SOME EXAMPLES OF MATHEMATICAL MODELS
IN FOREST MANAGEMENT

by

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ABSTRACT

The problem of the management of a forest subject to the risk of fire gives rise to some simple models which can be profitably used in a Mathematical Modelling course. The problem can be modelled at two levels:- (a) that of the single stand, and (b) that of the "whole-forest". The stand level model presented is in continuous time and provides some analytic results with interesting economic interpretations. The forest level model is in discrete-time and the results from it are numerical. The mathematical techniques used in the stand level analysis include simple probability (the Poisson process), geometric series and calculus. The forest level model is formulated using a stochastic difference equation and an approximately optimal feedback solution is found using linear programming. The results of the analyses at the two levels are complementary.

A real-life problem that lends itself to being modelled mathematically in more than one way is particularly useful in the teaching of the applications of mathematics. Not only may it provide examples of the use of several different mathematical techniques, but also the results of the different analyses can be compared. Serious discrepancies in the results can point the way to model inadequacies, while complementary results from two different models will lend considerable weight to their credibility and to the credibility of the conclusions that are drawn from them.

One such area of application that I have encountered in my own research concerns the problem of assessing the effects of forest fire on forest yield and forest harvesting policies. The problem can be modelled at two levels:- (a) that of the single stand, and (b) that of the "whole-forest" comprising many stands of different ages. I have found it convenient to model the single stand in continuous time and the whole-forest in discrete time. In teaching, this provides a useful illustration of how a discrete-time formulation is more convenient in some instances and a continuous-time formulation more convenient in others.

The mathematical techniques involved in the two levels of modelling are quite distinct. In the stand-level model the techniques associated with geometric series and simple probability theory (the Poisson process) are used. In the forest-level model dynamic equations for the evolution of the forest subject to fire, are used. These are formulated in matrix terms. The notion of feedback control for stochastic dynamic systems arises, and a procedure for determining approximately optimal harvests can be found through the use of linear programming.

The economic phenomenon of time discounting occurs in both models -- in continuous-time form for the stand-level model and in discrete-time form for the forest-level model. While mathematical analysis of the stand-level model leads to some general conclusions with economic interpretations, the results of the forest-level analysis are essentially numerical. Pedagogically this provides a useful illustration of the different kinds of output that can be expected from mathematical models. It turns out that the results of the two levels of modelling are complementary in the areas where one would expect.

In the next two sections a brief description of the models and some results are given. For the sake of brevity much of the mathematical detail is omitted. Emphasis is placed on the aspects of the models which are of pedagogic interest.

Stand-Level Analysis

Suppose that a stand of trees of age a growing on a site has a net stumpage value (value of timber net of cutting and transportation costs) of $V(a)$. Suppose further that a stand is cut whenever it reaches some rotation age, T, and that the site is then subsequently replanted with trees of similar growth characteristics. If costs of clearing and replanting the site after a cut are c_1, then the total present (discounted) value of the stream of revenues (and costs) from the site (starting with a newly planted site) is

$$\sum_{n=1}^{\infty} e^{-n\delta T}\left(V(T) - c_1\right) \tag{1}$$

where δ is the instantaneous discount rate related to the per annum discount rate i, by $\delta = \ln(1+i)$.[1] The above present value is known traditionally as the land expectation value (L.E.V.) (see e.g. Clark 1976, p. 259). The expression (1) is a geometric series and can be summed to give

$$\text{L.E.V.} = \frac{V(T) - c_1}{e^{\delta T} - 1} \tag{2}$$

The optimal rotation age can be determined by setting the derivative of (2) equal to zero. Thus the optimal rotation age T^* solves

$$V'(T) = \frac{\delta\left(V(T) - c_1\right)}{1 - e^{-\delta T}} \tag{3}$$

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[1] In the classroom some discussion of present value, time discounting and the relationship between the appropriate rate for annual compounding and instantaneous compounding (such as that in Clark 1976, p. 69) might be necessary at this point.

This result is known as the Faustmann formula (Faustmann, 1849). An economic interpretation of the result can be obtained by multiplying both sides of (3) by an infinitesimal time h and re-expressing it as

$$V'(T)h = \delta h\big(V(T) - c_1\big) + \delta h \sum_{n=1}^{\infty} e^{-n\delta T}\big(V(T) - c_1\big) \qquad (4)$$

again using the formula for the sum of a geometric series. The term on the left represents the growth in the value of a stand of age T during the infinitesimal time h if cutting does not take place. The first term on the right represents the growth in value of the revenue earned (interest) through cutting the stand at age T, while the second term (which is $\delta h \times$ (LEV)) is the incremental growth in the value of the site (i.e. the interest that could be earned on revenue obtained through selling the site). Thus, optimally, one sets the rotation age to the age where the growth in value of the stand through not cutting equals the growth in the value of revenue that could be earned through cutting the stand and selling the site. It should be pointed out in the classroom that results such as this, where marginal growth rates are equal at the optimum are common in the models arising in micro-economic theory.

Suppose now that stands are destroyed from time to time by fire. Suppose that subsequent to a fire, costs c_2 of clearing and replanting the site are incurred, and that after replanting, a new stand will grow. A typical evolution

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of site might look like Figure 1. We need some probabilistic model for the occurrence of fires. The simplest possible model is that fires occur independently of one another (in time) and at random (in time) i.e. that fires occur in a Poisson process (see e.g. Devore 1982, p. 118). If we denote the times between successive destructions of the stand (either by fire or by cutting) by $X_1, X_2, \ldots,$ then the X_i's are independently identically distributed random variables with cumulative distribution function (c.d.f.)[1] given by

[1]Most students with an introductory course in Probability or Statistics should be familiar with the notion of a c.d.f. and the fact that the interarrival times in a Poisson process have an exponential distribution. If not the derivation from the Poisson postulates provides a nice example of the use of the c.d.f. (see e.g. Devore 1982, p. 154).

$$F(x) = pr(X \le x) = \begin{cases} 1 - e^{-\lambda x}, & x < T \\ 1 & , \quad x \ge T \end{cases} \tag{5}$$

(where λ is the average rate of fires in the Poisson Process), i.e. the times between successive destructions have an exponential distribution, truncated at $X = T$ and with an atom of probability of size $e^{-\lambda T}$ at $X = T$.

Associated with each destruction of a stand there is a revenue, Y. For the n^{th} destruction the revenue is

$$Y_n = \begin{cases} -c_2 & \text{if } X_n < T \quad \text{(fire)} \\ V(T) - c_1 & \text{if } X_n = T \quad \text{(harvest)} \end{cases} \tag{6}$$

The expected (discounted) present value of the random stream of revenues from the site (the land expectation value) is

$$\text{L.E.V.} = E\left\{ \sum_{n=1}^{\infty} e^{-\delta(X_1+X_2+\ldots+X_n)} Y_n \right\} \tag{7}$$

Since Y_n is independent of $X_1, \ldots, X_{n-1}$ (but not of X_n), (7) can be written as

$$\text{LEV} = \sum_{n=1}^{\infty} E\left(e^{-\delta(X_1+\ldots+X_{n-1})}\right) E\left(e^{-\delta X_n} Y_n\right) \tag{8}$$

After some integration and again using the formula for the sum of a geometric series, (8) can be summed to give:-

$$\text{LEV} = \frac{(\lambda+\delta)\left(V(T) - c_1\right)e^{-(\lambda+\delta)T}}{\delta\left(1 - e^{-(\lambda+\delta)T}\right)} - \frac{\lambda}{\delta} c_2 \tag{9}$$

(See Reed 1984 for details.) The optimal cutting age T^* can be obtained by setting the derivative of (9) equal to zero. i.e. T^* satisfies

$$V'(T) = \frac{(\lambda+\delta)\left(V(T) - c_1\right)}{1 - e^{-(\lambda+\delta)T}} \tag{10}$$

This is of the same form as the Faustmann equation (3), with the discount rate δ replaced by $\lambda + \delta$. From this it can be seen that <u>the effect on the rotation period of a risk of destruction by fire is the same as that of adding a premium to the discount rate of an amount equal to the average rate at which fires occur.</u>

An economic interpretation to (10) similar to that in the no-fire case can be given. Multiplying both sides by an infinitesimal time h, (10) can be expressed as:-

$$V'(T)h(1-\lambda h) + [-V(T) + c_1 - c_2]\lambda h$$
$$= \delta h[V(T) - c_1] + \delta h \text{ L.E.V.}(T) + o(h) \qquad (11)$$

The r.h.s. is the same as the r.h.s. of (4) and represents the growth in revenue (interest) that could be earned, in the time increment h, through cutting the stand and selling the site, while the l.h.s. represents the expected growth in value of the stand given that there is no cut. To see this the l.h.s. can be written in terms of conditional expectation as

$$E(\text{growth in value of stand with no cut} \mid \text{no fire})\ \text{pr(no fire)}$$
$$+ E(\text{growth in value of stand with no cut} \mid \text{fire})\ \text{pr(fire)}$$

since in an infinitesimal time increment of length h the probability of a fire (an event in a Poisson Process) is λh.

The results above are all theoretical in nature and relate only to the optimal rotation age T^*. To determine the effect on land expectation value of the presence of the risk of fire, numerical methods must be used, since in general solution to (10) can only be carried out numerically. A suitable classroom approach is to plot the L.E.V. (10) as a function of T for various rates of fire λ.

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An example is shown in Figure 2. The growth curve used was that for a hectare of spruce growing in the Fort Nelson region of N.E. British Columbia (see Reed & Errico, 1985a, for details) and the per-annum discount rate was 3 percent. It can be seen how an increase in the rate of fires λ causes a small reduction in the optimal rotation age T^*, but a considerable reduction in the land expectation value. In particular with a fire rate of $\lambda = .005$ (on average one fire every 200 years) the LEV is reduced to approximately 50 percent of its value with no fires present. For fire rates in excess of $\lambda = .0104$ (one fire every 96 years) no net rent can be extracted from the resource in the long-run. The historical rate of fires in the region is estimated to be about $\lambda = .013$.

Of course the results above depend on the parameter values used. A useful homework exercise is to ask students to repeat the analysis for different values of the discount rate parameter δ and the cost parameters c_1 and c_2, and to determine the sensitivity of the results to these various parameters.

Another useful exercise is to ask students to criticize the model in terms of the realism of its assumptions and other shortcomings, such as important aspects omitted. Some of the things that they might come up with, such as age-dependent fire probabilities, the possibility of partial salvage after a fire etc., can quite easily be handled (see Reed & Errico 1985a). Other difficulties such as uncertainties over future stumpage values (dependent on price and demand for lumber), uncertainties (statistical and otherwise) over the growth characteristics of current ant future stands and uncertainties over future fire probabilities etc., can less easily be handled mathematically. This could lead to a useful discussion on the role and limitations of mathematical modelling in general.

If one is very lucky a student might criticize the model along the following lines: "A forest or timber supply area for a mill centre will typically comprise many stands. If each stand in the forest is independently managed in an optimal fashion, the flow of timber to the mill will likely be very erratic, especially in Western North America, where there is much old growth timber still to be cut. Clearly a more stable flow of timber would be preferred. How can one incorporate this into the model?" The answer to such a fortuitous question would be that one needs a forest-level model to accommodate this aspect. In the next section, such a model, formulated in discrete time is discussed.

A Forest Level Model and Analysis

Let the vector $\underset{\sim}{x}_t = \left(x_1^{\,t}, x_2^{\,t}, \ldots, x_k^{\,t}\right)'$ denote the areas in the forest with trees in age-classes 1, 2, ..., k at the start of period t, and let the vector

$\underset{\sim}{h}_t = \left(h_1^{\ t}, h_2^{\ t}, \ldots, h_k^{\ t}\right)'$ denote the areas harvested in each of the age-classes 1, 2, ..., k in period t. Let $\underset{\sim}{v}' = (v_1, v_2, \ldots, v_k)$ denote the average volume per hectare of stands in age-classes 1, 2, ..., k. Then the total volume harvested in period t will be $H_t = \underset{\sim}{v}'\underset{\sim}{h}_t$.

Suppose now that $\theta_1^{\ t}, \theta_2^{\ t}, \ldots, \theta_k^{\ t}$ are <u>random variables</u> representing the <u>proportions</u> of the areas destroyed by fire in age-classes 1, 2, ..., k in period t. A dynamic model for the evolution of the forest is given by the <u>stochastic difference equation</u>

$$\underset{\sim}{x}_{t+1} = R_t\, \underset{\sim}{x}_t - S_t\, \underset{\sim}{h}_t \tag{12}$$

where $\underset{\sim}{R}_t$ and $\underset{\sim}{S}_t$ are random matrices:

$$R_t = \begin{bmatrix} \theta_1^{\ t} & \theta_2^{\ t} & & \theta_k^{\ t} \\ 1-\theta_1^{\ t} & & & \\ & 1-\theta_2^{\ t} & & \\ & & & \\ & & 1-\theta_{k-1}^{\ t} & 1-\theta_k^{\ t} \end{bmatrix}, \quad S_t = \begin{bmatrix} -1+\theta_1^{\ t} & -1+\theta_2^{\ t} & & -1+\theta_k^{\ t} \\ 1-\theta_1^{\ t} & & & \\ & 1-\theta_2^{\ t} & & \\ & & & \\ & & 1-\theta_{k-1}^{\ t} & 1-\theta_k^{\ t} \end{bmatrix} \tag{13}$$

The appropriateness of this model can be checked by multiplying out the r.h.s. of (12). It will be seen that the area, $x_{t+1}^{\ 1}$, in age-class 1 at the start of period t + 1 comprises those areas harvested in period t, plus the parts of those areas not harvested, that are destroyed by fire. The area in age-class 2 at the start of period t + 1 is the area formerly in age-class 1 which was not harvested nor burnt. Note that it is assumed that age-class k comprises all stands of age greater than k periods.

The expected (discounted) present value of the stream of volumes harvested is

$$J = E\left\{\sum_{t=1}^{\infty} \alpha^t\ \underset{\sim}{v}'\underset{\sim}{h}_t\right\} \tag{14}$$

where α is the per period discount factor, and is related to the per annum discount rate by $\alpha = (1+i)^{-a}$, where a is the length in years of a period.

To ensure a degree of eveness in the flow of timber from the forest, harvest flow constraints could be imposed. For example we might have constraints

$$(1-\gamma_1)\underset{\sim}{v}'h_{t-1} \le \underset{\sim}{v}'\underset{\sim}{h}_t \le (1+\gamma_2)\underset{\sim}{v}'h_{t-1} \qquad t = 2, 3, \ldots \tag{16}$$

which would ensure that the percentage change in volume harvested from period to period would lie within specified bounds.

We could look now for an optimal policy to maximize (14) subject to (12), (16) and constraints of the form

$$\underset{\sim}{0} \le \underset{\sim}{h}_t \le \underset{\sim}{x}_t, \qquad t = 1, 2, \ldots \tag{17}$$

This is a problem in stochastic control since the dynamic equation (16) is stochastic. For students with some familiarity with control theory the difficulties of finding solutions to stochastic control problems could be discussed at this point. For example the use of Dynamic Programming and its limitation because of the problem of dimensionality could be discussed. In any case it should be pointed out that the optimal policy for this stochastic control problem will be of a <u>feedback</u> nature. i.e. that the optimal harvest $\underset{\sim}{h}_t^*$ in period t cannot be determined prior to period t, but will depend on the current state $\underset{\sim}{x}_t$ and on previous harvests and states (possibly).

Exact solution to the control problem cannot be obtained but an approximately optimal solution can be found using the principle of <u>certainty equivalence</u> (see e.g. Chow 1975). To do this one replaces the stochastic equation (12) by the <u>deterministic</u> equation

$$\underset{\sim}{x}_{t+1} = \bar{R}\, \underset{\sim}{x}_t - \bar{S}\, \underset{\sim}{h}_t \tag{18}$$

where $\bar{R}$ and $\bar{S}$ are the expected values of the random matrices R_t and S_t, and then solves the resulting <u>deterministic</u> control problem. The optimal first period harvest for this problem is determined and after this harvest has been made, and the random fires have occurred, the new state $\underset{\sim}{x}_2$ is observed. One then repeats the above procedure using $\underset{\sim}{x}_2$ as the initial state vector to determine the harvest $\underset{\sim}{h}_2$. One then continues to iterate the procedure solving a new deterministic control problem at each step. (see Reed & Errico 1985b for details).

The deterministic control problems that have to be solved are of the form, maximize (14) subject to (16), (17) and (18). To find a solution one can first replace the objective (14) by one with a finite time horizon

$$J' = \sum_{t=1}^{N} \alpha^t \, v' \underset{\sim}{h}_t + \alpha^{N+1} \, \underset{\sim}{r}' \underset{\sim}{X}_{N+1} \qquad (19)$$

where $r' = (r_1, r_2, \ldots, r_k)$ is a vector of expected present values of single hectares of forest in age-classes 1, 2, ..., k (determined from a single stand model).

The problem of maximizing (19) subject to (16) (17) and (18) is linear (in the $\underset{\sim}{x}_t$ and $\underset{\sim}{h}_t$) in both its objective and constraints and thus can be solved by linear programming (LP) using for example the Revised Simplex Algorithm (see e.g. Childress, 1974). As an exercise students can be asked to write out the tableau for the LP problem.

In Fig. 3 paths (a) and (b) show the sequence of values of $H_t = \underset{\sim}{v}' \underset{\sim}{h}_t$ for two such solutions corresponding to a zero fire probability (a) and a one percent per annum fire probability (b). The latter path can be regarded as an estimate or prediction of future volumes harvested under optimal management if in fact a one percent fire probability prevails. Path (c) shows an "actual" sequence of harvests obtained using the certainty equivalence procedure above, and using a random number generator to simulate fires with a one percent per annum probability of occurrence. Although this is only one possible sample path of many, it can be seen that in this case the solution to the deterministic problem (path (b)) provides a good prediction of future harvests.

Fig. 3 near here

Numerical comparisons between the results of the stand-level and forest-level analyses can be made. In the forest level model, in steady-state (after about 20 decades in the example) the harvest flow constraints are non-binding and each stand is effectively managed independently. The long-run average yield per annum per hectare can be compared to that for a single stand model. A very close agreement has been found (see Reed and Errico 1985(b)) the only differences being due to the different (discrete/continuous) time formulations. On examining the harvest schedules in steady-state in the forest-level model one finds that areas are harvested as soon as trees reach a given age. To the nearest decade (discrete time unit) this age agrees with the optimal single stand rotation age.

These points of agreement between the models should be emphasized in the classroom. While they do not guarantee that the models are an accurate representation of reality, they do offer reassurance that the mathematical techniques of analysis, and the subsequent calculations, are correct.

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FIGURE CAPTIONS

Figure 1. Possible evolution of a site. In rotations 1, 2, 4, 5 and 7 the stand grows until the cutting age T. In rotation 3 it is destroyed by fire at age x and no harvest is realized. Similarly in rotation 6 it is destroyed by fire at age y.

Figure 2. Land expectation value ($ per hectare) as a function of cutting age (years) for an annual discount rate of 3 percent and for fire rates $\lambda = 0$, .005, .0104 and .015. The costs of clearing/replanting are $c_1 = \$25/\text{ha}$ after a cut and $c_2 = \$50/\text{ha}$ after a fire.

Figure 3. Volumes harvested over time with a per annum discount rate of 3 percent. Path (a) shows the optimal harvest sequence when there is no risk of fire. Path (b) shows the predicted harvest sequence when there is an (age-dependent) probability of fire of one percent per annum. Path (c) shows one "actual" sequence of harvests using the certainty equivalence procedure, discussed in the text and using a random number generator to generate fires with a one percent per annum probability.