
Faculty of Science

Faculty Publications

Hankel and Toeplitz Determinants for a Subclass of q -Starlike Functions Associated with a General Conic Domain

Hari M. Srivastava, Qazi Zahoor Ahmad, Nasir Khan, Nazar Khan and Bilal Khan

February 2019

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).

This article was originally published at:

<http://dx.doi.org/10.3390/math7020181>

Citation for this paper:

Srivastava, H.M., Zahoor, Q., Khan, N., Khan, N. & Khan, B. (2019). Hankel and Toeplitz Determinants for a Subclass of q -Starlike Functions Associated with a General Conic Domain. *Mathematics*, 7(2), 181.

<https://doi.org/10.3390/math7020181>

Article

Hankel and Toeplitz Determinants for a Subclass of q -Starlike Functions Associated with a General Conic Domain

Hari M. Srivastava ^{1,2,*} , Qazi Zahoor Ahmad ³, Nasir Khan ⁴, Nazar Khan ³ and Bilal Khan ³ ¹ Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada² Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan³ Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan; zahoorqazi5@gmail.com (Q.Z.A.); nazarmaths@gmail.com (N.K.); bilalmaths789@gmail.com (B.K.)⁴ Department of Mathematics, FATA University, Akhorwal (Darra Adam Khel), FR Kohat 26000, Pakistan; dr.nasirkhan@fu.edu.pk

* Correspondence: harimsri@math.uvic.ca

Received: 29 January 2019; Accepted: 12 February 2019; Published: 15 February 2019



Abstract: By using a certain general conic domain as well as the quantum (or q -) calculus, here we define and investigate a new subclass of normalized analytic and starlike functions in the open unit disk $\mathbb{U}$. In particular, we find the Hankel determinant and the Toeplitz matrices for this newly-defined class of analytic q -starlike functions. We also highlight some known consequences of our main results.

Keywords: analytic functions; starlike and q -starlike functions; q -derivative operator; q -hypergeometric functions; conic and generalized conic domains; Hankel determinant; Toeplitz matrices

MSC: Primary 05A30, 30C45; Secondary 11B65, 47B38

1. Introduction and Definitions

Let the class of functions, which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

be denoted by $\mathcal{L}(\mathbb{U})$. Also let $\mathcal{A}$ denote the class of all functions f , which are analytic in the open unit disk $\mathbb{U}$ and normalized by

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Then, clearly, each $f \in \mathcal{A}$ has a Taylor–Maclaurin series representation as follows:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}). \quad (1)$$

Suppose that S is the subclass of the analytic function class $\mathcal{A}$, which consists of all functions which are also univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be starlike in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*$ the class of all such starlike functions in $\mathbb{U}$.

For two functions f and g , analytic in $\mathbb{U}$, we say that the function f is subordinate to the function g and write this subordination as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w which is analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$f(z) = g(w(z)).$$

In the case when the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see, for example, [1]; see also [2]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Next, for a function $f \in \mathcal{A}$ given by (1) and another function $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathbb{U}),$$

the convolution (or the Hadamard product) of f and g is defined here by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \quad (2)$$

Let $\mathcal{P}$ denote the well-known Carathéodory class of functions p , analytic in the open unit disk $\mathbb{U}$, which are normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (3)$$

such that

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}).$$

Following the works of Kanas et al. (see [3,4]; see also [5]), we introduce the conic domain Ω_k ($k \geq 0$) as follows:

$$\Omega_k = \left\{ u + iv : u > k \sqrt{(u-1)^2 + v^2} \right\}. \quad (4)$$

In fact, subjected to the conic domain Ω_k ($k \geq 0$), Kanas and Wiśniowska (see [3,4]; see also [6]) studied the corresponding class $k\text{-}\mathcal{ST}$ of k -starlike functions in $\mathbb{U}$ (see Definition 1 below). For fixed k , Ω_k represents the conic region bounded successively by the imaginary axis ($k = 0$), by a parabola ($k = 1$), by the right branch of a hyperbola ($0 < k < 1$), and by an ellipse ($k > 1$).

For these conic regions, the following functions play the role of extremal functions.

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \dots & (k=0) \\ 1 + \frac{2}{\pi^2} \left[\log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right]^2 & (k=1) \\ 1 + \frac{2}{1-k^2} \sinh^2 \left[\left(\frac{2}{\pi} \arccos k \right) \arctan (h\sqrt{z}) \right] & (0 \leq k < 1) \\ 1 + \frac{1}{k^2-1} \left[1 + \sin \left(\frac{\pi}{2K(\kappa)} \int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}} \right) \right] & (k > 1), \end{cases} \quad (5)$$

where

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa}z} \quad (z \in \mathbb{U}),$$

and $\kappa \in (0, 1)$ is so chosen that

$$k = \cosh \left(\frac{\pi K'(\kappa)}{4K(\kappa)} \right).$$

Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and

$$K'(\kappa) = K(\sqrt{1-\kappa^2}),$$

that is, $K'(\kappa)$ is the complementary integral of $K(\kappa)$ (see, for example, ([7], p. 326, Equation 9.4 (209))). Indeed, from (5), we have

$$p_k(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \quad (6)$$

The class $k\text{-}\mathcal{ST}$ is defined as follows.

Definition 1. A function $f \in \mathcal{A}$ is said to be in the class $k\text{-}\mathcal{ST}$ if and only if

$$\frac{zf'(z)}{f(z)} \prec p_k(z) \quad (\forall z \in \mathbb{U}; k \geq 0).$$

We now recall some basic definitions and concept details of the q -calculus which will be used in this paper (see, for example, ([7], p. 346 et seq.)). Throughout the paper, unless otherwise mentioned, we suppose that $0 < q < 1$ and

$$\mathbb{N} = \{1, 2, 3, \dots\} = \mathbb{N}_0 \setminus \{0\} \quad (\mathbb{N}_0 := \{0, 1, 2, \dots\}).$$

Definition 2. Let $q \in (0, 1)$ and define the q -number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \dots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 3. Let $q \in (0, 1)$ and define the q -factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1 & (n=0) \\ \prod_{k=1}^n [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 4 (see [8,9]). The q -derivative (or q -difference) operator D_q of a function f defined, in a given subset of $\mathbb{C}$, by

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases} \quad (7)$$

provided that $f'(0)$ exists.

From Definition 4, we can observe that

$$\lim_{q \rightarrow 1^-} (D_q f)(z) = \lim_{q \rightarrow 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for a differentiable function f in a given subset of $\mathbb{C}$. It is also known from (1) and (7) that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \quad (8)$$

Definition 5. The q -Pochhammer symbol $[\xi]_{n,q}$ ($\xi \in \mathbb{C}$; $n \in \mathbb{N}_0$) is defined as follows:

$$[\xi]_{n,q} = \frac{(q^\xi; q)_n}{(1-q)^n} = \begin{cases} 1 & (n = 0) \\ [\xi]_q [\xi + 1]_q [\xi + 2]_q \cdots [\xi + n - 1]_q & (n \in \mathbb{N}). \end{cases}$$

Moreover, the q -gamma function is defined by the following recurrence relation:

$$\Gamma_q(z+1) = [z]_q \Gamma_q(z) \quad \text{and} \quad \Gamma_q(1) = 1.$$

Definition 6 (see [10]). For $f \in \mathcal{A}$, let the q -Ruscheweyh derivative operator $\mathcal{R}_q^\lambda$ be defined, in terms of the Hadamard product (or convolution) given by (2), as follows:

$$\mathcal{R}_q^\lambda f(z) = f(z) * \mathcal{F}_{q,\lambda+1}(z) \quad (z \in \mathbb{U}; \lambda > -1),$$

where

$$\mathcal{F}_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\lambda+n)}{[n-1]_q! \Gamma_q(\lambda+1)} z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{q,n-1}}{[n-1]_q!} z^n.$$

We next define a certain q -integral operator by using the same technique as that used by Noor [11].

Definition 7. For $f \in \mathcal{A}$, let the q -integral operator $\mathcal{F}_{q,\lambda}$ be defined by

$$\mathcal{F}_{q,\lambda+1}^{-1}(z) * \mathcal{F}_{q,\lambda+1}(z) = z (D_q f)(z).$$

Then

$$\begin{aligned} \mathcal{I}_q^\lambda f(z) &= f(z) * \mathcal{F}_{q,\lambda+1}^{-1}(z) \\ &= z + \sum_{n=2}^{\infty} \psi_{n-1} a_n z^n \quad (z \in \mathbb{U}; \lambda > -1), \end{aligned} \quad (9)$$

where

$$\mathcal{F}_{q,\lambda+1}^{-1}(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} z^n$$

and

$$\psi_{n-1} = \frac{[n]_q! \Gamma_q(\lambda + 1)}{\Gamma_q(\lambda + n)} = \frac{[n]_q!}{[\lambda + 1]_{q, n-1}}.$$

Clearly, we have

$$\mathcal{I}_q^0 f(z) = z (D_q f)(z) \quad \text{and} \quad \mathcal{I}_q^1 f(z) = f(z).$$

We note also that, in the limit case when $q \rightarrow 1-$, the q -integral operator $\mathcal{F}_{q,\lambda}$ given by Definition 7 would reduce to the integral operator which was studied by Noor [11].

The following identity can be easily verified:

$$z D_q \left(\mathcal{I}_q^{\lambda+1} f(z) \right) = \left(1 + \frac{[\lambda]_q}{q^\lambda} \right) \mathcal{I}_q^\lambda f(z) - \frac{[\lambda]_q}{q^\lambda} \mathcal{I}_q^{\lambda+1} f(z). \quad (10)$$

When $q \rightarrow 1-$, this last identity in (10) implies that

$$z \left(\mathcal{I}^{\lambda+1} f(z) \right)' = (1 + \lambda) \mathcal{I}^\lambda f(z) - \lambda \mathcal{I}^{\lambda+1} f(z),$$

which is the well-known recurrence relation for the above-mentioned integral operator which was studied by Noor [11].

In geometric function theory, several subclasses belonging to the class of normalized analytic functions class $\mathcal{A}$ have already been investigated in different aspects. The above-defined q -calculus gives valuable tools that have been extensively used in order to investigate several subclasses of $\mathcal{A}$. Ismail et al. [12] were the first who used the q -derivative operator D_q to study the q -calculus analogous of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$ (see Definition 8 below). However, a firm footing of the q -calculus in the context of geometric function theory was presented mainly and basic (or q -) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, ([13], p. 347 et seq.); see also [14]).

Definition 8 (see [12]). A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*$ if

$$f(0) = f'(0) - 1 = 0 \quad (11)$$

and

$$\left| \frac{z}{f(z)} (D_q f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}. \quad (12)$$

It is readily observed that, as $q \rightarrow 1-$, the closed disk:

$$\left| w - \frac{1}{1-q} \right| \leq \frac{1}{1-q}$$

becomes the right-half plane and the class $\mathcal{S}_q^*$ of q -starlike functions reduces to the familiar class $\mathcal{S}^*$ of normalized starlike functions in $\mathbb{U}$ with respect to the origin ($z = 0$). Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (11) and (12) as follows (see [15]):

$$\frac{z}{f(z)} (D_q f)(z) \prec \hat{p}(z) \quad \left(\hat{p}(z) = \frac{1+z}{1-qz} \right). \quad (13)$$

The notation $\mathcal{S}_q^*$ was used by Sahoo and Sharma [16].

Now, making use of the principle of subordination between analytic functions and the above-mentioned q -calculus, we present the following definition.

Definition 9. A function p is said to be in the class $k\text{-}\mathcal{P}_q$ if and only if

$$p(z) \prec \frac{2p_k(z)}{(1+q) + (1-q)p_k(z)},$$

where $p_k(z)$ is defined by (5).

Geometrically, the function $p(z) \in k\text{-}\mathcal{P}_q$ takes on all values from the domain $\Omega_{k,q}$ ($k \geq 0$) which is defined as follows:

$$\Omega_{k,q} = \left\{ w : \Re \left(\frac{(1+q)w}{(q-1)w+2} \right) > k \left| \frac{(1+q)w}{(q-1)w+2} - 1 \right| \right\}.$$

The domain $\Omega_{k,q}$ represents a generalized conic region.

It can be seen that

$$\lim_{q \rightarrow 1^-} \Omega_{k,q} = \Omega_k,$$

where Ω_k is the conic domain considered by Kanas and Wiśniowska [3]. Below, we give some basic facts about the class $k\text{-}\mathcal{P}_q$.

Remark 1. First of all, we see that

$$k\text{-}\mathcal{P}_q \subseteq \mathcal{P} \left[\frac{2k}{2k+1+q} \right],$$

where $\mathcal{P} \left[\frac{2k}{2k+1+q} \right]$ is the well-known class of functions with real part greater than $\frac{2k}{2k+1+q}$. Secondly, we have

$$\lim_{q \rightarrow 1^-} k\text{-}\mathcal{P}_q = \mathcal{P}(p_k),$$

where $\mathcal{P}(p_k)$ is the well-known function class introduced by Kanas and Wiśniowska [3]. Thirdly, we have

$$\lim_{q \rightarrow 1^-} 0\text{-}\mathcal{P}_q = \mathcal{P},$$

where $\mathcal{P}$ is the well-known class of analytic functions with positive real part.

Definition 10. A function f is said to be in the class $\mathcal{ST}(k, \lambda, q)$ if and only if

$$\frac{z(D_q \mathcal{I}_q^\lambda f)(z)}{f(z)} \in k\text{-}\mathcal{P}_q \quad (k \geq 0; \lambda \geq 0),$$

or, equivalently,

$$\Re \left(\frac{(1+q) \frac{z(D_q \mathcal{I}_q^\lambda f)(z)}{f(z)}}{(q-1) \frac{z(D_q \mathcal{I}_q^\lambda f)(z)}{f(z)} + 2} \right) > k \left| \frac{(1+q) \frac{z(D_q \mathcal{I}_q^\lambda f)(z)}{f(z)}}{(q-1) \frac{z(D_q \mathcal{I}_q^\lambda f)(z)}{f(z)} + 2} - 1 \right|.$$

Remark 2. First of all, it is easily seen that

$$\mathcal{ST}(0, 1, q) = \mathcal{S}_q^*,$$

where $\mathcal{S}_q^*$ is the function class introduced and studied by Ismail et al. [12]. Secondly, we have

$$\lim_{q \rightarrow 1^-} \mathcal{ST}(k, 1, q) = k\text{-}\mathcal{ST},$$

where $k\text{-}\mathcal{ST}$ is a function class introduced and studied by Kanas and Wiśniowska [4]. Finally, we have

$$\lim_{q \rightarrow 1^-} \mathcal{ST}(0, 1, q) = \mathcal{S}^*,$$

where $\mathcal{S}^*$ is the well-known class of starlike functions in $\mathbb{U}$ with respect to the origin ($z = 0$).

Remark 3. Further studies of the new q -starlike function class $\mathcal{ST}(k, \lambda, q)$, as well as of its more consequences, can next be determined and investigated in future papers.

Let $n \in \mathbb{N}_0$ and $j \in \mathbb{N}$. The following j th Hankel determinant was considered by Noonan and Thomas [17]:

$$\mathcal{H}_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+j-1} \\ a_{n+1} & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n+j-1} & \cdot & \cdot & \cdot & \cdot & a_{n+2(j-1)} \end{vmatrix},$$

where $a_1 = 1$. In fact, this determinant has been studied by several authors, and sharp upper bounds on $\mathcal{H}_2(2)$ were obtained by several authors (see [18–20]) for various classes of functions. It is well-known that the Fekete–Szegő functional $|a_3 - a_2^2|$ can be represented in terms of the Hankel determinant as $\mathcal{H}_2(1)$. This functional has been further generalized as $|a_3 - \mu a_2^2|$ for some real or complex μ . Fekete and Szegő gave sharp estimates of $|a_3 - \mu a_2^2|$ for μ real and $f \in \mathcal{S}$, the class of normalized univalent functions in $\mathbb{U}$. It is also known that the functional $|a_2 a_4 - a_3^2|$ is equivalent to $\mathcal{H}_2(2)$ (see [18]). Babalola [21] studied the Hankel determinant $\mathcal{H}_3(1)$ for some subclasses of normalized analytic functions in $\mathbb{U}$. The symmetric Toeplitz determinant $\mathcal{T}_j(n)$ is defined by

$$\mathcal{T}_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+j-1} \\ a_{n+1} & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n+j-1} & \cdot & \cdot & \cdot & \cdot & a_n \end{vmatrix},$$

so that

$$\mathcal{T}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad \mathcal{T}_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \quad \mathcal{T}_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix},$$

and so on.

For $f \in \mathcal{S}$, the problem of finding the best possible bounds for $||a_{n+1}| - |a_n||$ has a long history (see, for details, [22]). It is a known fact from [22] that

$$||a_{n+1}| - |a_n|| < c$$

for a constant c . However, the problem of finding exact values of the constant c for $\mathcal{S}$ and its various subclasses has proved to be difficult. In a very recent investigation, Thomas and Abdul-Halim [23] succeeded in obtaining some sharp estimates for $\mathcal{T}_j(n)$ for the first few values of n and j involving symmetric Toeplitz determinants whose entries are the coefficients a_n of starlike and close-to-convex functions.

In the present investigation, our focus is on the Hankel determinant and the Toeplitz matrices for the function class $\mathcal{ST}(k, \lambda, q)$ given by Definition 10.

2. A Set of Lemmas

In order to prove our main results in this paper, we need each of the following lemmas.

Lemma 1 (see [20]). *If the function $p(z)$ given by (3) is in the Carathéodory class $\mathcal{P}$ of analytic functions with positive real part in $\mathbb{U}$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x^2|)z$$

for some $x, z \in \mathbb{C}$ with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 2 (see [24]). *Let the function $p(z)$ given by (3) be in the Carathéodory class $\mathcal{P}$ of analytic functions with positive real part in $\mathbb{U}$. Also let $\mu \in \mathbb{C}$. Then*

$$|c_n - \mu c_k c_{n-k}| \leq 2 \max(1, |2\mu - 1|) \quad (1 \leq k \leq n-1).$$

Lemma 3 (see [22]). *Let the function $p(z)$ given by (3) be in the Carathéodory class $\mathcal{P}$ of analytic functions with positive real part in $\mathbb{U}$. Then*

$$|c_n| \leq 2 \quad (n \in \mathbb{N}).$$

This last inequality is sharp.

3. Main Results

Throughout this section, unless otherwise mentioned, we suppose that

$$q \in (0, 1), \quad \lambda > -1 \quad \text{and} \quad k \in [0, 1].$$

Theorem 1. *If the function $f(z)$ given by (1) belongs to the class $\mathcal{ST}(k, \lambda, q)$, where $k \in [0, 1]$, then*

$$|a_2| \leq \frac{(1+q)p_1}{2q\psi_1},$$

$$a_3 \leq \frac{1}{2q\psi_2} \left(p_1 + \left| p_2 - p_1 + \frac{(q^2+1)p_1^2}{2q} \right| \right)$$

and

$$\begin{aligned} a_4 \leq & \frac{(1+q)}{4(q+q^2+q^3)\psi_3} \left(2p_1 + 4 \left| p_2 - p_1 + \frac{(2+q^2)p_1^2}{4q} \right| \right. \\ & + \left| 2p_3 + 2p_1 - 4p_2 - \frac{(2(1+q^2)-q)p_1^2}{q} + \frac{(4q^2-3q+2)}{q} p_1 p_2 \right. \\ & \left. \left. + \frac{(q^2+2q-1)}{2q^2} p_1^3 \right| \right), \end{aligned} \quad (14)$$

where p_j ($j = 1, 2, 3$) are positive and are the coefficients of the functions $p_k(z)$ defined by (6). Each of the above results is sharp for the function $g(z)$ given by

$$g(z) = \frac{2p_k(z)}{(1+q) + (1-q)p_k(z)}.$$

Proof. Let $f(z) \in \mathcal{ST}(k, \lambda, q)$. Then, we have

$$\frac{z(D_q f)(z)}{f(z)} = q(z) \prec S_k(z), \quad (15)$$

where

$$S_k(z) = \frac{2p_k(z)}{(1+q) + (1-q)p_k(z)},$$

and the functions $p_k(z)$ are defined by (6).

We now define the function $p(z)$ with $p(0) = 1$ and with a positive real part in $\mathbb{U}$ as follows:

$$p(z) = \frac{1 + S_k^{-1}(q(z))}{1 - S_k^{-1}(q(z))} = 1 + c_1 z + c_2 z^2 + \dots \quad (16)$$

After some simple computation involving (16), we get

$$q(z) = S_k\left(\frac{p(z)+1}{p(z)-1}\right).$$

We thus find that

$$\begin{aligned} & S_k\left(\frac{p(z)+1}{p(z)-1}\right) \\ &= 1 + \left(\frac{q+1}{2}\right) \left[\frac{p_1 c_1}{2} z + \left\{ \frac{p_1 c_2}{2} + \left(\frac{p_2}{4} - \frac{p_1}{4} + \left(\frac{(q-1)p_1^2}{8} \right) \right) c_1^2 \right\} z^2 \right. \\ & \quad + \left\{ \frac{p_1 c_3}{2} + \left(\frac{p_2}{2} - \frac{p_1}{2} + \left(\frac{(q-1)p_1^2}{4} \right) \right) c_1 c_2 \right. \\ & \quad \left. \left. + \left(\frac{p_1}{8} - \frac{p_2}{4} - \frac{(q-1)p_1^2}{8} + \frac{p_3}{8} - \frac{(q-1)p_1 p_2}{8} + \frac{(q-1)^2 p_1^3}{32} \right) c_1^3 \right\} z^3 \right] + \dots \quad (17) \end{aligned}$$

Now, upon expanding the left-hand side of (15), we have

$$\begin{aligned} & \frac{z(D_q \mathcal{I}_q^\lambda f)(z)}{f(z)} = 1 + q\psi_1 a_2 z + \left\{ (q+q^2)\psi_2 a_3 - q\psi_1^2 a_2^2 \right\} z^2 \\ & \quad + \left\{ (q+q^2+q^3)\psi_3 a_4 - (2q+q^2)\psi_1 \psi_2 a_2 a_3 + q\psi_1^3 a_2^3 \right\} z^3 + \dots \quad (18) \end{aligned}$$

Finally, by comparing the corresponding coefficients in (17) and (18) along with Lemma 3, we obtain the result asserted by Theorem 1. $\square$

Theorem 2. If the function $f(z)$ given by (1) belongs to the class $\mathcal{ST}(k, \lambda, q)$, then

$$\begin{aligned} \mathcal{T}_3(2) &\leq \left[\left(\frac{1+q}{2q\psi_1} \right) p_1^2 + \left(\frac{1+q}{4(q+q^2+q^3)\psi_3} \right) [\Omega_1 + \Omega_2] \right] \\ &\quad \cdot \left[4 \left(\frac{(1+q)^2}{16q^2\psi_1^2} \right) p_1^2 + 16|\Omega_3| + \frac{p_1^2}{4q^2\psi_2^2} + 2\Omega_5 p_1^2 \right] \left| 2 - \frac{\Omega_4}{\Omega_5 p_1^2} \right|, \end{aligned}$$

where

$$\begin{aligned}\Omega_1 &= 2p_1 + 4 \left| p_2 - p_1 + \frac{(2+q^2)}{4q} p_1^2 \right|, \\ \Omega_2 &= \left| 2p_3 + 2p_1 - 4p_2 - \left(2(1+q^2) - q \right) p_1^2 \right. \\ &\quad \left. + \left(\frac{4q^2 - 3q + 2}{q} \right) p_1 p_2 + \left(\frac{q^2 + q + 1}{2q^2} p_1^3 \right) \right|, \\ \Omega_3 &= \frac{1}{2q^2 \psi_2^2} \left(\frac{p_2}{4} - \frac{p_1}{4} + \frac{(q^2 + 1) p_1^2}{8q} \right)^2 - \Omega_5 \cdot \left[\frac{p_3}{4} + \frac{p_1}{4} - \frac{p_2}{2} \right. \\ &\quad \left. - \frac{[2(1+q^2) - q] p_1^2}{8q} + \frac{4q^2 - 3q + 2}{8q} p_1 p_2 + \left(\frac{q^2 + 2q - 1}{16q^2} \right) p_1^3 \right], \\ \Omega_4 &= \frac{p_1}{2q^2 \psi_2^2} \left(\frac{p_2}{4} - \frac{p_1}{4} + \frac{(q^2 + 1) p_1^2}{8q} \right) - \Omega_5 p_1 \left(p_2 - p_1 + \frac{(2+q^2) p_1^2}{4q} \right), \\ \Omega_5 &= \frac{(1+q)^2}{16q^2 (1+q+q^2) \psi_1 \psi_3}\end{aligned}$$

and p_j ($j = 1, 2$) are positive and are the coefficients of the functions $p_k(z)$ defined by (6).

Proof. Upon comparing the corresponding coefficients in (17) and (18), we find that

$$a_2 = \frac{(1+q) p_1 c_1}{4q \psi_1}, \quad (19)$$

$$a_3 = \frac{1}{2q \psi_2} \left[\frac{p_1 c_2}{2} + \left(\frac{p_2}{4} - \frac{p_1}{4} + \frac{(q^2 + 1) p_1^2}{8q} \right) c_1^2 \right], \quad (20)$$

$$\begin{aligned}a_4 &= \frac{(1+q)}{4(q+q^2+q^3) \psi_3} \left[p_1 c_3 + \left(p_2 - p_1 + \frac{(2+q^2) p_1^2}{4q} \right) c_1 c_2 \right. \\ &\quad \left. + \left(\frac{p_3}{4} + \frac{p_1}{4} - \frac{p_2}{2} - \frac{(2(1+q^2) - q) p_1^2}{8q} + \frac{(4q^2 - 3q + 2)}{8q} p_1 p_2 \right. \right. \\ &\quad \left. \left. + \frac{(q^2 + 2q - 1)}{16q^2} p_1^3 \right) c_1^3 \right]. \quad (21)\end{aligned}$$

By a simple computation, $\mathcal{T}_3(2)$ can be written as follows:

$$\mathcal{T}_3(2) = (a_2 - a_4) (a_2^2 - 2a_3^2 + a_2 a_4).$$

Now, if $f \in \mathcal{ST}(k, \lambda, q)$, then it is clearly seen that

$$\begin{aligned}|a_2 - a_4| &\leq |a_2| + |a_4| \\ &\leq \left(\frac{1+q}{2q \psi_1} \right) p_1^2 + \left(\frac{1+q}{4(q+q^2+q^3) \psi_3} \right) (\Omega_1 + \Omega_2).\end{aligned}$$

We need to maximize $|a_2^2 - 2a_3^2 + a_2a_4|$ for a function $f \in \mathcal{ST}(k, \lambda, q)$. So, by writing a_2, a_3 , and a_4 in terms of c_1, c_2 , and c_3 , with the help of (19)–(21), we get

$$\begin{aligned} & \left| a_2^2 - 2a_3^2 + a_2a_4 \right| \\ &= \left| \left(\frac{(1+q)^2}{16q^2\psi_1^2} \right) p_1^2 c_1^2 - \Omega_3 c_1^4 - \Omega_4 c_1^2 c_2 - \frac{p_1^2}{8q^2\psi_2^2} c_2^2 + \Omega_5 p_1^2 c_1 c_3 \right|. \end{aligned} \quad (22)$$

Finally, by applying the trigonometric inequalities, Lemmas 2 and 3 along with (22), we obtain the result asserted by Theorem 2. $\square$

As an application of Theorem 2, we first set $\psi_{n-1} = 1$ and $k = 0$ and then let $q \rightarrow 1^-$. We thus arrive at the following known result.

Corollary 1 (see [25]). *If the function $f(z)$ given by (1) belongs to the class $\mathcal{S}^*$, then*

$$\mathcal{T}_3(2) \leq 84.$$

Theorem 3. *If the function $f(z)$ given by (1) belongs to the class $\mathcal{ST}(k, \lambda, q)$, then*

$$\left| a_2a_4 - a_3^2 \right| \leq \frac{1}{4q^2\psi_2^2} p_1^2, \quad (23)$$

where $k \in [0, 1]$ and p_j ($j = 1, 2, 3$) are positive and are the coefficients of the functions $p_k(z)$ defined by (6).

Proof. Making use of (19)–(21), we find that

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{A(q)}{16q^2\psi_1\psi_3} p_1^2 c_1 c_3 + \left(\frac{A(q)\psi_2^2 - \psi_1\psi_3}{16q^2\psi_1\psi_2^2\psi_3} p_1 p_2 - \frac{A(q)\psi_2^2 - \psi_1\psi_3}{16q^2\psi_1\psi_2^2\psi_3} p_1^2 \right. \\ &\quad \left. + \frac{A(q)(2+q^2)\psi_2^2 - 2(1+q^2)\psi_1\psi_3}{64q^2\psi_1\psi_3} p_1^3 \right) c_1^2 c_2 + \frac{1}{16q^2\psi_2^2} p_1^2 c_2^2 \\ &\quad + \left[\frac{A(q)}{64q^2\psi_1\psi_3} p_1 p_3 + \left(\frac{A(q)\psi_2^2 - \psi_1\psi_3}{64q^2\psi_1\psi_2^2\psi_3} \right) p_1^2 + \left(\frac{\psi_1\psi_3 - A(q)\psi_2^2}{32q^2\psi_1\psi_2^2\psi_3} \right) p_1 p_2 \right. \\ &\quad \left. + \left(\frac{2(1+q^2)\psi_1\psi_3 - (2(1+q^2) - q)A(q)\psi_2^2}{128q^3\psi_1\psi_2^2\psi_3} \right) p_1^3 \right. \\ &\quad \left. + \left(\frac{A(q)(4q^2 - 3q + 2)\psi_2^2 - 2(1+q^2)\psi_1\psi_3}{128q^3\psi_1\psi_2^2\psi_3} \right) p_1^2 p_2 \right. \\ &\quad \left. + \left(\frac{A(q)(q^2 + 2q - 1)\psi_2^2 - (1+q^2)^2\psi_1\psi_3}{256q^4\psi_1\psi_2^2\psi_3} \right) p_1^4 - \frac{1}{64q^2\psi_2^2} p_2^2 \right] c_1^4, \end{aligned} \quad (24)$$

where

$$A(q) = \frac{(1+q)^2}{1+q+q^2}.$$

We substitute the values of c_2 and c_3 from the above Lemma and, for simplicity, take $Y = 4 - c_1^2$ and $Z = (1 - |x|^2)z$. Without loss of generality, we assume that $c = c_1$ ($0 \leq c \leq 2$), so that

$$\begin{aligned} a_2 a_4 - a_3^2 = & \left[\frac{q(1-q)A(q)\psi_2^2}{128q^2\psi_1\psi_3} p_1^3 + \frac{A(q)}{64q^2\psi_1\psi_3} p_1 p_3 \right. \\ & + \left(\frac{A(q)(4q^2 - 3q + 2)\psi_2^2 - 2(1+q^2)\psi_1\psi_3}{128q^3\psi_1\psi_2^2\psi_3} \right) p_1^2 p_2 \\ & + \left(\frac{A(q)(q^2 + 2q - 1)\psi_2^2 - (1+q^2)^2\psi_1\psi_3}{256q^4\psi_1\psi_2^2\psi_3} \right) p_1^4 - \frac{1}{64q^2\psi_2^2} p_2^2 \Big] c^4 \\ & + \left[\frac{A(q)\psi_2^2 - \psi_1\psi_3}{32q^2\psi_1\psi_2^2\psi_3} p_1 p_2 + \frac{A(q)(2+q^2)\psi_2^2 - 2(1+q^2)\psi_1\psi_3}{128q^2\psi_1\psi_3} p_1^3 \right] c^2 x Y \\ & \cdot \left[-\frac{A(q)}{64q^2\psi_1\psi_3} p_1^2 c^2 Y x^2 - \frac{1}{64q^2\psi_2^2} p_1^2 x^2 Y^2 + \frac{A(q)}{32q^2\psi_1\psi_3} p_1^2 c Y Z \right]. \end{aligned} \quad (25)$$

Upon setting $Z = (1 - |x|^2)z$ and taking the moduli in (25) and using trigonometric inequality, we find that

$$\begin{aligned} |a_2 a_4 - a_3^2| & \leq |\lambda_1| c^4 + |\lambda_2| |x| Y c^2 + \frac{A(q)}{64q^2\psi_1\psi_3} p_1^2 Y |x|^2 c^2 \\ & \quad + \frac{1}{64q^2\psi_2^2} p_1^2 |x|^2 Y^2 + \frac{A(q)}{32q^2\psi_1\psi_3} p_1^2 c^2 Y (1 - |x|^2) \\ & = \Lambda(c, |x|), \end{aligned} \quad (26)$$

where

$$\begin{aligned} \lambda_1 & = \frac{q(1-q)A(q)\psi_2^2}{128q^2\psi_1\psi_3} p_1^3 + \frac{A(q)}{64q^2\psi_1\psi_3} p_1 p_3 \\ & \quad + \left(\frac{A(q)(4q^2 - 3q + 2)\psi_2^2 - 2(1+q^2)\psi_1\psi_3}{128q^3\psi_1\psi_2^2\psi_3} \right) p_1^2 p_2 \\ & \quad + \left(\frac{A(q)(q^2 + 2q - 1)\psi_2^2 - (1+q^2)^2\psi_1\psi_3}{256q^4\psi_1\psi_2^2\psi_3} \right) p_1^4 - \frac{1}{64q^2\psi_2^2} p_2^2 \\ \lambda_2 & = \frac{A(q)\psi_2^2 - \psi_1\psi_3}{32q^2\psi_1\psi_2^2\psi_3} p_1 p_2 + \frac{A(q)(2+q^2)\psi_2^2 - 2(1+q^2)\psi_1\psi_3}{128q^2\psi_1\psi_3} p_1^3. \end{aligned}$$

Now, trivially, we have

$$\Lambda'(|x|) > 0$$

on $[0, 1]$, and so

$$\Lambda(|x|) \leq \Lambda(1).$$

Hence, by putting $Y = 4 - c_1^2$ and after some simplification, we have

$$\begin{aligned} |a_2 a_4 - a_3^2| & = \left(|\lambda_1| - |\lambda_2| + \frac{\psi_1\psi_3 - A(q)\psi_2^2}{64q^2\psi_1\psi_3} p_1^2 \right) c^4 \\ & \quad + \left(4|\lambda_2| + \left(\frac{A(q)\psi_2^2 - \psi_1\psi_3}{16q^2\psi_1\psi_3} p_1^2 \right) \right) c^2 + \frac{1}{4q^2\psi_2^2} p_1^2 \\ & = G(c). \end{aligned} \quad (27)$$

For optimum value of $G(c)$, we consider $G'(c) = 0$, which implies that $c = 0$. So $G(c)$ has a maximum value at $c = 0$. We therefore conclude that the maximum value of $G(c)$ is given by

$$\frac{1}{4q^2\psi_2^2}p_1^2,$$

which occurs at $c = 0$ or

$$c^2 = -\frac{128|\lambda_2|q^2\psi_1\psi_3 + 4A(q)\psi_2^2 - 2\psi_1\psi_3p_1^2}{(64q^2(|\lambda_1| - |\lambda_2|)\psi_1\psi_3 + \psi_1\psi_3 - A(q)\psi_2^2p_1^2)}.$$

This completes the proof of Theorem 3. $\square$

If we put $\psi_{n-1} = 1$ and let $q \rightarrow 1-$ in Theorem 3, we have the following known result.

Corollary 2 (see [26]). *If the function $f(z)$ given by (1) belongs to the class $k\text{-ST}$, where $k \in [0, 1]$, then*

$$|a_2a_4 - a_3^2| \leq \frac{p_1^2}{4}.$$

If we put

$$p_1 = 2 \quad \text{and} \quad \psi_{n-1} = 1,$$

by letting $q \rightarrow 1-$ in Theorem 3, we have the following known result.

Corollary 3 (see [18]). *If $f \in \mathcal{S}^*$, then*

$$|a_2a_4 - a_3^2| \leq 1.$$

By letting $k = 1$, $\psi_{n-1} = 1$, $q \rightarrow 1-$ and

$$p_1 = \frac{8}{\pi^2}, \quad p_2 = \frac{16}{3\pi^2} \quad \text{and} \quad p_3 = \frac{184}{45\pi^2}$$

in Theorem 3, we have the following known result.

Corollary 4 (see [27]). *If the function $f(z)$ given by (1) belong to the class $\mathcal{SP}$, then*

$$|a_2a_4 - a_3^2| \leq \frac{16}{\pi^4}.$$

4. Concluding Remarks and Observations

Motivated significantly by a number of recent works, we have made use of a certain general conic domain and the quantum (or q -) calculus in order to define and investigate a new subclass of normalized analytic functions in the open unit disk $\mathbb{U}$, which we have referred to as q -starlike functions. For this q -starlike function class, we have successfully derived several properties and characteristics. In particular, we have found the Hankel determinant and the Toeplitz matrices for this newly-defined class of q -starlike functions. We also highlight some known consequences of our main results which are stated and proved as theorems and corollaries.

Author Contributions: conceptualization, Q.Z.A. and N.K. (Nazar Khan); methodology, N.K. (Nasir Khan); software, B.K.; validation, H.M.S.; formal analysis, H.M.S.; writing—original draft preparation, H.M.S.; writing—review and editing, H.M.S.; supervision, H.M.S.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Miller, S.S.; Mocanu, P.T. Differential subordination and univalent functions. *Mich. Math. J.* **1981**, *28*, 157–171. [\[CrossRef\]](#)
2. Miller, S.S.; Mocanu, P.T. *Differential Subordination: Theory and Applications*; Series on Monographs and Textbooks in Pure and Applied Mathematics, No. 225; Marcel Dekker Incorporated: New York, NY, USA; Basel, Switzerland, 2000.
3. Kanas, S.; Wiśniowska, A. Conic regions and k -uniform Convexity. *J. Comput. Appl. Math.* **1999**, *105*, 327–336. [\[CrossRef\]](#)
4. Kanas, S.; Wiśniowska, A. Conic domains and starlike Functions. *Rev. Roum. Math. Pures Appl.* **2000**, *45*, 647–657.
5. Kanas, S.; Srivastava, H.M. Linear operators associated with k -uniformly convex functions. *Integral Transform. Spec. Funct.* **2000**, *9*, 121–132. [\[CrossRef\]](#)
6. Mahmood, S.; Jabeen, M.; Malik, S.N.; Srivastava, H.M.; Manzoor, R.; Riaz, S.M.J. Some coefficient inequalities of q -starlike functions associated with conic domain defined by q -derivative. *J. Funct. Spaces* **2018**, *2018*, 8492072. [\[CrossRef\]](#)
7. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Ellis Horwood Limited: Chichester, UK, 1985.
8. Jackson, F.H. On q -definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
9. Jackson, F.H. q -difference equations. *Am. J. Math.* **1910**, *32*, 305–314. [\[CrossRef\]](#)
10. Kanas, S.; Răducanu, D. Some class of analytic functions related to conic domains. *Math. Slov.* **2014**, *64*, 1183–1196. [\[CrossRef\]](#)
11. Noor, K.I. On new classes of integral operators. *J. Nat. Geom.* **1999**, *16*, 71–80.
12. Ismail, M.E.H.; Merkes, E.; Styer, D. A generalization of starlike functions. *Complex Var. Theory Appl.* **1990**, *14*, 77–84. [\[CrossRef\]](#)
13. Srivastava, H.M. Univalent functions, fractional calculus, and associated generalized hypergeometric functions. In *Univalent Functions, Fractional Calculus and Their Applications*; Srivastava, H.M., Owa, S., Eds.; Ellis Horwood Limited: Chichester, UK, 1989; pp. 329–354.
14. Srivastava, H.M.; Bansal, D. Close-to-convexity of a certain family of q -Mittag-Leffler functions. *J. Nonlinear Var. Anal.* **2017**, *1*, 61–69.
15. Uçar, H.E.Ö. Coefficient inequality for q -starlike Functions. *Appl. Math. Comput.* **2016**, *76*, 122–126.
16. Sahoo, S.K.; Sharma, N.L. On a generalization of close-to-convex functions. *Ann. Polon. Math.* **2015**, *113*, 93–108. [\[CrossRef\]](#)
17. Noonan, J.W.; Thomas, D.K. On the second Hankel derminant of areally mean p -valent functions. *Trans. Am. Math. Soc.* **1976**, *223*, 337–346.
18. Janteng, A.; Abdul-Halim, S.; Darus, M. Hankel determinant for starlike and convex functions. *Int. J. Math. Anal.* **2007**, *1*, 619–625.
19. Mishra, A.K.; Gochhayat, P. Second Hankel determinant for a class of analytic functions defined by fractional derivative. *Internat. J. Math. Math. Sci.* **2008**, *2008*, 153280. [\[CrossRef\]](#)
20. Singh, G.; Singh, G. On the second Hankel determinant for a new subclass of analytic functions. *J. Math. Sci. Appl.* **2014**, *2*, 1–3.
21. Babalola, K.O. On $H_3(1)$ Hankel determinant for some classes of univalent functions. *Inequal. Theory Appl.* **2007**, *6*, 1–7.
22. Duren, P.L. *Univalent Functions (Grundlehren der Mathematischen Wissenschaften 259)*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
23. Thomas, D.K.; Abdul-Halim, S. Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions. *Bull. Malays. Math. Sci. Soc.* **2017**, *40*, 1781–1790. [\[CrossRef\]](#)
24. Efraimidis, I. A generalization of Livingston’s coefficient inequalities for functions with positive real part. *J. Math. Anal. Appl.* **2016**, *435*, 369–379. [\[CrossRef\]](#)
25. Ali, M.F.; Thomas, D.K.; Vasudevarao, A. Toeplitz determinants whose element are the coefficients of univalent functions. *Bull. Aust. Math. Soc.* **2018**, *97*, 253–264. [\[CrossRef\]](#)

26. Ramachandran, C.; Annamalai, S. On Hankel and Toeplitz determinants for some special class of analytic functions involving conical domains defined by subordination. *Internat. J. Engrg. Res. Technol.* **2016**, *5*, 553–561.
27. Lee, S.K.; Ravichandran, V.; Supramaniam, S. Bounds for the second Hankel determinant of certain univalent functions. *J. Inequal. Appl.* **2013**, *2013*, 281–297. [[CrossRef](#)]



© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).