PROBABILITY OF ACUTE ANGLES

by

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1352. *Proposed by Mark Krusemeyer, Carleton College, Northfield, Minnesota.*

a. Suppose three lines are drawn independently and in random directions through the origin in the plane. (The lines will each extend in two opposite directions from the origin; "random" means that given two equal angles with vertex at the origin, each line is equally likely to be inside one as inside the other.) What is the probability that all the angles formed at the origin by adjacent pairs of lines will be acute? (For example, if the lines are $y = 0$, $y = x$, $y = 2x$, then the angle formed by $y = 2x$ and $y = 0$ as an adjacent pair of lines at the origin will not be acute. However, if the lines are $y = 0$, $y = 2x$, $y = -2x$, then all angles at the origin will be acute.)

b. Same question, with "three lines" replaced by "n lines".

Solution by Bruce R. Johnson, University of Victoria, Canada.

We will show that the answer for part b is $1 - n/2^{n-1}$. Evaluating this formula for $n = 3$ yields $1/4$ as the answer for part a.

With probability one no two lines will be collinear, so the n lines through the origin will create n pairs of nonoverlapping vertical angles that radiate from the origin and sum to 2π radians. To distinguish the n lines we arbitrarily label them from 1 to n, and for $j \in \{1,2,\cdots,n\}$ we let the pair of vertical angles formed by line j and the adjacent line on the counterclockwise side of line j be called vertical angle pair j. Since at most one vertical angle pair can be obtuse, the events $O_1, O_2, \cdots, O_n$ are mutually exclusive, where O_j denotes the event that vertical angle pair j

is obtuse. Therefore, by the additive property of probability

$$P(2n \text{ acute angles}) = 1 - P(O_1 \cup O_2 \cup \cdots \cup O_n)$$

$$= 1 - \sum_{j=1}^{n} P(O_j).$$

Also, for every j the event O_j will occur if and only if each of the $n-1$ angles measured counterclockwise from either ray of line j to the nearest ray of each of the other $n-1$ lines is between $\pi/2$ and π radians. Since these $n-1$ angles are distributed independently and uniformly over the interval $(0,\pi)$, it follows that

$$P(O_j) = (1/2)^{n-1}.$$

Hence,

$$P(2n \text{ acute angles}) = 1 - n/2^{n-1}.$$