ERGODIC BAKER'S TRANSFORMATIONS

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Abstract

Let T be a generalized baker's transformation on the unit square with cut function f. We show that if f is monotone non-increasing and bounded away from zero and one then T is ergodic. No topological conditions on f are assumed. Moreover, we prove in this case that T has a weak-Bernoulli generator. Both of these results follow from an exponential rate of contraction in variation by the related Perron-Frobenius operator. The connection between these results and similar facts for interval maps and g-measures is described.

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SECTION 1. Introduction and the Main Result

Let $(I, \mathcal{B}, m)$ denote the probability space consisting of the unit interval $I = [0,1]$ with Borel subsets $\mathcal{B}$ and Lebesgue measure m. Let $0 \le f \le 1$ be a measurable function on I. Define $a = \int_I f\, dm$, $I_0 = [0,a]$, $I_1 = [a,1]$ and maps $\phi_i : I_i \to I$, $i = 0,1$ by the formulas

$$\phi_0^*(x) = \int_0^x f\, dm \qquad \forall x \in I,$$

$$\phi_1^*(x) = 1 - \int_x^1 1 - f\, dm \qquad \forall x \in I,$$

$$\phi_0(x) = \inf\{t \mid \phi_0^*(t) \ge x\} \qquad \forall x \in I_0,$$

$$\phi_1(x) = \inf\{t \mid \phi_1^*(t) \ge x\} \qquad \forall x \in I_1.$$

By definition the ϕ_i are increasing and left continuous.

We define the generalized baker's transformation (associated to f) on the unit square $I \times I$ by

$$T(x,y) = \begin{cases} (\phi_0(x),\ f[\phi_0(x)]\, y), & (x,y) \in I_0 \times I \\ (\phi_1(x),\ 1 - (1 - f[\phi_1(x)])(1-y)), & (x,y) \in I_1 \times I. \end{cases}$$

It is easy to see that T is $\mathcal{B} \times \mathcal{B}$-measurable and preserves the measure $m \times m$.

f is called the *cut function* for the transformation T^{-1}. The action of T may be visualized as follows. The square is partitioned into two vertical columns over I_0 and I_1 respectively. The mass over I_0 is moved under the graph of f and the mass over I_1 is moved over the graph of f in such a way that vertical fibres are mapped into vertical fibres and measure is preserved. When $f \equiv \frac{1}{2}$, T is the (classical) baker's transformation (as described for example in Halmos [Ha] or Walters [Wa1]). For the benefit of the reader we shall state without proof a number of basic facts about this class of transformations. Details may be found in Bose [B]. We say that f has *margins* if there exists a constant

$c > 0$ satisfying $c \leq f \leq 1 - c$. We also define a two set partition of $I \times I$ as $P = \{P_0, P_1\}$ where

$$P_0 = \{(x, y) \in I \times I \mid 0 \leq y < f(x), \quad x \in I\},$$

$$P_1 = I \times I / P_0.$$

If f has margins, P generates under T. (Weaker sufficient conditions for P to be a generator are known: see Rahe [R] or Bose [B].) When P generates, T is the *natural-extension* of the 2-1 Lebesgue-measure-preserving endomorphism ϕ on I defined by

$$\phi(x) = \begin{cases} \phi_0(x) & x \in I_0 \\ \phi_1(x) & x \in I_1. \end{cases}$$

ϕ is an example of a piecewise monotone interval map. When f has margins, ϕ is *expanding*, with $|\phi'| \geq \frac{1}{1-c}$. In general, the *entropy of* T *with respect to* P is computed as

$$H(T, P) = \int_0^1 f \log_2 f \, dm + \int_0^1 (1 - f) \log_2 (1 - f) \, dm,$$

and this equals the *entropy of* T, $H(T)$ when P generates, in particular when f has margins. Although not necessary for the investigation at hand, one can show that every ergodic automorphism S with $0 < H(S) < 1$ is isomorphic to a generalized baker's transformation. We are interested in the question: Can one determine ergodicity of T from the function f?

Extensive literature on piecewise monotone and expanding interval maps provides motivation (and indeed, in many cases proof) for ergodic statements about T. For example, it follows from the work of Keller [Kel] that if ϕ is expanding and piecewise $C^{1+\delta}$ then its natural extension is Bernoulli, which translates as: If f is Hölder-continuous with margins then T is Bernoulli. This was also shown in Bose [B] under more general hypothesis. Recently Rahe [R] has obtained a similar result which allows $m\{x \mid f(x) = 0 \text{ or } f(x) = 1\} = 0$ but requires different continuity assumptions for the same conclusion. As another example, if f is continuous and of bounded variation with margins, T is Bernoulli. This follows from an analogous interval map result of Wong [Wo].

In this article, we take a different approach, studying the effect of monotonicity in f under no continuity assumptions.

Proposition 1.1. *Let f be monotone non-increasing with margins. Then T is ergodic. Moreover T is isomorphic to the Bernoulli shift with entropy*

$$H = \int_0^1 f \log_2 f \, dm + \int_0^1 (1-f) \log_2 (1-f) \, dm.$$

The second part of this result follows from the weak-Bernoulli property for the partition P.

As background and motivation we consider, given f with margins and associated T and P, an isomorphic shift system on $X = \prod_{\mathbb{Z}} \{0,1\}$, the isomorphism $\Psi : I \times I \to X$ being provided by the $P - T$ name of $(x,y) \in I \times I$. Put on X the product of the discrete topology on $\{0,1\}$. Ψ carries Lebesgue measure to a shift invariant Borel probability measure $\mu = (m \times m) \circ \Psi^{-1}$ on X.

Setting $X^- = \prod_{\mathbb{Z}^-} \{0,1\}$ and for $x^- \in X^-$, $g(x^-) = f(\Psi^{-1}(x^-))$ one easily verifies that μ is a g-measure in the sense of Keane [Kea], *i.e.* that it is a Borel probability measure on X satisfying

$$\mu\{x_0 = 0 \mid x_{-1}, x_{-2}, \ldots\} \stackrel{\text{a.e.}}{=} g(x_{-1}, x_{-2}, \ldots).$$

(Here we allow for a slight abuse of notation as $\Psi^{-1}(x^-)$ is a vertical fibre in the square rather than a point in I.) Conversely, given a measurable g on X^- with $0 < c \leq g \leq 1-c$ (g with *margins*) and a g-measure μ_g there is a generalized baker's transformation isomorphic to the shift on (X, μ_g) so that

$$g(x^-) = f(\Psi^{-1}(x^-)).$$

If g has a unique g-measure then T is ergodic; on the other hand from a non-ergodic μ_g one may construct a non-ergodic T as above.

Given a continuous g with margins on X^- let $\mathcal{M}_g$ denote the set of g-measures. $\mathcal{M}_g$ is a non-empty, convex, compact subset of the shift invariant probability measures on X. Walters [Wa2] has provided a "topological" criteria on g which is sufficient for $\mathcal{M}_g$ be a singleton, and proves that the shift is Bernoulli with respect to the unique g-measure. Incidentally, Walters' condition may also be used to show Hölder-continuous f with margins generate Bernoulli T.

Equip X^- (and X) with the partial order $x \leq y$ iff $x_i \leq y_i \quad \forall i \in \mathbb{Z}^- \quad (\forall i \in \mathbb{Z})$. Hulse [Hu] has shown that if g is continuous, monotone non-increasing in $\leq$ and has margins then $\mathcal{M}_g$ has a restricted structure: There exist two canonical g-measures μ_g^+ and μ_g^- so that $\mathcal{M}_g$ is a singleton if and only if $\mu_g^+ = \mu_g^-$. However, no examples where $\mu_g^+ \neq \mu_g^-$ were provided. Recently Bramson and Kalikow [B,K] have produced a continuous, monotone non-increasing g with margins for which $\mu_g^+ \neq \mu_g^-$. Their method is related to the production of phase transition examples in statistical mechanics and the monotonicity of g plays a crucial role in the success of their example.

When carried over to the generalized baker's transformation, the example of Bramson and Kalikow implies that there exists a cut function f with margins so that T is not ergodic. This was also observed in Bose [B] where a more explicit example was produced. Both of these non-ergodic examples have discontinuous (in interval topology) cut functions with unbounded variation. The production of a non-ergodic T with continuous f is an interesting and apparently difficult problem. Our present result may be viewed as a first step in understanding the restriction implied by bounded variation. We do not know at this time whether or not our results may generalize to bounded variation f. Curiously, our method does not work for *non-decreasing* f (see Remark 1.3).

We complete this section by establishing ergodicity of T. Mixing, and the weak-Bernoulli property for P are shown in Section 2.

The Perron-Frobenius operator P on $L_1(I)$ associated with ϕ is defined by

$$\int_{\phi^{-1}A} h \, dm = \int_A Ph \, dm \qquad \forall h \in L_1, \ \forall A \in \mathcal{B},$$

which in our setup may be written explicitly:

$$Ph(x) = f(x) \, h\left(\phi_0^{-1}(x)\right) + (1 - f(x)) \, h\left(\phi_1^{-1}(x)\right).$$

Lemma 1.2 *If $h \in L_1$ is monotone and $c > 0$ satisfies $c \leq f \leq 1-c$ then Ph is monotone and*

$$|Ph(1) - Ph(0)| \leq (1-c)\,|h(1) - h(0)|.$$

Proof. We assume h is non-decreasing, the argument in the other case being identical. Let $x \leq y$. Then

$$\begin{aligned} Ph(y) - Ph(x) &= f(y)\ h(\phi_0^{-1}\, y) + (1 - f(y))\ \ h(\phi_1^{-1}\, y) \\ &\quad - f(x)\ h(\phi_0^{-1}\, x) - (1 - f(x))\ \ h(\phi_1^{-1}\, x) \\ &\geq f(y)\ h(\phi_0^{-1}\, x) + (1 - f(y))\ \ h(\phi_1^{-1}\, x) \\ &\quad - f(x)\ h(\phi_0^{-1}\, x) - (1 - f(x))\ \ h(\phi_1^{-1}\, x) \\ &= (f(x) - f(y))\,(h(\phi_1^{-1}\, x) - h(\phi_0^{-1}\, x) \geq 0, \end{aligned}$$

where the first inequality uses the fact that the ϕ_i^{-1} are non-decreasing. For the estimate,

$$Ph(1) = f(1)\ h(a) + (1 - f(1))\ \ h(1) \leq c\ h(a) + (1 - c)\ h(1)$$

$$Ph(0) = f(0)\ h(0) + (1 - f(0))\ \ h(a) \geq (1 - c)\, h(0) + c\ h(a)$$

and the result follows. ∎

Remark 1.3 The above lemma is false for f non-decreasing.

We denote by $BV(I)$ and $C(I)$ classes of bounded variation and continuous functions on [0,1] respectively. Var_J denotes variation over an interval $J \subseteq I$.

Corollary 1.4 *If $h \in BV(I)$, $\|P^n h - \int_I\ h\ dm\|_\infty \leq (1-c)^n \,\mathrm{Var}_I\, h$. If $h \in C(I)$, $\|P^n\, h - \int_I\ h\ dm\|_\infty \longrightarrow 0$.*

Proof. Since

$$\int_I Ph\ dm = \int_I h\ dm$$

and $\mathrm{Var}_I\, Ph \leq (1-c)\,\mathrm{Var}_I\, h$ one has

$$\begin{aligned} \|P^n h - \int_I h\ dm\|_\infty &\leq \sup\, P^n h - \inf\, P^n h \\ &\leq (1-c)^n\, \mathrm{Var}_I\, h. \end{aligned}$$

The result for $h \in C(I)$ now follows by approximation with bounded variation functions and the fact that

$$\|Ph\|_\infty \leq \|h\|_\infty \qquad \forall h \in C(I).$$ ∎

We may now prove the first claim of Proposition 1.1. Let μ be any regular Borel probability measure on I which satisfies

$$\mu(Ph) = \mu(h) \qquad \forall h \in C(I). \tag{1.5}$$

Using Corollary 1.4 one has $\mu(h) = \lim_{n\to\infty} \mu(P^n h) = m(h) \quad \forall h \in C(I)$, so Lebesgue measure m is the unique (regular, Borel probability) measure satisfying (1.5). It follows that m is ergodic for ϕ. Conclude that $m \times m$ is ergodic for T since T is the natural extension of ϕ.

SECTION 2. T is Bernoulli

As in Section 1, $\bar{P} = T^{-1} P = \{I_0, I_1\}$ is a generator for ϕ on I. If $0 \le s \le t < \infty$ are integers we denote by $\bar{P}_{-t}^{-s}$ the partition $\phi^{-s}\bar{P} \vee \phi^{-s-1}\bar{P} \vee \cdots \vee \phi^{-t}\bar{P}$. Observe that if $J \in \bar{P}_{-t}^{0}$, $c^t \le m(J) \le (1-c)^t$. If $s \le 0$, $\bar{P}_{-\infty}^{-s}$ is the smallest σ-algebra containing all the $\bar{P}_{-t}^{-s}$, $t \le s$.

Lemma 2.1. *ϕ is mixing.*

Proof. Let $A \in \bar{P}_{-k}^{0}$ and $B \in \bar{P}_{-\ell}^{0}$. Then

$$\begin{aligned}
\left|m(\phi^{-t}A \cap B) - m(A)m(B)\right| &= \left|\int_I \chi_A(\phi^t(t))\chi_B(x)\,dm(x) - m(A)mB\right| \\
&= \left|\int_I \chi_A P^t \chi_B\,dm - m(A)m(B)\right| \\
&\le \int_I \chi_A \left|P^t\chi_B - m(B)\right| dm \overset{(*)}{\le} 2m(A)(1-c)^t,
\end{aligned}$$

where inequality $(*)$ follows from Corollary 1.3 and the fact that $\mathrm{Var}_I\,\chi_B = 2$. Mixing for arbitrary A and B in $\mathcal{B}$ now follows by standard arguments. ∎

Remark 2.2. One may easily modify the previous proof to show ϕ is exact, but this will follow from the next result.

Lemma 2.3. *P is weak-Bernoulli for T.*

Proof. We use the fact, due to del Junco, Rahe [dJ,R] that if T is mixing and the finite partition P satisfies

$$\sum_n h\left(P \mid P_{-n}^{-1}\right) - h\left(P \mid P_{-\infty}^{-1}\right) < \infty$$

then P is weak-Bernoulli. Set $\Gamma(t) = -t \log_2 t$, and define

$$f_n = E\left(f \mid P_{-n}^{-1}\right) = \sum_{J \in P_{-n}^{-1}} \chi_J \frac{1}{m(J)} \int_J f \, dm.$$

We estimate

$$\sum_n h\left(P \mid P_{-n}^{-1}\right) - h\left(P \mid P_{-\infty}^{-1}\right).$$

Each term in this sum may be written as

$$\begin{aligned} & \textstyle\int_I \Gamma(f_n) - \Gamma(f)\, dm \quad + \quad \int_I \Gamma(1-f_n) - \Gamma(1-f)\, dm \\ = \quad & A_n \quad + \quad B_n. \end{aligned}$$

For each A_n we may write

$$\begin{aligned} A_n &= \sum_{J \in P_{-n}^{-1}} m(J) \left\{ \frac{1}{m(J)} \int_J \Gamma(f_n) - \Gamma(f)\, dm \right\} \\ &= \sum_{J \in P_{-n}^{-1}} m(J) \beta(J) \\ &= \sum_{\{J \mid \beta(J) > \frac{A_n}{2}\}} m(J)\, \beta(J) + \sum_{\{J \mid \beta(J) \leq \frac{A_n}{2}\}} m(J)\, \beta(J) \\ &= \sum_{J \in \mathcal{J}_1} m(J)\, \beta(J) + \sum_{J \in \mathcal{J}_2} m(J)\, \beta(J). \end{aligned} \tag{2.4}$$

We now observe the following points.

1. $\sum_{J \in \mathcal{J}_1} m(J) \geq A_n$.
2. $\# \mathcal{J}_1 \geq A_n (1-c)^{-n}$.
3. If $\beta(J) = \gamma > 0$ then there exists $x_0 \in J$ satisfying $|f_n(x_0) - f(x_0)| \geq \frac{\gamma}{M}$ where

$$M = \max_{c \leq t \leq 1-c} \Gamma'(t).$$

1. follows from (2.4) and the fact that $0 \le \beta(J) \le \frac{1}{2}$. Since $m(J) \le (1-c)^n$, 2. follows from 1. 3. is elementary. From 3. it follows that if $J \in \mathcal{J}_1$, $\mathrm{Var}_J\, f \ge \frac{A_n}{2M}$, after which

$$\mathrm{Var}_I\, f \ge \sum_{J\in\mathcal{J}_1} \mathrm{Var}_J\, f \ge \sum_{J\in\mathcal{J}_1} \frac{A_n}{2M} \ge \frac{A_n^2}{2M(1-c)^n}$$

so $\sum A_n < \infty$. A similar argument shows $\sum B_n < \infty$ and the lemma is proved. ∎

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