

The Effect of Using Multiple Representations
on Student Success in Solving Rational, Radical,
and Absolute Value Equations and Inequalities

by

Kip Wood
B.Sc., University of Victoria, 1989

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Abstract

Because of an emerging body of research and a wealth of classroom experiences, the National Council of Teachers of Mathematics adopted representation as a process standard in 2000 to add to the four previously adopted (1989) process standards - problem solving, reasoning and proof, connections, and communication. The purpose of this study was to determine the effect of using multiple representations on learning to solve equations at the grade 11 level. The sources of data included three unit test scores for two different groups over a seven-week period prior to the treatment and a test score for the two groups after the treatment.

Because of small class sizes (23 and 26), the statistical results need to be verified in future studies with larger groups. However, the results of this study indicate that students benefit from a multi-representational approach to equation solving. Students in the experimental group chose the graphing method of solving equations more often than the algebraic method and had more success with graphing compared to the algebraic approach. However, on some questions, students in the experimental group scored lower with the graphical method than the algebraic method. These results indicate that teachers need to weigh the benefits of teaching mathematical concepts with more than one representation against the costs of the learning demands placed on the student.

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CHAPTER ONE INTRODUCTION

The National Council of Teachers of Mathematics (NCTM) identified five Content Standards (number and operations, algebra, geometry, measurement, and data analysis and probability) and five Process Standards (problem solving, reasoning and proof, connections, communication, and representation) in their landmark document *Principles and Standards for School Mathematics* (2000). Whether the language used is processes and content or abilities and knowledge, the identified Standards are inextricably linked and are interdependent (NCTM, 2000). Norris and Phillips (2003) argued that, historically, the content part of scientific literacy has been emphasized while the process part of scientific literacy has been neglected. In the development of mathematical literacy, as in scientific literacy, content and process are both essential parts of an interconnected curriculum. In this thesis, the Process Standard representation will be investigated as a component of mathematical literacy and a possible link to one of the unifying concepts of mathematics – the concept of function.

The Common Curriculum Framework for K-12 Mathematics (Western Canadian Protocol, 1996) used the Standards of the NCTM (originally published in 1989) to provide consistent learning outcomes for students in Western (and now Northern) Canada. Four of the five Process Standards identified by the NCTM (problem solving, communication, connections and reasoning) are mirrored in the *Framework* document as “Mathematical Processes”:

“Students are expected to:

- Relate and apply new mathematical knowledge through problem solving

- Reason and justify their thinking
- Connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines
- Communicate mathematically
- Use visualization to assist in processing information, making connections and solving problems.” (p. 5)

The fifth Process Standard, representation, is defined as an internal system in the *Framework* document (1996) under the heading “Visualization” (p. 11). Also, the *Framework* document includes a discussion entitled the “Nature of Mathematics” (p. 12), which identifies expectations related to the concepts of change, constancy, dimension, number, pattern, quantity, relationships, shape, and uncertainty. Specific to the representation Process Standard: “Relationships will be described visually, symbolically, orally and in written form” (p. 14). More recent research has shifted the focus from students becoming fluent users of representations to an awareness that representation is a cognitive and social process that is linked to generating understanding (Monk, 2003). The identification of representation as a meaning-making process rather than simply a vehicle for communicating ideas establishes an important link to help teachers facilitate students’ acquisition of mathematical content knowledge.

In the British Columbia *Integrated Resource Package (IRP)* (2000), four of the five Process Standards of the NCTM are again echoed with the following subheadings: “Becoming Mathematical Problem Solvers”, “Communicating Mathematically”, “Connecting and Applying Mathematical Ideas”, and

“Reasoning Mathematically” (p. 2-3). The *IRP* (2000) makes reference to the representation standard in their definition of numeracy (mathematical literacy): “Becoming numerate involves developing the ability to explore, conjecture, reason logically, and use a variety of mathematical methods to solve problems” (p. 1). The *IRP* (2000) also refers to representation when assessing the work of students with special needs: “Allow students to demonstrate their understanding of mathematical concepts in a variety of ways” (p. 10). Like the *Framework* document (1996), there is more emphasis in the *IRP* (2000) on using multiple representations for communicating mathematical ideas than there is on the use of representations to acquire conceptual understanding.

Purpose

The specific research question for the treatment groups is: In comparison to using an algebraic approach only, does the use of multiple representations (algebraic, numeric, and graphic) have an effect on students’ test scores for rational, radical, and absolute value equations and inequalities?

Definitions

This section provides a definition of multiple representations and transfer problems as they are to be used in the study. Also, professional associations and curriculum documents are defined.

Principles of Mathematics 11

An academic mathematics course in the British Columbia curriculum that covers topics such as functions (quadratic, polynomial, and rational), equations and inequalities (rational, radical and absolute value), circle geometry, and coordinate geometry.

Multiple Representations

The NCTM (2000) states that instructional programs should enable all students (prekindergarten to grade 12) to

- “Create and use representations to organize, record, and communicate mathematical ideas,
- Select, apply, and translate among mathematical representations to solve problems, and
- Use representations to model and interpret physical, social, and mathematical phenomena” (p. 360)

Outlining the purpose of using multiple representations in the classroom, the NCTM (2000) states that:

“Representations should be treated as essential elements in supporting students’ understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments and understandings to one’s self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling.” (p. 67)

Translation

The process of going from one mode of representation to another, for example, from an equation to a graph (Janvier, 1987).

Transfer Problems

An example of a transfer problem is one that is presented graphically, algebraically, numerically or verbally and the solution is presented using another representation. Cunningham (2005) used the term *transfer problem*.

National Council of Teachers of Mathematics (NCTM)

The NCTM is an organization dedicated to providing leadership toward its goal of improving the mathematics education of all students. The NCTM produced three standards documents – *Curriculum and Evaluation Standards for School Mathematics* (1989), *Professional Standards for Teaching Mathematics* (1991), and *Assessment Standards for School Mathematics* (1995). In 2000, the NCTM published *Principles and Standards for School Mathematics* and as a result, teachers, administrators, researchers, and policy makers have benefited from its coherent and focussed vision for mathematics education. *Principles and Standards* is the most influential document for curriculum design for school mathematics in North America as well as a comprehensive resource for anyone interested in school mathematics.

Western and Northern Canadian Protocol (WNCP)

The WNCP was formed in 1993 to provide a common curriculum framework for the western provinces (British Columbia, Alberta, Saskatchewan, and Manitoba) and the territories (Yukon, Northwest and Nunavut). Initially, it was called the Western Canadian Protocol (WCP) and the mathematics document

was the first to be released – *The Common Curriculum Framework for K-12 Mathematics* (1996). The *Framework* document of the WCP was influenced by the *Standards* documents of the NCTM.

Integrated Resource Package (IRP)

The *IRP* is the curriculum document published by the British Columbia Ministry of Education. The *IRP* includes suggested instructional strategies, prescribed learning outcomes, illustrative examples and various other resources for teachers. The *IRP* fits with the WCP and the NCTM vision for school mathematics. The *IRP* used in this study (Mathematics 10 to 12) was published in 2000.

Significance of the Study

In 1987, Claude Janvier of the University of Quebec at Montreal edited a book called *Problems of Representation in the Teaching and Learning of Mathematics*. The articles compiled in this volume are written by many of the researchers involved in investigating the role of representations in school mathematics. Janvier's book was a catalyst in this area of research and had a lasting impact on curriculum design. The contributors to this book and other researchers in this area come from a constructivist, rather than behaviourist, philosophy of learning. The shift from behaviourism to constructivism can be summarized by saying that students should be organizing their own experiences rather than replicating what the teacher demonstrates.

If the goal is to develop understanding rather than train specific performance, then teacher knowledge must consist of more than rules and procedures. The role of the teacher is not to dispense information; rather, it is to

guide the student in organizing their educational experiences (Von Glasersfeld, 1987). The constructivist philosophy does not treat knowledge as a transferable commodity and meaning is not simply given to students. Instead, students have a personal interaction with the concepts they are studying so that they can connect and broaden previous understanding (Ellis, 2003). Consequently the satisfaction of learning mathematics has little to do with rewards for specific performance outcomes and more to do with the experience of assimilating and organizing new concepts into what is already known.

The philosophy of constructivism goes back to Piaget in the 1930's and further back to the worldview of Descartes in the 17th century. However, research on the interplay between a student's internal representations and the conventional representations established throughout the history of mathematics has only received significant attention from researchers in the past two decades.

Much of the available research literature on the role of representation in mathematics education came the book *Problems of Representation in the Teaching and Learning of Mathematics* (Janvier, 1987) and after the NCTM published the *Curriculum and Evaluation Standards for School Mathematics* in 1989. The original *Standards* document (1989) did not contain the representation Process Standard. Also, the *Framework* document (1996) and the *IRP* (2000) were both written before the *Principles and Standards for School Mathematics* (2000) was available. When the Western and Northern Canadian Protocol and the British Columbia Ministry of Education publish their next curriculum documents, it is likely that all five Process Standards, including representation, will be clearly articulated.

This study will provide information to teachers who want to implement the NCTM Process Standard representation into their mathematics classes. It is hoped that the suggestions will be helpful to teachers and will provide insights into the obstacles that need to be overcome in order to use representations effectively. Finally, the study may support the need for more research in this area so that the representation Process Standard can be supported in future curriculum revisions.

CHAPTER TWO LITERATURE REVIEW

The NCTM guide that is now used to help various jurisdictions focus their curricula, *Principles and Standards for School Mathematics* (2000), states that:

“Instructional programs from prekindergarten through grade 12 should enable all students to -

- Create and use representations to organize, record, and communicate mathematical ideas
- Select, apply, and translate among mathematical representations to solve problems, and
- Use representations to model and interpret physical, social, and mathematical phenomena” (p. 360).

It is clear that the *Standards* (2000) identify using multiple representations as a priority to help students communicate mathematically but also to help students learn mathematics. Greeno and Hall (1997) stated that various forms of representations should not be taught as ends in themselves. Goldin (2003) stated that representation refers to process and to product and goes on to suggest that a representational system be considered because the relationship between representations gives meaning to mathematical situations. Monk (2003), in his paper *Learning to Graph and Graphing to Learn*, discussed two uses of graphs - a tool for communication and a tool for generating meaning through exploration and analysis. With this view of multiple representations as a bi-directional construct, the teacher of mathematics not only aims to

communicate specific characteristics of mathematical phenomena but also forces the student to examine familiar concepts in unfamiliar ways.

When solving mathematical problems, students choose specific representations in order to solve the problem but also to convey their mathematical ideas. The choice of representation may determine whether a student is successful in solving a problem but it may also determine if the student can communicate effectively. Stonewater (2002) found that the highest scoring writers in a college-level calculus class used appropriate language and notation, built context with the use of examples, and used a greater variety of representations to complement their written work. In doing so, these students were better able to express their ideas than students who scored lower on writing tasks.

The thoughtful use of multiple representations in the classroom helps students convey their understanding to other students and in doing so, shared understanding can be promoted. Lesser (2000) found that visual learners complement the perspectives of students that learn through verbal descriptions, by working with algebraic symbols, or by working with numbers in a table. If students do not have opportunities to investigate multiple representations, then the opportunity to share understanding is limited. Fisher and Hartmann (2005) found that blind students were able to give the same descriptions as sighted students when the chosen representation was appropriate for the problem situation. Comparing the descriptions given by blind and sighted students provided the authors with a test to determine if the representation was a powerful tool for communicating mathematical concepts.

How a student chooses to solve a mathematical problem is based on their prior knowledge and the expectations of their teacher. However, Friedlander and Tabach (2001) suggest that the choice of representation is based on the nature of the task, the personal preference and thinking style of the student, and attempts to overcome difficulty with another representation. It is the teacher's role to connect students' personal representations to conventional representations such as tables, graphs, and symbols. If the knowledge to connect the mathematical model to the intuitive model is not available, then the student will acquire separate formal and intuitive understanding (van Someren & Tabbers, 1998). Interpreting various representations and translating between them is a process that allows teachers to determine what students understand (Coulombe & Berenson, 2001). In this way, multiple representations are a powerful tool that teachers can use in their ongoing assessment of how students are learning mathematical concepts.

If students have a deficiency in computational ability and they believe that they have a deficiency, then they are less likely to pursue further study in mathematics. However, if mathematics is presented to students in problem situations that are familiar to them, then they do not need to be excluded from classes that investigate more advanced topics (Hadley, 1992). In order to access students' prior knowledge and motivate them to continue studying mathematics, teachers need to consider using multiple representations so that more learners can be reached.

Teachers, while using a variety of pedagogical approaches, must be able to distinguish between meaningful and rote learning of mathematics. In order to

implement strategies that incorporate the use of multiple representations, teachers must evaluate their approaches critically and distinguish between a classroom where the teacher states rules and imparts knowledge and an approach where the student detects patterns, makes conjectures, and investigates problems. In the constructivist view, meaningful mathematical knowledge is not simply communicated as a static body of facts and rules; instead, the individual organizes their own understanding and multiple representations are a tool to facilitate the process of construction.

The Role of Technology

Technology contributes to the development of mathematical literacy in terms of process and content. For instance, graphing calculators and graphing software allow teachers to communicate, and students to investigate concepts such as function behavior because the technology allows the user to represent functions numerically, algebraically, and graphically. Using graphing calculators allows more opportunities for “what if” questions and for exploration (Dunham & Dick, 1994).

Yerushalmy and Shternberg (2001) emphasize that technology allows the graph, rather than the algebraic symbols or numerical data, to be the leading representation and they state that this approach promotes experimentation by students. Technological tools such as the graphing calculator allow the student to move gradually to the algebraic method of solving a problem. Rojano (2002) suggests that technology is the intermediate language between the verbal and algebraic representations of mathematical concepts.

Technology allows teachers the opportunity to emphasize certain representations that may have been neglected in the past. Dickey (1993) states that teachers emphasize verbal and algebraic representations and neglect numerical and graphical representations. With the graphing calculator, translation to and between numerical and graphical representations is possible provided that the learning demands that come with the technology are met in the context of the activity.

With the graphing calculator, assessment has presented some new challenges for teachers. Tasks that students used to do have become trivial with the graphing calculator and preserving the meaning of the concept that teachers want their students to learn requires a new approach if the technology is to be used appropriately. Thompson, Beckman, and Senk (1997) suggest helpful ways that teachers can alter test items in order to use graphing calculators effectively. For instance, students can be asked to identify the equation of a given graph rather than graph a given equation.

Heid, Zbiek, and Blume (2004) explain how topics such as rate of change, the nature of linearity, the meaning of rational and negative exponents, and the comparison of function families take on new importance in technological settings. Technology should not be regarded as merely a tool for performing manipulations and producing graphical representations nor should technological devices be restricted to the purpose of tutoring students through tasks that get progressively more complex. The most powerful application of technology in mathematics education is when students are empowered to do the work of a

mathematician rather than simply acquiring mathematical knowledge (Jones et al., 2002).

However, technology does more than alter the process of acquiring procedural and conceptual understanding in mathematics; it changes the content of what can be investigated. With the graphing calculator and other display media, the type of problems that can be investigated by students is no longer limited to the set of problems that were investigated prior to the technology. For instance, graphing calculators allow students to find points of intersection much more quickly and accurately so that analyzing graphs and investigating function behaviour becomes the focus of the exercise rather than the tedious computation.

Kuhn (1996) suggests that the assimilation of new theory is not just an increment added to what is already known; rather, prior knowledge has to be re-evaluated and reconstructed. Kuhn (1996) goes on to say that the tradition that emerges from innovation is no longer compatible with what came before. Now that graphing technology is commonplace, teachers need a deeper knowledge of algebra concepts that they taught when the technology was not readily available.

Translation between the representations can be carried out with ease using graphing technology; however, teachers and students face new learning demands in order to effectively use the technology. In addition, technological support must be available, effective, and ongoing. Stenning (1998) states that the combinations of display media available now could not have been anticipated but he warns that the benefits may be overstated unless the technological methods stand up to evaluation.

Multiple Representations as Part of Instruction and Assessment

It is important that students learn to use conventional systems of representation to communicate ideas and to solve problems; however, the student must form a mental image, or internal representation, in order to use what was taught. For mathematics educators, the translation process between and among representational modes, internal and external, should be developed so that standard and external representations do not become isolated outcomes (Goldin, 2003). Part of the role of the teacher is to help the student connect their personal images with more conventional representations (NCTM, 2000). In this way, teachers gain insight into how students learn mathematics and how students connect their knowledge to what is expected of them.

Understanding a mathematical concept requires that a student have opportunities to experience a diversity of problems related to that concept. For instance, the concept of function is a unifying part of the algebra and geometry Content Standards identified by the NCTM (2000) and is developed throughout the school years (K-12). To achieve this ongoing curricular goal, students need opportunities to translate between representations of the same concept and mathematics teachers need to require that students translate from one representation to another while solving a problem (transfer problems). Clement and Sowder (2003) suggest that students must be able to connect the various representations of a function in order to understand functions.

Lesh, Post, and Behr (1987) identify five representation systems: experience-based knowledge, manipulatives, pictures and diagrams, spoken language, and written symbols. When a student understands an idea or a concept,

three conditions are met which relate to the five representation systems. The student can recognize the idea embedded in the representation, the student can manipulate the idea within the representation, and the student can translate the idea from one representation to another (Lesh, Post, & Behr, 1987). This research suggests that good problem solvers are flexible in their use of representations and can switch to a more suitable representation when the need arises.

In the classroom, teachers can present a problem in one representation (graphically for instance) and ask the student to solve the problem and provide a solution in another representation (perhaps algebraically). For a primary student, numerical data for the growth of a plant over time may be transferred to a line graph and a conclusion may be recorded in words. A grade twelve student may be presented with a verbal description of a population over time and be required to model the growth with an appropriate exponential function.

The literature shows that students struggle with the graphical representation and are reluctant to use graphs in order to solve problems (Kieran, 1992). Kerslake (1981) found that students, when working with linear functions, were more successful at recognizing the graph of a linear equation than they were at producing the equation of a given straight line. Stenning (1998) states that students tend to translate into the graphical mode and fail to translate in the reverse direction. Furthermore, the classroom experiences of Moore-Russo and Golzy (2005) suggest that students often use algebraic symbols prior to understanding the meaning behind the symbols. Polya (1957) suggests that understanding a problem is the first step in solving the problem. If algebra is

introduced first, then sometimes the meaning vanishes and the problem situation is reduced to a procedure rather than a transferable concept.

The literature indicates that teachers favour certain representations over others and emphasize certain translations between representations. Cunningham (2005) found that graphic to numeric transfer problems were presented less often by teachers and appeared less frequently on assessments. Piez and Voxman (1997) also found that students have difficulty in reading information from a graph. In order to overcome the difficulty in interpreting graphs, Bridger and Bridger (2001) suggest that teachers use mapping diagrams with a domain axis and a range axis (both vertical) in order to emphasize the correspondence between x and $f(x)$. The other five transfer problems in the study by Cunningham (2005) were graphic to algebraic, algebraic to graphic, algebraic to numeric, numeric to algebraic, and numeric to graphic. Choosing and building representations and translating between representations, together with interpretive skills, far outstrip computational skills in importance (Kaput, 1987a). This research indicates that teachers, in order to develop students' interpretive skills, not only need knowledge about representations but awareness of how the representations emphasize and suppress certain aspects of mathematical phenomena.

The representation that is used by a teacher influences how a student constructs meaning of a mathematical concept. The goal in using multiple representations, from a teacher's perspective, is to make mathematical ideas, abstract or otherwise, more accessible to students. In order to be effective with this strategy, the teacher must consider that the representation chosen should be

as close as possible to students' internal representations (Dufour-Janvier, Bednarz, & Belanger, 1987). Also, the teacher must be aware that some external representations may be inaccessible and perhaps counterproductive to learning because the students may not see that the same concept is embodied in all the representations presented.

Secondary school students usually define functions in terms of a rule of correspondence that is consistent over the entire domain of the function and is represented by a formula (Markovits, 1986). The trouble that students encounter is when they have to work with functions represented by discrete points. In teaching the concept of function, Hitt (1998) found that teachers prefer continuous functions that are represented by a single algebraic formula and prefer to define functions in terms of correspondence and ordered pairs rather than using the concept of variable. More research is needed to determine which representations should be chosen and how they can be used effectively for various age groups and in various mathematical contexts. Also, more research is needed to investigate the difficulties that teachers have in being able to translate and articulate between different representations.

One of the difficulties for mathematics teachers is determining when and how to use translation and when to avoid the use of translation altogether. Ainsworth (1999) described a functional taxonomy of multiple representations and argued that three distinct learning situations need to be considered before translation between representations is expected. If multiple representations are used to complement other information and past understanding, Ainsworth (1999) suggested that translation be avoided but if multiple representations are

used to constrain possible misconceptions then translation to a more familiar representation should be used. Prain (2004) cautioned that if children have relatively weak conceptual knowledge then translation to other representations reaffirmed those weaknesses. Ainsworth (1999) went on to state that if multiple representations are used to develop deeper understanding, then translation should be used to achieve abstraction and generalization.

Izsak and Sherin (2003) found when teachers learn to use translation as a pedagogical technique and are able to make connections between conventional representations and student-generated representations of the same concept, they gain access to student thinking. If students are only exposed to procedures and rules to solve problems, it is difficult to access their prior knowledge (Steele, 2005). If students are continually exposed to a single representation, they are forced to interpret the teacher's representational preference. However, in a multi-representational environment students are more likely to select or create a representation suitable to the problem situation (Aspinwall & Shaw, 2002). As well, students are more likely to demonstrate their prior knowledge and communicate their ideas if they are working with multiple representations.

Aside from the numeric, algebraic, and graphic representations, students can be encouraged to verbalize and write about their problem solving and level of understanding. Burton and Morgan (2000) argued that learning to think mathematically is facilitated by taking part in the discourse of mathematics. Writing is one way that students can construct mathematical knowledge by communicating their thoughts (Connolly & Vilardi, 1989). Bell and Bell (1985) conducted a study of grade nine students and found that writing in combination

with problem solving had a positive effect on achievement compared to students who did not write to solve problems.

Translating between representations is particularly difficult for students, even when the concepts are familiar to them. Even (1998) found that college mathematics students did not immediately switch to the graphic representation of a quadratic function when they were faced with a problem situation that lent itself to a graphic solution. Quadratic functions are part of a high school student's repertoire but college students in this study had difficulties linking different representations and analyzing the behaviour of the function over its entire domain (Even, 1998). This research reveals that students may be familiar with producing graphs of certain function families but may not know how to analyze the graph and extract the information that is unique to the representation.

It has been suggested that in order for learners to benefit from a multiple representation approach, three learning demands must be met. That is, the student must understand how a representation presents information, how the representation is related to that information, and how representations are related to each other (Ainsworth, 1998). Lesgold (1998) suggests that multiple representations should be used but cautions that the challenge with this approach is the coordination of those representations. In order to use multiple representations successfully the tradeoff between the benefits to support understanding and the cost of the learning demands placed on students must be evaluated (Ainsworth, 1998). This research points out the need to assess the difficulties that mathematics teachers need to overcome in order to use a multi-representational approach.

Cognition and Instruction in Mathematics

What students learn is connected to how they learn it (NCTM, 1991). A pedagogy that utilizes multiple representations allows the teacher to establish a discourse that is focused on exploring mathematical concepts, rather than just reporting answers and carrying out procedures. The teacher decides when it is appropriate to attach conventional mathematical ideas and notation to students' ideas during this ongoing discussion. In this way, the teacher's role is to help the student develop the concept prior to the traditional notation and procedures.

The role of language is important when considering cognitive development. Language supports the communication of mathematical ideas but it also supports the learning of new ideas. For children, ideas have more to do with the process of doing mathematics intuitively than with the proper use of terminology and notation. However, children need sufficient language to allow them to understand their peers and their teachers' explanations (Perry & Dockett, 2002). Eventually, language will allow the student to give their own explanations and share their understanding. Multiple representations help the student to link their intuitive knowledge to the language and conventions of mathematics.

It is possible to make complex mathematical ideas accessible to students at a variety of levels (Carpenter & Lehrer, 1999). A strategy teachers can use in order to make these ideas accessible to children is to use multiple representations in their mathematics lessons. Again, there is a recognition that children do mathematics intuitively in their daily lives and have prior knowledge that can be built upon (Jones, Langrall, Thornton, & Nisbet, 2002).

One of the aims of teaching is to reach as many learners as possible. Multiple representations allow the teacher to convey the same information in different ways. Since some learners assimilate certain representations more easily than others, communication is improved. The other purpose of using multiple representations is to enhance conceptual understanding. However, Yore and Treagust (2006) state that little research has been done on the enhanced cognition that occurs when students transfer from one representation to another. Dufour-Janvier, Bednarz, and Belanger (1987) demonstrated that students who were presented three different representations of the same problem did not see that the same concept was embodied in all three situations. Until more research is done, teachers can only rely on enhanced communication, and not necessarily enhanced cognition, if they adopt multiple representations as their pedagogical approach.

Curriculum Design

The role of representation is not restricted to the teaching and learning of mathematics; it could help to inform the direction of future research and alleviate the controversy that has surrounded mathematics education and subsequent curriculum design for decades. In this way, representation systems are educational goals as well as educational tools (Stenning, 1998). Deciding the best way to learn mathematics has devolved into a debate of ideologies and the result has been a cycle in which reforms are adopted, abandoned, and replaced with new reforms or traditional models, which in turn are abandoned and replaced. Goldin (2002) suggests that traditional and reform models have been inadequate

on their own but have both created helpful constructs that should not be dismissed entirely. The representational perspective that emphasizes the interplay between contextual mathematics and abstract mathematics allows us to see that context and abstraction are complementary rather than opposing orientations.

An aspect of the mathematics curriculum that is often excluded is a discussion about the historical development of ideas and concepts. Kaput (1987b) states that the historical growth of mathematics is due partly to the invention of new representations. Students appreciate how mathematical concepts developed through the use of new ways to represent ideas (Crites, 1995). When students consider problems and are prevented from using the best available notation in order to solve the problem they soon realize that developing new representations has facilitated innovations throughout the history of mathematics. With the latest technology, students get the opportunity to see that new representations of mathematical phenomena will continue to develop.

Goldin (1987) states that curriculum designers should not merely identify objectives in a way that suggests students should be able to solve certain types of problems. Instead, he suggests that the major goal of mathematics education is to foster the development of representational systems as cognitive processes. Goldin (1987) goes on to say that teachers should be encouraged to choose activities that facilitate the development of representational systems. If we continue to state our instructional goals as behavioural objectives and performance outcomes, then we are ignoring internal cognitive processes that lead to those outcomes. In other words, we need to teach students how to think rather than simply what to think.

The process of learning mathematics and the content that needs to be learned are not separate. Eisner (1997) suggests that the link between forms of representation and cognition have been neglected in the development of curricula. Different representations emphasize and suppress certain information but they also demand and develop different cognitive skills. Eisner (1997) emphasizes that the decisions that are made about what will be accessible to students influence the processes and products of thinking.

From the behaviourist point of view, debate around curriculum development in mathematics has focused on content knowledge and observable learning outcomes, instead of how the outcomes are achieved by students. Today's curriculum documents have statements about students' attitudes and habits but the prescribed learning outcomes and illustrative examples of what students should know are contained in a separate chapter or appendix. What is lacking is a clearly articulated link between the content and the processes of mathematical literacy. A possible path to establishing that link is teachers and curriculum developers using a multi-representational approach; as a result, more learners may be reached, more meaningful mathematics may be explored, and deeper conceptual understanding may be achieved.

Summary of Pedagogical Assumptions

In order to justify the design of this research, the philosophical orientation outlined in *Principles and Standards for School Mathematics* (NCTM, 2000) acts as a model for how mathematics should be taught and how it is learned. The teaching principle states that:

“Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well” (p. 16).

The learning principle states that:

“Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge” (p. 20).

The NCTM articulates a constructivist philosophy throughout *Principles and Standards* (2000) as well as in their previous *Standards* documents (1989, 1991, and 1995).

The NCTM (1991) states that teachers must help students construct their own representations and, in order to do this, need to be comfortable with representations that are familiar to students. Regarding assessment, multiple sources of evidence are needed in order to make inferences about students’ understanding of mathematical concepts. Using a variety of representations allows the teacher to assess understanding of concepts rather than just the mastery of procedures (NCTM, 1995). Accessing prior knowledge and allowing students to demonstrate their knowledge in a variety of ways are the guiding principles used to justify a pedagogical approach that utilizes multiple representations.

Based on the research, it is apparent that multiple representations are a powerful tool for communicating mathematical ideas and, to a lesser degree, there is also support for using multiple representations to enhance students’ conceptual understanding of mathematics. This study aims to investigate the benefits of using multiple representations in the mathematics classroom.

CHAPTER THREE METHOD

Design of the Study

This study used a two-group posttest non-equivalent groups research design. This approach is classified as a quasi-experimental design since the study used intact classes and participants were not randomly assigned to the experimental and control groups. Because groups may not be equivalent, analysis of covariance (ANCOVA), with the aggregate score from preceding unit exams as the covariate, was used. This method permits the researcher to statistically eliminate any initial differences between the experimental and control groups on the variable of interest (Gall, Gall, & Borg, 2005). However, as in all quasi-experimental designs, the results should be interpreted cautiously as there are considerably more threats to internal validity than with a true experimental design (Creswell, 2002) and the ANCOVA can over-estimate differences between the groups.

Location and Participants

This study took place in a Vancouver Island secondary school in a moderate size city. The socioeconomic status of the students was below average as the school is an inner-city school. Also contributing to the participants' backgrounds is the fact that the school has an accelerated mathematics program where 30 to 60 students (out of about 160) each year enter grade eight and enroll directly in mathematics at the grade nine level. Because of this program, the participants in this study are generally below average in achievement.

Two grade 11 classes of 26 and 23 students were participants in this study. Because of course selections dictating the construction of the timetable, the participants were not randomly selected and as a result, the two classes were not necessarily of the same ability. Students participated in the study on a voluntary basis and were recruited by a colleague who distributed and collected the consent forms (see Appendix A).

The two classes that made up the treatment groups in this study were all 15 to 17 years old. There was one international student (out of 26) registered in the control class and one international student (out of 23) in the experimental class. These two international students were the only students who had English as a second language. The control class had 12 girls and 14 boys; the experimental class had 18 girls and five boys.

Treatments

The control class and the experimental class were taught using the same lesson plans for the first three units of the course (September/ October, 2005). These three units covered the concepts of quadratic, polynomial, and rational functions. The fourth unit involved solving rational, radical, and absolute value equations and inequalities and was taught using two different methods. The control class was only exposed to an algebraic representation of the content while the experimental class was taught using algebraic, numeric, and graphic representations. The *Integrated Resource Package* (2000) summarized the prescribed learning outcomes for this unit in one statement. That is, students are expected to:

“formulate and apply strategies to solve absolute value equations, radical equations, rational equations and inequalities” (p. G-198).

The graphic method of solving problems was used before the algebraic method in the experimental class so those students could visualize and understand the problem prior to manipulating symbols. Graphing calculators were also used to present the numeric (tabular) representation in the experimental class. The numeric representation was not used as a method of solving problems but students in the experimental class were instructed to use the “table” function on their graphing calculators after they had solved the problem graphically.

An example of the algebraic method used to solve a radical inequality, as it was taught to the control class, is as follows:

$$\sqrt{2x+7} - 9 < 0$$

$$\sqrt{2x+7} < 9$$

$$2x + 7 < 81$$

$$2x < 74$$

$$x < 37$$

However, since the radicand cannot be negative:

$$2x + 7 \geq 0$$

$$2x \geq -7$$

$$x \geq \frac{-7}{2}$$

So, the solution is $\frac{-7}{2} \leq x < 37$

The same problem was taught to the experimental group by using the graph as the leading representation. The graph is shown below as it appeared to students on their graphing calculators:

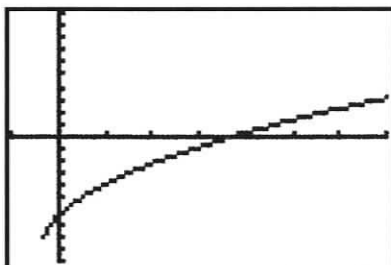


Figure 1. The graph of $y = \sqrt{2x + 7} - 9$.

This method of solving the problem involves identifying the domain (possible x-values) from the graph of $y = \sqrt{2x + 7} - 9$ and then identifying the x-values that correspond to negative y-values. Using the graphing calculator to find zeros and points of intersection was demonstrated to students so they could solve the problem correctly. The table function was used to reinforce the idea that the radical function was increasing and would eventually have a value of zero.

Since I taught both classes before, during, and after the treatment, teacher bias was considered and minimized where possible. The same three examples for each of the six types of problems were used in the lessons taught to both treatment groups and are included in Appendix B. The questions for the fourth unit test which was administered at the end of the treatment phase are included in Appendix C.

Data Collection

Prior to the research, the Human Research Ethics Board at the University of Victoria had to approve the study (see Appendix E). Also, School District 68 had to give permission for the study. See Appendix D for the letter to the Superintendent and his reply granting permission.

The pre-treatment phase was September 6 to October 25, 2005 and included the first three units of the Principles of Mathematics 11 course. Each student had a score that represented an average of his or her first three unit exams. The mean and standard deviation of the pre-treatment phase was calculated so that the two treatment groups could be compared statistically for the first three units of the course. The fourth unit was used as the treatment phase in the study and was taught October 26 to November 2, 2005. The mean and standard deviation of both groups was calculated for the fourth unit exam in order to examine the effect of using a multi-representational approach.

Data Analysis

The mean scores of the two treatment groups for the first three units combined were compared using a two-tailed t-test for independent samples in order to determine if the two classes were significantly different at the 0.05 level before the treatment. With the possibility of the two groups being significantly different prior to the treatment phase, an analysis of covariance was conducted using the aggregate score from the first three units as the covariate so that differences resulting from the treatment could be interpreted.

The test for the fourth unit was tested for reliability using the estimate called Coefficient Alpha (Cronbach, 1951).

Limitations of the Study

In this study, the participants were not randomly assigned to the groups because of timetable constraints. The control class consisted of 26 students and the experimental class consisted of 23 students. Rather than a pretest, three unit tests were used to establish baseline data for the pre-treatment phase.

Comparison of the treatment groups was carried out in the fourth unit – Rational, Radical and Absolute Value Equations and Inequalities. Because this study was quasi-experimental and involved a posttest only, it is difficult to make strong cause and effect inferences from the results (Gall, Gall, & Borg, 2005).

Despite the limitations, when the mean scores of the two classes were compared and analyzed statistically, information for educators and researchers emerged.

CHAPTER FOUR RESULTS

Comparison of Treatment Groups on Units 1-3

In order to determine if the scores achieved by the control group were significantly different than the scores achieved by the experimental group prior to the treatment, a two-directional t-test for independent samples was used. At the 0.05 level of significance, there was not sufficient evidence to conclude that the control group scored higher than the experimental group on the first three units of the Mathematics 11 course ($t(1,47)=1.45, p=0.15$). Since the two groups were different, although not significantly different, prior to the treatment, a conservative analysis of the data was done to address the problem of non-equivalence. An analysis of covariance was done with the aggregate score from units 1-3 as the covariate. The data for the pre-treatment phase are shown in TABLE 1.

TABLE 1. Achievement of Control Group vs. Experimental Group (Pre-Treatment).

<u>Statistics</u>	<u>Control</u>	<u>Experimental</u>
Number of Students	26	23
Mean (Units 1-3)	69.53	63.36
Standard Deviation	13.71	15.78

Comparison of Treatment Scores to Pre-Treatment Scores

Unit Four was the treatment phase of the study. Two-directional t-tests for related samples were used to determine if there was a significant difference between the scores of each group prior to the treatment compared to their scores on Unit Four. For the control group, it was found that there was no significant difference at the 0.05 level between the pre-treatment phase and the treatment phase ($t(25)=1.53$, $p=0.14$). Also for the experimental group, it was found that there was no significant difference at the 0.05 level between the pre-treatment phase and the treatment phase ($t(22)=0.48$, $p=0.63$). The data for this analysis are provided in TABLE 2.

TABLE 2. Achievement of Students on the Treatment Phase with Comparison to Pre-Treatment Phase.

<u>Statistics</u>	<u>Control</u>	<u>Experimental</u>
Number of Students	26	23
Mean (Units 1-3)	69.53	63.36
Mean (Unit 4)	63.78	61.60

Comparison of Treatment Groups on Unit 4

After the treatment was applied, the mean score of the control group and the experimental group were once again compared using a two-directional t-test for independent samples. At the 0.05 level, it was found that there was no

significant difference between the two groups ($t(47)=0.36, p=0.72$). The data for this analysis are summarized in TABLE 3.

TABLE 3. Control Group vs. Experimental Group on Treatment Phase.

Statistics	Control	Experimental
Number of Students	26	23
Mean (Unit 4)	63.78	61.60
Standard Deviation	22.48	20.23

In summary, the control group did not score significantly higher on the pre-treatment phase and there was no significant difference between the two mean scores on the treatment unit.

Reliability of Instruments

The reliability estimate, Coefficient Alpha, was calculated for the six test items that were used in the treatment phase on both groups. For the control group, the reliability estimate was 0.64 and for the experimental group, the reliability estimate was 0.62. This level of reliability is moderate and could have been improved if the unit four test had more questions and also if it had questions over a range of difficulty (Cronbach, 1951).

Analysis of Covariance

The mean squares between the groups for the pre-treatment (units 1-3) compared to the mean squares within the groups yielded an observed $F=0.25$ and $p=0.62$. Therefore, the ANCOVA confirmed that there was no significant difference in achievement between the two groups on the unit 4 exam.

Representations Used by Experimental Group

In the pre-treatment phase, students in both the control group and the experimental group were taught algebraic and graphical approaches in their work with quadratic, polynomial, and rational functions, equations and inequalities. Graphing calculators were available in both classes and students were encouraged to use the technology to verify their algebraic results and generate solutions when the algebraic approach was impractical.

In the treatment phase, the students in the control group were only taught to use an algebraic method to solve the six different types of problems that were required in the fourth unit. The 26 students who participated in the control group each completed a six-item test (see Appendix C). Of the 156 responses (26x6) on students' tests, 154 (99%) were answered algebraically and two items had no response.

The 23 students who participated in the experimental group were taught to use both the algebraic and graphical methods to solve the six problem types. The students in the experimental group completed the same test as the control group (see Appendix C). Of the 138 responses (23x6) on students' tests, 69 (50%)

were answered graphically, 62 (45%) were answered algebraically and seven items (5%) had no response.

Students in the experimental class chose to solve the rational problems (59% of responses) and the absolute value problems (67% of responses) by using graphs whereas algebra was the preferred method for the radical problems (70% of responses). Overall, the graphing method was preferred (50%) to the algebraic method (45%). No students chose the table of values (numeric) to solve the problems. Students' choices for each item are summarized in TABLE 4.

TABLE 4. Representation Chosen for Each Test Item (Experimental Group).

Question Type	Graphic	Algebraic	No response
Rational Equation	13	10	0
Rational Inequality	14	8	1
Rational (total)	27 (59%)	18 (39%)	1 (2%)
Radical Equation	7	16	0
Radical Inequality	4	16	3
Radical (total)	11 (24%)	32 (70%)	3 (7%)
Absolute Value Equation	16	5	2
Absolute Value Inequality	15	7	1
Absolute Value (total)	31 (67%)	12 (26%)	3 (7%)
Total of All Responses	69 (50%)	62 (45%)	7 (5%)

Success Rates for Each Method

Overall, students who answered the questions with a graphical method had a success rate of 69% whereas the students who answered questions algebraically got 60% correct. With the radical inequality (Appendix C: #4), students scored 81% with the graphing method but only 39% with the algebraic method. However, on two of the six questions, the experimental class was more successful using algebra to solve the problem. Students' success rates on each type of problem are summarized in TABLE 5.

TABLE 5. Success Rates for Each Representation on Each Question (Experimental Group).

<u>Question Type</u>	<u>Graphic</u>	<u>Algebraic</u>
Rational Equation	75%	63%
Rational Inequality	48%	44%
Radical Equation	82%	86%
Radical Inequality	81%	39%
Absolute Value Equation	92%	75%
Absolute Value Inequality	48%	54%
Total	69%	60%

CHAPTER FIVE DISCUSSION AND CONCLUSIONS

The purpose of this study was to investigate the effect of using multiple representations as opposed to using an algebraic approach only, on learning how to solve specific types of equations from the British Columbia Principles of Mathematics 11 course. To accomplish this, mean scores achieved by two classes were compared during a pre-treatment phase then compared again with the results of the treatment.

The results of the t-test (TABLE 1) indicate that the two mean scores were not significantly different prior to the treatment ($p=0.15$) and were not significantly different after the treatment phase ($p=0.72$). In order to address the possibility of having non-equivalent treatment groups to start with, an analysis of covariance was done. The ANCOVA did not reveal that the two groups were significantly different ($p=0.62$). However, the mean scores of the two classes did converge, indicating that the method of using multiple representations had a positive effect. However, it is impossible to make a strong case that multiple representations were the cause of any benefit to the scores observed in the experimental group. The mean scores of the two groups were further apart than on the sections of the course where the instruction did not differ. Although the sample sizes were small and the treatment period short (six days), the experimental group did seem to benefit from the multi-representational approach.

Both treatment groups scored lower on the treatment phase than on the pre-treatment phase. However, neither class dropped significantly in their

achievement from the pre-treatment to the treatment phase (TABLE 2). The treatment phase (unit four) is difficult for Grade 11 students traditionally, based on 16 years experience of teaching Mathematics 11. However, the mean score for the control class decreased more than the mean score for the experimental group. This indicates that the experimental class may have benefited from the opportunity to solve the equations in more than one way.

The students in both classes had the freedom to choose how they solved the problems on the test that was used in the treatment phase. In the control class, all the students chose an algebraic method to solve the equations. The students interpreted the algebraic method that was modeled to them in class and did not try a graphing method on any of the problems. Students in both classes have access to graphing calculators – most students own them – but the students in the control class were not encouraged to try alternative approaches and only did what was modeled to them.

The students in the experimental group chose a graphical method 50% of the time, an algebraic method 45% of the time, and 5% of the questions had no response. The radical equation and inequality required fewer algebraic steps than the other questions so students chose to do an algebraic method on 32 of the items versus 11 graphical solutions (TABLE 4). The questions involving radicals were answered correctly over 80% of the time except for the algebraic approach to the radical inequality that was only 39% correct (TABLE 5). The graph of the radical inequality reveals that the graph terminates on the left; however, if students solved the question algebraically they had to remember the restriction on the domain without seeing it. The discrepancy in success between the

graphical and algebraic methods for the radical inequality indicates that students can identify domain and range when they are presented with a graph but find it difficult to find the domain when given an algebraic equation.

With the rational and absolute value questions, students in the experimental class chose the graphing method most of the time (TABLE 4) but had more success with the graphing method on the equations than the inequalities (TABLE 5). When an equation was solved on the graphing calculator, the students were shown how to graph the left side and the right side of the equation separately and find the point of intersection. Solving an inequality graphically requires that the student identify the x -values where one graph is above the other. This process involves interpreting and analyzing the graph. The 48% success rate indicates that students had difficulty with this interpretation (TABLE 5).

Students in the experimental class who were taught graphing and algebraic methods did not always choose the method that led them to a higher success rate. With the absolute value inequality, 15 students chose the graphing method while only seven chose the algebraic approach yet the latter method was more successful than the former. Ainsworth (1998) reminds teachers to consider the balance between the benefits of using multiple representations and the costs of the learning demands placed on the student as a result of the multi-representational approach. Solving the absolute value inequality graphically required the student to interpret the respective y -values of the two images as well as find points of intersection. As a result of the learning demands of the graphing method, students were more successful solving this question algebraically.

Recommendations for Future Research

This study showed that using multiple representations is beneficial to students when solving equations and inequalities. Future studies need to investigate why some students choose representations that end up being less successful for them. Interviewing students may allow researchers to gain insights into why students choose certain representations when solving problems.

Several landmarks in the research have initiated a move toward a multi-representational approach to teaching mathematics. The group of researchers that Claude Janvier brought together in 1987 to contribute to the book *Problems of Representation in the Teaching and Learning of Mathematics* influenced the work of many others in the area of representation. Much of this practical knowledge for teachers was articulated in *Principles and Standards for School Mathematics* published by the National Council of Teachers of Mathematics in 2000. These documents clearly outline the benefits of using multiple representations and they identify the obstacles that need to be overcome in order to use representations effectively.

The body of knowledge that has been developed mainly by classroom teachers over the past two decades continues to grow. Teachers have written many articles for journals such as *Mathematics Teacher*, which support the use of multiple representations across the curriculum. This wealth of tacit knowledge needs to be developed further in the future and needs to be accessible to researchers. Conversely, the research in the area of representation has to be accessible to teachers of mathematics at all levels.

Teachers know that a multi-representational approach to teaching mathematics works for their students but they do not always understand why it works. The research indicates that using multiple representations increases teachers' ability to communicate mathematical concepts to their students and, as a result, students become better at communicating their mathematical ideas. An area that needs attention from researchers in the future is finding out which representations are preferred by students so that advantages and disadvantages of different representations can be shared with students. This will allow students to choose appropriate representations for various situations and become more powerful problem solvers. For now, the research indicates strongly that communication is enhanced when teachers use multiple representations. However, research that establishes the link between enhanced cognition and multiple representations needs further attention.

The benefits of using multiple representations are outlined in *Principles and Standards for School Mathematics* (NCTM, 2000). The four areas identified are supporting understanding, improving communication, recognizing connections, and applying concepts to problem situations. The latter three reasons in this list are related to reinforcing the other Process Standards of the NCTM (communication, connection, and problem solving). These standards are justified by a wealth of classroom experience and well documented by the NCTM in their journals such as *Mathematics Teacher*. Understanding mathematical concepts and the cognitive processes involved in acquiring conceptual understanding and the role that representations have in facilitating those processes is poorly understood and needs more work by researchers.

The other Process Standard identified by the NCTM is reasoning and proof. The role of representation as it relates to the process of doing mathematics needs to be investigated. Specifically, the strategies that are believed to improve cognitive processes need to be tested in future research projects.

The role of technology also needs attention in future research. Researchers such as Ainsworth have investigated the specific role that technology can play in utilizing multiple representations for specific purposes. These findings need to be confirmed and expanded to other grade levels so educators can use technology appropriately.

Recommendations for Educators

The experimental class in this study worked with graphing calculators every day during the treatment phase. Before this technology was available, graphing was a process that involved generating ordered pairs, predicting the behaviour of functions, and transforming those functions. With the graphing calculator, it was possible with the experimental group to start with the graph instead of a table of values or an algebraic formula. Immediately the students had a visual understanding of the problem to be solved before they proceeded to solve it. The students in the experimental group also had a tendency to experiment and ask “what if” questions because they had a broader view of the problems to be solved and did not view the problems as algorithms to be memorized.

Providing the opportunity to work with multiple representations of a mathematical concept made classroom discussion a richer experience for the students in the experimental group. The questions that students asked and the

comments that they made allowed me to assess their knowledge in a deeper way. For instance, solving a radical equation algebraically involves isolating the radical, squaring both sides, and checking for extraneous roots. Students generally accept this algorithm and set out to master it. When faced with a graphical representation of the same equation, the students were able to observe immediately if the two functions representing the two sides of the equation had a point of intersection or not. A connection was made to the importance of the extraneous root and the domain of the respective functions. When the students proceeded with the algebraic method, there was more appreciation for extraneous roots and why some equations have no solution. The advantage of seeing the graph first was that some students were able to anticipate the algebraic method. However, the link between representations must be reinforced otherwise students may not realize that the different representations embody the same concept.

Using multiple representations allows teachers to explain mathematical concepts in ways that are more accessible to students. If teachers are able to reach more students, then the effort is worthwhile. If students are experiencing a mathematics education that models the rote learning of procedures in a single representation, then the meaning for students may be lost. Educators may argue that learning outcomes are being met because students are performing well on tests that measure their proficiency in carrying out procedures. However, with multiple representations teachers can make connections between what a student already knows and the conventions of mathematics. With multiple representations, teachers are better equipped to assess the needs of their students

and make mathematical concepts meaningful for them. In doing so, flexible and powerful problem solvers can be developed. Mathematical knowledge has to be constructed by students and, in order to make the knowledge meaningful, students have to fit new ideas into an internal representational system. Teachers can move toward accessing this internal system and facilitating its development by using a multi-representational approach to teaching mathematics.

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Appendix A: Participant Consent Form

You are being invited to participate in a study called, “The Effect of Using Multiple Representations Versus Algebraic Representation on Learning the Concept of Function” that is being conducted by me (Mr. Wood). I am a graduate student in the Department of Curriculum and Instruction (Faculty of Education) at the University of Victoria. As a graduate student, I am required to conduct research as part of a thesis for my Master’s Degree. I can be contacted at (250) 753-2271 or kwood@uvic.ca. My research is being conducted under the supervision of Dr. Leslee Francis-Pelton; you may contact her at (250) 721-7794 or lfrancis@uvic.ca. You are being asked to participate in this study so that information can be obtained on the effect of using multiple representations in Mathematics 11.

Participation in the study is voluntary. If you agree to participate, your test scores for one unit will be used in the study; however, your identity and the identity of the school will not be revealed. Average scores will be compared and only these scores will be reported in my thesis. If you choose to withdraw from the study, contact the teacher who distributed the consent forms and your test scores will not be used (this does not mean you are withdrawing from the course!).

There are no anticipated risks to you by participating in this research but there may be some benefits. For instance, teachers in the future may gain some knowledge about how to enhance students’ understanding of functions.

In addition to being able to contact the supervisor, you may verify the ethical approval of this study, or raise concerns you might have, by contacting the Human Research Ethics Assistant at the University of Victoria at (250) 472-4545 or ovprhe@uvic.ca. Your signature below indicates that you understand the above conditions of participation in this study and that you have had the opportunity to have your questions answered.

Name of Participant

Signature

Signature of Parent

Date

Appendix B: Examples used in Lessons (Alexander & Kelly, 1998)

Rational Equations:

1. $\frac{16}{x-1} = -4$

2. $\frac{-2}{x+3} - \frac{5}{x} = 2$

3. $\frac{x-3}{x-4} + \frac{3x-5}{x-3} = x+2$

Rational Inequalities:

4. $\frac{-8}{x} < -2$

5. $\frac{-4}{3x-4} \geq x-4$

6. $\frac{2}{x-3} < \frac{1}{2x}$

Radical Equations:

7. $2\sqrt{x} = 12$

8. $\sqrt{2x+7} = 5$

9. $-2\sqrt{6x+1} = 14$

Radical Inequalities:

10. $3\sqrt{x} \leq 15$

11. $\sqrt{3x-2} > 10$

12. $\sqrt{2x+7} - 9 < 0$

Appendix B: Continued

Absolute Value Equations:

13. $|x - 4| = 2$

14. $|x + 2| = 1$

15. $|x + 1| = x - 1$

Absolute Value Inequalities:

16. $|x + 1| < 9$

17. $|x - 5| \geq 4$

18. $|3x - 2| \leq x + 1$

Appendix C: Unit Four Test Items

Students were required to solve the following equations and inequalities.

1. $\frac{x-5}{x+1} = x-3$

2. $\frac{7}{x} + 4 \geq 3x$

3. $\sqrt{5x+2} = 6$

4. $8 + \sqrt{4x+1} > 10$

5. $|3-2x| = x+4$

6. $|x-2| \geq 7$

Appendix D: Letter to Superintendent
July 25, 2005

Attention: Mr. Rick Borelli, Superintendent of Schools, SD#68

I am a graduate student in the Department of Curriculum and Instruction (Faculty of Education) at the University of Victoria. As a graduate student, I am required to conduct research as part of a thesis for my Master's Degree. My research is being conducted under the supervision of Dr. Leslee Francis-Pelton; you may contact her at (250) 721-7794 or lfrancis@uvic.ca. I am requesting permission to have my Mathematics 11 students participate in my research so that information can be obtained on the effect of using multiple representations in Mathematics 11. A consent form for the students is attached.

If students agree to participate, their test scores will be used in the study; however, the identity of the students and the identity of the school will not be revealed. There are no anticipated risks to students by participating in this research. If students choose to withdraw from the study, their test scores will not be used.

In addition to being able to contact the supervisor, you may verify the ethical approval of this study, or raise concerns you might have, by contacting the Human Research Ethics Assistant at the University of Victoria at (250) 472-4545 or ovprhe@uvic.ca.

Thank you for considering this research. I look forward to your response.

Sincerely,

Kip Wood

cc. Mr. Lee Venables (Principal)