Closure Operations and Hamiltonian Properties of Independent and Total Domination Critical Graphs

by

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Abstract

When an edge is added to a graph, each of the parameters γ, i, and γ_t may change. When the addition of any edge causes the parameter under consideration to decrease, such a graph is referred to as γ-critical (domination critical), i-critical (independent domination critical), or γ_t-critical (total domination critical), respectively. The graphs studied in this dissertation are the independent domination critical and total domination critical graphs.

For i-critical graphs G with $i = 3$, it is established that when $\delta \geq 3$, the graph G is hamiltonian, and when $\delta = 2$, there is exactly one family of non-hamiltonian graphs. In all cases, G has a Hamilton path provided it has more than six vertices. The hamiltonian properties of i-critical graphs are determined using a closure similar to one developed by Hanson. Furthermore, characterisations are given of the i-critical graphs with $i = 3$ that either contain a cut vertex or are 2-connected with $\delta = 2$.

Many properties of γ_t-critical graphs are established, and the γ_t-critical graphs with $\gamma_t = 3$ are studied in detail. It is known that all γ_t-critical graphs G with $\gamma_t = 3$ satisfy $2 \leq diam(G) \leq 3$, and hence the hamiltonian properties of the diameter two and diameter three cases are studied separately.

A new closure for total domination critical graphs is defined and used to study the hamiltonian properties of 2-connected diameter three γ_t-critical graphs with $\gamma_t = 3$. All such graphs are shown to contain a Hamilton path (and in most cases a Hamilton cycle), and several families of these graphs are characterised. The γ_t-critical graphs with $\gamma_t = 3$ that contain a cut vertex were characterised by Haynes, Mynhardt, and van der Merwe. All such graphs have diameter three and contain a Hamilton path.

In general, the diameter two γ_t-critical graphs with $\gamma_t = 3$ cannot be characterised in terms of a finite number of forbidden subgraphs. However, all such graphs are shown to be hamiltonian if $2 \leq \delta \leq 3$. A characterisation of several infinite families of diameter two γ_t-critical graphs with $\gamma_t = 3$ and $\delta = 3$ is given.

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Contents

List of Figures

Chapter 1

Introduction

1.1 Preview

The study of domination theory has grown to be a vast area of research in graph theory, giving rise to literally thousands of papers covering a broad range of related topics.

In graph theory, results are often obtained for specific families of graphs, that is, graphs that share some common properties. Insight can also be gained by studying the graphs within a family that are minimal or critical with respect to the defining property of the family. This work is important in that sometimes an observation about these minimal or critical graphs can lead to a more general result about the entire family of graphs.

In domination theory, the domination number γ, the independent domination number i, and the total domination number γ_t, are parameters that are defined for any graph (γ_t is defined only if $\delta > 0$). However, if a graph is altered by adding a missing edge, each of the parameters γ, i, and γ_t may change or may remain the same. It is interesting to study the family of graphs for which the addition of any edge causes the parameter under consideration to decrease. Such graphs will be referred to as critical (with respect to edge addition), and are the focus of this dissertation. Specifically, the hamiltonian properties of these critical graphs will be addressed, and for some families of critical graphs, a characterisation is given.

Section 1.2 will provide many of the definitions required in this dissertation. Other definitions specific to a particular section will be provided as needed.

For further discussion of basic graph theory not found here, the reader is referred to West [21]. If more background on the subject of domination theory is desired, the two volume reference by Haynes, Hedetniemi, and Slater [10, 11] is a good resource.

Section 1.3 provides an overview of what follows in each of the remaining chapters.

1.2 Definitions and Notation

All graphs $G = (V(G), E(G))$ will be simple, finite, and undirected. When it is clear from the context what graph is being discussed, the vertex set and edge set of G will be denoted simply by V and E, respectively.

For a vertex $v \in V$, the *neighbourhood* of v, $N(v)$, and *closed neighbourhood* of v, $N[v]$, are defined by $N(v) = \{w \in V : vw \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. We can also define the neighbourhood of a subset of V: If $S \subseteq V$, $N(S) = \{w \in V \mid w \in N(v),\ v \in S\}$ and $N[S] = N(S) \cup S$.

For any vertex $v \in V$, the *degree* of v, denoted $d(v)$, is the number of vertices adjacent to v, or simply $d(v) = |N(v)|$. The smallest (largest) value of $d(v)$, over all $v \in V$, is called the *minimum (maximum) degree of* G and is denoted $\delta(G)$ ($\Delta(G)$).

The *circumference* of a graph G, denoted $c(G)$, is the length of a longest cycle in G. A *Hamilton path (cycle)* in a graph G is a path (cycle) that passes through every vertex in V. A graph is called *hamiltonian* if it contains a Hamilton cycle, and hence if $c(G) = |V(G)|$.

A *connected graph* G is a graph for which there is a path from u to v (a *$u - v$ path*) for every pair of vertices $u, v \in V$. In general, G is called *k-connected* if $G - S$ is connected for any $S \subseteq V$ where $|S| < k$. The largest value of k for which

G is k-connected is called the *connectivity* of G, and is denoted $\kappa(G)$. Certainly if G is hamiltonian, then G is 2-connected. For a connected graph G, when $G-S$ is not connected for $S \subset V$, the set S is called a *vertex cut* of G. Furthermore, if $\{v\}$ is a vertex cut of G, then v is called a *cut vertex* of G. For any vertex set $S \subset V$, the number of components in $G - S$ is denoted by $\omega(G - S)$.

The subgraph of G induced by the nonempty vertex subset $S \subseteq V$ will be denoted by $G[S]$, or simply by $[S]$ when it is clear from the context what graph is being discussed.

A subset $D \subseteq V$ is called a *dominating set* of a graph G if for every $v \in V$, either $v \in D$ or v is adjacent to a vertex in D, that is, $N[D] = V$. The minimum cardinality of a dominating set in G is the *domination number* of G, $\gamma(G)$. An *independent set* in G is a set of pairwise nonadjacent vertices, and the *independence number* of G, $\beta(G)$, is the maximum cardinality of an independent set in G. A dominating set which is also an independent set is called an *independent dominating set.* The minimum cardinality of such a set in G is the *independent domination number* of G, $i(G)$. Lastly, a subset $D \subseteq V$ is called a *total dominating set* of a graph G if every vertex of G is adjacent to a vertex in D, or in other words, if $N(D) = V$. The minimum cardinality of a total dominating set in G is the *total domination number* of G, $\gamma_t(G)$. Note that $\gamma_t(G)$ is only defined when $\delta(G) > 0$. Since every independent dominating set is also a dominating set, and every total dominating set is also a dominating set, it is not hard to see that for any graph G in which all parameters are defined,

$$\gamma(G) \leq i(G) \leq \beta(G) \qquad \text{and}$$

$$\gamma(G) \leq \gamma_t(G).$$

When a pair of adjacent vertices u and v are discussed in the context of domination theory, it is often meaningful to say that u dominates v (or that v dominates u), and write $u \succ v$ (or $v \succ u$). Since a vertex u dominates every vertex in $N(u)$, we write $u \succ N(u)$. In general, for subsets S and T of V, if every vertex in T has a neighbour in S, we say that S dominates T and write $S \succ T$.

The graphs studied in this dissertation are those which are critical with respect to edge addition and one of the three graph parameters $\gamma, i,$ or γ_t. Specifically, if $e \notin E(G)$, then $\gamma(G+e) \leq \gamma(G)$. However, if $\gamma(G) = k$ and $\gamma(G+e) < k$ for every edge $e \notin E(G)$, then G is said to be *k-edge-γ-critical.* We say that G is *edge-γ-critical* if there exists k such that G is k-edge-γ-critical. Similarly, if $i(G) = k$ ($\gamma_t(G) = k$) and $i(G+e) < k$ ($\gamma_t(G+e) < k$) for every edge $e \notin E(G)$, then G is called *k-edge-i-critical* (*k-edge-γ_t-critical*). We say that G is *edge-i-critical* (*edge-γ_t-critical*) if there exists k such that G is k-edge-i-critical (k-edge-γ_t-critical). It is worth noting that for a graph G and $e \notin E(G)$, it may be that $i(G+e)$ is greater than, less than, or equal to $i(G)$. Also, it was first observed in [13] that $\gamma_t(G) - 2 \leq \gamma_t(G+e) \leq \gamma_t(G)$.

Since we discuss only graphs which are critical with respect to edge addition, the word *edge* will be dropped in each of the above definitions, for ease of notation. For example, a 3-edge-γ-critical graph will be referred to as a 3-γ-critical graph.

Various other notions of criticality of dominating sets are compared and contrasted in [2, 10, 19]. With regard to edge addition, there seems to be no general relationship between changes in $\gamma(G)$, changes in $i(G)$, and changes in $\gamma_t(G)$ resulting from joining a pair of non-adjacent vertices of G. Consequently, each of the families of graphs which are critical with respect to edge addition and the parameter γ, i, or γ_t, will be discussed separately.

1.3 An Overview

In the previous section, definitions were given for graphs which are critical with respect to edge addition and one of the graph parameters γ, i, or γ_t.

Several papers have been written on γ-critical graphs [10]. Chapter 2 is dedicated to summarizing some of the results that have been obtained for γ-critical graphs. The results are stated without proof, but are included because many of the results directly motivated the new results for i-critical and γ_t-critical graphs that will be developed in detail in Chapters 3 through 6.

Chapter 3 contains new results that have been obtained for i-critical graphs, and specifically, for 3-i-critical graphs. The hamiltonian properties of 3-i-critical graphs will be discussed, and a characterisation will be given for some families of 3-i-critical graphs.

Chapter 4 contains new results pertaining to γ_t-critical graphs. Results specific to 3-γ_t-critical graphs are given, and the 3-γ_t-critical graphs with a cut vertex are discussed in detail. As with the 3-i-critical graphs, the hamiltonian properties are discussed, and a characterisation is given.

In Chapters 5 and 6, all other 3-γ_t-critical graphs are discussed. Many families of 3-γ_t-critical graphs are characterised and shown to be hamiltonian.

Chapter 2

Domination Critical Graphs

This chapter provides a summary of many of the results that have previously been obtained for domination critical graphs, and in particular, those results that motivated the new results for independent domination critical graphs and total domination critical graphs which follow in Chapters 3 through 6. This dissertation does not contain any new results on domination critical graphs.

The study of domination critical graphs was initiated by Sumner and Blitch in [18]. Other results summarized here are taken from [7], [8], [17], and [22].

Section 2.1 contains general results on domination critical graphs, as well as results specific to 3-γ-critical graphs. Section 2.2 summarizes the results on the hamiltonian properties of 3-γ-critical graphs.

2.1 3-γ-critical Graphs

To begin the study of domination critical graphs, first consider the 1-γ-critical graphs. For any graph G, $\gamma(G) \geq 1$, so for a graph G that is not complete and has $\gamma = 1$, $\gamma(G + e) = 1$ for any $e \notin E$. Therefore, the 1-γ-critical graphs are precisely the complete graphs K_n, $n \geq 1$. The 2-γ-critical graphs are not much more difficult to characterise.

If G is a 2-γ-critical graph, then for any edge $uv \notin E(G)$, $\gamma(G + uv) = 1$. Without loss of generality, this means $u \succ V - \{v\}$ in G. Since this is true for any pair of nonadjacent vertices in G, every edge of $\overline{G}$ is incident with a vertex of degree one in $\overline{G}$. This gives the following characterisation of the 2-γ-critical graphs.

Theorem 2.1.1 *[18] A graph G is 2-γ-critical if and only if $\overline{G}$ is a disjoint union of nontrivial stars.*

Note that a nontrivial star is simply a complete bipartite graph of the form $K_{1,n}$, $n \geq 1$.

For $k \geq 3$, the problem of characterising the k-γ-critical graphs becomes much more complex. In fact, the problem of characterising the 3-γ-critical graphs is difficult.

The remainder of the results on domination critical graphs that will be given here are for 3-γ-critical graphs. Furthermore, all graphs will be assumed to be connected, since the disconnected 3-γ-critical graphs can be easily characterised:

Theorem 2.1.2 *[17] A disconnected graph G is 3-γ-critical if and only if either G is the disjoint union of a single vertex and a 2-γ-critical graph, or G is the disjoint union of a complete graph and a complete graph minus a perfect matching.*

A simple but important observation about 3-γ-critical graphs was first noted in [18]. The observation is that for any edge $uv \notin E(G)$, where G is a 3-γ-critical graph, there exists a vertex w in $V(G) - \{u, v\}$ such that either $\{u, w\}$ dominates every vertex in G except v, or $\{v, w\}$ dominates every vertex in G except u. Sumner and Blitch used the notation $[u, w] \to v$ and $[v, w] \to u$ to represent the two cases, respectively. This notation will be adopted here.

The first result in [18] proven for 3-γ-critical graphs follows. Analagous results will be shown to hold for 3-i-critical graphs and 3-γ_t-critical graphs in later chapters.

Lemma 2.1.3 *[18] Let G be a 3-γ-critical graph and I an independent set of $m \geq 4$ vertices. Then the vertices in I may be ordered as $x_1, x_2, \ldots, x_m$ in such a way that there exists a path $p_1, p_2, \ldots, p_{m-1}$ in $G - I$ with $[x_i, p_i] \to x_{i+1}$ for $i = 1, 2, \ldots, m - 1$.*

Arising directly from Lemma 2.1.3 are the following two results.

Lemma 2.1.4 *[18] If I is an independent set with $|I| = m$ in a connected 3-γ-critical graph G, then there exists $x \in I$ with $d(x) \geq m - 2$.*

Theorem 2.1.5 *[17] If G is a connected, 3-γ-critical graph, and S is a vertex cut in G, then $\omega(G - S) \leq |S| + 1$.*

The next result states a relationship between the independence number and the maximum degree of G, when G is a 3-γ-critical graph.

Theorem 2.1.6 *[17] If G is a 3-γ-critical graph, then $\beta(G) \leq \Delta(G)$.*

Furthermore, it is noted in [18] that there exist 3-γ-critical graphs with arbitrarily large independence number. In [17], a construction method is given, as well as the following result.

Theorem 2.1.7 *[17] For every $n \geq 3$, there exists a 3-γ-critical graph G with $3n$ vertices and $\beta(G) = n$.*

In [1], Allan and Laskar showed that if G has no induced $K_{1,3}$, then $\gamma(G) = i(G)$. Motivated by this, Sumner and Blitch conjectured that if G is a connected k-γ-critical graph, then $\gamma(G) = i(G)$. They proved the following:

Theorem 2.1.8 *[18] The diameter of a 3-γ-critical graph is at most three.*

Sumner and Blitch then proved their conjecture when $k = 3$ and $diam(G) = 3$. The conjecture, however, was shown not to be true by Ao, who constructed a 4-γ-critical graph G with $i(G) = 5$ [2]. The construction was later generalised in [3] to disprove the conjecture for $k \geq 4$, and finally disproven for all $k \geq 3$ in [12].

In addition to their conjecture, Sumner and Blitch also suggested that it is natural to ask when a k-γ-critical graph is hamiltonian. This suggestion motivated many papers on the topic, and related results are summarised in Section 2.2.

2.2 Hamiltonian properties of 3-γ-critical graphs

Sumner conjectured that every 3-γ-critical graph on more than 6 vertices has a Hamilton path. Wojcicka proved this conjecture in [22], making use of several results from [17] and [18] to prove that all 3-γ-critical graphs with at most two vertices of degree one have a Hamilton path. The 3-γ-critical graphs with at most one vertex of degree one are handled separately. First, the following results are shown. A *dominating cycle (dominating path)* is a cycle (path) whose vertices form a dominating set.

Theorem 2.2.1 *[22] If G is a connected 3-γ-critical graph, then G has a dominating cycle.*

Corollary 2.2.2 *[22] If G is a connected 3-γ-critical graph, then G has a dominating path.*

Since a connected 3-γ-critical graph G contains a dominating path, it must contain a longest such path. By showing that the longest dominating path in G is actually a Hamilton path, the main result of [22] is obtained:

Theorem 2.2.3 *[22] If G is a connected 3-γ-critical graph on more than 6 vertices, then G has a Hamilton path.*

As well as proving Sumner's conjecture, Wojcicka concluded with a conjecture of her own:

Conjecture 2.2.4 *[22] If G is a connected 3-γ-critical graph with $\delta(G) \geq 2$, then G is hamiltonian.*

In [8], some results are obtained regarding the structure of 3-γ-critical graphs. The first result requires the definition of a 1-tough graph: A graph G is said to be *t-tough* if for every vertex cut $S \subset V(G)$, $|S| \geq t\omega(G - S)$.

Theorem 2.2.5 *[8] Let G be a connected 3-γ-critical graph with $\delta(G) \geq 2$. Then G is 1-tough.*

Consequently, if G is a connected 3-γ-critical graph with $\delta(G) \geq 2$, then G does not have a cut vertex, and G is 2-connected.

Flandrin, Tian, Wei, and Zhang then used Theorem 2.2.5 and other results to prove the next Theorem involving the circumference, $c(G)$, of a 3-γ-critical graph.

Theorem 2.2.6 *[8] Let G be a connected 3-γ-critical graph with n vertices and C be a longest cycle of G. If there exists a vertex u of $G - V(C)$ that is adjacent to at least 2 vertices in C, then $c(G) \geq n - 1$. In particular, if G is 2-connected, then $c(G) \geq n - 1$.*

Favaron, Tian, and Zhang [7] then used Theorems 2.2.5 and 2.2.6 to tackle Wojcicka's conjecture. They do this by conditioning on the size of a maximum independent set in a 3-γ-critical graph. The following results were found.

Theorem 2.2.7 *[7] The independence number β of a 3-γ-critical graph G with minimum degree $\delta \geq 2$ satisfies $\beta \leq \delta + 2$. Moreover, if $\beta = \delta + 2$, then every maximum independent set contains every vertex of degree δ.*

Theorem 2.2.8 *[7] Every 3-γ-critical graph G of minimum degree $\delta \geq 2$ and independence number $\beta = \delta + 2$ satisfies $\beta(G) = \gamma(G)$.*

Theorem 2.2.9 *[7] Every 3-γ-critical graph with $\delta \geq 2$ and $\beta \leq \delta + 1$ is hamiltonian.*

In [20], Tian, Wei, and Zhang extended these results by working on the 3-γ-critical graphs with $\delta \geq 2$ that satisfy $\beta = \delta + 2$. They proved the following results.

Theorem 2.2.10 *[20] Let G be a 3-γ-critical graph with $\delta \geq 2$ and $\beta = \delta + 2$. Then G has only one vertex with degree δ.*

Theorem 2.2.11 *[20] Every 3-γ-critical graph with $\delta \geq 2$ and $\beta = \delta + 2$ is hamiltonian.*

Theorem 2.2.11, together with Theorem 2.2.9, completes the proof of the conjecture of Wojcicka:

Theorem 2.2.12 *[7, 20] If G is a connected 3-γ-critical graph with $\delta(G) \geq 2$, then G is hamiltonian.*

It will be shown in later chapters that analogous results to Theorems 2.2.7 through 2.2.10 hold for 3-γ_t-critical graphs.

When studying the hamiltonian properties of graphs, a well-known result of Ore [16] that gives sufficient conditions for a graph to be hamiltonian is often referred to:

Theorem 2.2.13 *Let G be a graph on n vertices. If $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices x and y, then G is hamiltonian.*

Bondy and Chvátal generalized Ore's theorem by defining the *closure* of a graph G to be the graph $cl(G)$ obtained from G by recursively joining nonadjacent vertices whose degrees sum to at least n. They proved that a graph G is hamiltonian if and only if its closure is hamiltonian.

Prior to Wojcicka's conjecture being proven, Hanson [9] defined a new closure concept that could be useful in the study of hamiltonian properties of 3-γ-critical graphs. It involves adding an edge uv to G whenever $\{u, v\} \succ V - \{w\}$ for some w with $d(w) \geq 3$. The obtained graph is denoted by $D^*(G)$. Hanson obtained the following result for 3-γ-critical graphs.

Theorem 2.2.14 *[9] Let G be a 2-connected 3-γ-critical graph. Then G is hamiltonian if and only if $D^*(G)$ is hamiltonian.*

An important consequence of using a closure is that often the obtained graph has large enough minimum degree to conclude that it is hamiltonian.

Although Hanson's closure concept was not used in the proof of Theorem 2.2.12, it motivated the new closures that will be defined for 3-i-critical and 3-γ_t-critical graphs in Chapters 3 and 4 which prove valuable in the study of the hamiltonian properties of these graphs.

Chapter 3

Independent Domination Critical Graphs

This chapter contains the following results for 3-i-critical graphs G:

(i) If G is 2-connected with $\delta \geq 3$, then G is hamiltonian (Corollary 3.1.4).

(ii) There is exactly one family of 2-connected non-hamiltonian graphs with $\delta = 2$ (Theorem 3.3.6).

(iii) If G is connected and $|V| > 6$, then G has a Hamilton path (Theorem 3.3.7).

(iv) A characterisation is given of the 2-connected, 3-i-critical graphs with $\delta = 2$ (Theorems 3.2.9 and 3.2.10), and

(v) A characterisation is given of the 3-i-critical graphs with a cut vertex (Theorems 3.2.6 - 3.2.8).

First, using a closure similar to the one developed by Hanson, we show that every 2-connected 3-i-critical graph with minimum degree at least three has a Hamilton cycle. We characterise the 2-connected, 3-i-critical graphs with $\delta = 2$, and determine which of these are hamiltonian. By combining these results with a complete characterisation of the 3-i-critical graphs with a cut-vertex, we establish that any connected 3-i-critical graph with more than six vertices has a Hamilton path.

3.1 Minimum degree at least three

In this section we prove that every 2-connected, 3-i-critical graph with $\delta \geq 3$ is hamiltonian. The main tool is a domination closure similar to that found in [9].

Let G be a 3-i-critical graph. If $uv \notin E(G)$, then $i(G + uv) = 2$ implies that there exists a vertex $x \notin N(u) \cup N(v)$ such that either $\{u, x\}$ dominates $G - v$ or $\{v, x\}$ dominates $G - u$. In the former case we write $[u, x] \rightarrow v$, and in the latter case we write $[v, x] \rightarrow u$.

Theorem 3.1.1 *Let G be a 2-connected graph. If $[u, v] \rightarrow w$ for some vertices u, v and w with $d(w) \geq 3$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Proof: First note that if G is hamiltonian then $G + uv$ is obviously also hamiltonian.

Now consider vertices u, v, and w such that $[u, v] \rightarrow w$ and $d(w) \geq 3$. Suppose $G+uv$ is hamiltonian while G is not. Then, G contains a Hamilton path $v_1 v_2 \dots v_n$ from $u = v_1$ to $v = v_n$ where $n = |V|$, and $N[v_1] \cup N[v_n] = V - \{v_p\}$ with $v_p = w$. Define $M = max\{i : v_1 v_i \in E\}$ and $m = min\{j : v_j v_n \in E\}$. If $v_1 v_i \in E$, then $v_n v_{i-1} \notin E$, otherwise $v_1 v_2 \dots v_{i-1} v_n v_{n-1} \dots v_i v_1$ is a Hamilton cycle in G. Therefore neither the case $p < m < M$ nor the case $m < M < p$ is possible. The remaining cases are $M \leq m < p$ (or $p < M \leq m$), $M < p < m$, and $m < p < M$.

Case 1: $M \leq m < p$.

Since $[v_1, v_n] \rightarrow v_p$, it follows that v_1 dominates $\{v_1, v_2, \dots, v_M\}$, v_n dominates $\{v_m, v_{m+1}, \dots, v_n\} - \{v_p\}$, and therefore $M \leq m \leq M + 1$.

We will first prove that $v_i v_j \notin E$ for all i and j, where $1 \leq i < M$ and $m < j \leq n$. Consider i and j such that $1 \leq i < M$ and $m < j \leq n$. If $v_i v_j \in E$, certainly $i \neq 1$ and $j \neq n$. If $j - 1 \neq p$, then

$$v_1 v_2 \dots v_i v_j v_{j+1} \dots v_n v_{j-1} v_{j-2} \dots v_{i+1} v_1$$

is a Hamilton cycle in G. Hence, assume $j - 1 = p$, that is, $j = p + 1$. We will obtain a contradiction by showing that v_p can have no neighbours other than $v_{p-1} = v_j$ and v_{p+1}. Suppose there exists $k \notin \{p-1, p+1\}$ such that $v_k v_p \in E$.

If $1 \le k \le M-1$, then

$$v_1 v_2 \ldots v_k v_p v_{p+1} \ldots v_n v_{p-1} v_{p-2} \ldots v_{k+1} v_1$$

is a Hamilton cycle in G. If $M \le k \le p-2$, then

$$v_1 v_2 \ldots v_i v_{p+1} v_{p+2} \ldots v_n v_{k+1} v_{k+2} \ldots v_p v_k v_{k-1} \ldots v_{i+1} v_1$$

is a Hamilton cycle in G. Finally, if $p+2 \le k \le n$, then

$$v_1 v_2 \ldots v_i v_{p+1} v_{p+2} \ldots v_{k-1} v_n v_{n-1} \ldots v_k v_p v_{p-1} \ldots v_{i+1} v_1$$

is a Hamilton cycle in G. Hence $d(v_p) = 2$, a contradiction. Therefore $v_i v_j \notin E$ for all i and j where $1 \le i < M$ and $m < j \le n$.

Now, since G is 2-connected, $m \ne M$. Thus, $m = M+1$. Since v_M and v_m are not cut-vertices, there must exist i and j with $1 \le i < M < m < j \le n$, such that $v_i v_m \in E$ and $v_M v_j \in E$. If $j - 1 \ne p$, then G contains the Hamilton cycle

$$v_1 v_2 \ldots v_i v_m v_{m+1} \ldots v_{j-1} v_n v_{n-1} \ldots v_j v_M v_{M-1} \ldots v_{i+1} v_1.$$

Hence, assume $j - 1 = p$. We will now obtain a contradiction by showing that v_p can have no neighbours other than v_{p-1} and $v_{p+1} = v_j$. Suppose $v_k v_p \in E$ for some $k \notin \{p-1, p+1\}$. If $1 \le k \le M-1$, then

$$v_1 v_2 \ldots v_k v_p v_{p+1} \ldots v_n v_{p-1} v_{p-2} \ldots v_{k+1} v_1$$

is a Hamilton cycle in G. If $k = M$, then G contains the Hamilton cycle

$$v_1 v_{i+1} v_{i+1} \dots v_M v_p v_{p+1} \dots v_n v_{p-1} v_{p-2} \dots v_m v_i v_{i-1} \dots v_1.$$

If $m \leq k \leq p-2$, then

$$v_1 v_2 \dots v_i v_m v_{m+1} \dots v_k v_p v_{p-1} \dots v_{k+1} v_n v_{n-1} \dots v_{p+1} v_M v_{M-1} \dots v_{i+1} v_1$$

is a Hamilton cycle in G. Finally, if $p+2 \leq k \leq n$, then

$$v_1 v_2 \dots v_i v_m v_{m+1} \dots v_p v_k v_{k+1} \dots v_n v_{k-1} v_{k-2} \dots v_{p+1} v_M v_{M-1} \dots v_{i+1} v_1$$

is a Hamilton cycle in G. It follows that $d(v_p) = 2$, a contradiction.

Since the case where $p < M \leq m$ is symmetrical to the above, the proof of Case 1 is complete.

Case 2: $M < p < m$.

In this case we must have $m = M + 2$. Also, $v_i v_j \notin E$ for all i and j such that $1 \leq i < M < m < j \leq n$, otherwise

$$v_1 v_2 \dots v_i v_j v_{j+1} \dots v_n v_{j-1} v_{j-2} \dots v_{i+1} v_1$$

is a Hamilton cycle in G. Since $d(v_p) \geq 3$, by symmetry we can assume that $v_k v_p \in E$ for some k with $1 < k < M$.

Suppose $v_p v_q \notin E$ for all q such that $m < q < n$. Then since v_m is not a cut-vertex, $v_M v_j \in E$ for some j with $m < j < n$. However, this implies

$$v_1 v_2 \dots v_k v_p v_m v_{m+1} \dots v_{j-1} v_n v_{n-1} \dots v_j v_M v_{M-1} \dots v_{k+1} v_1$$

is a Hamilton cycle in G, a contradiction. Therefore, $v_p v_q \in E$ for some q with $m < q < n$.

Since v_p is not a cut-vertex, $v_i v_m \in E$ for some i with $1 < i \leq M$. But now, if $i < M$ then

$$v_1 v_2 \ldots v_i v_m v_{m+1} \ldots v_{q-1} v_n v_{n-1} \ldots v_q v_p v_{p-1} \ldots v_{i+1} v_1$$

is a Hamilton cycle of G, and if $i = M$ then

$$v_1 v_2 \ldots v_k v_p v_q v_{q+1} \ldots v_n v_{q-1} v_{q-2} \ldots v_m v_M v_{M-1} \ldots v_{k+1} v_1$$

is a Hamilton cycle of G, a contradiction.

Case 3: $m < p < M$.

In this case we must have $N(v_1) \supseteq \{v_2, v_3, \ldots, v_{m-1}, v_{p+1}, v_{p+2}, \ldots, v_M\}$, and $N(v_n) \supseteq \{v_m, v_{m+1}, \ldots, v_{p-1}, v_{M+1}, v_{M+2}, \ldots, v_{n-1}\}$.

As in the previous cases, we obtain a contradiction by showing that v_p can have no neighbours other than v_{p-1} and v_{p+1}. Suppose there exists $k \notin \{v_{p-1}, v_{p+1}\}$ such that $v_k v_p \in E$. If $1 \leq k \leq m-2$, then

$$v_1 v_2 \ldots v_k v_p v_{p+1} \ldots v_n v_{p-1} v_{p-2} \ldots v_{k+1} v_1$$

is a Hamilton cycle in G. If $m \leq k \leq p-2$, then

$$v_1 v_2 \ldots v_k v_p v_{p-1} \ldots v_{k+1} v_n v_{n-1} \ldots v_{p+1} v_1$$

is a Hamilton cycle in G. If $k = m-1$, then

$$v_1v_2\dots v_{m-1}v_pv_{p-1}\dots v_mv_nv_{n-1}\dots v_{p+1}v_1$$

is a Hamilton cycle in G. By symmetry, if $p+2 \leq k \leq n$, then G contains a Hamilton cycle. Hence $d(v_p) = 2$, a contradiction.

Since all cases lead to a contradiction, we conclude that if $G + uv$ is hamiltonian, then G is hamiltonian. This completes the proof. ∎

If G is 2-connected, then we define the *domination closure* of G, denoted by $D^*(G)$, to be G together with all edges uv of $\overline{G}$ where u, v are such that, in G, $[u,v] \to w$ for some vertex w with $d(w) \geq 3$.

Corollary 3.1.2 *If G is 2-connected, then $D^*(G)$ is hamiltonian if and only if G is hamiltonian.*

Proof: If G is hamiltonian, then certainly $D^*(G)$ is hamiltonian.

Suppose the converse is false and choose a minimal subset $\{e_1, e_2, \dots, e_k\}$ of $E(D^*(G)) - E(G)$ such that $G + \{e_1, e_2, \dots, e_k\}$ has a Hamilton cycle but $G' = G + \{e_1, e_2, \dots, e_{k-1}\}$ does not. By Theorem 3.1.1, $k \geq 2$. Let $e_k = xy$. Then G' has a Hamilton path $P = v_1v_2\dots v_n$, where $x = v_1$ and $y = v_n$.

Since $e_k = xy$, $[x,y] \to w$ for some w with $d(w) \geq 3$ in G. By the minimality of $\{e_1, e_2, \dots, e_k\}$, all edges $e_1, e_2, \dots, e_{k-1}$ must be in P. If neither xw nor yw

are edges in P, then $[x,y] \to w$ in G' (as well as in G). But G' is 2-connected (since G is a subgraph of G'), and $G'+xy$ is hamiltonain, so Theorem 3.1.1 gives that G' is hamiltonian, a contradiction.

Therefore, without loss of generality, suppose xw is in P, that is, $w = v_2$ and $[x,w] \to y$. Consider the path $w = v_2v_3 \dots v_n = y$ in G'. If there exists k where $3 \leq k \leq n$ and $v_1v_k, v_2v_{k+1} \in E(G)$, then

$$v_2v_{k+1}v_{k+2}\cdots v_nv_1v_kv_{k-1}\cdots v_2$$

is a Hamilton cycle in $G + \{e_2, e_3, \dots, e_k\}$, a contradiction. Therefore, since $[x,w] \to y$, there exists p such that $N_{G'}(v_2) = \{v_3, v_4, \dots, v_p\} \cup \{v_1\}$ and $N_{G'}(v_1) = \{v_{p+1}, v_{p+2}, \dots, v_n\} \cup \{v_2\}$. But $[x,y] \to w$ gives $v_nv_p \in E(G)$, and hence

$$v_1v_{p+1}v_{p+2}\cdots v_nv_pv_{p-1}\cdots v_1$$

is a Hamilton cycle in G', a contradiction. The result now follows. ∎

We write $d^*(x)$ for the degree of vertex x in $D^*(G)$.

Theorem 3.1.3 *If G is a 2-connected, 3-i-critical graph with $\delta(G) \geq 3$, then $D^*(G)$ is hamiltonian.*

Proof: Let w be any vertex of G, and let $\overline{N}(w) = V(G) - N[w]$. Define the sets A_w and B_w by

$$A_w = \{x \in \overline{N}(w) : \exists y \in \overline{N}(w) \text{ s.t. } [x, y] \to w\}, \text{ and}$$

$$B_w = \overline{N}(w) - A_w.$$

Since G is 3-i-critical, for any $x \in B_w$ there exists y such that, in G, $[w, y] \to x$. This implies that y dominates every vertex in A_w, so $y \in B_w$. Hence, for each $x \in B_w$ there is a $y \in B_w$ so that $[w, y] \to x$. Furthermore, $[w, x] \to y$, and hence $[\overline{B_w}]$ is a matching and $wx \in E(D^*(G))$ for all $x \in B_w$. Now, for $x \in A_w$, there exists $y \in A_w$ such that $[x, y] \to w$, and hence $xy \in E(D^*(G))$.

From the above argument, it follows that each vertex of $\overline{N}(w)$ is incident with at least one edge of $D^*(G)$ which is not an edge of G. We call each of these edges a *new edge* with respect to w.

Consider now any edge in $E(D^*(G)) - E(G)$, say xy. Since $[x, y] \to w$ for some w, xy is a new edge with respect to w, x, and y (and no other vertex).

Each vertex x is in $\overline{N}(w)$ for $|\overline{N}(x)| = |V(G)| - |N[x]| = |V(G)| - d(x) - 1$ vertices w. Each of these choices for w leads to a new edge incident with x, and each new edge arises from exactly two choices of w (a new edge incident with x arrises from an independent dominating set $\{x, y, z\}$ and is new with respect to y and z). Thus,

$$d^*(x) \geq d(x) + |\overline{N}(x)|/2 = d(x) + |V(G)|/2 - d(x)/2 - 1/2 \geq |V(G)|/2.$$

Hence, by Dirac's theorem, $D^*(G)$ is hamiltonian. ∎

Corollary 3.1.4 *If G is 2-connected and 3-i-critical with $\delta \geq 3$, then G is hamiltonian.*

Proof: By Theorem 3.1.3, $D^*(G)$ is hamiltonian. Thus, by Corollary 3.1.2, G is also hamiltonian. ∎

3.2 Characterisations

In this section we give a complete description of the 3-i-critical graphs with a cut-vertex, and a complete description of the 2-connected, 3-i-critical graphs with $\delta = 2$.

The following three lemmata are similar to work found in [17, 18].

Lemma 3.2.1 *Let G be a 3-i-critical graph and I be an independent set in G with size $m \geq 4$. Then, the vertices in I may be ordered as $x_1, x_2, \ldots, x_m$ in such a way that there exists a path $p_1 p_2 \ldots p_{m-1}$ in $G - I$ with $[x_i, p_i] \rightarrow x_{i+1}$ for $i = 1, 2, \ldots, m-1$.*

Proof: We first define an orientation of the edges of $\overline{G}$ as follows: For any $uv \notin E(G)$, there exists x such that either $[u, x] \rightarrow v$ or $[v, x] \rightarrow u$ (or both). If $[u, x] \rightarrow v$, then orient uv as (u, v), and otherwise orient it as (v, u). (If both conditons hold, then the orientation (u, v) is chosen.) Since I is an independent set in G, this orientation of $\overline{G}$ induces a tournament with vertex set I. Since every tournament has a directed Hamilton path, we may label the vertices in I as $x_1, x_2, \ldots, x_m$ as determined by the Hamilton path. Hence, for $i = 1, 2, \ldots, m-1$, there exists p_i such that $[x_i, p_i] \rightarrow x_{i+1}$.

We claim that $p_1 p_2 \ldots p_{m-1}$ is the required path. Since $|I| = m \geq 4$, the vertex p_i dominates at least two vertices in I, and thus $p_i \notin I$. Furthermore, the vertices $p_1, p_2, \ldots, p_{m-1}$ are distinct: if $j \neq i, i+1$, then $p_j x_{i+1} \in E(G)$ and $p_i x_{i+1} \notin E(G)$, so $p_j \neq p_i$. Suppose $1 \leq i \leq m-2$. Then $p_i x_{i+2} \in E(G)$ and $p_{i+1} x_{i+2} \notin E(G)$, so $p_i \neq p_{i+1}$. Finally, for $i = 2, 3, \ldots, m-1$, we have that $p_{i-1} x_i \notin E(G)$ and $[x_i, p_i] \rightarrow x_{i+1}$, so $p_{i-1} p_i \in E(G)$. This proves the claim. ∎

Arising directly from Lemma 3.2.1 is the following result.

Lemma 3.2.2 *Let G be a connected 3-i-critical graph. If I is an independent set in G with $|I| = m$, then there exists $x \in I$ with $d(x) \geq m - 2$.*

Lemma 3.2.3 *If G is 3-i-critical, then no vertex of G has two neighbours of degree one.*

Proof: Suppose that z is adjacent to vertices u and v of degree one. Since $i = 3$ and $d(u) = 1$, there exists w such that $wz \notin E$ and $wu \notin E$. Hence there exists x such that either $[u, x] \rightarrow w$, or $[w, x] \rightarrow u$. In either case, v must be dominated, so $x = v$. If $[u, v] \to w$, then $V = \{u, v, z, w\}$ and $E = \{uz, vz\}$, contradicting $i = 3$. Therefore $[v, w] \to u$. Now, since $zw \notin E$, $\{z, w\}$ is an independent dominating set of size two, a contradiction. ∎

Lemma 3.2.4 *If G is a 3-i-critical graph and $S \subseteq V(G)$ is a vertex cut, then $\omega(G - S) \leq |S| + 1$.*

Proof: First suppose that $S = \{v\}$ and that $G - v$ has components C_1, C_2, C_3 (and possibly others). Then, by Lemma 3.2.3, at most one of these can be trivial. Assume that $|V(C_1)| \geq 2$ and $|V(C_2)| \geq 2$. For $i = 1, 2$, let $x_i \in V(C_i)$ be adjacent to v. Since $x_1x_2 \notin E$, without loss of generality we may assume that there exists $x \in V(G)$ such that $[x_1, x] \to x_2$. Since $x_1x \notin E$, the vertex $x \neq v$ and hence $x \in V(C_3)$. But no $u \in V(C_2) - \{x_2\} \neq \emptyset$, is dominated by $\{x_1, x\}$, a contradiction. Therefore, $G - v$ has at most $|S| + 1 = 2$ components.

Now assume that $|S| = n \geq 2$ and $G - S$ has components $A_1, A_2, \ldots, A_{n+2}$ (and possibly others). Let $I = \{x_1, x_2, \ldots, x_{n+2}\}$ be an independent set in G, where $x_i \in V(A_i)$ for $i = 1, 2, \ldots, n+2$. Then, $|I| = n + 2 \geq 4$. We may assume that the vertices in I are ordered as in Lemma 3.2.1, and $p_1p_2 \ldots p_{n+1}$ is a path in $G - I$ such that $[x_i, p_i] \to x_{i+1}$ for $i = 1, 2, \ldots, n+1$. Then, for $i = 1, 2, \ldots, n+1$, since x_i dominates only vertices in A_i and in S, we must have $p_i \in S$. But the vertices $p_1, p_2, \ldots, p_{n+1}$ are distinct, so this contradicts $|S| = n$. ∎

Lemma 3.2.5 *Let G be a connected, 3-i-critical graph. If v is a cut vertex of G, then $G - v$ has exactly two components C_1 and C_2, with C_1 complete and $i(C_2) = 2$.*

Proof: By Lemma 3.2.4, $G - v$ has exactly two components, C_1 and C_2. For $i = 1, 2$, let $x_i \in V(C_i)$ be adjacent to v. Since $x_1x_2 \notin E$, there exists x such that $[x_1, x] \to x_2$ or $[x, x_2] \to x_1$. In either case, the choice of x_1 and x_2 implies that $\{x, x_1, x_2\}$ is an independent dominating set not containing v. Without loss of generality, $x \in V(C_2)$, so that $i(C_1) = 1$ and $i(C_2) \leq 2$. Suppose there exists a vertex $z \in V(C_2)$ that dominates C_2. Then, since x_1 dominates $V(C_1) \cup \{v\}$ and $x_1x \notin E$, the set $\{x_1, z\}$ is an independent dominating set of G with size two, a contradiction. This gives $i(C_2) = 2$.

If $|V(C_1)| = 1$, certainly C_1 is complete. Otherwise, suppose $x, y \in V(C_1)$ and $xy \notin E$. Then there exists z such that (in G) either $[x, z] \to y$ or $[y, z] \to x$. Since $i(C_2) = 2$, $z = v$ and v dominates C_2. Now, for any $u \in V(C_2)$, since $xu \notin E$, there exists w such that either $[x, w] \to u$ or $[u, w] \to x$. Since $i(C_2) = 2$ and v dominates C_2, $w \in V(C_2)$. But then y can not be dominated by $\{x, w, u\}$, a contradiction. Therefore C_1 is complete. ∎

Let G and H be disjoint graphs. The *join* of G and H is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$.

The join of $n \geq 3$ vertex-disjoint graphs $G_1, G_2, \ldots, G_n$ is recursively defined to be the graph $G_1 + G_2 + \cdots + G_n = (G_1 + G_2 + \cdots + G_{n-1}) + G_n$.

Let $\mathcal{Q}_{n,p}$ $(p \geq n)$ denote the set of graphs $Q_{n,p}$ on $n + p$ vertices that can be obtained from $K_p \cup \overline{K}_n$ by adding, for each vertex $v \in K_p$, an edge from v to any one vertex in $\overline{K}_n$, such that $Q_{n,p}$ has no isolated vertices.

We use $T_{rn,n}$ to denote the complete n-partite graph in which each part contains r vertices.

For distinct vertices u and v, we use $N(u) \oplus N(v)$ to denote the symmetric difference of the neighbourhoods of u and v.

Theorem 3.2.6 *Let G be a connected 3-i-critical graph, and let v be a cut vertex of G. Then, $G - v$ has exactly two components C_1 and C_2 such that C_1 is complete and $i(C_2) = 2$. Furthermore, if $|V(C_1)| \geq 2$, then the following hold:*

1. *$C_2 = T_{2n,n}$ for some $n \geq 2$, and*

2. *for every $x \in V(C_1)$, $vx \in E$, and for any pair u, u' of nonadjacent vertices of C_2, $vu \in E$ if and only if $vu' \in E$.*

Proof: Since v is a cut vertex of G, by Lemma 3.2.5, $G - v$ has two components C_1 and C_2 with C_1 complete and $i(C_2) = 2$.

We first prove that $C_2 = \overline{K}_2 + \overline{K}_2 + \cdots + \overline{K}_2 = T_{2n,n}$. Let $x \in V(C_1)$ be adjacent to v. For every $u \in V(C_2)$, since $xu \notin E$, there exists u' such that either

$[u,u'] \to x$ or $[x,u'] \to u$. Since C_1 is complete and $xv \in E$, $u' \in V(C_2)$, and since $|V(C_1)| \geq 2$, $[x,u'] \to u$. So, $N[u'] \supseteq V(C_2) - \{u\}$. Now, since $xu' \notin E$, there exists w such that $[x,w] \to u'$ or $[u',w] \to x$. As above, $[x,w] \to u'$ and $w \in V(C_2)$, so $w = u$ and $N[u] \supseteq V(C_2) - \{u'\}$. Therefore, for every $u \in V(C_2)$, there exists $u' \in V(C_2)$ such that u and u' are both adjacent to every vertex of $V(C_2) - \{u,u'\}$. Hence $C_2 = \overline{K}_2 + \overline{K}_2 + \cdots + \overline{K}_2 = T_{2n,n}$ for some $n \geq 1$.

To prove statement 2, consider $u \in V(C_2)$ with $vu \notin E$. Then there exists y such that either $[u,y] \to v$ or $[v,y] \to u$. If $[u,y] \to v$, then $y \in V(C_1)$, and hence u dominates C_2, a contradiction. Thus $[v,y] \to u$. If $y \in V(C_1)$, then $yv \notin E$ and there exists w such that $[y,w] \to v$ or $[v,w] \to y$. In the first case, $w \in V(C_2)$ and w dominates C_2, a contradiction. In the second case, since C_1 is complete and $[v,y] \to u$, $w \in V(C_2)$, and $w = u$. Now, for the unique non-neighbour of u in C_2, u', since $yu' \notin E$, there exists z such that either $[z,y] \to u'$ or $[z,u'] \to y$. In both cases, $z = u$ (as y dominates C_1, u' dominates $C_2 - u$, and $u'v \in E$). But, since $vu, vy \notin E$, it is not possible that $[u,y] \to u'$. Therefore, $[u,u'] \to y$ and $V(C_1) = \{y\}$, contrary to $|V(C_1)| \geq 2$. Hence $y \in V(C_2)$. Since u' is the only vertex of C_2 not adjacent to u, $y = u'$ and $u'v \notin E$. Thus v is adjacent to either both u and u', or neither u nor u'.

Suppose v dominates $V(C_2)$. Then either $\{v\}$ is an independent dominating set of size one in G, or there exists $x \in V(C_1)$ such that $\{x,v\}$ is an independent dominating set of size two in G. Both cases contradict $i(G) = 3$, and therefore there exists a pair of vertices u and u' in $V(C_2)$ such that $[u, v] \rightarrow u'$. It follows that $vx \in E$ for every $x \in V(C_1)$, completing the proof of statement 2. Furthermore, since v does not dominate C_2, and G is connected, it follows that $C_2 = T_{2n,n}$ for some $n \geq 2$, completing the proof of statement 1. ∎

Corollary 3.2.7 *Let G be a connected 3-i-critical graph, and let v be a cut vertex of G. Let C_1 and C_2 denote the two components of $G - v$, where C_1 is complete and $i(C_2) = 2$. Then $|V(C_1)| \geq 2$ if and only if $\delta(G) \geq 2$.*

Proof: If $\delta(G) \geq 2$, certainly $|V(C_1)| \geq 2$. Conversely, suppose $|V(C_1)| \geq 2$. Then, by Theorem 3.2.6, $C_2 = T_{2n,n}$ for $n \geq 2$, and hence $d(x) \geq 2$ for all $x \in V(C_2)$. Also by Theorem 3.2.6, $vx \in E$ for every $x \in V(C_1)$, and hence $d(x) \geq 2$. Furthermore, $d(v) \geq 2$. Therefore $\delta(G) \geq 2$. ∎

Theorem 3.2.8 *Let G be a connected 3-i-critical graph with $\delta = 1$, and let v be a cut vertex of G. Then, $G - v$ has exactly two components C_1 and C_2 such that $|V(C_1)| = 1$ and $i(C_2) = 2$. Furthermore,*

1. *$C_2 = S_1 + S_2 + \cdots + S_m$, where $S_j = \overline{K}_2$ or a graph $Q_{2,p} \in \mathcal{Q}_{2,p}$, $1 \leq j \leq m$, and*

2. there exist nonadjacent vertices $u, u' \in V(C_2) - N[v]$ *such that*

(a) $N[u] \cup N[u'] = V(C_2)$, *and*

(b) for all $z \in N(u) \oplus N(u')$, $vz \in E$ *and* $N[z] \supseteq V(C_2) - \{u, u'\}$.

Proof: Since v is a cut vertex of G, by Lemma 3.2.5, $G - v$ has exactly two components C_1 and C_2, with C_1 complete and $i(C_2) = 2$. Furthermore, by Corollary 3.2.7, $|V(C_1)| = 1$.

Let $V(C_1) = \{x\}$. Then, since G is connected, $vx \in E$. Since $i(G) = 3$, there exists $u \in V(C_2)$ with $vu \notin E$, and u' such that $[v, u'] \to u$ or $[u, u'] \to v$. But $x \notin N[u] \cup N[u']$, so it must be that $[v, u'] \to u$, where $u' \in V(C_2)$. Furthermore, the fact that $vu' \notin E$ implies there exists y such that $[v, y] \to u'$ or $[u', y] \to v$. By the same reasoning, $[v, y] \to u'$, where $y \in V(C_2)$. Now, since $[v, u'] \to u$, it must be that $y = u$. Hence $[v, u] \to u'$ and $[v, u'] \to u$. We now prove statements (a) and (b) hold for these vertices.

Suppose there exists $w \in V(C_2) - N[u] \cup N[u']$. Then, from $wu \notin E$, there exists y such that $[w, y] \to u$ or $[u, y] \to w$. Since x must be dominated, y must be v or x. But $u' \notin N[w] \cup N[v] \cup N[u]$, a contradiction. Hence, (a) holds.

To prove (b), we first define S_1. Let $A_1 = N(u) \oplus N(u')$. If $A_1 = \emptyset$, then $S_1 = G[\{u, u'\}] = \overline{K}_2$. Otherwise, there exists $z \in A_1$. Consider $z \in N(u)$ such that $z \notin N(u')$. From the fact that $[v, u'] \to u$, $vz \in E$. Now, from $zu' \notin E$, we have t such that $[z,t] \to u'$ or $[u',t] \to z$. Since $vz \in E$ and $u'u \notin E$, we must have $t = x$. Thus $[z,x] \to u'$ and $N[z] = (V(C_2) - \{u'\}) \cup \{v\}$. Similarly, if $z \in N(u')$ and $z \notin N(u)$, then $N[z] = (V(C_2) - \{u\}) \cup \{v\}$. Therefore $G[A_1]$ is complete. In this case ($A_1 \neq \emptyset$), set $S_1 = G[A_1 \cup \{u,u'\}]$.

We now show that $S_1 \in \mathcal{Q}_{2,p}$. Suppose $A_1 \subseteq N(u)$ and $z \in A_1$. Then, $xu \notin E$ implies there exists y such that either $[x,y] \to u$ or $[u,y] \to x$. If $[x,y] \to u$, then $y \neq v$ as $xv \in E$, and $y \notin N(u)$ gives $y = u'$. But xz, $u'z \notin E$, a contradiction. Thus $[u,y] \to x$. Since $xv \in E$, $y \neq v$. Furthermore, since $uv \notin E$, $y \in N(v)$. But $vu' \notin E$ gives $y \neq u'$, and hence $y \notin N(u') - N(u)$, a contradiction. Therefore, $A_1 \not\subseteq N(u)$, and $S_1 \in \mathcal{Q}_{2,p}$.

Let $B_1 = N(u) \cap N(u')$. If $B_1 = \emptyset$, then $C_2 = S_1$. Otherwise, $B_1 \neq \emptyset$. If $u_1v \in E$ for all $u_1 \in B_1$, then $G[B_1] = \overline{K}_2 + \overline{K}_2 + \cdots + \overline{K}_2$, as in the proof of Theorem 3.2.6. Otherwise, there exists $u_1 \in B_1$ such that $u_1v \notin E$. Then, again as in Theorem 3.2.6, it can be shown that there exists $u_1' \in B_1$, such that $u_1u_1' \notin E$ and such that (a) and (b) hold with u_1 and u_1' in place of u and u'.

In general, if $B_k = N(u_{k-1}) \cap N(u_{k-1}') = \emptyset$, then $C_2 = S_1 + S_2 + \cdots + S_k$. Suppose $B_k \neq \emptyset$. If $u_kv \notin E$ for all $u_k \in B_k$, then $G[B_k] = \overline{K}_2 + \overline{K}_2 + \cdots + \overline{K}_2$. Otherwise, there exists $u_k \in B_k$ such that $u_kv \in E(B_k)$ and $u_kv \notin E$. Thus, there exists $u_k' \in B_k$ such that $u_ku_k' \notin E$, and such that (a) and (b) hold with u_k and u_k' in place of u and u'.

Since G is finite, we have $B_m = \emptyset$ for some m, and thus $C_2 = S_1 + S_2 + \cdots + S_m$, where $S_j = \overline{K}_2$ or $Q_{2,p}$ for all $1 \leq j \leq m$. The result now follows. ∎

Let $L_{p,q}$ denote the bipartite graph constructed from $K_{1,p-1} \cup K_{1,q-1}$ by adding an edge between the vertex of degree $p-1$ in $K_{1,p-1}$ and the vertex of degree $q-1$ in $K_{1,q-1}$. We will refer to the vertices incident with this new edge as the *centre vertices* of $K_{1,p-1}$ and $K_{1,q-1}$. Note that when $p = 1$ (or $q = 1$) $L_{p,q} = K_{1,q}$ (or $K_{1,p}$, respectively), and $L_{1,1} = K_{1,1} = K_2$.

Now, consider a 2-connected 3-i-critical graph G with $\delta = 2$. Let x be a vertex with $d(x) = 2$. Then $N(x)$ is a vertex cut of size two. By Lemma 3.2.4, the graph $G - N(x)$ has at most three components. The following two theorems give a description of such graphs.

Theorem 3.2.9 *Let G be a 2-connected, 3-i-critical graph with $\delta = 2$. If there exists a vertex x such that $N(x) = \{v, v'\} = S$, where $vv' \notin E$, then either 1 or 2 holds.*

1. *The graph $G - S$ has exactly two components C_1 and C_2 such that $V(C_1) = \{x\}$ and $C_2 = S_1 + S_2 + \cdots + S_m$, where $m \geq 1$ and $S_j = \overline{K}_{1,p}$ or $\overline{L}_{p,q}$ for all $1 \leq j \leq m$. Furthermore,*

   (a) $C_2 - (N(v) \cup N(v'))$ is a complete graph on $r \geq 1$ vertices, and

(b) for $j = 1, 2, \ldots, m$, one of the following holds:

i. $S_j = \overline{L}_{p,q}$ with centre vertices $\{w, w'\}$, where $w, w' \in N(v) \oplus N(v')$, and $V(S_j) - \{w, w'\} \subseteq N(v) \cap N(v')$;

ii. $S_j = \overline{K}_{1,p}$ with centre vertex u, where $u \in N(v) \cap N(v')$ or $V(S_j) - \{u\} \subseteq N(v) \cap N(v')$.

2. $G - S$ has exactly three components C_1, C_2 and C_3 such that $V(C_1) = \{x\}$, $C_2 = K_1$, and $C_3 = K_q$ with $q \geq 3$. Furthermore, $N(v) = N(v')$ and there exist vertices $z_1, z_2, z_3 \in V(C_3)$ that satisfy

$$V(C_1) \cup V(C_2) \cup \{z_1, z_2\} \subseteq N(v) \subseteq V(C_1) \cup V(C_2) \cup (V(C_3) - \{z_3\}).$$

Proof: By Lemma 3.2.4, $G - S$ has either two or three components. We consider these two cases separately.

<u>Case 1:</u> $G - S$ has exactly two components C_1 and C_2.

Without loss of generality, $V(C_1) = \{x\}$, and $i(C_2) \geq 2$.

We first show that for any $w \in N(v) \oplus N(v')$, there is a unique vertex $w' \in N(v) \oplus N(v')$ such that $ww' \notin E$, and furthermore, w dominates $V(C_2) - (N(v) \cup N(v'))$. For any $w \in N(v) - N(v')$, $wv' \notin E$ implies that there exists w' such that $[w, w'] \to v'$ or $[v', w'] \to w$. Since $\{w, w', v'\}$ must be an independent set, $w' \notin N[x]$, and hence $[v', w'] \to w$. Furthermore, $w' \in N(v) - N(v')$ and $N[w'] \supseteq V(C_2) - (N(v') \cup \{w\})$. Now, from $w'v' \notin E$ we must have $[v', w] \to w'$

and thus $N[w] \supseteq V(C_2) - (N(v') \cup \{w'\})$. Similarly, for any $u \in N(v') - N(v)$, we have $u' \in N(v') - N(v)$ such that $N[u] \supseteq V(C_2) - (N(v) \cup \{u'\})$ and $N[u'] \supseteq V(C_2) - (N(v) \cup \{u\})$. Now suppose there exists $w \in N(v) - N(v')$ and $u \in N(v') - N(v)$, such that $wu \notin E$. Then there exists t such that $[w,t] \to u$ or $[u,t] \to w$. In either case, since $t \notin \{v, v'\}$ and t must be dominated, $t = x$. However, neither $[w,x] \to u$ nor $[u,x] \to w$ is possible, a contradiction. Therefore $wu \in E$ for all $w \in N(v) - N(v')$ and $u \in N(v') - N(v)$.

Next, we prove that $C_2 - (N(v) \cup N(v'))$ is a complete graph with $r \geq 1$ vertices. Let $y \in V(C_2) - (N(v) \cup N(v'))$. Note that y exists, else $\{v, v'\}$ is an independent dominating set. Since $vy \notin E$, there exists t such that $[v,t] \to y$ or $[y,t] \to v$. In either case, $\{v,t,y\}$ must be an independent set, and $N(y) \supseteq N(v) \oplus N(v')$ by the first paragraph. Furthermore, neither v nor y dominate v'. It follows that if $[v,t] \to y$, then $t = v'$, and if $[y,t] \to v$, then $t = v'$. Now, if $[v,v'] \to y$, then $C_2 - (N(v) \cup N(v')) = \{y\}$, a complete graph on one vertex. Otherwise, $[y,v'] \to v$ (and by the same argument, $[y,v] \to v'$), and $N[y] \supseteq V(C_2) - (N(v) \cap N(v'))$. Since y is arbitrary, it follows that $C_2 - (N(v) \cup N(v'))$ is a complete graph with $r \geq 1$ vertices.

Finally, we prove that $C_2 = S_1 + S_2 + \cdots + S_m$, with $m \geq 1$ and $S_j = \overline{K}_{1,p}$ or $\overline{L}_{p,q}$ for all $1 \leq j \leq m$. Recall that $i(C_2) \geq 2$. First, consider any pair of nonadjacent vertices $w, w' \in N(v) \oplus N(v')$. If there exists a vertex $z \neq w'$ such that $wz \notin E$, then $z \in N(v) \cap N(v')$ and either $[w, x] \to z$ or $[z, x] \to w$. Since $\{w, x\}$ does not dominate w', we must have $[z, x] \to w$ and $N[z] \supseteq V(C_2) - \{w\}$. Therefore $V(C_2) - N(w) - \{w'\}$ induces a subgraph $\overline{K}_{1,p}$. Similarly, $V(C_2) - N(w') - \{w\}$ induces a subraph $\overline{K}_{1,q}$. Together, these two subgraphs induce a subgraph $\overline{L}_{p,q}$ with centre vertices w and w'. Thus the vertices in $N(v) \oplus N(v')$ together with the vertices of $N(v) \cap N(v')$ that do not dominate $N(v) \oplus N(v')$ induce $S_1 + S_2 + \cdots + S_i$, where $S_j = \overline{L}_{p,q}$ for all $1 \leq j \leq i = |N(v) \oplus N(v')|/2$.

It remains to be shown that $S_j = \overline{K}_{1,p}$ for $i + 1 \leq j \leq m$. The subgraphs $S_{i+1}, S_{i+2}, \ldots, S_m$ can be found recursively as follows. To find S_j, consider any pair of nonadjacent vertices y and z in $V(C_2) - V(S_1 + S_2 + \cdots + S_{j-1})$. Such vertices exist, else $\{x, y\}$ would be a dominating set for any $y \in V - V(S_1 + S_2 + \cdots + S_{j-1})$. By fact (a), either y or z (or both) is in $N(v) \cap N(v')$. Hence either $[y, x] \to z$ or $[z, x] \to y$, that is, either $N[y] \supseteq V(C_2) - \{z\}$ or $N[z] \supseteq V(C_2) - \{y\}$. Without loss of generality, suppose $[y, x] \to z$. Now consider all nonneighbours of z in $V(C_2)$. Specifically, if $wz \notin E$, then $[w, x] \to z$. Therefore, the graph induced by $V - N(z)$ is a graph $\overline{K}_{1,p}$. The result now follows.

<u>Case 2:</u> $G - S$ has exactly three components C_1, C_2 and C_3.

Without loss of generality, $V(C_1) = \{x\}$.

Since G is 2-connected, for $i = 2, 3$ there exist vertices $y_i \in V(C_i)$ such that $vy_2 \in E$ and $v'y_3 \in E$. Now, $y_2y_3 \notin E$ implies that there exists t such that

$[y_2, t] \to y_3$ or $[y_3, t] \to y_2$. Since $x \notin N[y_2] \cup N[y_3]$ and $t \notin \{v, v'\}$, we must have $t = x$. Without loss of generality, $[y_3, x] \to y_2$. Hence, $V(C_2) = \{y_2\}$, and hence C_2 is a complete graph on one vertex. For any $z \in V(C_3)$, $y_2 z \notin E$ gives $[z, x] \to y_2$ and hence C_3 is complete.

Furthermore, since $\delta = 2$, $v'y_2 \in E$. Also, for any $z \in V(C_3)$, if $v'z \in E$ then $vz \in E$ (otherwise $\{v, z\}$ is an independent dominating set) and if $vz \in E$ then $v'z \in E$ (otherwise $\{v', z\}$ is an independent dominating set). Therefore, $N(v) = N(v')$. Since $i = 3$, $\{v, v'\}$ is not an independent dominating set, and hence there exists $z \in V(C_3)$ such that $z \notin N(v) \cup N(v')$. Also, since y_3 is not a cut vertex, $V(C_3) \cap N(v) \neq \{y_3\}$. It follows that $C_3 = K_q$ for some $q \geq 3$. Specifically, there exist distinct vertices $z_1, z_2, z_3 \in V(C_3)$ such that $z_1, z_2 \in N(v)$, $z_3 \notin N(v)$, and

$$V(C_1) \cup V(C_2) \cup \{z_1, z_2\} \subseteq N(v) = N(v') \subseteq V(C_1) \cup V(C_2) \cup (V(C_3) - \{z_3\}).$$

This completes the proof. ∎

Let $\mathcal{R}_{3,p}$ be the set of graphs $R_{3,p}$ on $3 + p$ vertices with the form: $R_{3,p}$ can be obtained from $K_2 \cup K_1 \cup K_p$ by adding, for each $v \in K_p$, two edges from v to vertices not in K_p, such that the resulting graph is 2-connected.

Theorem 3.2.10 *Let G be a 2-connected 3-i-critical graph with $\delta = 2$, and a vertex x with $N(x) = \{v, v'\} = S$, where $vv' \in E$. Then, $G - S$ has exactly two components C_1 and C_2 such that $C_1 = \{x\}$ and $C_2 = S_1 + S_2 + \cdots + S_m$, where $S_j = \overline{K}_{1,p}$, a graph $Q_{2,p} \in \mathcal{Q}_{2,p}$, or a graph $R_{3,p} \in \mathcal{R}_{3,p}$, for all $1 \leq j \leq m$. Furthermore, there exist nonadjacent vertices $u, u' \in V(C_2) - N(v)$ such that*

1. *$N[u] \cup N[u'] = V(C_2)$, and*
2. *either*
   (a) *for all $z \in V(C_2) - (N(u) \cap N(u')) - \{u, u'\}$, $vz \in E$, $v'z \in E$, and $N[z] \supseteq V(C_2) - \{u, u'\}$, or*
   (b) *there exists $u'' \in N(v) - N(v')$ such that for all $z \in V(C_2) - (N(u) \cap N(u') \cap N(u'')) - \{u, u', u''\}$, $vz \in E$, $v'z \in E$, and $N[z] \supseteq V(C_2) - \{u, u', u''\}$.*

Proof: By Lemma 3.2.4, $G - S$ has either two or three components.

Suppose $G - S$ has three components C_1, C_2, C_3 where $V(C_1) = \{x\}$. Since G is 2-connected, there exist vertices $y_2 \in V(C_2)$ and $y_3 \in V(C_3)$ such that $vy_2 \in E$ and $v'y_3 \in E$. Since $y_2y_3 \notin E$, without loss of generality, there exists a vertex t such that $[y_2, t] \to y_3$. Since x must be dominated by t and $vy_2, v'y_3 \in E$, $t = x$. Hence $[y_2, x] \to y_3$, and $C_3 = \{y_3\}$. Since G is 2-connected, $vy_3 \in E$, and $i = 3$ implies there exists $z \in V(C_2)$ such that $vz \notin E$. Again, since $zy_3 \notin E$, there exists t such that either $[z,t] \to y_3$ or $[y_3,t] \to z$. In either case, x must be dominated by t as $xz, y_3z \notin E$, and $y_3v, y_3v' \in E$. Thus, $t = x$. Since $\{y_3, x\}$ does not dominate y_2, it is not possible that $[y_3, x] \to z$. Therefore, $[z,x] \to y_3$ and $\{z, v\}$ is an independent dominating set, a contradiction. Therefore $G - S$ has exactly two components C_1 and C_2, where $V(C_1) = \{x\}$.

Since $i = 3$, there exists $u \in V(C_2) - N(v)$. Now, $uv \notin E$ implies there exists u' such that either $[u,u'] \to v$ or $[v,u'] \to u$. Suppose $[u,u'] \to v$. Then, $u' \notin \{x, v, v'\}$, and x is not dominated. Therefore $[v,u'] \to u$. Since $u'v \notin E$, we have $u' \in V(C_2) - N(v)$ and $N[u'] \supseteq V(C_2) - N(v) - \{u\}$. Also, $u'v \notin E$ implies there exists w such that either $[u', w] \to v$ or $[v,w] \to u'$. As above, $[v,w] \to u'$ and $w \in V(C_2) - N(v)$. Now $N[u'] \supseteq V(C_2) - N(v) - \{u\}$ implies $w = u$. Hence $[v,u] \to u'$ and $N[u] \supseteq V(C_2) - N(v) - \{u'\}$. This argument shows that every $u \in V(C_2) - N(v)$ can be paired with a unique vertex $u' \in V(C_2) - N(v)$ such that $[v,u] \to u'$ and $[v,u'] \to u$. Similarly, every $u \in V(C_2) - N(v')$ can be paired with a unique vertex $u'' \in V(C_2) - N(v')$ such that $[v',u] \to u''$ and $[v', u''] \to u$.

Consider $u \in V(C_2) - (N(v) \cup N(v'))$. We now consider the possibilities for u and u' defined above. Suppose $u' \in V(C_2) - (N(v) \cup N(v'))$. Since $[v', u] \to u''$ and $uu', v'u' \notin E$, $u' = u''$. Furthermore, $[v,u] \to u'$, $[v',u] \to u'$, $[v,u'] \to u$, and $[v',u'] \to u$ gives $N[u] \supseteq V(C_2) - (N(v) \cap N(v')) - \{u'\}$ and $N[u'] \supseteq V(C_2) - (N(v) \cap N(v')) - \{u\}$. Therefore, if $u' \in N(v')$, then $u'' \in N(v)$.

We have now shown that $G - S$ has exactly two components C_1 and C_2 with $C_1 = \{x\}$, and that the vertices in $V(C_2) - (N(v) \cap N(v'))$ can be uniquely partitioned into pairs of the form $\{u, u'\}$ and triples of the form $\{u, u', u''\}$. If y is in a pair in the partition, we will refer to y as *Type I*, and if y is in a triple in the partition, we will refer to y as *Type II*.

Consider $u \in V(C_2) - N(v)$ (the same argument applies for $u \in V(C_2) - N(v')$). Suppose there exists $w \in V(C_2) - N[u] - N[u']$. Since $wu \notin E$, there exists t such that either $[w, t] \rightarrow u$ or $[u, t] \rightarrow w$. In either case, x must be dominated by t, and hence $t \in \{x, v, v'\}$. Also, u' must be dominated, and $u' \notin N[w] \cup N[u] \cup N[x] \cup N[v]$, so $t = v'$. Therefore $u \notin N(v')$ and $u' \in N(v')$. Similarly, $wu' \notin E$ gives $u' \notin N(v'), u \in N(v')$, a contradiction. Therefore, $N[u] \cup N[u'] = V(C_2)$.

We now show that if $y \in N(v) - N(v')$ and $u \in N(v') - N(v)$, then $uy \in E$. Suppose $uy \notin E$. Then there exists w such that $[u, w] \rightarrow y$ or $[y, w] \rightarrow u$. In either case, x must be dominated by w, and $uv', yv \in E$, so $w = x$. If $[u, x] \rightarrow y$, then $uu' \in E$, a contradiction. So $[y, x] \rightarrow u$ and $yy' \in E$, a contradiction. Hence $uy \in E$ for all $y \in N(v) - N(v')$, $u \in N(v') - N(v)$.

We have shown that statement 1 of the theorem holds. We now prove statement 2. There are two cases. After both have been considered, we subsequently show that C_2 has the structure claimed.

Case 1: u is Type I.

Let $A_1 = V(C_2) - (N(u) \cap N(u')) - \{u, u'\}$. Consider $z \in A_1$. If $u, u' \in N(v')$ (the case where $u, u' \in N(v)$ is similar), then $z \notin N(v) - N(v')$ since both u and u' dominate $N(v) - N(v')$, and $z \notin V(C_2) - N(v)$, so $z \in N(v) \cap N(v')$. If $u, u' \notin N(v) \cup N(v')$, then both u and u' dominate $V(C_2) - (N(v) \cap N(v')) - \{u, u'\}$ and hence $N(u) \cap N(u') \supseteq V(C_2) - (N(v) \cap N(v')) - \{u, u'\}$. Therefore $z \in N(v) \cap N(v')$. In either case, without loss of generality, let $uz \in E$ and $u'z \notin E$. Now $zu' \notin E$ implies there exists t such that $[z, t] \to u'$ or $[u', t] \to z$. Since $xz, xu' \notin E$ and $z \in N(v) \cap N(v')$, $t = x$. Hence $[u', x] \to z$ is not possible as neither u' nor x dominate u. Therefore, $[z, x] \to u'$ and $N[z] \supseteq V(C_2) - \{u'\}$. Similarly, if $u'z \in E$ and $uz \notin E$, then $N[z] \supseteq V(C_2) - \{u\}$. Therefore $N[z] \supseteq V(C_2) - \{u, u'\}$.

Case 2: u is Type II.

Let $A_1 = V(C_2) - (N(u) \cap N(u') \cap N(u'')) - \{u, u', u''\}$, and consider $z \in A_1$. From previous results, each of u, u', u'' dominates every vertex in

$$V(C_2) - (N(v) \cap N(v')) - \{u, u', u''\}.$$

Therefore, $z \in N(v) \cap N(v')$. Suppose $zu \notin E$. Then, there exists t such that $[z, t] \to u$ or $[u, t] \to z$. In order to dominate x, $t = x$. Since u' must be dominated, it is not possible that $[u, x] \to z$. Therefore, $[z, x] \to u$ and hence $N[z] \supseteq V(C_2) - \{u\}$. The same argument for $zu' \notin E$ or $zu'' \notin E$ gives

$N[z] \supseteq V(C_2) - \{u, u', u''\}$.

This completes the proof of 2. It remains to show that C_2 has the claimed structure. We begin by defining S_1, depending on whether u is Type I or Type II.

Suppose u is Type I. Let $A_1 = V(C_2) - (N(u) \cap N(u')) - \{u, u'\}$. If $A_1 = \emptyset$, then let $S_1 = G[\{u, u'\}] = \overline{K}_{1,1}$. If $A_1 \neq \emptyset$, let $S_1 = G[A_1 \cup \{u, u'\}]$. Since $N[z] \supseteq V(C_2) - \{u, u'\}$ for all $z \in A_1$, $G[A_1] = K_p$. Now consider any $z \in A_1$. Without loss of generality, $uz \in E$, and $u'z \notin E$. Also, $xu \notin E$ implies there exists y such that $[x, y] \to u$ or $[u, y] \to x$. In either case, u' must be dominated, so $y \in N[u']$. But $y \neq u'$ as neither $[x, u'] \to u$ nor $[u, u'] \to x$. Also, $y \in V(C_2)$ as $xy \notin E$. Therefore, $y \in N(u')$ and $y \notin N(u)$, so $y \in A_1$ and $A_1 \not\subseteq N(u)$. Therefore $S_1 \in \mathcal{Q}_{2,p}$.

Suppose u is Type II. Let $A_1 = V(C_2) - (N(u) \cap N(u') \cap N(u'')) - \{u, u', u''\}$. If $A_1 = \emptyset$, let $S_1 = G[\{u, u', u''\}] = S_1 = \overline{K}_{1,2}$. If $A_1 \neq \emptyset$, let $S_1 = G[A_1 \cup \{u, u', u''\}]$. Since $N[z] \supseteq V(C_2) - \{u, u', u''\}$ for all $z \in A_1$, again we have $G[A_1] = K_p$. By definintion of A_1, for any $z \in A_1$ there exists $w \in \{u, u', u''\}$ such that $wz \notin E$. Therefore there exists t such that either $[w, t] \to z$ or $[z, t] \to w$. Since $zx, wx \notin E$, x must be dominated by t, and z is adjacent to both v and v', so $t = x$. Now $[w, x] \not\to z$, as w does not dominate $\{u, u', u''\}$. Therefore

$[z, x] \to w$ and z dominates $\{u, u', u''\} - \{w\}$. This shows that every $z \in A_1$ is adjacent to exactly two vertices in $\{u, u', u''\}$. Suppose one of u, u', or u'' is adjacent to every vertex in A_1. Specifically, suppose $u'z \in E$ for all $z \in A_1$. Then $N[u'] = V - \{x, v, u\}$. Since $u'x \notin E$, there exists y such that either $[u', y] \to x$ or $[x, y] \to u'$. In either case, $\{u', y, x\}$ must be an independent set, and hence $y = u$. But neither $[u', u] \to x$ nor $[x, u] \to u'$ is true, a contradiction which implies u' does not dominate A_1. A similar argument can be used to show that neither u nor u'' dominate A_1. Therefore, every vertex in $\{u, u', u''\}$ has a neighbour in A_1, and specifically, u has at least two neighbours in A_1. This proves $S_1 \in \mathcal{R}_{3,p}$.

Proceed inductively as follows. If u is Type I, let $B_1 = N(u) \cap N(u')$, and if u is Type II, let $B_1 = N(u) \cap N(u') \cap N(u'')$. If $B_1 \neq \emptyset$ and there exists $y \in B_1$ such that $y \in (N(v) \cup N(v')) - (N(v) \cap N(v'))$, repeat the above procedure to find A_i, S_i, B_i for $i = 2, 3, \ldots, j$ (as in Theorem 3.2.6, Case 2), where j is the least i such that $B_i = \emptyset$ or there is no $y \in B_i$ such that $y \in (N(v) \cup N(v')) - (N(v) \cap N(v'))$. In the first case, $C_2 = S_1 + S_2 + \cdots + S_j$. In the second case, $y \in N(v) \cap N(v')$ for all $y \in B_j$. In this case, for any $y \in B_j$ there exists $y_1 \in B_j$ such that $yy_1 \notin E$, otherwise $\{y, x\}$ is an independent dominating set of G. Now $yy_1 \notin E$ implies there exists t such that, without loss of generality, $[y_1, t] \to y$. Either y_1 or t must dominate x, so $t = x$ and $[y_1, x] \to y$. This implies $N[y_1] \supseteq B_j - \{y\}$. For each vertex $w \in B_j - \{y, y_1\}$, either $yw \in E$ or $[w, x] \to y$

(as $[y, x] \to w$ would imply $yy_1 \in E$). So let $\{y_1, y_2, \ldots y_p\}$ be the set of vertices in B_j such that $[y_i, x] \to y$, $i = 1, 2, \ldots p$. Now let $S_{j+1} = G[\{y, y_1, \ldots y_p\}] = \overline{K}_{1,p}$, $p \geq 1$. Note that every vertex in $\{y, y_1, \ldots, y_p\}$ is adjacent to every vertex in $V(C_2) - \{y, y_1, \ldots y_p\}$. If $B_j - \{y, y_1, \ldots, y_p\} = \emptyset$ then $C_2 = S_1 + S_2 + \cdots + S_{j+1}$. Otherwise, set $B_{j+1} = B_j - \{y, y_1, \ldots, y_p\}$, and repeat this argument to find $S_{j+2}, S_{j+3}, \ldots, S_m$. It is important to note that at each step, $B_j, B_{j+1}, \ldots, B_m$ each contain at least two vertices, as if $B_m = \{y\}$ then $\{y, x\}$ is an independent dominating set of G. Therefore $C_2 = S_1 + S_2 + \cdots + S_m$, where S_j has one of the claimed structures, for $j = 1, 2, \ldots m$. ∎

3.3 Hamilton Paths and Cycles

Using the results of the previous sections, we give a characterisation of the 2-connected, 3-i-critical graphs that are hamiltonian. We then prove that every connected, 3-i-critical graph with more than six vertices has a Hamilton path.

Note that each graph in $\mathcal{Q}_{2,p}$ has a Hamilton path. By the definition of join we have the following lemmata. Several of the easy proofs are omitted.

Lemma 3.3.1 *If G has a Hamilton cycle, then so does $G + \overline{K}_{1,q}$, $G + \overline{L}_{p,q}$, $G + Q_{2,p}$ for any $Q_{2,p} \in \mathcal{Q}_{2,p}$, and $G + R_{3,p}$ for any $R_{3,p} \in \mathcal{R}_{3,p}$.*

Lemma 3.3.2 *If* $G = S_1 + S_2 + \cdots + S_k$*, where* $k \geq 2$ *and* $S_j = \overline{K}_{1,p}$ *or* $\overline{L}_{p,q}$*, for* $j = 1, 2, \ldots, k$*, then* G *has a Hamilton cycle.*

Lemma 3.3.3 *Every* $R_{3,p} \in \mathcal{R}_{3,p}$ *has a Hamilton cycle.*

Lemma 3.3.4 *If* $G = S_1 + S_2 + \cdots + S_k$ *where* $S_j = \overline{K}_{1,p}, Q_{2,p} \in \mathcal{Q}_{2,p}$*, or* $R_{3,p} \in \mathcal{R}_{3,p}$ *for* $j = 1, \ldots, k$*, then* G *has a Hamilton cycle, unless* $G = \overline{K}_{1,p}$ *or* $G \in \mathcal{Q}_{2,p}$*. If* $G \in \mathcal{Q}_{2,p}$*, then* G *has a Hamilton path.*

Proof: If $S_j = R_{3,p}$ for any j, then by Lemma 3.3.1 and Lemma 3.3.3, G has a Hamilton cycle. Otherwise, since $\overline{K}_{1,p} + \overline{K}_{1,q}$, $\overline{K}_{1,p} + Q_{2,q}$, and $Q_{2,p} + Q_{2,q}$ have Hamilton cycles, by Lemma 3.3.1 so does G, except possibly if $k = 1$. If $G = Q_{2,p}$, it is easy to see that G has a Hamilton path. ∎

For any $q \geq 3$, let $\mathcal{W}_{q,4}$ denote the set of graphs on $q + 4$ vertices constructed from $K_q \cup C_4$ such that two nonadjacent vertices of C_4 have the same neighbourhood in K_q of size greater than 1 and less than q, and the other two vertices of C_4 have no neighbours in K_q.

Lemma 3.3.5 *Each graph in* $\mathcal{W}_{q,4}$ *has a Hamilton path, but not a Hamilton cycle.*

Theorem 3.3.6 *If* G *is 2-connected and 3-i-critical, then* G *is hamiltonian, unless* $G \in \mathcal{W}_{q,4}$*, in which case it has a Hamilton path.*

Proof: If $\delta \geq 3$, then by Corollary 3.1.4, G is hamiltonian. Otherwise, suppose that there exists $x \in V$ such that $d(x) = 2$. Let $S = N(x) = \{v, v'\}$. Then by Theorem 3.2.9 and Theorem 3.2.10, $G - S$ has at most 3 components and one of 1 or 2 in the statement of Theorem 3.2.9 holds, or $vv' \in E$.

If 1 holds, then by Lemma 3.3.2, C_2 is hamiltonian. Notice that there exists some Hamilton path in C_2 that starts at a point of $N(v)$ and ends at a point of $N(v')$. Hence G is also hamiltonian.

If 2 holds, then $G \in \mathcal{W}_{q,4}$, and by Lemma 3.3.5, G has a Hamilton path, but not a Hamilton cycle.

If $vv' \in E$, by the construction of C_2 in Theorem 3.2.10,

$$S_1 \in \{Q_{2,p}, \overline{K}_{1,1}, \overline{K}_{1,2}, R_{3,p}\}.$$

Note that $C_2 \neq \overline{K}_{1,p}$, otherwise G is not 2-connected. Therefore, by Lemma 3.3.4, C_2 is either hamiltonian or $C_2 = Q_{2,p}$. If C_2 is hamiltonian and $S_j = Q_{2,p}$ or $R_{3,p}$ for some $1 \leq j \leq m$, then there exists a Hamilton path in C_2 from a vertex in $N(v)$ to a vertex in $N(v')$, and hence G is hamiltonian. If C_2 is hamiltonian and $C_2 = \overline{K}_{1,p_1} + \overline{K}_{1,p_2} + \cdots + \overline{K}_{1,p_m}$, where $p_i = 1$ or 2 for $i = 1, 2, \ldots, m$, it is not difficult to find a Hamilton path in C_2 from a vertex in $N(v)$ to a vertex in $N(v')$, and therefore G is hamiltonian. If $C_2 = Q_{2,p}$, then each of the vertices in the defining copy of $\overline{K}_2$ in C_2 have degree greater than one (or else G has a cut vertex) and there exists a Hamilton path in C_2 from a vertex in $N(v)$ to a vertex in $N(v')$. Hence there exists a Hamilton cycle in G. ∎

Theorem 3.3.7 *If G is connected and 3-i-critical with $|V(G)| > 6$, then G has a Hamilton path.*

Proof: If G is 2-connected, then by Theorem 3.3.6, G has a Hamilton path. Thus assume G has a cut vertex v. By Theorem 3.2.6, $G - v$ has exactly two components C_1 and C_2, such that C_1 is complete, $i(C_2) = 2$, and either 1 or 2 in the statement of Theorem 3.2.6 holds.

If 1 holds, then since $G[C_1 \cup \{v\}]$ is complete, it has a Hamilton path P that ends at v. Since $T_{2n,n}$ has a Hamilton path that starts at any vertex, $G[C_2 \cup \{v\}]$ has a Hamilton path Q that starts at v. Then the path PQ is a Hamilton path of G.

If 2 holds, then since G is connected and $i(G) = 3$, $C_2 \neq \overline{K}_2$. Thus, by Lemma 3.3.4, C_2 has a Hamilton path. Furthermore, $G[\{v\} \cup V(C_2)]$ has a Hamilton path Q that starts at v, except when $C_2 \in \mathcal{Q}_{2,2}$ (i.e. C_2 is a path with 4 vertices). Since $|V(G)| > 6$, this does not happen. Let $P = xv$. Then the path PQ is a Hamilton path of G. Therefore, if $|V(G)| > 6$ (thus, $C_2 \neq P_4$), then G has a Hamilton path. ∎

Corollary 3.3.8 *If G is connected and 3-i-critical with $|V(G)| > 6$ even, then G has a perfect matching.*

For any $k \geq 4$, there are arbitrarily large k-i-critical graphs with no Hamilton path: Let $m \geq 2k$, and $p = 1 + 1 + \cdots + 1 + (m - 2k + 1)$ be a partition of $p = m - k$. Then each graph in $\mathcal{Q}_{k,p}$ is k-i-critical but has $k - 1 \geq 3$ vertices of degree one, and hence no Hamilton path.

For $k \geq 4$, the question of when a k-i-critical graph G contains a Hamilton cycle is still open.

Chapter 4

Total Domination Critical Graphs

This chapter begins with many results of Haynes, Mynhardt, and van der Merwe. Sections 4.1 and 4.2 provide a summary of results from [13], where total domination critical graphs were introduced. These include properties of total domination critical graphs as well as properties specific to 3-γ_t-critical graphs. In section 4.3, a characterisation of the 3-γ_t-critical graphs with a cut vertex is given, and it is shown that all such graphs contain a Hamilton path. Furthermore, it may be seen that any 3-γ_t-critical graph G satisfies $2 \leq diam(G) \leq 3$. Section 4.4 marks the beginning of new work done on 3-γ_t-critical graphs, where the need for a new closure concept is motivated, and a closure is defined. It will be established that a

3-γ_t-critical graph G is hamiltonian if and only if the closure of G is hamiltonian. In addition, several results about the structure of G and its closure will be given.

4.1 Properties of γ_t-critical graphs

As in the cases of domination critical graphs and independent domination critical graphs, the first nontrivial case of a γ_t-critical graph G occurs when $\gamma_t(G) = 3$. Since the size of a total dominating set in any graph must be at least two, the only 2-γ_t-critical graphs are vaccuously the complete graphs K_n, with $n \geq 2$.

For any γ_t-critical graph G, since $\gamma_t(G+uv) < \gamma_t(G)$ for any $uv \notin E(G)$, it is not difficult to see that every minimum total dominating set of $G + uv$ contains at least one of u and v. Furthermore, the addition of the edge uv to G can change the total domination number by at most two:

Theorem 4.1.1 *[13] For any edge $uv \in E(\overline{G})$,*

$$\gamma_t(G) - 2 \leq \gamma_t(G+uv) \leq \gamma_t(G).$$

Theorem 4.1.2 *[13] For any connected graph G, $\Delta(G) \leq |V(G)| - \gamma_t(G)$.*

Theorem 4.1.3 *[13] For any graph G and any minimum total dominating set S of G, if $v \in S$ is in a component of $[S]$ of cardinality at least 3, then $N(v)$ is not complete.*

Theorem 4.1.4 *[13] Every vertex in a γ_t-critical graph is adjacent to at most one vertex of degree one.*

4.2 Introduction to 3-γ_t-critical graphs

What follows is a list of several results from [13] specific to 3-γ_t-critical graphs. First, a simple observation which is fundamental to all of the results on 3-γ_t-critical graphs is made.

Observation: For any 3-γ_t-critical graph G and any pair of nonadjacent vertices u and v, at least one of the following holds:

1. $\{u, v\}$ dominates G.

2. There exists $w \in N(u)$ such that $\{u, w\}$ dominates $G - v$, but not v.

3. There exists $w \in N(v)$ such that $\{v, w\}$ dominates $G - u$, but not u.

The second and third situations will be denoted by $uw \rightarrow v$ and $vw \rightarrow u$, respectively.

The first result is a corollary to Theorem 4.1.3 for graphs G with $\gamma_t(G) = 3$.

Corollary 4.2.1 *[13] If a graph G with $\gamma_t(G) = 3$ has a vertex v such that $N(v)$ is complete, then $v \notin S$, for any minimum total dominating set S of G.*

The next result states that there is only one 3-γ_t-critical graph on less than 6 vertices. All other results are for graphs with at least 6 vertices.

Theorem 4.2.2 *[13] The only 3-γ_t-critical graph with fewer than 6 vertices is C_5.*

Theorem 4.2.3 *[13] Every 3-γ_t-critical graph G has a minimum total dominating set S such that $[S] = P_3$.*

The next result provides some insight into one of the reasons why a characterisation of the 3-γ_t-critical graphs does not come easily.

Theorem 4.2.4 *[13] For any graph G, there is a 3-γ_t-critical graph H such that G is an induced subgraph of H.*

In other words, there is no finite forbidden subgraph characterisation for the 3-γ_t-critical graphs.

4.3 3-γ_t-critical graphs with a cutvertex

The following results of Haynes, Mynhardt, and van der Merwe lead to a characterisation of the 3-γ_t-critical graphs with a cut vertex. It will also be shown that all such graphs contain a Hamilton path.

Theorem 4.3.1 *[13] If G is a 3-γ_t-critical graph, then G has at most one vertex of degree one.*

Corollary 4.3.2 *[13] No tree is 3-γ_t-critical.*

Theorem 4.3.3 *[13] Let G be a graph with $\gamma_t(G) = 3$. If v is a cut vertex of G, then v is an element of every minimum total dominating set of G.*

Theorem 4.3.4 *[13] For any 3-γ_t-critical graph G with a cut vertex v, $G - v$ has exactly two components.*

Theorem 4.3.5 *[13] If G is a 3-γ_t-critical graph with a cut vertex v, then v is adjacent to a vertex of degree one.*

Corollary 4.3.6 *[13] If G is a 3-γ_t-critical graph, then G has at most one cut vertex.*

Theorem 4.3.7 *[13] If G is a 3-γ_t-critical graph, then*

$$2 \leq diam(G) \leq 3.$$

With use of the above results, a complete characterisation of the 3-γ_t-critical graphs with a cut vertex can be made. The characterisation also makes use of the following Lemma:

Lemma 4.3.8 *[13] If G is a 3-γ_t-critical graph that contains a cut vertex, then $diam(G) = 3$.*

Theorem 4.3.9 *[13] Let G be a graph with a vertex u of degree one, and let $N(u) = \{v\}$. Let $A = N(v) - \{u\}$ and $B = V - N[v]$. Then G is 3-γ_t-critical if and only if*

1. *$[A]$ is complete and $|A| \geq 2$,*
2. *$[B]$ is complete and $|B| \geq 2$, and*
3. *every vertex in A is adjacent to $|B| - 1$ vertices in B and every vertex in B is adjacent to at least one vertex in A.*

The next two corollories follow from Theorem 4.3.9 and Theorem 2.2.13 and address the hamiltonian properties of the 3-γ_t-critical graphs with a cut vertex.

Corollary 4.3.10 *[13] If G is a 3-γ_t-critical graph with a vertex u of degree one, then $G - u$ is hamiltonian.*

Corollary 4.3.11 *If G is a 3-γ_t-critical graph with a vertex u of degree one, then G has a Hamilton path.*

In [14] and [15], the 3-γ_t-critical graphs which do not have a cut vertex are considered. The first paper discusses the 3-γ_t-critical graphs with diameter three, and the second, the 3-γ_t-critical graphs with diameter two. We saw in

Lemma 4.3.8 that a 3-γ_t-critical graph with a cut vertex has diameter three, and in Theorem 4.3.9 these graphs were characterised. The two families of graphs that will be explored here are the 2-connected 3-γ_t-critical graphs with diameter three, and the 3-γ_t-critical graphs with diameter two (which are 2-connected by Lemma 4.3.8).

4.4 A new closure

The 3-γ_t-critical graphs with diameter two and diameter three will be studied separately in the chapters which follow. In both cases, when the hamiltonian properties of the graphs are discussed, we make use of a new closure: edges are added to a graph G in such a way that the obtained graph is hamiltonian precisely when G is hamiltonian.

Before the closure of a total domination critical graph is defined, a few definitions and observations about 3-γ_t-critical graphs need to be developed.

If G is a graph with $diam(G) = k$ and $d(u, v) = k$, then we say that u and v are *diametrical vertices.* Two subsets X and Y of V are called *diametrical sets* if $d(x, y) = diam(G)$ for each $x \in X$ and $y \in Y$. If X and Y are diametrical sets, then (X, Y) is a *maximal diametrical pair* if for each $z \in V - (X, Y)$, $d(x, z) < diam(G)$ for some $x \in X$ and $d(y, z) < diam(G)$ for some $y \in Y$.

Theorem 4.4.1 *[14] If G is a 3-γ_t-critical graph with diam$(G) = 3$, then G has a unique maximal diametrical pair (X, Y). Moreover, X (say) has cardinality one and $[Y]$ is complete.*

In a 3-γ_t-critical graph with diameter two, it is possible that every pair of nonadjacent vertices dominates the graph. This is not the case in a 3-γ_t-critical graph with diameter three:

Theorem 4.4.2 *[14] If G is a 3-γ_t-critical graph with diam$(G) = 3$, then G has a pair of nonadjacent vertices that does not dominate G.*

Furthermore, it is also possible in a 3-γ_t-critical graph with diameter two that for every pair of nonadjacent vertices u and v, there exist vertices x and y such that $ux \to v$ and $vy \to u$. This is not possible in the diameter three case:

Theorem 4.4.3 *[14] If G is a 3-γ_t-critical graph with diam$(G) = 3$, then G has a pair of nonadjacent vertices u and v such that $ux \to v$, for some $x \in V$, but there is no vertex y such that $vy \to u$.*

The following theorem marks the start of our developement of a total domination closure of a graph G.

Theorem 4.4.4 *If u and v are nonadjacent vertices in a 2-connected 3-γ_t-critical graph G and $\{u, v\} \succ G$, then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Proof: Let u and v be a pair of nonadjacent vertices that satisfy $\{u, v\} \succ G$. Certainly $G + uv$ is hamiltonian if G is hamiltonian.

Now suppose $G+uv$ is hamiltonian but G is not. Since $G+uv$ has a Hamilton cycle, G must have a Hamilton path from u to v, say $u = v_1v_2 \dots v_n = v$. For any i where $1 \leq i < n$, v_1v_{i+1} and v_iv_n cannot both be edges in G, otherwise

$$v_1v_2 \dots v_iv_nv_{n-1} \dots v_{i+1}v_1$$

is a Hamilton cycle in G.

Therefore, since $\{v_1, v_n\} \succ G$, and G is 2-connected, $v_1v_2, v_1v_3, \dots, v_1v_i \in E(G)$ for some $i \geq 3$, and $v_jv_n, v_{j+1}v_n, \dots, v_{n-1}v_n \in E(G)$ for $i \leq j \leq i + 1$.

Consider p and q such that $1 < p < i$ and $j < q < n$. It must be that $v_pv_q \notin E(G)$, otherwise

$$v_1v_{p+1}v_{p+2} \cdots v_{q-1}v_nv_{n-1} \cdots v_qv_pv_{p-1} \cdots v_1$$

is a Hamilton cycle in G.

We now consider two cases:

<u>Case 1</u>: $j = i$.

Since $\gamma_t(G) = 3$, $\{v_i, v_n\} \not\succ G$ (and $\{v_1, v_i\} \not\succ G$). Hence there exists $p \in \{2, 3, \dots, i-2\}$ such that $v_iv_p \notin E(G)$. This means that for $i + 1 \leq q \leq n$, $d(v_p, v_q) = 3$ and therefore $v_iv_{i+1}, v_iv_{i+2}, \dots, v_iv_n \in E(G)$. But now $\{v_1, v_i\} \succ G$, a contradiction.

Case 2: $j = i + 1$.

Since $d(v_1, v_n) = 3$, and G has a unique maximal diametrical pair $(\{x\}, Y)$ by Theorem 4.4.1, without loss of generality $v_1 = x$ and $v_n \in Y$. Now, since $v_2 \notin Y$, and by the uniqueness of $(\{x\}, Y)$, it must be that $d(v_2, v_n) = 2$ and hence $v_2 v_{i+1} \in E(G)$. Similarly, $v_3 v_{i+1}, v_4 v_{i+1}, \ldots, v_{i-1} v_{i+1} \in E(G)$. Now, $\{v_i, v_{i+1}\} \not\succ G$ implies that there exists $k \in \{i+3, i+4, \ldots, n-1\}$ such that v_k is not dominated by $\{v_i, v_{i+1}\}$. But now $d(v_2, v_k) = 3$ and $v_2 \notin \{x\} \cup Y$, contradicting the uniqueness of $(\{x\}, Y)$.

Therefore G is hamiltonian when $G + uv$ is hamiltonian. ∎

The closure of a 3-γ_t-critical graph G is defined as follows: Define $D^t(G)$ to be the graph obtained from G by adding the edge uv to G for each pair of nonadjacent vertices u and v that satisfy $\{u, v\} \succ G$.

From Theorem 4.4.2, we know that any 3-γ_t-critical graph G with diameter three has a pair of nonadjacent vertices u and v that does not dominate G (denoted $\{u, v\} \not\succ G$). Hence, for a 3-γ_t-critical graph G, $|E(D^t(G))| \geq |E(G)|$, but $D^t(G)$ is not complete. Theorem 4.4.4 showed that a 3-γ_t-critical graph G is hamiltonian precisely when $G + uv$ is hamiltonian, for nonadjacent vertices u and v that satisfy $\{u, v\} \succ G$. In fact, the following stronger result holds:

Theorem 4.4.5 *For any 2-connected 3-γ_t-critical graph G, G is hamiltonian if and only if $D^t(G)$ is hamiltonian.*

Proof: Certainly $D^t(G)$ is hamiltonian if G is hamiltonian.

To prove the converse, suppose $D^t(G)$ is hamiltonian while G is not. Consider a minimal subset $\{e_1, e_2, \ldots, e_k\} \subseteq E(D^t(G)) - E(G)$ for which $G + \{e_1, e_2, \ldots, e_k\}$ is hamiltonian but $G' = G + \{e_1, e_2, \ldots, e_{k-1}\}$ is not. By Theorem 4.4.4, $k \geq 2$. Let $e_k = xy$. Then G' has a Hamilton path $P : v_1 v_2 \ldots v_n$, where $v_1 = x$ and $v_n = y$. By minimality, each of $e_1, e_2, \ldots, e_{k-1}$ is on P. By definition of $D^t(G)$, since $e_k = xy$, $\{x, y\} \succ G$. As in the proof of Theorem 4.4.4, there is no i such that $v_1 v_{i+1}, v_i v_n \in E(G)$, so $v_1 v_2, v_1 v_3, \ldots, v_1 v_i, v_j v_n, v_{j+1} v_n, \ldots, v_{n-1} v_n \in E(G)$ for some $i \geq 3$ and $i \leq j \leq i+1$.

Now, for each $e_1, e_2, \ldots, e_{k-1}$, the pair of ends of each edge dominates G. But $v_p v_q \notin E(G')$ for $1 < p < i$ and $j < q < n$, otherwise

$$v_1 v_{p+1} v_{p+2} \ldots v_{q-1} v_n v_{n-1} \ldots v_q v_p v_{p-1} \ldots v_1$$

is a Hamilton cycle in G'. Therefore each of $e_1, e_2, \ldots, e_{k-1}$ must have an end in $\{v_i, v_j\}$, and hence $k \leq 3$.

Case 1: $j = i$.

If both $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$ dominate G, then v_i dominates G, a contradiction. Therefore without loss of generality, $k = 2$, $e_1 = v_j v_{j+1}$, and $e_2 = xy$.

Now $v_j v_{j-1} \dots v_1 v_n v_{n-1} \dots v_{j+1}$ is a Hamilton path in $G + e_2$. As before, $\{v_j, v_{j+1}\} \succ G$ gives $v_j v_{j-1}, v_j v_{j-2}, \dots, v_j v_1, v_j v_n \in E(G)$. But v_n dominates $\{v_j, v_{j+1}, \dots, v_{n-1}\}$, and hence $v_j v_n$ is a dominating edge of G, a contradiction.

Case 2: $j = i + 1$.

As in Case 1, at most one of $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$ dominates G, and similarly, at most one of $\{v_{j-1}, v_j\}$ and $\{v_j, v_{j+1}\}$ dominates G. If both $\{v_{i-1}, v_i\}$ and $\{v_j, v_{j+1}\}$ dominate G, then since $v_{i-1} v_{j+1} \notin E(G)$, $v_i v_{j+1}, v_j v_{i-1} \in E(G)$ and hence

$$v_1 v_2 \dots v_{i-1} v_j v_n v_{n-1} \dots v_{j+1} v_i v_1$$

is a Hamilton cycle, a contradiction. This implies $k = 2$ with $e_2 = xy$. As in Case 1, $e_1 \neq v_j v_{j+1}$ (and $e_1 \neq v_{i-1} v_i$). Therefore $e_1 = v_i v_j$.

Now $v_i v_{i-1} \dots v_1 v_n v_{n-1} \dots v_j$ is a Hamilton path in $G + e_2$. As in the argument used for $G + e_1$, $\{v_i, v_j\} \succ G$ gives $v_i v_{i-1}, v_i v_{i-2}, \dots, v_i v_1, v_n v_j, v_{n-1} v_j, \dots, v_{j+1} v_j \in E(G)$. Also, there is no p such that $v_i v_{p+1}, v_p v_j \in E(G)$. Hence $v_i v_p \notin E(G)$ for all $j \leq p \leq n$, $v_p v_j \notin E(G)$ for all $1 \leq p \leq i$, and $v_p v_q \notin E(G)$ for all $1 < p < i$, $j < q < n$. Hence G is not connected, a contradiction to G 2-connected.

Since both cases lead to a contradiction, G must be hamiltonian. ∎

What follows is a collection of new results analagous to ones found in Chapters 2 and 3, except that the results here are shown for the total domination closure of a 3-γ_t-critical graph G, rather than for G itself. Many of the results will be shown also to hold for G. The reason $D^t(G)$ is considered first is that all of our proofs of hamiltonicity use $D^t(G)$ and Theorem 4.4.5.

Theorem 4.4.6 *Let G be a 3-γ_t-critical graph and I an independent set in $D^t(G)$ with $|I| = m$. Then the vertices in I can be ordered as $x_1, x_2, \ldots, x_m$ in such a way that there exists a path $p_1p_2 \ldots p_{m-1}$ in $G - I$, where $x_ip_i \rightarrow x_{i+1}$ in G for $i = 1, 2, \ldots, m-1$.*

Proof: There is a natural orientation that can be given to the edges of $\overline{D^t(G)}$: For $uv \notin E(D^t(G))$, there exists w such that, in G, either $uw \rightarrow v$ or $vw \rightarrow u$. In the first case orient uv as (u, v), and in the second case orient uv as (v, u). Since I is independent in G, the subgraph of $\overline{G}$ induced by I is complete, and hence the orientation gives a tournament on I (with possibly extra arcs). Hence I contains a directed Hamilton path $x_1x_2 \ldots x_m$, and for each $i = 1, 2, \ldots, m-1$ there exists p_i such that $x_ip_i \rightarrow x_{i+1}$. Since $x_ip_i \in E(G)$, certainly $p_i \notin I$. Furthermore, p_i dominates $I - \{x_{i+1}\}$, so the vertices p_i are distinct. Finally, for $i = 1, 2, \ldots, m-2$, since $p_ix_{i+1} \notin E(G)$ and $x_{i+1}p_{i+1} \rightarrow x_{i+2}$, it follows that $p_ip_{i+1} \in E(G)$. Therefore $p_1p_2 \ldots p_{m-1}$ is the required path. ∎

Corollary 4.4.7 *Let G be a 3-γ_t-critical graph and I an independent set in G with $|I| = m \geq 3$. Then the vertices in I can be ordered as $x_1, x_2, \ldots, x_m$ in such a way that there exists a path $p_1 p_2 \ldots p_{m-1}$ in $G - I$, where $x_i p_i \rightarrow x_{i+1}$ in G for $i = 1, 2, \ldots, m-1$.*

Proof: If $m \geq 3$, then for any two vertices $x_i, x_j \in I$, $\{x_i, x_j\} \not\succ G$. Therefore I is also an independent set in $D^t(G)$, and the result follows directly from Theorem 4.4.6. ∎

Note that the paths described in Theorem 4.4.6 (and Corollary 4.4.7) in a 3-γ_t-critical graph G are not necessarily unique. The existence of these paths will be central to many arguments which follow, and will be referred to as follows: For an independent set I, the set of paths that satisfy the conditions of Theorem 4.4.6 will be denoted by $\mathcal{P}_I$. If P is a path given by I and Theorem 4.4.6, we will simply write $P \in \mathcal{P}_I$.

Lemma 4.4.8 *Let I be an independent set of m vertices in a 3-γ_t-critical graph G such that $I \cup \{v\}$ is independent in $D^t(G)$ (and therefore in G) for some vertex $v \in V - I$. Then the vertices of any path $P \in \mathcal{P}_I$ all lie in $N(v)$.*

Proof: Let $P \in \mathcal{P}_I$. Each vertex p_i of P is not v, as $x_i p_i \in E(G)$. Since $x_i p_i \rightarrow x_{i+1}$ and $x_i v \notin E(G)$, p_i must dominate v and hence be in $N(v)$. ∎

Corollary 4.4.9 *Let I be an independent set of $m \geq 2$ vertices in a 3-γ_t-critical graph G such that $I \cup \{v\}$ is independent in G for some vertex $v \in V - I$. Then for any $P \in \mathcal{P}_I$, the vertices of P all lie in $N(v)$.*

Proof: Since $m \geq 2$, $|I \cup \{v\}| \geq 3$, and hence $I \cup \{v\}$ is independent in $D^t(G)$. The result follows directly from Lemma 4.4.8. ∎

Lemma 4.4.10 *Let v be a vertex of degree $d \geq 2$ in a 3-γ_t-critical graph G and let I be a maximum independent set in $D^t(G)$. If $N(v) \subseteq I$, then $|I - N(v)| \leq 1$.*

Proof: Suppose $N(v) \subseteq I$ and there exist distinct vertices y, z in $I - N(v)$. By Theorem 4.4.6, there is an ordering $x_1, x_2, \ldots, x_{d+2}$ of the vertices in $N(v) \cup \{y, z\}$ and a path $p_1 p_2 \ldots p_{d+1}$ contained in $V(G) - N[v] - \{y, z\}$ such that $x_i p_i \rightarrow x_{i+1}$ for $1 \leq i \leq d+1$. In the ordering $x_1, x_2, \ldots x_{d+2}$, without loss of generality, $x_i = y$ for some $1 \leq i < d+2$. Since $y p_i \rightarrow x_{i+1}$, it follows that either p_i or y dominates v, which is not possible as neither p_i nor y are in $N(v)$. Therefore $|I - N(v)| \leq 1$.

∎

As with Lemma 4.4.8 and Corollary 4.4.9, the following corollary follows directly from Lemma 4.4.10 when the cardinality of a maximum independent set in G is at least three.

Corollary 4.4.11 *Let v be a vertex of degree $d \geq 2$ in a 3-γ_t-critical graph G and let I be a maximum independent set of cardinality at least three in G. If $N(v) \subseteq I$, then $|I - N(v)| \leq 1$.*

The following results relate several graph parameters in a 3-γ_t-critical graph G to graph parameters in $D^t(G)$. The first result shows that when the closure $D^t(G)$ is constructed, by adding to G every edge uv for which $\{u, v\} \succ G$, $D^t(G)$ will not contain a pair of nonadjacent vertices which dominate $D^t(G)$.

Theorem 4.4.12 *Let G be a 2-connected, 3-γ_t-critical graph. If u and v are nonadjacent vertices in $D^t(G)$, then $\{u, v\} \not\succ D^t(G)$.*

Proof: Suppose u and v are nonadjacent vertices in $D^t(G)$ and that $\{u, v\} \succ D^t(G)$. Then there exists a vertex w such that $uw, vw \notin E(G)$ (else uv would be in the closure). Since $\{u, v\} \succ D^t(G)$, either uw or vw is an edge in $D^t(G)$, and hence either $\{u, w\} \succ G$ or $\{v, w\} \succ G$. This gives a contradiction since uv, uw, and $vw \notin E(G)$. ∎

An important consequence of Theorem 4.4.12 is that for any pair of nonadjacent vertices in $D^t(G)$, there must exist a vertex w such that either $uw \rightarrow v$ or $vw \rightarrow u$ in G.

Theorem 4.4.13 *Let G be a 2-connected, 3-γ_t-critical graph. A set I of at least 3 vertices is independent in G if and only if it is independent in $D^t(G)$, and hence $\beta(G) = \beta(D^t(G))$ when $\beta(G) \geq 3$.*

Proof: Since G is a subgraph of $D^t(G)$, if I is independent in $D^t(G)$, then certainly I is independent in G.

Now assume I is an independent set in G with $|I| \geq 3$, and suppose I is not independent in $D^t(G)$. Then there are vertices $u, v \in I$ such that $uv \in E(D^t(G))$. It follows that $\{u, v\} \succ G$, contradicting $|I|$ being an independent set of size at least three in G. ∎

Theorem 4.4.14 *Let G be a 2-connected 3-γ_t-critical graph, and δ^t the minimum degree of $D^t(G)$. The independence number β^t of $D^t(G)$ satisfies $\beta^t \leq \delta^t + 2$. Moreover, if $\beta^t = \delta^t + 2$, then every maximum independent set contains all of the vertices of degree δ^t.*

Proof: Let v be any vertex of degree δ^t in $D^t(G)$, I any maximum independent set in $D^t(G)$, and $A = I - N[v]$. Notice that $|I \cap N[v]| \leq \delta^t$ and that if $v \in I$ then $I \cap N(v) = \emptyset$. The independence number is equal to $|N[v] \cap I| + |A|$, so $\beta^t = |A| + 1$ if $v \in I$ and $\beta^t = |A| + |N(v) \cap I|$ otherwise.

Consider first the case where $v \in I$, and hence $\beta^t = |A| + 1$. From the previous result, $\beta^t = 2$ is not possible. If $\beta^t = 1$, then $\delta^t = |V| - 1$, so $\beta^t < \delta^t + 2$ holds. If

$\beta^t \geq 3$, by Lemma 4.4.8 there exists an ordering $x_1, x_2, \ldots, x_k$ of the vertices of A and a path $P : p_1p_2 \ldots p_{k-1}$ in $N(v)$ such that $x_ip_i \to x_{i+1}$ for $1 \leq i \leq k-1$. This implies $\delta^t \geq k - 1 = |A| - 1$. Therefore $\beta^t = |A| + 1 \leq \delta^t + 2$.

Now consider the case where $v \notin I$. Consider any path $P \in \mathcal{P}_A$. Each vertex $p \in P$ must dominate v and hence lies in $N(v) - I$. It follows that

$$\delta^t + 1 = |N[v]| = |N(v) \cap I| + |N(v) - I| + 1 \geq |N(v) \cap I| + |A| - 1 + 1 = \beta^t.$$

In either case, $\beta^t \leq \delta^t + 2$, with equality only possible when $v \in I$ and $|A| = \delta^t + 1$. Since this is true for any minimum degree vertex v, if $\beta^t = \delta^t + 2$, then every maximum independent set must contain every vertex of degree δ^t. ∎

By the same reasoning as in the proof of Theorem 4.4.14, the following result is obtained:

Corollary 4.4.15 *Let G be a 2-connected 3-γ_t-critical graph, and δ the minimum degree of G. If the independence number β of G satisfies $\beta \geq 3$, then $\beta \leq \delta + 2$. Moreover, if $\beta = \delta + 2$, then every maximum independent set contains all the vertices of degree δ.*

When the upper bound in Theorem 4.4.14 for the independence number β^t of $D^t(G)$ is attained, more can be said about the structure of $D^t(G)$, and a lower bound on the maximum degree of $D^t(G)$ is found:

Theorem 4.4.16 *Let G be a 3-γ_t-critical graph for which $D^t(G)$ has minimum degree $\delta^t \geq 2$ and independence number $\beta^t = \delta^t + 2$. Let x be a vertex in $D^t(G)$ of degree δ^t. Then $D^t[N(x)]$ is complete and the maximum degree Δ^t of $D^t(G)$ satisfies $\Delta^t \geq 2\delta^t$.*

Proof: By Theorem 4.4.14, every maximum independent set in $D^t(G)$ is of the form $I \cup \{x\}$. Let $x_1, x_2, \ldots, x_{\delta^t+1}$ be an ordering of the vertices of I and P be a path $p_1p_2 \ldots p_{\delta^t}$ that satisfies $x_ip_i \to x_{i+1}$ for $1 \leq i \leq \delta^t$ (that is, $P \in \mathcal{P}_I$). By Lemma 4.4.8, the path P lies in $N(x)$, and since $d(x) = \delta^t$, $N(x) = \{p_1, p_2, \ldots, p_{\delta^t}\}$.

Now consider any two vertices x_{i+1} and x_{j+1} in I, where $1 \leq i \neq j \leq \delta^t$. Without loss of generality, there exists a vertex y such that $x_{i+1}y \to x_{j+1}$. The vertex y is in $N(x)$, and the only vertex in $N(x)$ that is not adjacent to x_{j+1} is p_j. Thus $y = p_j$. Now, since $x_{i+1}p_j \to x_{j+1}$ and $x_{i+1}p_i \notin E$, it must be that $p_jp_i \in E$. Since this holds for any pair i, j, it follows that $[N(x)]$ is complete. Moreover, every vertex $p_i \in N(x)$ is adjacent to every vertex in $(N[x] - \{p_i\}) \cup (I - \{x_{i+1}\})$. Hence $d(p_i) \geq \delta^t + 1 - 1 + \delta^t = 2\delta^t$ and therefore $\Delta^t \geq 2\delta^t$. ∎

Corollary 4.4.17 *If G is a 3-γ_t-critical graph for which $\delta(D^t(G)) \geq 2$ and $\Delta(D^t(G)) < 2\delta(D^t(G))$, then $\beta(D^t(G)) \leq \delta(D^t(G)) + 1$.*

Using the previous theorem, it can now be shown that when $\beta(D^t(G)) = \delta(D^t(G)) + 2$, as in Theorem 4.4.14, then the graph $D^t(G)$ has a unique vertex of minimum degree:

Theorem 4.4.18 *Let G be a connected 3-γ_t-critical graph. If the minimum degree δ^t and independence number β^t of $D^t(G)$ satisfy $\beta^t = \delta^t + 2$, then $D^t(G)$ has a unique vertex of degree δ^t.*

Proof: Let $I = \{x_1, x_2, \ldots, x_{\delta^t+2}\}$ be a maximum independent set in $D^t(G)$. By Theorem 4.4.14, I contains every vertex of degree δ^t. Consider any path $P : p_1p_2 \ldots p_{\delta^t+1} \in \mathcal{P}_I$, where $x_ip_i \to x_{i+1}$ for $1 \leq i \leq \delta^t + 1$.

Suppose $d(x_j) = \delta^t$ for some j with $1 \leq j \leq \delta^t + 1$. Since $p_j \in N(x_j)$ and by Theorem 4.4.16 $[N(x_j)]$ is complete, $p_j \succ N(x_j)$. Also, since $x_jp_j \to x_{j+1}$, $p_j \succ V - (N(x_j) - \{x_{j+1}\})$. Therefore, $p_j \succ V - \{x_{j+1}\}$. But then $\{p_j, x_{j+1}\} \succ D^t(G)$, contradicting Theorem 4.4.12. Thus x_{δ^t+2} is the only vertex of degree δ^t. ∎

The last result of this chapter is analogous to Lemma 3.2.4 for 3-i-critical graphs:

Theorem 4.4.19 *Let G be a connected 3-γ_t-critical graph. If S is a vertex cut of G, then $G - S$ has at most $|S| + 1$ components.*

Proof: Let m be the number of components of $G - S$. If $m = 2$, then clearly $m = 1 + 1 \leq |S| + 1$. If $m \geq 3$, let $C_1, C_2, \ldots, C_m$ denote the components of

$G-S$, and let x_i be any vertex in C_i, for $i = 1, 2, \ldots, m$. The set $\{x_1, x_2, \ldots, x_m\}$ is an independent set in G, and hence by Corollary 4.4.7, there exists a path $p_1p_2 \ldots p_{m-1}$ where $x_ip_i \rightarrow x_{i+1}$ for $i = 1, 2, \ldots, m-1$. For each i, $p_i \in S$ as p_i dominates $V - C_i$. Therefore $m - 1 \leq |S|$, which completes the proof. ∎

It should be noted that this chapter is not a complete summary of all of the general results which have been found for 3-γ_t-critical graphs. Several new general results arise when specific subfamilies of 3-γ_t-critical graphs are considered. Such results will be given as required, as some of them make use of notation that is not yet introduced.

Chapter 5

Diameter Three 3-γ_t-critical Graphs

In the previous chapter, we recalled (see Theorem 4.3.7) that in [13] it was proven that a 3-γ_t-critical graph G has diameter either two or three. In the same paper, it was shown that a 3-γ_t-critical graph with a cut vertex has diameter three, and these graphs were characterised.

This chapter will focus on the 2-connected 3-γ_t-critical graphs with diameter three. Specifically, in Section 5.1, results that hold for all 3-γ_t-critical graphs with diameter three will be given. These graphs will then be partitioned into four families of graphs based on the type of maximal diametrical pair the graph posesses. This will be explained in detail in Section 5.1.

The other sections in this chapter contain some characterisations of the families of graphs defined in section 5.1, as well as proof that all 3-γ_t-critical graphs with diameter three contain a Hamilton path (and in most cases contain a Hamilton cycle).

5.1 Properties when $diam(G) = 3$

The results which hold for all 3-γ_t-critical graphs with diameter three are now given. At the end of this section, the 3-γ_t-critical graphs with diameter three will be partitioned into four families which will be studied separately in the remaining sections of Chapter 5.

First, three results regarding 3-γ_t-critical graphs with diameter three that were given in Section 4.4 to motivate the need for a new closure concept are restated.

Theorem 5.1.1 *[14] If G is a 3-γ_t-critical graph with $diam(G) = 3$, then G has a unique maximal diametrical pair (X, Y). Moreover, X (say) has cardinality one and $[Y]$ is complete.*

Theorem 5.1.2 *[14] If G is a 3-γ_t-critical graph with $diam(G) = 3$, then G has a pair of nonadjacent vertices that does not dominate G.*

Theorem 5.1.3 *[14] If G is a 3-γ_t-critical graph with $diam(G) = 3$, then G has a pair of nonadjacent vertices u and v such that $ux \rightarrow v$, for some $x \in V$, but there is no vertex y such that $vy \rightarrow u$.*

In [14], the fact that every 3-γ_t-critical graph with diameter three has a unique maximal diametrical pair $(\{x\}, Y)$ is used to divide the 3-γ_t-critical graphs with diameter three into four subfamilies. The notation introduced in [14] will be adopted here:

Let $\mathcal{F}$ be the family of all graphs G with diameter three and maximal diametrical pair $(\{x\}, Y)$. For $G \in \mathcal{F}$, let $A = N(x)$, $B = \{b \mid b \notin Y \text{ and } b \succ Y\}$, and $C = V - (A \cup B \cup Y \cup \{x\})$. Note that at least one of B and C is not empty. Now $\mathcal{F}$ can be partitioned into $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$, where

$G \in \mathcal{F}_1$ if $C = \emptyset$ and $|Y| \geq 2$;

$G \in \mathcal{F}_2$ if $C = \emptyset$ and $|Y| = 1$;

$G \in \mathcal{F}_3$ if $B = \emptyset$;

$G \in \mathcal{F}_4$ if $B \neq \emptyset$ and $C \neq \emptyset$.

In [14], several results are given for the 3-γ_t-critical graphs in each of the above families. The results which hold for all families are now given. The results specific to a particular family will be given in the appropriate section of Chapter 5.

Lemma 5.1.4 *[14] Let $G \in \mathcal{F}$ be 3-γ_t-critical with $|Y| \geq 2$. If either $B = \emptyset$ or $C = \emptyset$, then $[A]$ is complete.*

Lemma 5.1.5 *[14] If $G \in \mathcal{F}$ is 3-γ_t-critical, then every vertex in C is adjacent to exactly $|Y| - 1$ vertices in Y.*

Lemma 5.1.6 *[14] If $G \in \mathcal{F}$ is 3-γ_t-critical and $C \neq \emptyset$, then $[C]$ is complete.*

When the hamiltonian properties of each of the families of 3-γ_t-critical graphs in $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$ are discussed in the remaining sections of Chapter 5, the closure $D^t(G)$ will be used extensively. Several results relating graph parameters in $D^t(G)$ to graph parameters in G were given in Section 4.4. A corollary to Theorem 4.4.13 holds for the diameter three case:

Corollary 5.1.7 *Let G be a 2-connected 3-γ_t-critical graph with $diam(G) = 3$. Then $\beta(G) = \beta(D^t(G))$.*

Proof: By Theorem 5.1.2, $\beta(G) \geq 3$, and the result follows directly from Theorem 4.4.13. ∎

Corollary 5.1.8 *Let G be a 2-connected 3-γ_t-critical graph with $diam(G) = 3$. If $\beta(D^t(G)) = \delta(D^t(G)) + 2$, then $\delta(G) = \delta(D^t(G))$.*

Proof: By Corollary 5.1.7, $\beta(G) = \beta(D^t(G))$, and $\beta(G) \leq \delta(G) + 2$ by Corollary 4.4.15. Therefore,

$$\beta(D^t(G)) = \beta(G) \leq \delta(G) + 2 \leq \delta(D^t(G)) + 2.$$

Since $\beta(D^t(G)) = \delta(D^t(G)) + 2$, it follows that $\delta(G) + 2 = \delta(D^t(G)) + 2$ and hence $\delta(G) = \delta(D^t(G))$. ∎

Throughout the remainder of Chapter 5 (and Chapter 6), the following notation will be used:

When $|A| = m$, the elements of A will be denoted by $\{a_1, a_2, \ldots, a_m\}$. The set B can be partitioned into sets of vertices with the same neighbourhood in A. The notation $B_{i_1, i_2, \ldots, i_p}$ will be used to denote the subset of B given by

$$B_{i_1, i_2, \ldots, i_p} = \{b \in B \mid N(b) \cap A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_p}\}\}.$$

Hence B is partitioned into at most $2^m - 1$ subsets. Furthermore, for any $a_i \in A$, we will denote by $N_B(a_i)$, the subset of vertices $b \in B$ that are adjacent to a_i. That is, $N_B(a_i) = N(a_i) \cap B$.

5.2 3-γ_t-critical graphs in $\mathcal{F}_1$

In this section, the 3-γ_t-critical graphs in $\mathcal{F}_1$ are shown to be hamiltonian.

As a starting point, [14] provides a collection of neccessary conditions for a 3-γ_t-critical graph to be in $\mathcal{F}_1$:

Theorem 5.2.1 *[14] A graph $G \in \mathcal{F}_1$ is 3-γ_t-critical if and only if the following conditions hold:*

1. *$(\{x\}, Y)$ is the unique maximal diametrical pair of G and $[Y]$ is complete.*
2. *$[A]$ is complete.*
3. *For every pair of nonadjacent vertices $u, v \in B$, there is a vertex $a \in A$ such that $ua \rightarrow v$. Also, no pair of adjacent vertices dominates G.*
4. *For every vertex $b \in B$, there is a vertex $d \in B \cup Y$ such that $bd \rightarrow x$.*
5. *For every pair $\{a, b\}$ of nonadjacent vertices where $a \in A$ and $b \in B$, $\{a, b\} \succ G$ or $aw \rightarrow b$ for some $w \in B$.*

In [14], a complete characterisation is given of the family of graphs in $\mathcal{F}_1$ that are 3-γ_t-critical and have $\delta(G) = 2$:

Theorem 5.2.2 *[14] A graph $G \in \mathcal{F}_1$ with $\delta(G) = 2$ is 3-γ_t-critical if and only if the following conditions hold:*

1. *$(\{x\}, Y)$ is the unique maximal diametrical pair of G and $[Y]$ is complete.*
2. *$d(x) = 2$ and $[A]$ is complete.*

3. $B_{1,2} = \emptyset$ *or* $[B_{1,2}]$ *is complete.*

4. $|B_i| \geq 2$ *and* $[B_i]$ *is complete, for* $i \in \{1, 2\}$.

5. $[\overline{B_1 \cup B_2}]$ *is the disjoint union of nontrivial stars.*

6. *If* $B_{1,2} \neq \emptyset$, *then every vertex in* $B_{1,2}$ *dominates exactly* $|B_i| - 1$ *vertices in* B_i, *for* $i \in \{1, 2\}$. *Also, if* $u \in B_1$ *(*$u \in B_2$*, respectively) does not dominate* $B_{1,2}$, *then* u *has a neighbour* $v \in B_{1,2} \cup B_2$ *(*$B_{1,2} \cup B_1$*, respectively) such that* $\{u, v\} \succ B$.

An example of a 3-γ_t-critical graph in $\mathcal{F}_1$ with $\delta = 2$ is given in [14]:

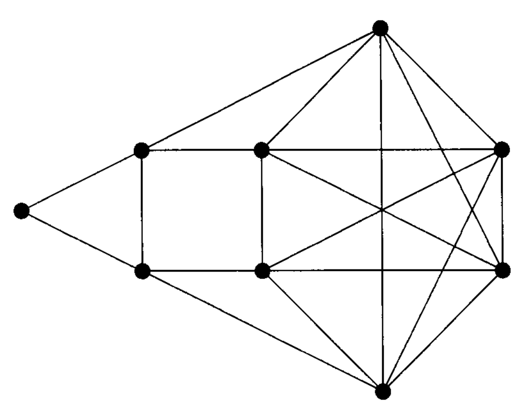

Figure 5.1: A 3-γ_t-critical graph $G \in \mathcal{F}_1$ with $\delta = 2$

For any 3-γ_t-critical graph $G \in \mathcal{F}_1$, since $C = \emptyset$, $\{x, y\} \succ G$ for any $y \in Y$. Thus, we have the following.

Corollary 5.2.3 *If $G \in \mathcal{F}_1$ is 3-γ_t-critical, then $\gamma(G) = 2$.*

We are now ready to prove the main result of this section, that every 3-γ_t-critical graph $G \in \mathcal{F}_1$ is hamiltonian. The proof makes use of the fact that a graph G is hamiltonian if the connectivity of G is not less than the independence number of G:

Theorem 5.2.4 *[6] For any graph G, if $\kappa(G) \geq \beta(G)$ (and $G \neq K_2$), then G is hamiltonian.*

Theorem 5.2.5 *If G is a 3-γ_t-critical graph in $\mathcal{F}_1$, then G is hamiltonian.*

Proof: Consider $cl(D^t(G))$ (as defined in Section 2.2). By Theorem 5.2.1, both $[Y]$ and $[A]$ are complete. For every $y \in Y$, since $\{x, y\} \succ G$, xy is an edge in $D^t(G)$. Also, by Theorem 5.2.1 condition 4, every vertex in A must be adjacent to a vertex in B. Therefore in $D^t(G)$, for any $a \in A$ and $y \in Y$,

$$d(a) + d(y) \geq |A| + 1 + |Y| + |B| = n,$$

so ay is an edge in $cl(D^t(G))$.

Now consider $a \in A$ and let $\overline{N_B}(a) = B - N_B(a)$. For each $b \in \overline{N_B}(a)$ there exists $b' \in N_B(a)$ such that $ab' \rightarrow b$ and $b' \succ \overline{N_B}(a) - b$ (and hence the b''s are distinct). Therefore $|N_B(a)| \geq |\overline{N_B}(a)|$ and each $b \in \overline{N_B}(a)$ has at least

$|\overline{N_B}(a)| - 1$ neighbours in $N_B(a)$. Hence in $cl(D^t(G))$,

$$d(a) + d(b) \geq (|A| + |N_B(a)| + |Y|) + (|Y| + |\overline{N_B}(a)| - 1) = n + |Y| - 2 \geq n.$$

So $ab \in cl(D^t(G))$ for all $a \in A$, $b \in B$. It has now been established that if $uv \notin E(cl(D^t(G)))$, then either $u = x$ and $v \in B$, or $u, v \in B$.

Now, let S be a vertex cut of $cl(D^t(G))$ and u, v be vertices in $V - S$ such that in $cl(D^t(G))$ there is no $u - v$ path. If $u = x$ and $v \in B$, then $A \cup Y \subseteq S$. If $u, v \in B$, again $A \cup Y \subseteq S$. Hence $|S| \geq |A| + |Y|$ and $\kappa = \kappa(cl(D^t(G))) \geq |A| + |Y|$.

Let I be an independent set of cardinality $\beta = \beta(cl(D^t(G)))$. If $\beta = 1$, then clearly $\kappa \geq \beta$. Otherwise, for any $v \in A \cup Y$, $v \notin I$, so $I \subseteq B \cup \{x\}$. Let I_B be a maximum independent set in $[B]$. We have

$$\beta - 1 = |I| - 1 \leq |I_B| \leq |I| = \beta.$$

By Theorem 4.4.6, the vertices of I_B can be ordered $b_1, b_2, \ldots, b_{|I_B|}$ in such a way that there is a path in A, $p_1 p_2 \ldots p_{|I_B - 1|}$, that satisfies $p_i b_i \rightarrow b_{i+1}$ for $1 \leq i \leq |I_B| - 1$. Hence $|A| \geq |I_B| - 1$ and

$$\kappa \geq |A| + |Y| \geq |I_B| - 1 + |Y| \geq |I_B| + 1 \geq |I| = \beta.$$

Since $\kappa \geq \beta$ in $cl(D^t(G))$, by Theorem 5.2.4, $cl(D^t(G))$ is hamiltonian. Therefore $D^t(G)$ is also hamiltonian, and by Theorem 4.4.5 G is hamiltonian. ∎

5.3 3-γ_t-critical graphs in $\mathcal{F}_2$

In this section, the 3-γ_t-critical graphs in $\mathcal{F}_2$ will be shown to be hamiltonian. The main method used is showing that $\kappa \geq \beta$, as in the proof of Theorem 5.2.5. This proves to be a much more difficult problem than it was for the 3-γ_t-critical graphs in $\mathcal{F}_1$, however.

Theorem 5.3.1 *[14] A graph $G \in \mathcal{F}_2$ is 3-γ_t-critical if and only if the following conditions hold:*

1. *$(\{x\},\{y\})$ is the unique maximal diametrical pair of G.*
2. *For each $a \in A$ and $b \in B$ with $ab \in E(G)$, $\{a,b\} \not\succ G$.*
3. *For each $a, a' \in A$ with $aa' \notin E(G)$, there exists $b' \in B$ such that $ab' \rightarrow a'$. A similar statement holds for each $b, b' \in B$ with $bb' \notin E(G)$.*
4. *For every $a \in A$, $\{a, y\} \succ G$ or there exists $a' \in A$ such that $aa' \rightarrow y$. A similar statement holds for every $b \in B$.*
5. *For every pair $\{a,b\}$ of nonadjacent vertices where $a \in A$ and $b \in B$, $\{a,b\} \succ G$ or there exists $b' \in B$ (or $a' \in A$) such that $ab' \rightarrow b$ ($a'b \rightarrow a$).*

What follows is a collection of results which will be used to prove that all 3-γ_t-critical graphs in $\mathcal{F}_2$ are hamiltonian. First, recall by Corollary 5.1.7 that

$\beta(G) = \beta(D^t(G))$. Furthermore, $\kappa(D^t(G)) \geq \kappa(G)$, so if $\kappa(G) \geq \beta(G)$, then $\kappa(D^t(G)) \geq \beta(D^t(G))$. However, it may be that $\kappa(D^t(G)) \geq \beta(D^t(G))$ when $\kappa(G) \not\geq \beta(G)$. Hence rather than attempting to show $\kappa(G) \geq \beta(G)$ for a 3-γ_t-critical graph in $\mathcal{F}_2$, we will show $\kappa(D^t(G)) \geq \beta(D^t(G))$.

For any 3-γ_t-critical graph in $\mathcal{F}_2$, consider a minimum vertex cut S and a maximum independent set I in $D^t(G)$. Since $D^t(G) - S$ is not connected, the vertices of $V - S$ can be partitioned into two sets $\{C_1, C_2\}$, so that no edge of $D^t(G)$ has one end in C_1 and the other in C_2. In what follows, the graphs in $\mathcal{F}_2$ are partitioned according to the size of the subsets C_1 and C_2. The graphs arising in each case are then shown to be hamiltonian. We first consider the case where both C_1 and C_2 contain at least two vertices.

Lemma 5.3.2 *Let G be a 3-γ_t-critical graph G in $\mathcal{F}_2$, S a minimum vertex cut in $D^t(G)$, and I a maximum independent set in $D^t(G)$. Let $\{C_1, C_2\}$ be a partition of $V - S$ such that no edge of $D^t(G)$ has one end in C_1 and the other in C_2. If $|C_1| \geq 2$ and $|C_2| \geq 2$, then $\kappa(D^t(G)) \geq \beta(D^t(G))$ and hence G is hamiltonian.*

Proof: Let $x_1, x_2, \ldots, x_\beta$ be an ordering of the vertices in I and $P : p_1 p_2 \ldots p_{\beta-1}$ be a path in $\mathcal{P}_I$. That is, $x_i p_i \rightarrow x_{i+1}$ in G for $i = 1, 2, \ldots, \beta - 1$.

If $x_i \in C_1$, where $i \neq \beta$, then since x_i has no neighbours in C_2, and $|C_2| \geq 2$, p_i must dominate $C_2 - \{x_{i+1}\}$. Also, $x_i p_i \in E(G)$, so $p_i \in S$. Similarly, if $x_i \in C_2$, where $i \neq \beta$, then $p_i \in S$. Let I_{C_1} denote $I \cap C_1$, I_{C_2} denote $I \cap C_2$, and I_S denote $I \cap S$. If $x_\beta \in I_S$, then $|S| \geq |I_S| + |I_{C_1}| + |I_{C_2}| = |I|$ and hence $\kappa \geq \beta$.

If $x_\beta \in I_{C_1} \cup I_{C_2}$, then $|S| \geq |I_S| + |I_{C_1}| + |I_{C_2}| - 1 = \beta - 1$. If there exists $x_i \in I_S$ such that $p_i \in S$ then this can be improved to $|S| \geq \beta - 1 + 1 = \beta$. Otherwise, $x_\beta \in I_{C_1} \cup I_{C_2}$ and for each $x_i \in S$, $p_i \notin S$. In this case, suppose $x \in I$. Then $y \notin I$ as $xy \in E(D^t(G))$. Also, $x = x_\beta$ as $xp_i \rightarrow x_{i+1}$ is not possible for any i, as $I - x \subseteq B$ and $N[x] \not\succ \{y\}$. Without loss of generality, let $x_\beta \in I_{C_1}$. Since $y \notin I$ and $\{x, y\} \succ G$, it follows that $y \in S - I_S$. Therefore $|S| \geq |I_{C_1} \cup I_{C_2}| - 1 + |I_S| + 1 = |I|$, as $y \neq p_i$ for any i, since $p_i \in A$ for $i = 1, 2, \ldots, \beta - 2$ and $p_{\beta-2}p_{\beta-1} \in E(G)$. Similarly, if $y \in I$, then $|S| \geq |I|$. Finally, suppose $I \subseteq A \cup B$. Then by Theorem 5.3.1, $p_i \in A \cup B$ for $i = 1, 2, \ldots, \beta - 1$. Now if x or y is in S, then since x and y are not on P, $|S| \geq |I_{C_1} \cup I_{C_2}| - 1 + |I_S| + 1 = |I|$. Otherwise, if $x, y \notin S$, then since $xy \in E(D^t(G))$, without loss of generality, $x, y \in C_1$. But then since $\{x, y\} \succ G$, it follows that $C_2 = \emptyset$, a contradiction.

Since $\kappa \geq \beta$ in all cases, $D^t(G)$ (and hence G) is hamiltonian. ∎

Lemma 5.3.3 *Let G be a 3-γ_t-critical graph in $\mathcal{F}_2$, S a minimum vertex cut in $D^t(G)$, and I a maximum independent set in $D^t(G)$. Let $\{C_1, C_2\}$ be a partition of $V - S$ such that no edge of $D^t(G)$ has one end in C_1 and the other in C_2. If $|C_2| = 1$ (or $|C_1| = 1$) and $I \subseteq C_1 \cup S$ ($I \subseteq C_2 \cup S$), then G is hamiltonian.*

Proof: Let I_{C_1} denote $I \cap C_1$ and I_S denote $I \cap S$. Since I is maximum, $I_S \neq \emptyset$. Let $x_1, x_2, \ldots, x_\beta$ be an ordering of I for which there is a path $P : p_1 p_2 \ldots p_{\beta-1} \in \mathcal{P}_I$. For any $x_i \in C_1$ where $i < \beta$, p_i must be in S since $\{x_i, p_i\} \succ \{c\}$ (where $\{c\} = C_2$). So if $x_\beta \notin C_1$, then $|S| \geq |I_S| + |I_{C_1}| = |I|$ and G is hamiltonian. If, however, $x_\beta \in C_1$, we will consider the possible locations for x and y (where $(\{x, y\})$ is the unique maximal diametrical pair of G).

If $x \in I$, then $I - \{x\} \subseteq B$, $x = x_\beta$, $p_i \in A$ for $1 \leq i \leq \beta - 2$, and $p_{\beta-1} \in B$. Since $xc \notin E$, $c \in B$ and $y \in S$. But y is not on P, so $|S| \geq |I_{C_1}| - 1 + |I_S| + 1 = |I|$. Similarly, if $y \in I$, then $|S| \geq |I|$. In either case, $\kappa \geq \beta$.

Otherwise, if $x, y \notin I$, then $I \subseteq A \cup B$ and $p_i \in A \cup B$ for $1 \leq i \leq \beta - 1$. Now, if x or y is in S, then $|S| \geq |I_{C_1}| - 1 + |I_S| + 1 = |I|$ (and again $\kappa \geq \beta$). If $x, y \notin S$, then since $\{x, y\} \succ G$, $xy \in E(D^t(G))$ and hence $x, y \in C_1$ and $C_2 = \emptyset$, a contradiction.

In all cases that do not lead to a contradiction, $\kappa(D^t(G)) \geq \beta(D^t(G))$. Therefore $D^t(G)$, and hence G, is hamiltonian. ∎

Lemma 5.3.4 *Let G be a 3-γ_t-critical graph in $\mathcal{F}_2$, S a minimum vertex cut in $D^t(G)$, and I a maximum independent set in $D^t(G)$. Let $\{C_1, C_2\}$ be a partition of $V - S$ such that no edge of $D^t(G)$ has one end in C_1 and the other in C_2. If $|C_2| = 1$ (or $|C_1| = 1$) and $I \subseteq C_1 \cup C_2$ and $I \subseteq A \cup B$, then G is hamiltonian.*

Proof: Since I is maximum, $|I| \geq 3$ and there exist vertices a and b such that $a \in A \cap I$ and $b \in B \cap I$. Without loss of generality, let $C_2 = \{a\}$. Then $x \in S$.

Suppose $y \in S$. Then there is a path P on $\beta - 2$ vertices in $\mathcal{P}_{I_{C_1}}$ which must lie in S. Since $I \subseteq A \cup B$, $p_i \in A \cup B$ for all i. Therefore $|S| \geq \beta - 2 + 2 = \beta$.

Now suppose $y \notin S$. Then $y \in C_1$ and $ay \notin E(D^t(G))$. So there exists $a' \in A$ such that $aa' \to y$. This implies $a' \in S$ and $a' \succ I$, which gives a' not on P. Therefore $|S| \geq \beta - 2 + 1 + 1 = \beta$. It follows that G is hamiltonian. ∎

Lemma 5.3.5 *Let G be a 3-γ_t-critical graph in $\mathcal{F}_2$, S a minimum vertex cut in $D^t(G)$, and I a maximum independent set in $D^t(G)$. Let $\{C_1, C_2\}$ be a partition of $V - S$ such that no edge of $D^t(G)$ has one end in C_1 and the other in C_2. If $|C_2| = 1$ (or $|C_1| = 1$) and $I \subseteq C_1 \cup C_2$ and $x \in I \cap C_1$ (or $y \in I \cap C_1$), then G is hamiltonian.*

Proof: Let $x_1, x_2, \ldots, x_\beta$ be an ordering of I for which there is a path $P : p_1p_2 \ldots p_{\beta-1} \in \mathcal{P}_I$. Then $x_ip_i \to x_{i+1}$ for $i = 1, 2, \ldots, \beta - 1$. Since $x \in I_{C_1}$, $I - \{x\} \subseteq B$ and $C_2 = I_{C_2} = \{c\}$ for some $c \in B$. Since $S = N(c)$, it follows that $y \in S$. For any $b \in B$, there is no w such that $xw \to b$, so $x = x_\beta$ and $p_i \in A$ for $i = 1, 2, \ldots, \beta - 2$. Also, $x_{\beta-1} \not\succ A$ as $x_{\beta-2}p_{\beta-2} \to x_{\beta-1}$ and $p_{\beta-2} \in A$. Therefore $p_{\beta-1} \in B$.

If P is contained in S, then $|S| \geq (\beta - 1) + 1 = \beta$. Otherwise, $c = x_{i+1}$ for some i where $1 \leq i \leq \beta - 2$ and $p_i \in C_1 \cap A$.

Now $x_{i+1}x \notin E(D^t(G))$ implies that there exists $b' \in B$ such that $x_{i+1}b' \to x$. Since $S = N(x_{i+1})$, $b' \in S$. If $S \cap B \neq \{p_{\beta-1}\}$, then $|S| \geq (\beta - 2) + 1 + 1 = \beta$. Otherwise $b' = p_{\beta-1}$ and $x_{i+1}p_{\beta-1} \to x$. Then $p_{\beta-1} \succ C_1 - \{x\}$ and hence $p_{\beta-1} \succ B$ in G. But now $\{x, p_{\beta-1}\} \succ G$, which contradicts $xp_{\beta-1} \notin E(D^t(G))$.

Therefore in all cases, $\kappa(D^t(G)) \geq \beta(D^t(G))$ and it follows that $D^t(G)$ (and hence G) is hamiltonian. ∎

Before the last lemma of this section is given, a definition is needed: A set of paths will be called *independent* if no vertex occurs on more than one path, and end vertices of distinct paths are not adjacent.

Lemma 5.3.6 *Let G be a 3-γ_t-critical graph in $\mathcal{F}_2$, S a minimum vertex cut in $D^t(G)$, and I a maximum independent set in $D^t(G)$. Let $\{C_1, C_2\}$ be a partition of $V - S$ such that no edge of $D^t(G)$ has one end in C_1 and the other in C_2. If $|C_2| = 1$ (or $|C_1| = 1$), $I \subseteq C_1 \cup C_2$, and $C_2 = \{x\}$ (or $\{y\}$), then G is hamiltonian.*

Proof: Since $C_2 = \{x\}$, $S = N(x)$ and $A \cup \{y\} \subseteq S$. Since $I \subseteq C_1 \cup C_2$ and $C_1 \subseteq B$, $I = B' \cup \{x\}$ for some $B' \subseteq B$. Consider an ordering $x_1, x_2, \ldots, x_\beta$ of the vertices in I for which there is a path $P : p_1p_2 \ldots p_{\beta-1}$ in $\mathcal{P}_I$. Note that $xw \to b$ is not possible for any $w \in V$ and $b \in B$, and $x_i p_i \to x_{i+1}$ for $i = 1, 2, \ldots, \beta - 1$. Therefore $x = x_\beta$ and $p_i \in A$ (and hence $p_i \in S$) for $i = 1, 2, \ldots, \beta - 2$. For $i = \beta - 1$, $x_{\beta-1} p_{\beta-1} \to x$ gives $p_{\beta-1} \in B$ since $x_{\beta-1} \not\succ A$. So far, $\{p_1, p_2, \ldots, p_{\beta-2}\} \cup \{y\} \subseteq S$.

If $B \cap S \neq \emptyset$, then $|S| \geq (\beta - 2) + 1 + 1 = \beta$ and $D^t(G)$ (and therefore G) is hamiltonian. Otherwise, $S = A \cup \{y\}$ and $C_1 = B$. If there exists $a \in A - \{p_1, p_2, \ldots, p_{\beta-2}\}$, then $|S| \geq (\beta - 2) + 1 + 1 = \beta$ and G is hamiltonian.

The only remaining case is when $S = A \cup \{y\}$ and $A = \{p_1, p_2, \ldots, p_{\beta-2}\}$. In this case, $\kappa(D^t(G)) < \beta(D^t(G))$, so G will be shown to be hamiltonian by directly finding a Hamilton cycle. First, we establish the following two facts:

Fact 1: $x_1 \succ B - B'$.

Proof: Consider any $b \in B - B'$. Since $x_1 \succ A$, if $x_1 b \notin E$, then $x_1 a \to b$ for some $a \in A$. Since $A = \{p_1, p_2, \ldots, p_{\beta-2}\}$, $a x_{i+1} \notin E$ for some $i \in \{1, 2, \ldots, \beta - 2\}$. But $x_1 x_{i+1} \notin E$ either, which contradicts $x_1 a \to b$. Therefore $x_1 \succ B - B'$.

Fact 2: For any $b \in B - B'$ and $i \in \{2, 3, \ldots, \beta - 2\}$, either $b x_i \in E$ or $b p_{i-1} \in E$.

Proof: Suppose $b x_i \notin E$ for some $i \in \{2, 3, \ldots, \beta - 2\}$. As in the proof of Fact 1, $x_i a \not\to b$ (for all $a \in A$) since this would imply $a \succ B'$. Therefore $ba \to x_i$ and $a = p_{i-1}$. Now $b p_{i-1} \to x_i$ gives $b p_{i-1} \in E$, which completes the proof of Fact 2.

Now, notice $\beta \geq 4$ and $|B'| \geq 3$, as $|A| \geq 2$ (there are no cut vertices). Consider a maximum independent set in $B - B'$. Since $xb \notin E(D^t(G))$ for all

$b \in B$, the independence number of $B - B'$ is no larger than $|B'| = \beta - 1$ (since I is maximum), and hence $B - B'$ can be partitioned into $m \leq \beta - 1$ independent paths.

Let $\{b_1, b_2, \ldots, b_m\}$ be a set obtained by taking one endvertex from each of the m paths in $B - B'$. Let Q be any nonempty subset of $\{b_1, b_2, \ldots, b_m\}$. Since Q is an independent set, the number of vertices in B' not dominated by Q is no more than $\beta - 1 - |Q|$. Therefore $|N_{B'}(Q)| \geq |Q|$. By Hall's Theorem, there is a matching of $\{b_1, b_2, \ldots, b_m\}$ into B'.

Using the edges defined by this matching, we can obtain a partition of B into $\beta - 1$ paths with endvertices $x_1, x_2, \ldots, x_{\beta-1}$. We will refer to these end vertices as the *heads* of the paths, and the other end vertices of the paths as the *tails.* If $m < \beta - 1$, then some paths will be a single vertex, which will be both the head and tail of its path.

By Fact 1, x_1 is adjacent to the tail of any path with more than one vertex. Adjoin x_1 to the tail of any such path. We know such a path exists as $B - B' \neq \emptyset$, since $p_{\beta-1} \in B - B'$. Continue to join two paths whenever they are not independent. Any new path created will still have its head in $\{x_2, x_3, \ldots, x_{\beta-1}\}$. This process yields a partition of B into r independent paths $P_1, P_2, \ldots, P_r$, where $1 \leq r \leq \beta - 2$. Denote the head of P_i by b'_i and the tail by b_i, for each $i = 1, 2, \ldots, r$. Note that $\{b'_1, b'_2, \ldots, b'_r\} \subseteq \{x_2, x_3, \ldots, x_{\beta-1}\}$. Without loss of generality, we can assume that the paths $P_1, P_2, \ldots, P_r$ are labelled such that if $1 \leq i < j \leq r$ and $b'_i = x_k$ and $b'_j = x_l$, then $k < l$. In other words, the paths are ordered according to the order in which their heads occur in $\{x_1, x_2, \ldots, x_{\beta-1}\}$.

Suppose $r = 1$ and $b'_1 = x_i$, $2 \leq i \leq \beta - 1$. Then since $x_i \succ A - \{p_{i-1}\}$, either $p_1 x_i \in E$ or $p_{\beta-2} x_i \in E$. So either

$$x p_{\beta-2} p_{\beta-3} \ldots p_1 b'_1 \ldots b_1 y x$$

or

$$x p_1 p_2 \ldots p_{\beta-2} b'_1 \ldots b_1 y x$$

is a hamilton cycle in $D^t(G)$, and hence G is hamiltonian.

Now suppose $r = 2$. Then $b'_1 = x_i$ and $b'_2 = x_j$ for some $2 \leq i < j \leq \beta - 1$. Now

$$y b_1 \ldots b'_1 p_i p_{i+1} \ldots p_{\beta-2} x p_1 p_2 \ldots p_{i-1} b'_2 \ldots b_2 y$$

is a Hamilton cycle in $D^t(G)$, and hence G is hamiltonian.

Now suppose $r \geq 3$. For each $i \in \{1, 2, \ldots, r\}$, let a_i be the unique vertex in A that is not adjacent to b'_i. For any b_i, b'_j where $i \neq j$, we know that $b_i b'_j \notin E$. By Fact 2, $b_i a_j \in E$. Since this is true for all $i \neq j$, it follows that for each i, b_i has at least $r - 1$ neighbours in $A' = \{a_1, a_2, \ldots, a_r\}$. Specifically, $b_i a_{i+1} \in E$ (subscripts are mod r) for $i \in \{1, 2, \ldots, r\}$. Now, since $b'_i \succ A - \{a_i\}$, certainly $b'_i a_{i+2} \in E$ for $i \in \{1, 2, \ldots, r\}$. We can join $P_1, P_2, \ldots, P_r$ together using the above edges to get the path

$$a_2b_1\dots b'_1a_3b_2\dots b'_2a_4\dots b'_{r-2}a_rb_{r-1}\dots b'_{r-1}a_1b_r\dots b'_r$$

in $D^t(G)$ which uses all vertices in $\{a_1, a_2, \dots, a_r\} \cup B$. We will now extend the path to include all of A.

Due to the way the P_i were labelled, if $a_i = p_k$ and $a_j = p_l$ where $i < j$, then $k < l$. If $a_2 = p_l$ and $a_1 = p_k \neq p_{l-1}$, then replace a_2 on the path by $p_{k+1}p_{k+2}\dots a_2$. If $a_1 = p_k \neq p_1$, replace $b'_{r-1}a_1$ on the path by $b'_{r-1}p_1p_2\dots a_1$. This is possible since $b'_{r-1} \succ A - \{a_{r-1}\}$ and $p_1 \neq a_{r-1}$. Finally, for $i \in \{3, 4, \dots, r\}$, if $a_i = p_l$ and $a_{i-1} = p_k \neq p_{l-1}$, replace $b'_{i-2}a_i$ on the path by $b'_{i-1}p_{k+1}p_{k+2}\dots a_i$. Call the obtained path P.

Since P has one end in A and the other end in B, if $a_r = p_{\beta-2}$ then $xPyx$ is a Hamilton cycle in $D^t(G)$. If $a_r = p_k \neq p_{\beta-2}$, then since b'_r is an end vertex of P and $b'_rp_{k+1} \in E$, $xPp_{k+1}\dots p_{\beta-2}x$ is a cycle in $D^t(G)$. Now since $r \leq \beta - 2$, some path P_i has at least two vertices, and each path P_i is a subpath of P. Therefore there is an edge $w_1w_2 \in P$ such that $w_1, w_2 \in B$. Replacing w_1w_2 on the cycle by w_1yw_2 gives a Hamilton cycle in $D^t(G)$. In either case, G is hamiltonian. ∎

Theorem 5.3.7 *If G is a 3-γ_t-critical graph in $\mathcal{F}_2$, then G is hamiltonian.*

Proof: Let S be a vertex cut of $D^t(G)$ of cardinality $\kappa(D^t(G))$, and let I be an independent set of $D^t(G)$ of cardinality $\beta(D^t(G))$. Let $\{C_1, C_2\}$ be a partition of

$V - S$ such that no edge of $D^t(G)$ has one end in C_1 and the other in C_2.

If $|C_1| \geq 2$ and $|C_2| \geq 2$, then by Lemma 5.3.2, G is hamiltonian. Otherwise, without loss of generality, assume $|C_2| = 1$.

If $I \subseteq C_1 \cup S$, then by Lemma 5.3.3, G is hamiltonian. If $I \nsubseteq C_1 \cup S$, then $C_2 = \{c\} \in I$. Since $S = N(c)$, $I \subseteq C_1 \cup C_2$.

Now if $I \subseteq A \cup B$, then by Lemma 5.3.4 G is hamiltonian. Otherwise, without loss of generality, assume $x \in I$. By Lemma 5.3.5 and Lemma 5.3.6, G is hamiltonian. ∎

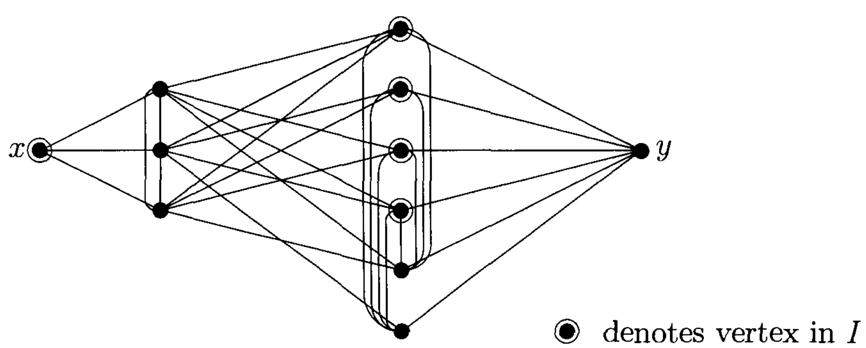

Figure 5.2: A 3-γ_t-critical graph $G \in \mathcal{F}_2$ with $\kappa(D^t(G)) = 4 < 5 = \beta(D^t(G))$

Notice that in only one of the five lemmata used to establish Theorem 5.3.7 does $\kappa(D^t(G)) \geq \beta(D^t(G))$ not hold. The example shown in the figure satisfies the conditions of Lemma 5.3.6, and verifies that there are in fact 3-γ_t-critical graphs in $\mathcal{F}_2$ with $\kappa(D^t(G)) = \beta(D^t(G)) - 1$.

5.4 3-γ_t-critical graphs in $\mathcal{F}_3$

In this section, the 3-γ_t-critical graphs in $\mathcal{F}_3$ are shown to be hamiltonian.

First, the following result characterizes the 3-γ_t-critical graphs in $\mathcal{F}_3$:

Theorem 5.4.1 *[14] A graph $G \in \mathcal{F}_3$ is 3-γ_t-critical if and only if the following conditions hold:*

1. *$(\{x\}, Y)$ is the unique maximal diametrical pair of G and $[Y]$ is complete.*
2. *$[A \cup C]$ is complete.*
3. *$|C| \geq 2$, $|Y| \geq 2$, and every vertex in C is adjacent to exactly $|Y| - 1$ vertices in Y.*

Theorem 5.4.2 *If G is a 3-γ_t-critical graph in $\mathcal{F}_3$, then G is hamiltonian.*

Proof: Since $B = \emptyset$ and $diam(G) = 3$, each $y \in Y$ has a neighbour in C. This fact, together with property 3 in Theorem 5.4.1 says that there must be two vertices $c_1, c_2 \in C$ such that $N_Y(c_1) \neq N_Y(c_2)$, and hence there exist vertices $y_1, y_2 \in Y$ where $c_1 \in N(y_1) - N(y_2)$ and $c_2 \in N(y_2) - N(y_1)$. Label the remaining vertices of C as $c_3, c_4, \ldots, c_{|C|}$, and label the vertices of A as $a_1, a_2, \ldots, a_{|A|}$. Finally, label the vertices of $Y - \{y_1, y_2\}$ as $y_3, y_4, \ldots, y_{|Y|}$. Then,

$$y_1 y_3 y_4 \cdots y_{|Y|} y_2 c_2 c_3 \cdots c_{|C|} a_1 a_2 \cdots a_{|A|-1} x a_{|A|} c_1 y_1$$

is a Hamilton cycle in G. ∎

5.5 3-γ_t-critical graphs in $\mathcal{F}_4$

As was done with the 3-γ_t-critical graphs in $\mathcal{F}_2$, the 3-γ_t-critical graphs in $\mathcal{F}_4$ will be partitioned into subfamilies based on the structure of $G - S$, where S is a minimum vertex cut. Again, $\{C_1, C_2\}$ will be used to denote any partition of the vertices of $V - S$ for which no edge of $D^t(G)$ has one end in C_1 and the other in C_2.

Before the results on the hamiltonian properties of the 3-γ_t-critical graphs in $\mathcal{F}_4$ are given, the following three results from [14] state some properties that hold for these graphs:

Theorem 5.5.1 *[14] A graph $G \in \mathcal{F}_4$ is 3-γ_t-critical if and only if the following hold:*

1. *$(\{x\}, Y)$ is the unique maximal diametrical pair of G and $[Y]$ is complete.*
2. *$[C]$ is complete and each $c \in C$ dominates exactly $|Y| - 1$ vertices in Y.*
3. *If $|Y| \geq 2$, then for every $y \in Y$, $\{x, y\} \succ G$ or there exists $w \in B \cup C$ such that $yw \rightarrow x$. If $|Y| = 1$, then $\{x, y\} \not\succ G$ and there exists $y' \in B$ such that $y' \succ A \cup C$ or $x' \in A$ such that $x' \succ B \cup C$.*
4. *For every $c \in C$, there exists $w \in B \cup C \cup Y$ such that $cw \rightarrow x$.*

5. *For every* $b \in B$, $\{x, b\} \succ G$ *or there exists* $w \in B \cup C \cup Y$ *such that* $bw \rightarrow x$.

6. *For every* $a \in A$ *and* $y \in Y$, $\{a, y\} \succ G$ *or there exists* $w \in A \cup C$ *such that* $aw \rightarrow y$.

7. *For each* $a \in A$ *and* $c \in C$ *with* $ac \notin E$, *there exists* $b \in B$ *such that* $ab \rightarrow c$.

8. *For each* $a \in A$ *and* $b \in B$, *if* $ab \notin E$, *then* $\{a, b\} \succ G$ *or there exists* $a' \in A$ *such that* $a'b \rightarrow a$ *or* $b' \in B$ *such that* $ab' \rightarrow b$. *If* $ab \in E$, *there exists* $w \in A \cup B \cup C$ *such that* $w \notin N(a) \cup N(b)$.

9. *For each* $b \in B$ *and* $c \in C$ *with* $bc \notin E$, *there exists* $a \in A$ *such that* $ab \rightarrow c$.

10. *For each* $c \in C$ *and* $y \in Y$ *with* $cy \notin E$, *there exists* $a \in A$ *such that* $ac \rightarrow y$.

Lemma 5.5.2 *[14] If* $G \in \mathcal{F}_4$ *is* 3-γ_t*-critical and* $[A]$ *is not complete, then* $|C| \geq |Y|$.

Lemma 5.5.3 *[14] If* $G \in \mathcal{F}_4$ *is* 3-γ_t*-critical and* $[A]$ *is not complete, then every* $y \in Y$ *dominates at most* $|C| - 1$ *vertices in* C.

Recall, for a maximum independent set I, we denote the sets $S \cap I$, $C_1 \cap I$, and $C_2 \cap I$ by I_S, I_{C_1}, and I_{C_2} respectively.

Note that the following lemma holds for all 3-γ_t-critical graphs, not specifically for graphs in $\mathcal{F}_4$:

Lemma 5.5.4 *Let G be a 3-γ_t-critical graph. If for every maximum independent set I in $D^t(G)$ none of the sets I_S, I_{C_1}, and I_{C_2} are empty, then G is hamiltonian.*

Proof: Let I be a maximum independent set in $D^t(G)$ such that all of the sets I_S, I_{C_1}, and I_{C_2} are nonempty. It follows that $|C_1| \geq 2$ and $|C_2| \geq 2$ (else $S = N(C_1)$ or $S = N(C_2)$). Let $\{x_1, x_2, \ldots, x_\beta\}$ be an ordering of I such that the path $P : p_1p_2 \ldots p_{\beta-1}$ is in $\mathcal{P}_I$. For $1 \leq i < \beta$, if $x_i \in C_1$, then since $x_ip_i \rightarrow x_{i+1}$ and $|C_2| \geq 2$, it follows that $p_i \in S$. Similarly, $p_i \in S$ when $x_i \in C_2$. Therefore, if $x_\beta \in I_S$, then $|S| \geq |I_{C_1} \cup I_{C_2}| + |I_S| = |I|$ and the result follows immediately. So assume $x_\beta \in C_1 \cup C_2$.

If $|I_S| \geq 2$, let $x_i, x_j \in I_S$ $(i < j < \beta)$. If either of p_i, p_j are in S, then $|S| \geq (|I_{C_1} \cup I_{C_2}| - 1) + (|I_S| + 1) = |I|$. Otherwise, without loss of generality, $p_i \in C_1$ and hence $x_ip_i \rightarrow x_{i+1}$ gives $x_{i+1} \in I_{C_2}$ and $I_{C_2} = \{x_{i+1}\}$. Furthermore, $i+1 \neq j$ and $i < j$, so $x_{i+1} \neq x_\beta$. Now, since $x_{i+1} \succ P - \{p_i\}$, $p_j \in C_2$ and hence $\{x_{j+1}\} = I_{C_1} = \{x_\beta\}$. But now $x_{i+1}x_{j+1} \notin E$ implies there exists $w \in S$ such that either $x_{i+1}w \rightarrow x_{j+1}$ or $x_{j+1}w \rightarrow x_{i+1}$. In either case, $w \neq p_{i+1}$ and hence $|S| \geq |I|$.

Now consider the case where $|I_S| = 1$. There are three subcases to consider:

<u>Case 1:</u> $|I_{C_1}| \geq 2$ and $|I_{C_2}| \geq 2$

For $1 \leq i \leq \beta - 1$, $p_i \in S$ and hence $|S| \geq (\beta - 1) + 1 = |I|$.

<u>Case 2:</u> $|I_{C_1}| = 1$ and $|I_{C_2}| = 1$

In this case, $\beta = 3$. Since the hypothesis states that there is no maximum independent set in $C_1 \cup C_2$, both $[C_1]$ and $[C_2]$ are complete. If $\{p_1, p_2\} \subseteq S$, then $|S| \geq |I|$ and G is hamiltonian. Otherwise, $|S| = 2$ and, without loss of generality, $p_i \in C_1$, $p_j \in S$ (where $\{i, j\} = \{1, 2\}$). Since $p_i \succ \{x_1, x_2, x_3\} - \{x_{i+1}\}$, $x_{i+1} \in C_2$. But now $\{p_i, x_{i+1}\} \succ D^t(G)$, a contradiction.

<u>Case 3:</u> $|I_{C_2}| = 1$ and $|I_{C_1}| \geq 2$ (or $|I_{C_1}| = 1$ and $|I_{C_2}| \geq 2$)

If $[C_2]$ is not complete, then there exists a maximum independent set in $C_1 \cup C_2$, a contradiction. Therefore $[C_2]$ is complete.

Suppose $|S| < |I|$. Then the vertices of P, namely $\{p_1, p_2, \ldots, p_{\beta-1}\}$ can not all be in S. Specifically, there exists $p_i \in C_1$ such that $x_i p_i \to x_{i+1}$, where $I_S = \{x_i\}$ and $I_{C_2} = \{x_{i+1}\}$. Furthermore, $S = (\{p_1, p_2, \ldots, p_{\beta-1}\} - \{p_i\}) \cup \{x_i\}$. Suppose $x_1 \in C_1$. Then for any $c \in C_2 - \{x_{i+1}\}$, $x_1 c \notin E(D^t(G))$ and there exists $w \in S$ such that either $x_1 w \to c$ or $cw \to x_1$. Every neighbour of x_1 in S is in P and hence has a nonneighbour in I, so $cw \to x_1$. But x_1 dominates every vertex in P, so $w = x_i$ and $cx_i \to x_1$. This is not possible as $x_i \not\succ I_{C_1} - \{x_1\}$, a contradiction.

Hence $I_S = \{x_1\}$, $I_{C_2} = \{x_2\}$, and $x_i \in C_1$ for $3 \leq i \leq \beta$. Specifically,

$x_3, x_4 \in C_1$ and $p_3 \in S$. Again consider any $c \in C_2 - \{x_2\}$, and conclude that there exists $w \in S$ such that $cw \to x_4$. Since $x_1 \not\succ C_1 - \{x_4\}$ and $x_4 \succ P - \{p_3\}$, it follows that $w = p_3$ and hence $cp_3 \to x_4$. It now follows that $p_3 \succ C_1 - \{x_4\}$. Also, $p_3 \succ C_2$ since $x_3p_3 \to x_4$, and $p_3x_1 \in E$. But $x_4 \succ P - \{p_3\}$, so $\{p_3, x_4\} \succ D^t(G)$, a contradiction. Therefore $|S| \geq |I|$ and G is hamiltonian. ∎

Before the cases where one of I_S, I_{C_1}, and I_{C_2} is empty are considered, we establish the following preliminary result:

Lemma 5.5.5 *Let G be a 3-γ_t-critical graph in $\mathcal{F}_4$ and I a maximum independent set in $D^t(G)$. If $I \cap (\{x\} \cup Y) \neq \emptyset$, then $x_\beta \in \{x\} \cup Y$. Furthermore, if for every maximum independent set I, both $x \in I$ and $Y \cap I \neq \emptyset$, then $\beta = 3$ and G is hamiltonian.*

Proof: If $x = x_i$ for $i < \beta$, then $xp_i \to x_{i+1}$ implies $p_i \in A$, and $x_{i+1} = y$ (and $Y = \{x_{i+1}\}$). If $y = x_j$ for $j < \beta$, then $yp_j \to x_{j+1}$ implies $p_j \in B \cup C$, and $x_{j+1} = x$. Combining these statements, it follows that if $x \in I$ or $Y \cap I \neq \emptyset$, then $x_\beta \in \{x, y\}$. In the case that every maximum independent set I contains x and an element in Y, by Theorem 5.5.1, I also contains exactly one vertex in C, and hence $\beta = 3$. Furthermore, if u, v are nonadjacent vertices in $A \cup B \cup C$, then $\{u, v\} \succ D^t(G)$. Thus $[A \cup B \cup C]$ is complete in $D^t(G)$, a Hamilton cycle can readily be found in $D^t(G)$, and hence G is hamiltonian. ∎

We will now consider the 3-γ_t-critical graphs in $\mathcal{F}_4$ that contain a maximum independent set I in $D^t(G)$ for which at least one of I_S, I_{C_1}, and I_{C_2} is empty.

Lemma 5.5.6 *Let G be a 3-γ_t-critical graph in $\mathcal{F}_4$ and I a maximum independent set in $D^t(G)$ such that $I \subseteq S$. Then G is hamiltonian.*

Proof: The result is immediate, since $\kappa \geq \beta$. ∎

Lemma 5.5.7 *Let G be a 3-γ_t-critical graph in $\mathcal{F}_4$ and I a maximum independent set in $D^t(G)$ such that $I \subseteq C_1 \cup S$ ($I \subseteq C_2 \cup S$). Then $x_1, x_2, \ldots, x_{|I_S|} \in I_S$ and $x_{|I_S|+1}, \ldots, x_\beta \in I_{C_1}$ (I_{C_2}), or G is hamiltonian.*

Proof: For $x_i \in I_{C_1}$ and $i < \beta$, p_i dominates C_2 and hence $p_i \in S$. If $|S| \geq |I|$, then G is hamiltonian. Otherwise, $S = I_S \cup \{p_i | x_i \in I_{C_1},\ i < \beta\}$ and $x_\beta \in I_{C_1}$.

Now, since $I_{C_1} \cup \{c_2\}$ is independent for any $c_2 \in C_2$, by Lemma 4.4.8, the vertices in I_{C_1} can be ordered $x'_1, x'_2, \ldots, x'_m$ (where $m = |I_{C_1}|$) such that there is a path $p'_1, p'_2, \ldots, p'_{m-1}$ entirely in S that satisfies $x'_i p'_i \to x'_{i+1}$. Certainly $p'_i \notin I_S$, so $\{p'_1, p'_2, \ldots, p'_{m-1}\} = \{p_i | x_i \in I_{C_1},\ i < \beta\}$. Therefore if $p_i \in S$, then $p_i \succ I_S$ and hence $x_{i+1} \in I_{C_1}$. This fact, together with $x_\beta \in I_{C_1}$ gives $x_1, x_2, \ldots, x_{|I_S|} \in I_S$ and $x_{|I_S|+1}, \ldots, x_\beta \in I_{C_1}$. ∎

Lemma 5.5.8 *Let G be a 3-γ_t-critical graph in $\mathcal{F}_4$ and I a maximum independent set in $D^t(G)$. If $I \subseteq C_1 \cup S$ (or $I \subseteq C_2 \cup S$) and $|C_2| \geq 2$ ($|C_1| \geq 2$), then G is hamiltonian.*

Proof: Let $x_1, x_2, \ldots, x_\beta$ be an ordering of I such that there is a path P : $p_1p_2\ldots p_{\beta-1}$ in $\mathcal{P}_I$. For any $x_i \in C_1$ where $1 \leq i < \beta$, p_i must dominate C_2 and hence $p_i \in S$. Suppose G is not hamiltonian. Then by Lemma 5.5.7, $x_1, x_2, \ldots, x_m \in S$ (where $|I_S| = m$) and $x_{m+1}, x_{m+2}, \ldots, x_\beta \in C_1$. Also, by Lemma 5.5.6, $1 \leq m < \beta$. In order to satisfy $x_ip_i \to x_{i+1}$ for all $1 \leq i < \beta$, $p_{m+1}, p_{m+2}, \ldots, p_{\beta-1} \in S$ and $p_1, p_2, \ldots, p_{m-1} \in C_1$. Also, $p_m \in C_1$ since $p_{m-1}p_m \in E$.

For any $x_i \in S$, $x_ip_i \to x_{i+1}$ and $p_i \in C_1$ implies $x_i \succ C_2$. Similarly, for any $p_i \in S$, $x_ip_i \to x_{i+1}$ and $x_i \in C_1$ implies $p_i \succ C_2$. In other words, every vertex in C_2 dominates S.

Consider $p_1 \in C_1$ and any vertex $c_2 \in C_2$. The vertices p_1 and c_2 are not adjacent, so there exist w such that either $p_1w \to c_2$ or $c_2w \to p_1$. Since $|C_2| \geq 2$ and $c_2 \succ S$, $p_1w \to c_2$ is not possible. So $c_2w \to p_1$ for some $w \in S$. If $w \in I_S$, then $\{c_2, w\} \not\succ I_{C_1}$. If $w = p_i$ for some $m+1 \leq i \leq \beta - 1$, then $\{c_2, w\} \not\succ x_{i+1}$. Therefore $|S| \geq |I|$, which contradicts G not hamiltonian. ∎

Lemma 5.5.9 *Let I be a maximum independent set in $D^t(G)$ and S a minimum vertex cut in a 3-γ_t-critical graph G. If $|C_1| \geq 2$, $|C_2| \geq 2$, and $I \subseteq C_1 \cup C_2$, then G is hamiltonian.*

Proof: Let $x_1, x_2, \ldots, x_m$ be an ordering of I_{C_1} for which there exists a path $P : p_1 p_2 \ldots p_{m-1}$ in $\mathcal{P}_{I_{C_1}}$ (where $m = |I_{C_1}|$). Similarly, let $y_1, y_2, \ldots, y_n$ be an ordering of I_{C_2} for which there exists a path $Q : q_1 q_2 \ldots q_{n-1}$ in $\mathcal{P}_{I_{C_2}}$ (where $n = |I_{C_2}|$). Since $x_i p_i \rightarrow x_{i+1}$, $p_i \succ C_2$ and $p_i \in S$, for all $1 \leq i < m$. Similarly, $y_i q_i \rightarrow y_{i+1}$ implies $q_i \succ C_1$ and $q_i \in S$, for all $1 \leq i < n$.

Since $x_1 y_1 \notin E$, there exists w such that, without loss of generality, $x_1 w \rightarrow y_1$. Since $|C_2| \geq 2$, $w \in S$ and $w \succ C_2 - \{y_1\}$. By the fact that $w y_1 \notin E$, w is neither on P nor Q.

Now consider $c \in C_2 - y_1$. Such a vertex exists, since otherwise C_2 is independent and $N(y_2) = S - \{q_1\}$ is a vertex cut, a contradiction. Since $x_1 c \notin E$, there exists z such that either $cz \rightarrow x_1$ or $x_1 z \rightarrow c$. In either case, $z \in S$. First suppose $cz \rightarrow x_1$. Since $x_1 \succ P \cup Q$ and $x_1 w \in E$, $z \notin P \cup Q \cup \{w\}$ and hence $|S| \geq (m-1) + (n-1) + 2 = |I|$. Suppose now that $x_1 z \rightarrow c$. Since each p_i dominates C_2 and each q_i has a nonneighbour in I_{C_2}, $z \notin P \cup Q$. Furthermore, $w y_1 \notin E$ gives $z \neq w$. Therefore in both cases $|S| \geq |I|$, and it follows that G is hamiltonian. ∎

The graphs that still need to be considered are those for which every minimum vertex cut in $D^t(G)$ is the neighbourhood of a single vertex. In this case (as in all previous cases), if $\kappa(D^t(G)) \geq \beta(D^t(G))$, then G is hamiltonian. Note that there are 3-γ_t-critical graphs in $\mathcal{F}_4$ for which $\kappa(G) < \beta(G)$, but which satisfy $\kappa(D^t(G)) \geq \beta(D^t(G))$. Figure 5.3 gives an example of such a graph.

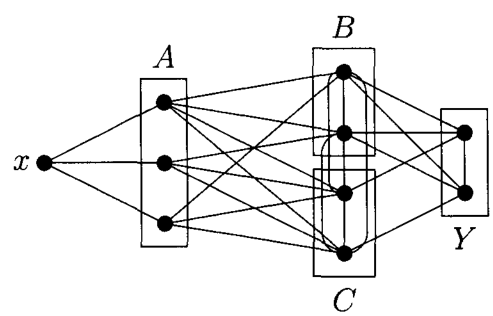

Figure 5.3: A 3-γ_t-critical graph $G \in \mathcal{F}_4$ with $\kappa(G) = 3$ and $\beta(G) = 4$ that satisfies $\kappa(D^t(G)) \geq \beta(D^t(G))$

Furthermore, there exist 3-γ_t-critical graphs $G \in \mathcal{F}_4$ for which $\kappa(D^t(G)) < \beta(D^t(G))$. An example of such a graph is in Figure 5.4.

Notice that although $\kappa(D^t(G)) < \beta(D^t(G))$ in the graph in Figure 5.4, the graph is in fact hamiltonian. Despite significant effort attempting to find a hamilton cycle in those graphs G which do not satisfy $\kappa(D^t(G)) \geq \beta(D^t(G))$, the best

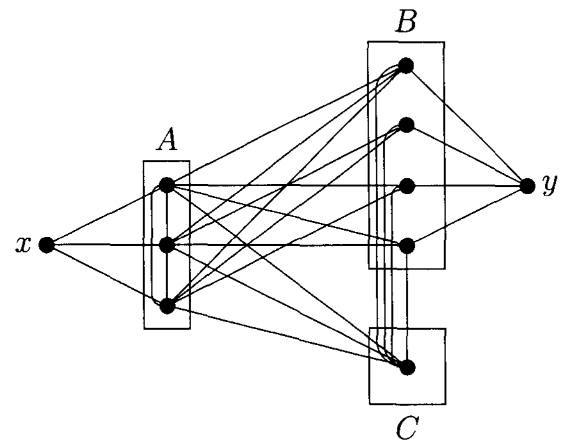

Figure 5.4: A 3-γ_t-critical graph $G \in \mathcal{F}_4$ with $\kappa(D^t(G)) = 3$ and $\beta(D^t(G)) = 5$

result we can obtain with our methods is that all such graphs contain a Hamilton path. Before this theorem is given, a corollary to Theorem 5.2.4 is needed. Although the following result is almost certainly known, we were not able to find a reference. Thus, the proof is included here.

Corollary 5.5.10 *For any graph G, if $\kappa(G) \geq \beta(G) - 1$, then G contains a Hamilton path.*

Proof: If $\kappa \geq \beta$, then G contains a Hamilton cycle, and hence a Hamilton path.

Suppose $\kappa = \beta - 1$. Consider the graph G' constructed by adding a new vertex v to G and the edge vx for each vertex $x \in V(G)$. A maximum independent set in G' does not include v, so $\beta(G') = \beta(G)$.

Consider now a minimum vertex cut S' in G'. Since $vx \in E(G')$ for all

$x \in V(G)$, it follows that $S' = S \cup \{v\}$ for some $S \subseteq V(G)$, and S is a vertex cut of G. Therefore, $\kappa(G') = \kappa(G) + 1 = \beta(G) - 1 + 1 = \beta(G) = \beta(G')$, and hence G' contains a Hamilton cycle. Such a Hamilton cycle in G' yields a Hamilton path in G when v is removed. ∎

Theorem 5.5.11 *A 3-γ_t-critical graph G contains a Hamilton path if and only if $D^t(G)$ contains a Hamilton path.*

Proof: If G contains a Hamilton path, certainly $D^t(G)$ contains a Hamilton path.

Suppose $D^t(G)$ contains a Hamilton path but G does not. Consider any minimal subset $\{e_1, e_2, \ldots, e_k\} \subseteq E(D^t(G)) - E(G)$ for which $G + \{e_1, e_2, \ldots, e_k\}$ has a Hamilton path, but $G' = G + \{e_1, e_2, \ldots, e_{k-1}\}$ does not. Let $v_1 v_2 \ldots v_n$ be a Hamilton path in $G + \{e_1, e_2, \ldots, e_k\}$. By the minimality of $\{e_1, e_2, \ldots, e_k\}$, all edges $e_1, e_2, \ldots, e_k$ are on the path, and specifically $e_k = v_i v_{i+1}$ for some $1 \leq i < n$.

Suppose $1 < i < n-1$. Since $D^t(G)$ is not hamiltonian (otherwise G is hamiltonian), $v_1 v_n \notin E(D^t(G))$. Since $\{v_i, v_{i+1}\} \succ G$, $\{v_i, v_{i+1}\} \succ \{v_1, v_n\}$. If $v_1 v_{i+1} \in E(G)$ (similarly for $v_i v_n$), then $v_i v_{i-1} \ldots v_1 v_{i+1} v_{i+2} \ldots v_n$ is a Hamilton path in G'. Therefore $v_1 v_{i+1}, v_i v_n \notin E(G)$ and hence $v_1 v_i, v_{i+1} v_n \in E(G)$. Suppose $v_j v_{i+1} \in E(G)$ for some $1 < j < i$. Then $v_{j+1} v_{j+2} \ldots v_i v_1 v_2 \ldots v_j v_{i+1} v_{i+2} \ldots v_n$ is a Hamilton path in G', a contradiction. A similar argument gives $v_iv_j \notin E(G)$ for $i+1 < j < n$. So $v_i \succ \{v_1, v_2, \ldots, v_{i-1}\}$ and $v_{i+1} \succ \{v_{i+2}, v_{i+3}, \ldots, v_n\}$. Now consider j and k where $1 \leq j < i$ and $i+1 < k \leq n$. If $v_jv_k \in E(G)$, then $v_{j+1}v_{j+2}\ldots v_iv_1v_2\ldots v_jv_kv_{k+1}\ldots v_nv_{i+1}v_{i+2}\ldots v_{k-1}$ is a Hamilton path in G', a contradiction. However, $v_iv_{i+1} \notin E(G)$ and hence G is not connected, a contradiction.

Now suppose $e_k = v_1v_2$ (or $v_{n-1}v_n$). Since $D^t(G)$ is not hamiltonian, $v_1v_n \notin E(D^t(G))$. Since $d_G(v_1) \neq 0$, $v_1v_i \in E(G)$ for some i, $2 < i < n$. Furthermore, $\{v_1, v_2\} \succ G$ gives $v_2v_n \in E(G)$. Now $v_1v_iv_{i+1}\ldots v_nv_2v_3\ldots v_{i-1}$ is a Hamilton path in G', a contradiction.

If follows that if $D^t(G)$ contains a Hamilton path, then so does G. ∎

The final result of this section states the hamiltonian properties of the 3-γ_t-critical graphs in $\mathcal{F}_4$.

Theorem 5.5.12 *Let G be a 3-γ_t-critical graph in $\mathcal{F}_4$. If there exists a minimum vertex cut S of $D^t(G)$ that is not the neighbourhood of a single vertex, then G is hamiltonian. Otherwise, G contains a Hamilton path.*

Proof: Suppose there exists a minimum vertex cut S of $D^t(G)$ such that for all $v \in V(D^t(G))$, $S \neq N(v)$. By Lemmata 5.5.4, 5.5.8, and 5.5.9, G is hamiltonian.

Now suppose S is a minimum vertex cut and I is a maximum independent set (in $D^t(G)$) and $S = N(v)$ for some vertex $v \in V(D^t(G))$. Let C_1 denote $V - N[v]$. Also, let $x_1, x_2, \ldots, x_\beta$ be an ordering of the vertices in I for which there exists a path $P \in \mathcal{P}_I$. There are two cases to consider: either $I \subseteq S \cup C_1$ or $I \subseteq C_1 \cup \{v\}$.

Case 1: $I \subseteq S \cup C_1$.

By Lemma 5.5.7, either G is hamiltonian or $I_S = \{x_1, x_2, \ldots, x_m\}$ for some m, $1 \leq m < \beta$, and $I_{C_1} = \{x_{m+1}, x_{m+2}, \ldots, x_\beta\}$. In the latter case, since $x_i p_i \rightarrow x_{i+1}$, $\{p_{m+1}, p_{m+2}, \ldots, p_{\beta-1}\} \subseteq S$. Therefore $\{x_1, x_2, \ldots, x_m, p_{m+1}, p_{m+2}, \ldots, p_{\beta-1}\} \subseteq S$ and hence $\kappa \geq \beta - 1$. By Corollary 5.5.10 and Theorem 5.5.11, G contains a Hamilton path.

Case 2: $I \subseteq C_1 \cup \{v\}$.

If $\kappa \geq \beta - 1$, then as in Case 1, G contains a Hamilton path. Otherwise, the vertices of I_{C_1} can be labelled $x_1, x_2, \ldots, x_{\beta-1}$ so that there exists a path $P \in \mathcal{P}_{I_{C_1}}$. Since $p_i \succ v$ for each $1 \leq i \leq \beta - 2$, P lies entirely in S.

Consider $C_1 - I_{C_1}$. As in the proof of Lemma 5.3.6, the graph induced by $C_1 - I_{C_1}$ can be partitioned into $m \leq \beta - 1$ independent paths. Let $\{y_1, y_2, \ldots, y_m\}$ be a set containing one end-vertex from each path. For any subset Q of size q, Q dominates at least q vertices in $\{x_1, x_2, \ldots, x_{\beta-1}\}$, otherwise there exists an independent set in C_1 of size greater than $q + (\beta - 1 - q) = \beta - 1$, a contradiction. Again, as in the proof of Lemma 5.3.6, by Hall's Theorem, the m paths can be extended by a matching to obtain a partition of C_1 into $\beta - 1$ paths, each with one end in $\{x_1, x_2, \ldots, x_{\beta-1}\}$. Denote the other end of each path by x_i'. Note that $\beta - 1 - m$ paths will be a single vertex x_i, in which case $x_i = x_i'$. Denote the path from x_i to x_i' by P_i^+ and the path from x_i' to x_i by P_i^-.

For $1 < i \leq \beta - 2$, $x_i p_i \to x_{i+1}$ gives $\{x_i, p_i\} \succ x_{i-1}'$. Therefore either $P_{i-1}^+ p_i x_i$ or $P_{i-1}^+ x_i$ is a path in $D^t(G)$. Also, $d(v) \geq 2$ gives $\beta \geq 4$, so $x_{\beta-1} p_1 \in E$ and

$$P_{\beta-1}^- p_1 P_1^+ (p_2) P_2^+ (p_3) \ldots (p_{\beta-2}) P_{\beta-2}^+$$

is a path in $D^t(G)$, where vertices (p_i) are only present in the path when $P_{i-1}^+ p_i x_i$ is a path in $D^t(G)$. Thus there exists a path that contains every vertex in C_1 and a subset of S. Either every vertex in S is on the path or there is some p_i not on the path, $1 < i \leq \beta - 2$.

Suppose not every vertex in S is on the path. Then for each i, $1 \leq i < \beta - 2$, if p_i is on the path and p_{i+1} is not, replace p_i by $p_i p_{i+1}$. This can be done recursively from $i = 1$ to $\beta - 3$ until the path contains all of $C_1 \cup S$. Now since the obtained path contains adjacent vertices $p_i p_{i+1}$ for some i, $p_i p_{i+1}$ can be replaced by $p_i v p_{i+1}$ to obtain a Hamilton path in $D^t(G)$.

Lastly, suppose the given path is in fact

$$P = P_{\beta-1}^- p_1 P_1^+ p_2 P_2^+ p_3 \ldots p_{\beta-2} P_{\beta-2}^+.$$

The path P contains every vertex in $C_1 \cup S$, but does not include v. To find a

Hamilton path, two cases will be handled separately: the case where the paths $P_1^+, P_2^+, \ldots, P_{\beta-1}^+$ are independent, and the case where they are not independent.

First consider the case where $P_1^+, P_2^+, \ldots, P_{\beta-1}^+$ are independent. That is, $x_i'x_j \notin E$ for all $i \neq j$. Consider any x_i such that $x_i' \neq x_i$(such an i exists). For $1 \leq j \leq \beta - 2$ and $i \neq j$, since $x_jp_j \to x_{j+1}$ and $x_jx_i' \notin E$, $p_jx_i' \in E$. Now consider the independent set $I \cup \{x_i'\} - \{x_i\}$. There exists an ordering of these vertices and a path in $\mathcal{P}_{I\cup\{x_i'\}-\{x_i\}}$ that lies in S. Since $p_jx_i' \in E$ for all $j \neq i$, if x_i' has a nonneighbour in S, then it must be p_i. But $p_ix_{i+1} \notin E$ and $x_ix_i' \notin E$, so $p_ix_i' \in E$. It follows that x_i' must be the first vertex in the ordering of $I \cup \{x_i'\} - \{x_i\}$. This argument holds for all $x_i' \neq x_i$, so at most one path P_i^+ has nonzero length (consider $I \cup \{x_i', x_j'\} - \{x_i, x_j\}$), and specifically we can assume P_1^+ is the path of nonzero length.

Now consider $\{x_1, v\}$. Since $vw \to x_1$ is not possible (as $x_1 \succ S$), $x_1w \to v$ for some $w \in C_1$. Since P_1^+ is the only path of nonzero length, w is on P_1^+. Furthermore, $w \succ I$. But since $I \cup \{x_1'\}$ is not independent, $x_1x_1' \in E$, and hence if $P_1^+ = x_1 \ldots w^-ww^+ \ldots x_1'$, then since P_2^+ is trivial, it follows that $P^* = x_2ww^- \ldots x_1x_1' \ldots w^+$ is a path that combines P_1^+ and P_2^+. Now since $x_3p_3 \to x_4$, either $x_3w \in E$ or $p_3w \in E$, and hence either $P_{\beta-1}^-p_1vp_2P^*p_3P_3^+ \ldots p_{\beta-2}P_{\beta-2}^+$ or $P_{\beta-1}^-p_1vp_2p_3P^*P_3^+ \ldots p_{\beta-2}P_{\beta-2}^+$ gives a Hamilton path in $D^t(G)$.

The case that remains is where there exist $i \neq j$ such that $x_i'x_j \in E$. As in the previous case, we will alter the path $P = P_{\beta-1}^{-}p_1P_1^{+}p_2P_2^{+}p_3\ldots p_{\beta-2}P_{\beta-2}^{+}$ to obtain a Hamilton path. The manner in which P is altered depends on whether $i < j$ or $j < i$.

Suppose $i < j$. Remove P_i^{+} from P and replace p_jx_j by $p_jP_i^{+}x_j$. Then replace p_ip_{i+1} by p_ivp_{i+1}. This is a Hamilton path in $D^t(G)$.

Suppose $i > j$. Remove P_j^{+} from P. If $i \neq \beta - 1$, replace p_j in the path with p_jp_{j+1} and replace $p_iP_i^{+}$ by $p_iP_i^{+}P_j^{+}p_{i+1}$ ($p_{i+1}x'_j \in E$, otherwise $x_{i+1}x'_j \in E$ and the method for $i < j$ can be applied). Now replace p_jp_{j+1} by p_jvp_{j+1}. If $i = \beta - 1$, replace $P_{\beta-1}^{-}$ by $P_j^{-}P_{\beta-1}^{-}$ and then replace p_jp_{j+1} by p_jvp_{j+1}. In either case, the obtained path is a Hamilton path in $D^t(G)$. ∎

We conjecture that all 3-γ_t-critical graphs in $\mathcal{F}_4$ are hamiltonian. In the subfamily of $\mathcal{F}_4$ that has been shown to contain a Hamilton path, it is possible that some additions to the proof of Theorem 5.5.12 would yield the desired result. It is more likely that additional information on the location of the maximum independent set and minimum vertex cut will need to be included. That is, for $S = N(v)$, the cases where v is a vertex in the unique diametrical pair of G and the cases where $v \in A \cup B \cup C$ may need to be handled individually.

Chapter 6

Diameter Two 3-γ_t-critical Graphs

In this chapter, the 3-γ_t-critical graphs with diameter two will be discussed. It will first be shown that this family of graphs cannot be characterised in terms of a finite number of forbidden subgraphs. This result also provides evidence that other types of characterisations of these graphs may be very difficult to find. However, when the minimum degree is small, some infinite families of 3-γ_t-critical graphs with diameter two can be described and shown to be hamiltonian.

For each of the four possibilities for the subgraph induced by the neighbourhood of a degree three vertex, a collection of conditions necessary for the graph to be 3-γ_t-critical are described. In the case where the neighbours of such a vertex are independent, the conditions are shown to be sufficient. In all cases, the necessary conditions are used to show that all such graphs (with minimum degree three) are hamiltonian.

6.1 Properties when $diam(G) = 2$

In this section, the results which hold for all 3-γ_t-critical graphs with diameter two will be given. It will also be shown why this family of graphs cannot easily be characterised. This same result shows that there is no characterisation of the 3-γ_t-critical graphs of diameter two in terms of a finite number of forbidden subgraphs.

Recall that in the previous chapter it was seen that in a 3-γ_t-critical graph with diameter three, there must be a pair of nonadjacent vertices that do not dominate the graph. This is not the case for the diameter two graphs, and hence it is worth mentioning this special family of graphs:

Theorem 6.1.1 *Let G be a 3-γ_t-critical graph with $diam(G) = 2$. If $\{u, v\} \succ G$ for every pair of nonadjacent vertices u and v, then G is hamiltonian.*

Proof: The proof follows directly from the fact that $D^t(G)$ is complete and hence hamiltonian. ∎

In [15], a characterisation of the 3-γ_t-critical graphs in Theorem 6.1.1 is given:

Theorem 6.1.2 *[15] A graph G is 3-γ_t-critical and satisfies $\{u,v\} \succ G$ for every pair of nonadjacent vertices u and v if and only if*

1. *$[V - N[v]] = K_m$, $m \geq 2$, for every $v \in V$, and*

2. *$X = V - (N[u] \cup N[v]) \neq \emptyset$ and $[X]$ is complete for every $uv \in E$.*

The following theorems and corollary demonstrate why a complete characterisation of the 3-γ_t-critical graphs with diameter two can not be expected. The first result is found in [13], while the latter two results provide a slightly stronger statement that will be needed in the next section.

Theorem 6.1.3 *[13] For any graph G, there is a 3-γ_t-critical graph H such that G is an induced subgraph of H.*

Theorem 6.1.4 *Let G be a graph that does not have a dominating edge. Then there exists a 3-γ_t-critical graph H with $diam(H) = 2$ such that G is the subgraph induced by $N(x)$, for some $x \in V(H)$.*

Proof:

Let G be such a graph, where $V(G) = \{a_1, a_2, \ldots, a_n\} = A$. The construction will require a copy of $\overline{G}$ that is vertex disjoint from G. Denote the vertices in $\overline{G}$ by $B = \{b_1, b_2, \ldots, b_n\}$, where $b_i = a_i$ for all i.

The graph H is obtained by adding to the graph $G \cup \overline{G}$ a new vertex x such that $N(x) = A$, and the set of edges $\{a_ib_j \mid a_i \in A,\ b_j \in B,\ i \neq j\}$. Certainly H has diameter two and $[N(x)] \cong G$. It remains to be shown that H is 3-γ_t-critical.

Consider an edge $xa_i \in E(H)$. Since $\{x, a_i\} \not\succ b_i$, xa_i is not a dominating edge. An edge a_ia_j is not dominating, as G does not have a dominating edge. Furthermore, a_ib_j is not dominating since $\{a_i, b_j\} \not\succ \{a_j, b_i\}$. Finally, no edge b_ib_j is a dominating edge, as $xb_i, xb_j \notin E(H)$. Hence $\gamma_t(H) \geq 3$.

Now consider $a_1 \in A$. Since $a_1 \succ B - b_1$ and $b_2 \succ A - a_2$, and either $a_1a_2 \in E$ or $b_1b_2 \in E$, it follows that either $\{a_1, a_2, b_2\}$ or $\{a_1, b_1, b_2\}$ is a total dominating set in H. Therefore, $\gamma_t(H) = 3$.

It remains to be shown that H is 3-γ_t-critical. Equivalently, it will be shown that for any pair of nonadjacent vertices u and v in H, either $\{u, v\} \succ H$ or, without loss of generality, $uw \rightarrow v$ for some vertex w.

For $xb_i \notin E(H)$, $xa_i \rightarrow b_i$. For $a_ia_j \notin E(H)$, $a_ib_j \rightarrow a_j$ and similarly, for $b_ib_j \notin E(H)$, $b_ia_j \rightarrow b_j$. Finally, $a_ib_i \notin E(H)$, but $\{a_i, b_i\} \succ H$. Therefore, H is 3-γ_t-critical. ∎

Corollary 6.1.5 *For any graph G, there is a diameter two 3-γ_t-critical graph H and a vertex $x \in V(H)$ such that G is the induced subgraph of some subset of $N(x)$.*

Proof: By Theorem 6.1.4, the result is true when G does not have a dominating edge. So consider any graph G that contains a dominating edge.

Let G' be the graph $G \cup K_1$. Then G' satisfies the hypothesis of Theorem 6.1.4 and the result follows. ∎

In [15], a characterisation of the 3-γ_t-critical diameter two graphs G with $\delta(G) = 2$ is given. Before this characterisation is stated, similar notation to that used in Chapter 5 is defined:

For any vertex v in a 3-γ_t-critical graph G with diameter two, let $A = N(v) = \{a_1, a_2, \ldots, a_m\}$ and $B = V - N[v]$.

The vertices in the set B can be partitioned according to their neighbourhood in A: The notation $B_{i_1,i_2,\ldots,i_p}$ will be used to denote the subset of B given by $B_{i_1,i_2,\ldots,i_p} = \{b \in B \mid N(b) \cap A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_p}\}\}$. Note that every vertex in B is adjacent to some vertex in A, as $diam(G) = 2$. For ease of notation, if $X = \{a_{i_1}, a_{i_2}, \ldots, a_{i_p}\}$, B_X may be used to denote $B_{i_1,i_2,\ldots,i_p}$.

As before, $N_B(a_i)$ denotes the subset of vertices in B that are adjacent to a_i. That is, $N_B(a_i) = N(a_i) \cap B$.

Theorem 6.1.6 *[15] A graph G with $diam(G) = 2$ and $\delta(G) = 2$ is 3-γ_t-critical if and only if for each vertex v of degree two and $N(v) = \{a_1, a_2\}$,*

1. $a_1a_2 \notin E$,

2. *either* $B_{1,2} = \emptyset$ *or* $[B_{1,2}]$ *is complete,*

3. $[B_1 \cup B_2]$ *is complete, and*

4. *if* $B_{1,2} \neq \emptyset$*, then every vertex in* $B_{1,2}$ *is adjacent to* $|B_1| - 1$ *vertices in* B_1 *and to* $|B_2| - 1$ *vertices in* B_2*, and* $|B_1| \geq 2$ *and* $|B_2| \geq 2$*. If* $B_{1,2} = \emptyset$*, then* $|B_1| \geq 1$ *and* $|B_2| \geq 1$*.*

It is not difficult to see that the graphs in Theorem 6.1.6 are hamiltonian:

Theorem 6.1.7 *If* G *is a* 3-γ_t*-critical graph with* $diam(G) = 2$ *and* $\delta(G) = 2$*, then* G *is hamiltonian.*

Proof: From the characterisation in Theorem 6.1.6, if $B_{1,2} = \emptyset$, then G contains the Hamilton cycle $va_1B_1B_2a_2v$ (where B_i in the cycle denotes the vertices in B_i in any order).

If $B_{1,2} \neq \emptyset$, then $va_1B_{1,2}B_1B_2a_2v$ is a Hamilton cycle (where if b_{12} is the last vertex listed from $B_{1,2}$, then the first vertex listed in B_1 is one of the $|B_1| - 1$ vertices in B_1 adjacent to b_{12}). ∎

Since the 3-γ_t-critical graphs with diameter two and $\delta = 2$ have a nice characterisation (and are easily seen to by hamiltonian), it seemed reasonable to attempt to characterise the subfamily with $\delta = 3$. This proved to be a much more difficult undertaking.

For diameter two 3-γ_t-critical graphs with $\delta = 3$, many necessary conditions on the structure of G can be given. However, meaningful sufficient conditions have not been found, that is, a characterisation that does not require resorting to the statement that either $\{u, v\} \succ G$ or $uw \rightarrow v$ for some vertex w for certain pairs of nonadjacent vertices u and v.

We have separated the diameter two 3-γ_t-critical graphs with $\delta = 3$ into several types based on the structure of the graph with respect to $N[v]$, where v is any minimum degree vertex. In some cases, such as when $N(v)$ is an independent set, the graphs can be characterised, providing some infinite families of diameter two 3-γ_t-critical graphs with $\delta = 3$. In other cases, much can be said about the structure of G, but the conditions found are not quite sufficient. These results will be the focus of the next section. First, some properties of all 3-γ_t-critical graphs with diameter two are given:

Theorem 6.1.8 *Let G be a diameter two 3-γ_t-critical graph. Let v be any vertex in G, $A = N(v)$, and $B = V - N[v]$. The graph G satisfies the following properties:*

1. *For every nonempty subset $X \subseteq A$, $[B_X]$ is complete or empty.*

2. *Let $X_1, X_2, \ldots, X_t$ be proper subsets of A, and suppose that for each i, $1 \leq i \leq t$, no vertex in X_i dominates $A - X_i$. Then $[B_{X_1} \cup B_{X_2} \cup \ldots \cup B_{X_t}]$*

is complete.

3. *For each $a \in A$, $a \not\succ B$.*

4. *For $X \subset Y \subseteq A$, if $b_x \in B_X$ and $b_y \in B_Y$ where $b_x b_y \notin E$, then there exists $a \in Y - X$ such that $b_y a \to b_x$, and hence $b_y \succ B_X - b_x$ and b_y dominates all nonneighbours of a in $B - b_x$.*

5. *If $X \subset Y \subseteq A$, then each $b_y \in B_Y$ has at most one nonneighbour in B_X.*

6. *If $X \subset Y \subseteq A$, then $[\overline{B_X \cup B_Y}]$ is a disjoint union of stars centred in B_X.*

7. *G is 2-connected.*

Proof:

1. Suppose $b_1, b_2 \in B_X$ and $b_1 b_2 \notin E$. Since $\{b_1, b_2\} \not\succ G$, there exists $a \in X$ (in order to dominate v) such that, without loss of generality, $ab_1 \to b_2$. But $ab_2 \in E$, a contradiction.

2. Let $b_1, b_2 \in B_{X_1} \cup B_{X_2} \cup \cdots \cup B_{X_t}$ and suppose $b_1 b_2 \notin E$. By the above, without loss of generality, $b_1 \in B_{X_1}$ and $b_2 \in B_{X_2}$. Since $\{b_1, b_2\} \not\succ G$, without loss of generality, there exists $a \in X_1$ such that $b_1 a \to b_2$. But $\{b_1, a\} \not\succ A$, a contradiction.

3. If not, av would be a dominating edge.

4. Since $\{b_x, b_y\} \not\succ G$, there exists $a \in Y$ such that either $b_x a \rightarrow b_y$ or $b_y a \rightarrow b_x$. Since $X \subset Y$, $a \in Y - X$ and $b_y a \rightarrow b_x$. Therefore $b_y \succ B_X - b_x$ and b_y dominates all nonneighbours of a in $B - b_x$.

5. Follows directly from 4.

6. Follows directly from 5.

7. Proven in [13]. ∎

Additional properties satisfied by 3-γ_t-critical graphs of diameter two and $\delta = 3$ can be described. Such results, as well as characterisations in special cases, will be given in the next section.

6.2 Necessary conditions when $\delta = 3$

The diameter two 3-γ_t-critical graphs with $\delta = 3$ can be divided into four types, depending on the structure of $A = N(v)$, where v is a vertex of minimum degree: $[A]$ is independent, $[A]$ contains a single edge, $[A]$ is a path, or $[A]$ is complete.

In all cases, recall $A = \{a_1, a_2, a_3\}$, and hence $B = B_1 \cup B_2 \cup B_3 \cup B_{1,2} \cup B_{1,3} \cup B_{2,3} \cup B_{1,2,3}$, where not all of the sets B_X need be nonempty. The only case in which the graphs have been completely characterised is when $[A]$ is independent.

These graphs will also be shown to be hamiltonian. In all other cases, necessary conditions for a graph G to be 3-γ_t-critical will be given.

In all cases, characterising the graphs becomes much more difficult when $B_{1,2,3} \neq \emptyset$. However, when $B_{1,2,3} \neq \emptyset$, the following lemma holds:

Lemma 6.2.1 *Let G be a 3-γ_t-critical graph with diam$(G) = 2$ and $\delta(G) = 3$. With respect to a minimum degree vertex v, if $B_{1,2,3} \neq \emptyset$, then for any $b_{123} \in B_{1,2,3}$, b_{123} has either 2 or 3 nonneighbours in B. Specifically, b_{123} has one nonneighbour in each of $B_{1,2}, B_{1,3}, B_{2,3}$, or b_{123} has one nonneighbour in each of $B_i, B_{j,k}$ for some $i \neq j \neq k$.*

Proof: Since for any $a \in A$ and $b_{123} \in B_{1,2,3}$, $b_{123}a$ is not a dominating edge, b_{123} has a nonneighbour in $b \in B - B_{1,2,3}$. Since $\{b_{123}, b\} \not\succ G$ and $ba \not\rightarrow b_{123}$ for all $a \in A$, there exists $a_i \in A$ such that $b_{123}a_i \rightarrow b$. Thus $b \in B_j \cup B_k \cup B_{j,k}$, $i \neq j \neq k$, and b_{123} dominates $B_j \cup B_k \cup B_{j,k} - \{b\}$. This holds for any $b \in B - B_{1,2,3}$, so b_{123} has at most one nonneighbour in $B_i \cup B_j \cup B_{i,j}$ for all $1 \leq i < j \leq 3$. Furthermore, for any $b_{123} \in B_{1,2,3}$ and $a_i \in A$, $b_{123}a_i \not\succ G$, so $b_{123} \not\succ B_j \cup B_k \cup B_{j,k}$ for all $1 \leq j < k \leq 3$. Therefore b_{123} has exactly one nonneighbour in each of $B_1 \cup B_2 \cup B_{1,2}$, $B_1 \cup B_3 \cup B_{1,3}$, and $B_2 \cup B_3 \cup B_{2,3}$. If b_{123} has a nonneighbour in B_i, then by above, $b_{123} \succ B_j \cup B_k \cup B_{i,j} \cup B_{i,k}$ and hence has exactly two nonneighbours: one in B_i and the other in $B_{j,k}$. Otherwise, if $b_{123} \succ B_1 \cup B_2 \cup B_3$, then b_{123} has exactly three nonneighbours: one in each of $B_{1,2}$, $B_{1,3}$, and $B_{2,3}$. This completes the proof. ∎

6.3 A characterisation when $\delta = 3$ and $[A]$ is independent

The main result of this section is a characterisation of all 3-γ_t-critical graphs with diameter two and $\delta = 3$ in which $[N(v)]$ is independent for some vertex v of minimum degree. Some conditions that such a graph must satisfy are established first:

Lemma 6.3.1 *Let G be a 3-γ_t-critical graph with diam$(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[N(v)]$ is independent, then G has the following properties:*

1. *$B_{1,2}$ and $B_{1,3}$ are nonempty (without loss of generality).*
2. *$B_2 \cup B_3 \cup B_{2,3} \neq \emptyset$.*
3. *$[B - B_{1,2,3}]$ is complete.*
4. *If $B_1 \cup B_2 \cup B_3 \neq \emptyset$, then $B_{1,2,3} \neq \emptyset$.*

Proof:

1. Since $\{a_1, a_2, a_3\}$ is independent, without loss of generality, there exists $b_{13} \in B_{1,3}$ and $b_{12} \in B_{1,2}$ such that $a_1 b_{13} \rightarrow a_2$ and $a_2 b_{12} \rightarrow a_3$. Specifically, $B_{1,3}$ and $B_{1,2}$ are nonempty.

2. Since $\gamma_t(G) = 3$, $\{a_1, v\} \not\succ G$. Hence $a_1 \not\succ B$, and $B_2 \cup B_3 \cup B_{2,3} \neq \emptyset$.

3. Suppose $x, y \in B - B_{1,2,3}$ and $xy \notin E$. Since $\{x, y\} \not\succ G$, without loss of generality, $xa_i \rightarrow y$ for some i, $1 \leq i \leq 3$. But $\{x, a_i\}$ does not dominate some $a_j \in A$, $i \neq j$ (since $x \notin B_{1,2,3}$). This contradicts $xa_i \rightarrow y$, and therefore $[B - B_{1,2,3}]$ is complete.

4. Consider $b_i \in B_i$, $1 \leq i \leq 3$. By property 1, $B_{1,2} \neq \emptyset$ and $B_{1,3} \neq \emptyset$. So $va_3 \not\rightarrow b_i$ and $va_2 \not\rightarrow b_i$. Without loss of generality, assume $i \neq 2$. Since $\{b_i, a_2\} \not\succ G$ and $b_i a_i \not\rightarrow a_2$, there exists w such that $a_2 w \rightarrow b_i$ and $w \in B_{1,2,3}$. Therefore if $B_1 \cup B_2 \cup B_3 \neq \emptyset$, then $B_{1,2,3} \neq \emptyset$. ∎

Lemma 6.3.2 *Let G be a 3-γ_t-critical graph with diam$(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[N(v)]$ is independent and $B_{1,2,3} = \emptyset$, then G has the following structure:*

1. *$B = B_{1,2} \cup B_{1,3} \cup B_{2,3}$ and $B_{i,j} \neq \emptyset$ for all $1 \leq i < j \leq 3$.*

2. $[B]$ is complete.

Furthermore, any graph G with the given structure has $diam(G) = 2$ and is 3-γ_t-critical.

Proof: By Lemma 6.3.1, $B_1 \cup B_2 \cup B_3 = \emptyset$ follows from $B_{1,2,3} = \emptyset$. Therefore $B = B_{1,2} \cup B_{1,3} \cup B_{2,3}$. Also by Lemma 6.3.1, each of $B_{1,2}$, $B_{1,3}$, and $B_{2,3}$ is nonempty, and $[B]$ is complete. This completes the proof that the claimed structure is necessary. It remains to be shown that the structure is sufficient.

Let G be any graph that satisfies both 1 and 2. It is easy to verify that $diam(G) = 2$ and that $\gamma_t(G) = 3$. To prove that G is 3-γ_t-critical, we will consider each pair of nonadjacent verticies $x, y \in V$ and show that either $\{x, y\} \succ G$ or there exists $w \in V$ such that $xw \rightarrow y$ or $yw \rightarrow x$.

For any $b_{ij} \in B_{i,j}$, $1 \leq i < j \leq 3$, $\{v, b_{ij}\} \succ G$ and $\{a_k, b_{ij}\} \succ G$ $(k \neq i$ and $k \neq j)$. For $i \neq j$, $\{a_i, a_j\} \not\succ G$, but $a_i b_{ik} \rightarrow a_j$ for any $b_{ik} \in B_{i,k}$, $k \neq i \neq j$. The result now follows. ∎

Lemma 6.3.3 *Let G be a 3-γ_t-critical graph with $diam(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[N(v)]$ is independent and $B_{1,2,3} \neq \emptyset$, then G has the following properties:*

1. $[B_{1,2,3}]$ is complete.

2. *$[B - B_{1,2,3}]$ is complete.*

3. *$B_{1,2}$ and $B_{1,3}$ are nonempty.*

4. *$B_2 \cup B_3 \cup B_{2,3} \neq \emptyset$.*

5. *If $b_{123} \in B_{1,2,3}$, then b_{123} has either 2 or 3 nonneighbours in B. Specifically, b_{123} has one nonneighbour in each of $B_{1,2}, B_{1,3}, B_{2,3}$ or b_{123} has one nonneighbour in each of $B_i, B_{j,k}$ for some $i \neq j \neq k$.*

6. *Each $b_{123} \in B_{1,2,3}$ has a neighbour in $B - B_{1,2,3}$.*

7. *For $1 \leq i \leq 3$, each $b_i \in B_i$ has at least one nonneighbour in $B_{1,2,3}$.*

Proof: Property 1 follows from Theorem 6.1.8. Properties 2 through 4 follow from Lemma 6.3.1. Property 5 follows from Lemma 6.2.1.

To verify property 6, suppose to the contrary that b_{123} does not have any neighbours in $B - B_{1,2,3}$. Consider the nonadjacent vertices v and b_{123}. Since $\{v, b_{123}\} \not\succ G$ and $va_i \not\to b_{123}$ for $a_i \in A$, it must be the case that $b_{123}b \to v$ for some $b \in B_{1,2,3}$. Furthermore, b and b_{123} do not have any common nonneighbours in B, and hence $|B - B_{1,2,3}| \geq 4$. But b_{123} has either 2 or 3 nonneighbours in $B - B_{1,2,3}$, and therefore has a neighbour in $B - B_{1,2,3}$.

Finally, consider $b_i \in B_i$, $1 \leq i \leq 3$. For any $j \neq i$, $\{a_j, b_i\} \not\succ G$ and $a_ib_i \not\to a_j$, so $a_jb_{123} \to b_i$ for some $b_{123} \in B_{1,2,3}$. Therefore b_i has a nonneighbour, b_{123}, in

$B_{1,2,3}$. This completes the proof of the final property. ∎

The necessary conditions given in Lemma 6.3.3 are not sufficient, in that a graph can have all of these properties and still not be 3-γ_t-critical. The problem arises in that there may exist vertices $b \in B - B_{1,2,3}$ such that $\{v, b\} \not\succ G$, and neither $bw \to v$ nor $vw \to b$ hold for any $w \in V$.

The 3-γ_t-critical graphs of diameter two with $\delta = 3$ that satisfy $B_{1,2,3} \neq \emptyset$ will be characterised as two subfamilies: those in which $va_i \to b$ for some $a_i \in A$ and $b \in B$, and all others.

Lemma 6.3.4 *Let G be a 3-γ_t-critical graph with $diam(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[N(v)]$ is independent, $B_{1,2,3} \neq \emptyset$, and $va_i \to b$ for some $a_i \in A$, $b \in B$, then G has one of the following structures:*

1. (a) *$B = B_2 \cup B_{1,2} \cup B_{1,3} \cup B_{1,2,3}$ and both $[B_{1,2,3}]$ and $[B_2 \cup B_{1,2} \cup B_{1,3}]$ are complete.*

   (b) *Each of B_2, $B_{1,2}$, $B_{1,3}$, and $B_{1,2,3}$ are nonempty.*

   (c) *At least one of B_2 and $B_{1,3}$ contains exactly one vertex.*

   (d) *Each $b_{123} \in B_{1,2,3}$ has exactly two nonneighbours in B: one in each of B_2 and $B_{1,3}$.*

   (e) *If $|B_2| \geq 2$, then every vertex in B_2 has both a neighbour and non-neighbour in $B_{1,2,3}$.*

2. *(a)* $B = B_{1,2} \cup B_{1,3} \cup B_{2,3} \cup B_{1,2,3}$ *and both* $[B_{1,2,3}]$ *and* $[B_{1,2} \cup B_{1,3} \cup B_{2,3}]$ *are complete.*

(b) Each of $B_{1,2}$, $B_{1,3}$, $B_{2,3}$, *and* $B_{1,2,3}$ *are nonempty.*

(c) $B_{2,3} = \{b_{23}\}$ *and either* $|B_{1,2}| > 1$ *or* $|B_{1,3}| > 1$.

(d) Each $b_{123} \in B_{1,2,3}$ *has exactly three nonneighbours in* B*: one in each of* $B_{1,2}$, $B_{1,3}$, *and* $B_{2,3}$.

(e) If there exists $b_{12} \in B_{1,2}$ *(or* $b_{13} \in B_{1,3}$*) that has no neighbour in* $B_{1,2,3}$, *then either* $|B_{1,2}| = 1$ *or there exists* $b_{13} \in B_{1,3}$ *(*$b_{12} \in B_{1,2}$ *respectively) that dominates* $B_{1,2,3}$.

3. *(a)* $B = B_1 \cup B_{1,2} \cup B_{1,3} \cup B_{2,3} \cup B_{1,2,3}$ *and both* $[B_{1,2,3}]$ *and* $[B_1 \cup B_{1,2} \cup B_{1,3} \cup B_{2,3}]$ *are complete.*

(b) Each of B_1, $B_{1,2}$, $B_{1,3}$, $B_{2,3}$, *and* $B_{1,2,3}$ *are nonempty.*

(c) $B_{2,3} = \{b_{23}\}$.

(d) For each $b_1 \in B_1$, b_1 *has both a neighbour and nonneighbour in* $B_{1,2,3}$.

(e) For each $b_{123} \in B_{1,2,3}$, $b_{23}b_{123} \notin E$ *and* b_{123} *has either exactly one other nonneighbour (which lies in* B_1*) or exactly two other nonneighbours (one in* $B_{1,2}$ *and one in* $B_{1,3}$*).*

Furthermore, any graph G *with one of the above structures has diameter two, and is* 3-γ_t*-critical.*

Proof: By Lemma 6.3.3, $B_{1,2} \neq \emptyset$ and $B_{1,3} \neq \emptyset$. Suppose $B_{2,3} = \emptyset$. Now either $va_1 \rightarrow b_2$ and hence $B_2 = \{b_2\}$ (the case where $va_1 \rightarrow b_3$ for some $b_3 \in B_3$ is symmetric), or $va_2 \rightarrow b_{13}$ for $B_{1,3} = \{b_{13}\}$ (the case where $va_3 \rightarrow b_{12}$ for $B_{1,2} = \{b_{12}\}$ is symmetric). These two cases together will make up the family of graphs defined in 1. We will call these graphs Type 1.

Type 1:

First suppose $va_1 \rightarrow b_2$ for some $b_2 \in B_2$, and hence $B_2 = \{b_2\}$ and $B_3 = \emptyset$. Since $B_{2,3} = \emptyset$, no $b_{123} \in B_{1,2,3}$ can have a nonneighbour in $B_{2,3}$, and hence must have exactly two nonneighbours (by Lemma 6.3.3). Since $B_{2,3} = \emptyset$ and $B_3 = \emptyset$, b_{123} has one nonneighbour b_2 and the other nonneighbour in $B_{1,3}$. Now since each $b_1 \in B_1$ must have a nonneighbour in $B_{1,2,3}$, $B_1 = \emptyset$. By Lemma 6.3.3, $[B_{1,2} \cup B_{1,3} \cup B_2]$ is complete, and $[B_{1,2,3}]$ is complete. Since $b_{123} \in B_{1,2,3}$ has nonneighbours b_2 and some $b_{13} \in B_{1,3}$, $b_{123} \succ B_{1,2}$.

Now consider the case where $va_2 \rightarrow b_{13}$, and hence $B_{1,3} = \{b_{13}\}$ and $B_1 \cup B_3 = \emptyset$. Since $B_2 \cup B_3 \cup B_{2,3} \neq \emptyset$, $B_2 \neq \emptyset$. By the same argument as above, each $b_{123} \in B_{1,2,3}$ has exactly two nonneighbours, one of which is in B_2 and the other is b_{13}. Again, $b_{123} \succ B_{1,2}$.

Suppose B_2 contains more than one vertex. We need to verify that for any $b_2 \in B_2$, b_2 has both a neighbour and a nonneighbour in $B_{1,2,3}$. Suppose to the contrary that some $b_2 \succ B_{1,2,3}$. Now $b_2a_1 \notin E$ and $\{b_2, a_1\} \not\succ G$, and $a_1w \not\rightarrow b_2$

for all $w \in V$. Therefore $b_2a_2 \rightarrow a_1$, a contradiction. Now suppose b_2 has no neighbour in $B_{1,2,3}$. Since $|B_2| \geq 2$, $va \not\rightarrow b_2$ for any $a \in A$. Therefore $b_2b_{123} \rightarrow v$ for some $b_{123} \in B_{1,2,3}$, a contradiction. This completes the proof that the conditions in 1 are necessary.

In all cases above, G does not have a dominating edge, but $\{a_1, b_{12}, b_{123}\} \succ G$ for any $b_{12} \in B_{1,2}$ and $b_{123} \in B_{1,2,3}$. So $\gamma_t(G) = 3$. To prove that any graph with the given structure is 3-γ_t-critical, each pair of nonadjacent vertices must be considered.

For any $b \in B$, $vb \notin E$, but $va_1 \rightarrow b_2$ if $B_2 = \{b_2\}$, $va_2 \rightarrow b_{13}$ if $B_{1,3} = \{b_{13}\}$, $b_{12}b_{123} \rightarrow v$ (for any $b_{12} \in B_{1,2}$ and $b_{123} \in B_{1,2,3}$), and $b_{13}b_{12} \rightarrow v$ (for any $b_{13} \in B_{1,3}$ and $b_{12} \in B_{1,2}$). If $|B_2| \geq 2$, then $b_2b_{123} \rightarrow v$ for each $b_2 \in B_2$ and some $b_{123} \in B_{1,2,3}$. We now check all other nonadjacent pairs of vertices involving a vertex in B: $a_1b_2 \notin E$ but either $va_1 \rightarrow b_2$ or $a_1b_{123} \rightarrow b_2$ for some $b_{123} \in B_{1,2,3}$, $a_2b_{13} \notin E$ but $\{a_2, b_{13}\} \succ G$, $a_3b_2 \notin E$ but $a_3b_{123} \rightarrow b_2$ for some $b_{123} \in B_{1,2,3}$, $a_3b_{12} \notin E$ but $b_{12}a_1 \rightarrow a_3$, when $b_2b_{123} \notin E$ $b_{123}a_3 \rightarrow b_2$, and if $b_{13}b_{123} \notin E$ then $b_{123}a_2 \rightarrow b_{13}$ (where $b_2 \in B_2$, $b_{12} \in B_{1,2}$, $b_{13} \in B_{1,3}$, and $b_{123} \in B_{1,2,3}$). Finally, consider the remaining pairs of nonadjacent vertices: $a_1a_2 \notin E$, $a_1a_3 \notin E$, and $a_2a_3 \notin E$, but $a_1b_{13} \rightarrow a_2$ for any $b_{13} \in B_{1,3}$, $a_1b_{12} \rightarrow a_3$ for any $b_{12} \in B_{1,2}$, and $a_2b_{12} \rightarrow a_3$ for any $b_{12} \in B_{1,2}$. Therefore such a graph is 3-γ_t-critical.

It can easily be checked that if G has the given structure, then $diam(G) = 2$. This completes the proof that all graphs defined in 1 are diameter two, 3-γ_t-critical.

Now suppose $B_{2,3} \neq \emptyset$. Since all of $B_{1,2}$, $B_{1,3}$, and $B_{2,3}$ are nonempty, by symmetry it can be assumed that $va_1 \rightarrow b$ for some $b \in B_{2,3}$. There are two cases to consider: the graphs in which $B_1 = \emptyset$ and the graphs in which $B_1 \neq \emptyset$. These two cases will give the Type 2 and Type 3 graphs which follow.

Type 2:

Suppose $B_1 = \emptyset$. Since $va_1 \rightarrow b$ for some $b \in B_{2,3}$, $B_{2,3} = \{b_{23}\}$, $B_2 \cup B_3 = \emptyset$, and $B = B_{1,2} \cup B_{1,3} \cup B_{2,3} \cup B_{1,2,3}$. It follows directly from Lemma 6.3.3 that $[B_{1,2} \cup B_{1,3} \cup B_{2,3}]$ is complete, $[B_{1,2,3}]$ is complete, and every vertex in $B_{1,2,3}$ has exactly one nonneighbour in each of the sets $B_{1,2}, B_{1,3}, B_{2,3}$. Furthermore, each $b_{123} \in B_{1,2,3}$ has a neighbour in $B - B_{1,2,3}$, so either $|B_{1,2}| > 1$ or $|B_{1,3}| > 1$. Finally, suppose there exists $b_{12} \in B_{1,2}$ such that $b_{12}b_{123} \notin E$ for all $b_{123} \in B_{1,2,3}$ and $|B_{1,2}| > 1$. Since $\{v, b_{12}\} \not\succ G$ and $va_3 \not\rightarrow b_{12}$, it must be that $b_{12}b_{13} \rightarrow v$ for some $b_{13} \in B_{1,3}$. Hence $b_{13} \succ B_{1,2,3}$. The same argument holds for $b_{1,3} \in B_{1,3}$ where $b_{13}b_{123} \notin E$ for all $b_{123} \in B_{1,2,3}$ and $|B_{1,3}| > 1$. Therefore all of the properties in 2 are necessary. To show that the properties are sufficient, it remains to show that such a graph is diameter two 3-γ_t-critical. It is easy to check that such a graph G has diameter two and has $\gamma_t(G) = 3$. By a similar argument used for the Type 1 graphs, the Type 2 graphs can be shown to be 3-γ_t-critical by checking that for every pair of nonadjacent vertices x and y, either $\{x, y\} \succ G$ or without loss of generality $xw \rightarrow y$ for some vertex w.

Type 3:

Suppose $B_1 \neq \emptyset$. Since $va_1 \rightarrow b$ for some $b \in B_{2,3}$, $B_{2,3} = \{b_{23}\}$, $B_2 \cup B_3 = \emptyset$, and $B = B_1 \cup B_{1,2} \cup B_{1,3} \cup B_{2,3} \cup B_{1,2,3}$. By Lemma 6.3.3, $[B_{1,2,3}]$ is complete, $[B_1 \cup B_{1,2} \cup B_{1,3} \cup B_{2,3}]$ is complete, each $b_{123} \in B_{1,2,3}$ has nonneighbours as described in 3e, and each $b_1 \in B_1$ has a nonneighbour in $B_{1,2,3}$. It remains to be shown that each $b_1 \in B_1$ also has a neighbour in $B_{1,2,3}$. If $|B_1| > 1$, then each $b_1 \in B_1$ has a neighbour in $B_{1,2,3}$ (any nonneighbour of another vertex in B_1), so suppose $|B_1| = 1$. It follows that each $b_{123} \in B_{1,2,3}$ has nonneighbours b_1 and b_{23}. But now consider $\{v, b_1\}$: None of $\{v, b_1\} \succ G$, $va_2 \rightarrow b_1$, $va_3 \rightarrow b_1$, and $b_1 b_{2,3} \rightarrow v$ are true, contradicting G 3-γ_t-critical. Therefore the properties in 3 are all necessary. As in the other two cases, sufficiency is shown by checking that such graphs G have diameter two, have $\gamma_t = 3$, and for every pair of nonadjacent vertices x and y, either $\{x, y\} \succ G$ or without loss of generality $xw \rightarrow y$ for some vertex w.

The result now follows. ∎

Lemma 6.3.5 *Let G be a 3-γ_t-critical graph with $diam(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[N(v)]$ is independent, $B_{1,2,3} \neq \emptyset$, and $va_i \not\to b$ for all $a_i \in A$, $b \in B$, then G has the following structure:*

1. *$[B_1 \cup B_2 \cup B_3 \cup B_{1,2} \cup B_{1,3} \cup B_{2,3}]$ is complete.*

2. *$[B_{1,2,3}]$ is complete.*

3. *Each of $B_{1,2}$, $B_{1,3}$, and $B_{1,2,3}$ are nonempty.*

4. *$|B_i \cup B_j \cup B_{i,j}| \geq 2$ for all $1 \leq i < j \leq 3$.*

5. *Each $b_{123} \in B_{1,2,3}$ has either 2 or 3 nonneighbours in $B - B_{1,2,3}$: one in each of $B_{1,2}, B_{1,3}, B_{2,3}$ or one in each of $B_i, B_{j,k}$ for some $i \neq j \neq k$.*

6. *Each $b_{123} \in B_{1,2,3}$ has a neighbour in $B - B_{1,2,3}$.*

7. *Each $b_i \in B_i$, $1 \leq i \leq 3$, has at least one nonneighbour in $B_{1,2,3}$.*

8. *If there is a vertex $b \in B - B_{1,2,3}$ with no neighbours in $B_{1,2,3}$, then there is also a vertex $b' \in B - B_{1,2,3}$ that dominates $B_{1,2,3}$, and $bb' \to v$.*

Furthermore, if a graph G has the given structure, then G is diameter two 3-γ_t-critical.

Proof: All of the claimed properties other than 4 and 8 follow directly from the hypothesis and Lemma 6.3.3.

First consider property 4. By Lemma 6.3.3, $|B_i \cup B_j \cup B_{i,j}| \geq 1$ for all $1 \leq i < j \leq 3$. However, since $va_i \not\rightarrow b$ for all $a_i \in A$ and $b \in B$, $|B_i \cup B_j \cup B_{i,j}| \geq 2$ for all $1 \leq i < j \leq 3$.

Now consider property 8. Let $b \in B - B_{1,2,3}$ be a vertex that has no nieghbours in $B_{1,2,3}$. Since $vb \notin E$ and $\{v, b\} \not\succ G$ and $va \not\rightarrow b$ for all $a \in A$, there must exist $b' \in B$ such that $bb' \rightarrow v$. It follows that $b' \succ B_{1,2,3}$. This completes the proof that all of the given properties are necessary. It remains to be shown that they are sufficient.

To prove sufficiency, all pairs of nonadjacent vertices x, y in a graph G that satisfies properties 1 through 8 must be checked, to verify that either $\{x, y\} \succ G$ or without loss of generality $xw \rightarrow y$ for some vertex w. For any $b \in B - B_{1,2,3}$, if b has a neighbour b_{123} in $B_{1,2,3}$, then $bb_{123} \rightarrow v$, and otherwise $bb' \rightarrow v$ (by property 8). For $b_{123} \in B_{1,2,3}$, $b_{123}b \rightarrow v$ for any neighbour of $b \in B - B_{1,2,3}$ (exists by property 6). For $a_1a_2 \notin E$, $a_1b_{13} \rightarrow a_2$ for any $b_{13} \in B_{1,3}$ (exists by property 3). Similarly, $a_1b_{12} \rightarrow a_3$ and $a_2b_{12} \rightarrow a_3$ for any $b_{12} \in B_{1,2}$. For $a_1b_2 \notin E$ ($b_2 \in B_2$), $a_1b_{123} \rightarrow b_2$ for some $b_{123} \in B_{1,2,3}$ (by properties 5 and 7). For $a_1b_{23} \notin E$ ($b_{23} \in B_{2,3}$), $\{a_1, b_{23}\} \succ G$. The same argument can be used for all other $ab \notin E$ where $a \in A$ and $b \in B$. If $b_1b_{123} \notin E$ (for $b_1 \in B_1$ and $b_{123} \in B_{1,2,3}$), then $b_{123}a_2 \rightarrow b_1$ (and similarly for b_2b_{123} and b_3b_{123}). Finally, if $b_{12}b_{123} \notin E$ (for $b_{12} \in B_{1,2}$ and $b_{123} \in B_{1,2,3}$), then $b_{123}a_3 \rightarrow b_{12}$ (and similarly for

$b_{13}b_{123}$ and $b_{23}b_{123}$).

Since any graph G that satisfies properties 1 through 8 has diameter two and does not have a dominating edge, by the above argument, G is 3-γ_t-critical. ∎

Theorem 6.3.6 *The 3-γ_t-critical graphs G with diam$(G) = 2$ and $\delta(G) = 3$ that satisfy $[N(v)]$ independent for some vertex v of minimum degree, can be characterised by the five families of graphs given in Lemmata 6.3.2, 6.3.4, and 6.3.5.*

In only two of the graphs characterised by Theroem 6.3.6 is it possible to have more than one vertex of minimum degree (6.3.2 and 6.3.4 Type 1). In each of those families, there are infinitely many graphs that have exactly one degree 3 vertex, and infinitely many that have exactly two degree 3 vertices. However, there are only two graphs that contain more than two minimum degree vertices. This fact is stated as a corollary, and the two mentioned graphs are shown in Figure 6.1. The proof is straightforward and omitted.

Corollary 6.3.7 *Let G be a 3-γ_t-critical graph with diam$(G) = 2$ and $\delta(G) = 3$ that has a minimum degree vertex v such that $[N(v)]$ is independent. Then G satisfies the following:*

1. $|V(G)| \geq 7$.

2. *If G has 4 vertices of degree 3, then G is the unique graph on 7 vertices defined by Lemma 6.3.2*

3. *If G has 3 vertices of degree 3, then G is the unique graph on 8 vertices defined by Lemma 6.3.4 (Type 1).*

4. *If $|V(G)| \geq 9$, then G has either 1 or 2 vertices of degree 3.*

Furthermore, for all $n \geq 8$, there exists such a graph G with $|V(G)| = n$ with exactly one degree 3 vertex, and such a graph with exactly two degree 3 vertices.

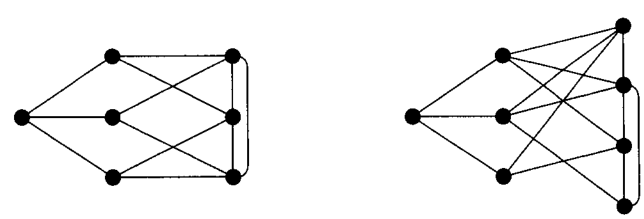

Figure 6.1: The only two graphs characterised by Theorem 6.3.6 that have more than two vertices of degree 3

Note that the first graph given in Figure 6.1 is the graph obtained by the construction in Theorem 6.1.4, where G consists of three isolated vertices.

The final result of this section shows that all 3-γ_t-critical graphs with diameter two that satisfy $[N(v)]$ independent for some vertex v are hamiltonian. Note that this result is not specific to $|N(v)| = 3$.

Theorem 6.3.8 *If G is a 3-γ_t-critical graph with $diam(G) = 2$, $\delta(G) \geq 3$, and $[N(v)]$ is independent for some $v \in V(G)$, then G is hamiltonian.*

Proof: Let G be a 3-γ_t-critical graph with $diam(G) = 2$ and $\delta(G) \geq 3$, and let v be a vertex for which $[N(v)]$ is independent. Let $A = N(v)$ and $B = V - N[v]$. Let $m = |A|$.

For some ordering $a_1, a_2, \ldots, a_m$ of A, there exists a path $p_1p_2 \ldots p_{m-1}$ in B such that $a_ip_i \rightarrow a_{i+1}$ for all $1 \leq i < m$. Consider an independent set I in B. Any path in $\mathcal{P}_I$ must lie in A. Since $[A]$ is independent, $|I| \leq 2$.

Consider a longest path P in $[B]$ that uses all of the edges p_1p_2, $p_2p_3, \ldots$, and $p_{m-2}p_{m-1}$. If P does not contain every vertex in B, then for b not on P, b is not adjacent to the endpoints of P. Since $\beta([B]) \leq 2$, the endpoints of P are adjacent and hence it can be assumed that $P = b_1b_2 \ldots b_kp_1p_2 \ldots p_{m-1}$. Now since p_{m-1} is not adjacent to any vertex in B not on P, $[B - V(P)]$ is complete and $a_{m-1} \succ B - \{p_{m-2}\}$. So

$$a_mva_{m-1}(B - V(P))\{a_{m-2}, p_{m-2}\}p_{m-1}a_{m-3}p_{m-3}a_{m-4}p_{m-4} \ldots p_1b_kb_{k-1} \ldots b_1$$

is a Hamilton path in G, where $(B - V(P))$ denotes the vertices of $B - V(P)$ listed in any order, and $\{a_{m-2}, p_{m-2}\}$ occurs in whichever order creates a path. If in fact P contains every vertex in B, then

$$a_mva_{m-1}p_{m-1}a_{m-2}p_{m-2} \ldots p_1b_kb_{k-1} \ldots b_1$$

is a Hamilton path. In either case, since $diam(G) = 2$, every $b \in B$ has a neighbour in A and, specifically, b_1 is adjacent to a_i for some i, $1 \leq i \leq m$.

If $b_1a_m \in E$, certainly G contains a Hamilton cycle. Otherwise, if $b_1a_i \in E$ for $i \leq m-2$, replace the portion of the Hamilton path $p_ia_{i-1}\ldots b_2b_1$ by $b_1b_2\ldots a_{i-1}p_i$. The obtained path together with p_ia_m is a Hamilton cycle. Finally, if $i = m-1$, remove a_mv from the path (to obtain a path from a_{m-1} to b_1) and replace the edge a_1p_1 by $a_1va_mp_1$. The obtained path, together with b_1a_{m-1} is a Hamilton cycle. In all cases, a Hamilton cycle can be found, and the result now follows. ∎

As a final note, all arguments which led to the characterisation in Theorem 6.3.6 relied on the fact that $|N(v)| = 3$. Specifically, when each vertex $b \in B = V - N[v]$ was considered, since $vb \notin E$ and $\{v, b\} \not\succ G$, it follows that either $va \to b$ for some $a \in N(v)$ or $bb' \to v$ for some $b' \in B$. Even with $|N(v)| = 3$, it was tedious to conclude the exact location of b', and many more necessary conditions were known. For example, when $|N(v)| > 3$, an argument such as that made in Lemma 6.3.3 can not be made to specify the exact locations of the nonneighbours of vertices in B_A.

What follows in the next sections is a collection of necessary conditions for diameter two 3-γ_t-critical graphs with $\delta = 3$ for which $[N(v)]$ is not independent for any minimum degree vertex v. In some cases, a characterisation can be given. The arguments are similar to those found in this section, and hence the results are stated without proof. Perhaps slightly counterintuitively, less can be said about the hamiltonian properties of such graphs than for the case with $[N(v)]$ independent. The reason for this is that all the properties obtained for each family of graphs is done by considering all possible pairs of nonadjacent vertices. Fewer nonadjacent vertices in $N(v)$ results in fewer necessary properties in $[V - N[v]]$.

6.4 Necessary conditions when $\delta = 3$ and $[A]$ contains one edge

This section contains a characterisation of the diameter two 3-γ_t-critical graphs with $\delta = 3$ that satisfy $[N(v)]$ contains a single edge (where v is a degree 3 vertex), and $B_{1,2,3} = \emptyset$. When $B_{1,2,3} \neq \emptyset$, a collection of neccessary conditions are given. In both cases, all such graphs will be shown to be hamiltonian.

The notation used is defined at the beginning of Chapter 6.

Theorem 6.4.1 *Let G be a 3-γ_t-critical graph with diam$(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[A]$ contains the single edge a_1a_2, then G has the following properties:*

1. *$[B_1 \cup B_2 \cup B_3 \cup B_{1,2}]$ is complete.*

2. *$[B_1 \cup B_{1,2} \cup B_{1,3}]$ is complete.*

3. *$[B_2 \cup B_{1,2} \cup B_{2,3}]$ is complete.*

4. $\overline{[B_1 \cup B_2 \cup B_3 \cup B_{1,3} \cup B_{2,3}]}$ *is a disjoint union of stars, and any trivial stars must be in* $B_1 \cup B_2 \cup B_3$*. Furthermore, for any nontrivial star, either*

   (a) *the centre is in* B_1 (B_2) *and the ends are in* $B_{2,3}$ $(B_{1,3})$,

   (b) *the centre is in* B_3 *and the ends are in* $B_{1,3} \cup B_{2,3}$, *or*

   (c) *the centre is in* $B_{1,3}$ $(B_{2,3})$ *and the ends are in* $B_{2,3}$ $(B_{1,3})$.

The necessary conditions given in Theorem 6.4.1 hold when $B_{1,2,3} = \emptyset$ as well as when $B_{1,2,3} \neq \emptyset$.

Despite the large amount of structure defined by Theorem 6.4.1, it is still possible to construct a graph which satisfies all of the given properties but is not 3-γ_t-critical.

When $B_{1,2,3} = \emptyset$, additional properties can be added to Theorem 6.4.1 to obtain a characterisation.

Theorem 6.4.2 *Let* G *be a 3-*γ_t*-critical graph with* $diam(G) = 2$ *and* $\delta(G) = 3$*. Let* v *be any minimum degree vertex in* G*. If* $[A]$ *contains the single edge* a_1a_2 *and* $B_{1,2,3} = \emptyset$*, then* G *is in one of the following three families of graphs:*

1. (a) $B = B_{1,2} \cup B_{1,3} \cup B_{2,3}$.

   (b) *Both* $[B_{1,2} \cup B_{1,3}]$ *and* $[B_{1,2} \cup B_{2,3}]$ *are complete, and* $B_{i,j} \neq \emptyset$ *for all* $i \neq j$.

(c) Either $|B_{1,3}| = 1$ *or* $|B_{2,3}| = 1$.

(d) $\overline{[B_{1,3} \cup B_{2,3}]}$ *is a disjoint union of nontrivial stars.*

2. (a) $B = B_3 \cup B_{1,2} \cup B_{1,3} \cup B_{2,3}$.

(b) Both B_3 *and* $B_{1,2}$ *are nonempty, and* $|B_3 \cup B_{1,3} \cup B_{2,3}| \geq 2$.

(c) For any $b_{12} \in B_{1,2}$, $b_{12} \succ B$.

(d) $\overline{[B_3 \cup B_{1,3} \cup B_{2,3}]}$ *is a disjoint union of stars. Any trivial star must be in* B_3. *If* $b_{13} \in B_{1,3}$ ($b_{23} \in B_{2,3}$) *is the centre of a star, then the ends must lie in* $B_{2,3}$ ($B_{1,3}$). *If* $b_3 \in B_3$ *is the centre of a nontrivial star, then the star must have ends in both* $B_{1,3}$ *and* $B_{2,3}$.

3. (a) $B = B_1 \cup B_3 \cup B_{1,3} \cup B_{2,3}$.

(b) Both $B_{1,3}$ *and* $B_{2,3}$ *are nonempty, and* $B_1 = \{b_1\}$.

(c) For any $b_3 \in B_3$, $b_3 \succ B$.

(d) $\overline{[B_1 \cup B_{1,3} \cup B_{2,3}]}$ *is a disjoint union of nontrivial stars. One star has centre* b_1 *and ends in* $B_{2,3}$. *For all other stars, if the centre is in* $B_{1,3}$ ($B_{2,3}$), *then the ends are in* $B_{2,3}$ ($B_{1,3}$).

Furthermore, any graph in one of the above families of graphs has diameter two and is 3-γ_t*-critical.*

Note that the graph obtained by the construction in Theroem 6.1.4 for $G = K_1 \cup K_2$ is the graph defined by Theorem 6.4.2 of Type 1, where $|B_{1,2}| = |B_{1,3}| = |B_{2,3}| = 1$.

The last result of this section shows that all diameter two 3-γ_t-critical graphs with $\delta = 3$ in which $[N(v)]$ contains a single edge (v a minimum degree vertex) are hamiltonian:

Theorem 6.4.3 *Let G be a 3-γ_t-critical graph with $diam(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[A]$ contains the single edge a_1a_2, then G is hamiltonian.*

Proof: Consider a longest path $P = p_1p_2 \dots p_m$ in $[B - B_{123}]$. We claim that P contains every vertex in $B - B_{1,2,3}$. Suppose to the contrary that there exists $b \in B - B_{1,2,3}$ such that b is not on P. Since P is a longest path, $p_1b, p_mb \notin E$, and therefore b is the centre of a star in $\overline{G}$ (by Theorem 6.4.2).

We will show that, without loss of generality, P contains precisely the vertices in $B_{2,3}$. If $b \in B_1$, then $p_1, p_m \in B_{2,3}$ and hence $p_1p_m \in E$. Thus b has no neighbour on P and hence $V(P) = B_{2,3}$. Similarly, if $b \in B_2$, then $V(P) = B_{1,3}$. Also, if $b \in B_{1,3}$, then $V(P) = B_{2,3}$, and if $b \in B_{2,3}$, then $V(P) = B_{1,3}$. Finally, if $b \in B_3$, then $V(P) = B_{1,3} \cup B_{2,3}$. But if $V(P) = B_{1,3} \cup B_{2,3}$, then $B_{1,2} = B_1 = B_2 = \emptyset$ and a_3v is a dominating edge, a contradiction. So either $V(P) = B_{1,3}$ or

$V(P) = B_{2,3}$. Without loss of generality, assume $V(P) = B_{2,3}$.

Since P is a longest path, there are no edges with one end in $B_{2,3}$ and the other end in $B_1 \cup B_2 \cup B_3 \cup B_{1,2} \cup B_{1,3}$. This implies $B_2 = B_{1,2} = \emptyset$. Also, since $b_{23} \in B_{2,3}$ has at most one nonneighbour in B_1 and $a_3 \not\to B$, $B_1 = \{b_1\}$. Furthermore, since P is a longest path, $b_1 b_{23} \notin E$ for all $b_{23} \in B_{2,3}$.

Now, since every $b_{23} \in B_{2,3}$ is in exactly one star (centred at b_1), $B_{2,3} \cup B_{1,3}$ is complete, as is $B_3 \cup B_{2,3}$. Since P is a longest path, $B_{1,3} = B_3 = \emptyset$. However, since $d(b_1, b_{23}) = 2$ for any $b_{23} \in B_{2,3}$, there exists $b \in B_{1,2,3}$ such that $b_1 b, b_{23} b \in E$. But every $b_{123} \in B_{1,2,3}$ has a nonneighbour in B_1 and $B_{2,3}$ by Lemma 6.2.1, so b_1 has no neighbour in $B_{1,2,3}$, a contradiction to $diam(G) = 2$.

It follows that P contains every vertex in $B - B_{1,2,3}$. Now, if the ends of P are adjacent, then P can be rewritten as a path with ends in different sets B_X, B_Y such that $a_3 \in X$, $Y \neq X$, and $Y \cap \{a_1, a_2\} \neq \emptyset$. If the ends of P are not adjacent, by the properties in Theorem 6.4.1, the ends of P are in B_X and B_Y for some X and Y, where $a_3 \in X$ and $Y \cap \{a_1, a_2\} \neq \emptyset$. In either case, assume $p_1 \in B_Y$ and $p_m \in B_X$. Without loss of generality, let $a_1 \in Y$. If $B_{1,2,3} \neq \emptyset$, then $a_1 p_1 p_2 \ldots p_m a_3 B_{1,2,3} a_2 v a_1$ is a Hamilton cycle (where the vertices in $B_{1,2,3}$ are listed in any order). If $B_{1,2,3} = \emptyset$, then $a_1 p_1 p_2 \ldots p_m a_3 v a_2 a_1$ is a Hamilton cycle. ∎

6.5 Necessary conditions when $\delta = 3$ and $[A]$ contains two edges

There are two results given in this section. The first is a list of properties that hold for all diameter two 3-γ_t-critical graphs with $\delta = 3$ for which $[N(v)]$ contains exactly two edges (where v is a minimum degree vertex). The second result shows that all such graphs are hamiltonian.

Theorem 6.5.1 *Let G be a 3-γ_t-critical graph with diam$(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[A]$ contains precisely the edges a_1a_2 and a_2a_3, then G has the following properties:*

1. *$[B_X]$ is complete for all $X \subseteq A$.*

2. *$[B_1 \cup B_3]$ is complete and $B_1 \neq \emptyset$, $B_3 \neq \emptyset$.*

3. *$[B_2 \cup B_{1,2} \cup B_{2,3}]$ is complete and nonempty.*

4. *$\overline{[B_1 \cup B_2 \cup B_3 \cup B_{1,2} \cup B_{2,3}]}$ is a disjoint union of stars such that nontrivial stars have centres in $B_1 \cup B_3$ and ends in $B_2 \cup B_{1,2} \cup B_{2,3}$.*

5. *For $b \in B_2 \cup B_{1,2} \cup B_{2,3}$, $b \not\succ B_1 \cup B_2 \cup B_3 \cup B_{1,2} \cup B_{2,3}$ if and only if $b \succ B_{1,3}$.*

6. *For $b_{13} \in B_{1,3}$, b_{13} has a nonneighbour in each of $B_2 \cup B_3 \cup B_{2,3}$ and $B_2 \cup B_1 \cup B_{1,2}$. Furthermore, if one of the nonneighbours b is in B_1 (or*

B_3), then $b_{13} \succ B_1 \cup B_2 \cup B_{1,2} - b$ ($b_{13} \succ B_2 \cup B_3 \cup B_{2,3} - b$).

7. *For $b_{123} \in B_{1,2,3}$, b_{123} either has exactly one nonneighbour in each of $B_{1,2}$, $B_{1,3}$, $B_{2,3}$ or b_{123} has exactly one nonneighbour in each of B_i, $B_{j,k}$ for some $i \neq j \neq k$.*

Theorem 6.5.2 *Let G be a 3-γ_t-critical graph with diam$(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[A]$ contains precisely the edges a_1a_2 and a_2a_3, then G is hamiltonian.*

Proof: Since both B_1 and B_3 are nonempty, and $[B_1 \cup B_3]$ is complete, there exists a path in G that starts in B_1, uses every vertex in $B_1 \cup B_3$, and ends in B_3. We will simply denote such a path by B_1B_3. Similar notation is used throughout the proof.

By Theorem 6.5.1, $B_2 \cup B_{1,2} \cup B_{2,3}$ is nonempty. For $b \in B_2 \cup B_{1,2} \cup B_{2,3}$, either $b \succ B_1$ or $b \succ B_3$ (by Theorem 6.5.1). If $B_{1,3} \cup B_{1,2,3} = \emptyset$, then without loss of generality, $a_2B_2B_{1,2}B_{2,3}B_1B_3a_3va_1a_2$ is a Hamilton cycle. So assume $B_{1,3} \cup B_{1,2,3} \neq \emptyset$.

First, suppose $B_{1,2,3} = \emptyset$ and hence $B_{1,3} \neq \emptyset$. If there exists $b \in B_2 \cup B_{1,2} \cup B_{2,3}$ such that b has a nonneighbour in $B_1 \cup B_3$, then $b \succ B_{1,3}$. Since every vertex in $B_2 \cup B_{1,2} \cup B_{2,3}$ has at most one nonneighbour in $B_1 \cup B_3$, without loss of generality, there is a path Q of the form $a_1B_1B_3(B_2 \cup B_{1,2} \cup B_{2,3})$ that ends at b. In this case, $QB_{1,3}a_3va_2a_1$ is a Hamilton cycle in G. Otherwise, every $b \in B_2 \cup B_{1,2} \cup B_{2,3}$ dominates $B_1 \cup B_3$. Any such vertex b has a nonneighbour $b_{13} \in B_{1,3}$ and without loss of generality, $b_{13}b_3 \in E$ for every $b_3 \in B_3$. Now $a_2B_2B_{1,2}B_{2,3}B_1B_3B_{1,3}a_3va_1a_2$ is a Hamilton cycle.

Finally, suppose $B_{1,2,3} \neq \emptyset$. As before, any $b \in B_2 \cup B_{1,2} \cup B_{2,3}$ dominates either B_1 or B_3. So without loss of generality, there is a path of the form $B_1B_3B_2B_{1,2}B_{2,3}$. If $B_{1,3} = \emptyset$, then $a_1B_1B_3B_2B_{1,2}B_{2,3}a_2B_{1,2,3}a_3va_1$ is a Hamilton cycle. If $B_{1,3} \neq \emptyset$, then since $b_{123} \in B_{1,2,3}$ can not have a nonneighbour in both B_1 and $B_{1,3}$, without loss of generality, either $a_1B_1B_3B_2B_{1,2}B_{2,3}a_2B_{1,2,3}B_{1,3}a_3va_1$ or $a_1B_{1,2,3}B_1B_3B_2B_{1,2}B_{2,3}a_2va_3B_{1,3}a_1$ is a Hamilton cycle.

Since all cases result in a Hamilton cycle, the result follows. ∎

6.6 Necessary conditions when $\delta = 3$ and $[A]$ is complete

In this final section of Chapter 6, necessary conditions for a graph G to be diameter two, 3-γ_t-critical with $\delta(G) = 3$, and satisfy $[N(v)]$ is complete (where v is a minimum degree vertex). All such graphs will be shown to be hamiltonian.

Theorem 6.6.1 *Let G be a 3-γ_t-critical graph with diam$(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[A]$ is complete, then G has the* *following properties:*

1. *$[B_X]$ is complete for all $X \subseteq A$.*

2. *Each of B_1, B_2, and B_3 are nonempty, and $[\overline{B_1 \cup B_2 \cup B_3}]$ is a disjoint union of stars.*

3. *$B_{1,2} \cup B_{1,3} \cup B_{2,3} \neq \emptyset$, and if $B_{1,2,3} = \emptyset$, then each of $B_{1,2}, B_{1,3}, B_{2,3}$ are nonempty.*

4. *For any $b \in B_{i,j}$, either $b \not\succ B_k$ and $b \succ B_i \cup B_j$ $(i \neq j \neq k)$, or b has exactly two nonneighbours, one in each of B_X and B_Y for some $X \neq Y$ and neither X nor Y equal to $\{a_1, a_2, a_3\}$, $a_i \in X$ and $a_j \in Y$.*

5. *For any $b_{123} \in B_{1,2,3}$, b_{123} has either 2 or 3 nonneighbours in B. Specifically, b_{123} has one nonneighbour in each of $B_{1,2}, B_{1,3}, B_{2,3}$ or b_{123} has one nonneighbour in each of $B_i, B_{j,k}$ for some $i \neq j \neq k$.*

Theorem 6.6.2 *Let G be a 3-γ_t-critical graph with diam$(G) = 2$ and $\delta(G) = 3$. Let v be any minimum degree vertex in G. If $[A]$ is complete, then G is hamiltonian.*

Proof: By Theorem 6.6.1, without loss of generality, $B_{1,2} \neq \emptyset$. Furthermore, any $b_{12} \in B_{1,2}$ either dominates $B_1 \cup B_2$ or has exactly one nonneighbour in each of $B_1 \cup B_{1,3}$ and $B_2 \cup B_{2,3}$.

Suppose that for all $i \neq j$, $b_{ij} \succ B_i \cup B_j$. Then $B_1 B_{1,2} B_2$ and $B_{2,3} B_3 B_{1,3}$ are paths (where $B_{1,3}$ and $B_{2,3}$ need not be nonempty). Note that, as in previous arguments, $B_X B_Y$ is used to denote any path that lists all vertices in B_X followed by all vertices in B_Y.

If $B_{1,3} = B_{2,3} = \emptyset$, then $B_{1,2,3} \neq \emptyset$ and every $b_{123} \in B_{1,2,3}$ dominates $B_1 \cup B_2$. Hence consider the paths B_3 and $B_1 B_{1,2} B_2 B_{1,2,3}$. Note that the ends of the second path are adjacent. Since a_3 is not a cut vertex, some $b_3 \in B_3$ is adjacent to a vertex in $B_1 \cup B_{1,2} \cup B_2 \cup B_{1,2,3}$. Thus there is a path that contains all vertices of B which starts in $B_1 \cup B_{1,2} \cup B_2 \cup B_{1,2,3}$ and ends in B_3, and hence a Hamilton cycle in G can be obtained.

If $B_{1,3}$ or $B_{2,3}$ is nonempty, without loss of generality, $B_{1,3} \neq \emptyset$. Then since $b_{13} \in B_{1,3}$ dominates $B_1 \cup B_3$, $B_{2,3} B_3 B_{1,3} B_1 B_{1,2} B_2$ is a path that contains all of the vertices of $B - B_{1,2,3}$. Hence $a_3 B_{2,3} B_3 B_{1,3} B_1 B_{1,2} B_2 a_2 v a_1 B_{1,2,3} a_3$ is a Hamilton cycle (where $B_{2,3}$ may be empty).

Now, without loss of generality, suppose there exists $b_{12} \in B_{1,2}$ with exactly two nonneighbours, one in $B_1 \cup B_{1,3}$ and one in $B_2 \cup B_{2,3}$. It follows that $b_{12} \succ B_3$ (and $b_{12} \succ B_{1,2,3}$).

Suppose that for all $b_{13} \in B_{1,3}$ and $b_{23} \in B_{2,3}$, $b_{13} \succ B_1 \cup B_3$ and $b_{23} \succ B_2 \cup B_3$. First consider the case where $B_{1,3} = B_{2,3} = \emptyset$. Then $B_{1,2,3} \neq \emptyset$ and $B_{1,2,3} \succ B_1 \cup B_2$. Therefore there is a Hamilton cycle of the form $a_2 B_{1,2} B_3 a_3 v a_1 B_1 B_{1,2,3} B_2 a_2$.

Second, consider the case (without loss of generality) where $B_{1,3} = \emptyset$ and $B_{2,3} \neq \emptyset$. Then $B_{1,2,3} \neq \emptyset$ and $B_{1,2,3} \succ B_2$. Consider the paths $B_{1,2}B_3B_{2,3}B_2B_{1,2,3}$ and B_1. Let the ends of the first path be $b'_{12} \in B_{1,2}$ and $b_{123} \in B_{1,2,3}$: If these vertices are not adjacent, then since $B_{1,3} = \emptyset$, and $b'_{12}b_{123} \notin E$, $b_{123} \succ B_1$ and $a_2B_{1,2}B_3B_{2,3}B_2B_{1,2,3}B_1a_1va_3a_2$ is a Hamilton cycle. If $b'_{12}b_{123} \in E$, then since a_1 is not a cut vertex, a Hamilton cycle can be found, as before. Finally, suppose both $B_{1,3}$ and $B_{2,3}$ are nonempty. Consider the paths $B_{1,2}B_3B_{1,3}B_1$ and $B_{2,3}B_2$. Since each $b_{123} \in B_{1,2,3}$ is adjacent to a vertex in $B_1 \cup B_{1,2}$, whether or not $B_{1,2,3} = \emptyset$, either $a_2B_{1,2,3}B_{1,2}B_3B_{1,3}B_1a_1va_3B_{2,3}B_2a_2$ or $a_2B_{1,2}B_3B_{1,3}B_1B_{1,2,3}a_1va_3B_{2,3}B_2a_2$ is a Hamilton cycle.

Next suppose that for all $b_{13} \in B_{1,3}$, $b_{13} \succ B_1 \cup B_3$ and there exists $b_{23} \in B_{2,3}$ such that $b_{23} \not\succ B_2 \cup B_3$. It follows that $B_{2,3} \neq \emptyset$ and $b_{23} \succ B_1$. First consider the case where $B_{1,3} = \emptyset$. It follows that $B_{1,2,3} \neq \emptyset$ and $B_{1,2,3} \succ B_2$. Consider the paths $B_{1,2}B_3$, $B_1B_{2,3}$, and $B_{1,2,3}B_2$. Since each $b_{123} \in B_{1,2,3}$ is nonadjacent to both ends of one path and neither end of the other, one of the following is a Hamilton cycle: $a_2B_2B_{1,2,3}B_{1,2}B_3a_3va_1B_1B_{2,3}a_2$ or $a_2B_2B_{1,2,3}B_{2,3}B_1a_1va_3B_3B_{1,2}a_2$. Secondly, consider the case where $B_{1,3} \neq \emptyset$. If $B_{1,2,3} = \emptyset$, then consider the paths $B_{1,2}B_3B_{1,3}B_1B_{2,3}$ and B_2. Let the ends of the first path be b'_{12} and b'_{23}. If $b'_{23}b'_{12} \in E$, then since a_2 is not a cut vertex, we are done, as before. Otherwise $b'_{12}b'_{23} \notin E$, and it follows that $b'_{12} \succ B_2$. As before, there is a

Hamilton cycle. If however $B_{1,2,3} \neq \emptyset$, then either there is a Hamilton cycle $a_1B_{1,2}B_3B_{1,3}B_1B_{2,3}a_3B_{1,2,3}B_2a_2va_1$, or every vertex in $B_{1,2,3}$ is nonadjacent to a vertex in B_2 and in $B_{1,3}$, but dominates $B_1 \cup B_3 \cup B_{2,3} \cup B_{1,2}$. In this case, $B_{1,2,3}B_{1,2}B_3B_{1,3}B_1B_{2,3}$ and B_2 is a partition of $[B]$ into a cycle and a path. Since a_2 is not a cut vertex, we are done, as before.

To complete the proof, we consider the remaining case, where there exist vertices $b_{12} \in B_{1,2}$, $b_{13} \in B_{1,3}$, and $b_{23} \in B_{2,3}$ such that $b_{12} \succ B_3$, $b_{13} \succ B_2$, and $b_{23} \succ B_1$. Consider the paths $B_{1,2}B_3$, $B_{1,3}B_2$, and $B_{2,3}B_1$. If $B_{1,2,3} \neq \emptyset$, then each of $b_{123} \in B_{1,2,3}$ is nonadjacent to at most one end of each path. Without loss of generality, $a_1B_{1,2}B_3B_{1,2,3}B_{1,3}B_2a_2va_3B_{2,3}B_1a_1$ or $a_3B_3B_{1,2}B_{1,2,3}B_{1,3}B_2a_2B_{2,3}B_1a_1va_3$ is a Hamilton cycle. If $B_{1,2,3} = \emptyset$, then since any $b_{12} \in B_{1,2}$ either dominates $B_1 \cup B_2$ or has exactly one nonneighbour in $B_1 \cup B_{1,3}$, one of the following is a Hamilton cycle: $a_2B_{2,3}B_1B_{1,2}B_3a_3va_1B_{1,3}B_2a_2$ or $a_2B_2B_{1,3}B_{1,2}B_3a_3B_{2,3}B_1a_1va_2$.

This completes the proof. ∎

Chapter 7

Concluding Remarks

In this dissertation, new results were obtained for 3-i-critical and 3-γ_t-critical graphs.

Using a closure operation, it was established that all 2-connected 3-i-critical graphs with $\delta \geq 3$ are hamiltonian. When $\delta = 2$, there is exactly one family of nonhamiltonian graphs. All connected 3-i-critical graphs on more than six vertices were shown to contain a Hamilton path. Characterisations were given of the 3-i-critical graphs which either have $\delta = 2$ or contain a cut vertex.

A closure operation was also used to study the hamiltonian properties of 3-γ_t-critical graphs. All 2-connected 3-γ_t-critical graphs with diameter three were shown to contain a Hamilton path and, for all but one family, they were shown to be hamiltonian. The 3-γ_t-critical graphs with diameter two were shown to be hamiltonian when $2 \le \delta \le 3$. Several infinite families of diameter two 3-γ_t-critical graphs with $\delta = 3$ were given.

The following are some suggestions for future research:

1. Find improved structural characterisations for the four families of diameter three 3-γ_t-critical graphs studied in Chapter 5.

2. Complete the proof that all diameter three 3-γ_t-critical graphs contain a Hamilton cycle (see Section 5.5).

3. Complete the characterisation of the diameter two 3-γ_t-critical graphs with $\delta = 3$ (see Chapter 6).

4. Use the structural results in Chapters 5 and 6 to find bounds on the number of edges in a 3-γ_t-critical graph. Such results could be used in the study of diameter 2-critical graphs, as the complement of a 3-γ_t-critical graph is diameter 2-critical.

5. For each of the parameters γ, i, and γ_t (and others), there are other notions of criticality that have been studied in the literature (e.g. see [10, 11]). One example is criticality with respect to vertex deletion. The structure and hamiltonian properties of such graphs could be investigated.

6. Consider the k-i-critical or k-γ_t-critical graphs for $k \geq 4$. It is known that for $k \geq 4$, not all k-i-critical graphs contain a Hamilton path (see page 50), so one problem would be to characterise the families which contain a Hamilton path or cycle.

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Place of Birth: Edmonton, Alberta, Canada

Educational Institutions Attended: University of Victoria 1991-1995, 1996-1998, 1999-2005

Degrees Awarded:

B.Sc.(Honours) University of Victoria 1996

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Academic Awards:

Andy Farquharson Award for Excellence in Graduate Student Teaching, 2004

Charles S. Humphrey Graduate Student Award, 2002

Departmental Fellowship, 2002

Petch Research Scholarship, 2000

President's Research Scholarship, 2000

NSERC Postgraduate Scholarship (PGS B), 2000-2002

Charles S. Humphrey Graduate Award, 1999

University of Victoria Fellowship, 1999

Petch Research Scholarship, 1997

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NSERC Postgraduate Scholarship (PGS A), 1996-1998

President's Research Scholarship, 1996

President's Scholarship, 1995

Victoria Real Estate Board Scholarship, 1992

NSERC Industrial Undergraduate Student Research Award, 1992

Grade Twelve Scholarship Award, 1991

Publications:

S. Ao, G. MacGillivray, and J. Simmons, Hamiltonian properties of independent domination critical graphs. Submitted to *Discrete Mathematics*, 2003.

E.J. Cockayne, G. MacGillivray, and J. Simmons, CO-irredundant Ramsey Numbers for Graphs. *Journal of Graph Theory* **34** (2000), 258-268.

E.J. Cockayne, C.M. Mynhardt, and J. Simmons, The CO-irredundant Ramsey Number $t(4,7)$. *Utilitas Mathematica* **LVII** (2000), 193-210.