ON NEW SUBCLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS*

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ABSTRACT

The object of the present paper is to prove several interesting results involving coefficient inequalities, distortion theorems, starlikeness and convexity for functions belonging to the subclasses $\mathcal{T}^*(\alpha,\lambda)$ and $\mathcal{C}(\alpha,\lambda)$ of analytic and univalent functions, which are introduced here in terms of fractional calculus. Some theorems for the classes $\mathcal{T}^*(\alpha,\lambda,k)$ and $\mathcal{C}(\alpha,\lambda,k)$ of functions with fixed finitely many coefficients are also proved, and the classes of univalent functions with quasiconformal extension are considered.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. Further, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of analytic and univalent functions in the unit disk $\mathcal{U}$. Then a function $f(z)$ of $\mathcal{S}$ is said to be starlike of order α if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some α $(0 \leqq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the class of all starlike functions of order α.

A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be convex of order α if and only if

$$\text{(1.3)} \qquad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U})$$

for some α $(0 \leqq \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the class of all convex functions of order α. We note that $f(z) \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$, and that

$$\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) \equiv \mathcal{S}^*, \quad \mathcal{K}(\alpha) \subset \mathcal{K}(0) \equiv \mathcal{K}, \quad \text{and} \quad \mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha)$$

for $0 \leqq \alpha < 1$.

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [10], and were studied subsequently by Schild [13], MacGregor [4], Pinchuk [9], and others.

Many essentially equivalent definitions of the fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [2, Chapter 13], [6], [7], [11], [12], [15, p. 28 et seq.], [17], and [18]). We find it to be convenient to recall here the following definitions which were used recently by Owa [8] (and by Srivastava and Owa [16]).

DEFINITION 1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$\text{(1.4)} \qquad D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\, d\zeta,$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is

removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

DEFINITION 2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$(1.5) \qquad D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta,$$

where $0 \leqq \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1 above.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $n + \lambda$ is defined by

$$(1.6) \qquad D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z),$$

where $0 \leqq \lambda < 1$, $n \in \mathcal{N}_0 = \{0,1,2,\ldots\}$.

Let $\mathcal{S}^*(\alpha,\lambda)$ denote the class of all functions $f(z)$ in $\mathcal{S}$ satisfying the inequality

$$(1.7) \qquad \operatorname{Re}\left(\frac{\Gamma(1-\lambda) z^{1+\lambda} D_z^{1+\lambda} f(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U})$$

for $\lambda < 1$ and $0 \leqq \alpha < 1$. Also let $\mathcal{K}(\alpha,\lambda)$ denote the class of all functions $f(z)$ in $\mathcal{S}$ such that

$$(1.8) \qquad \Gamma(1-\lambda) z^{1+\lambda} D_z^{1+\lambda} f(z) \in \mathcal{S}^*(\alpha,\lambda)$$

for $\lambda < 1$ and $0 \leqq \alpha < 1$. Clearly,

$$\mathcal{S}^*(\alpha,0) = \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{K}(\alpha,0) = \mathcal{K}(\alpha)$$

where we have set $\lambda = 0$. Thus $\mathcal{S}^*(\alpha,\lambda)$ and $\mathcal{K}(\alpha,\lambda)$ are generalizations of the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively.

Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of all functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0). \tag{1.9}$$

Denote by $\mathcal{T}^*(\alpha,\lambda)$ and $\mathcal{C}(\alpha,\lambda)$ the classes obtained by taking intersections, respectively, of the classes $\mathcal{S}^*(\alpha,\lambda)$ and $\mathcal{K}(\alpha,\lambda)$ with $\mathcal{T}$, that is,

$$\mathcal{T}^*(\alpha,\lambda) = \mathcal{S}^*(\alpha,\lambda) \cap \mathcal{T} \tag{1.10}$$

and

$$\mathcal{C}(\alpha,\lambda) = \mathcal{K}(\alpha,\lambda) \cap \mathcal{T}. \tag{1.11}$$

The special cases of the classes $\mathcal{T}^*(\alpha,\lambda)$ and $\mathcal{C}(\alpha,\lambda)$, when $\lambda = 0$, were studied by Silverman [14]. Thus the classes $\mathcal{T}^*(\alpha,\lambda)$ and $\mathcal{C}(\alpha,\lambda)$ provide interesting generalizations of the classes considered by Silverman [14].

In this paper, we first prove several interesting results for functions belonging to the general classes $\mathcal{T}^*(\alpha,\lambda)$ and $\mathcal{C}(\alpha,\lambda)$. We then introduce the subclasses $\mathcal{T}^*(\alpha,\lambda,k)$ and $\mathcal{C}(\alpha,\lambda,k)$ of functions with fixed finitely many coefficients. And, finally, we consider the classes of univalent functions with quasiconformal extension.

2. COEFFICIENT INEQUALITIES

THEOREM 1. *Let the function* $f(z)$ *be defined by* (1.1). *If*

$$\sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}|a_n| \leqq 1 - \alpha \tag{2.1}$$

for $\lambda < 1$ *and* $0 \leqq \alpha < 1$, *then* $f(z) \in \mathcal{S}^*(\alpha,\lambda)$. *The result* (2.1) *is sharp*.

PROOF. We need only prove that (2.1) implies (1.7). In order to prove (1.7), it suffices to show that

$$\left|\frac{\Gamma(1-\lambda)\, z^{1+\lambda}\, D_z^{1+\lambda} f(z)}{f(z)} - 1\right| < 1 - \alpha \qquad (z \in \mathcal{U}). \tag{2.2}$$

Note that

$$D_z^{1+\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} z^{-\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\lambda)} a_n z^{n-1-\lambda}, \tag{2.3}$$

which readily yields

$$\left|\frac{\Gamma(1-\lambda)\, z^{1+\lambda}\, D_z^{1+\lambda} f(z)}{f(z)} - 1\right| \tag{2.4}$$

$$= \left|\frac{\sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - 1\right\} a_n z^n}{z - \sum_{n=2}^{\infty} a_n z^n}\right|$$

$$\leqq \frac{\sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - 1\right\} a_n}{1 - \sum_{n=2}^{\infty} a_n}$$

$$\leqq 1 - \alpha,$$

provided that

$$\sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - 1\right\} a_n \leqq (1-\alpha)\left(1 - \sum_{n=2}^{\infty} a_n\right). \tag{2.5}$$

We note that (2.5) is equivalent to (2.1). Further, the result (2.1) is sharp for the functions

$$f(z) = z + \left[(1-\alpha)\Big/\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}\right] z^n \qquad (n \geqq 2). \tag{2.6}$$

Thus we complete the proof of Theorem 1.

REMARK 1. Letting $\lambda = 0$ in Theorem 1, we obtain the corresponding result for $\mathcal{S}^*(\alpha)$ due to Silverman [14].

THEOREM 2. <u>Let the function</u> $f(z)$ <u>be defined by</u> (1.1). <u>If</u>

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} |a_n| \leqq 1 - \alpha \tag{2.7}$$

<u>for</u> $\lambda < 1$ <u>and</u> $0 \leqq \alpha < 1$, <u>then</u> $f(z) \in \mathcal{K}(\alpha,\lambda)$. <u>The result</u> (2.7) <u>is sharp</u>.

PROOF. Note that $f(z) \in \mathcal{K}(\alpha,\lambda)$ if and only if

$$\Gamma(1-\lambda) z^{1+\lambda} D_z^{1+\lambda} f(z) \in \mathcal{S}^*(\alpha,\lambda).$$

Therefore, on replacing a_n by $[\Gamma(n+1)\Gamma(1-\lambda)/\Gamma(n-\lambda)]a_n$ in Theorem 1, we have Theorem 2. Further, the result (2.7) is sharp for the functions

$$f(z) = z + \left[(1-\alpha)\Big/\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}\right]z^n \qquad (n \geqq 2). \tag{2.8}$$

REMARK 2. Letting $\lambda = 0$ in Theorem 2, we have the corresponding result for $\mathcal{K}(\alpha)$ given by Silverman [14].

THEOREM 3. <u>Let the function</u> $f(z)$ <u>be defined by</u> (1.9). <u>Then</u> $f(z)$ <u>belongs to the class</u> $\mathcal{T}^*(\alpha,\lambda)$ <u>if and only if</u>

$$\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}a_n \leqq 1 - \alpha \tag{2.9}$$

<u>for</u> $\lambda < 1$ <u>and</u> $0 \leqq \alpha < 1$. <u>The result</u> (2.9) <u>is sharp</u>.

PROOF. By means of Theorem 1, (2.9) implies that $f(z) \in \mathcal{T}^*(\alpha,\lambda)$. Suppose that $f(z)$ is in the class $\mathcal{T}^*(\alpha,\lambda)$. Then

$$\begin{aligned} &\operatorname{Re}\left(\frac{\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z)}{f(z)}\right) \\ &= \operatorname{Re}\left(\frac{1 - \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}a_n z^{n-1}}{1 - \sum_{n=2}^{\infty}a_n z^{n-1}}\right) \\ &> \alpha \end{aligned} \tag{2.10}$$

for $\lambda < 1$, $0 \leqq \alpha < 1$, and $z \in \mathcal{U}$. Choose values of z on the real axis

so that $\Gamma(1-\lambda)z^{1+\lambda} D_z^{1+\lambda} f(z)/f(z)$ is real. Upon clearing the denominator in (2.10) and letting $z \to 1^-$, we obtain

$$(2.11) \qquad 1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} a_n \geqq \alpha\left(1 - \sum_{n=2}^{\infty} a_n\right)$$

which implies (2.9). The result (2.9) is sharp for the functions

$$(2.12) \qquad f(z) = z - \left[(1-\alpha)\Big/\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}\right] z^n \qquad (n \geqq 2).$$

This completes the proof of Theorem 3.

COROLLARY 1. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.9) <u>belong to the class</u> $\mathcal{T}^*(\alpha,\lambda)$. <u>Then</u>

$$(2.13) \qquad a_n \leqq (1-\alpha)\Big/\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} \qquad (n \geqq 2).$$

<u>The result</u> (2.13) <u>is sharp for the functions</u> $f(z)$ <u>given by</u> (2.12).

Next we have

THEOREM 4. <u>Let the function</u> $f(z)$ <u>be defined by</u> (1.9). <u>Then</u> $f(z)$ <u>belongs to the class</u> $\mathcal{C}(\alpha,\lambda)$ <u>if and only if</u>

$$(2.14) \qquad \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} a_n \leqq 1 - \alpha$$

<u>for</u> $\lambda < 1$ <u>and</u> $0 \leqq \alpha < 1$. <u>The result</u> (2.14) <u>is sharp for the functions</u>

$$(2.15) \qquad f(z) = z - \left[(1-\alpha)\Big/\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}\right] z^n \qquad (n \geqq 2).$$

COROLLARY 2. Let the function $f(z)$ defined by (1.9) belong to the class $\mathscr{C}(\alpha,\lambda)$. Then

$$a_n \leqq (1-\alpha) \Big/ \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{ \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha \right\} \qquad (n \geqq 2). \tag{2.16}$$

The result (2.16) is sharp for the functions $f(z)$ given by (2.15).

3. DISTORTION THEOREMS

By virtue of the coefficient inequalities, we can prove the following distortion theorems for functions $f(z)$ belonging to the classes $\mathscr{T}^*(\alpha,\lambda)$ and $\mathscr{C}(\alpha,\lambda)$.

LEMMA 1. The class $\mathscr{T}^*(\alpha,\lambda)$ is closed under linear combinations.

PROOF. Let the functions

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \qquad (i = 1,2;\ a_{n,i} \geqq 0) \tag{3.1}$$

be in the class $\mathscr{T}^*(\alpha,\lambda)$. Then we need only prove that the function

$$\delta\, f_1(z) + (1-\delta) f_2(z) \qquad (0 \leqq \delta \leqq 1)$$

is also in the class $\mathscr{T}^*(\alpha,\lambda)$. In fact, we have

$$\delta\, f_1(z) + (1-\delta) f_2(z) \tag{3.2}$$

$$= z - \sum_{n=2}^{\infty} \left(\delta\, a_{n,1} + (1-\delta) a_{n,2} \right) z^n.$$

Hence, with the aid of Theorem 3, we have

$$(3.3)\qquad \sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\left(\delta\, a_{n,1}+(1-\delta)a_{n,2}\right)$$

$$=\delta\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_{n,1}$$

$$+(1-\delta)\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_{n,2}$$

$$\leqq \delta(1-\alpha)+(1-\delta)(1-\alpha)$$

$$=1-\alpha,$$

which completes the proof of Lemma 1.

By means of Lemma 1, we know that $\mathcal{T}^*(\alpha,\lambda)$ is convex, and hence further that $\mathcal{T}^*(\alpha,\lambda)$ has some extreme points.

LEMMA 2. <u>Let</u>

$$(3.4)\qquad f_1(z)=z$$

<u>and</u>

$$(3.5)\qquad f_n(z)=z-\frac{1-\alpha}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha}\,z^n \qquad (n\geqq 2).$$

<u>Then</u> $f(z)$ <u>is in the class</u> $\mathcal{T}^*(\alpha,\lambda)$ <u>if and only if it can be expressed in the form</u>

$$(3.6)\qquad f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z),$$

<u>where</u> $\lambda_n \geqq 0 \quad (n \geqq 1)$ <u>and</u>

$$\sum_{n=1}^{\infty} \lambda_n = 1. \tag{3.7}$$

PROOF. We assume that $f(z)$ has the form (3.6). Then

$$f(z) = z - \sum_{n=2}^{\infty} \frac{(1-\alpha)\lambda_n}{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha} z^n. \tag{3.8}$$

Consequently, we obtain

$$\begin{aligned} \sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} \cdot \frac{(1-\alpha)\lambda_n}{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha} \qquad (3.9) \\ = (1-\alpha) \sum_{n=2}^{\infty} \lambda_n \\ = (1-\alpha)(1-\lambda_1) \\ \leqq 1 - \alpha, \end{aligned}$$

which implies that $f(z) \in \mathcal{T}^*(\alpha,\lambda)$.

For the converse, we assume that $f(z)$ is in the class $\mathcal{T}^*(\alpha,\lambda)$. Then, by setting

$$\lambda_n = \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} a_n/(1-\alpha) \qquad (n \geqq 2) \tag{3.10}$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n, \tag{3.11}$$

we have (3.6).

In view of Lemma 2, we have

THEOREM 5. <u>The extreme points of</u> $\mathcal{T}^*(\alpha,\lambda)$ <u>are the functions</u> $f_n(z)$ $(n \geqq 1)$ <u>given by</u> (3.4) <u>and</u> (3.5).

Similarly, we have

THEOREM 6. <u>The extreme points of</u> $\mathcal{C}(\alpha,\lambda)$ <u>are the functions</u> $f_n(z)$ $(n \geqq 1)$ <u>given by</u> (3.4) <u>and</u>

$$f_n(z) = z - \frac{1-\alpha}{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}} z^n \qquad (n \geqq 2). \tag{3.12}$$

THEOREM 7. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.9) <u>belong to the class</u> $\mathcal{T}^*(\alpha,\lambda)$. <u>Then, if</u> $-1 \leqq \lambda < 1$

$$|f(z)| \geqq |z| - \frac{(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)}|z|^2 \tag{3.13}$$

<u>and</u>

$$|f(z)| \leqq |z| + \frac{(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)}|z|^2 \tag{3.14}$$

<u>for</u> $z \in \mathcal{U}$. <u>Furthermore, if</u> $0 \leqq \lambda < 1$,

$$|f'(z)| \geqq 1 - \frac{2(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)}|z| \tag{3.15}$$

<u>and</u>

$$|f'(z)| \leqq 1 + \frac{2(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)}|z| \tag{3.16}$$

<u>for</u> $z \in \mathcal{U}$. <u>The bounds</u> (3.13) <u>to</u> (3.16) <u>are sharp</u>.

PROOF. Note that the extremal function is one of the extreme points. Therefore, we have

$$(3.17) \qquad |f(z)| \geqq |z| - \max_{n\geqq 2}\left\{\frac{1-\alpha}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha}\,|z|^n\right\}$$

and

$$(3.18) \qquad |f(z)| \leqq |z| + \max_{n\geqq 2}\left\{\frac{1-\alpha}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha}\,|z|^n\right\}.$$

Now define

$$(3.19) \qquad \mathcal{E}(\lambda,\alpha,n,|z|) = \frac{\Gamma(n-\lambda)\,|z|^n}{\Gamma(n+1)\Gamma(1-\lambda) - \alpha\,\Gamma(n-\lambda)}$$

for $-1 \leqq \lambda < 1$, $0 \leqq \alpha < 1$, $n \geqq 2$, and $z \in \mathcal{U}$. If

$$(3.20) \qquad \mathcal{E}(\lambda,\alpha,n,|z|) \geqq \mathcal{E}(\lambda,\alpha,n+1,|z|)$$

for $|z| \neq 0$, then we have the first half of the theorem.

Set

$$(3.21) \qquad \mathcal{E}_1(\lambda,\alpha,n,|z|) = \Gamma(n+1)\Gamma(1-\lambda)\{(n+1) - (n-\lambda)|z|\} - \alpha\,\Gamma(n+1-\lambda)(1-|z|).$$

If $\mathcal{E}_1(\lambda,\alpha,n,|z|) \geqq 0$, then we have (3.20). We note that $\mathcal{E}_1(\lambda,\alpha,n,|z|)$ is

a decreasing function of α. This implies that

$$(3.22)\qquad \mathcal{E}_1(\lambda,\alpha,n,|z|) \geqq \mathcal{E}_1(\lambda,1,n,|z|)$$

$$= \{\Gamma(n+2)\Gamma(1-\lambda) - \Gamma(n+1-\lambda)\}$$

$$- \{(n-\lambda)\Gamma(n+1)\Gamma(1-\lambda) - \Gamma(n+1-\lambda)\}|z|.$$

Since $\mathcal{E}_1(\lambda,1,n,|z|)$ is a decreasing function of $|z|$, we also have

$$(3.23)\qquad \mathcal{E}_1(\lambda,\alpha,n,|z|) \geqq \mathcal{E}_1(\lambda,1,n,1)$$

$$= (1+\lambda)\Gamma(n+1)\Gamma(1-\lambda)$$

$$\geqq 0$$

for $-1 \leqq \lambda < 1$, and $n \geqq 2$.

In order to establish the second half of Theorem 7, we note that

$$(3.24)\qquad |f'(z)| \geqq 1 - \max_{n\geqq 2} \frac{n(1-\alpha)}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha} |z|^{n-1}$$

and

$$(3.25)\qquad |f'(z)| \leqq 1 + \max_{n\geqq 2} \frac{n(1-\alpha)}{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha} |z|^{n-1}.$$

Define

$$(3.26)\qquad F(\lambda,\alpha,n,|z|) = \frac{n\,\Gamma(n-\lambda)|z|^{n-1}}{\Gamma(n+1)\Gamma(1-\lambda) - \alpha\,\Gamma(n-\lambda)}$$

and

$$(3.27)\qquad F_1(\lambda,\alpha,n,|z|) = \Gamma(n+2)\Gamma(1-\lambda)\{n - (n-\lambda)|z|\} - \alpha\,\Gamma(n+1-\lambda)\{n - (n-\lambda)|z|\}$$

for $0 \leqq \lambda < 1$, $0 \leqq \alpha < 1$, $n \geqq 2$, and $z \in \mathcal{U}$. Since $F_1(\lambda,\alpha,n,|z|)$ is a decreasing function of α,

$$(3.28)\qquad \begin{aligned} F_1(\lambda,\alpha,n,|z|) &\geqq F_1(\lambda,1,n,|z|) \\ &= n\{\Gamma(n+2)\Gamma(1-\lambda) - \Gamma(n+1-\lambda)\} \\ &\quad - (n-\lambda)\{\Gamma(n+2)\Gamma(1-\lambda) - \Gamma(n+1-\lambda)\}|z|. \end{aligned}$$

Further, since $F_1(\lambda,1,n,|z|)$ is a decreasing function of $|z|$, we get

$$(3.29)\qquad \begin{aligned} F_1(\lambda,\alpha,n,|z|) &\geqq F_1(\lambda,1,n,1) \\ &= \lambda\{\Gamma(n+2)\Gamma(1-\lambda) - \Gamma(n+1-\lambda)\} \\ &\geqq 0 \end{aligned}$$

for $0 \leqq \lambda < 1$ and $n \geqq 2$. Thus we have (3.15) and (3.16).

Finally, by taking the function

$$(3.30)\qquad f(z) = z - \frac{(1-\alpha)(1-\lambda)}{2 - \alpha(1-\lambda)} z^2,$$

we can show that all bounds given by Theorem 7 are sharp.

COROLLARY 3. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.9) <u>be in the class</u> $\mathcal{T}(\alpha,\lambda)$ <u>with</u> $-1 \leqq \lambda < 1$ <u>and</u> $0 \leqq \alpha < 1$. <u>Then the unit disk</u> $\mathcal{U}$ <u>is mapped</u>

onto a domain that contains the disk $|w| < (1+\lambda)/\{2 - \alpha(1-\lambda)\}$.

THEOREM 8. Let the function $f(z)$ defined by (1.9) be in the class $\mathscr{C}(\alpha,\lambda)$. Then, if $-1 \leqq \lambda < 1$,

$$|f(z)| \geqq |z| - \frac{(1-\alpha)(1-\lambda)^2}{2\{2 - \alpha(1-\lambda)\}} |z|^2 \tag{3.31}$$

and

$$|f(z)| \leqq |z| + \frac{(1-\alpha)(1-\lambda)^2}{2\{2 - \alpha(1-\lambda)\}} |z|^2 \tag{3.32}$$

for $z \in \mathscr{U}$. Furthermore, if $0 \leqq \lambda < 1$,

$$|f'(z)| \geqq 1 - \frac{(1-\alpha)(1-\lambda)^2}{2 - \alpha(1-\lambda)} |z| \tag{3.33}$$

and

$$|f'(z)| \leqq 1 + \frac{(1-\alpha)(1-\lambda)^2}{2 - \alpha(1-\lambda)} |z| \tag{3.34}$$

for $z \in \mathscr{U}$. The bounds (3.31) to (3.34) are sharp.

PROOF. By means of Theorem 6, we have

$$|f(z)| \geqq |z| - \max_{n\geqq 2}\left\{\frac{1 - \alpha}{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}} |z|^n\right\} \tag{3.35}$$

and

$$(3.36)\qquad |f(z)| \leqq |z| + \max_{n\geqq 2}\left\{\frac{1-\alpha}{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}}\,|z|^n\right\}.$$

Let

$$(3.37)\qquad H(\lambda,\alpha,n,|z|) = \frac{\Gamma(n-\lambda)^2|z|^n}{\Gamma(n+1)\Gamma(1-\lambda)\{\Gamma(n+1)\Gamma(1-\lambda) - \Gamma(n-\lambda)\alpha\}}$$

and

$$(3.38)\qquad H_1(\lambda,\alpha,n,|z|) = \Gamma(n+1)\Gamma(1-\lambda)\{(n+1)^2 - (n-\lambda)^2|z|\}$$
$$- \alpha(n-\lambda)\Gamma(n-\lambda)\{(n+1) - (n-\lambda)|z|\}$$

for $-1 \leqq \lambda < 1$, $0 \leqq \alpha < 1$, $n \geqq 2$, and $z \in \mathcal{U}$. Then, we know that $H(\lambda,\alpha,n,|z|)$ is a decreasing function of n if

$$H_1(\lambda,\alpha,n,|z|) \geqq 0.$$

In fact, since $H_1(\lambda,\alpha,n,|z|)$ is a decreasing function of α, and $H_1(\lambda,1,n,|z|)$ is a decreasing function of $|z|$, we can prove that

$$(3.39)\qquad \begin{aligned} H_1(\lambda,\alpha,n,|z|) &\geqq H_1(\lambda,1,n,|z|) \\ &\geqq H_1(\lambda,1,n,|z|) \\ &= (1+\lambda)\{(2n+1-\lambda)\Gamma(n+1)\Gamma(1-\lambda) - \Gamma(n+1-\lambda)\} \\ &\geqq 0 \end{aligned}$$

for $-1 \leqq \lambda < 1$ and $n \geqq 2$. Consequently, we have the first half of Theorem 8.

Next, we note that

$$(3.40)\qquad |f'(z)| \geqq 1 - \max_{n\geqq 2} \frac{1-\alpha}{\frac{\Gamma(n)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}} |z|^{n-1}$$

and

$$(3.41)\qquad |f'(z)| \leqq 1 + \max_{n\geqq 2} \frac{1-\alpha}{\frac{\Gamma(n)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}} |z|^{n-1}.$$

Define

$$(3.42)\qquad G(\lambda,\alpha,n,|z|) = \frac{\Gamma(n-\lambda)^2\ |z|^{n-1}}{\Gamma(n)\Gamma(1-\lambda)\{\Gamma(n+1)\Gamma(1-\lambda) - \alpha\ \Gamma(n-\lambda)\}}$$

and

$$(3.43)\qquad G_1(\lambda,\alpha,n,|z|) = \Gamma(n+1)\Gamma(1-\lambda)\{n(n+1) - (n-\lambda)^2|z|\}$$
$$- \alpha\ \Gamma(n+1-\lambda)\{n - (n-\lambda)|z|\}$$

for $0 \leqq \lambda < 1$, $0 \leqq \alpha < 1$, $n \geqq 2$, and $z \in \mathcal{U}$. Then it is sufficient to prove that

$$G_1(\lambda,\alpha,n,|z|) \geqq 0.$$

Note that $G_1(\lambda,\alpha,n,|z|)$ is a decreasing function of α, and $G_1(\lambda,1,n,|z|)$ is a decreasing function of $|z|$. Thus we have,

$$(3.44)\qquad G_1(\lambda,\alpha,n,|z|) \geqq G_1(\lambda,1,n,|z|)$$
$$\geqq G_1(\lambda,1,n,1)$$
$$= (n+2\lambda n-\lambda^2)\Gamma(n+1)\Gamma(1-\lambda) - \lambda\ \Gamma(n+1-\lambda)$$
$$\geqq 0$$

for $0 \leqq \lambda < 1$ and $n \geqq 2$, which implies the second half of Theorem 8.

Finally, all bounds asserted by Theorem 8 are sharp for the function

$$f(z) = z - \frac{(1-\alpha)(1-\lambda)^2}{2\{2 - \alpha(1-\lambda)\}} z^2. \tag{3.45}$$

COROLLARY 4. Let the function $f(z)$ defined by (1.9) be in the class $\mathscr{C}(\alpha,\lambda)$ with $-1 \leqq \lambda < 1$ and $0 \leqq \alpha < 1$. Then the unit disk $\mathscr{U}$ is mapped onto a domain that contains the disk $|w| < r_0$, where r_0 is given by

$$r_0 = 1 - \frac{(1-\alpha)(1-\lambda)^2}{2\{2 - \alpha(1-\lambda)\}}. \tag{3.46}$$

4. STARLIKENESS AND CONVEXITY

Silverman [14] proved the following lemmas.

LEMMA 3. Let the function $f(z)$ be defined by (1.9). Then $f(z)$ is starlike of order α if and only if

$$\sum_{n=2}^{\infty} (n-\alpha)a_n \leqq 1 - \alpha \tag{4.1}$$

for $0 \leqq \alpha < 1$.

LEMMA 4. Let the function $f(z)$ be defined by (1.9). Then $f(z)$ is convex of order α if and only if

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leqq 1 - \alpha \tag{4.2}$$

<u>for</u> $0 \leqq \alpha < 1$.

By applying the above lemmas, we now prove

THEOREM 9. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.9) <u>be in the class</u> $\mathcal{T}^*(\alpha,\lambda)$ <u>with</u> $0 \leqq \lambda < 1$ <u>and</u> $0 \leqq \alpha < 1$. <u>Then</u> $f(z)$ <u>is starlike of order</u> α.

PROOF. Note that

$$n \leqq \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \tag{4.3}$$

for $0 \leqq \lambda < 1$, $0 \leqq \alpha < 1$, and $n \geqq 2$. This shows that

$$\begin{aligned} \sum_{n=2}^{\infty} (n-\alpha)a_n &\leqq \sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}a_n \\ &\leqq 1 - \alpha, \end{aligned} \tag{4.4}$$

and we complete the proof of Theorem 9 in view of (4.1).

Similarly, by using Lemma 4, we have

THEOREM 10. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.9) <u>be in the class</u> $\mathcal{C}(\alpha,\lambda)$ <u>with</u> $0 \leqq \lambda < 1$ <u>and</u> $0 \leqq \alpha < 1$. <u>Then</u> $f(z)$ <u>is convex of order</u> α.

REMARK 3. For $\lambda = 0$, the classes $\mathcal{T}^*(\alpha,\lambda)$ and $\mathcal{C}(\alpha,\lambda)$ reduce to the classes $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively, which were introduced by Silverman [14]. It follows that

(4.5) $$\mathcal{T}^*(\alpha,0) = \mathcal{T}^*(\alpha)$$

and

(4.6) $$\mathcal{C}(\alpha,0) = \mathcal{C}(\alpha).$$

Hence, by means of Theorem 9 and Theorem 10, we have

(4.7) $$\mathcal{T}^*(\alpha,\lambda) \subset \mathcal{T}^*(\alpha,0) \qquad (0 \leqq \lambda < 1)$$

and

(4.8) $$\mathcal{C}(\alpha,\lambda) \subset \mathcal{C}(\alpha,0) \qquad (0 \leqq \lambda < 1).$$

Since

(4.9) $$\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha \leqq n - \alpha$$

and

(4.10) $$\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} \leqq n(n-\alpha)$$

for $\lambda < 0$, $0 \leqq \alpha < 1$, and $n \geqq 2$, we also have

(4.11) $$\mathcal{T}^*(\alpha,\lambda) \supset \mathcal{T}^*(\alpha,0) \qquad (\lambda < 0)$$

and

(4.12) $$\mathcal{C}(\alpha,\lambda) \supset \mathcal{C}(\alpha,0) \qquad (\lambda < 0).$$

THEOREM 11. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.9) <u>be in the class</u> $\mathcal{T}^*(\alpha,\lambda)$ <u>with</u> $\lambda < 1$ <u>and</u> $0 \leqq \alpha < 1$. <u>Then</u> $f(z)$ <u>is convex of order</u> δ

$(0 \leqq \delta < 1)$ <u>in the disk</u> $|z| < r_1$, where

$$(4.13) \qquad r_1 = \inf_{n \geqq 2} \left[\frac{(1-\delta)\{\Gamma(n+1)\Gamma(1-\lambda) - \alpha\,\Gamma(n-\lambda)\}}{n(n-\delta)(1-\alpha)\Gamma(n-\lambda)} \right]^{1/(n-1)} .$$

PROOF. It is sufficient to show that

$$(4.14) \qquad \left| \frac{zf''(z)}{f'(z)} \right| < 1 - \delta$$

for $|z| < r_1$. Note that

$$(4.15) \qquad \left| \frac{zf''(z)}{f'(z)} \right| \leqq \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} n\, a_n |z|^{n-1}}$$

$$\leqq 1 - \delta,$$

if and only if

$$(4.16) \qquad \sum_{n=2}^{\infty} n(n-\delta)a_n |z|^{n-1} \leqq 1 - \delta.$$

With the aid of Theorem 3, we need only find the values of $|z|$ for which

$$(4.17) \qquad |z|^{n-1} \leqq \frac{(1-\delta)\{\Gamma(n+1)\Gamma(1-\lambda) - \alpha\,\Gamma(n-\lambda)\}}{n(n-\delta)(1-\alpha)\Gamma(n-\lambda)} , \qquad (n \geqq 2)$$

and the proof of Theorem 11 is thus completed.

5. CLASSES OF FUNCTIONS WITH FIXED FINITELY MANY COEFFICIENTS

In view of Theorem 3, we denote by $\mathcal{T}^*(\alpha,\lambda,k)$ the subclass of $\mathcal{T}^*(\alpha,\lambda)$ consisting of all functions of the form

$$(5.1) \qquad f(z) = z - \sum_{i=2}^{k} A_i\, p_i z^i - \sum_{n=k+1}^{\infty} a_n z^n,$$

where

$$(5.2) \qquad A_i = (1-\alpha)\Big/\left\{\frac{\Gamma(i+1)\Gamma(1-\lambda)}{\Gamma(i-\lambda)} - \alpha\right\},$$

$0 \leqq p_i \leqq 1$, and $0 \leqq \sum_{i=2}^{k} p_i \leqq 1$.

Further, in view of Theorem 4, we denote by $\mathcal{C}(\alpha,\lambda,k)$ the sublcass of $\mathcal{C}(\alpha,\lambda)$ consisting of all functions of the form

$$(5.3) \qquad f(z) = z - \sum_{i=2}^{k} B_i\, p_i z^i - \sum_{n=k+1}^{\infty} a_n z^n,$$

where

$$(5.4) \qquad B_i = (1-\alpha)\Big/\frac{\Gamma(i+1)\Gamma(1-\lambda)}{\Gamma(i-\lambda)}\left\{\frac{\Gamma(i+1)\Gamma(1-\lambda)}{\Gamma(i-\lambda)} - \alpha\right\},$$

$0 \leqq p_i \leqq 1$, and $0 \leqq \sum_{i=2}^{k} p_i \leqq 1$.

Our first result on the classes of functions with fixed finitely many coefficients is contained in

THEOREM 12. <u>Let the function</u> $f(z)$ <u>be defined by</u> (5.1). <u>Then</u> $f(z)$ <u>is in the class</u> $\mathcal{T}^*(\alpha,\lambda,k)$ <u>if and only if</u>

$$(5.5)\qquad \sum_{n=k+1}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_n \leqq (1-\alpha)\left(1-\sum_{i=2}^{k}p_i\right)$$

<u>for</u> $\lambda < 1$ <u>and</u> $0 \leqq \alpha < 1$. <u>The result</u> (5.5) <u>is sharp</u>.

PROOF. Putting

$$(5.6)\qquad a_i = A_i p_i \qquad (i = 2,3,\ldots,k)$$

in Theorem 3, we have

$$(5.7)\qquad (1-\alpha)\sum_{i=2}^{k}p_i + \sum_{n=k+1}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_n \leqq 1-\alpha$$

which implies (5.5). Furthermore, the result is sharp for the functions

$$(5.8)\qquad f(z) = z - \sum_{i=2}^{k}A_i p_i z^i$$

$$- \left[(1-\alpha)\left(1-\sum_{i=2}^{k}p_i\right)\Big/\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}\right]z^n \quad (n \geqq k+1).$$

COROLLARY 5. <u>Let the function</u> $f(z)$ <u>defined by</u> (5.1) <u>be in the class</u> $\mathcal{T}^*(\alpha,\lambda,k)$. <u>Then</u>

$$(5.9)\qquad a_n \leqq (1-\alpha)\left(1-\sum_{i=2}^{k}p_i\right)\Big/\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\} \qquad (n \geqq k+1).$$

<u>The result</u> (5.9) <u>is sharp for functions</u> $f(z)$ <u>of the form</u> (5.8).

Similarly, we prove

THEOREM 13. Let the function $f(z)$ be defined by (5.3). Then $f(z)$ is in the class $\mathscr{C}(\alpha,\lambda,k)$ if and only if

$$\sum_{n=k+1}^{\infty} \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} a_n \leqq (1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right) \tag{5.10}$$

for $\lambda < 1$ and $0 \leqq \alpha < 1$. The result (5.10) is sharp for the functions

$$f(z) = z - \sum_{i=2}^{k} B_i p_i z^i - \left[(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right) \Big/ \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}\right] z^n \tag{5.11}$$

$$(n \geqq k + 1).$$

COROLLARY 6. Let the function $f(z)$ defined by (5.3) be in the class $\mathscr{C}(\alpha,\lambda,k)$. Then

$$a_n \leqq (1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right) \Big/ \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} \quad (n \geqq k + 1). \tag{5.12}$$

The result (5.12) is sharp for functions $f(z)$ of the form (5.11).

We find the extreme points of $\mathscr{T}^*(\alpha,\lambda,k)$ and $\mathscr{C}(\alpha,\lambda,k)$ by using the same technique as in Theorem 5.

THEOREM 14. The extreme points of $\mathscr{T}^*(\alpha,\lambda,k)$ are

$$f_k(z) = z - \sum_{i=2}^{k} A_i p_i z^i \tag{5.13}$$

and

$$(5.14)\qquad f_n(z) = z - \sum_{i=2}^{k} A_i p_i z^i - \left[(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right) \Big/ \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}\right] z^n \qquad (n \geqq k+1),$$

where A_i is given by (5.2).

THEOREM 15. The extreme points of $\mathscr{C}(\alpha,\lambda,k)$ are

$$(5.15)\qquad f_k(z) = z - \sum_{i=2}^{k} B_i p_i z^i$$

and

$$(5.16)\qquad f_n(z) = z - \sum_{i=2}^{k} B_i p_i z^i - \left[(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right) \Big/ \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}\right] z^n \qquad (n \geqq k+1),$$

where B_i is given by (5.4).

Next we determine the radius of convexity for functions belonging to the class $\mathscr{T}^*(\alpha,\lambda,k)$. Our result is given by

THEOREM 16. Let the function $f(z)$ defined by (5.1) be in the class $\mathscr{T}^*(\alpha,\lambda,k)$. Then $f(z)$ is convex in the disk $|z| < r_2$, where r_2 is the largest value for which

$$(5.17)\qquad \sum_{i=2}^{k} i^2 A_i p_i r^{i-1} + \frac{n^2(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)\Gamma(n-\lambda)}{\Gamma(n+1)\Gamma(1-\lambda) - \alpha\,\Gamma(n-\lambda)}\, r^{n-1} \leqq 1$$

<u>for</u> $n \geqq k + 1$. <u>The result</u> (5.17) <u>is sharp</u>.

PROOF. It suffices to prove that

$$(5.18)\qquad \left|\frac{zf''(z)}{f'(z)}\right| \leqq \frac{\sum_{i=2}^{k} i(i-1)A_i p_i r^{i-1} + \sum_{n=k+1}^{\infty} n(n-1)a_n r^{n-1}}{1 - \sum_{i=2}^{k} i\, A_i p_i r^{i-1} - \sum_{n=k+1}^{\infty} n\, a_n r^{n-1}}$$

$$\leqq 1$$

for $|z| \leqq r$. Note that (5.18) is true if and only if

$$(5.19)\qquad \sum_{i=2}^{k} i^2 A_i p_i r^{i-1} + \sum_{n=k+1}^{\infty} n^2 a_n r^{n-1} \leqq 1.$$

By virtue of Theorem 12, we can write

$$(5.20)\qquad a_n = \frac{(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)}{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha}\,\lambda_n \qquad (n \geqq k + 1),$$

where $\lambda_n \geqq 0$ $(n \geqq k + 1)$ and

$$(5.21)\qquad \sum_{n=k+1}^{\infty} \lambda_n \leqq 1.$$

For each fixed r, we choose the integer $n_0 = n_0(r)$ for which

$$n^2\, r^{n-1} \Big/ \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}$$

is a maximum. Then we have

$$(5.22)\qquad \sum_{n=k+1}^{\infty} n^2 a_n r^{n-1} \leqq \frac{n_0^2(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)}{\frac{\Gamma(n_0+1)\Gamma(1-\lambda)}{\Gamma(n_0-\lambda)}-\alpha}\, r^{n_0-1}.$$

Hence the function $f(z)$ is convex in $|z| \leqq r_2$ if

$$(5.23)\qquad \sum_{i=2}^{k} i^2 A_i p_i r^{i-1} + \frac{n_0^2(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)}{\frac{\Gamma(n_0+1)\Gamma(1-\lambda)}{\Gamma(n_0-\lambda)}-\alpha}\, r^{n_0-1} \leqq 1.$$

We find the value r_0 and the corresponding $n_0(r_0)$ so that

$$(5.24)\qquad \sum_{i=2}^{k} i^2 A_i p_i r_0^{i-1} + \frac{n_0^2(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)}{\frac{\Gamma(n_0+1)\Gamma(1-\lambda)}{\Gamma(n_0-\lambda)}-\alpha}\, r_0^{n_0-1} = 1.$$

Then this value r_0 is the radius of convexity of functions $f(z)$ in the class $\mathcal{T}^*(\alpha,\lambda,k)$. Furthermore, we know that the result of the theorem is sharp for the functions $f(z)$ given by (5.8).

6. QUASICONFORMAL EXTENSION

Let f be a sense-preserving homeomorphism of a domain $\mathcal{D}$. Then f is said to be a $\left(\frac{1+k}{1-k}\right)$-quasiconformal mapping of $\mathcal{D}$ if it is absolutely continuous on almost all lines parallel to the coordinate axes and

$$(6.1)\qquad \left|\frac{\partial}{\partial \bar{z}} f(z)\right| \leqq k\left|\frac{\partial}{\partial z} f(z)\right| \qquad (z \in \mathcal{D})$$

for $0 \leqq k < 1$. We say that a function $f(z)$ is in the class $\mathcal{S}_k$ if it is in the class $\mathcal{S}$ and has a $\left(\frac{1+k}{1-k}\right)$-quasiconformal extension defined on the extended complex plane (cf. [5]).

Let $\tilde{\mathcal{S}}^*(\alpha,\lambda)$ denote the subclass of $\mathcal{S}$ whose members satisfy the coefficient inequality (2.1) for $\lambda < 1$ and $0 \leqq \alpha < 1$. Then it follows that

$$\mathcal{T}^*(\alpha,\lambda) \subset \tilde{\mathcal{S}}^*(\alpha,\lambda) \subset \mathcal{S}^*(\alpha,\lambda). \tag{6.2}$$

Also let $\tilde{\mathcal{K}}(\alpha,\lambda)$ denote the subclass of $\mathcal{S}$ whose members satisfy the coefficient inequality (2.7) for $\lambda < 1$ and $0 \leqq \alpha < 1$. Then we have

$$\mathcal{C}(\alpha,\lambda) \subset \tilde{\mathcal{K}}(\alpha,\lambda) \subset \mathcal{K}(\alpha,\lambda). \tag{6.3}$$

We shall apply the following lemma due to Fait, Krzyz and Zygmunt [3] in order to prove Theorem 17 below.

LEMMA 5. <u>If the function</u> $f(z)$ <u>defined by</u> (1.1) <u>is in the class</u> $\mathcal{A}$ and satisfies

$$\sum_{n=2}^{\infty} n|a_n| \leqq k, \tag{6.4}$$

<u>then</u> $f(z) \in \mathcal{S}_k$.

THEOREM 17. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.1) <u>be in the class</u> $\tilde{\mathcal{S}}^*(\alpha,\lambda)$ <u>with</u> $0 < \lambda < 1$ <u>and</u> $0 \leqq \alpha < 1$. <u>Then</u> $f(z) \in \mathcal{S}_k$, <u>where</u>

$$k = \frac{2(1-\alpha)(1-\lambda)}{2 - \alpha(1-\lambda)}. \tag{6.5}$$

PROOF. In view of (2.1) and Lemma 5, we need to prove that (2.1) implies (6.4). Define $L(\alpha,\lambda,n)$ by

$$L(\alpha,\lambda,n) = \frac{1}{1-\alpha}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\} - \frac{n}{k} \tag{6.6}$$

for $0 < \lambda < 1$, $0 \leqq \alpha < 1$, and $n \geqq 2$. Then it suffices to show that $L(\alpha,\lambda,n) \geqq 0$ for $0 < \lambda < 1$, $0 \leqq \alpha < 1$, and $n \geqq 2$. Note that $L(\alpha,\lambda,n) \geqq 0$ if

$$k \geqq \frac{n(1-\alpha)\Gamma(n-\lambda)}{\Gamma(n+1)\Gamma(1-\lambda) - \alpha\,\Gamma(n-\lambda)} \tag{6.7}$$

for $0 < \lambda < 1$, $0 \leqq \alpha < 1$, and $n \geqq 2$. Since

$$\frac{n(1-\alpha)\Gamma(n-\lambda)}{\Gamma(n+1)\Gamma(1-\lambda) - \alpha\,\Gamma(n-\lambda)}$$

is a decreasing function of n, and

$$0 \leqq \frac{2(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)} < 1 \tag{6.8}$$

for $0 < \lambda < 1$ and $0 \leqq \alpha < 1$, we conclude that $f(z) \in \mathcal{S}_k$ for k given by (6.5).

COROLLARY 7. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.9) <u>be in the class</u> $\mathcal{T}^*(\alpha,\lambda)$ <u>with</u> $0 < \lambda < 1$ <u>and</u> $0 \leqq \alpha < 1$. <u>Then</u> $f(z) \in \mathcal{S}_k$ <u>for</u> k <u>given by</u> (6.5).

Recently, Brown [1] established the following results.

LEMMA 6. <u>If the function</u> $f(z)$ <u>defined by</u> (1.1) <u>is in the class</u> $\mathcal{A}$ <u>and satisfies</u>

$$\sum_{n=2}^{\infty} \left(n - \frac{1-k}{1+k}\right)|a_n| \leqq \frac{2k}{1+k}, \tag{6.9}$$

<u>then</u> $f(z) \in \mathcal{S}^* \cap \mathcal{S}_k$.

LEMMA 7. <u>If the function</u> $f(z)$ <u>defined by</u> (1.1) <u>is in the class</u> $\mathcal{A}$ <u>and satisfies</u>

$$\sum_{n=2}^{\infty} n\left(n - \frac{1-k}{1+k}\right)|a_n| \leqq \frac{2k}{1+k}, \tag{6.10}$$

<u>then</u> $f(z) \in \mathcal{K} \cap \mathcal{S}_k$.

THEOREM 18. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.1) <u>be in the class</u> $\tilde{\mathcal{S}}^*(\alpha,\lambda)$ <u>with</u> $0 \leqq \lambda < 1$ <u>and</u> $0 < \alpha < 1$. <u>Then</u> $f(z) \in \mathcal{S}^*(\alpha) \cap \mathcal{S}_k$, <u>where</u> $k = (1-\alpha)/(1+\alpha)$.

PROOF. Since

$$n \leqq \frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} \tag{6.11}$$

for $0 \leqq \lambda < 1$ and $n \geqq 2$, we have

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leqq \sum_{n=2}^{\infty} \left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)} - \alpha\right\}|a_n| \tag{6.12}$$

$$\leqq 1 - \alpha,$$

which implies that $f(z) \in \mathcal{S}^*(\alpha)$, by means of [14].

Further, replacing α by $(1-k)/(1+k)$ in (6.12), we get (6.9). Hence $f(z)$ is also in the class $\mathcal{S}^* \cap \mathcal{S}_k$ for $k = (1-\alpha)/(1+\alpha)$. We note that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*$ for $0 \leqq \alpha < 1$. Therefore, $f(z) \in \mathcal{S}^*(\alpha) \cap \mathcal{S}_k$.

COROLLARY 8. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.9) <u>be in the class</u> $\mathcal{T}^*(\alpha,\lambda)$ <u>with</u> $0 \leqq \lambda < 1$ <u>and</u> $0 < \alpha < 1$. <u>Then</u> $f(z) \in \mathcal{T}^*(\alpha,0) \cap \mathcal{S}_k$, <u>where</u> $k = (1-\alpha)/(1+\alpha)$.

By using Lemma 7, we prove

THEOREM 19. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.1) <u>be in the class</u> $\tilde{\mathcal{K}}(\alpha,\lambda)$ <u>with</u> $0 \leqq \lambda < 1$ <u>and</u> $0 < \alpha < 1$. <u>Then</u> $f(z) \in \mathcal{K}(\alpha) \cap \mathcal{S}_k$, <u>where</u> $k = (1-\alpha)/(1+\alpha)$.

COROLLARY 9. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.9) <u>be in the class</u> $\mathcal{C}(\alpha,\lambda)$ <u>with</u> $0 \leqq \lambda < 1$ <u>and</u> $0 < \alpha < 1$. <u>Then</u> $f(z) \in \mathcal{C}(\alpha,0) \cap \mathcal{S}_k$, <u>where</u> $k = (1-\alpha)/(1+\alpha)$.

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