q_\ell\delta^r_k\delta^s_i+\delta^q_k\delta^r_i\delta^s_\ell+\delta^q_i\delta^r_\ell\delta^s_k)C^b{}_j$$
$$+C^a{}_i\delta^b_p(\delta^q_j\delta^r_k\delta^s_\ell+\delta^q_k\delta^r_\ell\delta^s_j+\delta^q_\ell\delta^r_j\delta^s_k+\delta^q_\ell\delta^r_k\delta^s_j+\delta^q_k\delta^r_j\delta^s_\ell+\delta^q_j\delta^r_\ell\delta^s_k)]g_{ab}=0. \quad (2.21)$$

Now, considering the identity transform, so that $C^a{}_i=\delta^a_i$ and any remaining derivatives vanish, making use of the symmetry relation (2.6) and noting that, in general, $g_{ab}\neq 0$ we find

$$\Lambda^{aqrs}+\Lambda^{arsq}+\Lambda^{asqr}=0. \quad (2.22)$$

Similarly, differentiating by $C^p{}_{qr}$ and considering the identity transform yields

$$(\Lambda^{qbr}+\Lambda^{rbq})g_{pb}+(\Lambda^{aqr}+\Lambda^{arq})g_{ap}$$
$$+(\Lambda^{qbcr}+\Lambda^{rbcq}+\Lambda^{qbrc}+\Lambda^{rbqc})g_{pb,c}+(\Lambda^{aqcr}+\Lambda^{arcq}+\Lambda^{aqrc}+\Lambda^{arqc})g_{ap,c}$$
$$+(\Lambda^{abqr}+\Lambda^{abrq})g_{ab,p}+(\Phi^{qr}+\Phi^{rq})\phi_p=0 \quad (2.23)$$

where we have not simplified further, via the symmetry relations, as this form will be found more convenient in what follows.

Before deriving the third invariance relation we note that

$$\frac{\partial C}{\partial C^p{}_q}=C(C^{-1})^q{}_p \quad (2.24)$$

which, when we consider the identity transformation, reduces to δ^q_p. Thus, differentiating (2.8) by $C^p{}_q$ and considering the identity transform gives

$$\Lambda^{qb}g_{pb}+\Lambda^{aq}g_{ap}+\Lambda^{qbc}g_{pb,c}+\Lambda^{aqc}g_{ap,c}+\Lambda^{abq}g_{ab,p}$$
$$+\Lambda^{qbcd}g_{pb,cd}+\Lambda^{aqcd}g_{ap,cd}+\Lambda^{abqd}g_{ab,pd}+\Lambda^{abcq}g_{ab,cp}$$
$$+\Phi^q\phi_p+\Phi^{qb}\phi_{p,b}+\Phi^{aq}\phi_{a,p}=\delta^q_pL \quad (2.25)$$

where once again, for the sake of future convenience, we neglect to simplify further.

The first of the invariance relations may be used to derive further information about the symmetry properties of Λ^{abcd}. Repeated application of equation (2.22) gives, with (2.6), the relation

$$\Lambda^{abcd} = \Lambda^{cdab}. \tag{2.26}$$

With the derivation of this last symmetry relation we are in a position to introduce the quantity

$$\psi^{abcd} = \tfrac{1}{3}(\Lambda^{abcd} - \Lambda^{adcb}) \tag{2.27}$$

which will be found useful later. From this definition, the symmetry relations (2.6) and (2.26), and the invariance relation (2.22) we may show that ψ^{abcd} has the following properties[1]:

$$\Lambda^{abcd} = \psi^{abcd} + \psi^{abdc} \tag{2.28}$$

$$\psi^{abcd} = -\psi^{cbad} = -\psi^{adcb} = \psi^{badc} = \psi^{cdab} \tag{2.29}$$

$$\psi^{abcd} + \psi^{adbc} + \psi^{acdb} = 0. \tag{2.30}$$

The invariance relations will prove invaluable in the "integration", in the next chapter, of a general conserved quantity h^a for Lagrangians with the functional dependence (2.2). However, the fact that these relations contain non-tensorial quantities combined with the complications due to the presence of second derivatives of the potential g_{ab} results in difficulties in devising an unambiguous technique for the "integration" of a tensorial conserved quantity. Thus, in the next section, we introduce suitable tensor concomitants for Λ^{ab}, Λ^{abc} and Φ^a so that we may rewrite the invariance relations and derive conserved quantities wholly in terms of tensors and tensor densities.

[1]It is interesting that $r^{abcd} \equiv \psi^{acbd}$ and $s^{abcd} \equiv -3\Lambda^{abcd}$ have respectively the symmetries of the Riemann tensor and Synge's [30] symmetrized Riemann tensor, and also share the same interconnecting relationships (derivable from equations (2.27) and (2.28)). Synge's symmetrized Riemann tensor is derived from the coincidence limit of the fourth order covariant derivative of his two-point world function. Other derivatives of this two-point world function may also be used, with the Riemann tensor, to define tensorial (two-point) conserved quantites: the fluxes of total 4-momentum and angular momentum across an open 3-space relative to a base event P. Of course these conserved quantities are not derived via a variational principle.

2.4 Tensor Concomitants

We begin with the introduction of an arbitrary symmetric tensor field h_{ab} and define the quantity [8,20]

$$F = \Lambda^{abcd} h_{ab,cd} + \Lambda^{abc} h_{ab,c} + \Lambda^{ab} h_{ab}. \tag{2.31}$$

Using the transformation laws for Λ^{abcd}, Λ^{abc} and Λ^{ab}, and rearranging terms we have

$$\begin{aligned} CF &= \tilde{\Lambda}^{ijk\ell}\left(\frac{\partial \tilde{g}_{ij,k\ell}}{\partial g_{ab,cd}} h_{ab,cd} + \frac{\partial \tilde{g}_{ij,k\ell}}{\partial g_{ab,c}} h_{ab,c} + \frac{\partial \tilde{g}_{ij,k\ell}}{\partial g_{ab}} h_{ab}\right) \\ &\quad + \tilde{\Lambda}^{ijk}\left(\frac{\partial \tilde{g}_{ij,k}}{\partial g_{ab,c}} h_{ab,c} + \frac{\partial \tilde{g}_{ij,k}}{\partial g_{ab}} h_{ab}\right) + \tilde{\Lambda}^{ij} \frac{\partial \tilde{g}_{ij}}{\partial g_{ab}} h_{ab} \\ &= \tilde{\Lambda}^{ijk\ell} \tilde{h}_{ij,k\ell} + \tilde{\Lambda}^{ijk} \tilde{h}_{ij,k} + \tilde{\Lambda}^{ij} \tilde{h}_{ij} \end{aligned} \tag{2.32}$$

from which we conclude that F is a scalar density.

We now assume the existence of quantities Π^{abc} and Π^{ab} such that

$$F = \Lambda^{abcd} h_{ab;c;d} + \Pi^{abc} h_{ab;c} + \Pi^{ab} h_{ab}. \tag{2.33}$$

Rewriting, in equation (2.31), the partial derivatives of h_{ab} in terms of covariant derivatives and equating coefficients with equation (2.33) we find

$$\begin{aligned} \Pi^{abc} &= \Lambda^{abc} + (\Lambda^{nbcd} + \Lambda^{nbdc})\Gamma^{a}{}_{nd} \\ &\quad + (\Lambda^{ancd} + \Lambda^{andc})\Gamma^{b}{}_{nd} + \Lambda^{abnd}\Gamma^{c}{}_{nd} \end{aligned} \tag{2.34}$$

$$\begin{aligned} \Pi^{ab} &= \Lambda^{ab} + \Lambda^{nbc}\Gamma^{a}{}_{nc} + \Lambda^{anc}\Gamma^{b}{}_{nc} \\ &\quad + \Lambda^{nbcd}(\Gamma^{a}{}_{nc,d} + \Gamma^{m}{}_{nc}\Gamma^{a}{}_{md}) + \Lambda^{mncd}\Gamma^{a}{}_{mc}\Gamma^{b}{}_{nd} \\ &\quad + \Lambda^{ancd}(\Gamma^{b}{}_{nc,d} + \Gamma^{m}{}_{nc}\Gamma^{b}{}_{md}) + \Lambda^{mncd}\Gamma^{a}{}_{md}\Gamma^{b}{}_{nc} \\ &= \Lambda^{ab} + \Pi^{nbc}\Gamma^{a}{}_{nc} + \Pi^{anc}\Gamma^{b}{}_{nc} + \Lambda^{nbcd}\Gamma^{a}{}_{nc,d} + \Lambda^{ancd}\Gamma^{b}{}_{nc,d} \\ &\quad - \Lambda^{nbcd}\Gamma^{a}{}_{mc}\Gamma^{m}{}_{nd} - \Lambda^{mncd}\Gamma^{a}{}_{md}\Gamma^{b}{}_{mc} - \Lambda^{nbcd}\Gamma^{a}{}_{nm}\Gamma^{m}{}_{cd} \\ &\quad - \Lambda^{mncd}\Gamma^{a}{}_{mc}\Gamma^{b}{}_{nd} - \Lambda^{ancd}\Gamma^{b}{}_{mc}\Gamma^{m}{}_{nd} - \Lambda^{ancd}\Gamma^{b}{}_{nm}\Gamma^{m}{}_{cd}. \end{aligned} \tag{2.35}$$

Note that Π^{ab} and Π^{abc} also obey the symmetry relations

$$\Pi^{ab} = \Pi^{ba} \tag{2.36}$$

$$\Pi^{abc} = \Pi^{bac}. \tag{2.37}$$

In order to determine the tensorial nature of Π^{abc}, we substitute the transformation laws (2.16) and (2.17) and that of the connection

$$\Gamma^a{}_{bc} = \tilde{\Gamma}^i{}_{jk} C^a{}_i (C^{-1})^j{}_b (C^{-1})^k{}_c - C^a{}_{jk} (C^{-1})^j{}_b (C^{-1})^k{}_c \tag{2.38}$$

into equation (2.34) after which it is easy to see that Π^{abc} is a tensor density. This and the tensorial nature of F, Λ^{abcd} and h_{ab} and its covariant derivatives is enough to establish that Π^{ab} is also a tensor density.

Finally, we wish to show that the first Euler-Lagrange equation is tensorial. Rewriting equations (2.31) and (2.33) we have

$$\begin{aligned} F &= (\Lambda^{ab} - \Lambda^{abc}{}_{,c} + \Lambda^{abcd}{}_{,dc}) h_{ab} \\ &\quad + [(\Lambda^{abc} - \Lambda^{abcd}{}_{,d}) h_{ab} + \Lambda^{abcd} h_{ab,d}]_{,c} \\ &= (\Pi^{ab} - \Pi^{abc}{}_{;c} + \Lambda^{abcd}{}_{;d;c}) h_{ab} \\ &\quad + [(\Pi^{abc} - \Lambda^{abcd}{}_{;d}) h_{ab} + \Lambda^{abcd} h_{ab;d}]_{;c}. \end{aligned} \tag{2.39}$$

But is is easy to see, by direct substitution, that the two expressions in brackets are equal. Thus, since $V^a{}_{;a} = V^a{}_{,a}$ for a vector density, we have

$$\begin{aligned} E^{ab} &= -\Lambda^{ab} + \Lambda^{abc}{}_{,c} - \Lambda^{abcd}{}_{,dc} \\ &= -\Pi^{ab} + \Pi^{abc}{}_{;c} - \Lambda^{abcd}{}_{;d;c} \end{aligned} \tag{2.40}$$

and E^{ab} is clearly a tensor density.

Having defined tensor concomitants for Λ^{ab} and Λ^{abc}, we now wish to do the same for Φ^a. As the Lagrangian (2.2) contains only first derivatives of the potential

ϕ_a this task will prove considerably simpler than the foregoing analysis and we merely outline the procedure.

Introducing the arbitrary vector field k_a, we define the quantity

$$I = \Phi^{ab} k_{a,b} + \Phi^a k_a. \tag{2.41}$$

Substituting the transformation laws of Φ^{ab} and Φ^a into (2.41) and rearranging terms we find that I is a scalar density. We now write

$$I = \Phi^{ab} k_{a;b} + \Pi^a k_a \tag{2.42}$$

and, after rewriting $k_{a,b}$ in terms of $k_{a;b}$ and equating coefficients we have

$$\Pi^a = \Phi^a + \Phi^{nb} \Gamma^a{}_{nb}. \tag{2.43}$$

Noting that $I - \Phi^{ab} k_{a;b}$ is a scalar density we see that Π^a is clearly a vector density.

Finally, in order to show that the second Euler-Lagrange equation is tensorial, we rewrite (2.41) and (2.42) in the form

$$\begin{aligned} I &= (\Phi^a - \Phi^{ab}{}_{,b}) k_a + (\Phi^{ab} k_a)_{,b} \\ &= (\Pi^a - \Phi^{ab}{}_{;b}) k_a + (\Phi^{ab} k_a)_{;b}. \end{aligned} \tag{2.44}$$

But, since $\Phi^{ab} k_a$ is a vector density, this implies that

$$\begin{aligned} E^a &= -\Phi^a + \Phi^{ab}{}_{,b} \\ &= -\Pi^a + \Phi^{ab}{}_{;b} \end{aligned} \tag{2.45}$$

from which it is clear that E^a is a vector density.

2.5 Conversion of the Invariance Relations

Having derived tensor concomitants for the functional derivatives of L we now wish to rewrite the invariance relations in terms of them. This will be done by simple substitution, the conversion of partial derivatives to covariant derivatives, the occasional use of a previously rewritten invariance relation and, where convenient, the introduction of the quantity ψ^{abcd} in expressions in Λ^{abcd}.

The first invariance relation, equation (2.22), is already tensorial and needs no conversion. After the substitution of Π^{abc}, forΛ^{abc}, and the replacement of partial, by convariant, differentiation in equation (2.23) we have

$$\begin{aligned}&(\Pi^{qbr}+\Pi^{rbq})g_{pb}+(\Pi^{aqr}+\Pi^{arq})g_{ap}\\&+(\Lambda^{qbcr}+\Lambda^{rbcq}+\Lambda^{qbrc}+\Lambda^{rbqc})(g_{pb;c}+\Gamma^{n}{}_{pc}g_{nb})\\&+(\Lambda^{aqcr}+\Lambda^{arcq}+\Lambda^{aqrc}+\Lambda^{arqc})(g_{ap;c}+\Gamma^{n}{}_{pc}g_{an})\\&+(\Lambda^{abqr}+\Lambda^{abrq})g_{ab;p}+(\Phi^{qr}+\Phi^{rq})\phi_{p}\\&-[(\Lambda^{nbrd}+\Lambda^{nbdr}+\Lambda^{rbnd})\Gamma^{q}{}_{nd}+(\Lambda^{nbqd}+\Lambda^{nbdq}+\Lambda^{qbnd})\Gamma^{r}{}_{nd}]g_{pb}\\&-[(\Lambda^{anrd}+\Lambda^{andr}+\Lambda^{arnd})\Gamma^{q}{}_{nd}+(\Lambda^{anqd}+\Lambda^{andq}+\Lambda^{aqnd})\Gamma^{r}{}_{nd}]g_{ap}\\&-(\Lambda^{cbqr}+\Lambda^{cbrq})\Gamma^{n}{}_{cp}g_{nb}-(\Lambda^{acqr}+\Lambda^{acrq})\Gamma^{n}{}_{cp}g_{an}=0.\end{aligned}\qquad(2.46)$$

We now invoke equation (2.22) and apply it to each of the Γ-terms with the result

$$\begin{aligned}&(\Pi^{qbr}+\Pi^{rbq})g_{pb}+(\Pi^{aqr}+\Pi^{arq})g_{ap}\\&+(\Lambda^{qbcr}+\Lambda^{rbcq}+\Lambda^{qbrc}+\Lambda^{rbqc})g_{pb;c}\\&+(\Lambda^{aqcr}+\Lambda^{arcq}+\Lambda^{aqrc}+\Lambda^{arqc})g_{ap;c}\\&+(\Lambda^{abqr}+\Lambda^{abrq})g_{ab;p}+(\Phi^{qr}+\Phi^{rq})\phi_{p}=0.\end{aligned}\qquad(2.47)$$

For later convenience we define the intermediate quantity

$$\begin{aligned}A^{qr}{}_{p}&=\Pi^{qbr}g_{pb}+\Pi^{aqr}g_{ap}+(\Lambda^{qbcr}+\Lambda^{qbrc})g_{pb:c}\\&\quad+(\Lambda^{aqcr}+\Lambda^{aqrc})g_{ap:c}+\Lambda^{abqr}g_{ab;p}+\Phi^{qr}\phi_{p}\end{aligned}\qquad(2.48)$$

in which case the converted relation (2.47) is just the condition of antisymmetry,

$$A^{qr}{}_{p}+A^{rq}{}_{p}=0.\qquad(2.49)$$

The conversion of equation (2.25) proceeds similarly with an additional application of (2.47) and the introduction of the useful quantity ψ^{abcd}. As the calculation is quite lengthy, we leave the details to Appendix A and present the result:

$$\begin{aligned}
\delta^q_p L &= \Pi^{qb} g_{pb} + \Pi^{aq} g_{ap} + \Pi^{qbc} g_{pb;c} + \Pi^{aqc} g_{ap;c} + \Pi^{abq} g_{ab;p} \\
&\quad + \Lambda^{qbcd} g_{pb;c;d} + \Lambda^{aqcd} g_{ap;c;d} + \Lambda^{abqd} g_{ab;p;d} + \Lambda^{abcq} g_{ab;p;c} \\
&\quad + \Pi^{q} \phi_{p} + \Phi^{qb} \phi_{p;b} + \Phi^{aq} \phi_{a;p} \\
&\quad - \tfrac{1}{2}(\psi^{abdc} g_{pb} R^{q}{}_{cda} + \psi^{abcd} g_{ap} R^{q}{}_{cdb}) \\
&\quad + \tfrac{3}{2}(\psi^{abcq} g_{nb} R^{n}{}_{pca} + \psi^{abqd} g_{an} R^{n}{}_{pdb}).
\end{aligned} \tag{2.50}$$

It will also prove useful to rewrite the relation

$$L_{;p} = \Lambda^{ab} g_{ab;p} + \Lambda^{abc} g_{ab,c;p} + \Lambda^{abcd} g_{ab,cd;p} + \Phi^{a} \phi_{a;p} + \Phi^{ab} \phi_{a,b;p}. \tag{2.51}$$

The conversion of this equation is similar to the preceding and, as it too is lengthy and not particularly illuminating, we again relegate the details to Appendix A and proceed to the final result, which is

$$\begin{aligned}
L_{;p} &= \Pi^{ab} g_{ab;p} + \Pi^{abc} g_{ab;p;c} + \Lambda^{abcd} g_{ab;p;c;d} + \Pi^{a} \phi_{a;p} + \Phi^{ab} \phi_{a;p;b} \\
&\quad + \tfrac{1}{2} A^{qr}{}_{n} R^{n}{}_{prq} + \psi^{abcd} g_{nb} R^{n}{}_{pca;d} + \psi^{abdc} g_{an} R^{n}{}_{pcb;d}.
\end{aligned} \tag{2.52}$$

These relations, together with the symmetry relations, will greatly simplify the "integration" of the general conserved quantity h^a in the next chapter.

Chapter 3

The Variational Principle

3.1 Introduction

This chapter is the heart of the first part of the dissertation. We begin with a general variation of the action (2.1) and derive the resultant conserved quantities. As a consequence of the preparatory work of Chapter 2, the actual "integration" process will proceed smoothly—almost inevitably—and the intimate connection between the invariance relations and the variational principle will be readily apparent.

After varying the action we write the integrand of δS as the divergence of a vector density, in the form $h^a{}_{;a}$. At this point most authors specify the variation as an infinitesimal translation and use the properties of a specific Lagrangian to obtain a mixed energy-momentum complex $h^a{}_b$ which is then "integrated" to form the superpotential. Komar [15] derives an improved superpotential from the Hilbert Lagrangian $\sqrt{-g}\,R$ in terms of an arbitrary variation δx^a. In contrast, we use the invariance relations of Chapter 2 to "integrate" a strongly conserved quantity h^a, which is general in the choice of both the variation and the Lagrangian. The resulting expression has several advantages. It is mathematically simpler in that most of its properties may be deduced by inspection. Also, by specifying the appropriate variation, one may generate both a mixed and contravariant energy-momentum complex and an angular momentum complex for general scalar density Lagrangians. The complexes $h^a{}_b$ and h^{ab} are derived in Section 3.5. The angular momentum complex h^{abc} is presented in Section 3.6. We show that the moment of h^{ab} constitutes only part of h^{abc}. We then use the conservation of h^{abc} to find the unaccounted-for "spin" energy contribution of the remaining part of h^{abc}, which is added to h^{ab} to produce a symmetric total energy-momentum complex H^{ab}. Finally, in the following chapter Lovelock's Lagrangian is considered and new complexes are generated along with generalizations of those in common use.

3.2 Variation of the Action

In this section we take a general variation of the action (2.1). Rather than separately varying each of the fields we subject the total Lagrangian to a simultaneous variation of the coordinates x^a and potentials ϕ_a and g_{ab}. This permits the representation of the variation as a single infinitesimal coordinate transformation, which, in concert with the invariance relations derived in the previous chapter, permits the "integration" of a conserved quantity h^a in terms of both an arbitrary variation and scalar density Lagrangian. The resulting expression is a generalization of Komar's complex.

The general variation of the coordinates and potentials of the action (2.1) results in the expression [2]

$$\delta S = \int_R \delta L \, d^4x + \int_{\partial R} L \, \delta x^a \, dS_a \tag{3.1}$$

$$\delta L = \Lambda^{ab}\delta g_{ab} + \Lambda^{abc}\delta g_{ab,c} + \Lambda^{abcd}\delta g_{ab,cd} + \Phi^a\delta\phi_a + \Phi^{ab}\delta\phi_{a,b} \tag{3.2}$$

where ∂R denotes the boundary of the region R. If we (here) require that δx^a and its first and second derivatives vanish on ∂R, then the invariance of the action yields the Euler-Lagrange equations

$$E^{ab} \equiv -\Lambda^{ab} + \Lambda^{abc}{}_{,c} - \Lambda^{abcd}{}_{,dc} = 0 \tag{3.3}$$

$$E^a \equiv -\Phi^a + \Phi^{ab}{}_{,b} = 0 \tag{3.4}$$

which, with an appropriate Lagrangian, reduce to the Einstein-Maxwell equations. As we have shown previously, both E^i and E^{ij} are tensor densities.

3.3 "Integration" of the Conserved Quantity h^a

We now rewrite the variation of the action (3.1) in the form

$$\delta S = 2\int h^a{}_{;a}\, d^4x \tag{3.5}$$

and define the vector $\chi^a \equiv \delta x^a$. In the following we derive a strong conservation law by "integrating" the quantity h^a without making reference to equations (3.3) and (3.4).

The "integration" procedure is motivated by our objectives that the conserved quantity be covariant and its derivation unambiguous. Thus we will derive values for the varied potentials and substitute covariant for partial derivatives and tensor concomitants for the functional derivatives of the Lagrangian in the expression for δL. In doing so we will find that we have written δL in terms of products of equation (2.52) or the invariance relations, or segments therefrom, and the variation δx^a or its derivatives. When "integrating" we will eliminate only terms which would "integrate" to zero. The main consequence of this "rule" will be that in terms containing the Riemann tensor it will, if possible, be interpreted as acting on concomitants rather than δx^a or its derivatives.

Under the infinitesimal transformation $\tilde{x}^a = x^a + \chi^a$, the change in the potentials may be written

$$\begin{aligned} \tilde{\delta}\phi_a &\equiv \tilde{\phi}_a(\tilde{x}^k) - \phi_a(x^k) = -\phi_i\chi^i{}_{,a} \\ &= \delta\phi_a + \phi_{a,p}\chi^p \end{aligned} \tag{3.6}$$

$$\begin{aligned} \tilde{\delta}g_{ab} &\equiv \tilde{g}_{ab}(\tilde{x}^k) - g_{ab}(x^k) = -g_{ib}\chi^i{}_{,a} - g_{aj}\chi^j{}_{,b} \\ &= \delta g_{ab} + g_{ab,p}\chi^p. \end{aligned} \tag{3.7}$$

Thus

$$\begin{aligned}\delta\phi_a &= -(\phi_{a,p}\chi^p + \phi_p\chi^p{}_{,a}) \\ &= -(\phi_{a;p}\chi^p + \phi_p\chi^p{}_{;a}) \qquad (3.8)\\ \delta g_{ab} &= -(g_{ab,p}\chi^p + g_{pb}\chi^p{}_{,a} + g_{ap}\chi^p{}_{,b}) \\ &= -(g_{ab;p}\chi^p + g_{pb}\chi^p{}_{;a} + g_{ap}\chi^p{}_{;b}) \qquad (3.9)\end{aligned}$$

where both $\delta\phi_a$ and δg_{ab} are clearly tensors and we see that δL in equation (3.2) is of the form of a sum of equations (2.31) and (2.41). Therefore, we may write

$$\delta L = \Pi^a\delta\phi_a + \Phi^{ab}\delta\phi_{a;b} + \Pi^{ab}\delta g_{ab} + \Pi^{abc}\delta g_{ab;c} + \Lambda^{abcd}\delta g_{ab;c;d} \qquad (3.10)$$

a form in which all elements are tensors or tensor densities.

For the sake of notational convenience we now define, from equations (2.52) and (2.50) respectively,

$$\begin{aligned}\omega_p &= \psi^{abcd}g_{nb}R^n{}_{pca;d} + \psi^{abdc}g_{an}R^n{}_{pcb;d} \\ &= L_{;p} - \tfrac{1}{2}A^{qr}{}_nR^n{}_{prq} - \Pi^a\phi_{a;p} - \Phi^{ab}\phi_{a;p;b} \\ &\quad -\Pi^{ab}g_{ab;p} - \Pi^{abc}g_{ab;p;c} - \Lambda^{abcd}g_{ab;p;c;d} \qquad (3.11)\\ \Omega^q{}_p &= -\tfrac{1}{2}(\psi^{abdc}g_{pb}R^q{}_{cda} + \psi^{abcd}g_{ap}R^q{}_{cdb}) + \tfrac{3}{2}(\psi^{abcq}g_{nb}R^n{}_{pca} + \psi^{abqd}g_{an}R^n{}_{pdb}) \\ &= \delta^q_pL - \Pi^q\phi_p - \Phi^{qb}\phi_{p;b} - \Phi^{aq}\phi_{a;p} \\ &\quad -\Pi^{qb}g_{pb} - \Pi^{aq}g_{ap} - \Pi^{qbc}g_{pb;c} - \Pi^{aqc}g_{ap;c} - \Pi^{abq}g_{ab;p} \\ &\quad -\Lambda^{qbcd}g_{pb;c;d} - \Lambda^{aqcd}g_{ap;c;d} - \Lambda^{abqd}g_{ab;p;d} - \Lambda^{abcq}g_{ab;p;c}. \qquad (3.12)\end{aligned}$$

Now by substituting $\delta\phi_a$, δg_{ab} and their covariant derivatives into equation (3.10) we have

$$\begin{aligned}\delta L &= (\omega_p - L_{;p} + \tfrac{1}{2}A^{qr}{}_nR^n{}_{prq})\chi^p + (\Omega^q{}_p - \delta^q_pL)\chi^p{}_{;q} \\ &\quad -A^{qr}{}_p\chi^p{}_{;q;r} - \Lambda^{abcd}(g_{pb}\chi^p{}_{;a;c;d} + g_{ap}\chi^p{}_{;b;c;d}) \qquad (3.13)\end{aligned}$$

or, noting (3.1) and (3.5) and substituting (2.28) for Λ^{abcd},

$$\begin{aligned} 2h^a{}_{;a} &= (\omega_p + \tfrac{1}{2}A^{qr}{}_n R^n{}_{prq})\chi^p + \Omega^q{}_p \chi^p{}_{;q} + A^{qr}{}_p \chi^p{}_{;r;q} \\ &\quad -(\psi^{abcd} + \psi^{abdc})(g_{pb}\chi^p{}_{;a;c;d} + g_{ap}\chi^p{}_{;b;c;d}). \end{aligned} \tag{3.14}$$

This is the expression we must "integrate".

We first note that

$$\tfrac{1}{2}A^{qr}{}_n R^n{}_{prq}\chi^p = -A^{qr}{}_p \chi^p{}_{;r;q} \tag{3.15}$$

and so we could cancel the terms in $A^{qr}{}_p$ in equation (3.14). But we also have

$$\begin{aligned} A^{qr}{}_n R^n{}_{prq} &= A^{qr}{}_{p;r;q} - A^{qr}{}_{p;q;r} + A^{nr}{}_p R^q{}_{nrq} + A^{qn}{}_p R^r{}_{nrq} \\ &= 2A^{qr}{}_{p;q;r} \end{aligned} \tag{3.16}$$

so that

$$\begin{aligned} \tfrac{1}{2}A^{qr}{}_n R^n{}_{prq}\chi^p + A^{qr}{}_p \chi^p{}_{;r;q} &= (A^{qr}{}_p \chi^p)_{;r;q} - A^{qr}{}_{p;r}\chi^p{}_{;q} - A^{qr}{}_{p;q}\chi^p{}_{;r} \\ &= (A^{qr}{}_p \chi^p)_{;r;q} \end{aligned} \tag{3.17}$$

and we see that the terms in $A^{qr}{}_p$ do not "integrate" to zero and should be retained. Thus, written out in full,

$$\begin{aligned} 2h^a{}_{;a} &= (A^{ac}{}_p \chi^p)_{;c;a} + (\psi^{abcd} g_{nb} R^n{}_{pca;d} + \psi^{abdc} g_{an} R^n{}_{pcb;d})\chi^p \\ &\quad +(\tfrac{3}{2}\psi^{abcq} g_{nb} R^n{}_{pca} + \tfrac{3}{2}\psi^{abqd} g_{an} R^n{}_{pdb} \\ &\qquad -\tfrac{1}{2}\psi^{abdc} g_{pb} R^q{}_{cda} - \tfrac{1}{2}\psi^{abcd} g_{ap} R^q{}_{cdb})\chi^p{}_{;q} \\ &\quad -(\psi^{abcd} + \psi^{abdc})(g_{pb}\chi^p{}_{;a;c;d} + g_{ap}\chi^p{}_{;b;c;d}). \end{aligned} \tag{3.18}$$

In the terms in χ^p the Riemann tensor must first be partially integrated before it can be applied to anything. Hence, we write

$$\begin{aligned} &\psi^{abcd} g_{nb} R^n{}_{pca;d}\chi^p + \psi^{abcq} g_{nb} R^n{}_{pca}\chi^p{}_{;q} - 2\psi^{abcd} g_{pb}\chi^p{}_{;a;c;d} \\ &= (\psi^{abcd} g_{nb} R^n{}_{pca}\chi^p)_{;d} - (\psi^{abcd} g_{nb})_{;d} R^n{}_{pca}\chi^p \\ &\quad -(2\psi^{abcd} g_{pb}\chi^p{}_{;a;c})_{;d} + 2(\psi^{abcd} g_{pb})_{;d}\chi^p{}_{;a;c}. \end{aligned} \tag{3.19}$$

Noting that

$$\psi^{abcd}R^{n}{}_{pca}\chi^{p} = 2\psi^{abcd}\chi^{n}{}_{;a;c} \tag{3.20}$$

we see that the first and third terms of (3.19) have "integrated" to zero and may be cancelled. As to the remaining two terms,

$$\begin{aligned}
&-(\psi^{abcd}g_{nb})_{;d}R^{n}{}_{pca}\chi^{p} + 2(\psi^{abcd}g_{pb})_{;d}\chi^{p}{}_{;a;c} \\
&= [(\psi^{abcd}g_{nb})_{;d;a;c} - (\psi^{abcd}g_{nb})_{;d;c;a} + (\psi^{nbcd}g_{pb})_{;d}R^{a}{}_{nac} + (\psi^{abnd}g_{pb})_{;d}R^{c}{}_{nac}]\chi^{p} \\
&\quad -2(\psi^{abcd}g_{pb})_{;d}\chi^{p}{}_{;c;a} \\
&= -2[(\psi^{abcd}g_{pb})_{;d}\chi^{p}]_{;c;a} + 2(\psi^{abcd}g_{pb})_{;d;c}\chi^{p}{}_{;a} + 2(\psi^{abcd}g_{pb})_{;d;a}\chi^{p}{}_{;c} \\
&= -2[(\psi^{abcd}g_{pb})_{;d}\chi^{p}]_{;c;a}.
\end{aligned} \tag{3.21}$$

Similarly

$$\psi^{abdc}g_{an}R^{n}{}_{pcb;d}\chi^{p} + \psi^{abqd}g_{an}R^{n}{}_{pdb}\chi^{p}{}_{;q} - 2\psi^{abdc}g_{ap}\chi^{p}{}_{;b;c;d} = -2[(\psi^{abcd}g_{ap})_{;c}\chi^{p}]_{;d;b} \tag{3.22}$$

and we have

$$\begin{aligned}
2h^{a}{}_{;a} &= [A^{ac}{}_{p}\chi^{p} - 2(\psi^{abcd}g_{pb})_{;d}\chi^{p}]_{;c;a} - 2[(\psi^{abcd}g_{ap})_{;c}\chi^{p}]_{;d;b} \\
&\quad +\tfrac{1}{2}(\psi^{abcq}g_{nb}R^{n}{}_{pca} + \psi^{abqd}g_{an}R^{n}{}_{pdb} - \psi^{abdc}g_{pb}R^{q}{}_{cda} - \psi^{abcd}g_{ap}R^{q}{}_{cdb})\chi^{p}{}_{;q} \\
&\quad +(\psi^{abcd} - \psi^{abdc})g_{pb}\chi^{p}{}_{;a;c;d} - (\psi^{abcd} - \psi^{abdc})g_{ap}\chi^{p}{}_{;b;c;d}.
\end{aligned} \tag{3.23}$$

But

$$\begin{aligned}
&\tfrac{1}{2}(\psi^{abcq}g_{nb}R^{n}{}_{pca} - \psi^{abcn}g_{pb}R^{q}{}_{nca})\chi^{p}{}_{;q} \\
&= \tfrac{1}{2}[(\psi^{abcq}g_{nb})_{;c;a} - \psi^{abcq}g_{nb})_{;a;c} \\
&\qquad +\psi^{nbcq}g_{pb}R^{a}{}_{nca} + \psi^{abnq}g_{pb}R^{c}{}_{nca}]\chi^{p}{}_{;q} \\
&= (\psi^{abcd}g_{pb}\chi^{p}{}_{;d})_{;c;a} - (\psi^{abcd}g_{pb})_{;c}\chi^{p}{}_{;d;a} \\
&\qquad -(\psi^{abcd}g_{pb})_{;a}\chi^{p}{}_{;d;c} - \psi^{abcd}g_{pb}\chi^{p}{}_{;d;c;a} \\
&= (\psi^{abcd}g_{pb}\chi^{p}{}_{;d})_{;c;a} - \psi^{dbca}g_{pb}\chi^{p}{}_{;a;c;d}
\end{aligned} \tag{3.24}$$

and, for the corresponding terms in g_{ai},

$$\tfrac{1}{2}(\psi^{abqd}g_{an}R^{n}{}_{pdb} - \psi^{abnd}g_{ap}R^{q}{}_{ndb})\chi^{p}{}_{;q} = (\psi^{abcd}g_{ap}\chi^{p}{}_{;c})_{;d;b} - \psi^{adbc}g_{ap}\chi^{p}{}_{;b;c;d}. \quad (3.25)$$

We are just about done. Now

$$\begin{aligned} 2h^{a}{}_{;a} &= [A^{ac}{}_{p}\chi^{p} + \psi^{abcd}g_{pb}\chi^{p}{}_{;d} - 2(\psi^{abcd}g_{pb})_{;d}\chi^{p}]_{;c;a} \\ &\quad +[\psi^{abcd}g_{ap}\chi^{p}{}_{;c} - 2(\psi^{abcd}g_{ap})_{;c}\chi^{p}]_{;d;b} \\ &\quad -(\psi^{cbad} + \psi^{abdc} + \psi^{dbca})g_{pb}\chi^{p}{}_{;a;c;d} \\ &\quad -(\psi^{abcd} + \psi^{acdb} + \psi^{adbc})g_{ap}\chi^{p}{}_{;b;c;d}. \end{aligned} \quad (3.26)$$

The last two lines vanish by equation (2.30) leaving, after we apply the relation $\psi^{abcd} = \psi^{badc}$,

$$h^{a}{}_{;a} = [\tfrac{1}{2}A^{ac}{}_{p}\chi^{p} + \psi^{abcd}g_{pb}\chi^{p}{}_{;d} - 2(\psi^{abcd}g_{pb})_{;d}\chi^{p}]_{;c;a}. \quad (3.27)$$

Thus we may finally write

$$h^{a} = [\tfrac{1}{2}A^{ac}{}_{p}\chi^{p} + \psi^{abcd}g_{pb}\chi^{p}{}_{;d} - 2(\psi^{abcd}g_{pb})_{;d}\chi^{p}]_{;c}. \quad (3.28)$$

It is readily apparent from (3.28) that h^a is a vector density with vanishing divergence (if we wish, we may define h^a as the divergence of an antisymmetric superpotential, itself a tensor density). Thus, (3.28) constitutes a strong conservation law, general in the Lagrangian (2.2), which generates a conserved quantity for any specified variation χ^a. This new expression constitutes a generalization of Komar's complex.

3.4 Introduction of the Metric

It is possible, via the introduction of a general metric η_{ij}, to complete the analysis of this chapter with g_{ab} an arbitrary symmetric tensor field. However we now wish to explicitly consider the gravitational and electromagnetic fields. Thus we designate g_{ab} as our metric and restrict our Lagrangian to the form

$$L = {}_G L(g_{ab}, g_{ab,c}, g_{ab,cd}) + {}_{EM} L(g_{ab}, \phi_a, \phi_{a,b}) \tag{3.29}$$

so that the invariance relations split into gravitational and electromagnetic parts.

Now, from equation (2.47)

$$\Pi^q{}_p{}^r + \Pi^r{}_p{}^q + \Pi_p{}^{qr} + \Pi_p{}^{rq} = 0 \tag{3.30}$$

$$(\Phi^{qr} + \Phi^{rq})\phi_p = 0. \tag{3.31}$$

Equation (3.31) implies that Φ^{qr} is antisymmetric; from which, along with (2.18) and (2.43) we also see that $\Phi^a = \Pi^a$ is a vector density. Equations (3.30) and (2.37) imply

$$\Pi^{abc} = 0 \tag{3.32}$$

in which case equation (2.40) reduces to

$$A^{qr}{}_p = \bar{\Phi}^{qr}\phi_p. \tag{3.33}$$

Finally, we rewrite h^a in the form

$$\begin{aligned} h^a &= (\tfrac{1}{2}\Phi^{ac}\phi_p\chi^p + \psi^{abcd}\chi_{b;d} - 2\psi^{abcd}{}_{;d}\chi_b)_{;c} \\ &= (\tfrac{1}{2}\Phi^{ac}\phi_p\chi^p + \psi^{abcd}\chi_{b,d} - 2\psi^{abcd}{}_{;d}\chi_b)_{,c}. \end{aligned} \tag{3.34}$$

3.5 Derivation of Conserved Complexes from h^a

In order to generate physically interesting conserved quantities from the complex h^a we consider the variation of the previous section as given by an infinitesimal transformation defined in terms of an arbitrary set of independent parameters $\delta k^A{}_B$, where, here, the capitals represents sets of indices. That is, we write

$$\chi^a = f^a{}_A{}^B\,\delta k^A{}_B \tag{3.35}$$

where ${f^a}_A{}^B$ is some function of the coordinates and potentials and the $\delta k^A{}_B$, which are just the infinitesimal generators of the group whose "motion" represents the symmetry of the spacetime, are to be considered as arbitrary but predetermined, and thus constant with respect to the coordinates.

As in Hamilton-Jacobi theory we may then write

$$\frac{\delta S}{2\,\delta k^A{}_B} = \int h^a{}_A{}^B \, dS_a \tag{3.36}$$

(note that, up to a linear transformation, this fixes the coordinates). Now, by specifying the appropriate infinitesimal generators $\delta k^A{}_B$, we may write the "momenta" P_s, P^t and the "angular momentum" J^{ts} as surface integrals of the energy-momentum complexes $h^a{}_s$, h^{at} and the angular momentum complex h^{ats}. These $\delta k^A{}_B$ derive from the infinitesimal vectors

$$\xi^s = \gamma^s{}_n x^n + \zeta^s \tag{3.37}$$

$$\xi_t = \gamma_{tn} x^n + \zeta_t \tag{3.38}$$

where γ_{tn} is antisymmetric. But ξ^s, ξ_t are just the Killing vectors of Minkowski space and do not generally represent true symmetries of the spacetime. Only by integrating near infinity on an asymptotically flat spacetime may we be sure of obtaining valid results. Thus the motivation for the term "complex"; these objects are not true momentum densities and, in general, will not exhibit the corresponding local properties.

Further, as long as the sets of indices of $\delta k^A{}_B$ include at least one coordinate index, writing equation (3.36) will fix the coordinates, up to a linear transformation, and the conserved quantity will not, in general, be covariant. But if for the fundamental quantities of the theory we choose objects, such as spinors or twistors, which admit additional transformation groups whose infinitesimal generators $\delta k^A{}_B$ do not possess coordinate indices, the resulting conserved quantities will be coordinate (though not gauge) invariant.

We can now conclude that our goal of a covariant, localizable conserved quantity cannot be realized in a (general relativistic) theory that exhibits only coordinate invariance. However, the foregoing analysis has not been in vain. We are now in a postion to provide a compact derivation of a number of new and well-known momentum complexes and generalizations thereof. These objects are still in widespread use and, as they are likely to remain so, are not yet devoid of interest.

Letting

$$\chi^p = \sqrt{-g}^{(n-1)} \delta^p_s \, \delta k^s \tag{3.39}$$

we obtain from equation (3.34) the mixed complex

$$_{(n)}h^a{}_s = [\tfrac{1}{2}\sqrt{-g}^{(n-1)} \Phi^{ac} \phi_s + (\sqrt{-g}^{(n-1)} g_{bs})_{,d} \psi^{abcd} - 2\sqrt{-g}^{(n-1)} g_{bs} \psi^{abcd}{}_{;d}]_{,c} \tag{3.40}$$

and, if we let

$$\chi^p = \sqrt{-g}^{(n-1)} g^{pt} \, \delta k_t \tag{3.41}$$

we in turn generate the contravariant complex

$$_{(n)}h^{at} = [\tfrac{1}{2}\sqrt{-g}^{(n-1)} \Phi^{ac} \phi^t + \sqrt{-g}^{(n-1)}{}_{,d} \psi^{atcd} - 2\sqrt{-g}^{(n-1)} \psi^{atcd}{}_{;d}]_{,c} \tag{3.42}$$

where we have written $\sqrt{-g}^{(n-1)}$ to denote $\sqrt{-g}$ to the power $n-1$ and $_{(n)}h^{at}$ to denote the weight n complex.

With the introduction of a general relativistic Lagrangian, objects derived from equations (3.40) and (3.42) will, under the appropriate coordinate conditions, correctly give the global values for energy and momentum. However, equation (3.42) is not symmetric and thus the moment of $_{(n)}h^{at}$ does not define a conserved angular momentum complex. Further, both $_{(n)}h^a{}_s$ and $_{(n)}h^{at}$ contain a bothersome term in Φ^{ak}. With the introduction of the field equations (3.4) into these expressions we find those parts containing electromagnetic terms to be

$$_{(n)}t_{EM}{}^a{}_s = \sqrt{-g}^{(n-1)} {}_{EM}\Lambda^{ap} g_{ps} + [\tfrac{1}{2}\sqrt{-g}^{(n-1)} \Phi^{ac} \phi_s]_{,c} \tag{3.43}$$

$$_{(n)}t_{EM}{}^{at} = \sqrt{-g}^{(n-1)} {}_{EM}\Lambda^{at} + [\tfrac{1}{2}\sqrt{-g}^{(n-1)} \Phi^{ac} \phi^t]_{,c} \tag{3.44}$$

(recall that with an appropriate Lagrangian ${}_{EM}\Lambda^{at} = \sqrt{-g}\,T^{at}$), from which we see that the final terms must be eliminated if we are to correctly obtain the usual electromagnetic stress-energy tensor. These terms may be discarded ad hoc since they are divergenceless. However, it is instructive to seek a more illustrative basis for their elimination, which may lend itself to some physical interpretation. An appropriate procedure is suggested by appealing to electromagnetic field theory.

In the absence of a gravitational field, both equations (3.43) and (3.44) represent the same object

$$ {}_{(n)}t_{EM}{}^{at} = {}_{EM}\Lambda^{at} + (\tfrac{1}{2}\Phi^{ac}\phi^{t})_{,c} \tag{3.45}$$

which, for the usual Maxwell Lagrangian, reduces to the nonsymmetric stress-energy tensor [17],

$$ t_{EM}{}^{at} = \tfrac{1}{2}(-4\phi_{i,j}\eta^{jt}F^{ai} + \eta^{at}F_{ij}F^{ij}) \tag{3.46}$$

where η^{at} is the Minkowski metric. Equation (3.46) can be symmetrized, producing the normal electromagnetic stress-energy tensor, through the addition of a divergenceless term obtained via the conservation of the angular momentum density. This term is interpreted as the as yet uncounted energy-momentum contribution of that part of the angular momentum density not represented by the moment of (3.46). The presence of this anomalous electromagnetic term in our calculation in the presence of gravitation seems to imply a similarly uncounted gravitational energy-momentum contribution, which we may include via a symmetrization of the whole gravitational-electromagnetic complex. We perform this operation in the following section.

3.6 The Symmetrization of ${}_{(n)}h^{at}$

The angular momentum complex is generated from an infinitesimal rotation. Thus, we set

$$ \chi^{p} = \sqrt{-g}^{(n-1)}(g^{ps}x^{t} - g^{pt}x^{s})\,\delta k_{ts} \tag{3.47}$$

in which case (3.36) and (3.34) define the object

$$\begin{aligned}
{}_{(n)}h^{ats} &= {}_{(n)}h^{as}x^t - {}_{(n)}h^{at}x^s + [\sqrt{-g}^{(n-1)}(\psi^{asct} - \psi^{atcs})]_{,c} \\
&\quad + [\tfrac{1}{2}\sqrt{-g}^{(n-1)}(\Phi^{at}\phi^s - \Phi^{as}\phi^t) + \sqrt{-g}^{(n-1)}{}_{,d}(\psi^{astd} - \psi^{atsd}) \\
&\quad - 2\sqrt{-g}^{(n-1)}(\psi^{astd} - \psi^{atsd})_{;d}] \\
&= {}_{(n)}M^{ats} + {}_{(n)}S^{ats}
\end{aligned} \tag{3.48}$$

where ${}_{(n)}M^{ats}$ is the moment of the complex ${}_{(n)}h^{at}$, and ${}_{(n)}S^{ats}$ represents an intrinsic field momentum (in quantum mechanics this term is used to derive spin). The energy inherent in the ${}_{(n)}S^{ats}$ portion of ${}_{(n)}h^{ats}$ has not yet been accounted for; thus, we add an additional "spin" term ${}_{(n)}s^{at}$ to ${}_{(n)}h^{at}$ to obtain a total energy-momentum complex

$$ {}_{(n)}H^{at} = {}_{(n)}h^{at} + {}_{(n)}s^{at}. \tag{3.49}$$

The expression for ${}_{(n)}s^{at}$ is derived through the following set procedure (see Corson [7]).

We begin by writing the conservation law for the angular momentum complex

$$\begin{aligned}
h^{ats}{}_{,a} &= h^{ts} - h^{st} + S^{ats}{}_{,a} \\
&\equiv h^{ts} - h^{st} + (\mu^{tsa} - \mu^{sta})_{,a}
\end{aligned} \tag{3.50}$$

so that $h^{ts} + \mu^{tsa}{}_{,a}$ defines a symmetric object. Thus, if we let

$$ s^{ts} = \mu^{ats}{}_{,a} \tag{3.51}$$

$H^{at}{}_{,a}$ will vanish if and only if $\mu^{atb}{}_{,ba}$ vanishes; that is, if and only if

$$ \mu^{atb} = -\mu^{bta}. \tag{3.52}$$

From equations (3.50–3.52) we may infer

$$ s^{ts} = \tfrac{1}{2}(S^{ats} + S^{sta} + S^{tsa})_{,a} \tag{3.53}$$

which, after substitution of S^{ats} from (3.48), becomes

$$ {}_{(n)}s^{ts} = -2(\sqrt{-g}^{(n-1)}\psi^{tsac})_{,ca} - {}_{(n)}h^{ts}. \tag{3.54}$$

Substitution of (3.54) into (3.49) yields the total energy-momentum complex

$$\begin{aligned} {}_{(n)}H^{ts} &= -2(\sqrt{-g}^{(n-1)}\psi^{tsac})_{,ca} \\ &= -(\sqrt{-g}^{(n-1)}\Lambda^{tsac})_{,ca}. \end{aligned} \tag{3.55}$$

Reference to (2.29) shows that (3.55) is indeed symmetric and vanishes under a divergence of either index.

Chapter 4

Particular Energy-Momentum Complexes

4.1 Introduction

The formalism presented thus far has been sufficient to generate the field equations, the general conserved quantity h^a, the energy-momentum complexes ${}_{(n)}h^a{}_s$ and ${}_{(n)}h^{at}$, the angular momentum complex ${}_{(n)}h^{ats}$, and the symmetrized energy-momentum complex ${}_{(n)}H^{ts}$; all without reference to any particular Lagrangian. We will see that these quantities suffice to generate and generalize virtually all energy-momentum complexes currently known[1].

The physical situation in which we are interested is the usual one in Einstein-Maxwell theory: an electromagnetic field in the presence of gravity. In particular, the elimination of the electromagnetic field does not imply the elimination of the gravitational field and hence our Lagrangian takes the form (3.29). The most general such scalar density Lagrangian that generates the Einstein-Maxwell equations without the cosmological term is (Lovelock [18,19])

$$L = L_G + L_{EM} \tag{4.1}$$

[1]The notable exceptions are the complexes of Einstein [9] and Weinberg [32]. Weinberg's complex does not lend itself to derivation via a variational principle. However, our technique may be applied to the Einstein Lagrangian to derive Einstein's complex, although the nonscalar nature of this Lagrangian complicates the analysis and only the mixed weight-one complex ${}_{(1)}h^a{}_s$ (Einstein's) exists.

where[2]

$$\begin{aligned}
L_G &= L_H + \alpha L_\alpha + \beta L_\beta && (4.2)\\
L_{EM} &= L_M + \gamma L_\gamma && (4.3)\\
L_H &= \sqrt{-g}\, R && (4.4)\\
L_\alpha &= \varepsilon^{ijk\ell} R^{ab}{}_{ij} R_{abk\ell} && (4.5)\\
L_\beta &= \sqrt{-g}(RR - 4R_{ij}R^{ij} + R^{ij}{}_{k\ell}R^{k\ell}{}_{ij}) && (4.6)\\
L_M &= \sqrt{-g}\, F_{ij}F^{ij} && (4.7)\\
L_\gamma &= \varepsilon^{ijk\ell} F_{ij} F_{k\ell} && (4.8)
\end{aligned}$$

α, β, and γ are arbitrary constants, and

$$F_{ij} = \phi_{j,i} - \phi_{i,j}. \qquad (4.9)$$

From the above, and Section 3.4 we immediately have

$$\Pi^a = 0 \qquad (4.10)$$

$$\Pi^{abc} = 0. \qquad (4.11)$$

Also, by making use of the symmetry relations (including the antisymmetry of Φ^{ab}) we may rewrite the contributions of L_G and L_{EM} to equation (2.50) in the form

$$\begin{aligned}
{}_G\Pi^{pq} &= \tfrac{1}{2}(g^{pq}\, {}_GL + \psi^{pbcd} R^q{}_{cdb} - 3\psi^{qbcd} R^p{}_{cdb}) && (4.12)\\
{}_{EM}\Pi^{pq} &= {}_{EM}\Lambda^{pq} = \tfrac{1}{2}(g^{pq}\, {}_{EM}L + \Phi^{qb} F^p{}_b). && (4.13)
\end{aligned}$$

The direct calculation of Π^{pq} is quite lengthy and tedious. With equation (4.10–4.13) we can evaluate the Euler-Lagrange equations and the conserved quantities from a knowledge of Φ^{ab} and ψ^{abcd} alone.

[2]We set $c = G = 1$.

4.2 The Euler-Lagrange Equations

As an intermediate step in the calculation of the functional derivatives of the Lagrangian we consider the Lagrangian as a function of the quantities F_{ab} and

$$R_{abcd} = \tfrac{1}{2}(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}) + g_{mn}(\Gamma^m{}_{bc}\Gamma^n{}_{ad} - \Gamma^m{}_{bd}\Gamma^n{}_{ac}). \tag{4.14}$$

From equation (4.9) we can see

$$\frac{\partial F_{ab}}{\partial \phi_{p,q}} = -\delta^p_a\delta^q_b + \delta^p_b\delta^q_a. \tag{4.15}$$

In taking the functional derivative of R_{abcd} we must be careful to consider the symmetry of the metric and its second derivatives. Then

$$\begin{aligned}\frac{\partial R_{abcd}}{\partial g_{pq,rs}} &= \tfrac{1}{8}[(\delta^p_a\delta^q_d + \delta^p_d\delta^q_a)(\delta^r_b\delta^s_c + \delta^r_c\delta^s_b) + (\delta^p_b\delta^q_c + \delta^p_c\delta^q_b)(\delta^r_a\delta^s_d + \delta^r_d\delta^s_a) \\ &\quad -(\delta^p_a\delta^q_c + \delta^p_c\delta^q_a)(\delta^r_b\delta^s_d + \delta^r_d\delta^s_b) - (\delta^p_b\delta^q_d + \delta^p_d\delta^q_b)(\delta^r_a\delta^s_c + \delta^r_c\delta^s_a)].\end{aligned} \tag{4.16}$$

Thus

$$\begin{aligned}{}_M\Phi^{ab} &\equiv \frac{\partial L_M}{\partial \phi_{a,b}} = 2\sqrt{-g}\, g^{ik}g^{j\ell}F_{ij}\frac{\partial F_{k\ell}}{\partial \phi_{a,b}} \\ &= -4\sqrt{-g}\, F^{ab}\end{aligned} \tag{4.17}$$

and

$${}_\gamma\Phi^{ab} = -4\varepsilon^{abij}F_{ij}. \tag{4.18}$$

Similarly

$$\begin{aligned}{}_H\Lambda^{abcd} &\equiv \frac{\partial L_H}{\partial g_{ab,cd}} = \sqrt{-g}\, g^{ik}g^{j\ell}\frac{\partial R_{ijk\ell}}{\partial g_{ab,cd}} \\ &= -\sqrt{-g}(g^{ab}g^{cd} - \tfrac{1}{2}g^{ac}g^{bd} - \tfrac{1}{2}g^{ad}g^{bc})\end{aligned} \tag{4.19}$$

and

$$
\begin{aligned}
{}_{\alpha}\Lambda^{abcd} &= -(\varepsilon^{ijac}R^{bd}{}_{ij} + \varepsilon^{ijbd}R^{ac}{}_{ij} + \varepsilon^{ijad}R^{bc}{}_{ij} + \varepsilon^{ijbc}R^{ad}{}_{ij}) && (4.20)\\
{}_{\beta}\Lambda^{abcd} &= -2\sqrt{-g}[R^{adbc} + R^{acbd} - 2g^{ab}R^{cd} - 2g^{cd}R^{ab}\\
&\quad +g^{ad}R^{bc} + g^{bc}R^{ad} + g^{ac}R^{bd} + g^{bd}R^{ac}\\
&\quad +R(g^{ab}g^{cd} - \tfrac{1}{2}g^{ac}g^{bd} - \tfrac{1}{2}g^{ad}g^{bc})]. && (4.21)
\end{aligned}
$$

Hence, from equation (2.27) we have

$$
\begin{aligned}
{}_{H}\psi^{abcd} &= -\tfrac{1}{2}\sqrt{-g}(g^{ab}g^{cd} - g^{ad}g^{bc}) && (4.22)\\
{}_{\alpha}\psi^{abcd} &= -\tfrac{1}{3}(2\varepsilon^{ijac}R^{bd}{}_{ij} + 2\varepsilon^{ijbd}R^{ac}{}_{ij} + \varepsilon^{ijad}R^{bc}{}_{ij}\\
&\quad +\varepsilon^{ijbc}R^{ad}{}_{ij} + \varepsilon^{ijab}R^{cd}{}_{ij} + \varepsilon^{ijcd}R^{ab}{}_{ij}) && (4.23)\\
{}_{\beta}\psi^{abcd} &= -2\sqrt{-g}[R^{acbd} - (g^{ab}R^{cd} + g^{cd}R^{ab} - g^{ad}R^{bc} - g^{bc}R^{ad})\\
&\quad +\tfrac{1}{2}R(g^{ab}g^{cd} - g^{ad}g^{bc})]. && (4.24)
\end{aligned}
$$

In order to derive the Euler-Lagrange equations (2.40) and (2.45) we must first calculate the quantities Π^{ab}.

$$
{}_{H}\psi^{pbcd}R^{q}{}_{cdb} = \sqrt{-g}\,R^{qp} \qquad (4.25)
$$

so equation (4.12) gives us

$$
{}_{H}\Pi^{pq} = \sqrt{-g}(\tfrac{1}{2}g^{pq}R - R^{pq}) = -\sqrt{-g}\,G^{pq} \qquad (4.26)
$$

where G^{pq} is just the Einstein tensor.

$$
{}_{\alpha}\psi^{pbcd}R^{q}{}_{cdb} = (\varepsilon^{ijpb}R^{cd}{}_{ij} + \varepsilon^{ijcd}R^{pb}{}_{ij})R^{q}{}_{bcd} \qquad (4.27)
$$

and so appealing to (4.12) once more gives

$$
\begin{aligned}
{}_{\alpha}\Pi^{pq} &= \tfrac{1}{2}(g^{pq}\varepsilon^{ijk\ell}R^{ab}{}_{ij}R_{abk\ell} - 2\varepsilon^{ijk\ell}R^{pb}{}_{ij}R^{q}{}_{bk\ell}\\
&\quad +\varepsilon^{ijpb}R^{cd}{}_{ij}R^{q}{}_{bcd} - 3\varepsilon^{ijqb}R^{cd}{}_{ij}R^{p}{}_{bcd}). && (4.28)
\end{aligned}
$$

We can show that ${}_{\alpha}\Pi^{pq}$ is symmetric through the use of the following permutation identities, which result from the fact that $\delta^{pabcd}_{qijk\ell}$ vanishes identically in four dimensions:

$$\begin{aligned}
&\tfrac{1}{24}\delta^{pabcd}_{rijk\ell}g^{rq}\varepsilon^{ijk\ell}R^{mn}{}_{ab}R_{mncd} \equiv 0 \\
&= \quad g^{pq}\varepsilon^{ijk\ell}R^{ab}{}_{ij}R_{abk\ell} - 4\varepsilon^{pbij}R^{q}{}_{bk\ell}R_{ij}{}^{k\ell} \qquad (4.29)
\end{aligned}$$

$$\begin{aligned}
&\tfrac{1}{4}\delta^{pabcd}_{rijk\ell}g^{rq}g_{as}g_{bt}\varepsilon^{stk\ell}R^{mnij}R_{mncd} \equiv 0 \\
&= \quad g^{pq}\varepsilon^{ijk\ell}R^{ab}{}_{ij}R_{abk\ell} - 2\varepsilon^{pbij}R^{q}{}_{bk\ell}R_{ij}{}^{k\ell} - 2\varepsilon^{qbij}R^{p}{}_{bk\ell}R_{ij}{}^{k\ell}. \qquad (4.30)
\end{aligned}$$

Thus, we may write ${}_{\alpha}\Pi^{pq}$ in the symmetric form

$$\begin{aligned}
{}_{\alpha}\Pi^{pq} \quad = \quad & \tfrac{1}{2}(g^{pq}\varepsilon^{ijk\ell}R^{ab}{}_{ij}R_{abk\ell} - 2\varepsilon^{ijk\ell}R^{pb}{}_{ij}R^{q}{}_{bk\ell} \\
& -\varepsilon^{pbij}R^{q}{}_{bk\ell}R_{ij}{}^{k\ell} - \varepsilon^{qbij}R^{p}{}_{bk\ell}R_{ij}{}^{k\ell}). \qquad (4.31)
\end{aligned}$$

We can now further simplify ${}_{\alpha}\Pi^{pq}$ via the permutation identity

$$\begin{aligned}
&-\delta^{pabcd}_{rijk\ell}g^{rq}g_{as}\varepsilon^{sjmn}R^{i}{}_{ncd}R_{mb}{}^{k\ell} \equiv 0 \\
&= \quad g^{pq}\varepsilon^{ijk\ell}R^{ab}{}_{ij}R_{abk\ell} - 2\varepsilon^{ijk\ell}R^{pb}{}_{ij}R^{q}{}_{bk\ell} \\
&\qquad -(\varepsilon^{pbij}R^{q}{}_{bk\ell} + \varepsilon^{qbij}R^{p}{}_{bk\ell})R_{ij}{}^{k\ell} \\
&\qquad +2(\varepsilon^{pbij}R^{qd}{}_{ij} + \varepsilon^{qbij}R^{pd}{}_{ij})R_{bd} \qquad (4.32)
\end{aligned}$$

which, when substituted into (4.31) gives

$$ {}_{\alpha}\Pi^{pq} = -(\varepsilon^{pbij}R^{qd}{}_{ij} + \varepsilon^{qbij}R^{pd}{}_{ij})R_{bd}. \qquad (4.33)$$

In order to calculate ${}_{\beta}\Pi^{pq}$ we first note

$$ {}_{\beta}\psi^{pbcd}R^{q}{}_{cdb} = 2\sqrt{-g}(R^{pb}{}_{cd}R^{q}{}_{b}{}^{cd} - 2R^{pb}R^{q}{}_{b} - 2R^{pbqd}R_{bd} + R^{pq}R) \qquad (4.34)$$

which is clearly symmetric in the indices p, q. Thus, after substituting into equation (4.12) we have

$$\begin{aligned}
{}_{\beta}\Pi^{pq} \quad = \quad & \tfrac{1}{2}\sqrt{-g}[g^{pq}(R^{ab}{}_{cd}R_{ab}{}^{cd} - 4R^{ab}R_{ab} + RR) \\
& -4(R^{pb}{}_{cd}R^{q}{}_{b}{}^{cd} - 2R^{pb}R^{q}{}_{b} - 2R^{pbqd}R_{bd} + R^{pq}R)]. \qquad (4.35)
\end{aligned}$$

However, by another permutation identity, we have the Bach-Lanczos identity [1,16]

$$\begin{aligned}
&\tfrac{1}{4}\delta^{pijk\ell}_{rabcd}g^{rq}R^{ab}{}_{k\ell}R_{ij}{}^{cd} \equiv 0 \\
&= \; g^{pq}(R^{ab}{}_{cd}R_{ab}{}^{cd} - 4R^{ab}R_{ab} + RR) \\
&\quad -4(R^{pb}{}_{cd}R^{q}{}_{b}{}^{cd} - 2R^{pb}R^{q}{}_{b} - 2R^{pbqd}R_{bd} + R^{pq}R)
\end{aligned} \tag{4.36}$$

and hence ${}_{\beta}\Pi^{pq}$ vanishes

$${}_{\beta}\Pi^{pq} = 0. \tag{4.37}$$

The electromagnetic Lagrangians yield simpler concomitants. From equations (4.13) and (4.17) we have

$$\begin{aligned}
{}_{M}\Pi^{pq} &= \sqrt{-g}(\tfrac{1}{2}g^{pq}F^{ab}F_{ab} - 2F^{pb}F^{q}{}_{b}) \\
&= 8\pi\sqrt{-g}\,T^{pq}
\end{aligned} \tag{4.38}$$

where T^{pq} is just the usual electromagnetic stress-energy tensor.

$${}_{\gamma}\Pi^{pq} = \tfrac{1}{2}(g^{pq}\varepsilon^{abcd}F_{ab}F_{cd} - 4\varepsilon^{qbcd}F^{p}{}_{b}F_{cd}) \tag{4.39}$$

but, by a permutation identity analogous to equation (4.29) we see

$$\begin{aligned}
&\tfrac{1}{24}\delta^{qabcd}_{rijk\ell}g^{pr}\varepsilon^{ijk\ell}F_{ab}F_{cd} \equiv 0 \\
&= \; g^{pq}\varepsilon^{abcd}F_{ab}F_{cd} - 4\varepsilon^{qbcd}F^{p}{}_{b}F
\end{aligned} \tag{4.40}$$

so ${}_{\gamma}\Pi^{pq}$ also vanishes

$${}_{\gamma}\Pi^{pq} = 0. \tag{4.41}$$

It is clear from equation (4.19) that ${}_{H}\Lambda^{abcd}{}_{;m}$ is zero and as is evident from equation (4.36)

$$-\tfrac{1}{2}\sqrt{-g}\,\delta^{mnac}_{ijk\ell}g^{ib}g^{jd}R^{k\ell}{}_{mn} = {}_{\beta}\psi^{abcd} \tag{4.42}$$

which implies that ${}_\beta\psi^{abcd}{}_{;d}$, ${}_\beta\psi^{abcd}{}_{;c}$ and hence ${}_\beta\Lambda^{abcd}{}_{;d}$ vanish via the Bianchi identities. Taking covariant derivatives of (4.20) we have

$$
\begin{aligned}
{}_\alpha\Lambda^{abcd}{}_{;d} &= 2(\varepsilon^{ijca}R^{b}{}_{i;j} - \varepsilon^{ijbc}R^{a}{}_{i;j}) &\quad (4.43)\\
{}_\alpha\psi^{abcd}{}_{;d} &= \tfrac{2}{3}(2\varepsilon^{ijca}R^{b}{}_{i;j} - \varepsilon^{ijbc}R^{a}{}_{i;j} - \varepsilon^{ijab}R^{c}{}_{i;j}) &\quad (4.44)\\
{}_\alpha\Lambda^{pqcd}{}_{;d;c} &= (\varepsilon^{pbij}R^{qd}{}_{ij} + \varepsilon^{qdij}R^{pb}{}_{ij})R_{bd}. &\quad (4.45)
\end{aligned}
$$

From equations (4.9) and (4.18) it is also evident that ${}_\gamma\Phi^{ab}{}_{;b}$ is identically zero.

Thus the Euler-Lagrange equations (2.40) and (2.45) reduce to

$$\sqrt{-g}(G^{pq} - 8\pi T^{pq}) = 0 \qquad (4.46)$$

$$-4\sqrt{-g}\,F^{pq}{}_{;q} = 0, \qquad (4.47)$$

the Einstein-Maxwell equations.

4.3 The Energy-Momentum Complexes

Having calculated all of the relevant quantities we are now in a position to examine conserved quantities generated from the Lagrangian (4.1). Equations (3.34), (3.40), (3.42) and (3.55) now read:

$$
\begin{aligned}
h^{a} &= [\tfrac{1}{2}({}_M\Phi^{ac} + \gamma\,{}_\gamma\Phi^{ac})\phi_p\chi^p \\
&\qquad + ({}_H\psi^{abcd} + \alpha\,{}_\alpha\psi^{abcd} + \beta\,{}_\beta\psi^{abcd})\chi_{b,d} - 2\alpha\,{}_\alpha\psi^{abcd}{}_{;d}\chi_b]_{,c} \\
&= \{-2(\sqrt{-g}\,F^{ac} + \gamma\varepsilon^{acij}F_{ij})\phi_p\chi^p - (\tfrac{1}{2}\sqrt{-g}(g^{ab}g^{cd} - g^{ad}g^{bc}) \\
&\qquad + \tfrac{1}{3}\alpha(2\varepsilon^{ijac}R^{bd}{}_{ij} + 2\varepsilon^{ijbd}R^{ac}{}_{ij} + \varepsilon^{ijad}R^{bc}{}_{ij} \\
&\qquad + \varepsilon^{ijbc}R^{ad}{}_{ij} + \varepsilon^{ijab}R^{cd}{}_{ij} + \varepsilon^{ijcd}R^{ab}{}_{ij}) \\
&\qquad + 2\beta\sqrt{-g}[R^{acbd} - (g^{ab}R^{cd} + g^{cd}R^{ab} - g^{ad}R^{bc} - g^{bc}R^{ad}) \\
&\qquad + \tfrac{1}{2}R(g^{ab}g^{cd} - g^{ad}g^{bc})])\chi_{b,d}
\end{aligned}
$$

$$+\tfrac{4}{3}\alpha(2\varepsilon^{ijac}R^{b}{}_{i;j}+\varepsilon^{ijbc}R^{a}{}_{i;j}+\varepsilon^{ijab}R^{c}{}_{i;j})\chi_b\}_{,c} \tag{4.48}$$

$$\begin{aligned}
{}_{(n)}h^{a}{}_{s} &= [\tfrac{1}{2}\sqrt{-g}^{(n-1)}({}_{M}\Phi^{ac}+\gamma\,{}_{\gamma}\Phi^{ac})\phi_s \\
&\quad +(\sqrt{-g}^{(n-1)}g_{bs})_{,d}({}_{H}\psi^{abcd}+\alpha\,{}_{\alpha}\psi^{abcd}+\beta\,{}_{\beta}\psi^{abcd}) \\
&\quad -2\alpha\sqrt{-g}^{(n-1)}g_{bs}\,{}_{\alpha}\psi^{abcd}{}_{;d}]_{,c} \\
&= \{-2\sqrt{-g}^{(n-1)}(\sqrt{-g}\,F^{ac}+\gamma\varepsilon^{acij}F_{ij})\phi_s \\
&\quad -(\sqrt{-g}^{(n-1)}g_{bs})_{,d}(\tfrac{1}{2}\sqrt{-g}(g^{ab}g^{cd}-g^{ad}g^{bc}) \\
&\quad +\tfrac{1}{3}\alpha(2\varepsilon^{ijac}R^{bd}{}_{ij}+2\varepsilon^{ijbd}R^{ac}{}_{ij}+\varepsilon^{ijad}R^{bc}{}_{ij} \\
&\quad +\varepsilon^{ijbc}R^{ad}{}_{ij}+\varepsilon^{ijab}R^{cd}{}_{ij}+\varepsilon^{ijcd}R^{ab}{}_{ij}) \\
&\quad +2\beta\sqrt{-g}[R^{acbd}-(g^{ab}R^{cd}+g^{cd}R^{ab}-g^{ad}R^{bc}-g^{bc}R^{ad}) \\
&\quad +\tfrac{1}{2}R(g^{ab}g^{cd}-g^{ad}g^{bc})]) \\
&\quad +\tfrac{4}{3}\alpha\sqrt{-g}^{(n-1)}(2\varepsilon^{ijac}R_{si;j}+g_{bs}\varepsilon^{ijbc}R^{a}{}_{i;j}+g_{bs}\varepsilon^{ijab}R^{c}{}_{i;j})\}_{,c}
\end{aligned} \tag{4.49}$$

$$\begin{aligned}
{}_{(n)}h^{at} &= [\tfrac{1}{2}\sqrt{-g}^{(n-1)}({}_{M}\Phi^{ac}+\gamma\,{}_{\gamma}\Phi^{ac})\phi^t \\
&\quad +\sqrt{-g}^{(n-1)}{}_{,d}({}_{H}\psi^{atcd}+\alpha\,{}_{\alpha}\psi^{atcd}+\beta\,{}_{\beta}\psi^{atcd}) \\
&\quad -2\alpha\sqrt{-g}^{(n-1)}{}_{\alpha}\psi^{atcd}{}_{;d}]_{,c} \\
&= \{-2\sqrt{-g}^{(n-1)}(\sqrt{-g}\,F^{ac}+\gamma\varepsilon^{acij}F_{ij})\phi^t \\
&\quad -\sqrt{-g}^{(n-1)}{}_{,d}(\tfrac{1}{2}\sqrt{-g}(g^{at}g^{cd}-g^{ad}g^{tc}) \\
&\quad +\tfrac{1}{3}\alpha(2\varepsilon^{ijac}R^{td}{}_{ij}+2\varepsilon^{ijtd}R^{ac}{}_{ij}+\varepsilon^{ijad}R^{tc}{}_{ij} \\
&\quad +\varepsilon^{ijtc}R^{ad}{}_{ij}+\varepsilon^{ijat}R^{cd}{}_{ij}+\varepsilon^{ijcd}R^{at}{}_{ij}) \\
&\quad +2\beta\sqrt{-g}[R^{actd}-(g^{at}R^{cd}+g^{cd}R^{at}-g^{ad}R^{tc}-g^{tc}R^{ad}) \\
&\quad +\tfrac{1}{2}R(g^{at}g^{cd}-g^{ad}g^{tc})]) \\
&\quad +\tfrac{4}{3}\alpha\sqrt{-g}^{(n-1)}(2\varepsilon^{ijac}R^{t}{}_{i;j}+\varepsilon^{ijtc}R^{a}{}_{i;j}+\varepsilon^{ijat}R^{c}{}_{i;j})\}_{,c}
\end{aligned} \tag{4.50}$$

$$
\begin{aligned}
{}_{(n)}H^{at} &= -2[\sqrt{-g}^{(n-1)}({}_H\psi^{atcd} + \alpha\, {}_\alpha\psi^{atcd} + \beta\, {}_\beta\psi^{atcd})]_{,dc} \\
&= \{\sqrt{-g}^{(n)}(g^{at}g^{cd} - g^{ad}g^{tc}) + 2\alpha\sqrt{-g}^{(n-1)}(\varepsilon^{ijac}R^{td}{}_{ij} + \varepsilon^{ijtd}R^{ac}{}_{ij}) \\
&\quad +4\beta\sqrt{-g}^{(n)}[R^{actd} - (g^{at}R^{cd} + g^{cd}R^{at} - g^{ad}R^{tc} - g^{tc}R^{ad}) \\
&\quad +\tfrac{1}{2}R(g^{at}g^{cd} - g^{ad}g^{tc})]\}_{,dc}. \qquad (4.51)
\end{aligned}
$$

We have seen that neither L_α, L_β, nor L_γ has any effect on the Einstein-Maxwell equations. L_γ does not contribute to the usual electromagnetic stress energy tensor [see (4.51)] and hence is rarely encountered. L_α and L_β both contribute to the complexes (4.49–4.51) and other objects derived from (4.48) and neither contribution vanishes with the connection. At the same time they are nonzero in vacuum. The presence of the Riemann tensor in the α and β terms is also worthy of note. These may prove of interest in studies of gravitational radiation.

Having presented the general formulas, we now derive previously encountered energy-momentum complexes. Other than the weight-one α and β terms in ${}_{(n)}h^a{}_s$ (which can be found in Goenner and Kohler [12,13]), these all require $\alpha = \beta = \gamma = 0$.

Equation (4.48) now reads

$$
\begin{aligned}
h^a &= (\tfrac{1}{2}\,{}_M\Phi^{ac}\phi_p\chi^p + {}_H\psi^{abcd}\chi_{b,d})_{,c} \\
&= [\tfrac{1}{2}\sqrt{-g}(-4F^{ac}\phi_p\chi^p + \chi^{c;a} + \chi^{a;c})]_{,c} \qquad (4.52)
\end{aligned}
$$

which, aside from the electromagnetic term, is Komar's complex. Equations (4.49–4.51) become

$$
\begin{aligned}
{}_{(n)}h^a{}_s &= [\tfrac{1}{2}\sqrt{-g}^{(n-1)}{}_M\Phi^{ac}\phi_s + (\sqrt{-g}^{(n-1)}g_{bs})_{,d}\,{}_H\psi^{abcd}]_{,c} \\
&= \tfrac{1}{2}[-4\sqrt{-g}^{(n)}F^{ac}\phi_s + \sqrt{-g}^{(n)}g^{ap}g^{cq}(g_{ps,q} + g_{qs,p}) \\
&\quad +\sqrt{-g}^{(n-1)}{}_{,p}\sqrt{-g}(\delta^a_s g^{cp} - \delta^c_s g^{ap})]_{,c} \qquad (4.53)
\end{aligned}
$$

$$
\begin{aligned}
{}_{(n)}h^{at} &= [\tfrac{1}{2}\sqrt{-g}^{(n-1)}{}_M\Phi^{ac}\phi^t + \sqrt{-g}^{(n-1)}{}_{,d}\,{}_H\psi^{atcd}]_{,c} \\
&= -\tfrac{1}{2}[4\sqrt{-g}^{(n)}F^{ac}\phi^t + \sqrt{-g}^{(n-1)}{}_{,d}\sqrt{-g}(g^{at}g^{cd} - g^{ad}g^{tc})]_{,c} \qquad (4.54)
\end{aligned}
$$

$$
\begin{aligned}
{}_{(n)}H^{at} &= -2[\sqrt{-g}^{(n-1)}{}_H\psi^{atcd}]_{,dc} \\
&= [\sqrt{-g}^{(n)}(g^{at}g^{cd} - g^{ad}g^{tc})]_{,dc}. \qquad (4.55)
\end{aligned}
$$

Neglecting the electromagnetic terms, we see that (4.53) is a generalization of the ($n = 1$) Møller [22] complex to arbitrary weight. Equation (4.54) is the generalization to arbitrary weight of a complex briefly considered by Lorentz and Levi-Civita and later rejected by Einstein (see Pauli [25]). For $n = 1$ the nonelectromagnetic terms vanish. Thus, introducing the field equations, we may write the gravitational part of ${}_{(1)}h^{at}$ in the form

$$
{}_{(1)}t_G{}^{at} = \sqrt{-g}\, G^{at} \qquad (4.56)
$$

where G^{at} is the Einstein tensor. This complex was considered unsuitable because it permits the existence of nonempty spacetimes with zero total energy. Equation (4.55) gives the infinite family of complexes obtained by Goldberg, as generalizations of the Landau and Lifshitz complex, which is given by (4.55) with $n = 2$. To the best of the author's knowledge, this is the first time the Landau and Lifshitz complex has been derived via a variational principle.

The complex ${}_{(n)}H^{at}$ is of interest on another note. Setting $n = 1$ and substituting (3.55) into the Euler-Lagrange equation (2.40), we have (dropping the subscript)

$$
H^{ab} = \Lambda^{abcd}{}_{,dc} = \Lambda^{ab} - \Lambda^{abc}{}_{,c} \qquad (4.57)
$$

a relation which will also hold for more general Lagrangians including matter terms (as long as they do not contain derivatives of higher than second order in the metric). This is just the expression given by Landau and Lifshitz [17] for the energy-momentum density of a nongravitational field (note that our definition $c = G = 1$ differs from that of Landau and Lifshitz). Landau and Lifshitz, having generated the Einstein equations with the Einstein Lagrangian for which (4.57) vanishes identically, point out that their relation does not hold for gravitation. In fact, for the Einstein Lagrangian, equation (4.57) simply gives the nongravitational contribution to the Einstein equations: the stress-energy tensor. However the previous analysis has shown that, using (for instance) the Hilbert Lagrangian $\sqrt{-g}\,R$ with the appropriate interpretation of H^{ij} as a combined energy-momentum complex, this relation does, in fact, hold (in a global sense at least) for gravitation.

Chapter 5

A Palatini Variational Principle

5.1 Introduction

In this chapter we investigate the Palatini variation of a class of Lagrangians based on that of Nissani [23]. A Palatini variation differs from the Hilbert variation considered in the previous chapters in that the fundamental quantities chosen to represent the gravitational field are the metric, the connection and its first derivatives rather than the metric and its first and second derivatives.

The Palatini variational principle is sometimes considered to be superior to the Hilbert variation because, for the usual Lagrangian $\sqrt{-g}\,R$, the connection may be generalized from the Christoffel symbol to an arbitrary symmetric connection. Then the Euler-Lagrange equation resulting from the variation of the generalized connection requires that the connection be the Christoffel symbol. That is, the variation of the connection specifies the geometry to be Riemannian. In this sense the Palatini variation is more general than the Hilbert variation, in which Riemannian geometry must be assumed. However, if additional terms are added to the Lagrangian, to include matter for instance, then any further functional dependence on the connection will disrupt the Euler-Lagrange equation, once more relegating the Riemannian condition to the status of an assumption.

Nissani presents a Lagrangian which, he claims, generates Einstein's equations and specifies Riemannian geometry in the presence of a general matter tensor. But, as we will show, his latter claim is not justified; Riemannian geometry must, in fact, be assumed.

Nissani's variational strategy is quite different from the usual one, and is perhaps unique. He defines a Lagrangian with the functional form

$$L = L(g_{ab}, \Gamma^a{}_{bc}, \Gamma^a{}_{bc,d}, W_{abcd}, M_{abcd}) \tag{5.1}$$

where W_{abcd}, M_{abcd} are tensors satisfying

$$W_{abcd} = -W_{abdc} \tag{5.2}$$

$$M_{abcd} = -M_{abdc}. \tag{5.3}$$

The generalized matter tensor M_{abcd} is to be considered arbitrary during the variation but will later be restricted to the form

$$M_{abcd} = \tfrac{1}{2}(g_{ac}T_{bd} - g_{ad}T_{bc} - g_{bc}T_{ad} + g_{bd}T_{ac}) + \tfrac{1}{3}(g_{ad}g_{bc} - g_{ac}g_{bd})T \tag{5.4}$$

where T_{ab} is the stress-energy tensor and T is its trace. It is the Euler-Lagrange equations resulting from the variation of M_{abcd} and W_{abcd} which generate the Einstein equations. The first of these is

$$W_{abcd} + M_{abcd} = R_{abcd}. \tag{5.5}$$

The second, with (5.5) is

$$W^a{}_{bad} = 0 \tag{5.6}$$

which, after invoking equation (5.4), gives the Einstein equations and establishes W_{abcd} as the Weyl tensor.

Nissani's Lagrangian is of interest for several reasons. Unlike the usual formulations of general relativity, his Lagrangian follows the pattern of other field theories and permits the derivation of the field equations from a Lagrangian which is quadratic in the field quantities. Also Nissani's Lagrangian is based upon the Lagrangian [3,4,5] of Carmeli's $SL(2,C)$ gauge theory of gravitation.

We investigate this Lagrangian here in order to study the effects of an alternate set of fundamental quantities on both a new Lagrangian and, since Nissani's class contains the usual Palatini class, the gravitational Lagrangians considered previously. In particular we will be examining the Euler-Lagrange equation resulting from the variation of the connection. Further we will show that Nissani's Lagrangian is not unique.

5.2 The Variation

In this section we take the Palatini variation of the class of Lagrangians with the functional dependence (5.1) where $\Gamma^a{}_{bc}$ is an arbitrary symmetric connection. As the symmetry properties of W_{abcd} and M_{abcd} do not affect the analysis of this section we will, for the time being, consider them to be undetermined. When we consider Nissani's Lagrangian we will assume they obey equations (5.2) and (5.3), respectively. However, in order to derive the field equations of general relativity it will, as we shall see, be necessary to assume Riemannian geometry in which case both W_{abcd} and M_{abcd} must be antisymmetric in their first, as well as their second, pair of indices. Thereafter we shall assume

$$W_{abcd} = -W_{abdc} = -W_{bacd} \tag{5.7}$$

$$M_{abcd} = -M_{abdc} = -M_{bacd}. \tag{5.8}$$

Using the definitions (2.7) for a coordinate transformation $\tilde{x}^i = \tilde{x}^i(x^a)$ and noting that

$$(C^{-1})^i{}_{a,b} = -C^c{}_{jk}(C^{-1})^i{}_c(C^{-1})^j{}_a(C^{-1})^k{}_b \tag{5.9}$$

we have the following transformation laws for $\Gamma^a{}_{bc}$ and $\Gamma^a{}_{bc,d}$

$$\tilde{\Gamma}^i{}_{jk} = (C^{-1})^i{}_a C^b{}_j C^c{}_k \Gamma^a{}_{bc} + (C^{-1})^i{}_a C^a{}_{jk} \tag{5.10}$$

$$\begin{aligned}\tilde{\Gamma}^i{}_{jk,\ell} = {} & (C^{-1})^i{}_a C^b{}_j C^c{}_k C^d{}_\ell \Gamma^a{}_{bc,d} \\ & +[-C^d{}_{m\ell}(C^{-1})^i{}_d(C^{-1})^m{}_a C^b{}_j C^c{}_k + (C^{-1})^i{}_a C^b{}_{j\ell} C^c{}_k + (C^{-1})^i{}_a C^b{}_j C^c{}_{k\ell}]\Gamma^a{}_{bc} \\ & -C^d{}_{m\ell}(C^{-1})^i{}_d(C^{-1})^m{}_a C^a{}_{jk} + (C^{-1})^i{}_a C^a{}_{jk\ell}.\end{aligned} \tag{5.11}$$

We introduce the following notation for the functional derivatives of L

$$\Lambda^{ab} = \frac{\partial L}{\partial g_{ab}}, \qquad \tau_a{}^{bc} = \frac{\partial L}{\partial \Gamma^a{}_{bc}}, \qquad \tau_a{}^{bcd} = \frac{\partial L}{\partial \Gamma^a{}_{bc,d}}$$
$$\Omega^{abcd} = \frac{\partial L}{\partial W_{abcd}}, \qquad \mu^{abcd} = \frac{\partial L}{\partial M_{abcd}} \tag{5.12}$$

which obey the symmetry relations

$$\Lambda^{ab} = \Lambda^{ba} \tag{5.13}$$
$$\tau_a{}^{bc} = \tau_a{}^{cb} \tag{5.14}$$
$$\tau_a{}^{bcd} = \tau_a{}^{cbd} \tag{5.15}$$
$$\Omega^{abcd} = -\Omega^{abdc} = -\Omega^{bacd} \tag{5.16}$$
$$\mu^{abcd} = -\mu^{abdc} = -\mu^{bacd}. \tag{5.17}$$

Differentiating the transformation laws of $\tilde{L}(\tilde{g}_{ij}, \tilde{\Gamma}^i{}_{jk}, \tilde{\Gamma}^i{}_{jk,\ell}, \tilde{W}_{ijk\ell}, \tilde{M}_{ijk\ell})$ and its arguments with respect to the arguments of L, yields the following transformation laws for the quantities (5.12)

$$\tilde{\Lambda}^{ij} C^a{}_i C^b{}_j = C\Lambda^{ab} \tag{5.18}$$

$$\tilde{\tau}_i{}^{jk} (C^{-1})^i{}_a C^b{}_j C^c{}_k + \tilde{\tau}_i{}^{jk\ell} [(C^{-1})^i{}_a C^b{}_j C^c{}_k]_{,\ell} = C\tau_a{}^{bc} \tag{5.19}$$

$$\tilde{\tau}_i{}^{jk\ell} (C^{-1})^i{}_a C^b{}_j C^c{}_k C^d{}_\ell = C\tau_a{}^{bcd} \tag{5.20}$$

$$\tilde{\Omega}^{ijk\ell} C^a{}_i C^b{}_j C^c{}_k C^d{}_\ell = C\Omega^{abcd} \tag{5.21}$$

$$\tilde{\mu}^{ijk\ell} C^a{}_i C^b{}_j C^c{}_k C^d{}_\ell = C\mu^{abcd}. \tag{5.22}$$

Only $\tau_a{}^{bc}$ is not a tensor density.

We note that

$$\frac{\partial (C^{-1})^i{}_a}{\partial C^p{}_q} = -(C^{-1})^i{}_p (C^{-1})^q{}_a. \tag{5.23}$$

Then, differentiating the transformation laws of $\tilde{L}$ and its arguments with respect to the coordinate transform $C^p{}_q$ and its derivatives and considering the particular case of the identity transformation yields the invariance relations

$$\tau_p{}^{qrs} + \tau_p{}^{rsq} + \tau_p{}^{sqr} = 0 \tag{5.24}$$

$$\begin{aligned}
&\tau_p{}^{qr} + \tau_p{}^{rq} - \tau_p{}^{bcq}\Gamma^r{}_{bc} - \tau_p{}^{bcr}\Gamma^q{}_{bc} \\
&\quad +(\tau_a{}^{qcr} + \tau_a{}^{rcq})\Gamma^a{}_{pc} + (\tau_a{}^{bqr} + \tau_a{}^{brq})\Gamma^a{}_{bp} = 0
\end{aligned} \tag{5.25}$$

$$\begin{aligned}
&\mu^{qbcd}M_{pbcd} + \mu^{aqcd}M_{apcd} + \mu^{abqd}M_{abpd} + \mu^{abcq}M_{abcp} \\
&\quad +\Omega^{qbcd}W_{pbcd} + \Omega^{aqcd}W_{apcd} + \Omega^{abqd}W_{abpd} + \Omega^{abcq}W_{abcp} \\
&\quad -\tau_p{}^{bcd}\Gamma^q{}_{bc,d} + \tau_a{}^{qcd}\Gamma^a{}_{pc,d} + \tau_a{}^{bqd}\Gamma^a{}_{bp,d} + \tau_a{}^{bcq}\Gamma^a{}_{bc,p} \\
&\quad -\tau_p{}^{bc}\Gamma^q{}_{bc} + \tau_a{}^{qc}\Gamma^a{}_{pc} + \tau_a{}^{bq}\Gamma^a{}_{bp} + \Lambda^{qb}g_{pb} + \Lambda^{aq}g_{ap} = \delta^q_p L.
\end{aligned} \tag{5.26}$$

It is appropriate, here, to define the useful quantity $\nu_n{}^{abc}$, analogous to the quantity ψ^{abcd} introduced in Section (2.3),

$$\nu_n{}^{abc} \equiv \tfrac{1}{3}(\tau_n{}^{abc} - \tau_n{}^{cba}) = -\nu_n{}^{cba}. \tag{5.27}$$

The new quantity satisfies the relations

$$\nu_n{}^{abc} + \nu_n{}^{bac} = \tau_n{}^{abc} \tag{5.28}$$

$$\nu_n{}^{abc} + \nu_n{}^{bca} + \nu_n{}^{cab} = 0. \tag{5.29}$$

As $\tau_a{}^{bc}$ is non-tensorial, we now wish to derive its tensor concomitant. We introduce the arbitrary symmetric (in its lower indices) tensor $h^a{}_{bc}$ and define the quantity

$$F = \tau_a{}^{bcd}h^a{}_{bc,d} + \tau_a{}^{bc}h^a{}_{bc}. \tag{5.30}$$

Using the transformation laws for $\tau_a{}^{bcd}$ and $\tau_a{}^{bc}$ it is easy to show that F is a scalar density.

We now assume the existence of a quantity $\Pi_a{}^{bc}$ such that

$$F = \tau_a{}^{bcd} h^a{}_{bc;d} + \Pi_a{}^{bc} h^a{}_{bc}. \tag{5.31}$$

Comparing (5.30) and (5.31) we conclude that

$$\Pi_a{}^{bc} = \tau_a{}^{bc} - \tau_m{}^{bcd}\Gamma^m{}_{ad} + \tau_a{}^{mcd}\Gamma^b{}_{md} + \tau_a{}^{bmd}\Gamma^c{}_{md} = \Pi_a{}^{cb} \tag{5.32}$$

and, since $F - \tau_a{}^{bcd} h^a{}_{bc;d}$ is a scalar density, we see that $\Pi_a{}^{bc}$ is clearly a tensor density.

In order to show that the Euler-Lagrange equation which will result from the variation of the connection is a tensor equation we rewrite (5.30) and (5.31) in the form

$$\begin{aligned} F &= (\tau_a{}^{bc} - \tau_a{}^{bcd}{}_{,d}) h^a{}_{bc} + (\tau_a{}^{bcd} h^a{}_{bc})_{,d} \\ &= (\Pi_a{}^{bc} - \tau_a{}^{bcd}{}_{;d}) h^a{}_{bc} + (\tau_a{}^{bcd} h^a{}_{bc})_{;d}. \end{aligned} \tag{5.33}$$

But, since $\tau_a{}^{bcd} h^a{}_{bc}$ is a vector density, this implies that

$$\begin{aligned} E_a{}^{bc} &= -\tau_a{}^{bc} + \tau_a{}^{bcd}{}_{,d} \\ &= -\Pi_a{}^{bc} + \tau_a{}^{bcd}{}_{;d} \end{aligned} \tag{5.34}$$

from which it is clear that $E_a{}^{bc}$ is a tensor density.

The conversion of the invariance relations proceeds much more smoothly here than in Chapter 2. Equation (5.24) is, of course, already in tensor form. Substitution of $\Pi_a{}^{bc}$ for $\tau_a{}^{bc}$ in (5.25) yields, by (5.24) and the symmetry of the connection

$$\Pi_p{}^{qr} = 0. \tag{5.35}$$

In order to obtain a tensor form of (5.26) we must convert the expression

$$\begin{aligned} \zeta^p{}_q \equiv\; & -\tau_p{}^{bcd}\Gamma^q{}_{bc,d} + \tau_a{}^{qcd}\Gamma^a{}_{pc,d} + \tau_a{}^{bqd}\Gamma^a{}_{bp,d} + \tau_a{}^{bcq}\Gamma^a{}_{bc,p} \\ & -\tau_p{}^{bc}\Gamma^q{}_{bc} + \tau_a{}^{qc}\Gamma^a{}_{pc} + \tau_a{}^{bq}\Gamma^a{}_{bp}. \end{aligned} \tag{5.36}$$

As the connection is non-tensorial we should expect the resulting expression to consist of terms in the Riemann tensor. Again substituting $\Pi_a{}^{bc}$ for $\tau_a{}^{bc}$ leads, after applying (5.24) and (5.35) to

$$\begin{aligned}\zeta^p{}_q &= -\tau_p{}^{bcd}(\Gamma^q{}_{bc,d} - \Gamma^m{}_{bd}\Gamma^q{}_{mc} - \Gamma^m{}_{cd}\Gamma^q{}_{bm}) \\ &\quad +\tau_a{}^{qcd}(\Gamma^a{}_{pc,d} + \Gamma^a{}_{md}\Gamma^m{}_{pc} - \Gamma^m{}_{cd}\Gamma^a{}_{pm}) \\ &\quad +\tau_a{}^{bqd}(\Gamma^a{}_{bp,d} + \Gamma^a{}_{md}\Gamma^m{}_{bp} - \Gamma^m{}_{bd}\Gamma^a{}_{mp}) + \tau_a{}^{bcq}\Gamma^a{}_{bc,p}.\end{aligned} \tag{5.37}$$

Changing $\tau_p{}^{bcd}$ to ν's via (5.28) and applying the first invariance relation (5.24) to $\tau_a{}^{bcq}$ quickly reduces $\zeta_p{}^q$ to the form

$$\zeta^p{}_q = \nu_p{}^{bcd}R^q{}_{cbd} + \tau_a{}^{qcd}R^a{}_{cdp} + \tau_a{}^{bqd}R^a{}_{bdp}. \tag{5.38}$$

Thus the tensor form of the third invariance relation is

$$\begin{aligned}\delta^q_p L &= \Lambda^{qb}g_{pb} + \Lambda^{aq}g_{ap} + \nu_p{}^{bcd}R^q{}_{cbd} + \tau_a{}^{qcd}R^a{}_{cdp} + \tau_a{}^{bqd}R^a{}_{bdp} \\ &\quad +\Omega^{qbcd}W_{pbcd} + \Omega^{aqcd}W_{apcd} + \Omega^{abqd}W_{abpd} + \Omega^{abcq}W_{abcp} \\ &\quad +\mu^{qbcd}M_{pbcd} + \mu^{aqcd}M_{apcd} + \mu^{abqd}M_{abpd} + \mu^{abcq}M_{abcp}.\end{aligned} \tag{5.39}$$

Finally, for notational convenience, we *formally* extend the covariant differentiation operation to apply to the nontensorial quantities $\Gamma^a{}_{bc}$ and $\Gamma^a{}_{bc,d}$ and convert

$$\omega_p = \tau_a{}^{bc}\Gamma^a{}_{cb;p} + \tau_a{}^{bcd}\Gamma^a{}_{bc,d;p} \tag{5.40}$$

in order to write $L_{;p}$ entirely in terms of tensor quantities. Substituting $\Pi_a{}^{bc}$ for $\tau_a{}^{bc}$ and use of (5.24) and (5.35) gives

$$\omega_p = \tau_a{}^{bcd}(\Gamma^a{}_{cb,d} + \Gamma^a{}_{md}\Gamma^m{}_{bc})_{;p}. \tag{5.41}$$

Once again introducing ν's through equation (5.28) we have

$$\omega_p = -\nu_a{}^{bcd}R^a{}_{cbd;p} = -2\nu_a{}^{bcd}R^a{}_{cbp;d} \tag{5.42}$$

where, in anticipation of the next stage, we have made use of the Bianchi identities to obtain the second equality. Thus

$$L_{;p} = \Lambda^{ab} g_{ab;p} - 2\nu_a{}^{bcd} R^a{}_{cbp;d} + \Omega^{abcd} W_{abcd;p} + \mu^{abcd} M_{abcd;p}. \tag{5.43}$$

We now consider the action (2.1) where L is a scalar density with the functional dependence (5.1). The variation of the action is of the form (3.1) with

$$\begin{aligned} \delta L &= \Lambda^{ab}\, \delta g_{ab} + \tau_a{}^{bc}\, \delta\Gamma^a{}_{bc} + \tau_a{}^{bcd}\, \delta\Gamma^a{}_{bc,d} \\ &\quad + \Omega^{abcd}\, \delta W_{abcd} + \mu^{abcd}\, \delta M_{abcd}. \end{aligned} \tag{5.44}$$

The resulting Euler-Lagrange equations are

$$E^{ab} \equiv -\Lambda^{ab} = 0 \tag{5.45}$$

$$E_a{}^{bc} \equiv -\tau_a{}^{bc} + \tau_a{}^{bcd}{}_{,d} = \tau_a{}^{bcd}{}_{;d} = 0 \tag{5.46}$$

$${}_\Omega E^{abcd} \equiv -\Omega^{abcd} = 0 \tag{5.47}$$

$${}_\mu E^{abcd} \equiv -\mu^{abcd} = 0. \tag{5.48}$$

Under the infinitesimal transformation $\tilde{x}^a = x^a + \chi^a$ the varied potentials take the form

$$\delta g_{ab} = (g_{pb}\chi^p{}_{;a} + g_{ap}\chi^p{}_{;b} + g_{ab;p}\chi^p) \tag{5.49}$$

$$\delta\Gamma^a{}_{bc} = -\chi^a{}_{;b;c} + R^a{}_{bcp}\chi^p \tag{5.50}$$

$$\delta W_{abcd} = -W_{pbcd}\chi^p{}_{;a} - W_{apcd}\chi^p{}_{;b} - W_{abpd}\chi^p{}_{;c} - W_{abcp}\chi^p{}_{;d} - W_{abcd;p}\chi^p \tag{5.51}$$

$$\delta M_{abcd} = -M_{pbcd}\chi^p{}_{;a} - M_{apcd}\chi^p{}_{;b} - M_{abpd}\chi^p{}_{;c} - M_{abcp}\chi^p{}_{;d} - M_{abcd;p}\chi^p \tag{5.52}$$

all of which are clearly tensors. Thus $\tau_a{}^{bc}\, \delta\Gamma^a{}_{bc} + \tau_a{}^{bcd}\, \delta\Gamma^a{}_{bc,d}$ is of the form (5.30) and we may write

$$\delta L = \Lambda^{ab}\, \delta g_{ab} + \tau_a{}^{bcd}\, \delta\Gamma^a{}_{bc;d} + \Omega^{abcd}\, \delta W_{abcd} + \mu^{abcd}\, \delta M_{abcd}. \tag{5.53}$$

Substituting for the varied potentials above and making use of equations (5.39) and (5.43) then gives

$$\begin{aligned}\delta L &= -(L\chi^p)_{;p} - (2\nu_a{}^{cbd} - \tau_a{}^{bcd})R^a{}_{bcp;d}\chi^p \\ &\quad +[\nu_p{}^{bcd}R^q{}_{cbd} + (2\tau_a{}^{qbc} + \tau_a{}^{bcq})R^a{}_{bcp}]\chi^p{}_{;q} - \tau_p{}^{bcd}\chi^p{}_{;b;c;d}.\end{aligned} \tag{5.54}$$

As per equation (3.5) we write $2h^a{}_{;a}$ for $\delta L+(L\chi^p)_{;p}$. In equation (5.54) we simplify the term in χ^p by using equations (5.28) and (5.29). We convert τ's to ν's in the $\chi^p{}_{;q}$ term by applying (5.24) and (5.27) and apply (5.28) to the $\chi^p{}_{;b;c;d}$ term to obtain

$$\begin{aligned}2h^a{}_{;a} &= \nu_a{}^{bdc}R^a{}_{bcp;d}\chi^p + (\nu_p{}^{bcd}R^q{}_{cbd} + 3\nu_a{}^{bqc}R^a{}_{bcp})\chi^p{}_{;q} \\ &\quad -(\nu_p{}^{bcd} + \nu_p{}^{cbd})\chi^p{}_{;b;c;d}.\end{aligned} \tag{5.55}$$

As before we will assume, if possible, that the Riemann tensor operates on the functional derivatives of the Lagrangian rather than the variation χ^p. Thus, partially integrating the χ^p term is the obvious first step. We note

$$\nu_a{}^{bnc}R^a{}_{bcp} = -\tfrac{1}{2}\nu_a{}^{bnc}R^a{}_{pbc} \tag{5.56}$$

and then write

$$\begin{aligned}2h^a{}_{;a} &= -\tfrac{1}{2}(\nu_a{}^{bdc}R^a{}_{pbc}\chi^p)_{;d} + \tfrac{1}{2}\nu_a{}^{bdc}{}_{;d}R^a{}_{pbc}\chi^p \\ &\quad +(\nu_p{}^{bcd}R^q{}_{cbd} - \nu_a{}^{bqc}R^a{}_{pbc})\chi^p{}_{;q} - (\nu_p{}^{bcd} + \nu_p{}^{cbd})\chi^p{}_{;b;c;d}.\end{aligned} \tag{5.57}$$

Now

$$\begin{aligned}\tfrac{1}{2}\nu_a{}^{bdc}{}_{;d}R^a{}_{pbc}\chi^p &= \nu_p{}^{bdc}{}_{;d;b;c}\chi^p = (\nu_p{}^{bdc}{}_{;d}\chi^p)_{;b;c} - \nu_p{}^{bdc}{}_{;d}\chi^p{}_{;b;c} \\ &= (\nu_p{}^{bdc}{}_{;d}\chi^p)_{;b;c} - (\nu_p{}^{bdc}\chi^p{}_{;b;c})_{;d} + \nu_p{}^{bdc}\chi^p{}_{;b;c;d}\end{aligned} \tag{5.58}$$

and

$$\nu_p{}^{bmc}R^q{}_{mbc} - \nu_m{}^{bqc}R^m{}_{pbc} = 2\nu_p{}^{bqc}{}_{;c;b}. \tag{5.59}$$

Substituting these into equation (5.57) we have

$$\begin{aligned} 2h^a{}_{;a} &= -\tfrac{1}{2}(\nu_p{}^{bdc}R^p{}_{mbc}\chi^m)_{;d} + (\nu_p{}^{bdc}{}_{;d}\chi^p)_{;b;c} - (\nu_p{}^{bdc}\chi^p{}_{;b;c})_{;d} \\ &\quad +2\nu_p{}^{bqc}{}_{;c;b}\chi^p{}_{;d} + (\nu_p{}^{bdc} - \nu_p{}^{bcd} - \nu_p{}^{cbd})\chi^p{}_{;b;c;d}. \end{aligned} \tag{5.60}$$

Upon "integrating"[1] the first and third terms will cancel and may thus be eliminated. Also

$$(\nu_p{}^{bdc} - \nu_p{}^{bcd} - \nu_p{}^{cbd})\chi^p{}_{;b;c;d} = 2\nu_p{}^{bdc}\chi^p{}_{;d;c;b} \tag{5.61}$$

so we finally obtain

$$2h^a{}_{;a} = (2\nu_p{}^{adc}\chi^p{}_{;d} - \nu_p{}^{adc}{}_{;d}\chi^p)_{;c;a} \tag{5.62}$$

or

$$h^a = (\nu_p{}^{adc}\chi^p{}_{;d} - \tfrac{1}{2}\nu_p{}^{adc}{}_{;d}\chi^p)_{;c}. \tag{5.63}$$

Note the difference between (5.63) and the gravitation contribution to (3.34). The change in the coefficient of the χ^p term is a consequence of our adoption of the connection as a fundamental quantity.

As in Section (3.5) we may generate momentum complexes from (5.63) by considering the transformations

$$\chi^p = \sqrt{-g}^{(n-1)}\delta^p_s\,\delta k^s \tag{5.64}$$

$$\chi^p = \sqrt{-g}^{(n-1)}g^{pt}\,\delta k_t \tag{5.65}$$

$$\chi^p = \sqrt{-g}^{(n-1)}(g^{ps}x^t - g^{pt}x^s)\,\delta k_{ts}. \tag{5.66}$$

Then we have, respectively, the complexes

$${}_{(n)}h^a{}_s = [\sqrt{-g}^{(n-1)}{}_{,d}\nu_s{}^{adc} + \sqrt{-g}^{(n-1)}(\nu_m{}^{adc}\Gamma^m{}_{sd} - \tfrac{1}{2}\nu_s{}^{adc}{}_{;d})]_{,c} \tag{5.67}$$

$${}_{(n)}h^{at} = [(\sqrt{-g}^{(n-1)}g^{pt})_{,d}\nu_p{}^{adc} + \sqrt{-g}^{(n-1)}g^{pt}(\nu_m{}^{adc}\Gamma^m{}_{pd} - \tfrac{1}{2}\nu_p{}^{adc}{}_{;d})]_{,c} \tag{5.68}$$

$${}_{(n)}h^{ats} = {}_{(n)}h^{as}x^t - {}_{(n)}h^{at}x^s$$

[1] taking the antidivergence

$$
\begin{aligned}
&+[\sqrt{-g}^{(n-1)}(g^{ps}\nu_p{}^{atc} - g^{pt}\nu_p{}^{asc})]_{,c} \\
&+(\sqrt{-g}^{(n-1)}g^{ps})_{,d}\nu_p{}^{adt} + \sqrt{-g}^{(n-1)}g^{ps}(\nu_m{}^{adt}\Gamma^m{}_{pd} - \tfrac{1}{2}\nu_p{}^{adt}{}_{;d}) \\
&-[(\sqrt{-g}^{(n-1)}g^{pt})_{,d}\nu_p{}^{ads} + \sqrt{-g}^{(n-1)}g^{pt}(\nu_m{}^{ads}\Gamma^m{}_{pd} - \tfrac{1}{2}\nu_p{}^{ads}{}_{;d}). \qquad (5.69)
\end{aligned}
$$

We again form a symmetric complex by adding to ${}_{(n)}h^{at}$ a "spin" contribution derived from ${}_{(n)}h^{ats}$. We find

$$
{}_{(n)}s^{ts} = \tfrac{1}{2}[\sqrt{-g}^{(n-1)}(g^{pa}\nu_p{}^{stc} + g^{pa}\nu_p{}^{tsc} - g^{pt}\nu_p{}^{sac} - g^{ps}\nu_p{}^{tac})]_{,ca} - {}_{(n)}h^{ts} \qquad (5.70)
$$

from which we obtain

$$
\begin{aligned}
{}_{(n)}H^{ts} &= \tfrac{1}{2}[\sqrt{-g}^{(n-1)}(\nu^{astc} + \nu^{atsc} + \nu^{tcas} + \nu^{scat})]_{,ca} \\
&= \tfrac{1}{4}[\sqrt{-g}^{(n-1)}(\tau^{atsc} + \tau^{csta} + \tau^{tacs} + \tau^{scat})]_{,ca} \qquad (5.71)
\end{aligned}
$$

where we have made use of equation (5.28) in deriving the last line.

5.3 Lagrangians

We are now ready to apply the analysis of the previous section to particular Lagrangians. We will begin with Nissani's Lagrangian. Hence we adopt the symmetry relations (5.2) and (5.3) for W_{abcd} and M_{abcd} and write

$$
L = L_1 + L_2 + L_3 \qquad (5.72)
$$

where

$$
\begin{aligned}
L_1 &= \varepsilon^{cdgh}g^{af}g^{be}V_{abcd}V_{efgh} && (5.73) \\
V_{abcd} &= W_{abcd} + M_{abcd} - R_{abcd} && (5.74) \\
R^p{}_{bcd} &= \Gamma^p{}_{bd,c} - \Gamma^p{}_{bc,d} + \Gamma^p{}_{mc}\Gamma^m{}_{bd} - \Gamma^p{}_{md}\Gamma^m{}_{bc} && (5.75) \\
L_2 &= \sqrt{-g}\,g^{ac}g^{eg}g^{bf}g^{dh}W_{abcd}W_{efgh} && (5.76) \\
L_3 &= -\varepsilon^{cdgh}R^a{}_{bcd}R^b{}_{agh}. && (5.77)
\end{aligned}
$$

Following Nissani, we first consider the Euler-Lagrange equation resulting from the variation of M_{abcd}. We have

$$\mu^{pqrs} = {}_1\mu^{pqrs} = 2\varepsilon^{cdrs}V^{qp}{}_{cd} \tag{5.78}$$

which, with (5.48), implies that V_{abcd} is identically zero and

$$W_{abcd} + M_{abcd} = R_{abcd}. \tag{5.79}$$

As L_1 is now seen to be quadratic in the vanishing term V_{abcd} it will not contribute to the other functional derivatives.

Next we consider the Euler-Lagrange equation resulting from the variation of W_{abcd}. We have

$$\Omega^{pqrs} = {}_2\Omega^{pqrs} = \sqrt{-g}(g^{pr}W^{aq}{}_a{}^s - g^{ps}W^{aq}{}_a{}^r) \tag{5.80}$$

which, after contracting (5.47) with g_{pr}, implies

$$W^a{}_{bad} = 0. \tag{5.81}$$

Thus L_2 is quadratic in the vanishing term $W^a{}_{bad}$ and it, too, will not contribute further in the analysis. We now assume that M_{abcd} is of the form given by equation (5.4). Then, substitution of (5.79) into (5.81) results in

$$T_{bd} - \tfrac{1}{2}g_{bd}T = R^a{}_{bad}. \tag{5.82}$$

If the geometry is Riemannian these are just an alternate form of the Einstein equations.

Turning to the Euler-Lagrange equation resulting from the variation of the metric, we see that

$$\Lambda^{pq} = {}_3\Lambda^{pq} = 0 \tag{5.83}$$

so that equation (5.45) is identically satisfied.

Finally we consider the Euler-Lagrange equation resulting from the variation of the connection. We have

$$\tau_p{}^{qrs} = {}_3\tau_p{}^{qrs} = 2(\varepsilon^{cdrs}R^q{}_{pcd} + \varepsilon^{cdqs}R^r{}_{pcd}) \tag{5.84}$$

so that (5.46) vanishes via the Bianchi identities; providing no further information about the geometry. Nissani, in obtaining

$$\varepsilon^{cdis} R^{j}{}_{pcd;s} = 0 \tag{5.85}$$

for the Euler-Lagrange equation (5.46) seems to have functionally differentiated incorrectly: (5.85), which should represent $\tau_p{}^{ijs}{}_{;s}$, is not symmetric in the indices i, j. Regardless, Nissani writes:

$$\varepsilon^{abcd} R_{mnab;d} = 0 \tag{5.86}$$

"which are the Bianchi identities" give, with (5.85)

$$\varepsilon^{cdis} R_{mpcd} g^{mj}{}_{;s} = 0. \tag{5.87}$$

For a given j these are a set of sixteen homogeneous equations in the sixteen unknowns $g^{mj}{}_{;s}$ which implies

$$g^{mj}{}_{;s} = 0. \tag{5.88}$$

But, in fact, for an arbitrary connection the Bianchi identities are [20]

$$\begin{aligned} & R^{\ell}{}_{jhk;p} + R^{\ell}{}_{jkp;h} + R^{\ell}{}_{jph;k} \\ = \;& S^{m}{}_{hk} R^{\ell}{}_{jmp} + S^{m}{}_{kp} R^{\ell}{}_{jmh} + S^{m}{}_{ph} R^{\ell}{}_{jmk} \end{aligned} \tag{5.89}$$

where, here,

$$S^{\ell}{}_{hk} = \Gamma^{\ell}{}_{hk} - \Gamma^{\ell}{}_{kh} \tag{5.90}$$

is the torsion. When the torsion vanishes, as in our case, equation (5.90) clearly reduces to (5.85) rather than (5.86). Thus in writing equation (5.86) Nissani is assuming (5.88). His reasoning is circular and hence erroneous. Riemannian geometry must be *assumed* in order for equations (5.82) to correspond to the Einstein equations.

We will now make the assumption that the geometry is Riemannian. Thus we acknowledge that the components of the connection are given by the Christoffel symbols but still consider g_{ab} and $\Gamma^a{}_{bc}$ to be independent in the variation. Effectively we let g_{ab} be an arbitrary symmetric tensor and only later select it to be the metric. This does not affect the analysis of the previous section but *will* affect the results in this section. The Riemann tensor is now antisymmetric in both its first and last pair of indices which, with equation (5.4), implies the same must hold for both W_{abcd} and M_{abcd} if we are to obtain equation (5.79). Hence we will now assume that W_{abcd} and M_{abcd} obey equations (5.7) and (5.8). These new symmetries do not affect μ^{abcd}, so (5.79) holds as before; but they do change Ω^{abcd}. Recall that we must consider all possible permutations of indices when taking functional derivatives. We now have

$$\Omega^{pqrs} = {}_2\Omega^{pqrs} = \tfrac{1}{2}\sqrt{-g}(g^{pr}W^{aq}{}_a{}^s - g^{ps}W^{aq}{}_a{}^r - g^{qr}W^{ap}{}_a{}^s + g^{qs}W^{ap}{}_a{}^r). \qquad (5.91)$$

Contracting first with $g_{pr}g_{qs}$ and afterwards with g_{pr} again yields equation (5.81).

One might expect that we will also have to modify the derivations of Λ^{ab} and $\tau_a{}^{bcd}$ and, indeed, wonder why we have not explicitly considered the symmetries

$$R^a{}_{bcd} = -R^a{}_{bdc} \qquad (5.92)$$

$$\varepsilon^{abcd}R^e{}_{bcd} = 0 \qquad (5.93)$$

in the previous derivation of $\tau_a{}^{bcd}$. Of course (5.92) is clearly embodied in our definition of the Riemann tensor, equation (5.75). But (for a symmetric connection) (5.93) also follows from (5.75). It is a consequence of the functional form of $R^a{}_{bcd}(\Gamma^a{}_{bc}, \Gamma^a{}_{bc,d})$ and, as such, is already "built in" to the theory, as is evident from the first invariance relation. The new symmetries

$$\begin{aligned} R_{abcd} &= -R_{bacd} && (5.94) \\ R_{abcd} &= R_{cdab} && (5.95) \end{aligned}$$

which result from the restriction to Riemannian geometry are in fact equivalent and need not be considered separately. However, in order to write

$$R^a{}_{bcd} = \tfrac{1}{2}(R^a{}_{bcd} + g_{ci}g^{aj}R^i{}_{djb}) \tag{5.96}$$

which would result in changes to both Λ^{ab} and $\tau_a{}^{bcd}$, we must explicitly identify g_{ab} as the metric. But then g_{ab} and $\Gamma^a{}_{bc}$ would no longer be independent and we would be forced to resort to a Hilbert variation[2]. Thus the previous derivations of Λ^{ab} and $\tau_a{}^{bcd}$ remain valid and the corresponding Euler-Lagrange equations are satisfied. Once again equations (5.79) and (5.81) imply, with (5.4), equations (5.82) and that, with the assumption of Riemannian geometry in place, these are just the Einstein equations.

Examination of the Lagrangian (5.72), in light of the resulting Euler-Lagrange equations, shows that the only "surviving" term is L_3 which is just L_α of equation (4.5). As L_1 is also of this form, the analysis of Chaper 4 suggests that we may obtain results analogous to Nissani's by using Lagrangian terms based on the double-dual form, L_β. Thus we define

$$L = L_4 + L_2 + L_5 \tag{5.97}$$

where

$$\begin{aligned} L_4 &= \sqrt{-g}\,\delta^{cdgh}_{ijk\ell}\, g^{ak}g^{b\ell}g^{ei}g^{fj}V_{abcd}V_{efgh} && (5.98)\\ L_5 &= -4L_\beta = -\sqrt{-g}\,\delta^{cdgh}_{ijk\ell}\, g^{b\ell}g^{fj}R^k{}_{bcd}R^i{}_{fgh} && (5.99)\end{aligned}$$

Then

$$\mu^{pqrs} = {}_4\mu^{pqrs} = 2\sqrt{-g}\,\delta^{cdrs}_{ijk\ell}\, g^{pi}g^{qj}V^{k\ell}{}_{cd} \tag{5.100}$$

which, when substituted into the Euler-Lagrange equation (5.48), again implies that V_{abcd} vanishes and that (5.79) follows. Like L_1, L_4 is seen to be quadratic in the

[2] In the analysis of Chapter 4, when we used equation (4.14) to define R_{abcd}, all the symmetries evident in equations (5.92–5.95) are "built in". This is a reflection that theoretical changes occur when we switch fundamental quantities. The Palatini variational principle omits a direct consideration of the Christoffel symbols and, consequently, a certain amount of information is lost. This is particularly evident in the fact that the quantity ψ^{abcd} is, in general, much richer in symmetry properties than its Palatini analog, $\nu_a{}^{bcd}$.

vanishing term V_{abcd} and does not contribute further. Hence L_2 is the only remaining term in W_{abcd} and we again have (5.91), equations (5.81) and (5.82) following, and, since the geometry is Riemannian, equations (5.82) give the Einstein equations. For $\tau_p{}^{qrs}$ we have

$$\tau_p{}^{qrs} = {}_5\tau_p{}^{qrs} = 2\sqrt{-g}(\delta^{cdrs}_{pjk\ell}g^{qj} + \delta^{cdqs}_{pjk\ell}g^{rj})R^{k\ell}{}_{cd} \tag{5.101}$$

so that Euler-Lagrange equation (5.46) is satisfied by the Bianchi identities and our Riemannian geometry. Finally,

$$\Lambda^{pq} = {}_5\Lambda^{pq} = -\tfrac{1}{4}\sqrt{-g}(\delta^{pcdgh}_{rijk\ell}\,g^{qr} + \delta^{qcdgh}_{rijk\ell}\,g^{rp})R^{k\ell}{}_{cd}R^{ij}{}_{gh} \tag{5.102}$$

and once again vanishes identically. Thus we have shown that Nissani's Lagrangian is not unique. In fact any linear combination of the Lagrangians (5.73) and (5.97) will work. The only complication arises in the Euler-Lagrange equation for μ^{pqrs}, which is then equivalent to

$$(\alpha g_{ik}g_{j\ell} + \beta\sqrt{-g}\,\varepsilon_{ijk\ell})V^{k\ell}{}_{mn} = 0 \tag{5.103}$$

but this equation, as well, implies that V_{abcd} must vanish.

We now turn to the question of conserved quantities and, in particular, a comparison between quantities derived from equations (3.34) and (5.63). We will focus on the Lagrangian given by equation (4.1) as it contains the contributing terms of the Lagrangians presented in this chapter. But first we must discover the relationship between $\tau_a{}^{bcd}$ and $\nu_a{}^{bcd}$ and their Hilbert variation analogs Λ^{abcd} and ψ^{abcd}.

After having taken our Palatini variation and decided upon g_{ab} as the metric we may, in "hindsight", write

$$\Lambda^{pqrs} = \tau_a{}^{bcd}\frac{\partial\Gamma^a{}_{bc,d}}{\partial g_{pq,rs}} \tag{5.104}$$

from which we easily determine that

$$\Lambda^{pqrs} = -\tfrac{1}{4}(\tau^{psrq} + \tau^{qrsp} + \tau^{rqps} + \tau^{spqr}) \tag{5.105}$$

$$\psi^{pqrs} = \tfrac{1}{4}(\nu^{pqrs} - \nu^{qrsp} + \nu^{rspq} - \nu^{spqr}). \tag{5.106}$$

It is immediately apparent that, as a consequence of (5.105), our expressions for ${}_{(n)}H^{is}$, equations (3.55) and (5.71), are identical. Unfortunately, in the general case it is impossible to simplify (5.105) and (5.106) in order to establish a closer correspondence between equations (3.34) and (5.63). However if, for a particular Lagrangian, ν^{abcd} satisfies

$$\nu^{abcd} = -\nu^{adcb} = -\nu^{cbad} \tag{5.107}$$

then it immediately follows that

$$\psi^{abcd} = \nu^{abcd}. \tag{5.108}$$

In fact, from (5.101) and (4.42) is is easy to show that

$${}_5\nu^{abcd} = -4\,{}_\beta\psi^{abcd} \tag{5.109}$$

indicating that (5.108) holds for the double-dual Lagrangian L_β. Equation (5.108) is also true for L_H and, since $\psi^{abcd}{}_{;d}$ vanishes for these two Lagrangian terms, it is thus apparent that, for L_H and L_β, equations (3.34) and (5.63) produce identical results.

On the other hand, for the Lagrangian term L_α, equations (3.34) and (5.63) yield differing results. We have

$${}_\alpha\nu_p{}^{qrs} = \tfrac{2}{3}(2\varepsilon^{cdqs}R^r{}_{pcd} + \varepsilon^{cdqr}R^s{}_{pcd} + \varepsilon^{cdrs}R^q{}_{pcd}) \tag{5.110}$$

and so ${}_\alpha\nu^{pqrs} \neq -{}_\alpha\nu^{rqps}$ and equation (5.108) does not hold. Neither does ${}_\alpha\nu_p{}^{qrs}{}_{;r}$ vanish; we find

$${}_\alpha\nu_p{}^{qrs}{}_{;r} = \tfrac{8}{3}\varepsilon^{cdsq}R_{pc;d}. \tag{5.111}$$

Equations (5.63), (5.67) and (5.68) give

$${}_{\alpha(n)}h^a = \tfrac{2}{3}\{(2\varepsilon^{ijac}R^d{}_{pij} + \varepsilon^{ijad}R^c{}_{pij} + \varepsilon^{ijdc}R^a{}_{pij})\chi^p{}_{,d}$$

$$
\begin{aligned}
& +[(2\varepsilon^{ijac}R^{d}{}_{mij}+\varepsilon^{ijad}R^{c}{}_{mij}+\varepsilon^{ijdc}R^{a}{}_{mij})\Gamma^{m}{}_{pd} \\
& +2\varepsilon^{ijac}R_{pi;j}]\chi^{p}\} \qquad (5.112)
\end{aligned}
$$

$$
\begin{aligned}
{}_{\alpha(n)}h^{a}{}_{s} \;=\; & \tfrac{2}{3}\{\sqrt{-g}^{(n-1)}{}_{,d}(2\varepsilon^{ijac}R^{d}{}_{sij}+\varepsilon^{ijad}R^{c}{}_{sij}+\varepsilon^{ijdc}R^{a}{}_{sij}) \\
& +\sqrt{-g}^{(n-1)}[(2\varepsilon^{ijac}R^{d}{}_{mij}+\varepsilon^{ijad}R^{c}{}_{mij}+\varepsilon^{ijdc}R^{a}{}_{mij})\Gamma^{m}{}_{sd} \\
& +2\varepsilon^{ijac}R_{si;j}]\} \qquad (5.113)
\end{aligned}
$$

$$
\begin{aligned}
{}_{\alpha(n)}h^{at} \;=\; & \tfrac{2}{3}\{(\sqrt{-g}^{(n-1)}g^{pt})_{,d}(2\varepsilon^{ijac}R^{d}{}_{pij}+\varepsilon^{ijad}R^{c}{}_{pij}+\varepsilon^{ijdc}R^{a}{}_{pij}) \\
& +\sqrt{-g}^{(n-1)}[(2\varepsilon^{ijac}R^{d}{}_{mij}+\varepsilon^{ijad}R^{c}{}_{mij}+\varepsilon^{ijdc}R^{a}{}_{mij})g^{pt}\Gamma^{m}{}_{pd} \\
& +2\varepsilon^{ijac}R^{t}{}_{i;j}]\} \qquad (5.114)
\end{aligned}
$$

expressions which are somewhat simpler than their Hilbert analogs, the α terms of equations (4.48–4.50).

The fact that Lagrangians of the form of L_α yield differing conserved quantities in all but the symmetric case can be construed in either a positive or a negative sense in arguing for or against the use of the Lagrangians, the variational principles or the resulting conserved quantities. Yet it is probably relevant that L_α's field theoretic analogs such as L_γ given by equation (4.8), are only rarely encountered.

In this chapter we have seen that a change in fundamental quantities may yield new results. One of these results was the derivation of the Bianchi identities as an Euler-Lagrange equation. Since the Bianchi identities are the general relativistic analog of the second of Maxwell's equations, equation (1.2), one is led to speculate on the possibility of a general-relativistic analog to (1.1). Such an analog would constitute a matter current equation and would yield an entirely new approach to the idea of energy-momentum in general relativity. Unfortunately, the equations resulting from the Lagrangians we have so far considered have not led to any such relation. Still, in the presence of matter (introduced in the usual way by a matter Lagrangian L_M rather then via Nissani's generalized matter tensor) the

Euler-Lagrange equation (5.46) takes the form

$$\tau_a{}^{bcd}{}_{;d} = {}_M\Pi_a{}^{bc} \tag{5.115}$$

where, now, ${}_M\Pi_a{}^{bc}$ need not vanish. This is reminiscent of a current equation, especially when we consider that in a theory based on tetrads or spinors the indices a, b of (5.115) would become tetrad or spinor indices.

Part of Nissani's motivation for investigating his Lagrangian was that it was based on the Lagrangian of Carmeli's $SL(2,C)$ gauge theory of gravitation. Among the results of this chapter is the realization that Nissani's Lagrangian is not unique. In the following chapters we will be investigating $SL(2,C) \times U(1)$ gauge theories of gravitation and electromagnetism in the hope that more general Lagrangians, based in part on the new material presented thus far, will enable us to derive a general relativistic analog to the Maxwell equations (1.1), leading to some form of generalized matter current density.

Chapter 6

Theory of the $SL(2,C) \times U(1)$ Gauge Field

6.1 Introduction

In this chapter we present a variational principle on a class of gauge theories over the $SL(2,C) \times U(1)$ group. As the inclusion of the $U(1)$ symmetry requires some modification to the usual $SL(2,C)$ based spinor formalism, we will begin with an introduction to $SL(2,C) \times U(1)$ spinors. The material of this section follows fairly closely the $SL(2,C)$ presentation found in Carmeli [5] (alternatively see Pirani [28] or Wald [31]).

The $SL(2,C) \times U(1)$ formalism has several attractive properties. Usually, particularly in quantum field theory, gauge potentials are defined via ad hoc extensions (or introductions) of a spinor connection. For instance, the $SL(2,C)$ gauge theory of gravitation is also $U(1)$ invariant and thus permits such an ad hoc introduction of electromagnetism. However, in the $SL(2,C) \times U(1)$ formalism an ad hoc extension is unnecessary; an Abelian gauge potential is already present. As a result, variably charged matter may be cleanly and rigourously introduced through the use of spin densities—spinors of relative gauge weight. But a more immediate consequence lies in the fact that the gravitational and electromagnetic potentials have been naturally combined in a single object—the spinor connection. Not only does this greatly facilitate comparison between the two fields but, in a quantum mechanical sense at least, it constitutes a unification of gravity and electromagnetism. The obvious question arises: is this a coincidental mathematical artifact or a result with real physical implications?

In general relativity, the usual unification criteria seem to go back to Einstein [10].

> "If we speak about a unified theory we have two possible points of view, whose distinction is essential for the following:
>
> (1) That the field appear as a unified covariant entity. As an example I cite the unification of the electric and the magnetic fields by the special theory of relativity. The unification here consists in this that the entire field considered is described as a skew-symmetric tensor. The basic group of Lorentz transformations does not enable us to split this field independently of the system of coordinates, into an electric and a magnetic one.
>
> (2) Neither the field equations nor the Hamiltonian function can be expressed as the sum of invariant parts, but are formally unified entities. Also this (weaker) criterion of uniformity is satisfied in our example of the special relativistic description of Maxwell's equations."

These criteria deal with fields, which is quite unsurprising since in Einstein's day fields were considered to be the fundamental descriptors of physical phenomena; potentials were considered merely artificial mathematical constructions. However, today we know better. The Aharanov-Bohm experiment has radically altered our ideas about the reality of the electromagnetic potential [11] and gauge potentials in general. Thus it is interesting that in electromagnetic theory, the paradigm of unification and the model behind Einstein's reasoning, both conditions (1) and (2) follow from the unification of the electromagnetic potential under the group of Lorentz and $U(1)$ gauge transformations.

The theories we develop in the following will in no way satisfy either of these field-based conditions. However, in the $SL(2,C) \times U(1)$ formalism, the spinor connection does satisfy a potential-based formulation of the strong condition (1). It is unified in that the group of coordinate and $SL(2,C) \times U(1)$ gauge transformations does not enable us to split the potential, independently of the gauge and coordinate system, into gravitational and electromagnetic parts. If this unification has any real significance we may expect that, in the theories we develop, the Euler-Lagrange equation resulting from the variation of the spinor connection should be an equation in the graviational-electromagnetic field (unified or not) which, in the presence of matter, becomes a relation for a matter-charge current density. That is we expect a theory similar in form to a Yang-Mills theory.

With this in mind we apply, in the latter part of this chapter, our variational technique to a general class of Lagrangians based on the (cross) section (of the fiber bundle) and the spinor connection and its first derivatives. In anticipation of future needs we will also provide for the inclusion of both covariant and contravariant gauge weighted spinors.

6.2 $SL(2,C) \times U(1)$ Spinors

As a base space for our $SL(2,C) \times U(1)$ gauge theory we choose a manifold (henceforth referred to as spacetime) endowed with a symmetric connection $\Gamma^a{}_{bc}$ and a metric g_{ab} which vanishes under covariant differentiation. Thus our spacetime is Riemannian; the first condition implying that the Riemann tensor satisfies the Bianchi identities

$$\varepsilon^{abcd} R^m{}_{nbc;d} = 0 \tag{6.1}$$

and the second implying that the connection is just the Christoffel symbol[1].

For the structure group of our theory we have choosen $SL(2,C) \times U(1)$, the group of all two-by-two complex matrices whose determinant S is unimodular.

[1] We note the necessity of this assumption despite the fact that we will later be taking Palatini variations on our Lagrangians. With the incorporation of the electromagnetic potential into the spinor connection the resulting Euler-Lagrange equation will be of the form of a Maxwell equation and will not fix the connection.

The correspondence between spinors and tensors is established, as usual, via the 2×2 Hermitean matrices ${\sigma_a}^{AB'}$ (the *section*—primes denote conjugate indices) where

$$
\begin{aligned}
{\sigma_a}^{AB'} {\sigma^b}_{AB'} &= \delta_a^b & (6.2)\\
{\sigma_a}^{AB'} {\sigma^a}_{CD'} &= \delta_C^A \delta_{D'}^{B'}. & (6.3)
\end{aligned}
$$

Thus

$$
\begin{aligned}
V_a &= {\sigma_a}^{AB'} V_{AB'} & (6.4)\\
\eta_{AB'CD'} &= {\sigma^a}_{AB'} {\sigma^b}_{CD'} \eta_{ab}. & (6.5)
\end{aligned}
$$

We will choose a spinor metric f_{AC} proportional to the Levi-Civita spinor ε_{AC}

$$\varepsilon_{AC} = \varepsilon_{B'D'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon^{AC} = \varepsilon^{B'D'}. \tag{6.6}$$

Thus

$$
\begin{aligned}
& f_{AC} = f \varepsilon_{AC}, && f^{AC} = f^{-1} \varepsilon^{AC} \\
& \bar{f}_{B'D'} = \bar{f} \varepsilon_{B'D'}, && \bar{f}^{B'D'} = \bar{f}^{-1} \varepsilon^{B'D'} \\
& f = \sqrt{\det(f_{AC})} = e^{i\omega}
\end{aligned} \tag{6.7}
$$

where ω is real. The spinor and spacetime metrics satisfy the correspondence

$$g_{ab} {\sigma^a}_{AB'} {\sigma^b}_{CD'} = f_{AC} \bar{f}_{B'D'} = \varepsilon_{AC} \varepsilon_{B'D'}. \tag{6.8}$$

As f_{AC} is antisymmetric, we must contract it with care. We will lower and raise indices by contracting, respectively, on the first and second index of f_{AC}. Thus

$$
\begin{aligned}
& V_C = f_{AC} V^A, && V^A = f^{AC} V_C \\
& A^A B_A = -A_B B^B && \\
& \delta_C^A = {f_C}^A = -{f^A}_C. &&
\end{aligned} \tag{6.9}
$$

We also note that the relation

$$f_{AB}f_{CD} + f_{AC}f_{DB} + f_{AD}f_{BC} = 0 \tag{6.10}$$

leads to the frequently encountered identity

$$\eta_{AC} - \eta_{CA} = f_{AC}\eta_M{}^M. \tag{6.11}$$

A second useful identity, in the section $\sigma_a{}^{AB'}$, may also be derived as follows. We define $\varepsilon^{AB'CD'EF'GH'}$, the analog of the Levi-Civita tensor, to be $+1$ (-1) when AB', CD', EF', GH' is an even (odd) permutation of $00', 01', 10', 11'$ and zero otherwise. Then $\varepsilon^{AB'CD'EF'GH'}$ (a scalar under coordinate transformations) satisfies the relation

$$\varepsilon^{AB'CD'EF'GH'} = \varepsilon^{AG}\varepsilon^{B'F'}\varepsilon^{CE}\varepsilon^{D'H'} - \varepsilon^{AE}\varepsilon^{B'H'}\varepsilon^{CG}\varepsilon^{D'F'} \tag{6.12}$$

which may be verified via decomposition (as in Pirani [28]) or simply by checking components in the above for all possible values of the indices. Then, defining

$$\sigma \equiv \det(\sigma_a{}^{AB'}) = \sqrt{g} = -i\sqrt{-g} \tag{6.13}$$

where the sign of σ has been choosen to agree with the convention that the section of Minkowski space (in Cartesian coordinates) be just the identity and Pauli matrices, we have

$$\begin{aligned}
&\varepsilon_{abcd}\sigma^{aAB'}\sigma^{bCD'}\sigma^{cEF'}\sigma^{dGH'} \\
&= \; -i\sqrt{-g}^{(-1)}(\varepsilon^{AG}\varepsilon^{B'F'}\varepsilon^{CE}\varepsilon^{D'H'} - \varepsilon^{AE}\varepsilon^{B'H'}\varepsilon^{CG}\varepsilon^{D'F'}).
\end{aligned} \tag{6.14}$$

Note that, following relativistic convention, ε_{abcd} is defined via the relation

$$\varepsilon^{abcd}g_{ai}g_{bj}g_{ck}g_{d\ell} = -g\varepsilon_{ijk\ell} \tag{6.15}$$

so that $\varepsilon_{0123} = -1$. Multiplying (6.14) by $\sigma_{kAB'}\sigma_{\ell CD'}f_{EP}\bar{f}_{F'H'}$ yields

$$\sigma_{kPH'}\sigma_\ell{}^{GH'} - \sigma_k{}^{GH'}\sigma_{\ell PH'} = i\sqrt{-g}\,\varepsilon_{k\ell cd}\sigma^c{}_{PH'}\sigma^{dGH'} \tag{6.16}$$

and noting that, by (6.11)

$$\sigma_{aAB'}\sigma_b{}^{CB'} + \sigma_a{}^{CB'}\sigma_{bAB'} = g_{ab}\delta_A^C \tag{6.17}$$

we have, finally

$$\sigma_{aAB'}\sigma_b{}^{CB'} = \tfrac{1}{2}g_{ab}\delta_A^C + \tfrac{1}{2}i\sqrt{-g}\,\varepsilon_{abcd}\sigma^c{}_{AB'}\sigma^{dCB'}. \tag{6.18}$$

We extend the usual rules for covariant differentiation to spinor indices through the introduction of the spinor connections $\Gamma^A{}_{Ca}$ and $\bar{\Gamma}^{B'}{}_{D'a}$. A spinor $\eta_{aAB'}$ which, under a coordinate transformation $C^a{}_b$, $C = \det(C^a{}_b)$, $C^{(k)} \equiv C$ to the power k, satisfies

$$\tilde{\eta}_{bAB'} = C^{(k)}\eta_{aAB'}C'^a{}_b \tag{6.19}$$

and under a gauge transformation $S^A{}_C$, $S = \det(S^A{}_C) =: \bar{S}^{(-1)}$ satisfies

$$\tilde{\eta}_{aCD'} = S^{(m)}\bar{S}^{(n)}\eta_{aAB'}S^A{}_C\bar{S}^{B'}{}_{D'} \tag{6.20}$$

is said to be a relative spinor of coordinate weight k, gauge weight m and gauge antiweight n. The covariant derivative of such a spinor is defined to be

$$\begin{aligned} \eta_{aAB';p} &= \eta_{aAB',p} - \Gamma^i{}_{ap}\eta_{iAB'} - \Gamma^M{}_{Ap}\eta_{aMB'} - \bar{\Gamma}^{N'}{}_{B'p}\eta_{aAN'} - \Omega_p\eta_{aAB'} && (6.21)\\ \Omega_p &= k\Gamma^a{}_{pa} + m\Gamma^M{}_{Mp} + n\bar{\Gamma}^{N'}{}_{N'p}. && (6.22)\end{aligned}$$

Note that in accordance with our definition of gauge weight (for example consider a weighted scalar) we must require that $\Gamma^M{}_{Mp}$ be imaginary. That this is the case follows from the fact that

$$(f_{AC}\bar{f}_{B'D'})_{;p} = (\varepsilon_{AC}\varepsilon_{B'D'})_{;p} = 0. \tag{6.23}$$

Thus we write

$$\begin{aligned} \Gamma^M{}_{Mp} &= iA_p + f_{,p}/f && (6.24)\\ \Gamma^A{}_{Cp} &= \hat{\Gamma}^A{}_{Cp} + \tfrac{1}{2}\delta^A_C(iA_p + f_{,p}/f) && (6.25)\\ \bar{\Gamma}^{B'}{}_{D'p} &= \hat{\bar{\Gamma}}^{B'}{}_{D'p} - \tfrac{1}{2}\delta^{B'}_{D'}(iA_p + f_{,p}/f) && (6.26)\end{aligned}$$

where $\hat{\Gamma}_{ACp}$ and $\hat{\bar{\Gamma}}_{B'D'p}$ are symmetric in their spinor indices. $\Gamma^{A}{}_{Cp}$ is a vector under coordinate transformations. Under a gauge transformation we have

$$\tilde{\Gamma}^{C}{}_{Dp} = \Gamma^{A}{}_{Bp}(S^{-1})^{C}{}_{A}S^{B}{}_{D} + (S^{-1})^{C}{}_{A}S^{A}{}_{D,p}. \tag{6.27}$$

It now follows that

$$\begin{aligned}
&f_{AC;p} = -if_{AC}A_p, && f^{AC}{}_{;p} = if^{AC}A_p \\
&\bar{f}_{B'D';p} = i\bar{f}_{B'D'}A_p, && \bar{f}^{B'D'}{}_{;p} = -i\bar{f}^{B'D'}A_p \\
&f_{;p} = -ifA_p && \\
&\delta^{A}_{C;p} = 0, && \delta^{B'}_{D';p} = 0 \\
&\varepsilon_{AC;p} = 0, && \varepsilon^{AC}{}_{;p} = 0 \\
&\varepsilon_{B'D';p} = 0, && \varepsilon^{B'D'}{}_{;p} = 0.
\end{aligned} \tag{6.28}$$

In the following we will identify $A_p - if_{,p}/f = -i\Gamma^{M}{}_{Mp}$ with the vector potential of the electromagnetic field.

The relation between the spinor and spacetime connections is established by requiring

$$\sigma^{a}{}_{AB';p} = 0 \tag{6.29}$$

which implies

$$\begin{aligned}
\Gamma^{a}{}_{bp} &= \sigma^{a}{}_{AB'}(\sigma_{b}{}^{AB'}{}_{,p} + \sigma_{b}{}^{CB'}\Gamma^{A}{}_{Cp} + \sigma_{b}{}^{AD'}\bar{\Gamma}^{B'}{}_{D'p}) \\
&= \sigma^{a}{}_{AB'}(\sigma_{b}{}^{AB'}{}_{,p} + \sigma_{b}{}^{CB'}\hat{\Gamma}^{A}{}_{Cp} + \sigma_{b}{}^{AD'}\hat{\bar{\Gamma}}^{B'}{}_{D'p})
\end{aligned} \tag{6.30}$$

$$\hat{\Gamma}^{A}{}_{Cp} = \tfrac{1}{2}\sigma_{a}{}^{AD'}(\sigma^{a}{}_{CD',p} + \Gamma^{a}{}_{np}\sigma^{n}{}_{CD'}) \tag{6.31}$$

$$\hat{\bar{\Gamma}}^{B'}{}_{D'p} = \tfrac{1}{2}\sigma_{a}{}^{CB'}(\sigma^{a}{}_{CD',p} + \Gamma^{a}{}_{np}\sigma^{n}{}_{CD'}). \tag{6.32}$$

In analogy to the Riemann curvature tensor, we define the curvature spinor $F^{P}{}_{Qab}$:

$$\xi_{Q;a;b} - \xi_{Q;b;a} = F^{P}{}_{Qab}\xi_{P} + nF^{M}{}_{Mab}\xi_{Q} \tag{6.33}$$

$$F^P{}_{Qab} = \Gamma^P{}_{Qb,a} - \Gamma^P{}_{Qa,b} + \Gamma^P{}_{Na}\Gamma^N{}_{Qb} - \Gamma^P{}_{Nb}\Gamma^N{}_{Qa} \tag{6.34}$$

where $F^P{}_{Qab}$ is clearly antisymmetric in a, b. It is a tensor under coordinate transformations and transforms properly under gauge transformations

$$\tilde{F}^A{}_{Cab} = F^P{}_{Qab}(S^{-1})^A{}_P S^Q{}_C. \tag{6.35}$$

We may also write $F^P{}_{Qab}$ in the form

$$F^P{}_{Qab} = \hat{F}^P{}_{Qab} + \tfrac{1}{2}i\delta^P_Q F_{ab} \tag{6.36}$$

$$\hat{F}^P{}_{Qab} = \hat{\Gamma}^P{}_{Qb,a} - \hat{\Gamma}^P{}_{Qa,b} + \hat{\Gamma}^P{}_{Na}\hat{\Gamma}^N{}_{Qb} - \hat{\Gamma}^P{}_{Nb}\hat{\Gamma}^N{}_{Qa} \tag{6.37}$$

$$F_{ab} = A_{b,a} - A_{a,b} \tag{6.38}$$

where $\hat{F}_{PQab}$ is symmetric in its spinor indices. It is easy to show that $F^P{}_{Qab}$ (and both its gravitational and electromagnetic parts) satisfy the spinor Bianchi identities

$$\varepsilon^{abcd} F^P{}_{Qbc;d} = 0. \tag{6.39}$$

The relation between the curvature spinor and the Riemann tensor is also obtained via equation (6.29), when the identity

$$\begin{aligned} \eta_{b;p;q} - \eta_{b;q;p} &= R^{AB'}{}_{bpq}\eta_{AB'} \\ &= (\sigma_b{}^{CB'}F^A{}_{Cpq} + \sigma_b{}^{AD'}\bar{F}^{B'}{}_{D'pq})\eta_{AB'} \end{aligned} \tag{6.40}$$

implies

$$\begin{aligned} R^a{}_{bpq} &= \sigma^a{}_{AB'}\sigma_b{}^{CB'}F^A{}_{Cpq} + \sigma^a{}_{AB'}\sigma_b{}^{AD'}\bar{F}^{B'}{}_{D'pq} \\ &= \sigma^a{}_{AB'}\sigma_b{}^{CB'}\hat{F}^A{}_{Cpq} + \sigma^a{}_{AB'}\sigma_b{}^{AD'}\hat{\bar{F}}^{B'}{}_{D'pq} \end{aligned} \tag{6.41}$$

$$\hat{F}^A{}_{Cpq} = \tfrac{1}{2}\sigma_a{}^{AB'}\sigma^b{}_{CB'}R^a{}_{bpq} \tag{6.42}$$

$$\hat{\bar{F}}^{B'}{}_{D'} = \tfrac{1}{2}\sigma_a{}^{AB'}\sigma^b{}_{AD'}R^a{}_{bpq}. \tag{6.43}$$

Although we will be considering Lagrangians which are functions of the section and spinor connection it will be useful to examine both the Lagrangians and the resulting field equations on our spacetime. Thus we define

$$P_{abcd} = \sigma_{aAB'}\sigma_b{}^{CB'}F^A{}_{Ccd} \tag{6.44}$$

$$\bar{P}_{abcd} = \sigma_{aAB'}\sigma_b{}^{AD'}\bar{F}^{B'}{}_{D'cd} \tag{6.45}$$

and, noting (6.18) and (6.41–6.43), write

$$\begin{aligned} P_{abcd} &= \tfrac{1}{2}\sigma_{aAB'}(\sigma_b{}^{CB'}F^A{}_{Ccd} + \sigma_b{}^{AD'}\bar{F}^{B'}{}_{D'cd} + \sigma_b{}^{CB'}F^A{}_{Ccd} - \sigma_b{}^{AD'}\bar{F}^{B'}{}_{D'cd}) \\ &= \tfrac{1}{2}R_{abcd} + \tfrac{1}{4}i\sqrt{-g}\,\varepsilon_{abpq}R^{pq}{}_{cd} + \tfrac{1}{2}ig_{ab}F_{cd}. \end{aligned} \tag{6.46}$$

Finally, we should note that the condition $\sigma^a{}_{AB';p} = 0$ is analogous to the condition $g_{ab;p} = 0$. That is, given a Riemannian base space, we may always choose a connection such that equation (6.29) is true. Further, it is only with this connection that equations (6.41–6.43) hold.

6.3 A Variational Principle on the Spinor Manifold

In this section we apply our variational techniques to the class of scalar density Lagrangians with the functional dependence

$$L = L(\sigma_{aAB'}, \Gamma^A{}_{Ca}, \Gamma^A{}_{Ca,b}, \bar{\Gamma}^{B'}{}_{D'a}, \bar{\Gamma}^{B'}{}_{D'a,b}, \psi_A, \psi_{A,a}, \bar{\psi}_{B'}, \bar{\psi}_{B',a}, \phi^A, \phi^A{}_{,a}, \bar{\phi}^{B'}, \bar{\phi}^{B'}{}_{,a}) \tag{6.47}$$

where ψ_A and ϕ^A are of gauge weights $(n - \frac{1}{2})$ and $-(n - \frac{1}{2})$ respectively. Thus, for example,

$$\psi_{A;a} = \psi_{A,a} - \hat{\Gamma}^M{}_{Aa}\psi_M - n\Gamma^M{}_{Ma}\psi_A \tag{6.48}$$

and we may consider n to designate units of electric charge.

Despite the potential complexity of the Lagrangians of this class, the fact that some of the arguments are weighted and the fact that we must consider both coordinate and gauge transformations, the analysis of this chapter will prove to be relatively simple.

Under a gauge transformation $S^A{}_C$, as in equation (6.20), and noting that

$$(S^{-1})^I{}_{A,a} = -(S^{-1})^I{}_C(S^{-1})^K{}_A S^C{}_{K,a} \tag{6.49}$$

$$S^{(n-\frac{1}{2})}{}_{,a} = (n-\tfrac{1}{2})S^{(n-\frac{1}{2})}(S^{-1})^K{}_C S^C{}_{K,a} \tag{6.50}$$

we have for the transformation laws of L and its arguments

$$\tilde{L} = L \tag{6.51}$$

$$\tilde{\sigma}_{aIJ'} = S^A{}_I\bar{S}^{B'}{}_{J'}\sigma_{aAB'} \tag{6.52}$$

$$\tilde{\Gamma}^I{}_{Ka} = (S^{-1})^I{}_A S^C{}_K\Gamma^A{}_{Ca} + (S^{-1})^I{}_A S^A{}_{K,a} \tag{6.53}$$

$$\begin{aligned}\tilde{\Gamma}^I{}_{Ka,b} &= (S^{-1})^I{}_A S^C{}_K\Gamma^A{}_{Ca,b}\\ &\quad +[-(S^{-1})^I{}_E(S^{-1})^M{}_A S^E{}_{M,b}S^C{}_K + (S^{-1})^I{}_A S^A{}_{K,b}]\Gamma^A{}_{Ca}\\ &\quad -(S^{-1})^I{}_C(S^{-1})^M{}_A S^C{}_{M,b}S^A{}_{K,a} + (S^{-1})^I{}_A S^A{}_{K,ab}\end{aligned} \tag{6.54}$$

$$\tilde{\bar{\Gamma}}^{J'}{}_{L'a} = (\bar{S}^{-1})^{J'}{}_{B'}\bar{S}^{D'}{}_{L'}\bar{\Gamma}^{B'}{}_{D'a} + (\bar{S}^{-1})^{J'}{}_{B'}\bar{S}^{B'}{}_{L',a} \tag{6.55}$$

$$\begin{aligned}\tilde{\bar{\Gamma}}^{J'}{}_{L'a,b} &= (\bar{S}^{-1})^{J'}{}_{B'}\bar{S}^{D'}{}_{L'}\bar{\Gamma}^{B'}{}_{D'a,b}\\ &\quad +[-(\bar{S}^{-1})^{J'}{}_{F'}(\bar{S}^{-1})^{N'}{}_{B'}\bar{S}^{F'}{}_{N',b}\bar{S}^{D'}{}_{L'} + (\bar{S}^{-1})^{J'}{}_{B'}\bar{S}^{D'}{}_{L',b}]\bar{\Gamma}^{B'}{}_{D'a}\\ &\quad -(\bar{S}^{-1})^{J'}{}_{D'}(\bar{S}^{-1})^{N'}{}_{B'}\bar{S}^{D'}{}_{N',b}\bar{S}^{B'}{}_{L',a} + (\bar{S}^{-1})^{J'}{}_{B'}\bar{S}^{B'}{}_{L',ab}\end{aligned} \tag{6.56}$$

$$\tilde{\psi}_I = S^{(n-\frac{1}{2})}S^A{}_I\psi_A \tag{6.57}$$

$$\begin{aligned}\tilde{\psi}_{I,a} &= S^{(n-\frac{1}{2})}S^A{}_I\psi_{A,a}\\ &\quad +S^{(n-\frac{1}{2})}[(n-\tfrac{1}{2})(S^{-1})^K{}_C S^C{}_{K,a}S^A{}_I + S^A{}_{I,a}]\psi_A\end{aligned} \tag{6.58}$$

$$\tilde{\bar{\psi}}_{J'} = \bar{S}^{(n-\frac{1}{2})}\bar{S}^{B'}{}_{J'}\bar{\psi}_{B'} \tag{6.59}$$

$$\tilde{\bar{\psi}}_{J',a} = \bar{S}^{(n-\frac{1}{2})}\bar{S}^{B'}{}_{J'}\bar{\psi}_{B',a}$$

$$+\bar{S}^{(n-\frac{1}{2})}[(n-\tfrac{1}{2})(\bar{S}^{-1})^{L'}{}_{D'}\bar{S}^{D'}{}_{L',a}\bar{S}^{B'}{}_{J'}+\bar{S}^{B'}{}_{J',a}]\bar{\psi}_{B'} \quad (6.60)$$

$$\tilde{\phi}^{I} = S^{-(n-\frac{1}{2})}(S^{-1})^{I}{}_{A}\phi^{A} \quad (6.61)$$

$$\begin{aligned}\tilde{\phi}^{I}{}_{,a} &= S^{-(n-\frac{1}{2})}(S^{-1})^{I}{}_{A}\phi^{A}{}_{,a}\\ &\quad -S^{-(n-\frac{1}{2})}[(n-\tfrac{1}{2})(S^{-1})^{K}{}_{C}S^{C}{}_{K,a}(S^{-1})^{I}{}_{A}\\ &\qquad +(S^{-1})^{I}{}_{C}(S^{-1})^{K}{}_{A}S^{C}{}_{K,a}]\phi^{A}\end{aligned} \quad (6.62)$$

$$\tilde{\bar{\phi}}^{J'} = \bar{S}^{-(n-\frac{1}{2})}(\bar{S}^{-1})^{J'}{}_{B'}\bar{\phi}^{B'} \quad (6.63)$$

$$\begin{aligned}\tilde{\bar{\phi}}^{J'}{}_{,a} &= \bar{S}^{-(n-\frac{1}{2})}(\bar{S}^{-1})^{J'}{}_{B'}\bar{\phi}^{B'}{}_{,a}\\ &\quad -\bar{S}^{-(n-\frac{1}{2})}[(n-\tfrac{1}{2})(\bar{S}^{-1})^{L'}{}_{D'}\bar{S}^{D'}{}_{L',a}(\bar{S}^{-1})^{J'}{}_{B'}\\ &\qquad +(\bar{S}^{-1})^{J'}{}_{D'}(\bar{S}^{-1})^{L'}{}_{B'}\bar{S}^{D'}{}_{L',a}]\bar{\phi}^{B'}.\end{aligned} \quad (6.64)$$

We introduce the following notation for the functional derivatives of L

$$\begin{aligned}
&\Sigma^{aAB'} = \frac{\partial L}{\partial \sigma_{aAB'}} && \\
&\tau_{A}{}^{Ca} = \frac{\partial L}{\partial \Gamma^{A}{}_{Ca}}, && \bar{\tau}_{B'}{}^{D'a} = \frac{\partial L}{\partial \bar{\Gamma}^{B'}{}_{D'a}}\\
&\tau_{A}{}^{Cab} = \frac{\partial L}{\partial \Gamma^{A}{}_{Ca,b}}, && \bar{\tau}_{B'}{}^{D'ab} = \frac{\partial L}{\partial \bar{\Gamma}^{B'}{}_{D'a,b}}\\
&\Psi^{A} = \frac{\partial L}{\partial \psi_{A}}, && \bar{\Psi}^{B'} = \frac{\partial L}{\partial \bar{\psi}_{B'}}\\
&\Psi^{Aa} = \frac{\partial L}{\partial \psi_{A,a}}, && \bar{\Psi}^{B'a} = \frac{\partial L}{\partial \bar{\psi}_{B',a}}\\
&\Phi_{A} = \frac{\partial L}{\partial \phi^{A}}, && \bar{\Phi}_{B'} = \frac{\partial L}{\partial \bar{\phi}^{B'}}\\
&\Phi_{A}{}^{a} = \frac{\partial L}{\partial \phi^{A}{}_{,a}}, && \bar{\Phi}_{B'}{}^{a} = \frac{\partial L}{\partial \bar{\phi}^{B'}{}_{,a}}
\end{aligned} \quad (6.65)$$

where $\Sigma^{aAB'}$ is, of course, Hermitean.

Differentiating the transformation laws (6.51–6.64) of

$$\tilde{L}(\tilde{\sigma}_{aIJ'},\tilde{\Gamma}^{I}{}_{Ka},\tilde{\Gamma}^{I}{}_{Ka.b},\tilde{\bar{\Gamma}}^{J'}{}_{L'a},\tilde{\bar{\Gamma}}^{J'}{}_{L'a,b},\tilde{\psi}_{I},\tilde{\psi}_{I,a},\tilde{\bar{\psi}}_{J'},\tilde{\bar{\psi}}_{J',a},\tilde{\phi}^{I},\tilde{\phi}^{I,a},\tilde{\bar{\phi}}^{J'},\tilde{\bar{\phi}}^{J',a})$$

and its arguments with respect to the arguments of L yields the following gauge transformation laws for the functional derivatives of L

$$\tilde{\Sigma}^{aIJ'}=S^{A}{}_{I}\bar{S}^{B'}{}_{J'}=\Sigma^{aAB'} \tag{6.66}$$

$$\tilde{\tau}_{I}{}^{Ka}(S^{-1})^{I}{}_{A}S^{C}{}_{K}+\tilde{\tau}_{I}{}^{Kab}[(S^{-1})^{I}{}_{A}S^{C}{}_{K}]_{,b}=\tau_{A}{}^{Ca} \tag{6.67}$$

$$\tilde{\tau}_{I}{}^{Kab}(S^{-1})^{I}{}_{A}S^{C}{}_{K}=\tau_{A}{}^{Cab} \tag{6.68}$$

$$\tilde{\Psi}^{I}S^{(n-\frac{1}{2})}S^{A}{}_{I}+\tilde{\Psi}^{Ia}(S^{(n-\frac{1}{2})}S^{A}{}_{I})_{,a}=\Psi^{A} \tag{6.69}$$

$$\tilde{\Psi}^{Ia}S^{(n-\frac{1}{2})}S^{A}{}_{I}=\Psi^{Aa} \tag{6.70}$$

$$\tilde{\Phi}_{I}S^{-(n-\frac{1}{2})}(S^{-1})^{I}{}_{A}+\tilde{\Phi}_{I}{}^{a}[S^{-(n-\frac{1}{2})}(S^{-1})^{I}{}_{A}]_{,a}=\Phi_{A} \tag{6.71}$$

$$\tilde{\Phi}_{I}{}^{a}S^{-(n-\frac{1}{2})}(S^{-1})^{I}{}_{A}=\Phi_{A}{}^{a} \tag{6.72}$$

along with the conjugates of equations (6.67–6.72). We note that $\Sigma^{aAB'}$ and $\tau_{A}{}^{Cab}$ are "gauge tensors", Ψ^{Aa} and $\Phi_{A}{}^{a}$ are "gauge tensor densities" of gauge weights $-(n-\frac{1}{2})$ and $(n-\frac{1}{2})$ respectively while $\tau_{A}{}^{ca}$, Ψ^{A} and Φ_{A} transform improperly. The respective conjugates naturally possess appropriately conjugated properties.

Differentiating the transformation laws (6.51–6.64) of

$$\tilde{L}(\tilde{\sigma}_{aIJ'},\tilde{\Gamma}^{I}{}_{Ka},\tilde{\Gamma}^{I}{}_{Ka.b},\tilde{\bar{\Gamma}}^{J'}{}_{L'a},\tilde{\bar{\Gamma}}^{J'}{}_{L'a,b},\tilde{\psi}_{I},\tilde{\psi}_{I,a},\tilde{\bar{\psi}}_{J'},\tilde{\bar{\psi}}_{J',a},\tilde{\phi}^{I},\tilde{\phi}^{I,a},\tilde{\bar{\phi}}^{J'},\tilde{\bar{\phi}}^{J',a})$$

and its arguments with respect to $S^{A}{}_{C}$ and its derivatives and then considering the identity transform yields the following invariance relations

$$\tau_{P}{}^{Rpq}+\tau_{P}{}^{Rqp}=0 \tag{6.73}$$

$$\tau_P{}^{Rp} - \tau_P{}^{Cap}\Gamma^R{}_{Ca} + \tau_A{}^{Rap}\Gamma^A{}_{Pa}$$
$$+ \quad \Psi^{Rp}\psi_P + (n-\tfrac{1}{2})\delta^R_P\Psi^{Ap}\psi_A - \Phi_P{}^p\phi^R - (n-\tfrac{1}{2})\delta^R_P\Phi_A{}^p\phi^A = 0 \qquad (6.74)$$

$$\Sigma^{aRB'}\sigma_{aPB'} - \tau_P{}^{Ca}\Gamma^R{}_{Ca} + \tau_A{}^{Ra}\Gamma^A{}_{Pa} - \tau_P{}^{Cab}\Gamma^R{}_{Ca,b} + \tau_A{}^{Rab}\Gamma^A{}_{Pa,b}$$
$$+\Psi^R\psi_P + (n-\tfrac{1}{2})\delta^R_P\Psi^A\psi_A + \Psi^{Ra}\psi_{P,a} + (n-\tfrac{1}{2})\delta^R_P\Psi^{Aa}\psi_{A,a}$$
$$-\Phi_P\phi^R - (n-\tfrac{1}{2})\delta^R_P\Phi_A\phi^A - \Phi_P{}^a\phi^R{}_{,a} - (n-\tfrac{1}{2})\delta^R_P\Phi_A{}^a\phi^A{}_{,a} = 0 \qquad (6.75)$$

along with their complex conjugates.

Under a coordinate transformation $C^a{}_b$, as in equation (6.19), we have the following transformation laws for L and its arguments.

$$\tilde{L} = CL \qquad (6.76)$$
$$\tilde{\sigma}_{iAB'} = C^a{}_i\sigma_{aAB'} \qquad (6.77)$$
$$\tilde{\Gamma}^A{}_{Ci} = C^a{}_i\Gamma^A{}_{Ca} \qquad (6.78)$$
$$\tilde{\Gamma}^A{}_{Ci,j} = C^a{}_iC^b{}_j\Gamma^A{}_{Ca,b} + C^a{}_{ij}\Gamma^A{}_{Ca} \qquad (6.79)$$
$$\tilde{\bar{\Gamma}}^{B'}{}_{D'i} = C^a{}_i\bar{\Gamma}^{B'}{}_{D'a} \qquad (6.80)$$
$$\tilde{\bar{\Gamma}}^{B'}{}_{D'i,j} = C^a{}_iC^b{}_j\bar{\Gamma}^{B'}{}_{D'a,b} + C^a{}_{ij}\bar{\Gamma}^{B'}{}_{D'a} \qquad (6.81)$$
$$\tilde{\psi}_A = \psi_A \qquad (6.82)$$
$$\tilde{\psi}_{A,i} = C^a{}_i\psi_{A,a} \qquad (6.83)$$
$$\tilde{\bar{\psi}}_{B'} = \bar{\psi}_{B'} \qquad (6.84)$$
$$\tilde{\bar{\psi}}_{B',i} = C^a{}_i\bar{\psi}_{B',a} \qquad (6.85)$$
$$\tilde{\phi}^A = \phi^A \qquad (6.86)$$
$$\tilde{\phi}^A{}_{,i} = C^a{}_i\phi^A{}_{,a} \qquad (6.87)$$
$$\tilde{\bar{\phi}}^{B'} = \bar{\phi}^{B'} \qquad (6.88)$$
$$\tilde{\bar{\phi}}^{B'}{}_{,i} = C^a{}_i\bar{\phi}^{B'}{}_{,a}. \qquad (6.89)$$

Differentiating the transformation laws (6.76–6.89) of

$$\tilde{L}(\tilde{\sigma}_{aIJ'},\tilde{\Gamma}^{I}{}_{Ka},\tilde{\Gamma}^{I}{}_{Ka,b},\tilde{\bar{\Gamma}}^{J'}{}_{L'a},\tilde{\bar{\Gamma}}^{J'}{}_{L'a,b},\tilde{\psi}_{I},\tilde{\psi}_{I,a},\tilde{\bar{\psi}}_{J'},\tilde{\bar{\psi}}_{J',a},\tilde{\phi}^{I},\tilde{\phi}^{I,a},\tilde{\bar{\phi}}^{J'},\tilde{\bar{\phi}}^{J',a})$$

and its arguments with respect to the arguments of L yields the following coordinate transformation laws for the functional derivatives of L

$$\tilde{\Sigma}^{iAB'}C^{a}{}_{i}=C\Sigma^{aAB'} \tag{6.90}$$

$$\tilde{\tau}_{A}{}^{Ci}C^{a}{}_{i}+\tilde{\tau}_{A}{}^{Cij}C^{a}{}_{ij}=C\tau_{A}{}^{Ca} \tag{6.91}$$

$$\tilde{\tau}_{A}{}^{Cij}C^{a}{}_{i}C^{b}{}_{j}=C\tau_{A}{}^{Cab} \tag{6.92}$$

$$\tilde{\Psi}^{A}=C\Psi^{A} \tag{6.93}$$

$$\tilde{\Psi}^{Ai}C^{a}{}_{i}=C\Psi^{Aa} \tag{6.94}$$

$$\tilde{\Phi}_{A}=C\Phi_{A} \tag{6.95}$$

$$\tilde{\Phi}_{A}{}^{i}C^{a}{}_{i}=C\Phi_{A}{}^{a} \tag{6.96}$$

along with the conjugates of equations (6.91–6.96). After noting equation (6.73) and its conjugate we see that all of the functional derivatives of the Lagrangian are coordinate scalar or tensor densities.

Finally, differentiating the transformation laws (6.76–6.89) of

$$\tilde{L}(\tilde{\sigma}_{aIJ'},\tilde{\Gamma}^{I}{}_{Ka},\tilde{\Gamma}^{I}{}_{Ka,b},\tilde{\bar{\Gamma}}^{J'}{}_{L'a},\tilde{\bar{\Gamma}}^{J'}{}_{L'a,b},\tilde{\psi}_{I},\tilde{\psi}_{I,a},\tilde{\bar{\psi}}_{J'},\tilde{\bar{\psi}}_{J',a},\tilde{\phi}^{I},\tilde{\phi}^{I,a},\tilde{\bar{\phi}}^{J'},\tilde{\bar{\phi}}^{J',a})$$

and its arguments with respect to $C^{a}{}_{i}$ and its derivatives and then considering the identity transform, yields the additional invariance relations

$$(\tau_{A}{}^{Cpq}+\tau_{A}{}^{Cqp})\Gamma^{A}{}_{Ca}+(\bar{\tau}_{B'}{}^{D'pq}+\bar{\tau}_{B'}{}^{D'qp})\bar{\Gamma}^{B'}{}_{D'a}=0 \tag{6.97}$$

$$\begin{aligned}&\Sigma^{pAB'}\sigma_{qAB'}+\tau_{A}{}^{Cp}\Gamma^{A}{}_{Cq}+\bar{\tau}_{B'}{}^{D'p}\bar{\Gamma}^{B'}{}_{D'q}\\&\quad+\tau_{A}{}^{Cpb}\Gamma^{A}{}_{Cq,b}+\tau_{A}{}^{Cap}\Gamma^{A}{}_{Ca,q}+\bar{\tau}_{B'}{}^{D'pb}\bar{\Gamma}^{B'}{}_{D'q,b}+\bar{\tau}_{B'}{}^{D'ap}\bar{\Gamma}^{B'}{}_{D'a,q}\\&\quad+\Psi^{Ap}\psi_{A,q}+\bar{\Psi}^{B'p}\bar{\psi}_{B',q}+\Phi_{A}{}^{p}\phi^{A}{}_{,q}+\bar{\Phi}_{B'}{}^{p}\bar{\phi}^{B'}{}_{,q}=\delta^{p}_{q}L\end{aligned} \tag{6.98}$$

where (6.97) is identically satisfied by (6.73).

We now derive tensor concomitants for $\tau_A{}^{Ca}$, Ψ^A, Φ_A and their conjugates. We introduce arbitrary quantities $h^A{}_{Ca}$, $\bar{h}^{B'}{}_{D'a}$, k_A, $\bar{k}_{B'}$, t^A, $\bar{t}^{B'}$ which transform properly under gauge or coordinate transformations (whichever we are considering), k_A and t^A being of gauge weight $(n-\frac{1}{2})$, $-(n-\frac{1}{2})$ respectively and $\bar{k}_{B'}$ and $\bar{t}^{B'}$ of gauge antiweight $(n-\frac{1}{2})$, $-(n-\frac{1}{2})$ respectively. We define

$$H = \tau_A{}^{Cab}h^A{}_{Ca,b} + \tau_A{}^{Ca}h^A{}_{Ca} \tag{6.99}$$

$$K = \Psi^{Aa}k_{A,a} + \Psi^A k_A \tag{6.100}$$

$$T = \Phi_A{}^a t^A{}_{,a} + \Phi_A t^A \tag{6.101}$$

and their complex conjugates. It is easy to show that H, K, T and their respective conjugates are coordinate scalar densities and gauge scalars.

We now assume the existence of quantities $\Pi_A{}^{Ca}$, Π^A, P_A and the complex conjugates such that

$$H = \tau_A{}^{Cab}h^A{}_{Ca;b} + \Pi_A{}^{Ca}h^A{}_{Ca} \tag{6.102}$$

$$K = \Psi^{Aa}k_{A;a} + \Pi^A k_A \tag{6.103}$$

$$T = \Phi_A{}^a t^A{}_{;a} + P_A t^A \tag{6.104}$$

along with their conjugate equations. Comparing equations (6.99–6.101) we may conclude

$$\Pi_A{}^{Ca} \equiv \tau_A{}^{Ca} - \tau_M{}^{Cab}\Gamma^M{}_{Ab} + \tau_A{}^{Mab}\Gamma^C{}_{Mb} \tag{6.105}$$

$$\Pi^A \equiv \Psi^A + \Psi^{Ma}\Gamma^A{}_{Ma} + (n-\tfrac{1}{2})\Psi^{Aa}\Gamma^M{}_{Ma} \tag{6.106}$$

$$P_A \equiv \Phi_A - \Phi_M{}^a\Gamma^M{}_{Aa} - (n-\tfrac{1}{2})\Phi_A{}^a\Gamma^M{}_{Ma} \tag{6.107}$$

and similarly with their conjugates.

Noting that H, $\tau_A{}^{CaB}$, $h^A{}_{Ca}$, K, Ψ^{Aa}, k_A, T, $\Phi_A{}^a$, t^A and their conjugates are tensorial (under whichever type of transformation we are considering) establishes that $\Pi_A{}^{Ca}$, Π^A, P_A and their conjugates are also tensorial under both coordinate and gauge transformations.

These definitions also imply

$$
\begin{aligned}
E_A{}^{Ca} &\equiv -\tau_A{}^{Ca} + \tau_A{}^{Cab}{}_{,b} \\
&= -\Pi_A{}^{Ca} + \tau_A{}^{Cab}{}_{;b} && (6.108) \\
{}_\psi E^A &\equiv -\Psi^A + \Psi^{Aa}{}_{,a} \\
&= -\Pi^A + \Psi^{Aa}{}_{;a} && (6.109) \\
{}_\phi E_A &\equiv -\Phi_A + \Phi_A{}^a{}_{,a} \\
&= -P_A + \Phi_A{}^a{}_{;a} && (6.110)
\end{aligned}
$$

and similarly for their conjugates. It is now clear that $E_A{}^{Ca}$, ${}_\psi E^A$, ${}_\phi E_A$ and their conjugates are both coordinate and gauge tensorial.

The conversion of the invariance relations is straightforward. Of course (6.73) needs no conversion. Equations (6.74), (6.75) and (6.98) become

$$
\Pi_P{}^{Rp} + \Psi^{Rp}\psi_P + (n-\tfrac{1}{2})\delta^R_P\Psi^{Ap}\psi_A - \Phi_P{}^p\phi^R - (n-\tfrac{1}{2})\delta^R_P\Phi_A{}^p\phi^A = 0 \qquad (6.111)
$$

$$
\begin{aligned}
&\Sigma^{aRB'}\sigma_{aPB'} - \tfrac{1}{2}\tau_P{}^{Cab}F^R{}_{Cba} + \tfrac{1}{2}\tau_A{}^{Rab}F^A{}_{Pba} \\
&\quad + \Pi^R\psi_P + (n-\tfrac{1}{2})\delta^R_P\Pi^A\psi_A + \Psi^{Ra}\psi_{P;a} + (n-\tfrac{1}{2})\delta^R_P\Psi^{Aa}\psi_{A;a} \\
&\quad - P_P\phi^R - (n-\tfrac{1}{2})\delta^R_P P_A\phi^A - \Phi_P{}^a\phi^R{}_{;a} - (n-\tfrac{1}{2})\delta^R_P\Phi_A{}^a\phi^A{}_{;a} = 0 \qquad (6.112)
\end{aligned}
$$

$$
\begin{aligned}
&\Sigma^{pAB'}\sigma_{qAB'} + \tfrac{1}{2}\tau_A{}^{Cpb}F^A{}_{Cbq} + \tfrac{1}{2}\bar{\tau}_{B'}{}^{D'pb}\bar{F}^{B'}{}_{D'bq} \\
&\quad + \Psi^{Ap}\psi_{A;q} + \bar{\Psi}^{B'p}\bar{\psi}_{B';q} + \Phi_A{}^p\phi^A{}_{;q} + \bar{\Phi}_{B'}{}^p\bar{\phi}^{B'}{}_{;q} = \delta^p_q L. \qquad (6.113)
\end{aligned}
$$

We also wish to convert the expression

$$
L_{;p} = \Sigma^{aAB'}\sigma_{aAB';p} + \tau_A{}^{Ca}\Gamma^A{}_{Ca;p} + \tau_A{}^{Cab}\Gamma^A{}_{Ca,b;p}
$$

$$
\begin{aligned}
&+\bar{\tau}_{B'}{}^{D'a}\bar{\Gamma}^{B'}{}_{D'a;p}+\bar{\tau}_{B'}{}^{D'ab}\bar{\Gamma}^{B'}{}_{D'a,b;p}\\
&+\Psi^{A}\psi_{A;p}+\Psi^{Aa}\psi_{A,a;p}+\bar{\Psi}^{B'}\bar{\psi}_{B';p}+\bar{\Psi}^{B'a}\bar{\psi}_{B',a;p}\\
&+\Phi_{A}\phi^{A}{}_{;p}+\Phi_{A}{}^{a}\phi^{A}{}_{,a;p}+\bar{\Phi}_{B'}\bar{\phi}^{B'}{}_{;p}+\bar{\Phi}_{B'}{}^{a}\bar{\phi}^{B'}{}_{,a;p}
\end{aligned}
\tag{6.114}
$$

which yields with only slightly more difficulty

$$
\begin{aligned}
L_{;p} \;=\;& \Sigma^{aAB'}\sigma_{aAB';p}+\tau_{A}{}^{Cab}F^{A}{}_{Cpa;b}+\bar{\tau}_{B'}{}^{D'ab}\bar{F}^{B'}{}_{D'pa;b}\\
&+\Pi^{A}\psi_{A;p}+\Psi^{Aa}\psi_{A;p;a}+\bar{\Pi}^{B'}\bar{\psi}_{B';p}+\bar{\Psi}^{B'a}\bar{\psi}_{B';p;a}\\
&+P_{A}\phi^{A}{}_{;p}+\Phi_{A}{}^{a}\phi^{A}{}_{;p;a}+\bar{P}_{B'}\bar{\phi}^{B'}{}_{;p}+\bar{\Phi}_{B'}{}^{a}\bar{\phi}^{B'}{}_{;p;a}\\
&+[\Psi^{Ca}\psi_{A}+(n-\tfrac{1}{2})\delta^{C}_{A}\Psi^{Ma}\psi_{M}]F^{A}{}_{Cap}\\
&+[\bar{\Psi}^{D'a}\psi_{B'}+(n-\tfrac{1}{2})\delta^{D'}_{B'}\bar{\Psi}^{N'a}\bar{\psi}_{N'}]\bar{F}^{B'}{}_{D'ap}\\
&-[\Pi_{A}{}^{a}\phi^{C}+(n-\tfrac{1}{2})\delta^{C}_{A}\Phi_{M}{}^{a}\phi^{M}]F^{A}{}_{Cap}\\
&-[\bar{\Phi}_{B'}{}^{a}\phi^{D'}+(n-\tfrac{1}{2})\delta^{D'}_{B'}\bar{\Phi}_{N'}{}^{a}\bar{\phi}^{N'}]\bar{F}^{B'}{}_{D'ap}.
\end{aligned}
\tag{6.115}
$$

We now reconsider the action (2.1) where L is a scalar density with the functional dependence (6.47). Under an infinitesimal coordinate transformation the variation of the action is again of the form of (3.1) while under an infinitesimal gauge transformation we have

$$
\delta S=\int\delta L\,d^{4}x \tag{6.116}
$$

where, in both cases, δL is given by

$$
\begin{aligned}
\delta L \;=\;& \Sigma^{aAB'}\,\delta\sigma_{aAB'}+\tau_{A}{}^{Ca}\,\delta\Gamma^{A}{}_{Ca}+\tau_{A}{}^{Cab}\,\delta\Gamma^{A}{}_{Ca,b}\\
&+\bar{\tau}_{B'}{}^{D'a}\,\delta\bar{\Gamma}^{B'}{}_{D'a}+\bar{\tau}_{B'}{}^{D'ab}\,\delta\bar{\Gamma}^{B'}{}_{D'a,b}\\
&+\Psi^{A}\,\delta\psi_{A}+\Psi^{Aa}\,\delta\psi_{A,a}+\bar{\Psi}^{B'}\,\delta\bar{\psi}_{B'}+\bar{\Psi}^{B'a}\,\delta\bar{\psi}_{B',a}\\
&+\Phi_{A}\,\delta\phi^{A}+\Phi_{A}{}^{a}\,\delta\phi^{A}{}_{,a}+\bar{\Phi}_{B'}\,\delta\bar{\phi}^{B'}+\bar{\Phi}_{B'}{}^{a}\,\delta\bar{\phi}^{B'}{}_{,a}.
\end{aligned}
\tag{6.117}
$$

The resulting Euler-Lagrange equations are

$$E^{aAB'} \equiv -\Sigma^{aAB'} = 0 \tag{6.118}$$

$$E_A{}^{Ca} = -\tau_A{}^{Ca} + \tau_A{}^{Cab}{}_{,b} = 0 \tag{6.119}$$

$$\bar{E}_{B'}{}^{D'a} = -\bar{\tau}_{B'}{}^{D'a} + \bar{\tau}_{B'}{}^{D'ab}{}_{,b} = 0 \tag{6.120}$$

$$_{\psi}E^A = -\Psi^A + \Psi^{Aa}{}_{,a} = 0 \tag{6.121}$$

$$_{\psi}\bar{E}^{B'} = -\bar{\Psi}^{B'} + \bar{\Psi}^{B'a}{}_{,a} = 0 \tag{6.122}$$

$$_{\phi}E_A = -\Phi_A + \Phi_A{}^a{}_{,a} = 0 \tag{6.123}$$

$$_{\phi}\bar{E}_{B'} = -\bar{\Phi}_{B'} + \bar{\Phi}_{B'}{}^a{}_{,a} = 0. \tag{6.124}$$

Under the infinitesimal coordinate transformation $\tilde{x}^a = x^a + \chi^a$ the variations of the potentials take the form

$$\begin{aligned}\delta\sigma_{aAB'} &= -[\sigma_{mAB'}\chi^m{}_{;a} + \sigma_{aAB';m}\chi^m \\ &\quad +(\sigma_{aMB'}\Gamma^M{}_{Am} + \sigma_{aAN'}\bar{\Gamma}^{N'}{}_{B'm})\chi^m]\end{aligned} \tag{6.125}$$

$$\delta\Gamma^A{}_{Ca} = -[(\Gamma^A{}_{Cm}\chi^m)_{;a} - F^A{}_{Cam}\chi^m] \tag{6.126}$$

$$\delta\psi_A = -[\psi_{A;m} + \Gamma^M{}_{Am}\psi_M + (n - \tfrac{1}{2})\Gamma^M{}_{Mm}\psi_A]\chi^m \tag{6.127}$$

$$\delta\phi^A = -[\phi^A{}_{;m} - \Gamma^A{}_{Mm}\phi^M - (n - \tfrac{1}{2})\Gamma^M{}_{Mm}\phi^A]\chi^m \tag{6.128}$$

along with the usual conjugate equations. As these are coordinate vectors and scalars we may rewrite δL in the form

$$\begin{aligned}\delta L &= \Sigma^{aAB'}\,\delta\sigma_{aAB'} + \Pi_A{}^{Ca}\,\delta\Gamma^A{}_{Ca} + \tau_A{}^{Cab}\,\delta\Gamma^A{}_{Ca;b} \\ &\quad +\bar{\Pi}_{B'}{}^{D'a}\,\delta\bar{\Gamma}^{B'}{}_{D'a} + \bar{\tau}_{B'}{}^{D'ab}\,\delta\bar{\Gamma}^{B'}{}_{D'a;b} \\ &\quad +\Pi^A\,\delta\psi_A + \Psi^{Aa}\,\delta\psi_{A;a} + \bar{\Pi}^{B'}\,\delta\bar{\psi}_{B'} + \bar{\Psi}^{B'a}\,\delta\bar{\psi}_{B';a} \\ &\quad +P_A\,\delta\phi^A + \Phi_A{}^a\,\delta\phi^A{}_{;a} + \bar{P}_{B'}\,\delta\bar{\phi}^{B'} + \bar{\Phi}_{B'}{}^a\,\delta\bar{\phi}^{B'}{}_{;a}.\end{aligned} \tag{6.129}$$

Substitution of equations (6.125–6.128) and the relevant conjugates into equation (6.129) and application of the invariance relations and (6.115) immediately yields

$$\delta L + (L\chi^m)_{;m} = [(\tau_A{}^{Cab}\Gamma^A{}_{Cm} + \bar{\tau}_{B'}{}^{D'ab}\bar{\Gamma}^{B'}{}_{D'm})\chi^m]_{;b;a}. \tag{6.130}$$

Adopting the notation introduced in equation (3.5) then gives

$$h^a = \tfrac{1}{2}[(\tau_A{}^{Cab}\Gamma^A{}_{Cm} + \bar{\tau}_{B'}{}^{D'ab}\bar{\Gamma}^{B'}{}_{D'm})\chi^m]_{;b}. \tag{6.131}$$

Under the infinitesimal gauge transformation $S^A{}_C = \delta^A{}_C - \chi^A{}_C$ the variations of the potentials take the form

$$\delta\sigma_{aAB'} = -(\sigma_{aMB'}\chi^M{}_A + \sigma_{aAN'}\bar{\chi}^{N'}{}_{B'}) \tag{6.132}$$

$$\delta\Gamma^A{}_{Ca} = -\chi^A{}_{C;a} \tag{6.133}$$

$$\delta\psi_A = -[\psi_M\chi^M{}_A + (n-\tfrac{1}{2})\delta^K_I\psi_A\chi^I{}_K] \tag{6.134}$$

$$\delta\phi^A = \phi^M\chi^A{}_M + (n-\tfrac{1}{2})\delta^K_I\phi^A\chi^I{}_K \tag{6.135}$$

along with the appropriate conjugate equations. As these all transform properly under the gauge transformation we may again rewrite δL in the form (6.129). Now substitution of equations (6.132–6.135) and the relevant conjugates into equation (6.129) and application of the invariance relations gives

$$\begin{aligned}\delta L &= \tau_A{}^{Cab}{}_{;b;a}\chi^A{}_C + \tau_A{}^{Cab}\chi^A{}_{C;b;a} + \bar{\tau}_{B'}{}^{D'ab}{}_{;b;a}\bar{\chi}^{B'}{}_{D'} + \bar{\tau}_{B'}{}^{D'ab}\bar{\chi}^{B'}{}_{D';b;a} \\ &= (\tau_A{}^{Cab}\chi^A{}_C + \bar{\tau}_{B'}{}^{D'ab}\bar{\chi}^{B'}{}_{D'})_{;b;a}.\end{aligned} \tag{6.136}$$

In order to distinguish this result from the coordinate variation we write

$$\delta S = \int 2J^a{}_{;a}\, d^4x \tag{6.137}$$

in which case

$$J^a = \tfrac{1}{2}(\tau_A{}^{Cab}\chi^A{}_C + \bar{\tau}_{B'}{}^{D'ab}\bar{\chi}^{B'}{}_{D'})_{;b}. \tag{6.138}$$

Both h^a and J^a are vector densities under a coordinate transformation and J^a is invariant under a gauge transformation. Energy-momentum complexes generated by h^a will have all the coordinate dependent problems of those of classical general relativity. However this is not the case with J^a. Noting that

$$\chi^A{}_C = i \sum_\alpha k_\alpha T_\alpha{}^A{}_C \tag{6.139}$$

where the $T_\alpha{}^A{}_C$ are the generators of $SL(2,C) \times U(1)$ and the k_α are the parameters of the infinitesimal transformation, we see that currents derived from J^a will be coordinate invariant.

Chapter 7

Gauge Theories

7.1 The Einstein-Maxwell Equations

In this chapter we will be considering Lagrangians, with functional dependence given by equation (6.47), which generate gauge theories of gravitation and electromagnetism. In Sections 7.1 and 7.2 we will be concerned only with "pure" fields and thus the subclass of Lagrangians with the functional dependence

$$L = L(\sigma_{aAB'}, \Gamma^{A}{}_{Ca}, \Gamma^{A}{}_{Ca,b}, \bar{\Gamma}^{B'}{}_{D'a}, \bar{\Gamma}^{B'}{}_{D'a,b}). \tag{7.1}$$

In this section we will prove our previous claim; that, as a consequence of the fact that the connection already contains an Abelian gauge potential, our $SL(2,C) \times U(1)$ gauge theories will describe both gravitation and electromagnetism. We present three separate Lagrangians whose Euler-Lagrange equations are the Einstein-Maxwell equations.

Once we have discovered a Lagrangian which properly generates Einstein-Maxwell theory we may, of course, reconsider it to be a function of the metric, the electromagnetic potential and their derivatives. Recalling Lovelock's result, that the most general scalar density Lagrangian of the form (3.29) which generates the Einstein-Maxwell equation without the cosmological constant is given by equation (4.1), it thus appears expedient to look for $SL(2,C) \times U(1)$ Lagrangians which, on our spacetime, reduce to this expression.

Perhaps the most immediate way to construct such a Lagrangian is to discard the dual part of P_{abcd} (the second term of the final expression) in (6.46). So, we define

$$K_{abcd} = P_{abcd} - \bar{P}_{bacd} = R_{abcd} + i g_{ab} F_{cd} \tag{7.2}$$

$$K_{bd} = \tfrac{1}{2}(K^a{}_{bad} + K_b{}^a{}_{da}) = R_{bd} \tag{7.3}$$

$$K = K^{ab}{}_{ab} = R \tag{7.4}$$

where the symmetrized contraction appears in (7.3) because only these contractions occur in both the Einstein equations and the Bach-Lanczos identity for K_{abcd}[1].

Our Lagrangian is

$$L = \kappa L_1 + \alpha L_2 + \beta L_3 \tag{7.5}$$

$$L_1 = \sqrt{-g}\, K \tag{7.6}$$

$$L_2 = \varepsilon^{abcd} K^{mn}{}_{ab} K_{mncd} \tag{7.7}$$

$$L_3 = \sqrt{-g}\,(K^{ab}{}_{cd} K_{ab}{}^{cd} - 4K^a{}_b K_a{}^b + KK) \tag{7.8}$$

where κ, α and β are constants. The functional derivatives of the Lagrangian are

$${}_1\Sigma^{pPQ'} = \sqrt{-g}\,\sigma_q{}^{PQ'}(g^{pq}K - K^{pbq}{}_b - K^{ap}{}_a{}^q) \tag{7.9}$$

$${}_1\tau_P{}^{Rpq} = -\sqrt{-g}\,\sigma_{mPB'}\sigma_n{}^{RB'}(g^{mp}g^{nq} - g^{mq}g^{np}) \tag{7.10}$$

$${}_1\bar{\tau}_{Q'}{}^{S'pq} = \overline{{}_1\tau_Q{}^{Spq}} \tag{7.11}$$

$${}_2\Sigma^{pPQ'} = 0 \tag{7.12}$$

$${}_2\tau_P{}^{Rpq} = -4\sigma_{mPB'}\sigma_n{}^{RB'}\varepsilon^{abpq}K^{mn}{}_{ab} \tag{7.13}$$

$${}_2\bar{\tau}_{Q'}{}^{S'pq} = \overline{{}_2\tau_Q{}^{Spq}} \tag{7.14}$$

[1]These symmetrized contractions would also appear in the usual Bach-Lanczos identity but for the symmetry of the Ricci tensor.

$$
\begin{aligned}
{}_3\Sigma^{pPQ'} &= \sqrt{-g}\,\sigma_q{}^{PQ'}\{g^{pq}[K^{ab}{}_{cd}K_{ab}{}^{cd} - 4K^a{}_bK_a{}^b + KK] \\
&\quad -4[K^{abq}{}_dK_{ab}{}^{pd} - 2K^{ap}K_a{}^q \\
&\quad -K_a{}^b(K^{paq}{}_b - K^{apq}{}_b) + KK^{pq}]\} \qquad (7.15) \\
{}_3\tau_P{}^{Rpq} &= -4\sqrt{-g}\,\sigma_{mPB'}\sigma_n{}^{RB'}[K^{mnpq} \\
&\quad -(g^{mp}K^{nq} + g^{nq}K^{mp} - g^{mq}K^{np} - g^{np}K^{mq}) \\
&\quad +\tfrac{1}{2}(g^{mp}g^{nq} - g^{mq}g^{np})K] \qquad (7.16) \\
{}_3\bar{\tau}_{Q'}{}^{S'pq} &= \overline{{}_3\tau_Q{}^{Spq}} \qquad (7.17)
\end{aligned}
$$

which, after substituting for K_{abcd}, become

$$
\begin{aligned}
\Sigma^{pPQ'} &= \sqrt{-g}\,\sigma_q{}^{PQ'}[\kappa(g^{pq}R - 2R^{pq}) + \beta(-4g^{pq}F_{cd}F^{cd} + 16F^q{}_dF^{pd})] \\
&= \sqrt{-g}\,\sigma_q{}^{PQ'}(-2\kappa G^{pq} - 64\pi\beta T^{pq}) \qquad (7.18)
\end{aligned}
$$

$$
\begin{aligned}
\tau_P{}^{Rpq} &= -\sigma_{mPB'}\sigma_n{}^{RB'}[\kappa\sqrt{-g}\,(g^{mp}g^{nq} - g^{mq}g^{np}) \\
&\quad +4\alpha\varepsilon^{abpq}R^{mn}{}_{ab} + \beta\sqrt{-g}\,\delta^{ijpq}_{abcd}R_{ij}{}^{cd}g^{am}g^{bn}] \\
&\quad -8i\delta^R_P(\alpha\varepsilon^{abpq}F_{ab} + \beta\sqrt{-g}\,F^{pq}) \qquad (7.19)
\end{aligned}
$$

where use has been made of the Bach-Lanczos identity and G^{pq} and T^{pq} are, respectively, the Einstein tensor and the electromagnetic stress-energy tensor.

The Euler-Lagrange equations (6.118) and (6.108) now imply

$$G^{pq} = \frac{-32\pi\beta}{\kappa}T^{pq} \qquad (7.20)$$

$$(\sqrt{-g}\,F^{pq})_{;q} = 0 \qquad (7.21)$$

which are just the source-free Einstein-Maxwell equations.

While the "field" (7.2) does work, its definition is somewhat artificial. Forfunately, it is not unique. Just as Nissani's Lagrangian (5.72) was based on the Lagrangian of Carmeli's $SL(2, C)$ gauge theory, it is possible to design a Lagrangian for our $SL(2,C)\times U(1)$ gauge theory based on the double dual form of our alternate Nissani-class Lagrangian (5.97). We set $\alpha = 0$ and rewrite (7.5) in the form

$$L = \kappa L_4 + \beta L_5 \tag{7.22}$$

where

$$\begin{aligned} L_4 &= \sqrt{-g}(P+\bar{P}) \\ &= \sqrt{-g}\,R \end{aligned} \tag{7.23}$$

$$\begin{aligned} L_5 &= \sqrt{-g}[2(P^{ab}{}_{cd}P_{ab}{}^{cd} - 4P^{b}{}_{d}P_{b}{}^{d} + PP \\ &\quad +\bar{P}^{ab}{}_{cd}\bar{P}_{ab}{}^{cd} - 4\bar{P}^{b}{}_{d}\bar{P}_{b}{}^{d} + \bar{P}\bar{P}) \\ &\quad -\tfrac{1}{2}(P^{ab}{}_{cd} + \bar{P}^{ab}{}_{cd})(P_{ab}{}^{cd} + \bar{P}_{ab}{}^{cd}) \\ &= \sqrt{-g}(R^{ab}{}_{cd}R_{ab}{}^{cd} - 4R^{b}{}_{d}R_{b}{}^{d} + RR - 4F_{cd}F^{c\cdot\prime}) \end{aligned} \tag{7.24}$$

β and κ are, of course, real constants and we have defined

$$P^{b}{}_{d} = \tfrac{1}{2}(P^{ab}{}_{ad} + P^{ba}{}_{da}) = \tfrac{1}{2}R^{b}{}_{d} \tag{7.25}$$

$$P = P^{ab}{}_{ab} = \tfrac{1}{2}R \tag{7.26}$$

in analogy to equations (7.3) and (7.4).

The relevant functional derivatives are

$$\begin{aligned} {}_4\Sigma^{pPQ'} &= \sqrt{-g}\,\sigma_q{}^{PQ'}[g^{pq}(P+\bar{P}) - 2(P^{pq} + \bar{P}^{pq})] \\ &= -2\sqrt{-g}\,\sigma_q{}^{PQ'}G^{pq} \end{aligned} \tag{7.27}$$

$$\begin{aligned} {}_5\Sigma^{pPQ'} &= 2\sqrt{-g}\,\sigma_q{}^{PQ'}\{g^{pq}[P^{ab}{}_{cd}P_{ab}{}^{cd} - 4P^{b}{}_{d}P_{b}{}^{d} + PP] \\ &\quad -4[P^{ab}{}_{c}{}^{p}P_{ab}{}^{cq} - 2P^{bp}P_{b}{}^{q} - 2P^{b}{}_{d}(P^{p}{}_{b}{}^{qd} + P_{b}{}^{pdq}) + PP^{pq}] \\ &\quad +g^{pq}[\bar{P}^{ab}{}_{cd}\bar{P}_{ab}{}^{cd} - 4\bar{P}^{b}{}_{d}\bar{P}_{b}{}^{d} + \bar{P}\bar{P}] \\ &\quad -4[\bar{P}^{ab}{}_{c}{}^{p}\bar{P}_{ab}{}^{cq} - 2\bar{P}^{bp}\bar{P}_{b}{}^{q} - 2\bar{P}^{b}{}_{d}(\bar{P}^{p}{}_{b}{}^{qd} + \bar{P}_{b}{}^{pdq}) + \bar{P}\bar{P}^{pq}] \\ &\quad -\tfrac{1}{2}g^{pq}(P^{ab}{}_{cd} + \bar{P}^{ab}{}_{cd})(P_{ab}{}^{cd} + \bar{P}_{ab}{}^{cd}) \end{aligned}$$

$$
\begin{aligned}
& \quad +2(P^{ab}{}_{c}{}^{p} + \bar{P}^{ab}{}_{c}{}^{p})(P_{ab}{}^{cq} + \bar{P}_{ab}{}^{cq})\} \\
&= -64\pi\sqrt{-g}\,\sigma_{q}{}^{PQ'}T^{pq} \qquad (7.28) \\
{}_{4}\tau_{P}{}^{Rpq} &= -\sqrt{-g}\,\sigma_{aPQ'}\sigma_{b}{}^{PQ'}(g^{ap}g^{bq} - g^{aq}g^{bp}) \qquad (7.29) \\
{}_{5}\tau_{P}{}^{Rpq} &= -8\sqrt{-g}\,\sigma_{aPQ'}\sigma_{b}{}^{RQ'}[\tfrac{1}{2}(P^{abpq} + \bar{P}^{abpq}) \\
& \quad -(g^{ap}P^{bq} - g^{aq}P^{bp} + g^{bq}P^{ap} - g^{bp}P^{aq}) \\
& \quad +\tfrac{1}{2}(g^{ap}g^{bq} - g^{aq}g^{bp})P - \bar{P}^{abpq}] \\
&= -\sqrt{-g}(\sigma^{a}{}_{PQ'}\sigma^{bRQ'}\delta^{ijpq}_{abcd}R_{ij}{}^{cd} + 8i\delta^{R}_{P}F^{pq}) \qquad (7.30)
\end{aligned}
$$

and the Euler-Lagrange equations again reduce to the Einstein-Maxwell equations (7.20) and (7.21).

While the Lagrangian (7.22) also generates the Einstein-Maxwell equations, its complexity is somewhat of a handicap. However it is possible to define a much simpler Lagrangian, similar in form to a Yang-Mills Lagrangian, which reduces directly to the usual Maxwell Lagrangian. We write

$$L = \kappa L_4 + \beta L_6 \qquad (7.31)$$

where

$$
\begin{aligned}
L_6 &= -4\sqrt{-g}\,P^{ab}{}_{cd}\bar{P}_{ab}{}^{cd} \\
&= -4\sqrt{-g}\,F_{cd}F^{cd}. \qquad (7.32)
\end{aligned}
$$

The relevant functional derivatives are

$$
\begin{aligned}
{}_{6}\Sigma^{pPQ'} &= -4\sqrt{-g}\,\sigma_{q}{}^{PQ'}(g^{pq}P^{ab}{}_{cd}\bar{P}_{ab}{}^{cd} - 2P^{ab}{}_{c}{}^{p}\bar{P}_{ab}{}^{cq} - 2P^{ab}{}_{c}{}^{q}\bar{P}_{ab}{}^{cp}) \\
&= -64\pi\sqrt{-\quad}\;{}^{PQ'}T^{pq} \qquad (7.33) \\
{}_{6}\tau_{P}{}^{Rpq} &= 8\sqrt{-g}\,\delta_{P} \qquad {}_{N'}{}^{pq} \\
&= -8i\sqrt{-g}\,\delta^{R}_{P}F^{pq} \qquad (7.34)
\end{aligned}
$$

and the Euler-Lagrange equations yet again reduce to the Einstein-Maxwell equations (7.20) and (7.21).

7.2 The Generalized Einstein-Maxwell Equations

In the previous section we have proven our claim that, since the connection already contains an Abelian gauge potential, our $SL(2,C) \times U(1)$ gauge theories describe both gravitation and electromagnetism. However, despite the fact that this is a consequence of having a unified $SL(2,C) \times U(1)$ potential, we have, in these derivations, acted contrary to the spirit of this unification in the following sense. All three of the Lagrangians presented in the previous chapter consist, in whole or in part, of terms in which pieces of the natural "field" P_{abcd} (ie. the curvature spinor $F^A{}_{Ccd}$) have been discarded. In the first Lagrangian we discarded the dual part of P_{abcd} in defining K_{abcd}. The second Lagrangian contains terms in $P_{abcd} + \bar{P}_{abcd} = R_{abcd}$ and in the third Lagrangian, $P^{ab}{}_{cd}\bar{P}_{ab}{}^{cd} = F^M{}_{Mcd}\bar{F}^{N'}{}_{N'}{}^{cd}$ and we have retained only the Maxwell tensor part of P_{abcd}. Thus all three Lagrangians are subject to the criticism that they are somewhat artificial; each has been selectively modified in order to achieve the desired results.

This may be considered as mathematical support for our ideas about unification and its significance; that the natural consequence of a significant unification should be a theory in which the Euler-Lagrange equation resulting from the variation of the spinor connection would be an equation in the gravitational-electromagnetic field which, in the presence of matter, becomes a relation for a matter-charge current density. Such an equation would generalize the Maxwell current-equation and open up entirely new possibilities for the description of gravitational energy-momentum. We now present such a set of generalized Einstein-Maxwell equations through the introduction of a Yang-Mills Lagrangian. We write

$$L = \kappa L_4 + \beta L_7 \tag{7.35}$$

where

$$\begin{aligned} L_7 &= 2\sqrt{-g}(P^{ab}{}_{cd}P_{ab}{}^{cd} + \bar{P}^{ab}{}_{cd}\bar{P}_{ab}{}^{cd}) \\ &= -4\sqrt{-g}(F^A{}_{Ccd}F_A{}^{Ccd} + \bar{F}^{B'}{}_{D'cd}\bar{F}_{B'}{}^{D'cd}) \end{aligned}$$

$$= \ 2\sqrt{-g}(R^{ab}{}_{cd}R_{ab}{}^{cd} - 2F_{cd}F^{cd}). \tag{7.36}$$

We note that the Lagrangian (7.35) is not of the form of the general Lagrangian (4.1). Thus we might anticipate what we will shortly see; in generalizing the Maxwell equation we have "damaged" the Einstein equation.

The relevant functional derivatives of L_7 are

$$\begin{aligned} {}_7\Sigma^{pPQ'} &= 2\sqrt{-g}\,\sigma_q{}^{PQ'}[g^{pq}(P^{ab}{}_{cd}P_{ab}{}^{cd} + \bar{P}^{ab}{}_{cd}\bar{P}_{ab}{}^{cd}) \\ &\quad -4(P^{ab}{}_c{}^p P_{ab}{}^{cq} + \bar{P}^{ab}{}_c{}^p\bar{P}_{ab}{}^{cq})] \\ &= -64\pi\sqrt{-g}\,\sigma_q{}^{PQ'}(T^{pq} + {}_GT^{pq}) \end{aligned} \tag{7.37}$$

$${}_GT^{pq} = \frac{-1}{8\pi}(\tfrac{1}{4}g^{pq}R^{ab}{}_{cd}R_{ab}{}^{cd} - R^{ab}{}_c{}^pR_{ab}{}^{cq}) \tag{7.38}$$

$$\begin{aligned} {}_7\tau_P{}^{Rpq} &= -8\sqrt{-g}\,\sigma_{aPQ'}\sigma_b{}^{RQ'}P^{abpq} \\ &= 16\sqrt{-g}\,F_P{}^{Rpq} \end{aligned} \tag{7.39}$$

and, thus, the Euler-Lagrange equations take the form

$$G^{pq} = \frac{-32\pi\beta}{\kappa}(T^{pq} + {}_GT^{pq}) \tag{7.40}$$

$$\begin{aligned} &16\beta(\sqrt{-g}\,F_P{}^{Rpq})_{;q} = 0 \\ &= -8\beta\sqrt{-g}(\sigma_{aPQ'}\sigma_b{}^{RQ'}R^{abpq}{}_{;q} + i\delta_P^RF^{pq}{}_{;q}) \\ &= -8\beta\sqrt{-g}[(\sigma^a{}_{PQ'}\sigma^{bRQ'} - \sigma^b{}_{PQ'}\sigma^{aRQ'})R^p{}_{a;b} + i\delta_P^RF^{pq}{}_{;q}]. \end{aligned} \tag{7.41}$$

It is easy to show that $F_P{}^{Rpq}{}_{;q;p}$ vanishes so that, in the presence of matter, the current which will appear in equation (7.41) must be conserved. Thus we have accomplished our goal of generalizing the Maxwell equation in a gravitational setting. On the other hand, equation (7.40) is not the Einstein equation but instead includes an extra "gravitational stress-energy tensor" ${}_GT^{pq}$. This new tensor has an intriguing form—exactly mimicking the electromagnetic stress-energy tensor with Riemann tensor contractions replacing Maxwell tensor contractions. In fact

$$ {}_{G}T^{pq} = \frac{1}{16\pi} T^{pqr}{}_{r} \tag{7.42} $$

where

$$ T^{pqrs} = R^{par}{}_{b} R^{q}{}_{a}{}^{sb} + R^{pas}{}_{b} R^{q}{}_{a}{}^{rb} - \tfrac{1}{2} g^{pq} R^{acr}{}_{b} R_{ac}{}^{sb} \tag{7.43} $$

is the Bel-Robinson tensor, which has been found useful in studies of gravitational radiation.

Equations (7.40) and (7.41) are even more complicated than the Einstein-Maxwell equations. But, from the properties of the Bel-Robinson tensor [20], it follows that any vacuum solution of the Einstein equations will also satisfy equations (7.40) and (7.41). It is also interesting, though, that the Reissner-Nordström solution does not satisfy these equations; nor does it have a spherically symmetric analog. This need not be a fault, but could be considered a reflection of the physical fact that charged particles exhibit spin, implying that general relativistic charged solutions should possess at most cylindrical symmetry. However, at present there simply are no good reasons to displace the Einstein equation as the relation which best describes the gravitational field. Thus, unless such reasons eventually do appear, equation (7.40) must be considered only a potential alternative. Nevertheless, these equations and the reasoning which led to them are of some interest and in the following section we will explore some of the consequences of the introduction of matter and their implications concerning a generalized Maxwell equation like (7.41).

7.3 Matter

In this section we consider generalizations of the two most common sets of equations used in describing matter with spinors, the Dirac and Klein-Gordon equations. We will assume our Lagrangians are of the form

$$ L = L_G + \gamma L_M \tag{7.44} $$

where γ is a real constant,

$$L_M = L(\sigma_{aAB'}, \Gamma^A{}_{Ca}, \bar{\Gamma}^{B'}{}_{D'a}, \psi_A, \psi_{A,a}, \bar{\psi}_{B'}, \bar{\psi}_{B',a}, \phi^A, \phi^A{}_{,a}, \bar{\phi}^{B'}, \bar{\phi}^{B'}{}_{,a}) \tag{7.45}$$

and L_G is of the form (7.1). The Lagrangians (7.44) form a subclass of the class of Lagrangians (6.47) introduced in Section 6.3. While we shall consider L_G to be given by one of (7.31) or (7.35) we will not, here, explicitly choose either but tentatively assume the latter until the properties of L_M and the Euler-Lagrange equations rule otherwise.

In order to facilitate the study of the Euler-Lagrange equation (6.108) we define the quantity

$$\sigma^{ab}{}_A{}^C = \tfrac{1}{2}(\sigma^a{}_{AB'}\sigma^{bCB'} - \sigma^b{}_{AB'}\sigma^{aCB'}) = -\sigma^{ba}{}_A{}^C = \sigma^{abC}{}_A. \tag{7.46}$$

Noting equations (6.16) and (6.17), it is easy to show that

$$\sigma^{ab}{}_A{}^C\sigma^{cd}{}_C{}^A = -\tfrac{1}{2}(g^{ac}g^{bd} - g^{ad}g^{bc} + i\sqrt{-g}^{(-1)}\varepsilon^{abcd}). \tag{7.47}$$

Thus

$$\sigma^{pq}{}_P{}^R\sigma_{npR}{}^P = \tfrac{3}{2}\delta^q_n \tag{7.48}$$

and, using the contracted Bianchi identities with equation (7.41),

$$\sigma_{npR}{}^P{}_G\tau_P{}^{Rpq}{}_{;q} = -4\beta\sqrt{-g}\,R_{;n}. \tag{7.49}$$

As we also have

$$\delta^P_R\tau_P{}^{Rpq}{}_{;q} = -16i\beta\sqrt{-g}\,F^{pq}{}_{;q} \tag{7.50}$$

we are now in a position to separately examine the matter and charge current density components of ${}_M\Pi_P{}^{Rp}$.

The generalized Dirac Lagrangian takes the form

$$\begin{aligned} L_8 \;=\; & i\sqrt{-g}[\sigma^{aAB'}(\psi_A\bar{\psi}_{B';a} - \psi_{A;a}\bar{\psi}_{B'}) + \sigma^a{}_{AB'}(\phi^A{}_{;a}\bar{\phi}^{B'} - \phi^A\bar{\phi}^{B'}{}_{;a}) \\ & -2m(\psi_A\phi^A - \bar{\psi}_{B'}\bar{\phi}^{B'})] \end{aligned} \tag{7.51}$$

where ψ_A, $\bar{\psi}_{B'}$, ϕ^A and $\bar{\phi}^{B'}$ are gauge densities as described in Section 6.3. Then

$$\Psi^{Pa} = -i\sqrt{-g}\,\sigma^{aPB'}\bar{\psi}_{B'} \quad (7.52)$$

$$\Pi^P = i\sqrt{-g}(\sigma^{aPB'}\bar{\psi}_{B';a} - 2m\phi^P) \quad (7.53)$$

$$\Phi_P{}^a = i\sqrt{-g}\,\sigma^a{}_{PB'}\bar{\phi}^{B'} \quad (7.54)$$

$$P_P = -i\sqrt{-g}(\sigma^a{}_{PB'}\bar{\phi}^{B'}{}_{;a} + 2m\psi_P) \quad (7.55)$$

and the Euler-Lagrange equations (6.109) and (6.110) become

$$\sqrt{-g}(\sigma^{aPB'}\bar{\psi}_{B';a} - m\phi^P) = 0 \quad (7.56)$$

$$\sqrt{-g}(\sigma^a{}_{PB'}\bar{\phi}^{B'}{}_{;a} + m\psi_P) = 0 \quad (7.57)$$

and similarily for their complex conjugates. These equations imply

$$L_3 = 0. \quad (7.58)$$

Noting from equation (6.105) that ${}_M\Pi_p{}^{Ra} = {}_M\tau_p{}^{Ra}$ we have

$$\begin{aligned} {}_M\Pi_P{}^{Rp} = {} & i\sqrt{-g}[\sigma^{pRB'}\psi_P\bar{\psi}_{B'} + \sigma^p{}_{PB'}\phi^R\bar{\phi}^{B'} \\ & +(n - \tfrac{1}{2})\delta^R_P(\sigma^{pAB'}\psi_A\bar{\psi}_{B'} + \sigma^p{}_{AB'}\phi^A\bar{\phi}^{B'})]. \end{aligned} \quad (7.59)$$

Introducing the (real) null vectors

$$\psi_a = \sigma^{aAB'}\psi_A\bar{\psi}_{B'} \quad (7.60)$$

$$\phi_a = \sigma^a{}_{AB'}\phi^A\bar{\phi}^{B'} \quad (7.61)$$

we may rewrite equation (7.59) in the form

$${}_M\Pi_P{}^{Rp} = -i\sqrt{-g}[\sigma^{pq}{}_P{}^R(\psi_q - \phi_q) - n\delta^R_P(\psi_p + \phi_p)]. \quad (7.62)$$

Then, splitting the Euler-Lagrange equation into its gravitational and electromagnetic contributions, we have by equations (7.50) and (7.62)

$$8\beta F^{pq}{}_{;q} = -n\gamma(\psi_p + \phi_p) \quad (7.63)$$

and, from equations (7.48), (7.49) and (7.62)

$$4\beta R_{;q} = \tfrac{3}{2} i\gamma(\psi_q - \phi_q). \tag{7.64}$$

Since (7.64) must be real this implies

$$\psi_q = \phi_q \tag{7.65}$$

$$\psi_A \propto \phi^A \tag{7.66}$$

so the matter current density must vanish and the charge current density is null.

Finally, we have

$$\begin{aligned} {}_M\Sigma^{pPQ'} &= -i\sqrt{-g}\,\sigma^{aPQ'}[\sigma^{pAB'}(\psi_A\bar{\psi}_{B';a} - \psi_{A;a}\bar{\psi}_{B'}) \\ &\quad +\sigma^p{}_{AB'}(\phi^A{}_{;a}\bar{\phi}^{B'} - \phi^A\bar{\phi}^{B'}{}_{;a})]. \end{aligned} \tag{7.67}$$

Writing

$$\Sigma^{pq} = \Sigma^{pPQ'}\sigma^q{}_{PQ'} \tag{7.68}$$

and recalling that ${}_M\Sigma^p{}_p$ must be proportional to the Ricci scalar, we may substitute equations (7.56) and (7.57) into (7.67) to show that

$$R = 0. \tag{7.69}$$

The Euler-Lagrange equations are, of course, strongly coupled. It is interesting to examine some of the properties of ${}_M\Pi_P{}^{Rp}$ and ${}_M\Sigma^{pPQ'}$. To begin with, after substituting (7.65) in (7.62) it is easy to show that

$${}_M\Pi_P{}^{Rp}{}_{;p} = 0 \tag{7.70}$$

by the Euler-Lagrange equations (7.56), (7.57) and their complex conjugates. Thus the (electromagnetic) current density defined by ${}_M\Pi_P{}^{Rp}$ is conserved. Also, by writing equation (7.67) in form (7.68) we see that, in general, ${}_M\Sigma^{pq}$ and hence ${}_MT^{pq}$ need not be symmetric. Since the field portion of the Euler-Lagrange equation is symmetric, this constitutes a severe constraint on the matter spinors ψ_A, ϕ^A and their complex conjugates. Finally, with the use of the Euler-Lagrange equations (7.56) and (7.57), their complex conjugates and (7.62) and its complex conjugate we may, independently of (7.65), write

$$ {}_M\Sigma^{pPQ'}{}_{;p} = -\sigma^{aPQ'}(F^A{}_{Cpa}\,{}_M\Pi_A{}^{Cp} + \bar{F}^{B'}{}_{D'pa}\,{}_M\bar{\Pi}_{B'}{}^{D'p}). \tag{7.71} $$

This, with the Euler-Lagrange equation (6.108), illustrates beautifully the vanishing of the divergence of the total energy-momentum tensor, $T^{pq}{}_{;p}$.

While the Lagrangian (7.51) and its Euler-Lagrange equations may be consistently paired with either of the Lagrangians (7.31) or (7.35) its properties correspond more closely to those of the former. L_8 describes "massless" charged particles on the light cone. This is a reflection of the common difficulty associated with the introduction of "massive" particles in field theory; usually solvable only by breaking gauge invariance. However, we will now consider a gauge invariant, scalar density Lagrangian which can describe "massive" particles and for which the matter current density does not, generally, vanish.

We introduce the generalized Klein-Gordon Lagrangian which takes the form

$$ L_9 = \sqrt{-g}[\sigma^{aAB'}(\sigma^b{}_{CB'}\psi_{A;a}\phi^C{}_{;b} + \sigma^b{}_{AD'}\bar{\psi}_{B';a}\bar{\phi}^{D'}{}_{;b}) - m^2(\psi_A\phi^A + \bar{\psi}_{B'}\bar{\phi}^{B'})] \tag{7.72} $$

where, once again, ψ_A, $\bar{\psi}_{B'}$, ϕ^A and $\bar{\phi}^{B'}$ are gauge densities as described in Section 6.3. Then

$$ \begin{aligned} \Psi^{Pq} &= \sqrt{-g}\,\sigma^{qPB'}\sigma^b{}_{CB'}\phi^C{}_{;b} && (7.73)\\ \Pi^P &= -\sqrt{-g}\,m^2\phi^P && (7.74)\\ \Phi_P{}^q &= \sqrt{-g}\,\sigma^{aAB'}\sigma^q{}_{PB'}\psi_{A;a} && (7.75)\\ P_P &= -\sqrt{-g}\,m^2\psi_P && (7.76) \end{aligned} $$

and the Euler-Lagrange equations (6.109) and (6.110) become

$$ \begin{aligned} \sqrt{-g}(\sigma^{aPB'}\sigma^b{}_{CB'}\phi^C{}_{;b;a} + m^2\phi^P) &= 0 && (7.77)\\ \sqrt{-g}(\sigma^{aAB'}\sigma^b{}_{PB'}\psi_{A;a;b} + m^2\psi_P) &= 0 && (7.78) \end{aligned} $$

and similarly for their complex conjugates. Equations (7.77) and (7.78) may be rewritten in the form of generalized Pauli equations

$$\tfrac{1}{2}\sqrt{-g}[g^{ab}\psi_{P;a;b} - \sigma^{ab}{}_P{}^A(\hat{F}^M{}_{Aab}\psi_M + nF^M{}_{Mab}\psi_A) + 2m^2\psi_P] = 0 \quad (7.79)$$

$$\tfrac{1}{2}\sqrt{-g}[g^{ab}\phi^P{}_{;a;b} - \sigma^{ab}{}_C{}^P(\hat{F}^C{}_{Mab}\phi^M + nF^M{}_{Mab}\phi^C) + 2m^2\phi^P] = 0 \quad (7.80)$$

which, for chargeless particles in a vanishing gravitational field, clearly reduce to the Klein-Gordon equations.

Again noting that, here, ${}_M\Pi_P{}^{Ra} = {}_M\tau_P{}^{Ra}$ we have

$$\begin{aligned} {}_M\Pi_P{}^{Rp} = {} & -\sqrt{-g}[\sigma^{pRB'}\sigma^b{}_{CB'}\psi_P\phi^C{}_{;b} - \sigma^{aAB'}\sigma^p{}_{PB'}\psi_{A;a}\phi^R \\ & +(n-\tfrac{1}{2})\delta^R_P(\sigma^{pAB'}\sigma^b{}_{CB'}\psi_A\phi^C{}_{;b} - \sigma^{aAB'}\sigma^p{}_{CB'}\psi_{A;a}\phi^C)]. \end{aligned} \quad (7.81)$$

Introducing the shorthand notation

$$\kappa_A = \sigma^a{}_{AB'}\bar{\phi}^{B'}{}_{;a} \quad (7.82)$$

$$\lambda^A = \sigma^{aAB'}\bar{\psi}_{B';a} \quad (7.83)$$

and similarly for their complex conjugates, we may rewrite equation (7.81) in the form

$$\begin{aligned} {}_M\Pi_P{}^{Rp} = {} & \sqrt{-g}[\sigma^{pq}{}_P{}^R(\sigma_q{}^{AB'}\psi_A\bar{\kappa}_{B'} + \sigma_{qAB'}\phi^A\bar{\lambda}^{B'}) \\ & -n\delta^R_P(\sigma^{pAB'}\psi_A\bar{\kappa}_{B'} - \sigma^p{}_{AB'}\phi^A\bar{\lambda}^{B'})]. \end{aligned} \quad (7.84)$$

As before we may split the Euler-Lagrange equation (6.108) into its gravitational and electromagnetic contributions. Then, by (7.50) and (7.84) we have

$$8i\beta F^{pq}{}_{;q} = n\gamma(\sigma^{pAB'}\psi_A\bar{\kappa}_{B'} - \sigma^p{}_{AB'}\phi^A\bar{\lambda}^{B'}) \quad (7.85)$$

and, by equations (7.48), (7.49) and (7.84)

$$-4\beta R_{;n} = \tfrac{3}{2}\gamma(\sigma_n{}^{AB'}\psi_A\bar{\kappa}_{B'} + \sigma_{nAB'}\phi^A\bar{\lambda}^{B'}). \quad (7.86)$$

As the right hand sides of equations (7.85) and (7.86) must be, respectively, imaginary and real we see that $\sigma^{aAB'}\psi_A\bar{\kappa}_{B'}$ and $\sigma^a{}_{AB'}\phi^A\bar{\lambda}^{B'}$ must be complex conjugates. Hence

$$\sigma^{aAB'}\psi_A\bar{\kappa}_{B'} = \sigma^a{}_{AB'}\lambda^A\bar{\phi}^{B'} \tag{7.87}$$

and consequently, we may write

$$\lambda^A = \mu\varepsilon^{AC}\psi_C \tag{7.88}$$

$$\kappa_A = \bar{\mu}\phi^C\varepsilon_{CA} \tag{7.89}$$

where μ is of gauge weight $-2n$, with similar conjugate equations for $\bar{\kappa}_{B'}$, $\bar{\lambda}^{B'}$. Equations (7.87–7.89) have several immediate consequences.

The conjugates of equations (7.88) and (7.89), with the conjugates of equations (7.82) and (7.83), give the first order equations

$$\sigma^{aAB'}\psi_{A;a} - \bar{\mu}\varepsilon^{B'D'}\bar{\psi}_{D'} = 0 \tag{7.90}$$

$$\sigma^a{}_{AB'}\phi^A{}_{;a} + \mu\varepsilon_{B'D'}\bar{\phi}^{B'} = 0 \tag{7.91}$$

and, with equations (7.82), (7.83), their complex conjugates and the conjugates of the Euler-Lagrange equations (7.77) and (7.78), give the first order equations

$$\sigma^{aAB'}\kappa_{A;a} + m^2\mu^{(-1)}\varepsilon^{B'D'}\bar{\kappa}_{D'} = 0 \tag{7.92}$$

$$\sigma^a{}_{AB'}\lambda^A{}_{;a} - m^2\bar{\mu}^{(-1)}\varepsilon_{B'D'}\bar{\lambda}^{D'} = 0. \tag{7.93}$$

Also, since (7.88) and (7.89) imply that

$$\kappa_A\lambda^A = -|\mu|^2\psi_A\phi^A \tag{7.94}$$

we may take the divergence of equation (7.87) to find

$$(m^2 - |\mu|^2)(\psi_A\phi^A - \bar{\psi}_{B'}\bar{\phi}^{B'}) = 0. \tag{7.95}$$

Finally, using (7.94) we have

$${}_M\Pi_A{}^{Ap}g_{pq}\,{}_M\Pi_C{}^{Cq} = 4n^2(-g)|\mu|^2|\psi_A\phi^A|^2 \tag{7.96}$$

and, since $\mu = 0$ implies that the whole current density ${}_M\Pi_A{}^{Ca}$ vanishes, we deduce that the condition that the electromagnetic current be null is (neglecting the trivial chargeless case) the usual

$$\psi_A \phi^A = 0. \tag{7.97}$$

The last of the functional derivatives of the Lagrangian is

$$\begin{aligned}
{}_M\Sigma^{pPQ'} &= \sqrt{-g}\,\sigma_q{}^{PQ'}\{g^{pq}[\sigma^{aAB'}(\sigma^b{}_{CB'}\psi_{A;a}\phi^C{}_{;b} + \sigma^b{}_{AD'}\bar{\psi}_{B';a}\bar{\phi}^{D'}{}_{;b}) \\
&\quad -m^2(\psi_A\phi^A + \bar{\psi}_{B'}\bar{\phi}^{B'})] \\
&\quad -\sigma^{pAB'}(\sigma^b{}_{CB'}\psi_A{}^{;q}\phi^C{}_{;b} + \sigma^b{}_{AD'}\bar{\psi}_{B'}{}^{;q}\bar{\phi}^{D'}{}_{;b}) \\
&\quad -\sigma^{aAB'}(\sigma^p{}_{CB'}\psi_{A;a}\phi^{C;q} + \sigma^p{}_{AD'}\bar{\psi}_{B';a}\bar{\phi}^{D';q})\} \\
&= -\sqrt{-g}\,\sigma_q{}^{PQ'}[g^{pq}(m^2 + |\mu|^2)(\psi_A\phi^A + \bar{\psi}_{B'}\bar{\phi}^{B'}) \\
&\quad +\sigma^{pAB'}(\psi_A{}^{;q}\bar{\kappa}_{B'} + \kappa_A\bar{\psi}_{B'}{}^{;q}) + \sigma^p{}_{AB'}(\phi^{A;q}\bar{\lambda}^{B'} + \lambda^A\bar{\phi}^{B';q})].
\end{aligned} \tag{7.98}$$

Noting that, by equations (7.78) and (7.90), $\mu = 0$ implies $m = 0$ (the converse does not follow), we see that $\mu = 0$ implies that both ${}_M\Pi_A{}^{Ca}$ and ${}_M\Sigma^{aAB'}$ vanish and, hence, signals the absence of any real matter field.

We again investigate some additional properties of ${}_M\Pi_P{}^{Rp}$ and ${}_M\Sigma^{pPQ'}$. By the Euler-Lagrange equations (7.77) and (7.78) we find, for the divergence of ${}_M\Pi_P{}^{Rp}$,

$${}_M\Pi_P{}^{Rp}{}_{;p} = -\sqrt{-g}(\sigma^{pRB'}\psi_{P;p}\bar{\kappa}_{B'} - \sigma^p{}_{PB'}\phi^R{}_{;p}\bar{\lambda}^{B'}) \tag{7.99}$$

which, in general, will not vanish identically. But if the Euler-Lagrange equation is satisfied, (7.99) must vanish, implying

$$\psi_P{}^{;RB'}\bar{\kappa}_{B'} = \phi^R{}_{;PB'}\bar{\lambda}^{B'} \tag{7.100}$$

and the conservation of the generalized current density. Rewriting (7.98) in the form (7.68) we see that once again ${}_M\Sigma^{pq}$ and hence ${}_MT^{pq}$ need not, in general, be symmetric. However we may write

$$\begin{aligned}
&\sigma^{pAB'}\psi_A{}^{;q}\bar{\kappa}_{B'} + \sigma^p{}_{CB'}\phi^{C;q}\bar{\lambda}^{B'} \\
&= \tfrac{1}{2}(\sigma^{pAB'}\sigma^q{}_{PQ'} + \sigma^{qAB'}\sigma^p{}_{PQ'})\psi_A{}^{;PQ'}\bar{\kappa}_{B'} + \tfrac{1}{2}(\sigma^p{}_{CB'}\sigma^{qPQ'} + \sigma^q{}_{CB'}\sigma^{pPQ'})\phi^C{}_{;PQ'}\bar{\lambda}^{B'} \\
&\quad + \tfrac{1}{2}(\sigma^{pAB'}\sigma^q{}_{PQ'} - \sigma^{pA}{}_{Q'}\sigma^q{}_P{}^{B'} + \sigma^{pA}{}_{Q'}\sigma^q{}_P{}^{B'} - \sigma^p{}_{PQ'}\sigma^{qAB'})\psi_A{}^{;PQ'}\bar{\kappa}_{B'} \\
&\quad + \tfrac{1}{2}(\sigma^p{}_{CB'}\sigma^{qPQ'} - \sigma^{pP}{}_{B'}\sigma^q{}_C{}^{Q'} + \sigma^{pP}{}_{B'}\sigma^q{}_C{}^{Q'} - \sigma^{pPQ'}\sigma^q{}_{CB'})\phi^C{}_{;PQ'}\bar{\lambda}^{B'}. \qquad (7.101)
\end{aligned}$$

Then, given equation (7.99) and recalling that $\phi^C{}_{;CB'}$ and $\psi_A{}^{;AB'}$ are just $\bar{\kappa}_{B'}$ and $\bar{\lambda}^{B'}$ respectively we may easily show, using the relation (6.11) that the sum of the last two lines of (7.101) vanishes. Thus the conservation of the matter-charge current density implies that ${}_M\Sigma^{pq}$ is symmetric. Further, though the analysis is slightly more complex, we may, using the Euler-Lagrange equations (7.77) and (7.78), again show that the divergence of ${}_M\Sigma^{pPQ'}$ takes the form

$$ {}_M\Sigma^{pPQ'}{}_{;p} = -\sigma^{aPQ'}(F^A{}_{Cpa}\,{}_M\Pi_A{}^{Cp} + \bar{F}^{B'}{}_{D'pa}\,{}_M\bar{\Pi}_{B'}{}^{D'p}). \qquad (7.102)$$

The Euler-Lagrange equations derived from L_9 are subtly interconnected and are rich in interesting properties. As the gravitational contribution to ${}_9\Pi_P{}^{Rp}$ does not generally vanish, the properties of this system of equations clearly correspond most closely to the generalized Einstein-Maxwell equations derived from the Lagrangian (7.35). Thus we have arrived at a theory containing a fully generalized matter-charge current density, if only at the cost of "damaging" the Einstein equations.

It is likely that these properties are related. Certainly any Lagrangian generating a "damaged" field stress-energy tensor similar to (7.37) must also imply a current equation with a nonzero generalized matter-current term if the divergence of T^{pq} is to vanish. And while the converse is not so clear, the beautiful relations (7.71) and (7.102) suggest that a fully generalized matter-charge current density is likely to be accompanied by a correspondingly "damaged" stress-energy tensor. Thus it appears that the generalized Maxwell equations are inextricably bound to the "damaged" Einstein equations (7.40) and, in the absence of any real need to replace the Einstein equations, must be abandoned. Unless the introduction of some entirely new set of fundamental quantities permits the Einstein tensor to be represented in a form quadratic in some new field quantity, a generalized matter-charge current density seems incompatible with general relativity.

Chapter 8

Conclusion

In this dissertation we have presented new work on variational principles and gauge theories in general relativity. The general theme underlying the work has been a search for a "good" description of general-relativistic energy-momentum. While we cannot be said to have achieved this difficult objective, significant progress has been made in several areas.

In the first part of the dissertation we developed new variational techniques, acting on classes of Lagrangians with the same functional dependence but arbitrary functional form, for the derivation of general, strongly conserved quantities. Einstein's theory of general relativity provided an excellent example for the demonstration of these techniques both because of its difficulty and because of the opportunity to clarify the somewhat confused state of conserved quantities in the theory.

A potent tool in the development of this theory was the use of a seemingly little known resource: a set of invariance relations derived from the transformation laws of the Lagrangian and its arguments. Since the functional derivatives of the Lagrangian, in terms of which the invariance relations were written, were not generally the covariant quantities we wished to work with, we followed duPlessis and introduced their tensor concomitants with which we rewrote the invariance relations in tensor form. The intimate connection between the invariance relations and the variational principle became manifest in the manner in which the invariance relations guided and simplified the "integration" of the general conserved quantity.

Consideration of coordinate group invariances whose "motions" represent spacetime symmetries enabled us to generate both mixed and contravariant energy-momentum complexes. Further consideration of the angular-momentum complex enabled us to generalize the electromagnetic symmetrization process resulting in the symmetric energy-momentum complex H^{at}. By selecting particular Lagrangians, derived from Lovelock's general Einstein-Maxwell Lagrangian, we were then able to generate and generalize virtually all energy-momentum complexes currently known. In particular we presented, for the first time to the best of the author's knowledge, a derivation of the Landau and Lifshitz energy-momentum complex via a variational principle.

In the process of investigating our group invariances we discovered the reason for the difficulties associated with energy-momentum complexes in general relativity: the infinitesimal generators of our "motions" were, in fact, just the Killing vectors of Minkowski space and did not generally represent true symmetries of the spacetime. Thus the well known limitation: only by integrating near infinity on an asymptotically flat spacetime may one be sure of obtaining valid results. These complexes are not true momentum densities and, in general, will not exhibit the corresponding local properties.

These considerations pointed to the possiblity of formulating general relativity in terms of new quantities which possess group invariances on the tangent space. In order to investigate the manner in which an alternate set of fundamental quantities affect the theory we studied a Palatini variation of Nissani's Lagrangian which is based on the Lagrangian of Carmeli's $SL(2, C)$ gauge theory of gravitation. We found that Nissani's principal claim, that his Lagrangian specified Riemannian geometry in the presence of a general matter tensor, was not in fact justifiable. In addition we presented a new Lagrangian with properties analogous to Nissani's, proving that Nissani's Lagranian is not unique.

In studying Nissani's Lagrangian we saw that a change in fundamental quantities may indeed yield new results, one of which was the derivation of the Bianchi identities as an Euler-Lagrange equation. Since the Bianchi identities are the general-relativistic analog of the second of Maxwell's equations (1.2), we were led to speculate on the possibility of a general-relativistic analog to Maxwell's current equation, (1.1). Such an analog would constitute a matter current equation, yielding an entirely new approach to the idea of energy-momentum in general relativity.

In pursuit of this objective we developed the $SL(2,C)\times U(1)$ spinor formalism in which the gravitational and electromagnetic potentials naturally combine in a single object—the spinor connection. In consequence, variably charged matter could be cleanly and rigourously introduced through the use of spin densities and we were able to develop unified potential theories.

In order to demonstrate that this formalism did in fact describe both gravitation and electromagnetism, we presented three different Lagrangians which generated the Einstein-Maxwell equations. However, notably, all three Lagrangians suffered from the flaw that they could be considered somewhat artificial as, in all three, terms from the natural "field" P_{abcd} (ie. the curvature spinor $F^A{}_{Cab}$) were selectively extracted in order to achieve the desired results.

When we developed the theory in terms of the full field P_{abcd} ($F^A{}_{Cab}$) new results were obtained. The Euler-Lagrange equation corresponding to the spinor connection was indeed found to have generalized the Maxwell equation, taking a form which, in the presence of matter would constitute an equation in a matter-charge current density. However, the Euler-Lagrange equation corresponding to the variation of the (cross) section no longer gave the Einstein equation but also included a new term, the "gravitational stress-energy tensor". This new tensor, proportional to a contraction of the Bel-Robinson tensor, took an interesting and intriguing form, exactly mimicking the electromagnetic stress-energy tensor with Riemann tensor contractions replacing Maxwell tensor contractions. While there are at present no good reasons to replace the Einstein equation as the relation which best describes gravitation, should such reasons arise the equations we have derived herein may provide a viable alternative.

In order to attain a better understanding of the significance of these complex new equations we briefly considered the introduction of matter (described via spinors) to the theory. For a Lagrangian generating the generalized two spinor Dirac equations, we found that there could be no gravitational currents and that the electromagnetic currents must be on the light cone. For a Lagrangian generating the generalized Pauli equations, we found that both gravitational and electromagnetic currents were allowed. Both sets of equations, but especially the latter, demonstrate subtle connections between the (generalized) Einstein and Maxwell equations. In particular the unique results given by equations (7.71) and (7.102) beautifully illustrate the vanishing of the divergence of the total stress-energy tensor in this formalism. In the theory embodied by the generalized Einstein-Maxwell and Pauli equations we finally accomplished our goal of deriving an equation describing a generalized matter-charge current density.

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Appendix A

Conversions

In this appendix we convert the invariance relation (2.25) and the chain rule on the Lagrangian (2.51) to their concomitant forms (2.50) and (2.52).

Beginning with (2.25) we first note that

$$\begin{aligned}
&\Phi^q\phi_p + \Phi^{qb}\phi_{p,b} + \Phi^{aq}\phi_{a,p} \\
&= \Pi^q\phi_p + \Phi^{qb}\phi_{p;b} + \Phi^{aq}\phi_{a;p} - \Phi^{nd}\phi_p\Gamma^q{}_{nd} + (\Phi^{qr} + \Phi^{rq})\phi_n\Gamma^n{}_{rp}. \qquad \text{(A.1)}
\end{aligned}$$

The conversion of

$$a^q{}_p = \Lambda^{qb}g_{pb} + \Lambda^{aq}g_{ap} + \Lambda^{qbc}g_{pb,c} + \Lambda^{aqc}g_{ap,c} + \Lambda^{abq}g_{ab,p} \qquad \text{(A.2)}$$

will now complete the calculation.

Writing Λ^{ab} and Λ^{abc} in terms of their concomitants, (2.34) and (2.35), and converting partial to covariant derivatives in the Π^{abc} terms, we have

$$\begin{aligned}
a^q{}_p \;=\;& \Pi^{qb}g_{pb} + \Pi^{aq}g_{ap} + \Pi^{qbc}g_{pb;c} + \Pi^{aqc}g_{ap;c} + \Pi^{abq}g_{ab;p} \\
&+[(\Pi^{qbr} + \Pi^{rbq})g_{nb} + (\Pi^{aqr} + \Pi^{arq})g_{an}]\Gamma^n{}_{rp} \\
&-[\Pi^{nbd}g_{pb} + \Pi^{and}g_{ap} + (\Lambda^{nbcd} + \Lambda^{nbdc})g_{pb,c} + (\Lambda^{ancd} + \Lambda^{andc})g_{ap,c} \\
&\quad +\Lambda^{abnd}g_{ab,p} - (\Lambda^{nbcd} + \Lambda^{nbdc})\Gamma^m{}_{bc}g_{pm} - (\Lambda^{ancd} + \Lambda^{andc})\Gamma^m{}_{ac}g_{mp}]\Gamma^q{}_{nd} \\
&-\Lambda^{abcd}[(\Gamma^q{}_{ac,d} - \Gamma^q{}_{nc}\Gamma^n{}_{ad} - \Gamma^q{}_{an}\Gamma^n{}_{cd})g_{pb} \\
&\quad +(\Gamma^q{}_{bc,d} - \Gamma^q{}_{nc}\Gamma^n{}_{bd} - \Gamma^q{}_{bn}\Gamma^n{}_{cd})g_{ap}]
\end{aligned}$$

$$
\begin{aligned}
&-\Lambda^{qbcd}[(\Gamma^n{}_{bc,d} - \Gamma^n{}_{mc}\Gamma^m{}_{bd} - \Gamma^n{}_{bm}\Gamma^m{}_{cd})g_{pn} \\
&\quad +\Gamma^n{}_{bd}g_{pn,c} + \Gamma^n{}_{cd}g_{pb,n} + \Gamma^n{}_{bc}g_{pn,d}] \\
&-\Lambda^{aqcd}[(\Gamma^n{}_{ac,d} - \Gamma^n{}_{mc}\Gamma^m{}_{ad} - \Gamma^n{}_{am}\Gamma^m{}_{cd})g_{np} \\
&\quad +\Gamma^n{}_{ad}g_{np,c} + \Gamma^n{}_{cd}g_{ap,n} + \Gamma^n{}_{ac}g_{np,d}] \\
&-\Lambda^{abqd}(\Gamma^n{}_{ad}g_{nb,p} + \Gamma^n{}_{bd}g_{an,p}) - \Lambda^{abcq}(\Gamma^n{}_{ac}g_{nb,p} + \Gamma^n{}_{bc}g_{an,p}).
\end{aligned}
\tag{A.3}
$$

We now convert partial to covariant derivatives in the $\Gamma^q{}_{nd}$ term (lines 3–4) to obtain

$$
\begin{aligned}
a^q{}_p \quad = \quad & \Pi^{qb}g_{pb} + \Pi^{aq}g_{ap} + \Pi^{qbc}g_{pb;c} + \Pi^{aqc}g_{ap;c} + \Pi^{abq}g_{ab;p} \\
&+[(\Pi^{qbr} + \Pi^{rbq})g_{nb} + (\Pi^{aqr} + \Pi^{arq})g_{an}]\Gamma^n{}_{rp} \\
&-[\Pi^{nbd}g_{pb} + \Pi^{and}g_{ap} + (\Lambda^{nbcd} + \Lambda^{nbdc})g_{pb;c} \\
&\quad +(\Lambda^{ancd} + \Lambda^{andc})g_{ap;c} + \Lambda^{abnd}g_{ab;p}]\Gamma^q{}_{nd} \\
&-\Lambda^{abcd}[(\Gamma^q{}_{ac,d} - \Gamma^q{}_{nc}\Gamma^n{}_{ad} - \Gamma^q{}_{an}\Gamma^n{}_{cd})g_{pb} \\
&\quad +\Gamma^q{}_{ad}\Gamma^n{}_{pc}g_{nb} + \Gamma^q{}_{ac}\Gamma^n{}_{pd}g_{nb} + \Gamma^q{}_{cd}\Gamma^n{}_{ap}g_{nb} \\
&\quad +(\Gamma^q{}_{bc,d} - \Gamma^q{}_{nc}\Gamma^n{}_{bd} - \Gamma^q{}_{bn}\Gamma^n{}_{cd})g_{ap} \\
&\quad +\Gamma^q{}_{bd}\Gamma^n{}_{pc}g_{an} + \Gamma^q{}_{bc}\Gamma^n{}_{pd}g \quad + \Gamma^q{}_{cd}\Gamma^n{}_{bp}g_{an}] \\
&-\Lambda^{qbcd}[(\Gamma^n{}_{bc,d} - \Gamma^n{}_{mc}\Gamma^m{}_{bd} - \Gamma^n{}_{bm}\Gamma^m{}_{cd})g_{pn} \\
&\quad +\Gamma^n{}_{bd}g_{pn,c} + \Gamma^n{}_{cd}g_{pb,n} + \Gamma^n{}_{bc}g_{pn,d}] \\
&-\Lambda^{aqcd}[(\Gamma^n{}_{ac,d} - \Gamma^n{}_{mc}\Gamma^m{}_{ad} - \Gamma^n{}_{am}\Gamma^m{}_{cd})g_{np} \\
&\quad +\Gamma^n{}_{ad}g_{np,c} + \Gamma^n{}_{cd}g_{ap,n} + \Gamma^n{}_{ac}g_{np,d}] \\
&-\Lambda^{abqd}(\Gamma^n{}_{ad}g_{nb,p} + \Gamma^n{}_{bd}g_{an,p}) - \Lambda^{abcq}(\Gamma^n{}_{ac}g_{nb,p} + \Gamma^n{}_{bc}g_{an,p}).
\end{aligned}
\tag{A.4}
$$

After using the invariance relation (2.22) on the Λ^{abcd} term (lines 5–8) it reduces to

$$
-\Lambda^{abcd}[(\Gamma^q{}_{ac,d} + \Gamma^q{}_{nd}\Gamma^n{}_{ac})g_{pb} + (\Gamma^q{}_{bc,d} + \Gamma^q{}_{nd}\Gamma^n{}_{bc})g_{ap}]
\tag{A.5}
$$

but, by the same relation and $\Lambda^{abcd} = \Lambda^{abdc}$, we may write

$$\begin{aligned}
\Lambda^{abcd} &= \tfrac{1}{3}(\Lambda^{abcd} - \Lambda^{cbad}) + \tfrac{1}{3}(\Lambda^{abdc} - \Lambda^{dbac}) \\
&= \tfrac{1}{3}(\Lambda^{abcd} - \Lambda^{acbd}) + \tfrac{1}{3}(\Lambda^{abcd} - \Lambda^{adcb}).
\end{aligned} \tag{A.6}$$

Thus, the Λ^{abcd} term is just

$$-\tfrac{1}{3}\Lambda^{abcd}(R^q{}_{cda}g_{pb} + R^q{}_{cdb}g_{ap}) = -\tfrac{1}{2}(\psi^{abdc}R^q{}_{cda}g_{pb} + \psi^{abcd}R^q{}_{cdb}g_{ap}). \tag{A.7}$$

In order to simplify the Λ^{qbcd} term [lines 9–10 of equation (A.4)] we first substitute

$$\Gamma^n{}_{bc}g_{pn} = g_{pb,c} - g_{pb;c} - \Gamma^n{}_{pc}g_{nb} \tag{A.8}$$

into $(\Gamma^n{}_{bc}g_{pn})_{,d}$, then convert all partial to covariant derivatives—excepting only $g_{pb,cd}$—after which this term reduces to

$$\begin{aligned}
&(\Lambda^{qbcr} + \Lambda^{qbrc})g_{nb;c}\Gamma^n{}_{rp} \\
&\quad + \Lambda^{qbcd}[g_{pb;c;d} - g_{pb,c,d} + (\Gamma^n{}_{pc;d} + \Gamma^m{}_{pd}\Gamma^n{}_{mc})g_{nb}] \\
= \ & (\Lambda^{qbcr} + \Lambda^{qbrc})g_{nb;c}\Gamma^n{}_{rp} + \Lambda^{qbcd}(g_{pb;c;d} - g_{pb,cd}) \\
&\quad -\Lambda^{abqd}(\Gamma^n{}_{pd;a} + \Gamma^m{}_{pa}\Gamma^n{}_{md})g_{nb} - \Lambda^{abcq}(\Gamma^n{}_{pa;c} + \Gamma^m{}_{pc}\Gamma^n{}_{ma})g_{nb}.
\end{aligned} \tag{A.9}$$

Similarly, the Λ^{aqcd} term [lines 11–12 of equation (A.4)] reduces to

$$\begin{aligned}
&(\Lambda^{aqcr} + \Lambda^{aqrc})g_{an;c}\Gamma^n{}_{rp} + \Lambda^{aqcd}(g_{ap;c;d} - g_{ap,cd}) \\
&\quad -\Lambda^{abqd}(\Gamma^n{}_{pd;b} + \Gamma^m{}_{pb}\Gamma^n{}_{md})g_{an} - \Lambda^{abcq}(\Gamma^n{}_{pb;c} + \Gamma^m{}_{pc}\Gamma^n{}_{mb})g_{an}.
\end{aligned} \tag{A.10}$$

Thus, writing

$$b^q{}_p = a^q{}_p + \Lambda^{qbcd}g_{pb,cd} + \Lambda^{aqcd}g_{ap,cd} + \Lambda^{abqd}g_{ab,pd} + \Lambda^{abcq}g_{ab,cp} \tag{A.11}$$

we have

$$b^q{}_p = \Pi^{qb}g_{pb} + \Pi^{aq}g_{ap} + \Pi^{qbc}g_{pb;c} + \Pi^{aqc}g_{ap;c}$$

$$
\begin{aligned}
&+\Pi^{abq}g_{ab;p} + \Lambda^{qbcd}g_{pb;c;d} + \Lambda^{aqcd}g_{ap;c;d} \\
&+[(\Pi^{qbr} + \Pi^{rbq})g_{nb} + (\Pi^{aqr} + \Pi^{arq})g_{an} \\
&\quad +(\Lambda^{qbcr} + \Lambda^{qbrc})g_{nb;c} + (\Lambda^{aqcr} + \Lambda^{aqrc})g_{an;c}]\Gamma^{n}{}_{rp} \\
&-[\Pi^{nbd}g_{pb} + \Pi^{and}g_{ap} + (\Lambda^{nbcd} + \Lambda^{nbdc})g_{pb;c} \\
&\quad +(\Lambda^{ancd} + \Lambda^{andc})g_{ap;c} + \Lambda^{abnd}g_{ab;p}]\Gamma^{q}{}_{nd} \\
&-\tfrac{1}{2}(\psi^{abdc}g_{pb}R^{q}{}_{cda} + \psi^{abcd}g_{ap}R^{q}{}_{cdb}) \\
&+\Lambda^{abqd}[g_{ab,pd} - \Gamma^{n}{}_{ad}g_{nb,p} - \Gamma^{n}{}_{bd}g_{an,p} \\
&\quad -(\Gamma^{n}{}_{pd;a} + \Gamma^{m}{}_{pa}\Gamma^{n}{}_{md})g_{nb} - (\Gamma^{n}{}_{pd;b} + \Gamma^{m}{}_{pb}\Gamma^{n}{}_{md})g_{an}] \\
&+\Lambda^{abcq}[g_{ab,cp} - \Gamma^{n}{}_{ac}g_{nb,p} - \Gamma^{n}{}_{bc}g_{an,p} \\
&\quad -(\Gamma^{n}{}_{pa;c} + \Gamma^{m}{}_{pc}\Gamma^{n}{}_{ma})g_{nb} - (\Gamma^{n}{}_{pb;c} + \Gamma^{m}{}_{pc}\Gamma^{n}{}_{mb})g_{an}]. \qquad \text{(A.12)}
\end{aligned}
$$

To simplify the Λ^{abqd} terms (lines 8–9) we convert all partial to covariant derivatives, when this term reduces to

$$
\begin{aligned}
&(\Lambda^{rbqc}g_{nb;c} + \Lambda^{arqc}g_{an;c} + \Lambda^{abqr}g_{ab;n})\Gamma^{n}{}_{rp} + \Lambda^{abqd}g_{ab;p;d} \\
&\quad +\Lambda^{abqd}(g_{nb}R^{n}{}_{pda} + g_{an}R^{n}{}_{pdb}) \\
&= \Lambda^{rbqc}g_{nb;c} + \Lambda^{arqc}g_{an;c} + \Lambda^{abqr}g_{ab;n})\Gamma^{n}{}_{rp} + \Lambda^{abqd}g_{ab;p;d} \\
&\quad +\tfrac{3}{2}(\psi^{abcq}g_{nb}R^{n}{}_{pca} + \psi^{abqd}g_{an}R^{n}{}_{pdb}). \qquad \text{(A.13)}
\end{aligned}
$$

Similarly, the Λ^{abcq} term [lines 10–11 of equation (A.12)] reduces to

$$
(\Lambda^{rbcq}g_{nb;c} + \Lambda^{arcq}g_{an;c} + \Lambda^{abrq}g_{ab;n})\Gamma^{n}{}_{rp} + \Lambda^{abcq}g_{ab;p;c}. \qquad \text{(A.14)}
$$

Now

$$
\begin{aligned}
\delta^{q}_{p}L &= \Pi^{qb}g_{pb} + \Pi^{aq}g_{ap} + \Pi^{qbc}g_{pb;c} + \Pi^{aqc}g_{ap;c} + \Pi^{abq}g_{ab;p} \\
&\quad +\Lambda^{qbcd}g_{pb;c;d} + \Lambda^{aqcd}g_{ap;c;d} + \Lambda^{abqd}g_{ab;p;d} + \Lambda^{abcq}g_{ab;p;c} \\
&\quad +\Pi^{q}\phi_{p} + \Phi^{qb}\phi_{p;b} + \Phi^{aq}\phi_{a;p}
\end{aligned}
$$

$$
\begin{aligned}
&+[(\Pi^{qbr}+\Pi^{rbq})g_{nb}+(\Pi^{aqr}+\Pi^{arq})g_{an}+(\Lambda^{abqr}+\Lambda^{abrq})g_{ab;n}\\
&\quad+(\Lambda^{qbcr}+\Lambda^{rbcq}+\Lambda^{qbrc}+\Lambda^{rbqc})g_{nb;c}\\
&\quad+(\Lambda^{aqcr}+\Lambda^{arcq}+\Lambda^{aqrc}+\Lambda^{arqc})g_{an;c}+(\Phi^{qr}+\Phi^{rq})\phi_n]\Gamma^n{}_{rp}\\
&-[\Pi^{nbd}g_{pb}+\Pi^{and}g_{ap}+(\Lambda^{nbcd}+\Lambda^{nbdc})g_{pb;c}\\
&\quad+(\Lambda^{ancd}+\Lambda^{andc})g_{ap;c}+\Lambda^{abnd}g_{ab;p}+\Phi^{nd}\phi_p]\Gamma^q{}_{nd}\\
&-\tfrac{1}{2}(\psi^{abdc}g_{pb}R^q{}_{cda}+\psi^{abcd}g_{ap}R^q{}_{cdb})\\
&+\tfrac{3}{2}(\psi^{abcq}g_{nb}R^n{}_{pca}+\psi^{abqd}g_{an}R^n{}_{pdb}).
\end{aligned}
\tag{A.15}
$$

But the terms in $\Gamma^n{}_{rp}$ and $\Gamma^q{}_{nd}$ are just $(A^{qr}{}_n+A^{rq}{}_n)\Gamma^n{}_{rp}$ and $-A^{nd}{}_p\Gamma^q{}_{nd}$, both of which vanish leaving us with equation (2.50).

We now wish to convert equation (2.51). We first note that

$$\Phi^a\phi_{a;p}+\Phi^{ab}\phi_{a,b;p}=\Pi^a\phi_{a;p}+\Phi^{ab}\phi_{a;p;b}+\Phi^{qr}\phi_n\gamma^n{}_{pqr} \tag{A.16}$$

where, for the sake of convenience, we have introduced the notation

$$\gamma^n{}_{pqr}=\Gamma^n{}_{pq,r}-\Gamma^n{}_{mq}\Gamma^m{}_{pr}. \tag{A.17}$$

We now write $g_{ab,c;p}$ in the form

$$g_{ab,c;p}=g_{ab;p;c}+\Gamma^n{}_{ac}g_{nb;p}+\Gamma^n{}_{bc}g_{an;p}+\gamma^n{}_{pac}g_{nb}+\gamma^n{}_{pbc}g_{an} \tag{A.18}$$

so that

$$
\begin{aligned}
\Lambda^{abc}g_{ab,c;p} &= \Lambda^{abc}g_{ab;p;c}+(\Lambda^{nbc}\Gamma^a{}_{nc}+\Lambda^{anc}\Gamma^b{}_{nc})g_{ab;p}\\
&\quad+(\Lambda^{qbr}g_{nb}+\Lambda^{aqr}g_{an})\gamma^n{}_{pqr}.
\end{aligned}
\tag{A.19}
$$

It is also fairly easy, if somewhat tedious, to write $g_{ab,cd;p}$ in the form

$$
\begin{aligned}
g_{ab,cd;p} &= g_{ab;p;c;d}+\Gamma^n{}_{ad}g_{nb;p;c}+\Gamma^n{}_{ac}g_{nb;p;d}\\
&\quad+\Gamma^n{}_{bd}g_{an;p;c}+\Gamma^n{}_{bc}g_{an;p;d}+\Gamma^n{}_{cd}g_{ab;p;n}
\end{aligned}
$$

$$
\begin{aligned}
&+\Gamma^n{}_{ac,d}g_{nb;p} + \Gamma^m{}_{ac}\Gamma^n{}_{md}g_{nb;p} + \Gamma^m{}_{ac}\Gamma^n{}_{bd}g_{mn;p} \\
&+\Gamma^n{}_{bc,d}g_{an;p} + \Gamma^m{}_{ad}\Gamma^n{}_{bc}g_{mn;p} + \Gamma^m{}_{bc}\Gamma^n{}_{md}g_{an;p} \\
&+\gamma^n{}_{pad}g_{nb;c} + \gamma^n{}_{pac}g_{nb;d} + \gamma^n{}_{pbd}g_{an;c} + \gamma^n{}_{pbc}g_{an;d} + \gamma^n{}_{pcd}g_{ab;n} \\
&+\gamma^n{}_{pmc}\Gamma^m{}_{ad}g_{nb} + \gamma^n{}_{pmd}\Gamma^m{}_{ac}g_{nb} + \gamma^n{}_{pac}\Gamma^m{}_{bd}g_{nm} \\
&+\gamma^n{}_{pad}\Gamma^m{}_{bc}g_{nm} + \gamma^n{}_{pam}\Gamma^m{}_{cd}g_{nb} + \gamma^n{}_{pbc}\Gamma^m{}_{ad}g_{mn} \\
&+\gamma^n{}_{pbd}\Gamma^m{}_{ac}g_{nm} + \gamma^n{}_{pmc}\Gamma^m{}_{bd}g_{an} + \gamma^n{}_{pmd}\Gamma^m{}_{bc}g_{an} + \gamma^n{}_{pbm}\Gamma^m{}_{cd}g_{an} \\
&+B^n{}_{pacd}g_{nb} + B^n{}_{pbcd}g_{an} \qquad (A.20)
\end{aligned}
$$

where

$$
\begin{aligned}
B^n{}_{pacd} &\equiv \gamma^n{}_{pac;d} - \gamma^n{}_{pmd}\Gamma^m{}_{ac} + \gamma^m{}_{pcd}\Gamma^n{}_{ma} + \gamma^m{}_{pad}\Gamma^n{}_{mc} \\
&= \Gamma^n{}_{pa,cd} - \Gamma^n{}_{ma,d}\Gamma^m{}_{pc} - \Gamma^n{}_{ma,c}\Gamma^m{}_{pd} - \gamma^n{}_{pam}\Gamma^m{}_{cd} \\
&\quad -\gamma^n{}_{pmc}\Gamma^m{}_{ad} - \gamma^n{}_{pmd}\Gamma^m{}_{ac} + \gamma^m{}_{pac}\Gamma^n{}_{md} + \gamma^m{}_{pad}\Gamma^n{}_{mc} \qquad (A.21)
\end{aligned}
$$

is clearly symmetric in c and d . Thus, using (2.28) we have

$$
\begin{aligned}
\Lambda^{abcd}B^n{}_{pacd} &= 2\psi^{abcd}B^n{}_{pacd} \qquad (A.22) \\
\Lambda^{abcd}B^n{}_{pbcd} &= 2\psi^{abdc}B^n{}_{pbcd} \qquad (A.23)
\end{aligned}
$$

and, hence

$$
\begin{aligned}
\Lambda^{abcd}g_{ab,cd;p} = {} & \Lambda^{abcd}g_{ab;p;c;d} \\
&+[(\Lambda^{nbcd} + \Lambda^{nbdc})\Gamma^a{}_{nd} + (\Lambda^{ancd} + \Lambda^{andc})\Gamma^b{}_{nd} + \Lambda^{abnd}\Gamma^c{}_{nd}]g_{ab;p;c} \\
&+(\Lambda^{nbcd}\Gamma^a{}_{nc,d} + \Lambda^{nbcd}\Gamma^m{}_{nc}\Gamma^a{}_{md} + \Lambda^{mncd}\Gamma^a{}_{mc}\Gamma^b{}_{nd} \\
&\quad +\Lambda^{ancd}\Gamma^b{}_{nc,d} + \Lambda^{mncd}\Gamma^a{}_{md}\Gamma^b{}_{nc} + \Lambda^{ancd}\Gamma^m{}_{nc}\Gamma^b{}_{md})g_{ab;p} \\
&+\{(\Lambda^{qbcr} + \Lambda^{qbrc})g_{nb;c} + (\Lambda^{aqcr} + \Lambda^{aqrc})g_{an;c} + \Lambda^{abqr}g_{ab;n} \\
&\quad +[(\Lambda^{mbrd} + \Lambda^{mbdr})\Gamma^q{}_{md} + (\Lambda^{qmrd} + \Lambda^{qmdr})\Gamma^b{}_{md} \\
&\quad +\Lambda^{qbmd}\Gamma^r{}_{md}]g_{nb} + [(\Lambda^{mqrd} + \Lambda^{mqdr})\Gamma^a{}_{md}
\end{aligned}
$$

$$
\begin{aligned}
&\quad +(\Lambda^{amrd} + \Lambda^{amdr})\Gamma^{q}{}_{md} + \Lambda^{aqmd}\Gamma^{r}{}_{md}]g_{an}\}\gamma^{n}{}_{pqr} \\
&+2\psi^{abcd}g_{nb}\gamma^{n}{}_{pac;d} + 2\psi^{abdc}g_{an}\gamma^{n}{}_{pbc;d} \\
= \;& \Lambda^{abcd}g_{ab;p;c;d} + (\Pi^{abc} - \Lambda^{abc})g_{ab;p;c} \\
&+(\Pi^{ab} - \Lambda^{ab} - \Lambda^{nbc}\Gamma^{a}{}_{nc} - \Lambda^{anc}\Gamma^{b}{}_{nc})g_{ab;p} \\
&+[(\Pi^{qbr} - \Lambda^{qbr})g_{nb} + (\Pi^{aqr} - \Lambda^{aqr})g_{an} \\
&\quad +(\Lambda^{qbcr} + \Lambda^{qbrc})g_{nb;c} + (\Lambda^{aqcr} + \Lambda^{aqrc})g_{an;c} + \Lambda^{abqr}g_{ab;n}]\gamma^{n}{}_{pqr} \\
&+\psi^{abcd}g_{nb}R^{n}{}_{pca;d} + \psi^{abdc}g_{an}R^{n}{}_{pcb;d}. \qquad (A.24)
\end{aligned}
$$

But the term in $\gamma^{n}{}_{pqr}$ is just $A^{qr}{}_{n}\gamma^{n}{}_{pqr}$ or, noting (2.49) and (A.17), just $\frac{1}{2}A^{qr}{}_{n}R^{n}{}_{prq}$. Therefore we may finally write

$$
\begin{aligned}
L_{;p} = \;& \Pi^{ab}g_{ab;p} + \Pi^{abc}g_{ab;p;c} + \Lambda^{abcd}g_{ab;p;c;d} + \Pi^{a}\phi_{a;p} + \Phi^{ab}\phi_{a;p;b} \\
&+\tfrac{1}{2}A^{qr}{}_{n}R^{n}{}_{prq} + \psi^{abcd}g_{nb}R^{n}{}_{pca;d} + \psi^{abdc}g_{an}R^{n}{}_{pcb;d} \qquad (A.25)
\end{aligned}
$$

which is equation (2.52).