

**A CERTAIN SUBCLASS OF ANALYTIC
FUNCTIONS ASSOCIATED WITH
OPERATORS OF FRACTIONAL CALCULUS**

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Abstract

In the present paper we introduce a new class of functions, defined by using certain fractional calculus operators, which are analytic in the open unit disk. A necessary and sufficient condition for a function to belong to such a class is obtained. The various other results presented here include the radii of close-to-convexity, starlikeness, and convexity, and some distortion theorems involving fractional integrals and fractional derivatives. Several special cases of our results are also pointed out.

1. Introduction and Definitions

Denote by $\mathcal{F}(n)$ the class of functions of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0; n \in \mathbb{N}), \quad (1.1)$$

which are analytic in the *open* unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Let $\mathcal{S}_{\lambda, \mu, \eta}(n, \sigma, \alpha)$ be the subclass of functions $f(z)$ in $\mathcal{F}(n)$ which also satisfy the inequality:

$$\Re \left[\phi_1(\lambda, \mu, \eta) z^{\mu-1} \left((1 - \sigma) J_{0,z}^{\lambda, \mu, \eta} f(z) + \sigma z J_{0,z}^{\lambda+1, \mu+1, \eta+1} f(z) \right) \right] > \alpha \quad (1.2)$$

$$(0 \leq \lambda < 1; 0 \leq \alpha < 1; 0 \leq \sigma \leq 1; \mu, \eta \in \mathbb{R}; \mu < 2; \lambda - \eta < 2; \mu - \eta < 2),$$

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where, for convenience,

$$\phi_m(\lambda, \mu, \eta) = \frac{\Gamma(1 - \mu + m)\Gamma(1 + \eta - \lambda + m)}{\Gamma(1 + m)\Gamma(1 + \eta - \mu + m)}. \quad (1.3)$$

The operator $J_{0,z}^{\lambda,\mu,\eta}$, occurring in the defining relation (1.2), is the fractional derivative operator given by Definition 2 below (see Srivastava *et al.* [9]).

Definition 1. Let $\alpha \in \mathbb{R}_+$ and $\beta, \eta \in \mathbb{R}$. Then, in terms of the familiar (Gauss's) hypergeometric function ${}_2F_1$, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z - \zeta)^{\alpha-1} f(\zeta) {}_2F_1 \left(\alpha + \beta, -\eta; \alpha; 1 - \frac{\zeta}{z} \right) d\zeta, \quad (1.4)$$

where the function $f(z)$ is analytic in a simply-connected region of the z -plane containing the origin, with the order

$$f(z) = O(|z|^r) \quad (z \rightarrow 0) \quad (1.5)$$

for

$$r > \max\{0, \beta - \eta\} - 1, \quad (1.6)$$

and the multiplicity of $(z - \zeta)^{\alpha-1}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

Definition 2. The fractional derivative operator $J_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$\begin{aligned} J_{0,z}^{\alpha,\beta,\eta} f(z) &= \frac{d}{dz} \left(\frac{z^{\alpha-\beta}}{\Gamma(1-\alpha)} \int_0^z (z - \zeta)^{-\alpha} f(\zeta) \right. \\ &\quad \left. {}_2F_1 \left(\beta - \alpha, 1 - \eta; 1 - \alpha; 1 - \frac{\zeta}{z} \right) d\zeta \right) \\ &\quad (0 \leq \alpha < 1; \beta, \eta \in \mathbb{R}), \end{aligned} \quad (1.7)$$

where the function $f(z)$ is analytic in a simply-connected region of the z -plane containing the origin, with the same order as given by (1.5), and the multiplicity of $(z - \zeta)^{-\alpha}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

The operators $I_{0,z}^{\alpha,\beta,\eta}$ and $J_{0,z}^{\alpha,\beta,\eta}$ include (as their special cases) the Riemann-Liouville and Erdélyi-Kober operators of fractional calculus ([4],[5]). Indeed, we have

$$I_{0,z}^{\alpha,-\alpha,\eta} f(z) = {}_0D_z^{-\alpha} f(z) \quad (\alpha \in \mathbb{R}_+); \quad (1.8)$$

$$J_{0,z}^{\alpha,\alpha,\eta} f(z) = {}_0D_z^\alpha f(z) \quad (0 \leq \alpha < 1); \quad (1.9)$$

$$I_{0,z}^{\alpha,0,\eta} f(z) = E_{0,z}^{\alpha,\eta} f(z) \quad (\alpha \in \mathbb{R}_+; \eta \in \mathbb{R}); \quad (1.10)$$

$$J_{0,z}^{\alpha,1,\eta} {}_z f(z) = E_{0,z}^{-\alpha,\eta} f(z) + (\alpha - \eta) E_{0,z}^{1-\alpha,\eta} f(z) \quad (\alpha < 0; \eta \in \mathbb{R}). \quad (1.11)$$

The theory of fractional calculus operators has found deep access into the realm of the theory of analytic functions. The Riemann-Liouville fractional calculus operators [3] and their various other generalizations ([4],[5]) have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates and distortion inequalities for various subclasses of analytic and univalent functions. For numerous references on the subject, one may refer to the recent works by Srivastava and Owa ([7] and [8]). This paper is devoted to the study of a new class $\mathcal{S}_{\lambda,\mu,\eta}(n, \sigma, \alpha)$ of functions which we have defined above. Some results connected with this class of functions, including the characterization property, the radii of close-to-convexity, starlikeness, convexity, and distortion inequalities, are obtained. In the process of our investigation, we are led also to the corrected forms of some results given recently by Altıntaş *et al.* [1] for the class $\mathcal{F}_\lambda(n, \sigma, \alpha)$, where

$$\mathcal{F}_\lambda(n, \sigma, \alpha) = \mathcal{S}_{\lambda,\lambda,\eta}(n, \sigma, \alpha) \quad (1.12)$$

$$(0 \leq \alpha < 1; 0 \leq \lambda < 1; 0 \leq \sigma \leq 1; n \in \mathbb{N})$$

2. A Characterization Property

We begin by proving

Theorem 1. *A function $f(z)$ in $\mathcal{F}(n)$ belongs to the class $\mathcal{S}_{\lambda,\mu,\eta}(n, \sigma, \alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{1 + \sigma(k - \mu - 1)}{\phi_k(\lambda, \mu, \eta)} a_k \leq \frac{1 - \mu\sigma - \alpha}{\phi_1(\lambda, \mu, \eta)}, \quad (2.1)$$

where $\phi_m(\lambda, \mu, \eta)$ is given by (1.3). The result is sharp.

Proof. Let $f(z) \in \mathcal{S}_{\lambda,\mu,\eta}(n, \sigma, \alpha)$. Then, by applying the formula (*cf.* Srivastava *et al.* [9, p. 415, Lemma 3]):

$$J_{0,z}^{\lambda,\mu,\eta} z^\kappa = \frac{\Gamma(1 + \kappa)\Gamma(1 - \mu + \eta + \kappa)}{\Gamma(1 - \mu + \kappa)\Gamma(1 - \lambda + \eta + \kappa)} z^{\kappa - \mu} \quad (2.2)$$

$$(0 \leq \lambda < 1; \mu, \eta \in \mathbb{R}; \kappa > \max\{0, \mu - \eta\} - 1),$$

and the inequality (1.2), and then performing some elementary calculations as in the work of Altıntaş *et al.* [1], we are led to the assertion (2.1) of Theorem 1.

Conversely, suppose that the inequality (2.1) holds true. Then we obtain

$$\begin{aligned}
& \left| \phi_1(\lambda, \mu, \eta) z^{\mu-1} \left\{ (1 - \sigma) J_{0,z}^{\lambda, \mu, \eta} f(z) + \sigma z J_{0,z}^{\lambda+1, \mu+1, \eta+1} f(z) \right\} + \mu\sigma - 1 \right| \\
&= \left| - \sum_{k=n+1}^{\infty} \frac{\phi_1(\lambda, \mu, \eta)}{\phi_k(\lambda, \mu, \eta)} [1 + \sigma(k - \mu - 1)] a_k z^{k-1} \right| \\
&\leq \sum_{k=n+1}^{\infty} \frac{\phi_1(\lambda, \mu, \eta)}{\phi_k(\lambda, \mu, \eta)} [1 + \sigma(k - \mu - 1)] a_k |z|^{k-1} \\
&\leq 1 - \mu\sigma - \alpha,
\end{aligned}$$

under the conditions stated with (1.2). This implies that $f(z) \in \mathcal{S}_{\lambda, \mu, \eta}(n, \sigma, \alpha)$.

It is easy to observe that the assertion (2.1) of Theorem 1 is sharp, the extremal function being given by

$$f(z) = z - \frac{(1 - \mu\sigma - \alpha) \phi_{n+1}(\lambda, \mu, \eta)}{[1 + \sigma(n - \mu)] \phi_1(\lambda, \mu, \eta)} z^{n+1} \quad (n \in \mathbb{N}). \quad (2.3)$$

In view of the relationship (1.12), a special case of Theorem 1 when $\lambda = \mu$ can be rewritten in the following corrected form:

Corollary 1 (*cf.* Altıntaş *et al.* [1, p. 2, Theorem 1]). *A function $f(z) \in \mathcal{F}(n)$ is in the class $\mathcal{F}_\delta(n, \lambda, \alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1) [1 + \lambda(k-1-\delta)]}{\Gamma(k+1-\delta)} a_k \leq \frac{1 - \lambda\delta - \alpha}{\Gamma(2-\delta)}. \quad (2.4)$$

The result is sharp.

Our next result is contained in

Theorem 2. *Let $f(z)$ defined by (1.1) and $g(z)$ defined by*

$$g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \quad (b_k \geq 0; n \in \mathbb{N}) \quad (2.5)$$

be in the same class $\mathcal{S}_{\lambda, \mu, \eta}(n, \sigma, \alpha)$.

Then the function $h(z)$ defined by

$$h(z) = (1 - \theta) f(z) + \theta g(z) = z - \sum_{k=n+1}^{\infty} c_k z^k \quad (2.6)$$

$$(c_k := (1 - \theta) a_k + \theta b_k \geq 0; \quad 0 \leq \theta \leq 1; \quad n \in \mathbb{N})$$

is also in the class $\mathcal{S}_{\lambda, \mu, \eta}(n, \sigma, \alpha)$.

Proof. The result follows easily by using (2.5), (2.6), and (2.1).

3. Distortion Theorems

We prove two distortion theorems (Theorems 3 and 4 below) involving the fractional calculus operators $I_{0,z}^{\alpha, \beta, \eta}$ and $J_{0,z}^{\alpha, \beta, \eta}$, respectively.

Theorem 3. *Let $\beta \in \mathbb{R}_+$ and $\gamma, \eta \in \mathbb{R}$ such that $\gamma < 2$, $\beta + \eta > -2$, and $\gamma - \eta < 2$. If n is a positive integer such that*

$$n \geq \frac{\gamma(\beta + \eta)}{\beta} - 2,$$

and if $f(z) \in \mathcal{F}(n)$ is in the class $\mathcal{S}_{\lambda, \mu, \eta}(n, \sigma, \alpha)$, then

$$\begin{aligned} \left| I_{0,z}^{\beta, \gamma, \eta} f(z) \right| &\leq \frac{|z|^{1-\gamma}}{\phi_1(-\beta, \gamma, \eta)} \left(1 + \frac{1 - \mu\sigma - \alpha}{1 + \sigma(n - \mu)} \right. \\ &\quad \left. \cdot \frac{\phi_1(-\beta, \gamma, \eta) \phi_{n+1}(\lambda, \mu, \eta)}{\phi_1(\lambda, \mu, \eta) \phi_{n+1}(-\beta, \gamma, \eta)} |z|^n \right) \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} \left| I_{0,z}^{\beta, \gamma, \eta} f(z) \right| &\geq \frac{|z|^{1-\gamma}}{\phi_1(-\beta, \gamma, \eta)} \left(1 - \frac{1 - \mu\sigma - \alpha}{1 + \sigma(n - \mu)} \right. \\ &\quad \left. \cdot \frac{\phi_1(-\beta, \gamma, \eta) \phi_{n+1}(\lambda, \mu, \eta)}{\phi_1(\lambda, \mu, \eta) \phi_{n+1}(-\beta, \gamma, \eta)} |z|^n \right) \end{aligned} \quad (3.3)$$

$$(z \in \mathcal{U} \quad \text{if } \gamma \leq 1; \quad z \in \mathcal{U} - \{0\} \quad \text{if } \gamma > 1),$$

where $\phi_m(\lambda, \mu, \eta)$ is given by (1.3).

Proof. Under the hypotheses of Theorem 3, it follows from (2.1) that

$$\begin{aligned} & [1 + \sigma(n - \mu)] \frac{\phi_1(\lambda, \mu, \eta)}{\phi_{n+1}(\lambda, \mu, \eta)} \sum_{k=n+1}^{\infty} a_k \\ & \leq \sum_{k=n+1}^{\infty} \frac{\phi_1(\lambda, \mu, \eta)}{\phi_k(\lambda, \mu, \eta)} [1 + \sigma(k - \mu - 1)] a_k, \end{aligned} \quad (3.4)$$

which readily yields

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{1 - \mu\sigma - \alpha}{1 + \sigma(n - \mu)} \frac{\phi_{n+1}(\lambda, \mu, \eta)}{\phi_1(\lambda, \mu, \eta)} \quad (n \in \mathbb{N}). \quad (3.5)$$

From (1.1), (1.4), and a known result due to Srivastava *et al.* [9, p. 415, Lemma 3], we have

$$I_{0,z}^{\beta, \gamma, \eta} f(z) = \frac{z^{1-\gamma}}{\phi_1(-\beta, \gamma, \eta)} \left(1 - \sum_{k=n+1}^{\infty} \frac{\phi_1(-\beta, \gamma, \eta)}{\phi_k(-\beta, \gamma, \eta)} a_k z^{k-1} \right). \quad (3.6)$$

Next we observe that the function $\Theta(k)$ defined by

$$\Theta(k) = \frac{\phi_1(-\beta, \gamma, \eta)}{\phi_k(-\beta, \gamma, \eta)} = \frac{(2)_{k-1} (2 - \gamma + \eta)_{k-1}}{(2 - \gamma)_{k-1} (2 + \beta + \eta)_{k-1}} \quad (3.7)$$

is non-increasing for integers $k \geq n + 1$, under the hypotheses of Theorem 3 including the constraint (3.1). Thus we obtain

$$0 < \Theta(k) \leq \Theta(n + 1) = \frac{\phi_1(-\beta, \gamma, \eta)}{\phi_{n+1}(-\beta, \gamma, \eta)}. \quad (3.8)$$

The desired distortion inequality (3.2) follows now from (3.5), (3.6), (3.7), and (3.8).

The assertion (3.3) can be proved in a similar manner.

The proof of the following distortion theorem would run parallel to that of Theorem 3.

Theorem 4. *Let $0 \leq \beta < 1$ and $\gamma, \eta \in \mathbb{R}$ such that $\gamma < 2$, $\eta - \beta > -2$, and $\gamma - \eta < 2$. If n is a positive integer satisfying*

$$n \geq \frac{\gamma(\beta - \eta)}{\beta} - 2, \quad (3.9)$$

and if $f(z) \in \mathcal{F}(n)$ is in the class $\mathcal{S}_{\lambda, \mu, \eta}(n, \sigma, \alpha)$, then

$$\begin{aligned} \left| J_{0,z}^{\beta, \gamma, \eta} f(z) \right| & \leq \frac{|z|^{1-\gamma}}{\phi_1(\beta, \gamma, \eta)} \left(1 + \frac{1 - \mu\sigma - \alpha}{1 + \sigma(n - \mu)} \right. \\ & \quad \left. \cdot \frac{\phi_1(\beta, \gamma, \eta) \phi_{n+1}(\lambda, \mu, \eta)}{\phi_{n+1}(\beta, \gamma, \eta) \phi_1(\lambda, \mu, \eta)} |z|^n \right) \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} |J_{0,z}^{\beta,\gamma,\eta} f(z)| &\geq \frac{|z|^{1-\gamma}}{\phi_1(\beta,\gamma,\eta)} \left(1 - \frac{1-\mu\sigma-\alpha}{1+\sigma(n-\mu)} \right. \\ &\quad \left. \cdot \frac{\phi_1(\beta,\gamma,\eta) \phi_{n+1}(\lambda,\mu,\eta)}{\phi_{n+1}(\beta,\gamma,\eta) \phi_1(\lambda,\mu,\eta)} |z|^n \right), \end{aligned} \quad (3.11)$$

$(z \in \mathcal{U} \text{ if } \gamma \leq 1; \quad z \in \mathcal{U} - \{0\} \text{ if } \gamma > 1),$

where $\phi_m(\lambda, \mu, \eta)$ is given by (1.3).

In the special case when $\lambda = \mu$ and $\gamma = -\beta$, Theorems 3 and 4 would correspond, respectively, to Corollaries 2 and 3 below.

Corollary 2 (cf. Altıntaş et al. [1, p. 4, Theorem 3]). *If $f(z) \in \mathcal{F}_\delta(n, \lambda, \alpha)$, then*

$$|D_z^{-\mu} f(z)| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left(1 + \frac{(1-\lambda\delta-\alpha)\Gamma(2+\mu)\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\Gamma(n+2+\mu)\Gamma(2-\delta)} |z|^n \right) \quad (3.12)$$

and

$$|D_z^{-\mu} f(z)| \geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left(1 - \frac{(1-\lambda\delta-\alpha)\Gamma(2+\mu)\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\Gamma(n+2+\mu)\Gamma(2-\delta)} |z|^n \right) \quad (3.13)$$

for $\mu > 0$ and $n \in \mathbb{N}$, and for all $z \in \mathcal{U}$.

Corollary 3 (cf. Altıntaş et al. [1, p. 5, Theorem 4]). *If $f(z) \in \mathcal{F}_\delta(n, \lambda, \alpha)$, then*

$$|D_z^\mu f(z)| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left(1 + \frac{(1-\lambda\delta-\alpha)\Gamma(2-\mu)\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\Gamma(n+2-\mu)\Gamma(2-\delta)} |z|^n \right) \quad (3.14)$$

and

$$|D_z^\mu f(z)| \geq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left(1 - \frac{(1-\lambda\delta-\alpha)\Gamma(2-\mu)\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\Gamma(n+2-\mu)\Gamma(2-\delta)} |z|^n \right) \quad (3.15)$$

for $0 \leq \mu < 1$ and $n \in \mathbb{N}$, and for all $z \in \mathcal{U}$.

4. Radii of Close-to-Convexity, Starlikeness, and Convexity

A function $f(z)$ in $\mathcal{F}(n)$ is said to be *close-to-convex of order ρ* in \mathcal{U} if

$$\Re\{f'(z)\} > \rho \quad (4.1)$$

for some ρ ($0 \leq \rho < 1$) and for all $z \in \mathcal{U}$.

If $f(z) \in \mathcal{F}(n)$ satisfies the inequality:

$$\Re \left\{ \frac{z f'(z)}{f(z)} \right\} > \rho \quad (4.2)$$

for some ρ ($0 \leq \rho < 1$) and for all $z \in \mathcal{U}$, then $f(z)$ is said to be *starlike of order ρ* in \mathcal{U} . On the other hand, if $f(z) \in \mathcal{F}(n)$ satisfies the inequality:

$$\Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \rho \quad (4.3)$$

for some ρ ($0 \leq \rho < 1$) and for all $z \in \mathcal{U}$, then $f(z)$ is said to be *convex of order ρ* in \mathcal{U} .

It follows at once that $f(z) \in \mathcal{F}(n)$ is convex of order ρ in \mathcal{U} if and only if $z f'(z)$ is starlike of order ρ in \mathcal{U} (see, for details, Duren [2]).

We now prove

Theorem 5. *If $f(z) \in \mathcal{S}_{\lambda, \mu, \eta}(n, \sigma, \alpha)$, then $f(z)$ is close-to-convex of order ρ in $|z| < r_1(\lambda, \mu, \eta, \sigma, \alpha, \rho)$, where*

$$r_1(\lambda, \mu, \eta, \sigma, \alpha, \rho) = \inf_k \left[\frac{(1 - \rho)[1 + \sigma(k - \mu - 1)] \phi_1(\lambda, \mu, \eta)}{k(1 - \mu\sigma - \alpha) \phi_k(\lambda, \mu, \eta)} \right]^{1/(k-1)} \quad (4.4)$$

$(k \geq n + 1; \quad n \in \mathbb{N}),$

where $\phi_1(\lambda, \mu, \eta)$ is given by (1.3).

Proof. Let $f(z) \in \mathcal{S}_{\lambda, \mu, \eta}(n, \sigma, \alpha)$. Then, by virtue of (4.1), the function $f(z)$ is close-to-convex of order ρ in \mathcal{U} , provided that

$$\left| \sum_{k=n+1}^{\infty} k a_k z^{k-1} \right| \leq \sum_{k=n+1}^{\infty} k a_k |z|^{k-1} \leq 1 - \rho \quad (4.5)$$

$(0 \leq \rho < 1; \quad z \in \mathcal{U}).$

In view of (2.1), the assertion (4.5) is true if

$$\frac{k |z|^{k-1}}{1 - \rho} \leq \frac{[1 + \sigma(k - \mu - 1)] \phi_1(\lambda, \mu, \eta)}{(1 - \mu\sigma - \alpha) \phi_k(\lambda, \mu, \eta)} \quad (k \geq n + 1; \quad n \in \mathbb{N}). \quad (4.6)$$

Upon solving (4.6) for $|z|$, we get (4.4).

Theorem 6. If $f(z) \in \mathcal{S}_{\lambda, \mu, \eta}(n, \sigma, \alpha)$, then $f(z)$ is starlike of order ρ in $|z| < r_2(\lambda, \mu, \eta, \sigma, \alpha, \rho)$, where

$$r_2(\lambda, \mu, \eta, \sigma, \alpha, \rho) = \inf_k \left[\frac{(1-\rho)[1+\sigma(k-\mu-1)]\phi_1(\lambda, \mu, \eta)}{(k-\rho)(1-\mu\sigma-\alpha)\phi_k(\lambda, \mu, \eta)} \right]^{1/(k-1)} \quad (4.7)$$

$$(k \geq n+1; \quad n \in \mathbb{N}),$$

and $\phi_m(\lambda, \mu, \eta)$ is given by (1.3).

Proof. Under the hypothesis of Theorem 6, $f(z)$ is starlike of order ρ in \mathcal{U} , provided that

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{k=n+1}^{\infty} (k-1) a_k |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k |z|^{k-1}} \leq 1 - \rho \quad (4.8)$$

$$(0 \leq \rho < 1; \quad z \in \mathcal{U}).$$

In view of (2.1), the assertion (4.8) is true if

$$\frac{(k-\rho)|z|^{k-1}}{1-\rho} \leq \frac{[1+\sigma(k-\mu-1)]\phi_1(\lambda, \mu, \eta)}{(1-\mu\sigma-\alpha)\phi_k(\lambda, \mu, \eta)} \quad (k \geq n+1; \quad n \in \mathbb{N}), \quad (4.9)$$

which obviously leads to (4.7).

Theorem 7. If $f(z) \in \mathcal{S}_{\lambda, \mu, \eta}(n, \sigma, \alpha)$, then $f(z)$ is convex of order ρ in $|z| < r_3(\lambda, \mu, \eta, \sigma, \alpha, \rho)$, where

$$r_3(\lambda, \mu, \eta, \sigma, \alpha, \rho) = \inf_k \left[\frac{(1-\rho)[1+\sigma(k-\mu-1)]\phi_1(\lambda, \mu, \eta)}{k(k-\rho)(1-\mu\sigma-\alpha)\phi_k(\lambda, \mu, \eta)} \right]^{1/(k-1)} \quad (4.10)$$

$$(k \geq n+1; \quad n \in \mathbb{N}),$$

and $\phi_m(\lambda, \mu, \eta)$ is given by (1.3).

Proof. Under the hypothesis of Theorem 7, $f(z)$ is convex of order ρ in \mathcal{U} , provided that

$$\left| \frac{z f''(z)}{f'(z)} \right| \leq \frac{\sum_{k=n+1}^{\infty} k(k-1) a_k |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} k a_k |z|^{k-1}} \leq 1 - \rho \quad (0 \leq \rho < 1; \quad z \in \mathcal{U}). \quad (4.11)$$

By means of (2.1), it is easily seen that (4.11) holds true if

$$\frac{k(k-\rho)|z|^{k-1}}{1-\rho} \leq \frac{[1+\sigma(k-\mu-1)]\phi_1(\lambda,\mu,\eta)}{(1-\mu\sigma-\alpha)\phi_k(\lambda,\mu,\eta)} \quad (k \geq n+1; n \in \mathbb{N}), \quad (4.12)$$

which yields (4.7).

In their special case when $\lambda = \mu$, Theorems 5, 6 and 7 would correspond to the corrected versions of the analogous results of Altıntaş *et al.* [1] for the class $\mathcal{F}_\delta(n, \lambda, \alpha)$ given by (1.12).

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