THE DECISION TO CONSERVE OR HARVEST OLD–GROWTH FOREST

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DMS–534–IR February 1990

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Abstract

The decision as to whether to harvest or conserve old–growth forest is formulated as a stochastic decision problem in continuous time. Uncertainty in future amenity values for standing forest and in future timber revenues for harvested forest are included in the model, along with the risk of catastrophic destruction by fire, pest infestation, etc. It is shown how the decision problem can be expressed as an optimal stopping problem for Brownian motion processes, and the optimal decision rule is shown to depend on how the ratio of current timber value to amenity benefit flow compares with some critical level. The effects of changes in uncertainty and other parameters on the optimal rule are discussed. Also it is shown how the cost–benefit analysis and certainty–equivalence procedures lead to premature harvesting. The expected survival time of the forest using the optimal decision rule and other sub–optimal rules is discussed.

Keywords: amenity values, conservation, uncertainty, catastrophic risk, optimal stopping, geometric Brownian motion, expected survival time.

1. INTRODUCTION

As old–growth forest becomes increasingly scarce the conflict between user groups intensifies. To the logging industry old–growth forest is a source of cheap high–quality timber, while to environmentalists and recreational users it represents an irreplaceable natural treasure which should be preserved indefinitely. Often the arguments put forward in favour of logging by the industry and its supporters are expressed in economic terms (*e.g.* the wealth generated and the number of jobs preserved through logging, *etc.*). Environmentalists on the other hand often tend to reject economic arguments, basing their claims on philosophical and moral grounds.

However, if economics can be defined as the study of "how men and society end up choosing, with or without the use of money, to employ scarce productive resources, which could have alternative uses..." (Samuelson and Scott, 1975), then it would appear that the question facing society as to whether to harvest or conserve old–growth forest is preeminently an economic one. Once one recognizes that standing forest has a value *per se*, other than as a source of timber, then the problem falls squarely into the framework of economic analysis.

While forest–level harvest scheduling procedures such as FORPLAN (Johnson, Jones and Kent, 1980) have always allowed for the explicit inclusion of amenity values for standing forest, the first analytical treatment to include amenity values appears to be the stand–level model of Hartman (1976). Hartman shows that if amenity values are increasing with age, then the optimal rotation age for the stand will exceed the Faustmann rotation age. Thus harvesting will be delayed.

There are two major shortcomings with the Hartman model. The first is that it deals with a single stand in isolation, and the second is that it is deterministic, regarding future timber values and amenity values as known. Attempts to address the first of these shortcomings lead back to the forest–level harvesting models of the

FORPLAN type, for which analytic solutions appear impossible (see Bowes and Krutilla (1985) for an economic forest–level harvesting model which includes amenity values).

The problem of future uncertainty in timber and amenity values has not been addressed explicitly in the framework of the Hartman model. However, there is now a considerable body of literature which deals in general with the problem of irreversible decisions in the face of uncertainty. The pioneering paper is that of Arrow and Fisher (1974) who considered the problem of developing wilderness when the future benefits from conservation and development are uncertain. They show that since development is irreversible, while conservation is not, there is a value, which they called the *quasi–option value*, associated with the reversible decision to conserve. Simply put, with uncertainty present, there is some value associated with keeping one's options open.

This literature for the most part has dealt with highly stylized two–period models with linear benefit functions, leading therefore to total development or total conservation in period 1 or period 2. A more general model which allows for partial development at any time over an infinite time horizon is that of Clarke and Reed (1990). This model assumes uncertain future amenity values for wilderness which depend on the amount of wilderness surviving, but it assumes that the costs (or immediate revenues) from developing are known. The model might thus be applicable to the problem of harvesting old–growth forest if one could reasonably assume that future timber prices were fixed and known, and that the forest exhibited no growth nor was subject to catastrophic destruction.

It is the purpose of this paper to address this problem, when both future amenity values and future timber values are uncertain. Growth in the volume of timber is not explicitly included, it being assumed that biologically the old–growth forest has reached something close to a steady–state. However, the possibility of unpredictable catastrophic destruction via fire, tempest or pest infestation is included. Also, the model is formulated for a single stand of old–growth. The possibility of the amenity value of the stand depending on the amount of old–growth forest remaining globally, or regionally, has not been considered. This is undoubtedly a shortcoming of the model, which can be defended only on the grounds that to include this aspect would lead to a model for which analytic results would not be forthcoming.

In Section 2 the model is developed. The objective considered is the maximization of the expected present value of the flow of amenity services plus timber revenues up until the time of destruction of the stand, whether through harvesting or through catastrophe. If the latter occurs, of course, there will be no timber revenues. The decision problem faced by the managers of the resource at each instant in time is whether to conserve (no harvest) or to harvest. It is assumed that current timber and amenity values are known but that future values are not known with certainty. Both are assumed to follow (possibly correlated) geometric Brownian motion processes. Thus both timber values and amenity values are assumed to grow at a proportional rate, which at all times is made up of a fixed component and a random white–noise component. The problem of whether to harvest or not is formulated as an optimal stopping problem. Conditions which guarantee conservation at all times are given.

In Section 3, the solution to the optimization problem is presented. In Section 4 the sensitivity of the solution to small changes in model parameters is considered and some numerical results given. Section 5 discusses other possible (sub–optimal) harvest rules and compares them analytically and numerically with the optimal rule. It is shown that a rule based on a simple cost–benefit analysis, as well as the myopic–look–ahead and certainty–equivalence rules, all lead to premature harvesting of the forest. In Section 6 the expected survival time of the forest under various harvest rules is considered.

2. A MODEL FOR THE HARVEST–CONSERVATION DECISION PROBLEM

Consider an area of old–growth forest which can be either conserved *in toto*(1) or clear–cut harvested. Clearly the decision to conserve is reversible, in that the forest can be harvested at a later date, while the decision to harvest is irreversible. Suppose that if the forest is harvested at time t a net revenue $V(t)$ is realized. This includes the revenue from the sale of timber net of harvest costs, plus the value of the bare land on which future rotations may possibly be grown. We shall assume that $V(t)$ is an observable stochastic process following *geometric Brownian motion*, governed by the stochastic differential equation

$$\frac{dV}{V} = b\ dt + \sigma_1 dw_1 \tag{1}$$

where b is a drift parameter, $\{w_1(t)\}$ is a standard Wiener process, and σ_1^2 is a variance parameter. Thus we assume that the proportional change in net value over time dt is a constant $b\ dt$ plus or minus a random component $\sigma_1 dw_1(t)$. Large values of the parameter σ_1 will correspond to a large amount of uncertainty in future timber values. If harvest and replanting costs are neglected, $V(t)$ should be proportional to the current price of timber, since both the revenues from the sale of timber and the land value will then be proportional to timber price (see Clarke and Reed, 1989). The assumption that a commodity price follows geometric Brownian motion is very reasonable. A similar assumption for asset prices lies at the heart of much of the recent literature in financial economics.

The realization of the immediate benefit $V(T)$ resulting from harvesting at time T is offset by the foregoing of the option to harvest at a later date when timber prices might conceivably be higher, as well as by the foregoing of the flow of amenity services from the old–growth forest at all times beyond T.

Suppose that the flow of amenity services at time t has social valuation $A(t)$. It should be emphasized that $A(t)$ is a flow and has units such as dollars per month. It represents the rent that society is willing to pay to preserve the old-growth forest. To reflect the fact that future valuations of amenity services are uncertain, we shall assume that $A(t)$ is a stochastic process. Specifically we shall assume it follows geometric Brownian motion

$$\frac{dA}{A} = a\ dt + \sigma_2 dw_2 \ . \tag{2}$$

As before, a and σ_2^2 are drift (mean growth rate) and variance (uncertainty) parameters while $\{w_2(t)\}$ is another standard Wiener process. To reflect the fact that future changes in amenity value and timber value may be correlated, we shall assume that the white-noise processes $\{dw_1(t)\}$ and $\{dw_2(t)\}$ have correlation ρ. Positive values of ρ will reflect a positive association between fluctuations in timber and amenity values, and negative values of ρ a negative association.

The fact that a harvest does not take place does not guarantee the survival of the forest forever. There is always the possibility that it will be destroyed by some natural catastrophe such as fire, tempest or pest infestation. In assessing the total value of amenity services obtained from the forest before a scheduled clear-cut harvest, this fact must be taken into account.

If there were to be no harvest, amenity services would accrue up until the time at which catastrophic destruction occurred, if indeed it ever did. Thus the expected present value of amenity services, using an *instantaneous discount rate,* δ, would be

$$E\left\{\int_0^{\tau} e^{-\delta t}\ A(t)dt\right\} \tag{3}$$

where τ is a random variable denoting the time of destruction (τ would be infinity if destruction never occurs). The expectation in (3) is taken with respect to both τ and the stochastic process $\{A(t)\}$. The integral can be re–expressed as

$$(4) \qquad E\left\{\int_0^{\infty} e^{-\delta t}\ A(t)S(t)dt\right\}$$

where $S(t)$ is the *survivor function* denoting the probability that the stand survives catastrophic destruction until time t.

If a clear–cut harvest is scheduled to take place at some time T, then amenity services will accrue only up until that time, or until the time of catastrophic destruction, if that occurs sooner. Thus the total expected present value of amenity services will be

$$(5) \qquad E\left\{\int_0^{T} e^{-\delta t}\ A(t)S(t)dt\right\}$$

which can be re–written as

$$(6) \qquad E\left\{\int_0^{\infty} e^{-\delta t}\ A(t)S(t)dt\right\} - E\left\{\int_T^{\infty} e^{-\delta t}\ A(t)S(t)dt\right\},$$

and the total expected present value of timber revenues and amenity services will be

$$(7) \qquad e^{-\delta T}\ V(T)S(T) - E\left\{\int_T^{\infty} e^{-\delta t}\ A(t)S(t)dt\right\} + E\left\{\int_0^{\infty} e^{-\delta t}\ A(t)S(t)dt\right\}.$$

The second term represents the expected present value of amenity services foregone by undertaking a harvest at time T.

We shall seek a harvest rule (which will determine when, if ever, the stand is clear–cut harvested) to maximize the total expected present value (7).

The survivor function $S(t)$ in (7) is determined by the *hazard–rate.* Any functional form could be used here (see *e.g.* Thompson, 1988). For simplicity, and in absence of any direct indication to the contrary, we shall assume a constant hazard rate λ, *i.e.* assume that the hazard of catastrophic destruction is time–independent. In this case the survivor function can be written as

$$S(t) = e^{-\lambda t}$$

and the total expected present value written as

$$\text{(8)} \qquad e^{-(\delta+\lambda)T} V(T) - E\left\{\int_T^\infty e^{-(\delta+\lambda)t} A(t)dt\right\} + E\left\{\int_0^\infty e^{-(\delta+\lambda)t} A(t)dt\right\}.$$

Note that the effect of the hazard λ is equivalent to adding a premium λ to the discount rate δ (see Reed, 1984).

The expression (8) can be simplified by solving (2) explicitly. However, a technical problem arises here as to which stochastic integral (Itô or Stratonovich) should be used for the solution (see Reed and Clarke, 1990 for a discussion of this issue in a forestry context). Rather than entering into further discussion here, we shall employ the Itô calculus and point out how the results would differ if the Stratonovich calculus had been used instead. It is shown in Appendix 1 that the expectations in (8) can be evaluated explicitly as

$$\text{(9)} \qquad E\left\{\int_0^\infty e^{-(\delta+\lambda)t} A(t)dt\right\} - E\left\{\int_T^\infty e^{-(\delta+\lambda)t} A(t)dt\right\}$$

$$= \frac{A(0)}{\delta + \lambda - a} - \frac{e^{-(\delta+\lambda)T} A(T)}{\delta + \lambda - a} .$$

(If the Stratonovich calculus is used, a is replaced by $a' = a + \frac{1}{2}\sigma_2^2$ in the right–hand side of (9).) Note that to ensure convergence we require the condition

(10) $$\delta + \lambda - a > 0.$$

We shall assume that this condition, and in addition

(11) $$\delta + \lambda - b > 0,$$

holds. If either of these conditions is not met it will be optimal to never harvest the forest. Thus sufficient conditions for *conservation in perpetuity* to be optimal are that $\delta + \lambda < a$ *or* $\delta + \lambda < b$. If the former is met, amenity services are growing faster in expectation than the risk–adjusted discount rate and it therefore pays to conserve forever. If the latter condition is met, timber values are growing faster in expectation than the risk–adjusted discount rate, and so at all times it will pay to postpone a clear–cut harvest.

The problem of determining an optimal harvest rule can be expressed as that of finding a *stopping time* T to maximize the total expected present value:

(12) $$e^{-(\delta+\lambda)T}\left[V(T) - \frac{A(T)}{\delta + \lambda - a}\right] + \frac{A(0)}{\delta + \lambda - a} .$$

This can be formulated as a problem of *optimal stopping* (see *e.g.* Brock, Rothschild and Stiglitz, 1988) with *intrinsic value* function

$$R(V,A) = V(T) - \frac{A(T)}{\Delta - a} \tag{13}$$

and the objective of maximizing the expected present value

$$E\left\{e^{-\Delta T} R(V(T),A(T))\right\} \tag{14}$$

where $\Delta = \delta + \lambda$ is the *risk–adjusted discount rate.* (Note that (14) omits the last term on the right–hand side of (12) which is a constant independent of T, A(T), R(T).) Note also that in general a stopping time T may depend on past and current values of the bivariate stochastic process $\{V(t),A(t)\}$, and that the expectation in (14) depends on the initial values (V_0,A_0) of this process.

In expressing the harvesting decision as an optimal stopping problem for the $\{V(t),A(t)\}$ process, it is assumed that V(t) and A(t) are *observable.* This is something of a simplification since societal valuations of amenity services can be determined only through sampling methods and thus will contain an element of sampling error. Similarly, even though current timber prices may be known exactly, the net revenue V(t) to be realized through an immediate clear–cut harvest of the old growth cannot be known exactly prior to the harvest. For example, harvesting costs can only be estimated and the volume of usable timber is usually known only via sampling methods. However, uncertainty in future values of V(t) and A(t) is much greater than uncertainty concerning current values. Abstracting from the former source of uncertainty captures the essence of the problem of when, if ever, a clear–cut harvest should take place.

3. THE OPTIMAL DECISION RULE.

Since the stochastic processes defined by (1) and (2) are stationary we can confine our attention to *stationary* decision or *stopping rules*, which in this case are simply partitions of the V – A space into two regions – a *stopping region* and a *continuation region.* The process is stopped (forest clear–cut harvested) when the $\{V(t),A(t)\}$ process leaves the continuation region for the first time.

For purposes of analysis it is convenient to consider a transformation of variables V and A. Let

$$(15) \qquad x(t) = \ln V(t) \qquad \text{and} \qquad y(t) = \ln A(t).$$

Then from (1) and (2) using Itô's lemma

$$(16) \qquad dx = (b-\tfrac{1}{2}\sigma_1^2)dt + \sigma_1 dw_1$$

$$dy = (a-\tfrac{1}{2}\sigma_2^2)dt + \sigma_2 dw_2.$$

(If the Stratonovich calculus were used, b would be replaced by $b' = b + \frac{1}{2}\sigma_1^2$, and a would be replaced by $a' = a + \frac{1}{2}\sigma_2^2$ leading to the coefficients of dt being b and a respectively.)

Consider now any stopping rule S which partitions the V – A space (and hence the x – y space) into stopping and continuation regions. For such a stopping rule we define a *value function*

$$(17) \qquad W^S(x,y) = E\left\{e^{-\Delta\tau_S} R(\exp(x(\tau_S)), \exp(y(\tau_S)))\,|\,x(0) = x, \quad y(0) = y\right\}$$

where τ_s is a random variable denoting the time at which the process first leaves the continuation region, and the expectation is taken with respect to τ_s, $x(\tau_s)$ and $y(\tau_s)$. The function R is as specified in (7). The function W^s simply represents the expected present value of timber revenues net of foregone amenity services, given that initially the timber and amenity service values are $A(0) = e^x$ and $V(0) = e^y$, and using the stopping rule S. Adding $A(0)/(\Delta - a)$ to W^s gives the expected present value of all benefits from the forest (see (12)).

It is easy to show that for (x,y) belonging to the continuation region, the function $W^s(x,y)$ satisfies the following *Hamilton–Jacobi–Bellman* equation

$$(18)\ \Delta W^s = (b-\tfrac{1}{2}\sigma_1^2)\, W^s_x + (a-\tfrac{1}{2}\sigma_2^2)\, W^s_y + \tfrac{1}{2}\sigma_1^2\, W^s_{xx} + \rho\sigma_1\sigma_2\, W^s_{xy} + \tfrac{1}{2}\sigma_2^2\, W^s_{yy}$$

where subscripts denote partial derivatives. Also, on the boundary of the stopping region,

$$(19)\qquad W^s(x,y) = R(e^x,e^y).$$

In addition, if the stopping rule S is optimal (*i.e.* if $W^s(x,y) \geq W^u(x,y)$ for all stopping rules U), then W^s satisfies the so-called "*smooth– pasting*" conditions on the boundary of the stopping region

$$(20)\qquad W^s_x(x,y) = \frac{\partial}{\partial x} R(e^x,e^y) = e^x$$

$$(21)\qquad W^s_y(x,y) = \frac{\partial}{\partial y} R(e^x,e^y) = -e^y/(\Delta-a)$$

(using (13)). It can be shown (see *e.g.* Brock, Rothschild and Stiglitz (1988)) that the conditions (10), (19), (20) and (21), along with the condition that

$W^s(x,y) > R(e^x,e^y)$ on the continuation region, determine an optimal stopping rule provided certain regularity conditions are satisfied. Thus the optimal stopping problem can be solved as a *free–boundary problem* given by the partial differential equation (18) with free–boundary conditions (19), (20) and (21).

In general, solution of free–boundary problems in two or more dimensions is very difficult. However, the problem above turns out to have a particularly simple solution, *viz.* a stopping rule which prescribes stopping whenever $x - y$ exceeds some critical level. In other words *the old–growth forest is optimally clear–cut harvested as soon as the ratio* $V(t)/A(t)$ *of timber value to amenity value exceeds some critical level.* The details are given in Appendix 2. There it is shown that the critical level for the ratio $V(t)/A(t)$ is:

(22) $$C^* = \frac{1}{\Delta - a}\left[\frac{1+\theta}{\theta}\right]$$

where θ is the *positive* root of the quadratic equation

(23) $$\tfrac{1}{2}\sigma_z^2\theta^2 + (b-a+\tfrac{1}{2}\sigma_z^2)\theta + (b-\Delta) = 0$$

and σ_z^2 is the variance of the $\{x-y\}$ process, *i.e.*

(24) $$\sigma_z^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 .$$

The associated value function is

(25) $$W(x,y) = \frac{1}{(1+\theta)C^{*\theta}} \exp\{(1+\theta)x - \theta y\}$$

which in terms of a current timber value V_0 and current amenity value A_0 gives the maximum expected net present social value of the asset (see (12)) as:

$$U(V_0,A_0) = \frac{V_0}{(1+\theta)}\left[\frac{V_0}{C^*\ A_0}\right]^{\theta} + \frac{A_0}{\Delta - a}\ . \tag{26}$$

The optimal barrier C^* in (22) can be interpreted in the following way, using (8) and (12). At the boundary, $V(t)/A(t) = C^*$ or

$$V(t) = \left[1 + \frac{1}{\theta}\right]\frac{A(t)}{\Delta - a} = q^*\,\frac{A(t)}{\Delta - a}\ , \tag{27}$$

say where

$$q^* = 1 + \frac{1}{\theta} = (\Delta - a)C^*. \tag{28}$$

The term $A(t)/(\Delta - a)$ is the expected present value of amenity benefits foregone through harvesting. Thus *optimally one harvests when the ratio of immediate timber benefits to the expected present value of amenity benefits foregone exceeds unity by a suitably large amount,* q^*.

4. NUMERICAL RESULTS AND SENSITIVITY ANALYSIS

The effects of changes in parameter values on the optimal harvest rule can be determined by finding the sign of the derivative of the optimal barrier q^* for the ratio of immediate timber values to expected present value of amenity services foregone. It is shown in Appendix 3 that increases in

(a) the expected growth rate a of amenity values,

(b) the discount rate δ,

(c) the hazard rate λ,

(d) the correlation ρ between amenity service values and timber values,

all lead to a *decrease* in q^*. Conversely increases in

(e) the expected growth rate b of timber values,

(f) the uncertainty in the growth of amenity values σ_2,

(g) the uncertainty in the growth of timber values σ_1,

lead to an *increase* in q^*. Note in particular that increased uncertainty leads to a more conservative policy in that optimally one must wait until the ratio of immediate timber benefits to the expected present value of amenity services foregone attains a larger value before a harvest takes place.

In terms of the barrier C^* of the ratio of current timber value to current flow of amenity services, increases in (a), (e), (f) and (g) lead to an increase in C^*, while (b), (c) and (d) lead to a decrease in C^*.(2)

Numerical calculations of q^* and C^* have been carried out for various values of the parameters a, b, σ_z^2 and δ. The results are presented in Tables 1 and 2. In all cases the hazard rate for catastrophic destruction was set at $\lambda = .005$ corresponding closely to a 1 in 200 chance of destruction in any given year. The blank entries in the table (where a or b exceed $\Delta = \delta + \lambda$) correspond to cases where it is never optimal to harvest the resource. Note, in Table 1, how the critical ratio q^* is large when the difference $b - a$ in expected growth rates for timber values and amenity value is large, and how q^* increases with the variance σ_z^2. For example with $a = 0$, $b = 0.05$, $\delta = .05$ and $\sigma_z^2 = .02$ the current timber value would have to exceed the expected present value of amenity services foregone by a factor of more than 13 before, optimally, a harvest would be undertaken.

With σ_z^2 set at 0.02 one could be fairly confident ($\simeq$ 95% confidence) that within one year the ratio of timber value to amenity flow $V(t)/A(t)$ would lie within between 0.75 and 1.33 times its initial value;(3) with $\sigma_z^2 = .01$ the corresponding range would be 0.82 and 1.22 times the initial value. Such fluctuation seems well within the range of possibility.

5. OTHER HARVEST RULES

In this section the optimal harvest rule developed in Section 3 is compared with other policies that might be employed in making a decision as to whether to harvest or not.

A traditional tool of economic policy–making is that of *cost–benefit analysis.* In the current context one could consider the benefit to be the immediate net timber revenue $V(t)$ obtained from harvesting, and consider the cost to be the expected present value of amenity services foregone through harvesting. Thus the expected benefit net of costs of undertaking a harvest at time T is

$$V(T) - \frac{A(T)}{\Delta - a}\,. \tag{29}$$

Cost–benefit analysis would indicate that a harvest be undertaken if this quantity is positive; or in other words if the ratio $V(T)/A(T)$ exceeds the barrier

$$C_0 = \frac{1}{\Delta - a}\,, \tag{30}$$

or if the ratio of timber benefits to expected present value of amenity services foregone exceeds $q_0 = 1$. It can be seen from (28) that whenever there is uncertainty present $(\sigma_z^2 > 0)$ such a policy will always prescribe a premature harvest.(4) It will be most in error when the difference in mean growth rates $b - a$ is large, and the degree of uncertainty large.

Another possible harvest rule which could be adopted is a stochastic version of the Wicksell rule (Wicksell, 1934), which would advocate harvesting when the intrinsic net revenue from harvesting $(R(V,A)$ in (13)) is growing in expectation at the risk–adjusted discount rate Δ. Thus one sets $E(dR) = \Delta R\, dt$. From (13),

(1) and (2) this yields the rule: harvest as soon as the ratio $V(t)/A(t)$ attains the barrier level

$$C_1 = \frac{1}{\Delta - b}, \tag{31}$$

or alternatively as soon as the ratio of timber value to expected present value of amenity benefits foregone exceeds

$$q_1 = \frac{\Delta - a}{\Delta - b}. \tag{32}$$

This Wicksell–type rule has been referred to as the myopic–look–ahead (MLA) rule (Clarke and Reed, 1989). It is easily shown (see Appendix 4) that $C_1 < C^*$, $q_1 < q^*$. Thus the MLA rule prescribes a premature harvest.(5) The degree to which q_1 (and C_1) are too small is:

$$r_1 = \frac{q^* - q_1}{q^*} = \frac{C^* - C_1}{C^*} = \frac{1 + \alpha\theta}{1 + \theta} \tag{33}$$

where $\alpha = (a-b)/(\Delta-b)$ (greater than one), which is decreasing in θ. Since θ decreases with σ_z^2 it follows that an increase in the uncertainty of either future timber values or future amenity values increases the degree to which the MLA rule prescribes a premature harvest. Thus *if there is a high degree of uncertainty the* MLA *rule may be considerably sub–optimal.*

A third possible harvest rule is the *certainty–equivalence* rule, which is obtained by replacing random variables by their expected value and solving the resulting deterministic optimization problem. The nature of the certainty–equivalence rule depends on whether timber values are growing in expectation faster than amenity values $(b > a)$ or *vice–versa* $(b \leq a)$. In the former case the certainty–equivalence rule is identical to the MLA rule, while in the latter case it is identical to the cost–benefit rule.[6] Thus the critical level of the ratio of timber value to amenity benefits foregone for the certainty–equivalence rule is

(34)
$$q_2 = \begin{cases} \dfrac{\Delta - a}{\Delta - b}, & b > a \\ 1, & b \leq a \end{cases} .$$

Since both the MLA and cost–benefit rules prescribe premature harvesting it follows that the certainty–equivalence rule will do likewise. The degree to which the critical level q_2, of the certainty equivalence rule, is too small is:

(35)
$$r_2 = \frac{q^* - q_2}{q^*} = \begin{cases} \dfrac{1 + \alpha\theta}{1 + \theta}, & a < b \\ \dfrac{\theta}{1 + \theta}, & a \geq b \end{cases} .$$

Since r_2 decreases with θ it follows, as before, that the degree to which the certainty–equivalence rule prescribes premature harvesting, will increase with increasing uncertainty. This concurs with the results of Arrow and Fisher (1974) in their much simpler two–period model.

Numerical calculations of q_2 and r_2 (in brackets) are presented in Table 3. It can be seen how the error index r_2 increases with σ_z^2 and is largest when growth rates a and b are equal. Thus if, *as seems plausible, there is a great deal of uncertainty in future timber and amenity values, the use of a certainty–equivalence procedure may be considerably sub–optimal.*

6. EXPECTED SURVIVAL TIME

In this section we consider the expected survival time of the old–growth forest under various harvest rules.

Let τ be a random variable denoting the time at which the forest stand is destroyed either by fire or other catastrophe or by a clear–cut harvest. We shall consider harvest rules of the type discussed in Section 5, *viz.* harvest as soon as the ratio $V(t)/A(t)$ reaches some critical level C. If T_C is the first passage time to this boundary with distribution function $F(t)$ and probability density function $f(t)$, then the probability that the stand is still surviving t periods into the future is

$$\text{(36)} \qquad \begin{aligned} P(\tau > t) &= S(t)[1 - F(t)] \\ &= e^{-\lambda t}[1 - F(t)] \end{aligned}$$

for a constant hazard, λ.

The *expected survival time* is (using integration by parts)

$$\text{(37)} \quad E(\tau) = \int_0^\infty P(\tau > t)dt = \frac{1}{\lambda}\left[1 - \int_0^\infty e^{-\lambda t}\ f(t)dt\right]$$

$$= \frac{1}{\lambda}\left[1 - E(e^{-\lambda T_C})\right].$$

The right–hand side of (37) involves the Laplace transform of the first–passage time of Brownian motion with drift to a fixed barrier. Using well–known results for this (*e.g.* Karlin and Taylor (1975), p. 362) gives:

$$\text{(38)} \qquad E(\tau) = \frac{1}{\lambda}\left[1 - \exp\{-(k-z_0)\varphi\}\right]$$

where φ is the *positive* root to

$$\tfrac{1}{2}\sigma_z^2\varphi^2 + (b-a+\tfrac{1}{2}\sigma_z^2)\varphi - \lambda = 0, \tag{39}$$

$k = \ln c$ and $z_0 = x_0 - y_0 = \ln(V_0/A_0)$ which depends on the current (initial) ratio of timber to amenity values.

An alternative form of (38) is

$$E(\tau) = \frac{1}{\lambda}\left[1 - \left[\frac{V_0}{C\ A_0}\right]^{\varphi}\right]. \tag{40}$$

It is assumed here that $V_0/A_0 < C$; if not, a harvest would take place immediately and τ would be zero.

It was shown in Section 5 that the cost–benefit, MLA and certainty–equivalence harvesting rules lead to premature harvesting. This can be expressed quantitatively by comparing expected survival times under the various harvest policies. Using a superscript * to refer to the optimal policy, and subscripts 0 and 1 to refer to the cost–benefit and MLA policies respectively, one gets from (40)

$$E(\tau^*-\tau_0) = \frac{1}{\lambda}\left[\frac{V_0}{A_0}\right]^{\varphi}\left[\frac{1}{C_0{}^{\varphi}} - \frac{1}{C^{*\varphi}}\right] > 0 \tag{41}$$

since $C^* > C_0$ and $\varphi > 0$. Also

$$E(\tau^*-\tau_1) = \frac{1}{\lambda}\left[\frac{V_0}{A_0}\right]^{\varphi}\left[\frac{1}{C_1{}^{\varphi}} - \frac{1}{C^{*\varphi}}\right] > 0. \tag{42}$$

The corresponding value for the certainty–equivalence policy will be (41) or (42) depending on whether $a \geq b$ or $a < b$.

We consider now the effects on expected survival time of changes in certain model parameters.

It was seen in Section 4 that an increase in the parameter a or a decrease in parameter δ leads to an increase in the optimal barrier C^*. It can also be shown (see Appendix 3) that an increase in a leads to an increase in C^*, while φ is unaffected by changes in δ. It follows from (40) that *an increase in the expected growth rate,* a, *of amenity values or a decrease in the discount rate both lead unambiguously to an increase in the expected survival time of the forest under optimal management.* The effects of changes in other parameters is complicated and ambiguous, depending on the initial values V_0 and A_0.

For the cost–benefit rule, the barrier C_0 increases with a and decreases with δ and λ. It does not depend on other parameters. From the results in Appendix 3 it follows that the expected survival time $E(\tau_0)$ increases with increases in a and ρ, but decreases with increases in b, σ_1, σ_2 and δ.

For the MLA rule, the expected survival time $E(\tau_1)$ increases with increases in a and ρ, but decreases with increases in σ_1, σ_2 and δ.

Finally, the loss in expected lifetime using the suboptimal cost–benefit or MLA rules (41) and (42) *increases* with the initial ratio V_0/A_0. This result (which follows directly from equations (41) and (42), on observing that $\varphi > 0$) seems at first sight somewhat surprising. It is due however to the fact that when V_0/A_0 is small, the process has a long way to travel to either of the critical boundaries, and therefore has a high probability of catastrophic destruction before reaching either boundary. The difference in expected lifetimes is therefore small. However when V_0/A_0 is close to the lower (sub–optimal) critical boundary there is a small chance of catastrophic destruction before reaching that boundary and the difference in expected lifetimes is larger. When $\lambda = 0$, the difference in expected lifetimes is independent of the initial ratio V_0/A_0, as a glance ahead at (43) will verify.

Some sample numerical results on expected survival time using the optimal rule and the certainty–equivalence rule are given in Table 4. Two cases are used:

(a) $a = b = .025, \quad \sigma_2^2 = .02, \quad \delta = .05, \quad \lambda = .005$

(b) $a = 0, \quad b = .025, \quad \sigma_2^2 = .02, \quad \delta = .05, \quad \lambda = .005.$

In case (a) the optimal critical ratio of timber value to expected present value of amenity services foregone is $q^* = 1.77$, while the certainty–equivalence critical ratio is given by the cost benefit rule and $q_2 = q_0 = 1.0$ ($C^* = 58.92$, $C_2 = C_0 = 33.33$).

In case (b) the optimal critical ratio of timber value to expected present value of amenity services foregone is $q^* = 2.40$ while the certainty–equivalence critical ratio is given by the MLA rule and is $q_2 = q_1 = 1.83$ ($C^* = 43.71$, $C_2 = C_1 = 33.33$). In both cases in the absence of harvesting the expected lifetime of the stand would be 200 years ($= 1/\lambda$).

Table 2 gives the expected lifetime of the stand in each case for various values of the initial ratio $R_0 = \frac{V_0}{A_0}$ of timber value to amenity flow.

It can be seen that difference in the expected survival time increases with the ratio R_0 up until the critical level for the certainty–equivalence rule. It can be quite large (up to 70 years in case (a)). These results confirm the importance of including uncertainty in the analysis of the harvesting decision problem.

Letting the hazard rate tend to zero in (38) or (40) leads to different limits depending on whether $b - a + \frac{1}{2}\sigma_z^2$ is positive or negative. In the former case $E(\tau)$ tends to

(43)
$$\frac{\ell n(C\ A_0/V_0)}{b - a + \frac{1}{2}\sigma_z^2}$$

while in the latter case it tends to infinity, reflecting the fact that in this case there is a positive probability that the critical barrier level may never be attained. Indeed from well–known results (*e.g.* Karlin and Taylor (1975), p. 362) this probability can be computed as

(44)
$$1 - \left[\frac{V_0}{C\ A_0}\right]^{2|b-a+\frac{1}{2}\sigma_z^2|/\sigma_z^2} > 0.$$

Thus, in the absence of the risk of catastrophic destruction — provided the expected growth rate of amenity values exceeds that of timber values by a suitable amount — there is a positive probability that, under optimal management, the stand will never be harvested.

7. CONCLUDING REMARKS

It has been argued in the Introduction that the decision as to whether to conserve or harvest old–growth forest is a problem amenable to economic analysis. However, in order to address the problem one needs to be able to attach an amenity value to standing old–growth forest. Such forest has value in many ways. It can serve as a locus for recreational activity and tourism; it provides a refuge for wildlife and bestows positive environmental benefits, both local (such as regulation of water flow) and global (such as absorbing carbon dioxide). Apart from these fairly tangible benefits, society is coming to recognize the fact that old–growth forest is valuable simply because it is a part of a vanishing pristine Nature. Like diamonds, or any other economic good, it has value simply because it is simultaneously wanted and scarce.

However, since old–growth forest is a purely public good, there is no natural market mechanism for pricing it. Thus economists have had to turn to more subtle means of determining amenity values (see *e.g.* Price (1989), Chapters 25 and 26). In this article these issues have not been discussed, except to note that there is likely to be considerable uncertainty as to what exactly is the amenity value of a particular area of old–growth forest, and *even more uncertainty as to its future value.*

In the model employed in the paper we have assumed that the current amenity value is known, but future values are unknown, following a stochastic process with known parameter values. A similar assumption has been made with respect to the timber value of the forest. Future uncertainty as to these values lies at the heart of questions of conservation and harvesting, and it has been seen in the paper that increasing levels of uncertainty raise the critical level of the ratio of timber–value to amenity–value which determines in an ongoing fashion whether the forest should be harvested or not. Since there is likely to be a great deal of uncertainty in both future timber values and future amenity values, harvesting may not be optimal until the cash benefits from logging exceed the expected present value of amenity benefits foregone by a large factor.

Another source of uncertainty which has been explicitly included in the model is the possibility that old–growth forest may be destroyed by natural agents such as fire and pest infestations. The presence of such risks has often been used as an argument against any delay in logging ("use it now or lose it to Nature"). The way in which this uncertainty interacts with other uncertainties in determining optimal policy is made explicit in the paper. As one might expect, the presence of a time–independent hazard rate has exactly the same effect in determining the optimal policy as the addition of a premium, equal to the hazard rate, to the effective rate discount. This in turn reduces the expected present value of amenity services foregone through harvesting, thereby lowering the ratio of timber value to amenity value at which harvesting becomes optimal.

In order to implement the decision rule developed in the paper, estimates of model parameters are required. Obtaining such estimates, especially of the uncertainty parameters, presents formidable problems. Policy–wise, probably the best that one can do is to consider various feasible ranges for these parameters and determine optimal policies over the whole range. Another issue which has not been addressed in the paper is how the overall quantity of remaining old–growth forest will affect the amenity value of the area in question. To do this would require a forest–level model for which analytic results are unlikely to be obtainable. However, if one assumes that the total amount of old–growth forest is going to diminish over time, then one can crudely incorporate this effect by assuming a suitable positive value for the mean growth parameter of amenity values.

Footnotes

1. We do not consider the possibility of harvesting part of the old-growth area. For a treatment of that problem with known future timber revenues but uncertain future amenity values see Clarke and Reed (1990).

2. Similar results hold in the case where the Stratonovich calculus is used except possibly when the correlation term ρ is very large (see Clarke and Reed [1988]).

3. These calculations are based on a two standard deviation band for $\ell n(V(t)/A(t))$ after one year which would exhibit a normal distribution with variance σ_z^2.

4. Indeed the cost–benefit analysis decision rule could prescribe a harvest when it is in fact never optimal for one to take place. Such would be the case with probability one if $b > \Delta > a$. If $a > b$, then there is a non–zero probability that a harvest should never take place in cases where the cost–benefit rule prescribes one.

5. This in fact follows directly from a theorem of Miroshnichenko, which states that under fairly general conditions, the MLA stopping region contains the optimal stopping region (see Brock, Rothschild & Stiglitz, 1988, for references).

6. This rule can be derived by simple calculus. Also the same rule arises as the limit of the optimal rule as $\sigma_z^2 \longrightarrow 0$, since in that case, $\theta \longrightarrow \frac{\Delta - b}{\Delta - a}$ if $b > a$ and $\theta \longrightarrow \infty$ if $b \leq q$.

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Appendix 1

Evaluation of the Expected Present Value of Amenity Services

The expected present value (at time 0) of amenity services from time T over an infinite time horizon is (see (9))

$$\text{(A1.1)}\qquad E\left\{\int_T^\infty e^{-(\delta+\lambda)t}\,A(t)dt\right\} = \int_T^\infty e^{-(\delta+\lambda)t}\,E(e^{y(t)})dt$$

where $y(t) = \ln A(t)$; the interchange of the order of the expectation and integration operators is justified by the convergence of the right-hand side.

Now $\{y(t)\}$ is Brownian motion with drift $a - \frac{1}{2}\sigma_2^2$ (see (16)), and thus for $t > T$, $y(t) = y(T) + (a-\frac{1}{2}\sigma_2^2)(t-T) + \sigma_2\sqrt{t - T}\ Z$, where Z is a standard normal random variable. From well-known results on the log-normal distribution it follows that

$$\text{(A1.2)}\qquad E(e^{y(t)}) = \exp\left\{y(T) + (a-\tfrac{1}{2}\sigma_2^2)(t-T) + \tfrac{1}{2}\sigma_2^2(t-T)\right\}$$

$$= A(T)\ \exp\{a(t-T)\}.$$

Substituting this in the right-hand side of (A1.1) and performing the integration gives

$$\text{(A1.3)}\qquad E\left\{\int_T^\infty e^{-(\delta+\lambda)t}\,A(t)dt\right\} = \frac{e^{-(\delta+\lambda)T}\,A(T)}{\delta + \lambda - a}\ .$$

Equation (9) follows.

Appendix 2

Solution of the Optimal Stopping Problem of Section 3

Consider a stopping rule which partitions $x - y$ space into the continuation region $\{x - y < k\}$ and the stopping region $\{x - y \geq k\}$ where k is some, as yet unspecified, constant. This stopping rule prescribes stopping as soon as the ratio $V(t)/A(t)$ reaches, from below, a threshold level, $C = e^k$. Consider also a value function $W(x,y)$ of the form

$$\text{(A2.1(a))} \qquad W(x,y) = B\exp\{(1+\theta)x - \theta y - \theta k\}$$

on the continuation region $\{x - y < k\}$ (where B and θ are as yet unspecified constants), and of the form

$$\text{(A2.1(b))} \qquad W(x,y) = R(e^x, e^y)$$

on the stopping region $\{x - y \geq k\}$, where R is as specified in (13). We shall show that there are particular values of the constants k, B and θ for which W satisfies the conditions set out in Section 3 as sufficient for an optimum (*i.e.* conditions (18), (19), (20), (21) and the condition that $W(x,y) \geq R(x,y)$ on the continuation region).

Before proceeding with the analysis we discuss briefly the reasoning behind specifying a value function of the form A2.1. It can be shown that for a stopping rule of the above type which is a simple barrier rule on $z(t) = x(t) - y(t)$, the associated value function must be of this form. This follows from writing the expectation in (11) as

(A2.2) $$E_\tau\left\{e^{-\delta\tau}\ E_{x(\tau)|x(\tau)-y(\tau)=k}\ [R(e^{x(\tau)},e^{y(\tau)})]\right\},$$

and evaluating the inner conditional expectation using well–known results on Brownian motion and the log–normal distribution. The outside expectation can then by evaluated using, again well–known, results on the moment generating function of the first–passage time of Brownian motion with drift. Full details are given for a similar problem in a paper of Clarke and Reed (1988). Rather than repeat all this detailed analysis (which is in fact unnecessary), we shall start with a value function of the form (A2.1) and find values of the unspecified constants which allow it to satisfy the conditions for an optimum.

Firstly, to satisfy the Hamilton–Jacobi–Bellman equation (18), θ must satisfy the quadratic equation

(A2.3) $$\tfrac{1}{2}\sigma_z^2\theta^2 + (b-a+\tfrac{1}{2}\sigma_z^2)\theta + (b-(\delta+\lambda)) = 0$$

where

(A2.4) $$\sigma_z^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

is the variance parameter for the (Brownian motion with drift) process $\{z(t) = x(t) - y(t)\}$.

Next, to satisfy the continuity condition (19) at the stopping boundary, the condition

$$B\ \exp\{x + \theta(x-y-k)\} = e^x - \frac{e^y}{\Delta - a}$$

must be satisfied on the stopping boundary $x - y = k$.

This reduces to the condition

(A2.5) $$B = 1 - \frac{e^{-k}}{\Delta - a} .$$

To satisfy the smooth–pasting conditions (20) and (21) the condition

$$B(1+\theta) \exp\{x - \theta(x-y-k)\} = e^{x}$$

and

$$-B\theta \exp\{x + \theta(x-y-k)\} = \frac{-e^{y}}{\Delta - a}$$

must be satisfied on $x - y = k$. These reduce to

(A2.6) $$(1+\theta)B = 1$$

(A2.7) $$\theta B = \frac{e^{-k}}{\Delta - a}.$$

Note that (A2.6) is the sum of (A2.5) and (A2.7), so we have only two independent conditions on the two unknowns B and k (θ is determined by (A2.3)). Dividing (A2.6) by (A2.7) gives the optimal barrier

(A2.8) $$C^* = e^{k} = \frac{1}{\Delta - a} \left[\frac{1+\theta}{\theta}\right]$$

and

(A2.9) $$B = \left[1 - \frac{1}{C(\Delta-a)}\right] = \frac{1}{1 + \theta} .$$

To complete the proof of optimality we need only show that the condition $W(x,y) \geq R(e^x,e^y)$ on the continuation region $\{x - y < k\}$ is met. This can be confirmed provided we choose θ to be the *positive* root to the quadratic (A2.3). (The fact that (A2.3) has real roots follows from the assumptions that $\delta + \lambda > a$ and $\delta + \lambda > b$; that they are of different signs follows similarly.)

Using (A2.9), (A2.8) and (A2.5) the value function on $\{x - y < k\}$ can be expressed as

(A2.10) $$W(x,y) = \left[1 - \frac{e^{-k}}{\Delta - a}\right] e^x \, e^{\theta(x-y-k)}.$$

This will be strictly greater than

(A2.11) $$R(e^x,e^y) = e^x - \frac{e^y}{\Delta - a}$$

if

$$e^{\theta(x-y-k)} - 1 > \frac{e^{-k}}{\Delta - a}\left[e^{\theta(x-y-k)} - e^{y-x+k}\right]$$

which on using (A2.8), and some simplification leads to the condition

(A2.12) $$\frac{\theta}{1 - e^{-\theta(y-x+k)}} > \frac{1}{e^{(y-x+k)} - 1}\,.$$

Now since on $x - y \leq k$ the right hand side is strictly bounded above by $1/(y-x+k)$, and the left hand side bounded below by the same quantity (this follows from showing that for $\theta > 0$ the left hand side is increasing in θ, and by

L'Hôpital's rule, that in the limit as $\theta \longrightarrow 0$, it assumes the value $1/(y-x+k))$, it follows that the condition will always be met for $\theta > 0$.

Since θ is the positive root of (A2.3) it follows that the condition $W(x,y) > R(e^x,e^y)$ is met on the continuation region $\{x - y < k\}$ and hence that stopping rule derived above is optimal.

Appendix 3

Effects of Parameter Changes on the Optimal Critical Levels q^* and C^*, and the Expected Survival Time

The optimal barrier C^* for $V(t)/A(t)$ is given by (see (22)):

$$C^* = \frac{1}{\Delta - a}\left[\frac{1+\theta}{\theta}\right] \tag{A3.1}$$

while q^* is given by (see (28))

$$q^* = 1 + \frac{1}{\theta} \tag{A3.2}$$

where θ is the *positive* root to the quadratic equation (23)

$$\tfrac{1}{2}\sigma_z^2\theta^2 + (b-a+\tfrac{1}{2}\sigma_z^2)\theta + b - \Delta = 0. \tag{A3.3}$$

We shall consider the partial derivative of θ with respect to various parameters. First we note that

$$D = b - a + \tfrac{1}{2}\sigma_z^2 + \sigma_z^2\theta > 0. \tag{A3.4}$$

This follows from the fact that from (A3.3)

$$D = b - a + \tfrac{1}{2}\sigma_z^2 + \sigma_z^2\theta = \frac{1}{\theta}\,(\Delta - b + \tfrac{1}{2}\sigma_z\theta^2) > 0.$$

Differentiating (A3.3) implicitly with respect to a, b, σ_z^2 and Δ gives:

(A3.5)

$$\frac{\partial\theta}{\partial a} = \frac{\theta}{D} > 0$$

$$\frac{\partial\theta}{\partial b} = \frac{-(1+\theta)}{D} < 0$$

$$\frac{\partial\theta}{\partial\sigma_z^2} = \frac{-\theta(\theta+1)}{2D} < 0$$

$$\frac{\partial\theta}{\partial\Delta} = \frac{1}{D} > 0.$$

Since $\sigma_z^2 = \sigma_1^2 - 2\rho,\ \sigma_1\sigma_2 + \sigma_z^2$ and $\Delta = \delta + \lambda$, it follows that q* decreases with small increases in a, λ, δ and ρ and decreases with small increases in b, σ_1^2, σ_2^2.

Differentiating C* totally with respect to a, b, σ_z^2 and Δ using the above results gives

(A3.6)

$$\frac{\partial C^*}{\partial a} = \frac{\frac{1}{2}\sigma_z^2(\theta+1)}{\theta(\Delta-a)^2 D} > 0$$

$$\frac{\partial C^*}{\partial b} = \frac{-1}{\theta^2(\Delta-a)}\ \frac{\partial\theta}{\partial b} > 0$$

$$\frac{\partial C^*}{\partial\sigma_z^2} = \frac{-1}{\theta^2(\Delta-a)}\ \frac{\partial\theta}{\partial\sigma_z^2} > 0$$

$$\frac{\partial C^*}{\partial\Delta} = \frac{-1}{(\Delta-a)^2}\left[\frac{1+\theta}{\theta}\right] - \frac{1}{\theta^2(\Delta-a)}\ \frac{\partial\theta}{\partial\Delta} < 0.$$

It follows that C* will increase with small increases in a, b, σ_1, σ_2 and decrease with small increases in ρ, δ and λ.

The expected survival time using a harvest rule determined by a barrier C on $V(t)/A(t)$ is (see (40))

(A3.7)
$$E(\tau) = \frac{1}{\lambda}\left[1 - \left[\frac{V_0}{C\ A_0}\right]^{\varphi}\right]$$

where φ is the positive root to the quadratic equation

$$\tfrac{1}{2}\sigma_z^2\varphi^2 + (b-a+\tfrac{1}{2}\sigma_z^2)\varphi - \lambda = 0.$$

By the same arguments as above (for θ) we obtain

(A3.8)
$$\frac{\partial\varphi}{\partial a} > 0, \quad \frac{\partial\varphi}{\partial b} < 0, \quad \frac{\partial\varphi}{\partial\sigma_z^2} < 0, \quad \frac{\partial\varphi}{\partial\lambda} > 0$$

while, since φ does not depend on δ, $\frac{\partial\varphi}{\partial\delta} = 0$.

Since $(V_0/CA_0) < 1$ we see that $E(\tau)$ will increase with the increase in any parameter (save λ) which simultaneously increases (or leaves unchanged) C and increases (or leaves unchanged) φ.

Similarly an increase in any parameter which decreases (or leaves unchanged) C while decreasing (or leaving unchanged) φ will result in a decrease in $E(\tau)$.

Thus for the optimal barrier, $E(\tau^*)$ increases with a and decreases with δ, while for the cost benefit barrier, $E(\tau_0)$ increases with increases in a and ρ but decreases with increases in b, σ_1, σ_2 and δ_2. For the MLA rule, $E(\tau_1)$ increases with increases in a and ρ, but decreases with increases in σ_1, σ_2 and δ.

Appendix 4

Comparison of the MLA and Optimal Harvesting Rules

The optimal barrier, C^*, for $V(t)/A(t)$ is (from (22))

$$C^* = \left[1 + \frac{1}{\theta}\right]/(\Delta - a), \tag{A4.1}$$

while the MLA barrier is (from (31))

$$C_1 = 1/(\Delta - b). \tag{A4.2}$$

It follows that C^* will be $> C_1$ provided

$$\theta(a-b) + (\Delta - b) > 0. \tag{A4.3}$$

If $a > b$, then this condition is clearly met since $\theta > 0$. If $a < b$, then from the defining equation of θ (25) it follows that

$$(a-b)\theta + (\Delta - b) = \tfrac{1}{2}\sigma_z^2\theta^2 + \tfrac{1}{2}\sigma_z^2\theta > 0. \tag{A4.4}$$

Thus (A4.3) is always met.

		δ = .03 σ_z^2			δ = .05 σ_z^2			δ = .07 σ_z^2		
		0	.01	.02	0	.01	.02	0	.01	.02
a = 0	b = 0	1.00	1.46	1.70	1.00	1.35	1.53	1.00	1.29	1.44
	b = .025	3.50	4.16	4.77	1.83	2.15	2.40	1.50	1.73	1.91
	b = .05	—	—	—	11.00	12.09	13.16	3.00	3.24	3.51
a = .025	b = 0	1.00	1.19	1.36	1.00	1.17	1.31	1.00	1.16	1.28
	b = .025	1.00	2.00	2.62	1.00	1.50	1.77	1.00	1.37	1.55
	b = .05	—	—	—	6.00	7.16	8.27	2.00	2.32	2.62
a = .05	b = 0	—	—	—	1.00	1.10	1.20	1.00	1.10	1.19
	b = .025	—	—	—	1.00	1.19	1.38	1.00	1.17	1.32
	b = .05	—	—	—	1.00	2.62	3.73	1.00	1.55	1.86

TABLE 1. Values of the critical ratio, q^*, of timber value to expected present value of amenity benefits foregone. In all cases the hazard rate was $\lambda = .005$. Blank entries indicate that it is never optimal to harvest.

		δ								
		.03 σ_z^2			.05 σ_z^2			.07 σ_z^2		
		0	.01	.02	0	.01	.02	0	.01	.02
a = 0	b = 0	28.57	41.60	48.46	18.18	24.55	27.76	13.33	17.25	19.17
	b = .025	100.00	118.81	136.16	33.33	39.01	43.71	20.00	23.15	25.57
	b = .05	—	—	—	200.00	219.82	239.35	40.00	43.83	47.42
a = .025	b = 0	100.00	118.81	136.16	33.33	39.01	43.71	20.00	23.14	25.57
	b = .025	100.00	200.00	261.80	33.33	50.00	58.92	20.00	27.40	31.17
	b = .05	—	—	—	200.00	238.74	275.83	40.00	46.96	52.87
a = .05	b = 0	—	—	—	200.00	219.82	239.35	40.00	43.83	47.42
	b = .025	—	—	—	200.00	238.74	275.83	40.00	46.96	52.87
	b = .05	—	—	—	200.00	523.61	746.41	40.00	62.33	74.53

TABLE 2. Values of the critical ratio C^* of timber value to current amenity flow value $(V(t)/A(t))$ for the optimal policy. In all cases the hazard rate was $\lambda = .005$. Blank entries indicate that it is never optimal to harvest.

		δ = .03, σ_z^2 = 0	.03, .01	.03, .02	δ = .05, σ_z^2 = 0	.05, .01	.05, .02	δ = .07, σ_z^2 = 0	.07, .01	.07, .02
a = 0	b = 0	1.0 (0%)	1.0 (31.5%)	1.0 (41.2%)	1.0 (0%)	1.0 (25.9%)	1.0 (34.6%)	1.0 (0%)	1.0 (22.5%)	1.0 (30.1%)
	b = .025	3.50 (0%)	3.50 (15.9%)	3.50 (26.6%)	1.83 (0%)	1.83 (14.9%)	1.83 (31.1%)	1.50 (0%)	1.50 (13.3%)	1.50 (21.5%)
	b = .05	1.17 (–)	1.17 (–)	1.17 (–)	11.00 (0%)	11.00 (9.0%)	11.00 (16.4%)	3.00 (0%)	3.00 (7.4%)	3.00 (14.5%)
a = .025	b = 0	1.0 (0%)	1.0 (16.0%)	1.0 (26.5%)	1.0 (0%)	1.0 (14.5%)	1.0 (23.7%)	1.0 (0%)	1.0 (13.8%)	1.0 (21.9%)
	b = .025	1.0 (0%)	1.0 (50.0%)	1.0 (61.8%)	1.0 (0%)	1.0 (33.3%)	1.0 (43.4%)	1.0 (0%)	1.0 (27.0%)	1.0 (35.5%)
	b = .05	1.08 (–)	1.08 (–)	1.08 (–)	6.00 (0%)	6.00 (16.2%)	6.00 (27.4%	2.00 (0%)	2.00 (13.8%)	2.00 (23.7%)
a = .05	b = 0	1.0 (–)	1.0 (–)	1.0 (–)	1.0 (0%)	1.0 (9.1%)	1.0 (16.7%)	1.0 (0%)	1.0 (9.1%)	1.0 (16.0%)
	b = .025	1.0 (–)	1.0 (–)	1.0 (–)	1.0 (0%)	1.0 (16.0%)	1.0 (27.5%)	1.0 (0%)	1.0 (14.5%)	1.0 (24.2%)
	b = .05	1.0 (–)	1.0 (–)	1.0 (–)	1.0 (0%)	1.0 (61.8%)	1.0 (73.2%)	1.0 (0%)	1.0 (35.5%)	1.0 (46.3%)

TABLE 3. Values of the critical ratio q_2, of timber value to expected present value of amenity benefits foregone for the certainty–equivalence harvest rule and in brackets the degree, r_2, to which q_2 is too small compared with the optimal q^*. Dashes in brackets occur in cases when it is never optimal to harvest.

	$R_0 = V_0/A_0$	5	10	15	20	25	30	35	40	45	50	55
(a)	$E(\tau^*)$	175.8	156.2	138.0	120.7	104.0	87.8	71.9	56.4	41.2	26.2	11.5
	$E(\tau_2)$	160.6	128.6	99.0	70.8	43.7	17.2	0	0	0	0	0
	difference	15.2	27.5	39.0	49.9	60.3	70.5	71.9	56.4	41.2	26.2	11.5
(b)	$E(\tau^*)$	176.9	154.0	131.1	108.2	85.3	62.5	39.7	16.9			
	$E(\tau_2)$	169.8	139.7	109.7	79.7	49.8	19.9	0	0			
	difference	7.2	14.3	21.4	28.5	35.5	42.6	39.7	16.9			

TABLE 4. Expected survival times in years using the optimal rule $(E(\tau^*))$ and using the certainty–equivalence harvest rule $(E(\tau_2))$ for two cases (a) and (b) described in the text. In case (a) the certainty–equivalence rule corresponds to the cost–benefit rule while in case (b) it corresponds to the MLA rule.