

SOME APPLICATIONS OF THE GENERALIZED
HYPERGEOMETRIC FUNCTION INVOLVING CERTAIN
SUBCLASSES OF ANALYTIC FUNCTIONS[†]

by

H.M. SRIVASTAVA* and SHIGEYOSHI OWA

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ABSTRACT

Two familiar subclasses of analytic functions are considered here: the class $\mathcal{S}(\alpha)$ of analytic functions $f(z)$ satisfying the inequality

$$\operatorname{Re}\{f(z)/z\} > \alpha \quad (0 \leq \alpha < 1),$$

and the class \mathcal{R} of analytic functions whose derivative has a positive real part. The object of this paper is to present several interesting applications of the generalized hypergeometric function, which involve the classes $\mathcal{S}(\alpha)$ and \mathcal{R} , and the concept of subordination between analytic functions. A theorem on the radius of univalence for a certain class of generalized hypergeometric functions is also established.

1. INTRODUCTION AND DEFINITIONS

Let \mathcal{A} denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. A function $f(z)$ belonging to the class \mathcal{A} is said to be in the class $\mathcal{S}(\alpha)$ if it satisfies the following inequality:

$$(1.2) \quad \operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \alpha \quad (z \in \mathcal{U})$$

for $0 \leq \alpha < 1$.

The class $\mathcal{S}(\alpha)$ was introduced by Goel [5], and was studied subsequently by Chen ([2], [3]). In particular, the class $\mathcal{S}(0)$ was studied by Goel [6] and Yamaguchi [18].

A function $f(z)$ belonging to the class \mathcal{A} is said to be in the class \mathcal{R} if it satisfies the following inequality:

$$(1.3) \quad \operatorname{Re}\{f'(z)\} > 0 \quad (z \in \mathcal{U}).$$

The class \mathcal{R} was introduced by MacGregor [11].

In order to recall the concept of subordination between analytic functions, let $f(z)$ and $g(z)$ be analytic in the unit disk \mathcal{U} . The function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $h(z)$ analytic in the unit disk \mathcal{U} , with $h(0) = 0$ and $|h(z)| < 1$, such that

$$(1.4) \quad f(z) = g(h(z))$$

for $z \in \mathcal{U}$. We denote this subordination by

$$(1.5) \quad f(z) \prec g(z).$$

In particular, if $g(z)$ is univalent in the unit disk \mathcal{U} , the subordination (1.5) is equivalent to (cf. [4], p. 190)

$$(1.6) \quad f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

The concept of subordination can be traced back to Lindelöf [8], although Littlewood ([9], [10]) and Rogosinski ([14], [15]) introduced the term and established the basic results involving subordination. More recently, Suffridge [17], and Hallenbeck and Ruscheweyh [7], studied various families of subordinate functions.

Finally, let α_j ($j = 1, \dots, p$) and β_j ($j = 1, \dots, q$) be complex numbers with

$$\beta_j \neq 0, -1, -2, \dots; \quad j = 1, \dots, q.$$

Also let $(\lambda)_n$ denote the Pochhammer symbol defined by

$$(1.7) \quad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1) \dots (\lambda+n-1), & \text{if } n \in \mathcal{N} = \{1, 2, 3, \dots\}. \end{cases}$$

Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by (cf., e.g., [16], p. 33 et seq.)

$$(1.8) \quad {}_pF_q(z) \equiv {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n} \frac{z^n}{n!} \quad (p \leq q + 1).$$

It may be recalled that the ${}_pF_q(z)$ series in (1.8) converges absolutely for $|z| < \infty$ if $p < q + 1$, and for $z \in \mathcal{U}$ if $p = q + 1$. Furthermore, if we set

$$(1.9) \quad \omega = \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j,$$

then the ${}_pF_q(z)$ series (1.8), with $p = q + 1$, is absolutely convergent for

$$(1.10) \quad |z| = 1 \quad \text{if } \operatorname{Re}(\omega) > 0,$$

and conditionally convergent for

$$(1.11) \quad |z| = 1, \quad z \neq 1, \quad \text{if } -1 < \operatorname{Re}(\omega) \leq 0.$$

We now introduce the following class of generalized hypergeometric functions:

DEFINITION. The generalized hypergeometric function ${}_pF_q(z)$ defined by (1.8) is said to be in the class $\mathcal{A}(p;q;\alpha)$ if it satisfies the following inequality:

$$(1.12) \quad \operatorname{Re} \left\{ {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \right\} > \alpha \quad (z \in \mathcal{U})$$

for $0 \leq \alpha < 1$.

Making use of the above definitions, we present several interesting applications of the generalized hypergeometric function ${}_pF_q(z)$, which involve the classes $\mathcal{S}(\alpha)$ and \mathcal{H} , and the concept of subordination between analytic functions. We also establish a theorem on the radius of univalence for a certain class of generalized hypergeometric functions.

2. APPLICATION INVOLVING SUBORDINATION BETWEEN ANALYTIC FUNCTIONS

Our first application of the generalized hypergeometric function ${}_pF_q(z)$ depends upon a result due to Nehari [12, p. 168], which we recall here as

LEMMA 1. Let $\phi(z)$ be analytic in the unit disk \mathcal{U} and satisfy $|\phi(z)| \leq 1$ for $z \in \mathcal{U}$. Then

$$(2.1) \quad |\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \quad (z \in \mathcal{U}).$$

By using Lemma 1 and the concept of subordination, we shall prove

THEOREM 1. Let the generalized hypergeometric function ${}_p F_q(z)$ defined by
 (1.8) belong to the class $\mathcal{A}(p; q; \alpha)$. Then

$$(2.2) \quad \left| {}_p F_q(\alpha_1+1, \dots, \alpha_p+1; \beta_1+1, \dots, \beta_q+1; z) \right| \\
 \leq \frac{\left| \prod_{j=1}^q \beta_j \right|}{\left| \prod_{j=1}^p \alpha_j \right|} \left(\frac{2(1-\alpha)}{(1-|z|)^2} \right),$$

where

$$(2.3) \quad \prod_{j=1}^p \alpha_j \prod_{j=1}^q \beta_j \neq 0.$$

The result (2.2) is sharp.

PROOF. We note that

$$(2.4) \quad \operatorname{Re} \left(\frac{1 + (1-2\alpha)z}{1-z} \right) > \alpha \quad (z \in \mathcal{U}).$$

Hence, by virtue of the definition of subordination, we obtain

$$(2.5) \quad {}_p F_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) < \frac{1 + (1-2\alpha)z}{1-z}$$

for $z \in \mathcal{U}$. Thus we may write

$$(2.6) \quad {}_p F_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \frac{1 + (1-2\alpha)w(z)}{1-w(z)},$$

where $w(z)$ is analytic in the unit disk \mathcal{U} , with

$$(2.7) \quad w(0) = 0 \quad \text{and} \quad |w(z)| < 1.$$

Differentiating both sides of (2.6), we get

$$(2.8) \quad {}_pF'_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \frac{2(1-\alpha)w'(z)}{[1-w(z)]^2}.$$

Note that

$$(2.9) \quad {}_pF'_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \\ = \frac{\prod_{j=1}^p \alpha_j}{\prod_{j=1}^q \beta_j} {}_pF_q(\alpha_1+1, \dots, \alpha_p+1; \beta_1+1, \dots, \beta_q+1; z),$$

and that

$$(2.10) \quad |w(z)| \leq |z| \quad \text{for} \quad z \in \mathcal{U},$$

by the Schwarz lemma.

Applying Lemma 1 to $w(z)$, we find that

$$(2.11) \quad \left| {}_pF_q(\alpha_1+1, \dots, \alpha_p+1; \beta_1+1, \dots, \beta_q+1; z) \right| \\ \leq \frac{\prod_{j=1}^q \beta_j}{\prod_{j=1}^p \alpha_j} \cdot \frac{2(1-\alpha)}{[1-|w(z)|]^2} \cdot \frac{1-|w(z)|^2}{1-|z|^2} \\ \leq \frac{\prod_{j=1}^q \beta_j}{\prod_{j=1}^p \alpha_j} \left(\frac{2(1-\alpha)}{(1-|z|)^2} \right)$$

provided that the condition (2.3) holds true.

Finally, by taking the generalized hypergeometric function defined by

$$(2.12) \quad {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \frac{1 + (1-2\alpha)z}{1-z},$$

we readily verify that the result (2.2) is sharp.

3. APPLICATION INVOLVING THE CLASS $\mathcal{S}(\alpha)$

We need the following result given by Chen [3]:

LEMMA 2. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}(\alpha)$.

Then, for $0 \leq |z| < \frac{1}{2}$,

$$(3.1) \quad \operatorname{Re}\{f'(z)\} \geq \frac{1 + 2(2\alpha-1)|z| + (2\alpha-1)|z|^2}{(1+|z|)^2},$$

and, for $\frac{1}{2} \leq |z| < 1$,

$$(3.2) \quad \operatorname{Re}\{f'(z)\} \geq \frac{\alpha - 2\alpha|z|^2 + (2\alpha-1)|z|^4}{(1-|z|^2)^2}.$$

The results (3.1) and (3.2) are sharp.

By using Lemma 2, we now prove

THEOREM 2. Let the generalized hypergeometric function ${}_pF_q(z)$ defined by (1.8) belong to the class $\mathcal{A}(p;q;\alpha)$. Then, for $0 \leq |z| < \frac{1}{2}$,

$$(3.3) \quad \operatorname{Re} \left\{ {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 2; \beta_1, \dots, \beta_q, 1; z) \right\} \\ \cong \frac{1 + 2(2\alpha-1)|z| + (2\alpha-1)|z|^2}{(1+|z|)^2},$$

and, for $\frac{1}{2} \cong |z| < 1$,

$$(3.4) \quad \operatorname{Re} \left\{ {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 2; \beta_1, \dots, \beta_q, 1; z) \right\} \\ \cong \frac{\alpha - 2\alpha|z|^2 + (2\alpha-1)|z|^4}{(1-|z|^2)^2}.$$

The results (3.3) and (3.4) are sharp.

PROOF. Define a function $H(z)$ by

$$(3.5) \quad H(z) = z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)$$

for $z \in \mathcal{U}$. Since ${}_pF_q(z)$ is in the class $\mathcal{A}(p; q; \alpha)$, we have

$$(3.6) \quad \operatorname{Re} \left\{ \frac{H(z)}{z} \right\} > \alpha \quad (z \in \mathcal{U}),$$

which implies that $H(z) \in \mathcal{S}(\alpha)$.

Applying Lemma 2 to $H(z)$, we find that

$$(3.7) \quad \operatorname{Re} \{ H'(z) \} \cong \frac{1 + 2(2\alpha-1)|z| + (2\alpha-1)|z|^2}{(1+|z|)^2}$$

for $0 \cong |z| < \frac{1}{2}$, and that

$$(3.8) \quad \operatorname{Re}\{H'(z)\} \geq \frac{\alpha - 2\alpha|z|^2 + (2\alpha-1)|z|^2}{(1-|z|^2)^2}$$

for $\frac{1}{2} \leq |z| < 1$.

Now it is not difficult to verify that

$$(3.9) \quad \begin{aligned} H'(z) &= z {}_pF'_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \\ &\quad + {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \\ &= {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 2; \beta_1, \dots, \beta_q, 1; z), \end{aligned}$$

which, in conjunction with (3.7) and (3.8), yields the assertions (3.3) and (3.4) of Theorem 2.

Finally, by taking the generalized hypergeometric function defined by

$$(3.10) \quad {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) * {}_1F_0(2; -; z) = \begin{cases} \frac{1 + 2(\alpha-1)z + (2\alpha-1)z^2}{(1+z)^2} & (0 \leq |z| < \frac{1}{2}), \\ \frac{\alpha - 2\alpha z^2 + (2\alpha-1)z^4}{(1-z^2)^2} & (\frac{1}{2} \leq |z| < 1), \end{cases}$$

we can show that the results (3.3) and (3.4) are sharp; here $f(z) * g(z)$ denotes the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$.

4. APPLICATION INVOLVING THE CLASS \mathcal{R}

In order to apply the generalized hypergeometric function ${}_pF_q(z)$ to the class \mathcal{R} , we require the following lemma due to MacGregor [11]:

LEMMA 3. Let the function $f(z)$ defined by (1.1) be in the class \mathcal{H} .

Then

$$(4.1) \quad |a_n| \leq \frac{2}{n} \quad (n \geq 2),$$

$$(4.2) \quad |f'(z)| \leq \frac{1 + |z|}{1 - |z|} \quad (z \in \mathcal{U}),$$

$$(4.3) \quad \operatorname{Re}\{f'(z)\} \geq \frac{1 - |z|}{1 + |z|} \quad (z \in \mathcal{U}),$$

$$(4.4) \quad |f(z)| \geq -|z| + 2 \log(1 + |z|) \quad (z \in \mathcal{U}),$$

and

$$(4.5) \quad |f(z)| \leq -|z| - 2 \log(1 - |z|) \quad (z \in \mathcal{U}).$$

We now establish

THEOREM 3. Let the generalized hypergeometric function ${}_pF_q(z)$ defined by (1.8) belong to the class $\mathcal{A}(p; q; \alpha)$. Then

$$(4.6) \quad \left| \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \right| \leq 2(1 - \alpha) \cdot n! \quad (n \geq 1),$$

$$(4.7) \quad \left| {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \right| \leq \frac{1 + (1 - 2\alpha)|z|}{1 - |z|} \quad (z \in \mathcal{U}),$$

$$(4.8) \quad \operatorname{Re}\left\{ {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \right\} \geq \frac{1 - (1 - 2\alpha)|z|}{1 + |z|} \quad (z \in \mathcal{U}),$$

$$(4.9) \quad \left| {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 1; \beta_1, \dots, \beta_q, 2; z) - \alpha \right| \\ \cong (\alpha-1) + \frac{2(1-\alpha) \log(1+|z|)}{|z|} \quad (z \in \mathcal{U} - \{0\}),$$

and

$$(4.10) \quad \left| {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 1; \beta_1, \dots, \beta_q, 2; z) - \alpha \right| \\ \cong (\alpha-1) + \frac{2(\alpha-1) \log(1-|z|)}{|z|} \quad (z \in \mathcal{U} - \{0\}).$$

The results (4.6) to (4.10) are sharp.

PROOF. We introduce a function $G(z)$ defined by

$$(4.11) \quad G(z) = \int_0^z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; t) dt \\ = z {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 1; \beta_1, \dots, \beta_q, 2; z)$$

for $z \in \mathcal{U}$. It follows from (4.11) that

$$(4.12) \quad \operatorname{Re}\{G'(z)\} = \operatorname{Re}\left\{{}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)\right\} \\ > \alpha.$$

Next we define a function $G_1(z)$ by

$$(4.13) \quad G_1(z) = \frac{G(z) - \alpha z}{1 - \alpha}$$

for $z \in \mathcal{U}$. Then it is easily verified that

$$(4.14) \quad \operatorname{Re}\{G_1'(z)\} = \operatorname{Re}\left\{\frac{G'(z) - \alpha}{1 - \alpha}\right\} > 0, \quad z \in \mathcal{U},$$

which, by the definition (1.3), implies that

$$G_1(z) \in \mathcal{R}.$$

Noting that

$$(4.15) \quad G_1(z) = z + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n (1)_n}{(\beta_1)_n \cdots (\beta_q)_n (2)_n (1-\alpha)} z^{n+1},$$

and applying Lemma 3 to $G_1(z)$, we immediately get the assertions (4.6) to (4.10) of Theorem 3.

Finally, taking the generalized hypergeometric function defined by

$$(4.16) \quad {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = 1 + 2(1-\alpha) \sum_{n=1}^{\infty} z^n$$

we see that each of the results (4.6) to (4.10) is sharp.

5. UNIVALENCE OF THE GENERALIZED HYPERGEOMETRIC FUNCTION

Carlson and Shaffer [1] presented a study of various interesting classes of starlike, convex, and prestarlike hypergeometric functions by applying a linear operator defined by a certain convolution. Recently, Owa and Srivastava [13] derived several interesting results concerning univalent generalized hypergeometric functions, starlike generalized hypergeometric functions of order α , and convex generalized hypergeometric functions of order α . In this section we determine the radius of univalence for the generalized hypergeometric functions belonging to the class $\mathcal{A}(p; q; \alpha)$ with the aid of the following lemma due to Chen [3]:

LEMMA 4. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}(\alpha)$ with $0 \leq \alpha < \frac{1}{10}$. Then $f(z)$ is univalent in $|z| < r_1$, where r_1 is given by

$$(5.1) \quad r_1 = \sqrt{\frac{2(1-\alpha)}{1-2\alpha}} - 1.$$

The result is sharp.

By using Lemma 4, we shall derive

THEOREM 4. Let the generalized hypergeometric function ${}_pF_q(z)$ defined by (1.8) belong to the class $\mathcal{A}(p;q;\alpha)$ with $0 \leq \alpha < \frac{1}{10}$. Then the function

$$z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)$$

is univalent in $|z| < r_1$, where r_1 is given by (5.1). The result is sharp.

PROOF. The hypothesis that ${}_pF_q(z)$ is in the class $\mathcal{A}(p;q;\alpha)$ implies that

$$(5.2) \quad z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \in \mathcal{S}(\alpha),$$

and the proof of Theorem 4 follows easily from Lemma 4.

The assertion of Theorem 4 is sharp for the generalized hypergeometric function defined by (3.10).

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Department of Mathematics
University of Victoria
Victoria, British Columbia V8W 2Y2
Canada

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan