CONVEXITY OF MINIMAL TOTAL DOMINATING FUNCTIONS OF GRAPHS

by

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ABSTRACT

A total dominating function (TDF) of a graph is a function from its vertex set to the unit interval such that the sum of function values, taken over the open neighbourhood of each vertex, is at least one. The thesis studies some basic properties of minimal total dominating functions (MTDFs) of graphs and in particular the question of when convex combinations of MTDFs are themselves MTDFs, especially on trees. We give a necessary and sufficient condition for graphs to have a unique MTDF and various conditions for general graphs and trees to have a universal MTDF. We characterize universal MTDFs of a certain class of trees. Several classes of trees with a universal MTDF or without a universal MTDF are given.

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To my parents

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Chapter 1

Introduction

This thesis is concerned with convexity properties of total dominating functions of graphs. Such functions generalize the notion of total dominating sets of a graph. The definition of these concepts are given in Section 1.1, motivation for the work is the subject of Section 1.2 and the contents of the remainder of the thesis are summarized in Section 1.3.

1.1 Notation and Definitions

In this section, we list definitions and notations used in this thesis. For the basic graph-theoretic terminology, the reader is referred to [4].

A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite set of *vertices* and $E(G)$ is a set of unordered pairs, called *edges*, of distinct vertices. If $\{u, v\} \in E(G)$, we denote $\{u, v\}$ by uv.

A graph G is called *a complete graph* if $uv \in E(G)$ for every pair of distinct vertices $u, v \in V(G)$. The complete graph with n vertices is denoted by K_n.

A graph G is a *complete bipartite graph* if $V(G)$ can be partitioned into nonempty subsets X and Y such that $E(G) = \{uv | u \in X, v \in Y\}$. If $|X| = n$ and $|Y| = m$, we denote G by $K_{n,m}$.

The *path* P_n is the graph with vertex set $V(P_n) = \{v_1, \ldots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$. The integer $n-1$ is called the *length* of the path.

The *cycle* C_n has vertex set $\{v_1, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} | 1 \leq i \leq n\}$ (where addition is modulo n). The integer n is the *length* of C_n.

A graph G with $n+1$ vertices is called an $(n+1)$ *-vertex wheel*, and denoted by W_n, if $V(G) = V(C_n) \bigcup \{v_0\}$ and $E(G) = E(C_n) \bigcup \{v_0 v_i | i = 1, \ldots, n\}$. The vertex v_0 is called the *center* of W_n.

A *tree* is a connected acyclic graph.

A graph G is *k-regular* if every vertex has degree k.

A graph G is *vertex-transitive* if for any two vertices $u, v \in V(G)$, there is an automorphism f such that $f(u) = v$. We note that a *vertex-transitive* graph is also a *regular* graph.

Let $W \subseteq V(G)$. The *subgraph of G induced by W* is the subgraph of G whose vertex set is W and whose edge set is the set of all edges in $E(G)$ which have both incident vertices in W.

A *rooted graph* is an ordered pair (G, v) where G is a graph and v is a vertex of G called the *root*. Let T be a tree rooted at v and $w \in V(T)$, T_w will denote the *subtree* of T induced by w and all descendants of w.

The *open neighbourhood* of a vertex v in a graph $G = (V, E)$, denoted by $N(v)$, is the set $\{u \in V \mid uv \in E\}$.

The *closed neighbourhood* of a vertex v, denoted by $N[v]$, is the set $N(v) \bigcup \{v\}$.

The *distance* between two vertices u, v, denoted by $d(u, v)$, is the *smallest* length of those paths connecting u and v.

The *diameter* of a graph G, denoted by $diam(G)$, is $\max_{u,v \in V(G)} d(u, v)$.

The following are some concepts related to domination in graphs.

A *total dominating set*(TDS) is a subset X of the vertex set V such that any $v \in V$ is adjacent to at least one $x \in X$.

A *minimal total dominating set*(MTDS) is a TDS X such that any proper subset of X is not a TDS.

A *dominating set* is a subset X of the vertex set V such that any $v \in V$ is either in X or adjacent to at least one $x \in X$.

For subsets A, B of vertex set V, we say A *totally dominates* B and write $A \rightarrow B$ if $N(v) \bigcap A \neq \emptyset$ for each $v \in B$. We say A *dominates* B and write $A \succ B$ if $N[v] \bigcap A \neq \emptyset$ for each $v \in B$.

A *total dominating function*(TDF) of $G = (V, E)$ is a function $f : V \rightarrow$

[0,1] such that

$$\sum_{u \in N(v)} f(u) \geq 1$$

for each $v \in V$.

A *dominating function*(DF) of $G = (V, E)$ is a function $f : V \rightarrow [0,1]$ such that

$$\sum_{u \in N[v]} f(u) \geq 1$$

for each $v \in V$.

We use the same notation $f[v]$ for $\sum_{u \in N[v]} f(u)$ if f is a DF and for $\sum_{u \in N(v)} f(u)$ if f is a TDF. When the notation $f[v]$ is used , the ambiguity will be clarified by the context.

Let f be a TDF or a DF. The *boundary* of f is denoted by $B_f = \{v \in V \mid f[v] = 1\}$. The *positive set* of f is denoted by $P_f = \{v \in V \mid f(v) > 0\}$.

The *aggregate* of a TDF (DF) f is the quantity

$$\sum_{u \in V} f(u).$$

For any two functions f, $g : V \rightarrow [0,1]$ of $G = (V, E)$, we write $f \leq g$ if for all $v \in V$, $f(v) \leq g(v)$. Further, we write $f < g$ if $f \leq g$ and for some $v \in V$, $f(v) < g(v)$.

A TDF (DF) g of $G = (V, E)$ is *minimal*, denoted by MTDF (MDF), if for all functions $f : V \rightarrow [0,1]$ such that $f < g$, f is not a total dominating function (a dominating function). If an MTDF (MDF) f has integer values (i.e., 0 or 1), then we call f a 0-1 MTDF (MDF). An MTDF (MDF) f of $G = (V, E)$ is called *positive* if $f(v) > 0$ for each $v \in V$.

A *convex combination* of the TDFs (DFs) f and g of $G = (V, E)$ is a function $h_t : V \to [0,1]$, where $t \in (0,1)$ and $h_t(v) = tf(v) + (1-t)g(v)$ for each $v \in V$. For MTDFs (MDFs) f, g of G, we write $f\Re g$ if h_t is an MTDF (MDF) for all $t \in (0,1)$ and $f\bar{\Re}g$ otherwise.

The following are some notations:

$C_0(G) = \{v \in V \mid f(v) = 0$ for all MTDF f of $G\}$.

$C_1(G) = \{v \in V \mid f(v) = 1$ for all MTDF f of $G\}$.

$L(G) = \{v \in V \mid d(v) = 1\}$, the set of *leaves* of G

$R(G) = \{v \in V \mid v \in N(u)$ for some $u \in L\}$, the set of *remote vertices* of G.

$S(G) = \{v \in V \mid v$ is not a leaf and $N(v) \bigcap R \neq \emptyset\}$, the set of *short vertices* of G.

Example 1: In Fig 1, f and g are TDFs of the graph G with vertex set {1,2,3,4,5}. Since $f < g$, g is not minimal. A later result (Theorem 27) will demonstrate that f is minimal, i.e., is a MTDF. The vertices of boundaries B_f, B_g are depicted by solid squares. For this graph $L(G) = \{4,5\}, R(G) = \{2,3\}$ and $S(G) = \{1,2,3\}$.

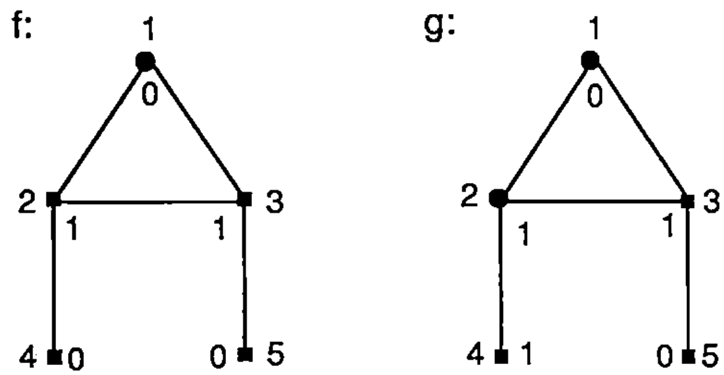

Fig 1 Two TDFs of a graph G

1.2 Motivation for the Thesis

There has been a vast amount of mathematical literature recently concerning dominating sets in graphs. Dominating functions (DFs) generalize the concept of dominating sets because the integer valued (i.e., 0 or 1) dominating functions are precisely the characteristic functions of dominating sets. The reader is referred to [16] for an excellent bibliography concerning these topics.

In particular, the convexity of minimal dominating functions (MDFs) has been considered in [7,8-12]. The original reason for this was an attempt to answer the following interpolation problem raised by Hedetniemi[15].

Given MDFs f, g of G with aggregates a, b respectively and any c satisfying $a < c < b$, does there exist an MDF of G with aggregate c ?

Suppose that f and g are DFs of G. For $t \in (0,1)$, define the convex combination $h_t : V \to [0,1]$ of f and g by:

$$h_t(v) = tf(v) + (1-t)g(v)$$

for each $v \in V$.

It is elementary to show that h_t is a DF and that if f, g have aggregates a, b respectively and $a < c < b$, then by a suitable choice of $t \in (0,1)$, h_t has aggregate c. Hence the answer to the above question is "yes", provided that h_t is minimal. However this is not always the case and this fact led to the study of the binary relation $\Re$ on the set $\Im$ of all MDFs of G, defined by: $f \Re g$ if and only if h_t is an MDF for all $t \in (0,1)$. In particular, some graphs have a universal MDF, i.e., an MDF f satisfying $f \Re g$ for all MDF g.

In this thesis, we develop an analogous convexity theory for total dominating functions which arise when we simply change "closed" neighbourhood in the definition of dominating functions to "open".

1.3 Outline

It is the purpose of Chapter 2 to give a survey of results on minimal dominating functions (MDFs) of graphs, which motivated this thesis.

Our main results about minimal total dominating functions (MTDFs) of graphs are discussed in Chapter 3, 4 and 5.

In Chapter 3, some general results on MTDFs of graphs are given. In Section 3.1, we present some basic results about MTDFs which give a necessary and sufficient condition for a total dominating function (TDF) to be minimal and a necessary and sufficient condition for convex combinations of two MTDFs to be MTDFs themselves. Also, several classes of graphs which have a universal MTDF are given. The section ends with a condition for the non-existence of universal MTDFs which is then applied to vertex-transitive graphs. In Section 3.2, we characterize the graphs which have a unique MTDF. Short vertices and hot vertices of graphs, which are very important in the question of the existence of universal MTDFs, are defined and discussed in Section 3.3.

In Chapter 4, we turn our attention to trees. Hot vertices for trees are characterized in Section 4.1. These vertices play a central role in the convexity of MTDFs of trees and are heavily involved in questions of existence of universal MTDFs of U-trees, which is the main topic of Section 4.2.

In Chapter 5, we use the analysis of Section 4.1 to exhibit three classes of trees which have universal MTDFs and one class of trees which does not have universal MTDFs.

Finally, in Chapter 6, we compare and contrast our results on MTDFs with those for MDFs, give the conclusions of this thesis and state some open problems.

Chapter 2

Convexity of Minimal Dominating Functions of Graphs

In this chapter, we will review recent results of [7-12] on convexity of MDFs of graphs. As stated above, the purpose of this thesis is to develop an analogous theory for MTDFs. Similarities between the two theories will be apparent in later chapters. Further details and illustrations may be found in the references.

2.1 Basic Results

Firstly, we give a necessary and sufficient condition for a DF to be minimal.

Theorem 1 (Fricke [13]) *A DF f of a graph G is an MDF if and only if $B_f \succ P_f$.* ∎

The following example shows that convex combinations of MDFs are not always themselves MDFs.

Example 2[12] Consider the tree T^* of Fig 2. Let $f(1), \ldots, f(6) = 0,0,1,0,1,0$ and $g(1), \ldots, g(6) = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 1$. It is easy to verify that f, g are MDFs. However, $h_{1/4}(1), \ldots, h_{1/4}(6) = \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{3}{8}, \frac{1}{4}, \frac{3}{4}$. This is a DF whose boundary {1,2,6} does not dominate its positive set {1,2,3,4,5,6}. Therefore by Theorem 1, $h_{1/4}$ is not an MDF.

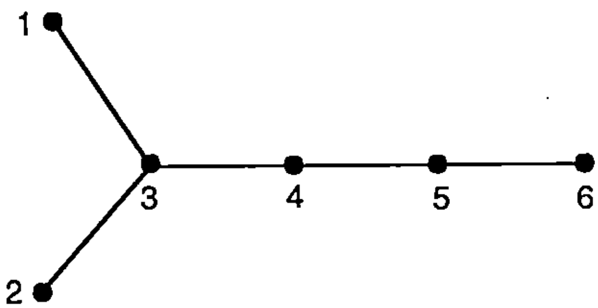

Fig 2 Tree T^*

The next theorem shows that either all convex combinations of MDFs f, g are minimal or none of them are. Recall that for MTDFs f, g, we write $f \Re g$ if the convex combinations of f and g are MTDFs.

Theorem 2 [7] *For any MDFs f, g of a graph,* $f \Re g$ *if and only if* $B_f \cap B_g \succ P_f \cup P_g$. ∎

Theorem 3 [7] *Any graph G has a positive MDF.* ∎

2.2 Universal MDFs of Graphs

This section considers the question of the existence of MDFs which relate in $\Re$ to all the other MDFs.

A *universal* MDF g is an MDF whose convex combinations with any other MDFs are MDFs, i.e., for all $f \in \Im$ and all $t \in (0,1)$, the convex combination $h_t \in \Im$.

Not all graphs have universal MDFs. Examples will be given in a later section. The next result gives a simple criterion for existence.

Proposition 4 [7] *If the MDF g satisfies $B_g = V$ and for all $f \in \Im$, $B_f \succ V$, then g is a universal MDF.* ∎

This result can be used to prove the existence of universal MDFs in several classes of graphs.

Theorem 5 [7] *The path $P_n (n \geq 1)$, the cycle $C_n (n \geq 3)$, the complete graph $K_n (n \geq 1)$, the complete bipartite graph $K_{m,n} (m, n \geq 1)$ and the n-vertex wheel $W_n (n \geq 4)$ all have universal MDFs.* ∎

Proposition 6 [7] *If g is a universal MDF, then $B_g \succ V$.* ∎

The next result demonstrates the existence of universal MDFs whose boundaries do not contain all vertices (as required by Proposition 4). It also provides examples of graphs, all of whose MDFs are universal.

Let u be a vertex of graph H. By a *complete addition to H at u*, we mean the identification of u and a vertex of some complete graph with at least two vertices.

Proposition 7 [7] *Let H be any graph. Form G from H by making one or more complete additions to H at u, for each vertex u of H. Then each MDF of G is universal.* ∎

2.3 Cool Vertices and Loose Vertices

In this section, we will introduce two types of vertices which play an important role in the existence of universal MDFs.

Definition: Let f be an MDF of G. Vertex v is called *f-loose* if $B_f \bigcap N[v] = \emptyset$ and is called *loose* if v is f-loose for some MDF f.

Theorem 8 [7] *If G has a vertex v such that each $u \in N[v]$ is loose, then G has no universal MDF.* ∎

Corollary 9 [7] *If G is vertex-transitive, then G has a universal MDF if and only if for every MDF f of G, $B_f \succ V$.* ∎

The following example gives a vertex-transitive graph which does not have universal MDFs.

Example 3 [7] Let G be the circulant formed by adding edges $\{i, i+5\}$ for $i = 1, \ldots, 5$ to the cycle with vertex sequence $1, \ldots, 10$. Then, for example, the function f which is 1 on $\{1,3,6,8\}$ and 0 elsewhere, is an MDF with $B_f = \{4,5,9,10\}$ which does not dominate V. By the corollary, G has no universal MDF.

Definition: Let f be an MDF of G. Vertex v is called *f-cool* if and only if $B_f \bigcap N[v] \subseteq R$ and is called *cool* if v is f-cool for some MDF f of G. We observe that a loose vertex is a special case of a cool vertex.

Example 4[12] In the tree T^* of Fig 2, $B_f = \{1,2,3,5,6\}$, $N[4] = \{3,4,5\}$ and $R = \{3,5\}$. Since $B_f \bigcap N[4] \subseteq R$, vertex 4 is f-cool.

Proposition 10 [8] *(a) If v is a cool vertex of a graph G, then $v \notin L \bigcup R$.* *(b) If v is f-cool in a graph G, then $f(v) = 0$ and $v \notin B_f$.* ∎

The next three theorems indicate the relevance of cool vertices to the existence of universal MDFs in graphs.

Theorem 11 [8] *If g is a universal MDF of G, then $g(v) = 0$ for each cool vertex v of G.* ∎

Theorem 12 [8] *If the MDF g satisfies (i) $g(v) = 0$ for each cool vertex v and (ii) $V - R \subseteq B_g$, then g is a universal MDF.* ∎

Theorem 13 [8] *If G has a vertex v such that each $u \in N[v]$ is cool, then G has no universal MDF.* ∎

We note that this theorem generalizes Theorem 8.

2.4 Universal MDFs of Trees

This section contains characterizations of cool and loose vertices of trees, properties of such vertices, a characterization of MDFs of trees which are universal and a variety of existence results for universal MDFs of trees.

Theorem 14 [8] *Vertex v is a cool vertex of a tree if and only if*
(a) $d(u) \geq 3$ for each $u \in N(v) - R$ and
(b) $N(v)$ contains at least two vertices, each of which is adjacent to at least two vertices of $V - R$. ∎

Theorem 15 [8] *Vertex v is a loose vertex of a tree if and only if*
(a) $d(u) \geq 3$ for each $u \in N(v)$ and
(b) $N(v)$ contains at least two vertices, each of which is adjacent to at least two vertices of $V - R$. ∎

When proving facts concerning cool vertices and the existence of universal MDFs in trees, it is often important to have information about the function f for which a certain vertex v is f-cool. We present one result of this type.

Theorem 16 [8] *Let the tree T be rooted at vertex v. Then v is cool if and only if v is g-cool where*
(a) $B_g \bigcap N[v] = R \bigcap N[v]$,
(b) $g(u) = 1$ for all $u \in R \bigcap N[v]$ and
(c) g has integer values(i.e., 0 or 1) except perhaps on T_w, where $w \in N(v) - R$ and $g(w) > 0$. ∎

Now we give some results concerning universal MDFs of trees. The first one is an immediate corollary of Proposition 7.

Corollary 17 [7] *Any MDF of a tree whose leaves dominate the tree, is universal.* ∎

The next theorem is a characterization of universal MDFs of trees.

Theorem 18 [8] *The MDF g is a universal MDF for a tree T if and only if (a) $g(v) = 0$ for all cool vertices v of T and (b)$V - R \subseteq B_g$.* ∎

The remaining results which assert the existence of universal MTDFs depend on the successful completion of one of two algorithms called UNICAND and UNICAND1. Details of these two algorithms are given in [8].

Theorem 19 [8] *If T may be rooted at v such that for all $u \in V - (L \bigcup R)$, u has a non-cool child, then T has a 0-1 universal MDF.* ∎

Corollary 20 [8] *In a tree T, if each $u \in V - (L \bigcup R)$ has at least two non-cool neighbours, then T has a 0-1 universal MDF.* ∎

Theorem 21 [8] *If the tree T is a generalized star or a caterpillar, then T has a 0-1 universal MDF.* ∎

Theorem 22 [8] *If T is a tree with $diam(T) \leq 7$, then T has a 0-1 universal MDF.* ∎

Theorem 23 [8] *If $diam(T) = 8$ and the central vertex of T is not cool, then T has a 0-1 universal MDF.* ∎

It will have been noticed that all the above existence theorems guarantee the existence of 0-1 universal MDFs. A corollary of the next theorem states that a tree with a universal MDF always has a 0-1 universal MDF.

Theorem 24 [9] *Let f be an MDF of the tree T and let* $v \in A_f = \{v \in V | 0 < f(v) < 1\}$. *Then T has an MDF g satisfying*
(a) $g(v) = 1$, *(b)* $B_g \supseteq B_f$, *(c)* $P_g \subseteq P_f$, *(d)* $A_g \subset A_f$. ∎

Corollary 25 [9] *If T has a universal MDF, then T has a 0-1 universal MDF.* ∎

The final result concerns the non-existence of universal MDFs of trees.

Theorem 26 [8] *Let x and y be distinct vertices of a tree T satisfying*
(a) $N[x] \bigcap N[y] = \{v\}$ *where* $v \neq x, y$ *and*
(b) each $u \in N(x) \bigcup N(y)$ *is cool.*
Then T has no universal MDF. ∎

Chapter 3

Convexity of Minimal Total Dominating Functions of Graphs

We now commence the original work of this thesis, i.e., we develop an analogous convexity theory for total dominating functions (TDFs) which arise when we simply change "closed" neighbourhood in the definition of dominating functions (DFs) to "open". In order to ensure the existence of a TDF, we restrict our attention to graphs without isolated vertices. The integer-valued TDFs are precisely the characteristic functions of TDSs. TDFs and TDSs have been studied in [1-3, 5, 6, 14, 17-21].

There are some potential applications of TDFs. Suppose the graph G models a network of people all of whom use and can usually supply for themselves, one unit of commodity X. Occasionally a member of the network is unable to supply X for himself and hence must be supplied by his neighbours in the network. Therefore , in addition to supplying his own needs, person v must have available an extra $f(v)$ units of X in case of an emergency in his neighbourhood. If for each v, $0 \leq f(v) \leq 1$, then in order that any single emergency may be dealt with, f must be a TDF of G.

To be specific, suppose X is security. Usually a person is present and physically able to take care of his own property. However in his absence or illness, various neighbours collectively look after his security needs. Let $f(v)$ be the proportion of time that person v is able to guard a neighbour's property in an emergency. Provided that f is a TDF of the network, the property of any single absentee may be protected by his neighbours.

Now we give some basic results about MTDFs of graphs.

3.1 Universal MTDFs of Graphs

Theorem 27 *The TDF f is minimal if and only if $B_f \to P_f$.*

Proof. If f is minimal and $B_f \not\to P_f$, then there exists $v \in P_f$ such that $N(v) \bigcap B_f = \emptyset$. Let $N(v) = \{v_1, \ldots, v_n\}$. For $i \in \{1, \ldots, n\}, f[v_i] > 1$. Let ϵ satisfy $0 < \epsilon \leq f(v)$ and

$$\epsilon \leq \min_i \{f[v_i] - 1\}.$$

Define a new function g by $g(v) = f(v) - \epsilon$ and $g(u) = f(u)$ for each $u \in V - \{v\}$. For each $u \notin \{v_1, \ldots, v_n\}, g[u] = f[u] \geq 1$, and for $i \in \{1, \ldots, n\}$, $g[v_i] = f[v_i] - \epsilon > f[v_i] - (f[v_i] - 1) = 1$. Therefore the function g is a TDF satisfying $g < f$, a contradiction.

Conversely, suppose that $B_f \rightarrow P_f$ but f is not minimal and there exists a TDF $g < f$. For some vertex $v, g(v) < f(v)$, hence $f(v) > 0$. It follows that $B_f \rightarrow \{v\}$, i.e., there exists a vertex $u \in B_f \bigcap N(v)$. But $v \in N(u)$, hence $g[u] < f[u] = 1$. Therefore g is not a TDF, a contradiction. ∎

In the graph G of Fig 1, we observe that $B_f \rightarrow P_f$ and hence f is an MTDF.

Theorem 28 *Let f, g be MTDFs and $h_t = tf + (1-t)g$ where $t \in (0,1)$. Then h_t is an MTDF if and only if $B_f \bigcap B_g \rightarrow P_f \bigcup P_g$.*

Proof. We will prove that $B_{h_t} = B_f \bigcap B_g$ and $P_{h_t} = P_f \bigcup P_g$. The result is then immediate from Theorem 27. If $v \notin P_f \bigcup P_g$, then $f(v) = g(v) = h_t(v) = 0$. If, say, $v \in P_f$, then $h_t(v) \geq tf(v) > 0$. Thus $P_{h_t} = P_f \bigcup P_g$. Suppose $v \in B_f \bigcap B_g$. Then

$$\sum_{u \in N(v)} (h_t(u)) = \sum_{u \in N(v)} (tf(u) + (1-t)g(u))$$

$$= t \sum_{u \in N(v)} (f(u)) + (1-t) \sum_{u \in N(v)} (g(u)) = t + (1-t) = 1.$$

A similar calculation shows

$$\sum_{u \in N(v)} (h_t(u)) > 1$$

for $u \notin B_f \bigcap B_g$ and hence $B_{h_t} = B_f \bigcap B_g$. ∎

We emphasize that this theorem shows that the minimality of h_t is independent of t, i.e., either all nontrivial convex combinations of f, g are minimal or none are minimal.

A *universal* MTDF g is an MTDF whose convex combinations with any other MTDFs are MTDFs, i.e., for all MTDF f and all $t \in (0,1), h_t$ is an MTDF.

Proposition 29 *If the MTDF g satisfies $B_g = V$and for all MTDFs f, $B_f \to V$, then g is a universal MTDF.*

Proof. For any MTDF $f, B_g \bigcap B_f = B_f$ which totally dominates $V \supseteq P_f \bigcup P_g$. By Theorem 28, h_t is an MTDF. Therefore g is a universal MTDF. ∎

The existence of universals in certain classes of graphs can be proved using Proposition 29.

Theorem 30 *The cycle* $C_n(n \geq 3)$*, the complete bipartite graph* $K_{m,n}(m, n \geq 1)$*, the (n+1)-vertex wheel* $W_n(n \geq 3)$ *and the complete graph* $K_n(n \geq 2)$ *all have universal MTDFs.*

Proof. (1) The cycle C_n :

The function g which assigns $1/2$ to each vertex is an MTDF with $B_f = V$.

Let f be any MTDF of G such that $B_f \not\rightarrow V$, i.e., there exists $v_0 \in V$ such that $B_f \bigcap N(v_0) = \emptyset$. Since $B_f \rightarrow P_f$, $f(v_0) = 0$. Suppose $N(v_0) = \{v_1, v_2\}$, and v_1 is also adjacent to v_3 (for $n = 3, v_3 = v_2$). Since $v_1 \notin B_f$, we have $f[v_1] = f(v_0) + f(v_3) = f(v_3) > 1$, a contradiction. Hence by Proposition 29, g is a universal MTDF of C_n.

(2) The complete graph K_n :

The function g which assigns $1/(n-1)$ to each vertex is an MTDF with $B_f = V$.

Suppose K_n has an MTDF f whose boundary B_f does not totally dominate vertex a. Trivially $n > 2$. Then no vertex b of $V - \{a\}$ is in B_f and $f(a) = 0$ (since $B_f \rightarrow P_f$). Hence

$$f[b] = f(a) + \sum_{u \in V - \{a,b\}} f(u) > 1.$$

Therefore

$$\sum_{u \in V - \{a,b\}} f(u) > 1.$$

But

$$f[a] \geq \sum_{u \in V-\{a,b\}} f(u) > 1.$$

We deduce $a \notin B_f$ and $B_f = \emptyset$, a contradiction.

Then by Proposition 29, g is a universal MTDF of K_n.

(3) The (n+1)-vertex wheel $W_n (n \geq 3)$:

This graph has (n+1)-vertices and is the join of K_1 and C_n. Let the central vertex be v_0 and the cycle have vertex sequence $v_1, \ldots, v_n$. The function g which assigns $1/n$ to each $v_i, i \geq 1$ and $g(v_0) = 1 - 2/n$ is an MTDF with $B_g = V$.

Suppose W_n has an MTDF f such that $B_f \not\rightarrow V$. There are two cases to consider:

Case 1: $B_f \not\rightarrow \{v_0\}$:

Then no cycle vertex is in B_f and since $B_f \rightarrow P_f, f(v_0) = 0$.

Since $1 < f[v_2] = f(v_1) + f(v_3) + f(v_0)$, we deduce $f(v_1) + f(v_3) > 1$.

Therefore $f[v_0] \geq f(v_1) + f(v_3) > 1$ and so $B_f = \emptyset$, a contradiction.

Case 2: $B_f \not\rightarrow \{v_2\}$ (without loss of generality):

Then none of v_0, v_1, v_3 is in B_f and since $B_f \rightarrow P_f, f(v_2) = 0$.

If $n = 3$, since $v_0 \notin B_f, f(v_1)+f(v_2)+f(v_3) > 1$, i.e., $f(v_1)+f(v_3) > 1$.

Then $f[v_2] \geq f(v_1) + f(v_3) > 1, v_2 \notin B_f$ and $B_f = \emptyset$, a contradiction.

If $n \geq 4$, then $f[v_3] = f(v_2)+f(v_0)+f(v_4)$ and since $v_3 \notin B_f$, this gives $f(v_0) + f(v_4) > 1$ and it follows that $f(v_4) > 0$. Now $f[v_5] \geq f(v_0) + f(v_4) > 1$ (for $n = 4, v_5 = v_1$). Therefore $v_5 \notin B_f$ and $B_f \not\rightarrow \{v_4\}$ which contradicts $B_f \rightarrow P_f$.

By Proposition 29, g is a universal MTDF for W_n.

(4)The complete bipartite graph $K_{m,n}$:

Let the defining independent sets of $K_{m,n}$ be $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$. The function g which satisfies $g(a_i) = 1/m$ and $g(b_j) = 1/n$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ is an MTDF with $B_g = V$.
For any MTDF f of $K_{m,n}$, if (say) $B_f \not\rightarrow \{a_1\}$, then $f[b_i] > 1$ for each $i = 1, \ldots, n$. Therefore there exists a_k such that $f(a_k)$ is positive.
But $N(a_k) \bigcap B_g = \emptyset$, i.e., $B_f \not\rightarrow P_f$, a contradiction. By Proposition 29, g is a universal MTDF for $K_{m,n}$.
This completes the proof of Theorem 30. ∎

The next result will be used to prove the non-existence of universal MTDFs for some graphs.

Proposition 31 *Let $v \in V$ be such that for each $u \in N(v)$ there exists an MTDF f_u where $B_{f_u} \not\rightarrow \{u\}$. Then G has no universal MTDF.*

Proof. Suppose G has a universal MTDF g. Then for any MTDF f,

$B_g \bigcap B_f \rightarrow P_g \bigcup P_f$. But by the hypothesis, for each $u \in N(v), B_g \bigcap B_{f_u} \not\rightarrow \{u\}$ and we deduce $u \notin P_g \bigcup P_{f_u}$ and $g(u) = 0$. Therefore $g[v] = 0$ and g is not a TDF, a contradiction. ∎

Corollary 32 *If G is vertex-transitive, then G has a universal MTDF if and only if for each MTDF f, $B_f \rightarrow V$.*

Proof. Let G be vertex-transitive and r-regular. If $B_f \rightarrow V$ for any MTDF f, the function g with value $1/r$ on each vertex satisfies $B_g = V$ and is universal by Proposition 29.

Conversely, suppose $B_f \not\rightarrow V$ for some MTDF f. Then $B_f \not\rightarrow \{v\}$ for some $v \in V$. Since G is vertex-transitive, there exists for every $u \in V(G)$ and in particular for every $u \in N(v)$, an MTDF f_u for which $B_{f_u} \not\rightarrow \{u\}$. By Proposition 31, G has no universal MTDF. ∎

The following example shows a vertex-transitive graph which does not have a universal MTDF.

Example 5: Let G be the circulant formed by adding edges $\{i, i+2\}$ for $i = 1, \ldots, 11$ (addition is modulo 11) to the cycle with vertex sequence $\{1, \ldots, 11\}$. Then the function f, which is 1 on {2,4,9,11} and 0 elsewhere, is an MTDF with $B_f = \{4, 5, 6, 7, 8, 9\}$ which does not totally dominate $\{1\}$. By Corollary 32, G has no universal MTDF.

3.2 Graphs Having a Unique MTDF

The main purpose of this section is to characterize those graphs which have a unique MTDF. This is done in Theorem 37.

Recall that $C_0(G)$ is the set of vertices whose function values are zero for all MTDFs of G and that $C_1(G)$ is the set of vertices whose function values are one for all MTDFs of G.

We first establish some properties of $C_0(G)$ and $C_1(G)$.

Proposition 33 *For any graph $G, C_1(G) = R$.*

Proof. Let $v \in C_1(G)$ and $v \notin R$ where $N(v) = \{v_1, \ldots, v_n\}$. Since $v \notin R$, no v_i is a leaf and therefore $N(v_i) - \{v\} \neq \emptyset$ for $i = 1, \ldots, n$. Now we define a function $g : V \rightarrow [0, 1]$ by:

$$g(u) = \begin{cases} 0 & \text{if } u = v \\ 1 & \text{otherwise.} \end{cases}$$

For $v_i \in N(v)$, since $N(v_i) - \{v\} \neq \emptyset$,

$$g[v_i] = g(v) + \sum_{w \in N(v_i) - \{v\}} g(w) \geq 1.$$

For $u \notin N(v)$, $d(u) \geq 1$, hence $g[u] \geq 1$.

Therefore g is a TDF and there exists an MTDF g' such that $g' \leq g$. However $g'(v) \leq g(v) = 0$ and so $v \notin C_1(G)$.

If $v \in R$, then there exists $l \in L$ adjacent to v. For any MTDF f, $1 \leq f[l] = f(v)$, i.e., $f(v) = 1$ and $v \in C_1(G)$. ∎

Proposition 34 *The vertex $v \in C_0(G)$ if and only if for any $u \in N(v)$ there exists a vertex w such that $N(w) \subseteq N(u) - \{v\}$.*

Proof. Let $v \in C_0(G)$ and let U be the set of vertices $x \in N(v)$ such that for all vertices $w \in V$, $N(w) \not\subseteq N(x) - \{v\}$. The result states $U = \emptyset$. Suppose not and choose $u \in U$ with minimum degree k. By Proposition 33, $v \notin R$ and hence $k \geq 2$. Define $g : V \to [0, 1]$ by:

$$g(x) = \begin{cases} 1/k & \text{for } x \in N(u) \\ 1 & \text{otherwise.} \end{cases}$$

We note $g[u] = 1$. For $t \in V$ satisfying $N(t) \not\subseteq N(u)$,

$$g[t] \geq \sum_{x \in N(t) - N(u)} g(x) \geq 1.$$

If $N(t) \subseteq N(u)$, since $N(t) \not\subseteq N(u) - \{v\}$, we deduce $v \in N(t)$ and therefore $t \in N(v)$. If a vertex w satisfies $N(w) \subseteq N(t) - \{v\}$, then $N(w) \subseteq N(u) - \{v\}$ which would contradict $u \in U$. It follows that $t \in U$ and so by the minimality of k, $N(t) = N(u)$ and $d(t) = k$ which implies $g[t] = 1$. Therefore g is a TDF and there exists an MTDF h with $h \leq g$. If $h(v) < g(v)$, then $h[u] < g[u] = 1$ and h is not a TDF. We deduce $h(v) = 1/k \neq 0$, contrary to the hypothesis.

Conversely, suppose that v is a vertex for which the set U, as defined above, is empty and that f is an MTDF with $f(v) \neq 0$. Let $g(v) = 0$ and $g(x) = f(x)$ for all $x \in V - \{v\}$. For any $u \in N(v)$, there exists w with $N(w) \subseteq N(u) - \{v\}$. Therefore $g[u] \geq g[w] = f[w] \geq 1$. For any $t \in V - N(v)$, we have $g[t] = f[t] \geq 1$. It follows that g is a TDF and $g < f$, contrary to the minimality of f. ∎

Example 6: In the graph G of Fig 3, $C_1(G) = R(G) = \{5\}$. The vertex $v = 1$ satisfies the hypothesis of Proposition 34. Observe that $N(1) = \{2, 3\}$.
For $u = 2, w = 8$ satisfies $N(w) \subseteq N(u) - \{v\} = N(2) - \{1\}$.
For $u = 3, w = 7$ satisfies $N(w) \subseteq N(u) - \{v\} = N(3) - \{1\}$.
Hence by Proposition 34, $1 \in C_0(G)$ and in fact one may verify (using this Proposition) that $C_0(G) = \{1\}$.

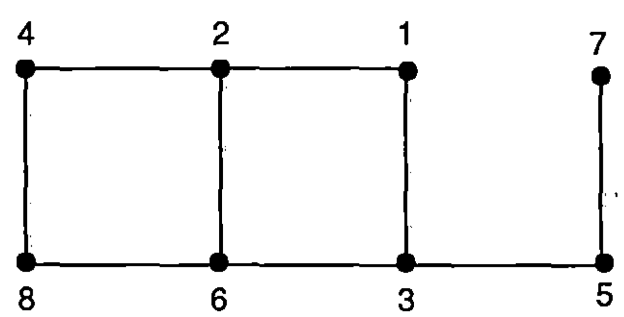

Fig 3 Illustration of $C_0(G), C_1(G)$

The next result gives characterization of $C_0(G), C_1(G)$ in terms of MTDSs of the graph G.

Proposition 35 *For any graph G and vertex v,*

(a) $v \in C_0(G)$ if and only if v is in no MTDS of G.

(b) $v \in C_1(G)$ if and only if v is in every MTDS of G.

Proof. (a) Let $v \in C_0(G)$ and suppose, contrary to the result, $v \in B$, an MTDS of G. The characteristic function f of B is an MTDF with $f(v) = 1$. Hence $v \notin C_0(G)$, a contradiction.

Conversely, suppose $v \notin C_0(G)$. Then by Proposition 34 there exists $u \in N(v)$ such that for any $w \in V, N(w) \not\subseteq N(u) - \{v\}$. It follows that $X = \{v\} \bigcup \{V - N(u)\}$ is a TDS which contains an MTDS Y. The only vertex of X which totally dominates u is vertex v. Therefore $v \in Y$.

(b) Let $v \in C_1(G) = R$ (Proposition 33) and A be any MTDS. There exists $l \in L \bigcap N(v)$. Since A totally dominates $l, v \in A$. Conversely, if $v \notin R$, then $Y - \{v\}$ is a TDS which contains an MTDS, i.e., there exists an MTDS not containing v. ∎

Here, we mention that the above characterization theorems of $C_0(G)$ and $C_1(G)$ have a big influence on later work.

Note that for any graph $G(V, E)$ without isolated vertices, the vertex set V is a TDS of G and thus G has an MTDS. The characteristic function of this MTDS is a 0-1 MTDF. Therefore the next result is an immediate consequence of this observation.

Proposition 36 *If G has a unique MTDF g, then g is a 0-1 MTDF.*

We now establish the main results of this section.

Theorem 37 *The graph G has a unique MTDF if and only if every vertex of G is adjacent to a remote vertex.*

Proof. Suppose that G has a unique MTDF g. For any $v \in V$, since $g[v] \geq 1$, by Proposition 36, there exists $u \in N(v)$ with $f(u) = 1$. Since g is unique, $u \in C_1(G) = R$ (Proposition 33) and v is adjacent to a remote vertex.

Conversely, suppose each $v \in V$ is adjacent to a remote vertex. Let g be the characteristic function of R. Since $C_1(G) = R$, it remains to show $V - R \subseteq C_0(G)$ and it will follow that g is unique. Let $x \in V - R$ and $u \in N(x)$. By the hypothesis, u is adjacent to $r \in R (r \neq x)$ and r is adjacent to $w \in L$. Then

$$N(w) = \{r\} \subseteq N(u) - \{x\}$$

and $x \in C_0(G)$ by Proposition 34, as required. ∎

The following example uses Theorem 37 to show the existence of a universal MTDF f whose boundary does not totally dominate V. This situation differs from that of MDF theory, in which the boundary of every universal MDF dominates V (Proposition 6).

Example 7: Let G have $V = \{1,2,3,4,5,6\}$ and $E = \{12, 23, 13, 15, 34, 26\}$. Then $R = \{1,2,3\}$. By Theorem 37, G has a unique MTDF f which has value 1 on {1,2,3} and 0 elsewhere. Since f is the unique MTDF of G, f is universal. However, $B_f = \{4,5,6\}$ does not totally dominate {4,5,6}.

Theorem 38 *A graph G has either a unique MTDF or infinitely many MTDFs.*

Proof. If G has more than one MTDF then, by Theorem 37,

$$M = \{x \in V \mid N(x) \cap R = \emptyset\} \neq \emptyset.$$

Let $u \in M$ such that $d(u) \leq d(x)$ for all $x \in M$. Since leaves are only adjacent to remote vertices, $k = d(u) \geq 2$. Let $v \in N(u)$ and for $n \geq 2$, define $g_n : V \to [0,1]$ by:

$$g_n(x) = \begin{cases} 1/n & x = v \\ (1-1/n)/(k-1) & x \in N(u) - \{v\} \\ 1 & \text{otherwise.} \end{cases}$$

For x such that $N(x) - N(u) \neq \emptyset$, $g_n[x] \geq 1$. If $N(x) \subseteq N(u)$, then $N(x) \bigcap R \subseteq N(u) \bigcap R = \emptyset$ and therefore $x \in M$. By the minimum property of u, $N(x) = N(u)$ and $g_n[x] = g_n[u] = 1$. Thus g_n is a TDF and there exists an MTDF $h_n \leq g_n$. Moreover $h_n(y) = g_n(y)$ for $y \in N(u)$ (otherwise $h_n[u] < g_n[u] = 1$). Further $h_n(v) \neq h_m(v)$ if $n \neq m$ and we have an infinite set of MTDFs. ∎

3.3 Short Vertices and Hot Vertices

In this section we introduce two types of vertices of graphs which are very important in the question of the existence of universal MTDFs.

A vertex v is called *short* if $v \notin L$ and v is adjacent to a remote vertex. Let $S(G)$ or S (if confusion is unlikely) denote the set of short vertices of G.

Let f be an MTDF of G. Vertex v is called *f-hot* if $B_f \bigcap N(v) \subseteq S$. Further, v is called *hot* if v is f-hot for some MTDF f. Let $H(G)$ or H (if confusion is unlikely) denote the set of hot vertices of G.

Example 8: In Fig 4, we depict a 0-1 MTDF g of a 12-vertex graph G with $V = \{2, \ldots, 13\}, B_g = V - \{6, 9\}$ (solid squares) and $S = \{6, 8, 9, 10, 11\}$. We observe that $B_g \bigcap N(7) = \{8\} \subseteq S$, $B_g \bigcap N(12) = \{10\} \subseteq S$ and $B_g \bigcap N(13) = \{11\} \subseteq S$. Therefore vertices 7, 12, 13 are g-hot.

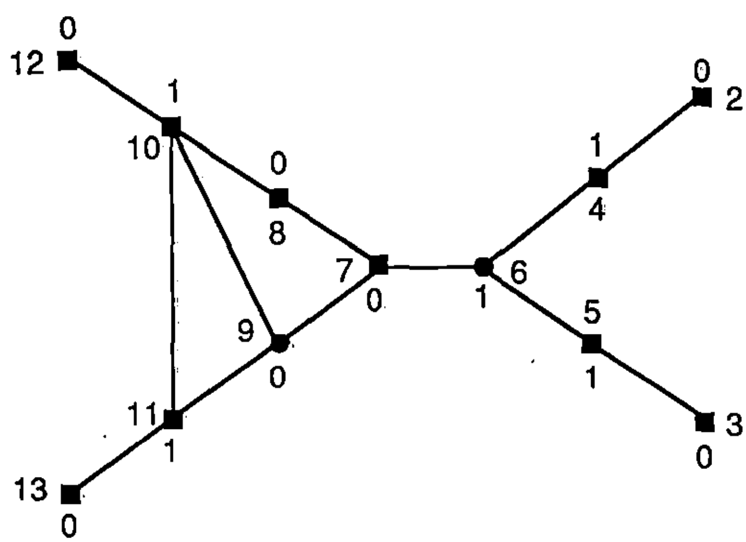

Fig 4 Short vertices and hot vertices

We now establish a few properties of hot and short vertices and finally in Theorem 45, relate them to universal MTDFs.

Proposition 39 *If* $N(v) \subseteq S$, *then* $v \in C_0(G)$.

Proof. Suppose $N(v) \subseteq S$. Since $L \bigcap S = \emptyset$, $v \notin R$. For any $u \in N(v) \subseteq S$, there exists a remote vertex $r(\neq v)$ adjacent to u and a leaf l adjacent to r. We have $N(l) = \{r\} \subseteq N(u) - \{v\}$. By Proposition 34, $v \in C_0(G)$. ∎

Proposition 40 *Let* f *be any MTDF of* G *. Then*

(a) $L \subseteq B_f$ *and*

(b) A vertex $v \in S \bigcap B_f$ *if and only if* $v \notin L$ *and* v *is adjacent to exactly one remote vertex* r *and* $f(x) = 0$ *for all* $x \in N(v) - \{r\}$.

Proof. (a) Let $v \in L$ be adjacent to $r \in R$. By Proposition 33, $r \in C_1(G)$ and so $f[v] = f(r) = 1$ and $v \in B_f$.

(b) If $v \in S \bigcap B_f$, then v is adjacent to exactly one $r \in R$, otherwise Proposition 33 asserts $f[v] \geq 2$. Since $v \in B_f$, we have

$$1 = f[v] = f(r) + \sum_{x \in N(v)-\{r\}} f(x).$$

Since $f(r) = 1$ by Proposition 33, it follows that $f(x) = 0$ for all $x \in N(v) - \{r\}$.

The converse is obvious. ∎

Proposition 41 *Let u be a vertex of a graph G. If there is at most one vertex $x \in N(u)$ such that $N(x) - \{u\} \not\subseteq S$, then $u \in B_f$ for any MTDF f.*

Proof. Suppose that there is at most one vertex $x \in N(u)$ such that $N(x) - \{u\} \not\subseteq S$, but $u \notin B_f$, i.e., $f[u] > 1$. Then u has at least two neighbours with positive values of f, and by the hypothesis, one of these, say w, satisfies $N(w) - \{u\} \subseteq S$. Let $\epsilon > 0$ satisfy $f[u] - \epsilon \geq 1$ and $f(w) - \epsilon \geq 0$. Define $g : V \to [0,1]$ by $g(w) = f(w) - \epsilon$ and $g(x) = f(x)$ on $V - \{w\}$. Note that

$g[u] = f[u] - \epsilon \geq 1$. For $y \notin N(w)$, $g[y] = f[y] > 1$. For $y \in N(w) - \{u\}$, since $y \in S$, there exists $r \in R$ which is adjacent to y. Therefore $g[y] \geq g(r) = 1$. We have proved that $g < f$ is a TDF, contrary to the minimality of f. ∎

Corollary 42 *Let r be a remote vertex of a graph G. If $|N(r) - L| \leq 1$, then $r \in B_f$ for any MTDF f of G.*

Proof. Let $l \in N(r)$ and $l \in L$. Then $N(l) - \{r\} = \emptyset \subseteq S$. Since $|N(r) - L| \leq 1$, there is at most one $x \in N(r)$ such that $N(x) - \{r\} \not\subseteq S$. By Proposition 41, $r \in B_f$ for any MTDF f. ∎

Proposition 43 *(a) If v is a hot vertex , then $v \notin R$.*

(b) If v is f-hot, then $f(v) = 0$.

Proof. (a) If $v \in R$, then there exists a leaf l adjacent to v. Suppose v were f-hot. Since $l \in B_f$ (Proposition 40(a)), $l \in B_f \bigcap N(v) \subseteq S$. But $S \bigcap L = \emptyset$, a contradiction.

(b) If $f(v) > 0$, then by Theorem 27, $B_f \to \{v\}$ and there exists $u \in B_f \bigcap N(v) \subseteq S$. By Proposition 40(b), u is adjacent to exactly one $r \in G$ and by (a), $r \neq v$. Since $u \in B_f$ and $f(r) = 1$ (Proposition 33), we have

$$1 = f[u] \geq f(r) + f(v) = 1 + f(v).$$

It follows that $f(v) = 0$, a contradiction. ∎

Theorem 44 *If v is a hot vertex of a graph G, then for each $u \in N(v) - S$, there are at least two vertices $x \in N(u) - \{v\}$ with $N(x) - \{u\} \not\subseteq S$.*

Proof. Suppose that v is f-hot where f is an MTDF of G. Let $u \in N(v) - S$, then $u \notin B_f$ (otherwise $u \in B_f \bigcap N(v) \subseteq S$, a contradiction). Therefore, $f[u] > 1$ and u has at least two neighbours with positive values of f. But $v \in N(u)$ and $f(v) = 0$ by Proposition 43(b). Hence u has at least three neighbours v, x, and y where $f(x)$ and $f(y)$ are positive.

Suppose x satisfies $N(x) - \{u\} \subseteq S$. Choose $\epsilon > 0$ so that $f[u] - \epsilon \geq 1$ and $f(x) - \epsilon \geq 0$. Define $g : V \to [0,1]$ by $g(x) = f(x) - \epsilon$ and $g(t) = f(t)$ for each $t \in V - \{x\}$.

If $t \notin N(x), g[t] = f[t] \geq 1$.

If $t \in N(x) - \{u\}$, then $t \in S$ and t is adjacent to $r \in R = C_1(G)$. Hence $g[t] \geq g(r) = f(r) = 1$. Also $g[u] = f[u] - \epsilon \geq 1$.

Therefore $g < f$ is a TDF, contrary to the minimality of f. Therefore $N(x) - \{u\} \not\subseteq S$. Similarly $N(y) - \{u\} \not\subseteq S$. Therefore there are at least two vertices $x \in N(u) - \{v\}$ with $N(x) - \{u\} \not\subseteq S$. ∎

Now, we are ready to relate hot vertices to universal MTDFs.

Theorem 45 *(a) If the MTDF g of a graph G satisfies*

(i) $V - S \subseteq B_g$ and (ii) $g(v) = 0$ for each hot vertex v,

then g is a universal MTDF of G.

(b) If g is a universal MTDF of G,

then $g(v) = 0$ for each hot vertex v of G.

Proof. (a) Let f be any MTDF and $v \in P_f \bigcup P_g$. We claim that v is not f-hot, for otherwise $f(v) = 0$ by Proposition 43(b) and $g(v) = 0$ by the hypothesis. Therefore $v \notin P_f \bigcup P_g$, a contradiction. Since v is not f-hot, there exists $x \in B_f \bigcap N(v)$ such that $x \notin S$. Since $V - S \subseteq B_g$, it follows that $x \in (B_g \bigcap B_f) \bigcap N(v)$ and so $B_f \bigcap B_g \to P_f \bigcup P_g$. By Theorem 28, g is a universal MTDF.

(b) Suppose v is f-hot and $g(v) > 0$. Since $v \in P_f \bigcup P_g$ and g is universal, $B_f \bigcap B_g \to \{v\}$, i.e., there exists $x \in B_f \bigcap B_g \bigcap N(v) \subseteq S$ (since v is f-hot). Since x is short, there exists $r \in N(x) \bigcap R$ and by Proposition 43(a) $r \neq v$. It follows that

$$1 = g[x] \geq g(r) + g(v) = 1 + g(v) > 1,$$

a contradiction. ∎

Theorem 46 *If G has a vertex v such that each $u \in N(v)$ is hot, then G has no universal MTDF.*

Proof. Suppose G has a universal MTDF g. By Theorem 45(a)(ii), $g(u) = 0$ for each $u \in N(v)$. Therefore $g[v] = 0$, i.e., g is not a TDF, a contradiction. ∎

In Chapter 5, Theorem 46 will be invoked to show non-existence of universal MTDFs in a class of trees.

Chapter 4

Convexity of Minimal Total Dominating Functions of Trees

In this chapter, we apply the analysis of Chapter 3 to prove two principal results concerning the convexity of MTDFs of trees. Firstly, in Section 4.1 we obtain a necessary and sufficient condition for a vertex of a tree to be hot. This result will permit the solution of the existence problem for universal MTDFs in several classes of trees (Chapter 5). Then in Section 4.2, we obtain a characterization of universal MTDFs for a special class of trees known as U-trees.

4.1 Characterization of Hot Vertices of Trees

In this section, a characterization of hot vertices of trees will be obtained (Theorem 49) by showing that the converse of Theorem 44 holds for trees. We need some preliminary work.

Theorem 47 *For any vertex v of tree T, $v \in C_0(T)$ if and only if $N(v) \subseteq S$.*

Proof. Let $v \in C_0(T)$. By Proposition 34, for any $u \in N(v)$, there exists a vertex w such that $N(w) \subseteq N(u) - \{v\}$. Since $N(w) \neq \emptyset$, it follows that $N(u) - \{v\} \neq \emptyset$, i.e., $u \notin L$. If $w \notin L$, then T would have a cycle, which is impossible. Therefore $w \in L$ and there exists a remote vertex r such that $\{r\} = N(w) \subseteq N(u) - \{v\}$. Thus u is adjacent to r and, consequently, $u \in S$ which implies that $N(v) \subseteq S$.
The converse is a special case of Proposition 34. ∎

Corollary 48 *(a) For any MTDF f of a tree T and $v \in C_0(T)$, v is f-hot.*
(b) For any tree T and any vertex v of T, if $N(v) \bigcap C_0(T) \neq \emptyset$, then $N(v) \bigcap R \neq \emptyset$.

Proof. (a) Let $v \in C_0(T)$. By Theorem 47, $N(v) \subseteq S$ and so $N(v) \bigcap B_f \subseteq S$, i.e., v is f-hot.
(b) Let $x \in N(v) \bigcap C_0(T)$. By Theorem 47, $N(x) \subseteq S$ and in particular, $v \in S$. Thus there exists a vertex $y \in N(v) \bigcap R$ and $N(v) \bigcap R \neq \emptyset$. ∎

The next result shows that the converse of Theorem 44 holds for trees and thus we have a characterization of hot vertices of trees.

Throughout the remainder of this chapter, we repeatedly use the notation which is illustrated by the following three examples: Let T be a tree rooted at v and $w \in V(T)$.

$f|_{T_w} = \chi(\text{MTDS},w)$ means the function f restricted to T_w is the characteristic function of an MTDS of T_w which contains w.

$f|_{T_w} = \chi(\text{MTDS},\bar{w})$ means the function f restricted to T_w is the characteristic function of an MTDS of T_w which does not contain w.

$f|_{T_w} = \chi(\text{MTDS})$ means the function f restricted to T_w is the characteristic function of an MTDS of T_w.

Theorem 49 *The vertex v of the tree T is hot if and only if for each $u \in N(v)-S$, there are at least two vertices $x \in N(u)-\{v\}$ with $N(x)-\{u\} \not\subseteq S$.*

Proof. The condition is illustrated in Example 9, Fig 5 following this proof.

Suppose the condition holds. We will define a 0-1 MTDF f so that v is f-hot. If $N(v) \subseteq S$, then for any MTDF f, $B_f \bigcap N(v) \subseteq S$ and v is f-hot. Hence we assume $N(v) - S \neq \emptyset$.

The definition of f will have two stages. We first define a function $g : V \to [0,1]$ and then under certain conditions g will be amended to form the required MTDF f.

Let $g(v) = 0$ and define g on $V - \{v\}$ by **G1**,**G2**,**G3** below. Assume that the tree T is rooted at v.

G1: For $w \in N(v) \bigcap S$, define

$$g|_{T_w} = \begin{cases} \chi(\text{MTDS},\bar{w}) & \text{if } w \in C_0(T_w) \\ \chi(\text{MTDS},w) & \text{otherwise} \end{cases}$$

Let $W = \{w \in N(v) \bigcap S \ \mid \ g(w) = 1\}$.

G2: Assignment of g values for vertices of $N(v) - S$.

(i) If $N(v) \bigcap (W \bigcup R) \neq \emptyset$, then for each $u \in N(v) - S$, let

$$g(u) = \begin{cases} 1 & \text{if } u \in C_1(T) = R \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If $N(v) \bigcap (W \bigcup R) = \emptyset$, then by Corollary 48(b), $N(v) \bigcap C_0(T) = \emptyset$. Thus we select any $\lambda \in N(v) - S \neq \emptyset$ and define

$g(\lambda) = 1$ and

$g(u) = 0$ for each $u \in N(v) - (S \bigcup \{\lambda\})$.

G3: Assignment of g values for descendents of $u \in N(v) - S$.

Let $u \in N(v) - S$. There exist children x_1, x_2 of u such that $N(x_i) - \{u\} \not\subseteq S$, $i = 1, 2$.

Let $g|_{T_{x_i}} = \chi(\text{MTDS}, x_i)$ for i = 1, 2.

For $z \in N(u) - \{v, x_1, x_2\}$, let $g(z) = 0$ for $z \in L$, and $g|_{T_z} = \chi(\text{MTDS})$ for $z \notin L$.

In order to show that g is well-defined, we must prove that **G3** is possible, i.e., that T_{x_i} possesses an MTDS containing x_i. Suppose the contrary. By Proposition 35(a), $x_i \in C_0(T_{x_i})$ and by Theorem 47, each neighbour y of x_i in T_{x_i} is a short vertex of T_{x_i}, i.e., y is not a leaf of T_{x_i} and y is adjacent to a remote vertex r_y of T_{x_i}. It follows that y is not a leaf of T and r_y is also remote in T. Hence y is a short vertex of T. We have $N(x_i) - \{u\} \subseteq S$, contrary to the definition of x_i.

The above assignments have defined $g : V \to [0,1]$. This function will now be amended to form an MTDF f for which v is f-hot.

Let $U = \{u \in N(v) - S \mid g(u) = 1\}$ and $A = \{t \mid t$ is a grandchild of $u \in U, g(t) = 1$ and $N(t) \bigcap B_g = \emptyset\}$.

Define $f : V \to [0,1]$ as follows,

F1 :

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ g(x) & \text{otherwise.} \end{cases}$$

Now we show f is a TDF.

The assignment **G1**, **G2**, **F1** ensure that $f[x] = g[x] \geq 1$ for

$$x \in \{v\} \bigcup (\bigcup_{w \in N(v) \cap S} V(T_w)).$$

If $u \in N(v) - S$, where $f(u) = g(u) = 0$, then **G2**, **G3**, **F1** ensure that $f[x] = g[x] \geq 1$ for all $x \in V(T_u)$.

Suppose that $u \in U$ and $w \in V(T_u)$. If $w \notin N(t)$ for $t \in A$, then $f[w] = g[w] \geq 1$ by **G3**.

Suppose that w is adjacent to $t \in A$. Either $w \in N(u)$ and

$f[w] \geq f(u) = g(u) = 1$ or w is a child of t. In the latter case, by the definition of A, $B_g \cap N(t) = \emptyset$, i.e., $g[w] \geq 2$. Therefore $f[w] = g[w] - 1 \geq 1$. We conclude that f is a TDF.

We now demonstrate the minimality of f. Let $x \in P_f$ (i.e., $f(x) = 1$). Several cases must be considered.

Case 1: If $x \in T_w$ for $w \in N(v) \cap S$, **G1** and **F1** ensure that $B_f \to \{x\}$.

Case 2: If $x \in T_u$ where $u \in N(v) - S$ and $g(u) = 0$, **G3** and **F1** give $B_f \to \{x\}$.

In the rest of the cases, we assume that $x \in T_u$ where $u \in U$ (i.e., $f(u) = g(u) = 1$).

Case 3: $x = u$ where $u \in R$. In this case, **G3** assigns $g(l) = 0$ for a leaf l adjacent to x. Then $l \in B_f$ and so $B_f \to \{x\}$.

Case 4: $x = u = \lambda$, the selected vertex of **G2**(ii). In this case, $g(w) = 0$ for all $w \in N(v) - \{\lambda\}$ and $f[v] = g[v] = g[\lambda] = 1$. Therefore $v \in B_f$ and $B_f \to \{x\}$.

Case 5: Let $x \in T_u$, where $u \in U$ and $x \neq u$. Then $x \in T_z$ for some $z \in N(u)$. Notice that since the TDF f satisfies $f \leq g, B_g \subseteq B_f$. Since $f(x) = 1, g(x) = 1$ and by **G3**, there exists y adjacent to x in T_z such that

$$\sum_{w \in N_{T_z}(y)} g(w) = 1.$$

If $y \neq z, N_T(y) = N_{T_z}(y)$ and

$$\sum_{w \in N_T(y)} g(w) = 1.$$

Therefore $y \in B_g \subseteq B_f$ and $B_f \to \{x\}$.

If $y = z$, then x is a grandchild of u. Since $f(x) = 1, x \notin A$ and hence $N(x) \bigcap B_g \neq \emptyset$. Therefore $B_f \supseteq B_g \to \{x\}$.

It follows from **Cases 1—5** that f is an MTDF.

Finally, if $u \in N(v) - S$, then by **G3**, $f[u] = g[u] \geq g(x_1) + g(x_2) = 2$. Therefore $N(v) \bigcap B_f \subseteq S$ and v is f-hot.

The converse is a special case of Theorem 44. ∎

Example 9: In the tree of Fig 5, vertices x, u, v are indicated which satisfy the condition of Theorem 49. Therefore v is hot.

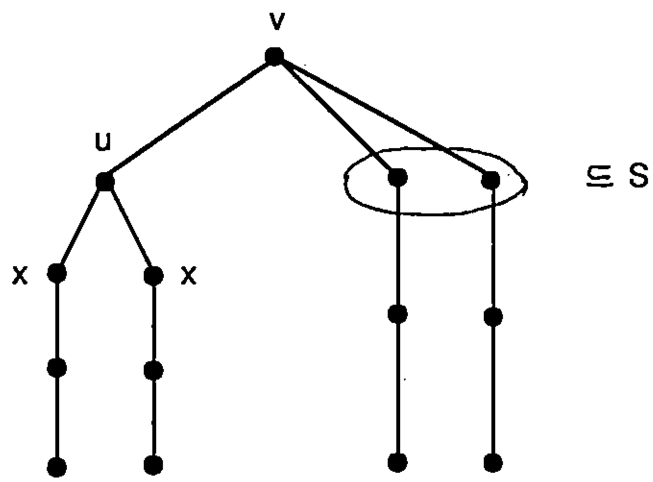

Fig 5 Illustration of Theorem 49

Corollary 50 *Vertex v in a tree T is hot if and only if there exists a 0-1 MTDF g such that v is g-hot.*

Proof. Let g be the 0-1 MTDF f defined in the proof of Theorem 49. ∎

The hypothesis of Proposition 31 involves vertices u, for each of which there exists an MTDF f_u with $B_{f_u} \not\succ \{u\}$. This clearly implies

$$B_{f_u} \cap N(u) = \emptyset \subseteq S,$$

i.e., such vertices form a subset of the hot vertices. We call these vertices *hard.* (In the MDF theory of Chapter 2, loose vertices analogously form a subset of the cool vertices.)

There is a characterization of hard vertices of trees which is very similar to that of hot vertices (Theorem 49). The only difference is that "$u \in N(v) - S$" is replaced by "$u \in N(v)$". The proof, which closely resembles that of Theorem 49, is omitted.

4.2 Universal MTDFs of U-Trees

In Theorem 45(a), sufficient conditions for an MTDF of a graph G to be a universal MTDF were given. In this section (Theorem 53), we prove that these conditions are also necessary if G is a special kind of tree which we will call a U-tree. The next result is involved in the U-tree definition.

Proposition 51 *Let v be a vertex of a tree T. Then vertex $v \in B_f$ for any MTDF f of T if and only if there exists at most one vertex $u \in N(v)$ such that $N(u) - \{v\} \not\subseteq S$.*

Proof. Suppose $v \in B_f$ for any MTDF f and assume , contrary to the result, there exist at least two vertices $x_1, x_2 \in N(v)$ such that $N(x_1) - \{v\} \not\subseteq S$ and $N(x_2) - \{v\} \not\subseteq S$. Therefore by Theorem 47, $x_1 \notin C_0(T_{x_1})$ and $x_2 \notin C_0(T_{x_2})$.

We will define an MTDF f such that $v \notin B_f$. The idea is almost the same as that used in the proof of Theorem 49. We first define a function g and then amend g to form the required MTDF f. Assume that the tree is rooted at v.

Define a function $g : V \rightarrow [0,1]$ by:

G1: **(i):** If $v \in R$ then let

$g(v) = 1$ and $g(l) = 0$ for $l \in N(v) \bigcap L$.

(ii): If $v \notin R$ then let $g(v) = 0$.

G2: $g|_{T_{x_1}} = \chi(\text{MTDS},x_1)$.

$g|_{T_{x_2}} = \chi(\text{MTDS},x_2)$.

G3: $g|_{T_x} = \chi$ (MTDS), $x \in N(v) - L \bigcup \{x_1, x_2\}$.

Since $x_1 \notin C_0(T_{x_1})$ and $x_2 \notin C_0(T_{x_2})$, **G2** can be achieved. Therefore g is well-defined.

We now amend g to form the required MTDF f.

Let $A = \{t \mid t$ is a child of $u \in N(v) - L$, $g(t) = 1$ and $N(t) \bigcap B_g = \emptyset \}$.

Define $f : V \rightarrow [0,1]$ as follows:

F1:

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ g(x) & \text{otherwise} . \end{cases}$$

The assignments **G1, G2** ensure that $f[x] = g[x] \geq 1$ for $x \in \{v\} \bigcup \{N(v) \bigcap L\}$.

The assignments **G2, G3, F1** ensure that if $w \notin N(t)$ for $t \in A$, then $f[w] = g[w] \geq 1$.

Now suppose w is adjacent to $t \in A$. By the definition of A, $B_g \bigcap N(t) = \emptyset$, i.e., $g[w] \geq 2$. Hence $f[w] = g[w] - 1 \geq 1$.

We conclude that f is a TDF.

We now show the minimality of f, i.e., $B_f \rightarrow P_f$. Let $x \in P_f$ (i.e., $f(x) = 1$). Several cases must be considered:

Case 1: If $x = v$, then by **G1(i)**, x is totally dominated by a leaf $l \in B_f$.

Case 2: If x is not a child of w for $w \in N(v) - L$, by **G2, G3, F1**, $B_f \rightarrow x$.

Case 3: If x is a child of $w \in N(v) - L$, since $f(x) = 1 = g(x)$, by **F1**, $x \notin A$, then there exists $t \in N(x)$ such that $t \in B_g$. By **F1**, $t \in B_f$. Therefore $B_f \rightarrow x$.

It follows from **Cases 1–3** that f is an MTDF. But,

$$f[v] \geq f(x_1) + f(x_2) = 2.$$

Therefore $v \notin B_f$, a contradiction.

The converse is Proposition 41. ∎

By Theorem 47, we obtain the following corollary.

Corollary 52 *Let the tree T be rooted at v. The vertex $v \in B_f$ for any MTDF f of T if and only if there is at most one vertex $u \in N(v)$ such that $u \notin C_0(T_u)$.* ∎

Definition: A tree T is called a *U-tree* if and only if

(a) for any $r \in R$ there is at most one vertex $u \in N(r)$ such that

$N(u) - \{r\} \not\subseteq S$ and

(b) for $v \notin R \bigcup L, |N(v) - S| \geq 2$.

Example 10: The tree T in Fig 6 is a U-tree with $S = \{s_1, s_2\}$. It is easy checked that the vertices not in $R \bigcup L$ satisfy condition (b) and to illustrate condition (a), we observe that there is exactly one vertex $s_1 \in N(r)$ such that $N(s_1) - \{r\} = \{r', s_2\} \not\subseteq S$.

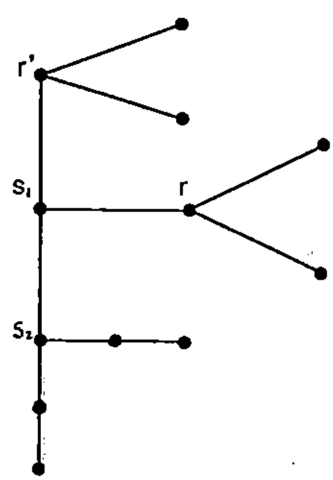

Fig 6 A U-tree

We now show that the conditions of Theorem 45(a) characterize universal MTDFs of U-trees.

Theorem 53 *The MTDF g of a U-tree T is a universal MTDF if and only if (i) $g(v) = 0$ for each hot-vertex v and (ii) $V - S \subseteq B_g$.*

Proof. By Theorem 45, it suffices to prove that if g is a universal MTDF of a U-tree, then $V - S \subseteq B_g$.

Suppose this is false and there exists $v \in V - S$ which is not in B_g. The definition of T and Proposition 51 imply that $R \subseteq B_g$ and for any graph, $L \subseteq B_g$; hence $v \in V - (S \bigcup L \bigcup R)$. Choose any vertex $w \in N(v) - S$ ($\neq \emptyset$ by the definition of T). Since $v \notin R, w \notin L$ and so $w \in N(v) - (S \bigcup L)$. It is easily verified from the above that for any $l \in L$, both distances $d(v, l)$ and $d(w, l)$ are at least three. Consider T to be rooted at v.

Define the function $h : V \rightarrow \{0, 1\}$ as follows:

C1: $h(w) = h(v) = 1$.

C2: For $u \in N(v) - \{w\}$, $h|_{T_u} = \chi(\text{MTDS}, \bar{u})$.

Since $d(v, l) \geq 3$ for $l \in L$, $u \notin R(T_u)$, i.e., $u \notin C_1(T_u)$. Therefore, by Proposition 35(b), the assignment of **C2** is possible.

C3: For any $y \in N(w) - \{v\}, h(y) = 0$.

C4: For any $x \in N(y) - \{w\}$ where $y \in N(w) - \{v\}$, let $h|_{T_x} = \chi\,(\text{MTDS}, x)$.

To show that this assignment is possible, we must show $x \notin C_0(T_x)$.

This is certainly true if $x \in R$ (Proposition 33). Otherwise $x \notin R \bigcup L$ and, by the definition of T, $|N(x) - S| \geq 2$. Therefore $N(x) - \{y\} \nsubseteq S(T)$ and hence $N(x) - \{y\} \nsubseteq S(T_x)$. By Theorem 47, $x \notin C_0(T_x)$.

Now, we show that h is a TDF.

By **C1** and w adjacent to v, we have $h[v] \geq 1$ and $h[w] \geq 1$.
By **C2**, $h[z] \geq 1$ for $z \in T_u$ where $u \in N(v) - \{w\}$. Since $h(w) = 1$, $h[y] \geq 1$ for $y \in N(w) - \{v\}$.
Finally by **C4**, we have $h[u] \geq 1$ for $u \in T_x$ where $x \in N(y) - \{w\}$ and $y \in N(w) - \{v\}$.
We have proved that h is a TDF.

It is easily verified that if $h(t) = 1$ and $t \in \{v\} \bigcup V(T_w)$ then $B_h \rightarrow \{t\}$. This may not be the case for all vertices s such that $h(s) = 1$ where $s \in V(T_u)$ for $u \in N(v) - \{w\}$, i.e., h may not be minimal. If h is not minimal, there exists an MTDF $h_1 < h$ where $h_1(z) = h(z)$, except possibly for vertices $z \in V(T_u) - \{u\}$ where $u \in N(v) - \{w\}$. It follows that

$$B_{h_1} \cap N(w) = B_h \cap N(w) = \{v\}.$$

Recall that $v \notin B_g$. If h is an MTDF, then $B_g \bigcap B_h \bigcap N(w) = \emptyset$ and so $B_g \bigcap B_h \not\rightarrow \{w\}$, where $w \in P_g \bigcup P_h$. Otherwise the same holds when the

MTDF h_1 replaces h. Hence by Theorem 28, we have contradicted the universal property of g. ∎

Chapter 5

More Special Classes of Trees

In this section, we will use Theorem 45(a) and Theorem 49 to obtain several classes of trees which have a universal MTDF or have no universal MTDF.

Let T_n denote the the set of trees rooted at v with the following properties. If H_k denotes the set of vertices at distance k from v ($H_0 = \{v\}$), then for $k = 0, \ldots, n-1$, each $u \in H_k$ has at least two children and $H_n = L$.

Theorem 54 *If $n \geq 8$ and $T \in T_n$, then T has no universal MTDF.*

Proof. First observe that for $n \geq 2$ and $T \in T_n, S = H_{n-2}$. It is then easy to check the conditions of Theorem 49 to show that for $n \geq 8$ and $T \in T_n$, each vertex of $H_2 \bigcup \ldots \bigcup H_{n-6} \bigcup H_{n-4}$ is a hot vertex of T. In addition, if $n = 8$ or 9, then H_4 or H_5 also consists of hot vertices respectively. Hence for $n = 8$ or $n \geq 10$, $H_2 \bigcup H_4$ contains only hot vertices. For $u \in H_3, N(u)$ contains only hot vertices. By Theorem 46, T has no universal MTDF. If

$n = 9$, for $u \in H_4, N(u)$ contains only hot vertices. By Theorem 46, T has no universal MTDF. ∎

If the root v of $T \in T_n$ has degree at least three, then Theorem 49 shows that each vertex of H_1 is also hot for $n = 5$ or $n \geq 7$, and when $n = 6$, $H_0 \bigcup H_2$ consists of hot vertices. A similar proof then establishes:

Theorem 55 *If $T \in T_n$ where $n \geq 6$, and $d(v) \geq 3$, then T has no universal MTDF.*

Definition : A *caterpillar* is a tree consisting of a path with vertex sequence $v_1, \ldots, v_t$ and each v_i is adjacent to a set L_i (possibly empty) of leaves. Without loss of generality, we assume $L_1, L_t \neq \emptyset$.

We now show that all caterpillars have 0-1 universal MTDFs (Theorem 57).

Proposition 56 *Let g be a 0-1 MTDF of a graph G, $l \in L, r \in R$ and l is adjacent to r. If $g(l) = 1$, then $g(v) = 0$ for all $v \in N(r) - \{l\}$, i.e., $r \in B_g$.*

Proof. If not, since g is a 0-1 MTDF, there exists $u \in N(r) - \{l\}$ such that $g(u) = 1$ then $g[r] \geq 2$. Define a function h by:

$$h(w) = \begin{cases} 0 & w = l \\ g(w) & \text{otherwise.} \end{cases}$$

Then $h[r] = g[r] - 1 \geq 1$. Therefore h is a TDF and $h < g$, which contradicts the minimality of g. ∎

Theorem 57 *If T is a caterpillar, then T has a 0-1 universal MTDF.*

Proof. We require some preliminary results which we state and prove as lemmas. Recall that H is the set of hot vertices.

Lemma 58 *Let T be a caterpillar, then*

(a) If $v_i \in H$, then $N(v_i) \subseteq S$, i.e., $v_i \in C_0(T)$.

(b) If $l \in L_i$ and $l \in H$, then (i) $N(l) \subseteq S$, i.e., $l \in C_0(T)$ or (ii)$3 \leq i \leq t-2$ and $v_{i\pm 2} \notin S$.

Proof. (a) If $v_i \in H$:

Since $H \bigcap R = \emptyset$ by Proposition 43(a), $i \neq 1, t$ and $v_i \notin R$, i.e., $L_i = \emptyset$. Then, if $N(v_i) \not\subseteq S$, say $v_{i-1} \notin S$, by Theorem 49, $d(v_{i-1}) \geq 3$, so $v_{i-1} \in R$. By Theorem 49, there are at least two vertices $y \in N(v_{i-1}) - \{v_i\} = \{v_{i-2}\} \bigcup L_{i-1}$ (or simply L_{i-1} if $i = 2$) such that $N(y) - \{v_{i-1}\} \not\subseteq S$. Therefore there exists a leaf $l \in L_{i-1}$ such that $N(l) - \{v_i\} \not\subseteq S$. But $N(l) - \{v_{i-1}\} = \emptyset \subseteq S$, a contradiction. Therefore v_{i-1} (and similarly v_{i+1}) $\in S$ and $N(v_i) \subseteq S$. By Theorem 47, $v_i \in C_0(T)$.

(b) If $l \in L_i$ and $l \in H$:

Suppose $l \in L_1$ and $N(l) \not\subseteq S$, i.e., $v_1 \notin S$. Since $v_1 \in N(l) - S$, by Theorem 49 there are two vertices $x \in N(v_1) - \{l\}$ with $N(x) - \{v_1\} \not\subseteq S$. We obtain a similar contradiction as in the proof of (a). Hence $N(l) \subseteq S$. An identical proof shows $N(l) \subseteq S$ if $l \in L_t$.

So now assume $l \in L_i$ where $1 < i < t$. If $N(l) \not\subseteq S$, i.e., $v_i \notin S$, then, $v_{i\pm1} \notin R$ which implies $3 \leq i \leq t-2$. By Theorem 49, there are at least two vertices $\{x, y\} \subseteq N(v_i) - \{l\}$ such that $N(x) - \{v_i\} \not\subseteq S$ and $N(y) - \{v_i\} \not\subseteq S$. Therefore $\{x, y\} = \{v_{i-1}, v_{i+1}\}$. Hence $N(v_{i\pm1}) - \{v_i\} = \{v_{i\pm2}\} \not\subseteq S$. ∎

Lemma 59 *Let T be a caterpillar, then there exists a 0-1 MTDF g such that $g(v) = 0$ for all $v \in H$.*

Proof. Let Z_f be the set of vertices v with $f(v) = 0$ for an MTDF f. Choose a 0-1 MTDF f such that $|Z_f \cap H|$ is maximum. If $H \subseteq Z_f$, then we are finished. Otherwise let $v \in H - Z_f$ (i.e., $f(v) = 1$).

By Lemma 58, $v = l \in L_i$, $N(l) \not\subseteq S$, $3 \leq i \leq t-2$ and $v_{i\pm2} \notin S$.

By Proposition 56, we have $f(v_{i\pm1}) = 0$ and $v_i \in B_f$. Now we define $g_1 : V \to \{0, 1\}$ by:

$$g_1(u) = \begin{cases} 0 & u = l \\ 1 & u = v_{i+1} \\ f(u) & \text{otherwise.} \end{cases}$$

Let $U = \{u \in N(v_{i+2}) \mid g_1(u) = 1\}$ and $A = \{y \in U \mid N(y) \cap B_{g_1} = \emptyset\}$. Define $g : V \to [0, 1]$ as follows:

$$g(x) = \begin{cases} 0 & x \in A \\ g_1(x) & x \in V - A. \end{cases}$$

We claim that g is a 0-1 MTDF of T. It is easily verified that g_1 is a 0-1 TDF.

If, for all $a \in A$, $y \notin N(a)$, then $g[y] = g_1[y] \geq 1$.

Suppose $y \in N(a)$ where $a \in A$. If $y \in N(a_1) \bigcap N(a_2)$ where $\{a_1, a_2\} \subseteq A$, then $y = v_{i+2}$. Since v_{i+1} is totally dominated by $v_i \in B_{g_1}, v_{i+1} \notin A$. Hence (say) $a_1 \in L_{i+2}$ and $f[v_{i+2}] \geq f(a_1) + f(a_2) = 2$. It follows that B_f does not totally dominate $a_1 \in P_f$, a contradiction which shows $y \neq v_{i+2}$. Further, by the definition of A, $g_1[y] \geq 2$. Therefore

$$g[y] = g_1[y] - 1 \geq 1.$$

We conclude that g is a TDF.

We now show that g is minimal. Clearly $v_i \in B_g \bigcap B_{g_1}$. Next observe that v_{i+1} is the only vertex whose function value has been increased during the construction of g, i.e., $f(v_{i+1}) = 0, g_1(v_{i+1}) = 1$ and $g(v_{i+1}) = 1$. Because of these facts and since g, g_1 are TDFs, we have

$$B_f - \{v_{i+2}\} \subseteq B_{g_1} \subseteq B_g.$$

Let $x \in P_g$, i.e., $g(x) = 1$.

If $x = v_{i+1}$, then $v_i \in B_g \bigcap N(x)$.

If $x \notin U$, then $f(x) = 1$ and $B_g \supseteq B_f - \{v_{i+2}\} \rightarrow \{x\}$.

Finally, let $x \in U - \{v_{i+1}\}$. Since $g(x) = 1, x \notin A$ and hence $N(x) \bigcap B_{g_1} \neq \emptyset$, i.e., $B_g \supseteq B_{g_1} \rightarrow \{x\}$.

Therefore g is an MTDF. But l is hot and v_{i+1} is not hot (by Lemma 58(a), note $g(v_{i+1}) = 1$). Hence $|Z_g \bigcap H| \geq |Z_f \bigcap H| + 1$, contrary to the maximum property of f. This concludes the proof of Lemma 59. ∎

Lemma 60 *Let T be a caterpillar, then*

(a) $\{v_1, v_t\} \subseteq B_f$ for any MTDF f.

(b) If f is a 0-1 MTDF and $v_i \notin B_f$, then $f(v_{i\pm1}) = 1$.

(c) If $f(v_{i-1}) = 0$ or $f(v_{i+1}) = 0$ for some 0-1 MTDF f, then $v_i \in B_f$.

Proof. Easy and omitted. ∎

We now continue with the proof of Theorem 57. By Theorem 45(a) and Lemma 59, it is sufficient to show that in the set M of 0-1 MTDFs which are zero on all hot vertices , there is a function g satisfying $V - S \subseteq B_g$. Suppose this statement is false and define

$$\emptyset \neq M^\star = \{m \in M \quad | \quad |(V - S) - B_m| \text{ is minimum}\}.$$

For $m \in M^\star$, let $i(m)$ be the largest integer such that $v_{i(m)} \in (V - S) - B_m$ and choose g_1 such that $i = i(g_1)$ is maximum. By Lemma 60, $2 \leq i \leq t - 1$ and $g_1(v_{i\pm1}) = 1$, while the minimality of g_1 (as a TDF) ensures that $g_1(l) = 0$ for each $l \in L_i$. Note that since $v_i \notin S, v_{i\pm1} \notin R$, i.e., $L_{i\pm1} = \emptyset$.
Form g_2 from g_1 by:

$$g_2(u) = \begin{cases} 0 & u = v_{i+1} \\ g_1(u) & \text{otherwise.} \end{cases}$$

By the minimality of g_1, g_2 is not a TDF. The construction of g_2 and the fact that $L_{i+1} = \emptyset$ imply that $g_2[u] = g_1[u]$ for $u \notin \{v_i, v_{i+2}\}$. Also, $g_2[v_i] = 1$. Hence we deduce that $g_2[v_{i+2}] = g_1[v_{i+2}] - 1 = 0$ since $v_{i+2} \in B_{g_1} \to \{v_{i+1}\}$. It follows that $g_1(u) = 0$ for $u \in N(v_{i+2}) - \{v_{i+1}\}$.

Since $v_{i+1} \notin R$, $N(v_{i+2}) \bigcap R = \emptyset$ and so $v_{i+2} \notin S$.

If $i + 2 = t$, then $L_{i+2} \neq \emptyset$. Choose any $l \in L_{i+2}$. By Lemma 58(b), $l \notin H$. Define g_3 by:

$$g_3(u) = \begin{cases} g_2(u) & u \neq l \\ 1 & u = l. \end{cases}$$

It is easy to see that g_3 is an MTDF, hence $g_3 \in M$. But $g_3[v_i] = 1$ and $g_3[v_j] = g_1[v_j]$ for $j \neq i$, contradicting $g_1 \in M^\star$.

Thus $t \geq i + 3$ and since $g_1(v_{i+3}) = 0$, it follows that $v_{i+3} \notin R = C_1(T)$. Therefore $t > i + 3$ and by Lemma 60(c), $v_{i+4} \in B_{g_1}$. Thus $g_1(x) = 1$ for precisely one vertex $x \in N(v_{i+4}) - \{v_{i+3}\}$.

Define g_4 by:

$$g_4(u) = \begin{cases} 0 & u = v_{i+1} \\ 1 & u = v_{i+3} \\ g_1(u) & \text{otherwise.} \end{cases}$$

Clearly, g_4 is a 0-1 TDF. Further, since $v_{i+2} \notin S, N(v_{i+3}) \nsubseteq S$ and by Lemma 58(a), v_{i+3} is not hot. But $v_i \in B_{g_4}$ and hence, by the choice of g_1, g_4 is not an MTDF. Since g_1 is an MTDF and $v_{i+2} \in B_{g_1} - \{v_{i+4}\} \subseteq B_{g_4}$, it follows that $B_{g_4} \not\to \{x\}$, i.e., $g_4[u] \geq 2$ for all $u \in N(x)$. But then the function g_5 defined by

$$g_5(u) = \begin{cases} 0 & u = x \\ g_4(u) & \text{otherwise} \end{cases}$$

can easily be seen to be a 0-1 MTDF with $g_5(u) = 0$ for all hot vertices, $v_i \in B_{g_5}$ and $B_{g_1} \subset B_{g_5}$, contradicting the choice of g_1. This completes the proof of Theorem 57. ∎

Corollary 61 *The path $P_n (n \geq 2)$ has a 0-1 universal MTDF.*

Proof. The path is the special caterpillar with $L_i = \emptyset$ for $2 \leq i \leq n-3$ and $|L_1| = 1, |L_{n-2}| = 1$. ∎

Definition: Suppose that Q_1, Q_2, Q_3 are paths with vertex sequences $(v_1, \ldots, v_{k_1+k_2}), (u_1, \ldots, u_{k_3})$ and $(w_1, \ldots, w_{k_4+k_5})$ respectively, where $k_1, k_3, k_4 \geq 2$ and $k_2, k_5 \geq 1$. The *H-tree* $T(k_1, k_2, k_3, k_4, k_5)$ is formed from Q_1, Q_2, Q_3 by two vertex identifications. Firstly u_1 and v_{k_1} are identified and secondly u_{k_3} and w_{k_4} are identified.

The next result will be needed to show that certain H-trees $T(k_1, k_2, k_3, k_4, k_5)$ possess universal MTDFs.

Proposition 62 *The path with vertex sequence $(v_1, \ldots, v_n)(n \neq 4, 7)$ has two 0-1 MTDFs f_1, f_2 such that $V - S \subseteq B_{f_1}, V - S \subseteq B_{f_2}$ and $f_1(v_i) = 1, f_2(v_i) = 0$ for $v_i \notin R$.*

Proof. We define f_1 and f_2 by:

$$f_1(v_i) = \begin{cases} 0 & i \equiv 0,1(\bmod 4) \text{ and } v_i \notin R \\ 1 & \text{otherwise,} \end{cases}$$

and

$$f_2(v_i) = \begin{cases} 0 & i \equiv 2,3(\bmod 4) \text{ and } v_i \notin R \\ 1 & \text{otherwise.} \end{cases}$$

It is easily verified that f_1, f_2 satisfy the conclusions of the proposition. ∎

Theorem 63 *The H-tree* $T(k_1, k_2, k_3, k_4, k_5)$ *with* $k_1, k_4 \geq 6, k_2, k_5 \geq 5$ *and* $k_3 \not\equiv 1(\bmod 4)$ *has a universal 0-1 MTDF.*

Proof. The set of short vertices of $T(k_1, k_2, k_3, k_4, k_5)$ is the set $S = \{v_3, w_3, v_{k_1+k_2-2}, w_{k_4+k_5-2}\}$ and by Theorem 49, when $k_3 \neq 3$, $H = \emptyset$. It is therefore sufficient to construct a 0-1 MTDF g with $V - S \subseteq B_g$ (Theorem 45(a)).

By Proposition 62, there exist MTDFs f_1, f_2 of Q_1 such that $f_1(u_1) = 1, f_2(u_1) = 0$ and $V(Q_1) - S(Q_1) \subseteq B_{f_i}(i = 1,2)$. Also there exist MTDFs h_1, h_2 of Q_3 such that $h_1(u_{k_3}) = 1, h_2(u_{k_3}) = 0$ and $V(Q_3) - S(Q_3) \subseteq B_{h_i}(i = 1,2)$.

There are some cases to be considered:

Case 1: $k_3 \equiv 0(\text{mod}4)$:

Define g by:

$$g(v) = \begin{cases} f_1(v) & v \in V(Q_1) \\ h_1(v) & v \in V(Q_3) \\ 0 & v = u_i \text{ for } i \equiv 2,3(\text{mod}4) \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to show that g is an MTDF and $V(Q_2) \subseteq B_g$. Since $S = S(Q_1) \bigcup S(Q_3)$, $g|Q_1 = f_1$ and $g|Q_3 = h_1$, it follows that $V - S \subseteq B_g$ and g is universal.

Case 2: $k_3 \equiv 2(\text{mod}4)$:

Define g by:

$$g(v) = \begin{cases} f_2(v) & v \in V(Q_1) \\ h_2(v) & v \in V(Q_3) \\ 0 & v = u_i \text{ for } i \equiv 1,2(\text{mod}4) \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, we can show that g is universal.

Case 3: $k_3 \equiv 3(\text{mod}4)$:

Define g by:

$$g(v) = \begin{cases} f_1(v) & v \in V(Q_1) \\ h_2(v) & v \in V(Q_3) \\ 0 & v = u_i \text{ for } i \equiv 2,3(\text{mod}4) \\ 1 & \text{otherwise.} \end{cases}$$

When $k_3 \neq 3$, similarly, we can show that g is universal.

When $k_3 = 3$, $H \subseteq \{u_2\}$ by Theorem 49. Also we have $g(u_2) = 0$. Therefore $g|_H = 0$. Moreover $V - S \subseteq B_g$. Hence g is a universal MTDF (Theorem 45(a)).

Definition: A *generalized star* is a tree formed from $k(\geq 3)$ disjoint paths with vertex sequences $v_{i1}, \ldots, v_{it(i)}, i = 1, \ldots, k$ (each $t(i) \geq 2$), by identifying the end vertices $v_{11}, v_{21}, \ldots, v_{k1}$. This identified vertex is denoted by v.

Proposition 64 *Let T be the generalized star with $t(i) \geq 3$ for each $i = 1, \ldots, k$. Then*

either (i) $H = \{v\}$ and $t(i) = 4$ for all $i = 1, \ldots, k$,

or (ii) $H \subset \{v_{i2} | i = 1, \ldots, k\}$.

Proof. (i) If $v \in H$, then $d(u) \geq 3$ for each $u \in N(v) - S$ by Theorem 49(a). But $d(w) \leq 2$ for all $w \in N(v)$. Therefore $N(v) \subseteq S$, i.e., for all $i = 1, \ldots, k$, $v_{i2} \in S$, $v_{i3} \in R$ and $v_{i4} \in L$. Hence $t(i) = 4$ for all i and Theorem 49 implies that $H = \{v\}$.

(ii) Suppose $v \notin H$. If $v_{in} \in H$ where $n \geq 3$ and $1 \leq i \leq k$, then $v_{in} \notin R$ by Proposition 43(a). Since $t(i) \geq 3, v \notin R$. Therefore $v_{i(n-1)} \notin S$. But $v_{i(n-1)} \in N(v_{in})$, $d(v_{i(n-1)}) = 2$ and hence by Theorem 49, $v_{in} \notin H$, a contradiction. Hence $H \subseteq \{v_{i2} | i = 1, \ldots, k\}$.

Now suppose, without loss of generality, $v_{12} \in H$.
If $v \in S$, then there exists $v_{m2} \in R$. By Proposition 43(a), $v_{m2} \notin H$.
If $v \notin S$, by Theorem 49(b) there exist two vertices $\{x, y\} \subseteq N(v) - \{v_{12}\}$ such that $N(x) - \{v\} \not\subseteq S$ and $N(y) - \{v\} \not\subseteq S$. If, say, $x = v_{22}$, then

$N(x) - \{v\} = v_{23} \notin S$. Since $d(v_{23}) = 2$ or $1 < 3$, $v_{22} \notin H$ by Theorem 49. Therefore $H \subset \{v_{i2} | i = 1, \ldots, k\}$. ∎

Theorem 65 *The generalized star T has a 0-1 universal MTDF.*

Proof. Let the path of T with vertex sequence $v_{i1}, \ldots, v_{it(i)}$ be denoted by Q_i.

Case 1: $v \in H$:

By Proposition 64(i), we have $t(i) = 4$ for $i = 1, \ldots, k$. Define a function g by:

$$g(u) = \begin{cases} 0 & u = v \\ 1 & u = v_{12} \\ 0 & u = v_{14} \\ 0 & u = v_{i2} \text{ for } i = 2, \ldots, k \\ 1 & u = v_{i4} \text{ for } i = 2, \ldots, k \\ 1 & u = v_{i3} \text{ for } i = 1, \ldots, k. \end{cases}$$

It is easy to verify that g is an MTDF and $B_g = V \supseteq V - S$. Moreover $g|H = g(v) = 0$. Therefore g is universal by Theorem 45(a).

Case 2: $t(i) \geq 4$ for all $i = 1, \ldots, k$ and $v \notin H$:

By Proposition 64(ii), we have $H \subset \{v_{i2} | i = 1, \ldots, k\}$ and there exists a vertex, say v_{12}, such that $v_{12} \notin H$. Define g on $V(Q_1)$ by:

$$g(v_{1j}) = \begin{cases} 1 & j = t(1) - 1 \\ 1 & j \equiv 1, 2 (\text{mod} 4) \\ 0 & \text{otherwise.} \end{cases}$$

Define g on $V(Q_i)$ for $i = 2, \ldots, k$ by:

$$g(v_{ij}) = \begin{cases} 1 & j = t(i) - 1 \\ 1 & j \equiv 0, 1(\text{mod} 4) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that g is a 0-1 MTDF satisfying $V - S \subseteq B_g$. Moreover by Proposition 64, $H \subseteq \{v_{22}, \ldots, v_{k2}\}$ and $g|_{\{v_{22},\ldots,v_{k2}\}} = 0$. Hence $g|_H = 0$ and g is universal by Theorem 45(a).

Case 3: $t(m) = 3$ for some $m \in \{1, \ldots, k\}$ and $t(i) \geq 3$ for $i = 1, \ldots, k$.

This case implies $v \in S$ and $v \notin H$ (Proposition 64(i)).

Define g on $V(Q_i)$ for $i = 1, \ldots, k$ by:

$$g(v_{ij}) = \begin{cases} 1 & j = t(i) - 1 \\ 1 & j \equiv 0, 1(\text{mod} 4) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that g is a 0-1 universal MTDF using Theorem 45(a).

Case 4: $v \in R$, i.e., for some $m \in \{1, \ldots, k\}, t(m) = 2$.

In this case, one may prove from Theorem 49 that $H \subseteq \{v_{i2}, v_{i3} | i = 1, \ldots, k\}$. We claim that for some $q \in \{1, \ldots, k\}, v_{q2} \notin H$. By symmetry, either all leaf neighbours of v are hot or none are.

In the former case, since $v_{m2} \in H$, there exists $x \in N(v) - \{v_{m2}\}$ (say $x = v_{q2}$) with $N(x) - \{v\} \not\subseteq S$. It follows that v_{q3} exists and is not short. Thus vertex v_{q3} is a non-short neighbour of v_{q2} of degree less than three and we deduce $v_{q2} \notin H$.

In the latter case, take $q = m$.

Define g on V by:

$$g(v_{ij}) = \begin{cases} 1 & j = t(i) - 1 \\ 1 & j \equiv 0, 1(\text{mod} 4) \\ 1 & v_{ij} = v_{q2} \\ 0 & \text{otherwise.} \end{cases}$$

One further change must be made if $t(q) \geq 3$, specifically, in this case, values of $g(v_{qj})$ must be assigned by:

$$g(v_{qj}) = \begin{cases} 1 & j = t(i) - 1 \\ 1 & j \equiv 1, 2(\text{mod} 4) \\ 0 & \text{otherwise.} \end{cases}$$

Function g is a 0-1 MTDF satisfying $V - S \subseteq B_g$ and $g|_H = 0$. Thus g is universal by Theorem 45(a).

Chapter 6

Conclusions: Open Problems

In this thesis, motivated by work of [7–12] on convexity of MDFs, which was summarized in Chapter 2, we have begun to develop an analogous theory for MTDFs.

Many facts about minimality and convexity in the TDF theory (Section 3.1) closely resemble results in the DF theory . However there are also basic differences. It was showed that some graphs have vertices which have the value zero in any MTDF (Proposition 34). Thus unlike MDFs, not all graphs have MTDFs with all positive function values. We have been unable to characterize such graphs but make the following:

Conjecture: A graph G has a positive MTDF if and only if $C_0(G) = \emptyset$.

A further difference is that there exist graphs with a unique MTDF (Theorem 37) while for graphs with at least one edge, there are infinitely many MDFs.

Our main work was on the existence of universal MTDFs. It turns out that all paths, cycles, complete graphs, complete bipartite graphs and wheels have both universal MTDFs and MDFs.

A basic difference here is that the boundary of any universal MDF g of G dominates the vertex set V of G (i.e., $B_g \succ V$, (Proposition 6)), but there exist universal MTDFs whose boundaries do not totally dominate V (Example 7).

In Section 3.3, we introduced short vertices and hot vertices , the counterparts of remote vertices and cool vertices in MDF theory. These were found to play an important role in the existence of universal MTDFs (Theorem 45).

Hot vertices of trees were characterized in Chapter 4 (Theorem 49) and this result enabled us to solve the universal MTDF existence problem for various classes of trees (Chapter 5).

If v is a hot vertex of a tree, there is always a 0-1 MTDF f such that v is f-hot (Corollary 50). The analogous statement for cool vertices of trees in MDF theory is false.

A characterization of MTDFs of trees which are universal, has not yet been obtained. The corresponding problem for MDFs has been solved. We were only able to find such a result for U-trees (Theorem 53).

We are far from characterizing the graphs which have universal MTDFs and in fact (as in the MDF theory) this problem has not even been solved for trees. The analysis developed in Chapter 3 and 4 might ultimately lead to a solution of the tree characterization problem.

In conclusion, we note that we have only studied one topic concerning the convexity relation on the set of MTDFs of a graph, namely the existence of universals. Further investigation of this relation will undoubtedly lead to interesting results.

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