# A CHARACTERIZATION OF STRONG HALL CRITICAL GRAPHS

by

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# A Characterization of Strong Hall Critical Graphs

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**Abstract**

A bipartite graph $G = (X, Y)$ is *strong Hall* with respect to $Y$ if for all $S \subset Y$, $S \neq \emptyset$,$|S| < |N_G(S)|$. A bipartite graph $G = (X, Y)$ is *strong Hall critical* with respect to $Y$ if it is strong Hall with respect to $Y$ but it is no longer once any edge is removed. In this paper strong Hall critical graphs are shown to be characterized by those strong Hall bipartite graphs $G = (X, Y)$ whose vertices in $Y$ have only degree two. It is also shown that strong Hall graphs $G = (X, Y)$ have the property that between any two vertices in $X$ which are connected in $G$, there is a path between them and a matching which saturates the remaining vertices in $Y$.

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# 1 Introduction

The reader is referred to [1] for the basic graph theory notation and definitions. We will also use the following. If $P_1$ and $P_2$ are paths in a graph $G$, and $P_1$ ends in a vertex adjacent in $G$ to the vertex that $P_2$ starts with, then $P_1P_2$ denotes the juxtaposition of $P_1$ and $P_2$ as alternating sequences of vertices and edges so that the edge from the end of $P_1$ to the beginning of $P_2$ is added giving a single path. Let $G = (X, Y)$ be a bipartite graph. $G$ is *strong Hall* with respect to $Y$ if for all $S \subset Y$, $S \neq \emptyset$, $|S| < |N_G(S)|$. $G$ is *strong Hall critical* if $G$ is strong Hall and $G - e$ is not strong Hall for every edge $e$ in $G$. The focus of this paper is to characterize those graphs that are strong Hall critical.

Let $G$ be strong Hall with respect to $Y$. If for all $y \in Y$, $d_G(y) = 2$, then it is clear that $G$ is strong Hall critical with respect to $Y$. The converse is shown to be true in Theorem 3.

Strong Hall graphs have been used in combinatorial matrix theory. The vertices in one part of the bipartite graph correspond to the rows of a matrix, and the vertices in the other, the columns of the matrix. Two vertices are adjacent if and only if the entry indexed by the two vertices in the matrix is nonzero. With the property that the constructed graph is strong Hall one can accurately predict the sign pattern of the upper triangular part of symbolic Gaussian elimination of the sign pattern of the original matrix (see [2]). More recently, the sign patterns of $Q$ and $R$ in the $QR$ factorization of a matrix have been determined given only the sign pattern of the matrix if the matrix's graph defined above is strong Hall. To this end, Theorem 4 gives a useful property of strong Hall graphs needed for this process (see [3]).

## 2 Theorems

Before proving Theorem 3 we need some preliminary results. The first lemma characterizes those sets that are satisfy the strong Hall property in $G$ but not in $G - e$ for some edge $e$ of $G$.

**Lemma 1** *Suppose $G = (X, Y)$ is strong Hall with respect to $Y$ and there exists $y \in Y$ with $d_G(y) \geq 3$ and an $x \in N_G(\{y\})$ such that $H = G - \{xy\}$ is not strong Hall with respect to $Y$. If $S \subseteq Y$, $S \neq \emptyset$, is such that $|N_H(S)| \leq |S|$, then:*

- $y \in S$,
- $|N_G(S)| = |S| + 1$,
- $|N_G(S \setminus \{y\})| = |S|$, *and*
- $N_G(\{y\}) \setminus \{x\} \subseteq N_G(S \setminus \{y\})$.

**Proof:** If $y \notin S$, then $N_H(S) = N_G(S)$ and hence $|N_H(S)| = |N_G(S)| > |S|$ since $G$ is strong Hall with respect to $Y$. Therefore $y \in S$. Also, if $x \in N_G(S \setminus \{y\})$ then again $N_H(S) = N_G(S)$, a contradiction, and so $x \notin N_G(S \setminus \{y\})$.

Thus $|N_G(S)| = |N_H(S)|+1 \leq |S|+1$ and since $G$ is strong Hall with respect to $Y$, $|N_G(S)| = |S| + 1$.

Now $S \setminus \{y\} \neq \emptyset$ since $d_H(y) = d_G(y) - 1 \geq 2$. Thus $|N_G(S \setminus \{y\})| \geq |S|$ because $G$ is strong Hall with respect to $Y$. Since $x \notin N_G(S \setminus \{y\})$,

$$|N_G(S \setminus \{y\})| \leq |N_G(S)| - 1 = (|S| + 1) - 1 = |S|.$$

Therefore $|N_G(S \setminus \{y\})| = |S|$.

If there exists a $z \in N_G(\{y\})$, $z \neq x$, such that $z \notin N_G(S \setminus \{y\})$, then

$$\begin{aligned} |N_G(S)| &= |N_G(\{y\}) \setminus N_G(S \setminus \{y\})| + |N_G(S \setminus \{y\})| \\ &\geq 2 + |S|, \end{aligned}$$

a contradiction. Thus $N_G(\{y\}) \setminus \{x\} \subseteq N_G(S \setminus \{y\})$. ■

**Lemma 2** *Suppose $G = (X,Y)$ is strong Hall with respect to $Y$ and there exists $y \in Y$ with $d_G(y) \geq 3$ and $x_1, x_2 \in N_G(\{y\})$, $x_1 \neq x_2$, such that for $i = 1, 2$, $H_i = G - \{x_i y\}$ is not strong Hall with respect to $Y$ and $S_i \subseteq Y$ is such that $|N_{H_i}(S_i)| \leq |S_i|$. Then $S_1 \cap S_2 = \{y\}$.*

**Proof:** Suppose $(S_1 \cap S_2) \setminus \{y\} = S \neq \emptyset$. For each $z \in S$ and $i = 1, 2$, $zx_i$ is not an edge in $G$ since $x_i \notin N_G(S_i \setminus \{y\})$ by Lemma 1.

Thus

$$\begin{aligned} |N_G(S_1 \cap S_2)| &= |N_G(\{y\}) \setminus N_G(S)| + |N_G(S)| \\ &\geq 2 + |N_G(S)| \qquad \text{since } x_1, x_2 \notin N_G(S) \end{aligned}$$

$$\begin{aligned} &> 2+|S| \qquad \text{since } S \neq \emptyset \text{ and } G \text{ is strong Hall} \\ &= 2+(|S_1 \cap S_2|-1) \\ &= |S_1 \cap S_2|+1. \end{aligned}$$

Therefore

$$|N_G(S_1) \cap N_G(S_2)| \geq |N_G(S_1 \cap S_2)| \geq |S_1 \cap S_2| + 2. \qquad (1)$$

Now consider $S_1 \cup S_2$. By applying Lemma 1 and equation 1 we have:

$$\begin{aligned} |S_1 \cup S_2| &= (|S_1|+1)+(|S_2|+1)-(|S_1 \cap S_2|+2) \\ &\geq |N_G(S_1)|+|N_G(S_2)|-|N_G(S_1) \cap N_G(S_2)| \\ &= |N_G(S_1) \cup N_G(S_2)| \\ &= |N_G(S_1 \cup S_2)|. \end{aligned}$$

This contradicts that $G$ is strong Hall with respect to $Y$. Hence $S = \emptyset$. By Lemma 1 $y \in S_1 \cap S_2$ and therefore $S_1 \cap S_2 = \{y\}$. ∎

We are now ready to prove one of the main theorems. The first theorem characterizes strong Hall critical graphs. The second gives a matching property of strong Hall graphs.

**Theorem 3** *Let $G = (X, Y)$ is strong Hall with respect to $Y$. If there exists a vertex $y \in Y$ such that for all $x \in N_G(\{y\})$, $G - \{xy\}$ is not strong Hall with respect to $Y$, then $d_G(y) = 2$.*

**Proof:** Suppose such a $y$ exists. Let $t = d_G(y)$, let $N_G(\{y\}) = \{x_1, x_2, \ldots, x_t\}$ and suppose, to the contrary, that $t \geq 3$. For each $i = 1, 2, \ldots, t$, since

$H_i = G - \{x_i y\}$ is not strong Hall with respect to $Y$, there exists $S_i \subseteq Y$ such that $|N_{H_i}(S_i)| \leq |S_i|$. Then

$$\begin{aligned} |N_G(S_1 \cup S_2 \cup \cdots \cup S_t)| &= |N_G(\{y\})| + \left| \bigcup_{i=1}^{t} (N_G(S_i \setminus \{y\}) \setminus N_G(\{y\})) \right| \\ &\leq t + \sum_{i=1}^{t} |N_G(S_i \setminus \{y\}) \setminus N_G(\{y\})| \\ &\leq t + \sum_{i=1}^{t} (|N_G(S_i \setminus \{y\})| - (t-1)) \end{aligned}$$

since by Lemma 1 $N_G(\{y\}) \setminus \{x_i\} \subseteq N_G(S_i \setminus \{y\})$.

Thus

$$\begin{aligned} |N_G(S_1 \cup S_2 \cup \cdots \cup S_t)| &= t - t(t-2) + \sum_{i=1}^{t} (|N_G(S_i \setminus \{y\})| - 1) \\ &= 3t - t^2 + \sum_{i=1}^{t} (|S_i| - 1) \qquad \text{by Lemma 1} \\ &= 3t - t^2 + (|S_1 \cup S_2 \cup \cdots \cup S_t| - 1) \text{ by Lemma 2} \\ &\leq |S_1 \cup S_2 \cup \cdots \cup S_t| \qquad \text{since } t \geq 3. \end{aligned}$$

This last inequality contradicts that $G$ is strong Hall with respect to $Y$. Therefore $d_G(y) = t = 2$. ■

The next theorem is partly a consequence of the previous theorem, however, other work is needed.

**Theorem 4** *Let $G = (X, Y)$ be a bipartite graph such that $G$ is strong Hall with respect to $Y$. Suppose there is a path $P$ from $x_1$ to $x_2$ for some $x_1, x_2 \in X$ ($x_1$ may equal $x_2$). Then for $i = 1, 2$, there exists a matching $M_i$ in $G$*

*which saturates $Y$ and an $M_i$-alternating path $Q$ from $x_1$ to $x_2$. Moreover, if $x_1 \neq x_2$, then $M_i$ saturates $x_i$.*

**Proof:** If $x_1 = x_2$, then the induced bipartite graph $H = (X \setminus \{x_1\}, Y)$ satisfies Hall's condition with respect to $Y$ and hence has a matching saturating $Y$. Thus the theorem is true for this case.

Suppose the statement of the theorem is false in general and let $G = (X, Y)$ be a counterexample with the fewest edges. Then there exists two distinct vertices $x_1$ and $x_2$ in $X$ and a path $P$ in $G$ from $x_1$ to $x_2$. Now each path from $x_1$ to $x_2$ induces two different matchings $M_1'$, $M_2'$, such that the set of edges of the path is $M_1' \cup M_2'$, and for $i = 1, 2$, $M_i'$ saturates $x_i$. The path with either of these matchings is alternating. Thus to prove the theorem by contradiction, it is sufficient to find a path $Q$ in $G$ from $x_1$ to $x_2$, and a matching $M''$ in $G - Q$ such that $M''$ saturates $V(G - Q) \cap Y$ for then $M_i = M_i' \cup M''$ saturates $Y$ and $x_i$, and $Q$ is $M_i$-alternating.

Thus, since $G$ is a counterexample there must exist a vertex $y \in Y$ not on $P$. If $d_G(y) \geq 3$, then by Theorem 3, there exists $x \in N_G(\{y\})$ such that $H = G - \{xy\}$ is strong Hall with respect to $Y$. Now $P$ is a path in $H$ also, and since $G$ is a counterexample with the fewest edges, $H$ has a matching which saturates $V(H - P) \cap Y$. But this matching and path are also in $G$, a contradiction.

Therefore $d_G(y) = 2$ ($d_G(y) > 1$ since $G$ is strong Hall with respect to $Y$). Let $a$ and $b$ be the neighbors of $y$ and let $H = (U, V)$ be the graph obtained from $G$ by identifying $a$ with $b$ and by removing $y$. That is, $U = (X \setminus \{a, b\}) \cup \{\alpha\}$ where $\alpha \notin V(G)$ and $V = Y \setminus \{y\}$. Moreover, for all $u \in U$ and $v \in V$, $uv \in E(H)$ if and only if

- $u \in X \setminus \{a, b\}$ and $v \in Y$; or
- $u = \alpha$, $v \in V$, and $av \in E(G)$ or $bv \in E(G)$.

**Claim:** *$H$ is strong Hall with respect to $V$.*

Proof of Claim: Let $S \subseteq V$, $S \neq \emptyset$. If $\alpha \notin N_H(S)$, then $a \notin N_G(S)$ and $b \notin N_G(S)$. Thus $N_H(S) = N_G(S)$ and since $G$ is strong Hall with respect to $Y$, $|S| < |N_G(S)| = |N_H(S)|$.

Suppose there exists $S \subseteq V$ such that $|S| \geq |N_H(S)|$. Then by the above $\alpha \in N_H(S)$ and hence at least one of $a$ or $b$ is contained in $N_G(S)$. If only one is contained in $N_G(S)$, then since it is only replaced with $\alpha$ in $H$, $|N_G(S)| = |N_H(S)|$, a contradiction to $|S| < |N_G(S)|$. Thus both $a$ and $b$ are contained in $N_G(S)$. Since $a$ and $b$ are replace with $\alpha$ in $H$, $|N_G(S)| = |N_H(S)| + 1$ and $N_G(S \cup \{y\}) = N_G(S)$. But then

$$|S \cup \{y\}| = |S| + 1 \geq |N_H(S)| + 1 = |N_G(S)| = |N_G(S \cup \{y\})|$$

contradicts that $G$ is strong Hall with respect to $Y$. This completes the proof of the claim.

We now construct a path $P'$ in $H$ from the path $P$ in $G$. The following table describes how $P'$ is defined in $H$ given the structure of $P$. In the table, $A$, $B$ and $C$ are nonempty paths (they each contain at least a vertex).

| Case | $P$ is | new path $P'$ in $H$ is |
|---|---|---|
| 1 | $P$ ($a$ and $b$ not on $P$) | $P$ |
| 2 | $AaB$ or $AbB$ | $A\alpha B$ |
| 3 | $AaBbC$ or $AbBaC$ | $A\alpha C$ |
| 4 | $aAbB$ or $bAaB$ | $\alpha B$ |
| 5 | $aA$ or $bA$ | $\alpha A$ |
| 6 | $AaBb$ or $AbBa$ | $A\alpha$ |
| 7 | $Aa$ or $Ab$ | $A\alpha$ |
| 8 | $aAb$ or $bAa$ | $\alpha$ |

Let $P'$ start in $z_1$ and end in $z_2$ (note that $z_1, z_2 \in U$). Since $H$ is strong Hall with respect to $Y$, and $H$ has fewer edges than $G$, there exists a path $R$ in $H$ from $z_1$ to $z_2$ and matching $N$ in $H - R$ which saturates $V(H - R) \cap U$. We will use this path and matching to construct similar ones in $G$. The constructions will differ depending on the structure of $P$ in $G$, but each will contradict that $G$ is a counterexample to the theorem. The cases below are described in the table above.

**Cases 1–3.** Here we have $\{a, b\} \cap \{x_1, x_2\} = \emptyset$. If $\alpha$ is not on $R$ and $\alpha$ is not saturated by $N$, then $R$ is a path in $G$ from $x_1$ to $x_2$ and $M'' = N \cup \{ay\}$ is a matching in $G$ which saturates $V(G - R) \cap Y$, a contradiction.

If there exists a $z \in V$ such that $\alpha z \in N$, then $\alpha$ is not on $R$ and hence $R$ is a path in $G$ from $x_1$ to $x_2$. Moreover, $cz \in E(G)$ for some $c \in \{a, b\}$. Thus $M'' = (N \setminus \{\alpha z\}) \cup \{cz\}$ is a matching in $G$ which saturates $V(G - R) \cap Y$, a contradiction.

Therefore $\alpha$ is on $R$ and $N$ is a matching in $G$. Let $y_1$ and $y_2$ be the neighbors of $\alpha$ on $R$ and let $R = R_1 \alpha R_2$ where $R_1$ ends in $y_1$ and $R_2$

starts in $y_2$.

If $\{a, b\} \subseteq N_G(\{y_1, y_2\})$, then at least one of $Q_1 = R_1 ayb R_2$ and $Q_2 = R_1 bya R_2$ is a path in $G$ that starts in $x_1$ and ends in $x_2$. Thus for some $i \in \{1, 2\}$, $N$ saturates $V(G - Q_i) \cap Y$, a contradiction.

Hence $\{a, b\} \not\subseteq N_G(\{y_1, y_2\})$. Thus since $\alpha$ is adjacent to $y_1$ and $y_2$, they are both adjacent to $a$ or to $b$ in $G$. If $y_1$ and $y_2$ are neighbors of $a$, then $Q = R_1 a R_2$ is a path in $G$ from $x_1$ to $x_2$, and $M'' = N \cup \{by\}$ is a matching in $G$ which saturates $V(G - Q) \cap Y$, a contradiction. Similarly, if $y_1$ and $y_2$ are neighbors of $b$, then $Q = R_1 b R_2$ is a path in $G$ from $x_1$ to $x_2$, and $M'' = N \cup \{ay\}$ is a matching in $G$ which saturates $V(G - Q) \cap Y$, another contradiction. Therefore, Cases 1–3 cannot occur.

**Cases 4–7.** We give the details for Cases 4 and 5. Cases 6 and 7 can be handled similarly. Note that $z_1 = \alpha$ and hence $R$ starts in $\alpha$. Moreover, $N$ is a matching in $G$. Let $R = \alpha R_1$ and let $y_1$ be the first vertex of the path $R_1$. Since $y_1$ is adjacent to $\alpha$ in $H$, it is adjacent to either $a$ or $b$ in $G$. Thus either $Q_1 = aR_1$, $Q_2 = aybR_1$, $Q_3 = bR_1$, or $Q_4 = byaR_1$ is a path from $x_1$ to $x_2$ in $G$. If $Q_2$ is such a path in $G$, then $N$ saturates $V(G - Q_2) \cap Y$. Similarly if $Q_4$ is a path from $x_1$ to $x_2$ in $G$. Otherwise, $N \cup \{by\}$ saturates $V(G - Q_1) \cap Y$ or $N \cup \{ay\}$ saturates $V(G - Q_3) \cap Y$, a contradiction. Therefore Cases 4–7 cannot occur.

**Case 8.** In this case, $R = \alpha$ and so $N$ is a matching in $G$ and it saturates $V = Y \setminus \{y\}$. Thus letting $Q$ be the path $x_1 y x_2$ we have a contradiction. Therefore Case 8 cannot occur.

Since one of the cases must occur, we have a contradiction. Therefore no such counterexample exists and the theorem is proved. ■

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