GENERALIZED LIE IDEALS IN $\star$–PRIME RINGS

by

W.S. MARTINDALE, 3RD AND C. ROBERT MIERS

DMS–550–IR July 1990

GENERALIZED LIE IDEALS IN $\star$–PRIME RINGS

by

W.S. Martindale, 3rd

Department of Mathematics
University of Massachusetts,
Amherst, Massachusetts 01003

and

C. Robert Miers[1]

Department of Mathematics and Statistics
University of Victoria,
Victoria, B.C. V8W 3P4 Canada

Abstract

Let R be a $\star$-prime ring with skew elements K, extended centroid C, and central closure RC. For $U,W \subseteq R$ we define $U^{(n)}(W)$ inductively: $U^{(1)}(W) = [U,W]$, $U^{(n+1)}(W) = [U,U^{(n)}(W)]$. An additive subgroup V of K is called a generalized Lie ideal (GLI) of K of index $\leq n$ if $V^{(n)}(K) \subseteq V$. The notion of a GLI includes that of a Lie ideal of K, a Lie inner ideal of K, and an additive subgroup of K which generates a nilpotent subring of R. Theorem. *If* char. $R = 0$ *and* $V \subseteq K$ *such that* $V^{(n)}(K) = 0$, *then* $V \subseteq C + B$, *where* B *is a nilpotent subring of* RC. Theorem. *Suppose* $\star$ *is an involution of the first kind*, V *is a* GLI *of* K *of index* $\leq n$, *and* $T = \{t \in K \mid [V,[t,K]] \subseteq V\}$. *If* char. $R \neq 2$ *and* $[T,T] \neq 0$ *then* $[I \cap K,K] \subseteq V$ *for some nonzero* $\star$*-ideal* I *of* R. *If* char. $R = 0$ *and* $[T,T] = 0$ *then* $V^{6n-7} = 0$. A similar result is obtained when $\star$ is of the second kind.

1980 *Mathematics Subject Classification* (1985 *Revision*): 16A12, 16A28, 16A72.

---

[1] The second author was partially supported by NSERC of Canada and a University of Victoria research grant.

1. INTRODUCTION

Throughout this paper (unless explicitly stated otherwise) R will denote a *$\star$-prime ring*, *i.e.*, a ring with involution $\star$ in which the product of any two nonzero $\star$-ideals is never zero. We shall assume throughout that R admits the operator $\frac{1}{2}$, which necessarily implies that char. $R \neq 2$. R may be written as $S \oplus K$, where S is the set of symmetric elements $s^\star = s$ and K is the set of skew elements $k^\star = k$. S forms a Jordan ring under $x \circ y = xy + yx$ and K forms a Lie ring under $[x,y] = xy - yx$.

Our paper is primarily concerned with the Lie ring K and relies heavily on a recent paper of the authors, "Herstein's Lie Theory Revisited" [9]. It makes significant use of the extended centroid C and the central closure RC of R, and we very briefly recall these and related notions (for details see [1, Section 2]).

In a fashion similar to the construction of the extended centroid of an ordinary prime ring, the filter of all nonzero $\star$-ideals is used to construct the *extended centroid* C of R, and one is able to form the overring $A = RC$, called the *central closure* of R. C is a commutative ring with 1 containing the center Z of R and has the important property that given $\lambda \in C$ there exists a $\star$-ideal $I \neq 0$ in R such that $\lambda I \subseteq R$. Furthermore the involution $\star$ may be extended in a natural way to A. The symmetric elements of C are denoted by $C_\star$, and we remark that $C_\star$ is a field.

We define the involution $\star$ of R to be *of the first kind* if $C = C_\star$ and *of the second kind* if $C_\star \neq C$ (*i.e.*, if C contains a skew element). The ring $RC_\star$ is called the *$\star$-closure* of R and, if F is the algebraic closure of $C_\star$, we can form the *super $\star$-closure* $\tilde{R} = RC_\star \otimes_{C_\star} F$. $\tilde{R}$ is again a $\star$-prime ring and is prime if and only if $\star$ is of the first kind [9, Lemma 5.1]. If R satisfies a polynomial identity (PI) and $\star$ is of the first kind then it is known that $\tilde{R}$ is either *of class* BD(n) (*i.e.*, $\tilde{R} = M_n(F)$ under transpose involution) or *of class* C(n) (*i.e.*, $\tilde{R} = M_n(F)$ under symplectic involution). If R is PI and $\star$ is of the second kind it is known that $\tilde{R}$ is *of class* A(n) (*i.e.*, $\tilde{R} = T \oplus T^o$, T^o the opposite ring of T, $\star$ the exchange involution $(x,y) \longrightarrow (y,x)$, $T = M_n(F)$).

Let V and W be subsets of any ring R. We define $V^{(m)}(W)$ inductively as follows: $V^{(1)}(W) = [V,W]$, $V^{(m+1)}(W) = [V,V^{(m)}(W)]$. The notation $V^{(m)}$ will mean $V^{(m-1)}(V)$. Occasionally we shall need the derived series of V: $V^{[1]} = [V,V]$, $V^{[m+1]} = [V^{[m]},V^{[m]}]$. V^m will have its usual associative meaning. We also introduce the terminology $\bar{V}$ to mean the associative subring of R generated by V.

Definition. *Let* R *be a ring with involution, with skew elements* K. *An additive subgroup* V *of* K *is a* _generalized Lie ideal (GLI) of K of index $\leq n$_ *if* $V^{(n)}(K) \subseteq V$.

The case $n = 1$ coincides with the notion of a Lie ideal of K. The case $n = 2$ is precisely that of a *Lie inner ideal* of K (first introduced and studied by Benkart [2]). But the notion of a GLI of K of index $\leq n$ also includes that of an additive subgroup of K which generates associatively in R a nilpotent subring of index $\left[\frac{n+1}{2}\right]$.

We are now in a position to state the two main results of this paper. For simplicity of presentation we do not state them in the fullest generality (accurate statements are to be found in Sections 5 and 6).

THEOREM 5.2. *Let* R *be* $\star$*-prime*, $\star$ *of the first kind*, char. R = 0, *and let* V *be a* GLI *of* K *of index* $\leq n$. *Then, unless* $\tilde{R}$ *is of class* BD(2), BD(4), *or* C(2), *one of the following holds*:

(*a*) $V \supseteq [J \cap K, K]$ *for some nonzero* $\star$*-ideal* J *of* R,

(*b*) $V^{6n-7} = 0$.

THEOREM 6.4. *Let* R *be* $\star$*-prime*, $\star$ *of the second kind*, char. R = 0, *and let* V *be a* GLI *of* K *of index* $\leq n$. *Then, unless* $\tilde{R}$ *is of class* A(2), *one of the following holds*:

(*a*) *There exists* $a \in C_*$ *such that* $V[a] \supseteq [J \cap K, K]$ *for some nonzero* $\star$*-ideal* J *of* R,

(*b*) $V \subseteq C + B$, B *a subset of* RC, $B^{3n-3} = 0$.

A very special case occurs when V is taken to be generated by a single element, a, of K and we have $(\operatorname{ad} a)^n(K) = 0$. In this regard we recently proved:

THEOREM 1.1 ([11], Main Theorem). *Let* R *be prime with* $\star$, *of* char. 0, *and let* $a \in K$ *be such that* $(\operatorname{ad} a)^n(K) = 0$.

(1) *If* $\star$ *is of the second kind*, $(a-\lambda)^{\left[\frac{n+1}{2}\right]} = 0$ *for some skew element* λ *in* C.

(2) *If* $\star$ *is of the first kind* $a^{\left[\frac{n+1}{2}\right]} = 0$ *unless*

(*a*) $\tilde{R}$ *is of class* BD(2).

(*b*) $n \equiv 0 \pmod 4$ *or* $n \equiv 3 \pmod 4$, *in which case*

$a^{[\frac{n+1}{2}]+1} = 0$.

Theorem 1.1 will be used significantly in the proof of our present results. We also mention that Theorem 6.4 generalizes the following recent result of ours:

THEOREM 1.2 ([10], Theorem 3). *Let* R *be prime of* char. 0, *and let* V *be a* GLI *of* R *of index* $\leq n$ (*considering* R *as a Lie ring under* $[x,y]$). *Then either*

(*a*) $V \supseteq [J,R]$ *for some nonzero ideal* J *of* R, *or*

(*b*) $V \subseteq C + B$, B *a subset of* RC, $B^{3n-3} = 0$.

As we have already stated, the goal of the paper is to analyze the generalized Lie ideals V of the skew elements K of a $\star$-prime ring R. The plan of the paper is to treat separately the cases when $\star$ is of the first kind and of the second kind. Let us indicate briefly the plan of attack when $\star$ is of the first kind (a similar plan is followed when $\star$ is of the second kind). A useful tool for studying V is the set $T = \{t \in K \mid [V,[t,K]] \subseteq V\}$ (this idea originated with Herstein and was subsequently exploited by Benkart [2]). It can be shown that T is both a Lie inner ideal of K (*i.e.*, $T^{(2)}(K) \subseteq T$) and a Lie subring of K (*i.e.*, $[T,T] \subseteq T$). Some useful lemmas concerning Lie inner ideals and Lie subrings, along with other preliminary material, are developed in Section 2. It is then natural to separately follow through the cases when $T^{(2)} = 0$ and $T^{(2)} \neq 0$. Since it is readily shown that $V^{(2n-1)}(K) \subseteq T$, when $T^{(2)} = 0$ we have $[V^{(2n-2)}(K), V^{(2n-2)}(K)] = 0$.

This situation is then treated in Section 3, the conclusion of Theorem 3.3 being that $V^{(6n-7)}(K) = 0$. In Section 4 Theorem 4.4 is then invoked in order to conclude that $V^{6n-7} = 0$, which is precisely conclusion (b) of Theorem 5.2. The condition $T^{(2)} \neq 0$ is then analyzed in Section 5 in order to obtain part (a) of Theorem 5.2 and thereby complete its proof. In a fashion analogous to Section 5 the proof of Theorem 6.4 for involutions of the second kind is completed in Section 6.

2. PRELIMINARIES

In this section we first continue with more introductory material concerning the Lie structure of K in $\star$-prime rings, referring primarily to our paper [9]. After that we prove some general lemmas and derive further properties of Lie inner ideals of K.

We remind the reader of our standing assumption that R is a $\star$-prime ring. Following [9] we recall that for a subset X of R $X \equiv 0$ means $[X,K] = 0$. By [9, Lemma 5.5] for $X \subseteq K$, $X \equiv 0$ implies that X is central in R unless $\tilde{R}$ is of class BD(2). (This latter situation will not be of great interest to us since KC is 1-dimensional over C and hence K is "trivial" in the sense that $[K,K] = 0$.

Any additive subgroup U of K such that $U \equiv 0$ is clearly a Lie ideal of K (of a rather trivial sort). On the other hand if I is any $\star$-ideal of R then $[I \cap K,K]$ is always a Lie ideal of K and furthermore by [9, Corollary 5.3] $[I \cap K,K] \not\equiv 0$ provided $I \neq 0$ and $K \not\equiv 0$. Such Lie ideals will be called *standard* [9, p. 14].

If V is an additive subgroup of a Lie ideal U of K such that

$[V,U] \subseteq V$ then V is called a *Lie subideal* of K. In this paper Lie subideals of K will turn out to play an important role and fortunately in [9] Lie subideals of K have been as thoroughly analyzed as have Lie ideals of K. For V a Lie subideal of K we define J_V to be the $\star$-ideal $R[V^{[2]} \circ V^{[2]}, V^{[1]}]^2 R$ and remark [9, Theorem 3.5] that $[J_V \cap K, K] \subseteq V$. We say that V is *exceptional* if $V \not\equiv 0$ and $J_V = 0$ (equivalently, $[J_V \cap K, K] \equiv 0$). One of the main results of [9] is that exceptional Lie subideals can occur very rarely: by [9, Theorem 5.10 and Theorem 5.7] if V is exceptional then $\tilde{R}$ is of class $A(2)$, $C(2)$, or $BD(4)$.

When R is not prime we summarize a few facts about R in the form of a remark (see [1, p. 860]).

Remark 2.1. If R is not a prime ring (in which case $\star$ is necessarily of the second kind) then there exists a nonzero $\star$-ideal of R of the form $J \oplus J^\star$, with $J \neq 0$ an ideal of R such that $J \cap J^\star = 0$. This ideal can be used to construct idempotents ϵ_1, ϵ_2 in C such that $\epsilon_1 + \epsilon_2 = 1$, $\epsilon_1\epsilon_2 = 0$, $\epsilon_1^\star = \epsilon_2$. We observe that $\beta = \epsilon_1 - \epsilon_2$ is a skew element of C such that $\beta^2 = 1$. For $i = 1,2$ $\epsilon_i R$ is a prime ring with extended centroid $\epsilon_i C$, and we have $R \subseteq \epsilon_1 R \oplus \epsilon_2 R$. If I is any nonzero $\star$-ideal of R then $\epsilon_i I$ is a nonzero ideal of $\epsilon_i R$, $i = 1,2$.

In the course of this paper we shall sometimes come across so-called "differential identities" (see [5] for a definition of these), and we shall need some very special cases of the following general result in this regard.

Remark 2.2 ([3], Theorem 1). Let R be a prime ring and let f be a "differential polynomial" vanishing on a nonzero ideal of R. Then f must vanish on $A = RC$.

A well-known result of Herstein [4, p. 5] states that if R is prime and $a \in R$ is such that $[a,[a,R]] = 0$ then $a \in Z$ (the center of R).

Lemma 2.3. *Let* R *be a* $\star$*-prime ring and suppose* $[a,[a,I]] = 0$ *for some nonzero* $\star$*-ideal* I. *Then* $a \in Z$.

Proof. If R is already prime then by Remark 2.2 $[a[a,R]] = 0$ and we are finished by Herstein's result. If R is not prime by Remark 2.1 $[\epsilon_i a,[\epsilon_i a,\epsilon_i I]] = 0$ in the prime ring $\epsilon_i R$, $i = 1,2$, and by Remark 2.2 $[\epsilon_i a,[\epsilon_i a,\epsilon_i R]] = 0$ whence a is central by Herstein's result again. ■

At this point we prove a very easy but very useful lemma which slightly generalizes [9, Lemma 3.1].

Lemma 2.4. *Let* R *be any ring,* W *an additive subgroup of* R, *and* I *an ideal of* R. *Then* $[\overline{W},I] = [W,I]$ (*where* $\overline{W}$ *is the subring generated by* W).

Proof. Let $x_1,x_2,\cdots,x_n \in W$, $r \in I$. Then

$$[x_1x_2\cdots x_n,r] = [x_1x_2\cdots x_{n-1},x_nr] + [x_n,rx_1x_2\cdots x_{n-1}],$$

and so by induction we are done. ■

If $r_1,r_2,\cdots,r_n$ are elements of a ring R the notation $[r_1,r_2,\cdots,r_n]$ will designate any one of the various n-fold Lie products of $r_1,r_2,\cdots,r_n$. An obvious induction combined with the Jacobi identity yields the following result, and we leave the details of the proof to the reader.

Lemma 2.5. *Let* R *be any ring and let* $v_1, v_2, \cdots, v_n$, $r \in R$. *Then*

(*a*) $[[v_1, v_2, \cdots, v_n], r] = \Sigma \pm [w_1, \cdots, [w_n, r] \cdots]$

(*b*) $[v_1, v_2, \cdots, v_n] = \Sigma [w_1, \cdots, [w_{n-1}, w_n] \cdots]$

where $w_1, w_2, \cdots, w_n$ *is a permutation of* $v_1, v_2, \cdots, v_n$.

We now return to a ring R with involution. For the remainder of this section we will be interested in the Lie inner ideals T of K. We recall their definition: $[T,[T,K]] \subseteq T$. It follows by the Jacobi identity that $[[T,T],K] \subseteq T$, but some caution must be exercised since if W is an additive subgroup of K satisfying $[W^{(2)}, K] \subseteq W$ it does not necessarily follow that W is a Lie inner ideal of K.

Lemma 2.6. *Let* R *be a ring with* $\star$ *and let* T *be both a Lie inner ideal of* K *and a Lie subring of* K. *Then*

(*a*) $T^{(n)} \subseteq T^{(n-1)} \quad n \geq 2$,

(*b*) $T^{(n)}$ *is a Lie subring of* K,

(*c*) $T^{(n)}$ *is a Lie inner ideal of* K.

Proof. The proof of (a) is immediate by induction on n and (b) follows easily from (a) and the fact that T is a Lie subring. To prove (c) let $x, y \in T^{(n)}$, $k \in K$, and write $y = [t_1, t_2, \cdots, t_n]$. Using Lemma 2.5(a) we have

$$\begin{aligned}[x[y,k]] &= \Sigma [x, [w_1, \cdots, [w_{n-1}, [w_n, k]] \cdots]] \\ &= \Sigma [x, [w_1, \cdots, [w_{n-2}, w] \cdots]] = [x,t] \in [T^{(n)}, T] \\ &\subseteq [T^{(n-1)}, T] = T^{(n)},\end{aligned}$$

where $w_1, w_2, \cdots, w_n$ is a permutation of $t_1, t_2, \cdots, t_n$, $t = [w_{n-1}, [w_n, k]] \in T$, and where we have used the fact that T is a Lie subring. ■

An induction in conjunction with the Jacobi identity will yield the following result and again we leave the details to the reader.

Lemma 2.7. *Let* R *be a ring with* $\star$ *and let* T *be a Lie inner ideal of* K. *Then* $[T^{(n)}, K] \subseteq T^{(n-1)}$ *for all* $n \geq 2$.

For T a Lie inner ideal of K and a Lie subring of K we define I_T to be the $\star$-ideal $R[T^{(3)} \circ T^{(3)}, T^{(2)}]^2 R$.

Lemma 2.8. $[I_T \cap K, K] \subseteq T$.

Proof. Let $a, b \in T^{(3)}$, $k \in K$, $s \in S$. With the help of Lemma 2.7 the two calculations

$$[a \circ b, k] = [a,k] \circ b + a \circ [b,k] \in [T^{(3)}, K] \circ T^{(3)} \subseteq T^{(2)} \circ T^{(2)}$$

$$[a \circ b, s] = [a, b \circ s] + [b, a \circ s] \in [T^{(3)}, K] \subseteq T^{(2)}$$

show that $[T^{(3)} \circ T^{(3)}, R] \subseteq \overline{T^{(2)}}$ (the subring generated by $T^{(2)}$). Now let $a \in T^{(3)} \circ T^{(3)}$, $b \in T^{(2)}$, and $r \in R$. From what we have just shown $[a,b]r = [a,br] + b[a,r] \in \overline{T^{(2)}}$, *i.e.*, $[T^{(3)} \circ T^{(3)}, T^{(2)}]R \subseteq \overline{T^{(2)}}$. Likewise $R[T^{(3)} \circ T^{(3)}, T^{(2)}]^2 \subseteq \overline{T^{(2)}}$, and so $I_T \subseteq \overline{T^{(2)}}$. Making use of Lemma 2.4 we have

$$[I_T \cap K, K] \subseteq [\overline{T^{(2)}}, R] = [T^{(2)}, R] = [T^{(2)}, K] + [T^{(2)}, S]$$

whence

$$[I_T \cap K, K] \subseteq [T^{(2)}, K] \subseteq T,$$

and the proof is complete. ■

We close this section with a result of Benkart.

Lemma 2.9 ([2, Lemma 4.4]). *Let* R *be a ring with involution of the first kind and let* T *be a Lie inner ideal of* K *and a Lie subring of* K. *Then for all* $n \geq 2$, $T^{(2)} = 0$ *if and only if* $T^{(n)} = 0$.

Proof. We may assume that $\tilde{R}$ is not of class $BD(2)$ since the result is trivially true in that situation. By Lemma 2.6(a) the implication $T^{(2)} = 0$ implies $T^{(n)} = 0$ is obvious. Now assume $T^{(n)} = 0$ with $n \geq 3$. By Lemma 2.5(a) $[T^{(n-1)}, K] \subseteq T^{(n-1)}(K) = T^{(n-3)}(T^{(2)}(K)) \subseteq T^{(n-3)}(T) \subseteq T$, since T is a Lie subring. Therefore

$$[T^{(n-1)}, [T^{(n-1)}, K]] \subseteq [T^{(n-1)}, T] = T^{(n)} = 0.$$

Choosing $a \in T^{(n-1)}$, we have $[a, [a, K]] = 0$. By Theorem 1.1(2) we conclude that $a = 0$, and so we have shown that $T^{(n-1)} = 0$. Continuing in this fashion, we finally obtain $T^{(2)} = 0$.

3. COMMUTING PRODUCTS OF DERIVATIONS

Let P be a ring, not necessarily associative, and let $\mathrm{Der}\,P$ be the set of all derivations of P. Later in this section we shall take P to be an associative ring R and also take P to be the Lie ring K of skew elements of a ring with involution. We let $\Delta = \delta_1\delta_2\cdots\delta_n$ denote the product of $\delta_1,\delta_2,\cdots,\delta_n \in \mathrm{Der}\,P$ and let $|\Delta| = n$. For $x,y \in P$ we have the useful Leibnitz Formula:

$$(xy)^{\Delta} = \sum_{\Delta_i} x^{\Delta_i} y^{\Delta_{n-i}}$$

where

$$\Delta_i = \delta_{j_1}\delta_{j_2}\cdots\delta_{j_i} \qquad j_p < j_{p+1}$$

$$\Delta_{n-i} = \delta_{k_1}\delta_{k_2}\cdots\delta_{k_{n-i}} \qquad k_q < k_{q+1}$$

$$\{j_1,j_2,\cdots,j_i,\ k_1,k_2,\cdots,k_{n-i}\} = \{1,2,\cdots,n\}$$

$$\{j_1,j_2,\cdots,j_i\} \cap \{k_1,k_2,\cdots,k_{n-i}\} = \emptyset .$$

We define Δ_0, the product of no derivations, to be the identity map on P.

Let D be a nonempty subset of $\mathrm{Der}\,P$, fix a positive integer n, and let W be the subring of P generated by all elements of the form u^{Δ}, $\Delta = \delta_1\delta_2\cdots\delta_m$, $|\Delta| \geq n$, where u varies over P and $\delta_1,\delta_2,\cdots,\delta_m$ vary over D. The δ_i are not necessarily distinct in these expressions. We first state a lemma which appeared in [10]:

Lemma 3.1 ([10, Lemma A]). *For* $\ell = 1,2,\cdots,n$ $u^{\Omega_{n-\ell}}v^{\Lambda_{2n+\ell-1}} \in W$, *where* $u,v \in P$ *and* $\Omega_{n-\ell}$ *and* $\Lambda_{2n+\ell-1}$ *are products of elements of* D *of lengths* $n - \ell$ *and* $2n + \ell - 1$ *respectively.*

By relabeling $P = R$, $u = x$, $v = y$ the proof of Lemma 3.1 is the same proof verbatim as that of [10, Lemma A], since the associative law is never used. Therefore we omit the proof, referring the reader to [10] for details.

As a first application of Lemma 3.1 we restate a result from [10], again omitting the proof.

Theorem 3.2 ([10], Theorem 2). *Let* R *be a prime associative ring, let* $n > 0$ *be fixed, let* D *be a nonempty subset of* $\mathrm{Der}\, R$, *and suppose* $[u^{\Delta},v^{\Omega}] = 0$ *for all* $u,v \in R$ *and all* Δ, Ω *such that* $|\Delta| \geq n$ *and* $|\Omega| \geq n$. *Then either* R *is commutative or* $u^{\Gamma} = 0$ *for all* $u \in R$ *and all* Γ *for which* $|\Gamma| \geq 3n - 1$.

As a corollary of Theorem 3.2 we have

Theorem 3.3. *Let* R *be a* $\star$*-prime ring with involution of the second kind, let* $n > 0$ *be fixed, let* D *be a nonempty subset of* $\mathrm{Der}\, R$, *and let* $I \neq 0$ *be a* $\star$*-ideal of* R. *Suppose* $[x^{\Delta},y^{\Omega}] = 0$ *for all* $x,y \in I$ *and all* Δ, Ω *such that* $|\Delta| \geq n$ *and* $|\Omega| \geq n$. *When either* R *is commutative or* $x^{\Gamma} = 0$ *for all* $x \in R$ *and all* Γ *for which* $|\Gamma| \geq 3n - 1$.

Proof. If R is prime then by Remark 2.2 the "differential identity" $[x^{\Delta},y^{\Omega}] = 0$ holds for all $x,y \in R$ and the result follows from Theorem 3.2.

If R is not prime we write $R \subseteq \epsilon_1 R \oplus \epsilon_2 R$ according to the machinery of Remark 2.1. $\mathrm{Der}\, R \subseteq \mathrm{Der}(RC)$ in a natural way, and it is easy to see that for $\delta \in D$ $\epsilon_i^\delta = 0$, $i = 1,2$, whence $(\epsilon_i x)^\delta = \epsilon_i x^\delta$, $x \in I$. Clearly

$$[(\epsilon_i x)^\Delta, (\epsilon_i y)^\Omega] = \epsilon_i [x^\Delta, y^\Omega] = 0$$

for all $x,y \in I$. By Remark 2.2 again we conclude that $[(\epsilon_i x)^\Delta, (\epsilon_i y)^\Omega] = 0$ for all $x,y \in R$, $i = 1,2$. Then by Theorem 3.2 $(\epsilon_i x)^\Gamma = 0$ for all $x \in R$ and all Γ such that $|\Gamma| \geq 3n - 1$, $i = 1,2$, whence $x^\Gamma = 0$ for all $x \in R$. ■

For the remainder of this section we assume R is a prime ring with involution of the first kind.

Lemma 3.4. *If* $u,v \in K$ *are such that* $[u[v,k]] = 0$ *for all* $k \in K$ *then* $[u,v] = 0$.

Proof. For $x \in R$, $x - x^\star \in K$, so that $[u[v,x]] = [u,[v,x^\star]]$. Now, for $k \in K$, $[u[v,kv]] = [u,[v,k]v] = [v,k][u,v]$, since $[u[v,k]] = 0$. On the other hand

$$[u,[v,kv]] = [u,[v,(kv)^\star]] = [u,[v,vk]] = [u,v[v,k]] = [u,v][v,k]$$

for the same reason. Thus $[[u,v],[v,k]] = 0$ for all $k \in K$. Replacing k by $[k,u]$ we have $0 = [[u,v],[v,[k,u]]] = [[u,v],[[u,v],k]]$, again since $[u,[v,k]] = 0$. By Theorem 1.1(2) either $[u,v] = 0$ or $\tilde{R}$ is of class BD(2). However in the latter case $[K,K] = 0$ and so $[u,v] = 0$. ■

As a corollary to Lemma 3.4 we have

Lemma 3.5. *If* $u \in K$ *is such that* $[[K,u],[K,u]] = 0$ *then either* $u = 0$ *or* $\tilde{R}$ *is of class* BD(2).

Proof. For $k \in K$ set $v = [k,u]$. Then $[v,[u,K]] = 0$, which implies that $[u,v] = 0$ by Lemma 3.4, *i.e.*, $[u[u,K]] = 0$ for all $k \in K$. Again by Theorem 1.1(2) this forces $u = 0$ unless $\tilde{R}$ is of class BD(2). ■

Theorem 3.6. *Let* R *be a prime ring with involution of the first kind such that* $\tilde{R}$ *is not of class* BD(2), *let* $n > 0$ *be fixed, and let* D *be a nonempty subset of* Der K *(the set of derivations of the Lie ring* K). *Suppose that* $[u^\Delta, v^\Omega] = 0$ *for all* $u,v \in K$ *all* Δ, Ω *which are products of elements of* D *with* $|\Delta| \geq n$ *and* $|\Omega| \geq n$. *Then* $u^\Gamma = 0$ *for all* $u \in K$ *and all* Γ *for which* $|\Gamma| \geq 3n - 1$.

Proof. Let W be the Lie subring of K generated by all v^Δ, $v \in K$, $|\Delta| \geq n$. By hypothesis W is commutative. In Lemma 3.1 by letting P be the Lie ring K, interpreting the operation as the bracket product, and setting $\ell = n$, we conclude that $[K,u^\Gamma] \subseteq W$ for all $u \in K$ and all Γ such that $|\Gamma| \geq 3n - 1$. Since W is commutative $[[K,u^\Gamma],[K,u^\Gamma]] = 0$, and so by Lemma 3.5 $u^\Gamma = 0$. ■

4. LIE NILPOTENCY

Let R be any ring and let R^o denote the opposite ring, with multiplication in R^o given by $x \circ y = yx$. Putting mappings on the left, we see that R_ℓ (resp. R_r), the ring of left (resp. right) multiplications of R, is isomorphic to R (resp. R^o). The inner derivation ad a, $a \in R$, may be written as $a_\ell - a_r$. We fix a sequence of m elements $a_1, a_2, \cdots, a_m \in R$ (not necessarily distinct). For $\mathbb{N} = \{0,1,2,\cdots\}$ we let $\mathbb{N}^{2m-1}$ denote the additive semigroup of all (2m-1)-sequences with components in $\mathbb{N}$. If $x = (\alpha_m, \cdots, \alpha_1, \beta_2, \cdots, \beta_m) \in \mathbb{N}^{2m-1}$ we define the products:

$$\bar{x} = (a_m^{\alpha_m} \cdots a_1^{\alpha_1} a_2^{\beta_2} \cdots a_m^{\beta_m})_\ell, \quad \tilde{x} = (a_m^{\alpha_m} \circ \cdots \circ a_1^{\alpha_1} \circ a_2^{\beta_2} \circ \cdots \circ a_m^{\beta_m})_r.$$

Also for x we make the definitions:

$$\ell(x) = \alpha_m + \cdots + \alpha_1 + \beta_2 + \cdots + \beta_m$$

$$\ell_1(x) = \alpha_1$$

$$\ell_i(x) = \alpha_i + \beta_i, \qquad i = 2,3,\cdots,m$$

$$h_i(x) = \max(\alpha_i, \beta_i) \qquad i = 2,3,\cdots,m.$$

We then define a partial ordering on $\mathbb{N}^{2m-1}$ in the following way. For any two elements $x,y \in \mathbb{N}^{2m-1}$ we compute $\ell, \ell_1, \ell_2, \cdots, \ell_m, h_2, \cdots, h_m$ (in that order). At the first place where they disagree (if at all) we say accordingly whether $x > y$ or $x < y$. If $\ell, \ell_1, \cdots, h_m$ all agree on x and y then x and y are not comparable and we just say that $x \sim y$ (x is equivalent to y). For example, if $m = 3$, we have $(11,2,3,4,1) < (4,7,3,1,6)$ whereas $(1,4,2,5,9) \sim (9,5,2,4,1)$.

Now for $w = (\sigma_m, \cdots, \sigma_1, \tau_2, \cdots, \tau_m) \in \mathbb{N}^{2m-1}$ we define

$$\varphi(w) = (\mathrm{ad}\ a_m)^{\sigma_m} \cdots (\mathrm{ad}\ a_1)^{\sigma_1} (\mathrm{ad}\ a_2)^{\tau_2} \cdots (\mathrm{ad}\ a_m)^{\tau_m},$$

in other words,

$$(1) \qquad \varphi(w) = (a_{m\ell} - a_{mr})^{\sigma_m} \cdots (a_{1\ell} - a_{1r})^{\sigma_1} (a_{2\ell} - a_{2r})^{\tau_2} \cdots (a_{m\ell} - a_{mr})^{\tau_m}.$$

Expanding (1) we have

$$\varphi(w) = \Sigma(-1)^h \epsilon_k (a_m^{\alpha_m} \cdots a_1^{\alpha_1} a_2^{\beta_2} \cdots a_m^{\beta_m})_\ell (a_m^{\gamma_m} \circ \cdots \circ a_1^{\gamma_1} \circ a_2^{\delta_2} \circ \cdots \circ a_m^{\delta_m})_r$$

where $\alpha_i + \gamma_i = \sigma_i$, $\beta_i + \delta_i = \tau_i$, $h = \gamma_m + \cdots + \gamma_1 + \delta_2 + \cdots + \delta_m$

$$\epsilon_k = \begin{bmatrix} \sigma_m \\ \alpha_m \end{bmatrix} \cdots \begin{bmatrix} \sigma_1 \\ \alpha_1 \end{bmatrix} \begin{bmatrix} \tau_2 \\ \beta_2 \end{bmatrix} \cdots \begin{bmatrix} \tau_m \\ \beta_m \end{bmatrix}.$$

Setting $x = (\alpha_m, \cdots, \alpha_1, \beta_2, \cdots, \beta_m)$ and $u = (\gamma_m, \cdots, \gamma_1, \delta_2, \cdots, \delta_m)$ we may in turn express (1) in the more compact form

$$\varphi(w) = \sum_{x+u=w} (-1)^{\ell(u)} \epsilon(x) (\bar{x})_\ell (\tilde{u})_r, \qquad \epsilon(x) \text{ as above.}$$

Theorem 4.1. *Let* R *be any ring and let* $a_1, a_2, \cdots, a_m \in R$. *Suppose there exists* $z \in \mathbb{N}^{2m-1}$ *such that* $\bar{z} \neq 0$ *or* $\tilde{z} \neq 0$, *and suppose there exists* $t \in \mathbb{N}^{2m-1}$ *such that for all* $x \gtrsim t$, $\bar{x} = \tilde{x} = 0$. *Then there exists* $w \in \mathbb{N}^{2m-1}$ *such that* $\varphi(w) = (-1)^{\ell(y)}\,\epsilon(z)(\bar{z})_{\ell}(\tilde{y})_{r}$, *where* $\bar{z} \neq 0$, $\tilde{y} \neq 0$, *and* $\ell(z) = \ell(y)$.

Proof. Because of the existence of the element t in the hypothesis we choose $z = (\alpha_m, \cdots, \alpha_1, \beta_2, \cdots, \beta_m)$ maximal re $\bar{z} \neq 0$ or $\tilde{z} \neq 0$. Without loss of generality we may assume $\bar{z} \neq 0$ (just replace z by

$$z_0 = (\beta_m, \cdots, \beta_2, \alpha_1, \alpha_2, \cdots, \alpha_m)).$$

We proceed to pick $y = (\epsilon_m, \cdots, \epsilon_1, \rho_2, \cdots, \rho_m)$ by the following inductive process. Set $Y_1 = \{y \mid \tilde{y} \neq 0,\ \ell_i(y) = \ell_i(z),\ i = 1,2,\cdots,m\}$. Y_1 is nonempty since $z_0 \in Y_1$. We also note that for $y \in Y_1$, we have $\epsilon_1 = \alpha_1$. For $i = 2,3,\cdots,m$ we define Y_i inductively as follows:

If $h_i(z) = \alpha_i$ (*i.e.*, $\alpha_i \geq \beta_i$) then $Y_i = \{y \in Y_{i-1} \mid \epsilon_i \text{ maximal}\}$. We note that $\rho_i = \ell_i(y) - \epsilon_i = \ell_i(z) - \epsilon_i$ is then determined.

If $h_i(z) = \beta_i$ (*i.e.*, $\beta_i > \alpha_i$) then $Y_i = \{y \in Y_{i-1} \mid \rho_i \text{ maximal}\}$. We note that $\epsilon_i = \ell_i(y) - \rho_i = \ell_i(z) - \rho_i$ is then determined.

Clearly Y_m consists of a single element $(\epsilon_m, \cdots, \epsilon_1, \rho_2, \cdots, \rho_m)$ which we now fix as y.

We let $\sigma_i = \alpha_i + \epsilon_i$, $\tau_i = \beta_i + \rho_i$, and set

$$w = (\sigma_m, \cdots, \sigma_1, \tau_2, \cdots, \tau_m) = z + y.$$

Referring to the material at the beginning of this section we write

$$(2) \qquad \varphi(w) = \sum_{x+u=w} (-1)^{\ell(u)} \epsilon(x)(\bar{x})_\ell(\tilde{u})_r.$$

We shall show that for any pair $x,u \neq x,y$ appearing in (2) either $\bar{x} = 0$ or $\tilde{u} = 0$, thereby proving that $\varphi(w) = \epsilon(\bar{z})_\ell(\tilde{y})_r$.

We first claim that $\ell(x) = \ell(z) = \ell(u)$. Indeed, from $x + u = z + y$ we have $\ell(x) + \ell(u) = 2\ell(z)$. If either $\ell(x) > \ell(z)$ or $\ell(u) > \ell(z)$ then $x > z$ or $u > z$, whence $\bar{x} = 0$ or $\tilde{u} = 0$. Therefore $\ell(x) \leq \ell(z)$ and $\ell(u) \leq \ell(z)$, forcing $\ell(x) = \ell(u) = \ell(z)$.

A similar argument shows that for all i $\ell_i(x) = \ell_i(u) = \ell_i(z)$. Indeed, let i be the first subscript (if any) for which the above fails. From $x + u = z + y$ we have $\ell_i(x) + \ell_i(u) = 2\ell_i(z)$. If either $\ell_i(x) > \ell_i(z)$ or $\ell_i(u) > \ell_i(z)$ we see from the definition of the partial ordering that $x > z$ or $u > z$, whence $\bar{x} = 0$ or $\tilde{u} = 0$. It follows that $\ell_i(x) = \ell_i(u) = \ell_i(z)$.

We are finally ready to claim that, unless $x = z$ (or equivalently $u = y$), $\bar{x} = 0$ or $\tilde{u} = 0$. We write $x = (\gamma_m, \cdots, \gamma_1, \delta_2, \cdots, \delta_m)$ and $u = (\lambda_m, \cdots, \lambda_1, \mu_2, \cdots, \mu_m)$. From $\ell_1(x) = \ell_1(u) = \ell_1(z) = \ell_1(y)$ we see that $\gamma_1 = \alpha_1$ and $\lambda_1 = \epsilon_1$. Let i be the first subscript among $i = 2,3,\cdots,m$ for which the pair γ_i, δ_i is unequal to the pair α_i, β_i. Suppose first that $h_i(z) = \alpha_i \geq \beta_i$. If $\gamma_i > \alpha_i$ then $x > z$ and so $\bar{x} = 0$. If $\gamma_i < \alpha_i$ then from $x + u = z + y$ we have $\gamma_i + \lambda_i = \alpha_i + \epsilon_i$, whence $\lambda_i > \epsilon_i$. Since $u \in Y_{i-1}$ we may conclude that $\tilde{u} = 0$ by the maximality of ϵ_i. Therefore $\gamma_i = \alpha_i$ and from $\ell_i(x) = \ell_i(z)$ we obtain $\delta_i = \beta_i$. Suppose next that $h_i(z) = \beta_i > \alpha_i$. If $\delta_i > \beta_i$ then $x > z$ and so $\bar{x} = 0$. If $\delta_i < \beta_i$ then from $x + u = z + y$ we have $\delta_i + \mu_i = \beta_i + \rho_i$, whence $\mu_i > \rho_i$. Again, since $u \in Y_{i-1}$, we conclude that $\tilde{u} = 0$ by the maximality of ρ_i.

Therefore we must have $\delta_i = \beta_i$ and so from $\ell_i(x) = \ell_i(z)$ we also have $\gamma_i = \alpha_i$. The proof of Theorem 4.1 is now complete. ■

As a first application of Theorem 1 we rederive one of the main results of [10].

THEOREM 4.2 ([10, Theorem 1). *Let* R *be a prime ring of* char. 0 *and let* V *be a nonempty subset satisfying*

$$V^{(n)}(R) = 0$$

for some fixed positive integer n. *Then* $V \subseteq C + B$, B *a subset of* $A = RC$, $B^m = 0$ *where* $m = \left[\frac{n+1}{2}\right]$.

Proof. We first note that $V^{(q)}(A) = 0$ for all $q \geq n$. In particular $(\operatorname{ad} v)^n = 0$ and so by [8, Corollary 1(b)] $(v-\lambda)^m = 0$, $v \in V$, $\lambda = \lambda(v) \in C$, $m = \left[\frac{n+1}{2}\right]$. Setting $B = \{v - \lambda(v) \mid v \in V\}$ we note that B also satisfies

$$a^m = 0, \qquad a \in B \tag{3}$$

as well as

$$B^{(q)}(A) = 0 \qquad q \geq n. \tag{4}$$

Suppose there exist $a_1, a_2, \cdots, a_m \in B$ for which $a_1 a_2 \cdots a_m \neq 0$. We now follow the terminology preceding Theorem 4.1 based on this fixed choice of $a_1, a_2, \cdots, a_m$, with R replaced by $A = RC$. For $v = (0, \cdots, 0, 1, \cdots, 1)$

($m-1$ 0's and m 1's) we have $\bar{v} = a_1a_2\cdots a_m \neq 0$. On the other hand in view of (3) and the nature of the partial ordering of $\mathbb{N}^{2m-1}$ the element $t = (m,m,\cdots,m)$ is such that $\bar{x} = \tilde{x} = 0$ for all $x \underset{\sim}{>} t$. We then apply Theorem 4.1 to find $w = (\sigma_m,\cdots,\sigma_1,\tau_2,\cdots,\tau_m)$ such that

$$\varphi(w) = (\mathrm{ad}\ a_m)^{\sigma_m}\cdots(\mathrm{ad}\ a_1)^{\sigma_1}(\mathrm{ad}\ a_2)^{\tau_2}\cdots(\mathrm{ad}\ a_m)^{\tau_m}$$

$$= \epsilon(\bar{z})_{\ell}(\tilde{y})_{r}, \qquad \bar{z} \neq 0, \quad \tilde{y} \neq 0.$$

Since char. $R = 0$ we have $\epsilon \neq 0$. Since $\ell(v) = m$ we know that $\ell(z) \geq m$. Therefore $\ell(w) \geq 2m = n$ since $w = z + y$ and $\ell(z) = \ell(y)$. It follows from $\ell(w) \geq n$ and from (4) that $\varphi(w) = 0$, resulting in the contradiction $\bar{x}A\tilde{y} = 0$.

As a corollary to Theorem 4.2 we have

Theorem 4.3. *Let* R *be a* $\star$*-prime ring of* char. 0 *with involution of the second kind, and let* V *be a nonempty subset of* R *satisfying*

$$V^{(n)}(R) = 0$$

for some fixed positive integer n. *Then* $V \subseteq C + B$, B *a subset of* RC, $B^m = 0$, *where* $m = \left[\frac{n+1}{2}\right]$.

Proof. We may assume that R is not prime, so we may once again invoke the machinery of Remark 2.1 to write $R \subseteq \epsilon_1R \oplus \epsilon_2R$, recalling that each ϵ_iR is prime with extended centroid ϵ_iC. Clearly the condition $(\epsilon_iV)^{(n)}(\epsilon_iR) = 0$

prevails and so by Theorem 4.2 we have $\epsilon_i V \subseteq \epsilon_i C + \epsilon_i B_i$, $\epsilon_i B_i$ a subset of $\epsilon_i A$, $(\epsilon_i B)^{\left[\frac{n+1}{2}\right]} = 0$. Therefore

$$V \subseteq \epsilon_1 V + \epsilon_2 V \subseteq \epsilon_1 C + \epsilon_1 B_1 + \epsilon_2 C + \epsilon_2 B_2 \subseteq C + B, \quad B^{\left[\frac{n+1}{2}\right]} = 0. \quad \blacksquare$$

As a second application of Theorem 4.1 we turn our attention to the situation where R is a ring with involution of the first kind.

Theorem 4.4. *Let* R *be a prime ring with involution* $\star$ *of the first kind, of* char. 0, *and* $\tilde{R}$ *not of class* BD(2). *Let* V *be a nonempty subset of the skew elements* K *satisfying*

$$V^{(n)}(K) = 0. \tag{5}$$

Then $V^n = 0$.

Proof. From (5) we see in particular that $(\mathrm{ad}\ a)^n(K) = 0$ and so by Theorem 1.1(2) $a^{\left[\frac{n+1}{2}\right]+1} = 0$ for all $a \in V$, which implies

$$a^n = 0, \qquad a \in V. \tag{6}$$

We next claim that

$$\prod_{i=1}^{2n-1} (\mathrm{ad}\ a_i)(k \circ r) = 0 \tag{7}$$

for all $a_i \in V$, $k \in K$, $r \in R$, where $k \circ r = kr + rk = (k_\ell + k_r)(r)$. Indeed, for $r = s \in S$ (the symmetric elements of R) $k \circ s \in K$ and the result follows directly from (5). If $r = j \in K$ then the Leibnitz Formula, with $\Delta = \prod_{i=1}^{2n-1} (\operatorname{ad} a_i)$ applied to the product kj (and similarly to the product jk), yields $(kj)^\Delta = \sum_{\Delta_i} k^{\Delta_i} j^{\Delta_{2n-i}}$. In any summand either $|\Delta_i| \geq n$ or $|\Delta_{2n-i}| \geq n$ and so from (5) again we see that $(kj+jk)^\Delta = 0$ and our claim is established.

We rewrite (7) in the form

$$\prod_{i=1}^{q} (a_{i\ell} - a_{ir})(k_\ell + k_r) = 0, \qquad q \geq 2n - 1 \tag{8}$$

where $a_i \in V$ and $k \in K$. Suppose there exist $a_1, a_2, \cdots, a_n \in V$ such that $a_1 a_1 \cdots a_n \neq 0$. We now follow the terminology preceding Theorem 4.1 based on the fixed choice $a_1, a_2, \cdots, a_n$, with n of course now playing the role of m. For $v = (0, \cdots, 0, 1, \cdots, 1)$ ($n-1$ 0's and n 1's) we have $\bar{v} = a_1 a_2 \cdots a_n \neq 0$. On the other hand, in view of (6), $t = (n, n, \cdots, n)$ is such that $\bar{x} = \tilde{x} = 0$ for all $x \underset{\sim}{>} t$. We then apply Theorem 4.1 to find $w = (\sigma_n, \cdots, \sigma_1, \tau_2, \cdots, \tau_n)$ for which

$$\varphi(w) = (\operatorname{ad} a_n)^{\sigma_m} \cdots (\operatorname{ad} a_1)^{\sigma_1} (\operatorname{ad} a_2)^{\tau_2} \cdots (\operatorname{ad} a_n)^{\tau_n}$$

$$= \epsilon(\bar{z})_\ell (\tilde{y})_r, \qquad \bar{z} \neq 0, \quad \tilde{y} \neq 0, \quad \epsilon \neq 0$$

since char. $R = 0$. Since $\ell(v) = n$ we know that $\ell(z) \geq n$ and since $\ell(y) = \ell(z)$ and $w = z + y$ we see that $\ell(w) \geq 2n$. As a result we see from

(8) that

(9) $$(\bar{z})_{\ell}(\tilde{y})_{r}(k_{\ell}+k_{r}) = 0$$

for all $k \in K$. Using the isomorphism $A_{\ell}A_{r} \cong A \underset{C}{\otimes} A^{0}$ we rewrite (9) as

$$(\bar{z} \otimes \tilde{y})(k \otimes 1 + 1 \otimes k) = 0$$

which in turn can be written as

(10) $$\bar{z}k \otimes \tilde{y} + \bar{z} \otimes (\tilde{y} \circ k) = 0$$

for all $k \in K$. Since $\tilde{y} \neq 0$ we see from (10) that $\bar{z}k$, $\bar{z}$ are C–dependent for all $k \in K$, and since $\bar{z} \neq 0$ we conclude that $\bar{z}K \subseteq C\bar{z}$ and by reiteration we have $\bar{z}\bar{K} \subseteq C\bar{z}$, where $\bar{K}$ is the associative subring of R generated by K. Since $\tilde{R}$ is not of class BD(2) we see from [9, Theorem 3.3 and Remark 4.5(d)] that $\bar{K}$ contains a nonzero ideal U of R, and we reach the contradiction that $\bar{z}UC$ is a 1–dimensional right ideal of $A = RC$.

5. INVOLUTIONS OF THE FIRST KIND

Throughout this section R will denote a $\star$-prime ring with involution of the first kind (necessarily R must be a prime ring). As usual K will denote the skew elements and $A = RC$ the central closure (again a prime ring with $\star$ given by $(r\lambda)^\star = r^\star\lambda$, $r \in R$, $\lambda \in C$).

Now let V be a generalized Lie ideal of K of index $\leq n$. We define $T = T_V = \{t \in K \mid [V,[t,K]] \subseteq V\}$, a set that will prove useful in the study of V.

Lemma 5.1. T_V *is* (a) *a Lie subring of* K *and* (b) *a Lie inner ideal of* K.

Proof. Let $t,u \in T = T_V$, $k,j \in K$, and $v \in V$. The proof of (a) follows from the observation that

$$[v,[[t,u],k]] = [v,[[t,k],u]] + [v,[t,[u,k]]] \in V.$$

To prove (b) we write

$$\begin{aligned}[v,[[t,[u,k]],j]] &= [v,[[t,j],[u,k]]] + [v,[t,[[u,k],j]]]\\ &= [[v,[t,j]],[u,k]] + [[t,j],[v,[u,k]]]\\ &\quad + [v,[t,[[u,k],j]]] \in [V,[u,k]] + [[t,j],V]\\ &\quad + [v,[t,K]] \subseteq V.\end{aligned}$$

We are now ready to prove one of the main results of this paper.

THEOREM 5.2. *Let* R *be a* $\star$*-prime ring with involution of the first kind, let* $V \neq 0$ *be a generalized Lie ideal of* K *of index* $\leq n$, *and set*

$$T = T_V = \{t \in K \mid [V,[t,K]] \subseteq V\}.$$

(a) *Suppose* $\text{char. } R \neq 2$, $T^{(2)} \neq 0$, *and* $\tilde{R}$ *is not of class* $C(2)$ *or* $BD(4)$. *Then* $[J \cap K, K] \subseteq V$ *for some nonzero* $\star$*-ideal* J *of* R.

(b) *Suppose* $\text{char. } R = 0$ *and* $T^{(2)} = 0$. *Then* $V^{6n-7} = 0$.

REMARKS. Since $V \neq 0$ it necessarily follows that $\tilde{R}$ cannot be of class $BD(2)$. The two exceptional cases where $\tilde{R}$ is of class $C(2)$ or $BD(4)$ will be discussed at the end of this section.

Proof. (a) Suppose first that $[T^{(3)} \circ T^{(3)}, T^{(2)}]^2 = 0$. Since we are assuming here that $T^{(2)} \neq 0$ we know by Lemma 2.9 that $T^{(4)} \neq 0$. We thereby choose $a \neq 0 \in T^{(4)}$ and form the element

$$[[a,x_1]^2,[a,x_2]]\,[[a,x_3]^2,[a,x_4]] \tag{11}$$

in the free product $A_C\langle x_1,x_2,x_3,x_4\rangle$ of A and the free algebra $C\langle x_1,x_2,x_3,x_4\rangle$ over C. By Lemma 2.7 $[a,K] \subseteq T^{(3)}$ and so (11) is a generalized polynomial identity (GPI) on K, *i.e.*, vanishes on K. Furthermore it is nontrivial (*e.g.*, the term $ax_1ax_1ax_2ax_3ax_3ax_4$ cannot be canceled). Therefore by [7, Theorem 4.9] R, and hence $A = RC$, is GPI and by [6, Theorem 3] A has a nonzero socle H.

Suppose R is not PI. Now H is a simple ring with $\star$, with skew elements $KC \cap H$ and it is known (*e.g.*, [9, Theorem 3.3 and Theorem 5.11(d)]) in this case that $KC \cap H$ generates H as a ring. By Lemma 2.9 $T^{(3)}C \neq 0$ and therefore $[T^{(3)}C,KC\cap H] \neq 0$. But $[T^{(3)}C,KC\cap H] \subseteq T^{(2)}C \cap H$ by Lemma 2.7, and so $T^{(2)}C \cap H \neq 0$. Set $T_1 = TC \cap H$ and suppose $T_1^{(2)} = 0$. Then $[T^{(2)}C\cap H,KC\cap H]^{(2)} \subseteq (TC\cap H)^{(2)} = 0$, which violates Lemma 3.5 (with H playing the role of R, $KC \cap H$ playing the role of K, and $0 \neq T^{(2)}C \cap H$). Therefore $T_1^{(2)} \neq 0$. We then select $u_1,v_1 \in T_1$ such that $[u_1,v_1] \neq 0$. By [9, Corollary 2.9] there exists a symmetric idempotent e in H such that eAe contains u_1 and v_1 and also $(eAe:C) > 16$. The set $T_2 = T_1 \cap eAe$ is clearly both a Lie subring and a Lie inner ideal of $eKCe$. Furthermore $T_2^{(2)} \neq 0$ since $u_1,v_1 \in T_2$. By Lemma 2.9 (applied to the ring eAe) we note that $T_2^{(m)} \neq 0$ for all m. We consider the chain of C–spaces:

$$T_2 \supseteq T_2^{(2)} \supseteq \cdots \supseteq T_2^{(m)} \supseteq \cdots.$$

Because $(eAe:C) < \infty$ the sequence levels off, say, $T_2^{(m)} = T_2^{(m+1)} \neq 0$. From Lemma 2.7 we see from $[T_2^{(m)},eKCe] = [T_2^{(m+1)},eKCe] \subseteq T_2^{(m)}$ that $0 \neq T_2^{(m)}$ is a Lie ideal of $eKCe$. Since $(eAe:C) > 16$ $T_2^{(m)}$ is not an exceptional Lie subideal of $eKCe$ and so from the definition of being exceptional we have

$$[T_2^{(m)[2]} \circ T_2^{(m)[2]},\ T^{(m)[1]}]^2 \neq 0.$$

By Lemma 2.6(b) $T_2^{(m)}$ is a Lie subring and so $T_2^{(m)[2]} \subseteq T_2^{(m)} \subseteq T_2^{(3)}$ and $T_2^{(m)[1]} \subseteq T_2^{(m)} \subseteq T_2^{(2)}$. Therefore we have arrived at the contradiction $[T^{(3)}C \circ T^{(3)}C,\ T^{(2)}C]^2 \neq 0$, and so we must conclude that R is PI.

Since R is PI we know that $A = RC$ is finite dimensional over C. Again we form a sequence

$$TC \supseteq T^{(2)}C \supseteq \cdots \supseteq T^{(m)}C \supseteq \cdots$$

which must level off, say, $T^{(m)}C = T^{(m+1)}C \neq 0$. As above we conclude that $T^{(m)}C$ is a Lie ideal of KC. Since we are assuming $\tilde{R}$ is not of class $C(2)$ or $BD(4)$, $T^{(m)}C$ cannot be an exceptional Lie subideal and we reach the same contradiction as in the preceding paragraph. Therefore we must conclude that $[T^{(3)} \circ T^{(3)}, T^{(2)}]^2 \neq 0$.

The $\star$-ideal $I = I_T = R[T^{(3)} \circ T^{(3)}, T^{(2)}]R$ is therefore nonzero, by Lemma 2.8 $[I \cap K, K] \subseteq T$, and by [9, Corollary 5.3] we know that $[[I \cap K, K], K] \neq 0$. From the definition of T we have $[V, [[I \cap K, K], K]] \subseteq V$. Setting $U = [[I \cap K, K], K]$ we have $[V, U] \subseteq V$, $U \neq 0$, from which it follows that $[V, U]$ is a Lie subideal of K. By [9, Lemma 5.2(b)] $[V, U] \neq 0$ and by assumption $[V, U]$ is not exceptional. Therefore $[J \cap K, K] \subseteq [V, U] \subseteq V$ for an appropriate $\star$-ideal $J \neq 0$ (namely, the ideal J_V defined in Section 2), and the proof of (a) is now complete.

(b) We now suppose that $T^{(2)} = 0$. Using Lemma 2.5(a) we see from $[V, [V^{(n-1)}, K]] \subseteq [V, V^{(n-1)}(K)] = V^{(n)}(K) \subseteq V$ that $V^{(n-1)} \subseteq T$, whence $[V^{(n-1)}, V^{(n-1)}] = 0$. We also note that

$$V^{(2n-2)}(K) = V^{(n-2)}(V^{(n)}(K)) \subseteq V^{(n-2)}(V) = V^{(n-1)}.$$

We set $D = \{\mathrm{ad}\, v \mid v \in V\} \subseteq \mathrm{Der}\, K$ and let Δ and Ω be products of elements of D such that $|\Delta| \geq 2n - 2$ and $|\Omega| \geq 2n - 2$. Putting these observations together we see that $[x^{\Delta}, y^{\Omega}] = 0$ for all $x, y \in K$, and so by Theorem 3.6 $V^{(6n-7)}(K) = 0$. Then by Theorem 4.4 $V^{6n-7} = 0$ and the proof is complete. ∎

Suppose $\tilde{R}$ is of class $C(2)$ and $V \neq 0$ is a GLI of K. The skew elements KC of RC are a 3-dimensional simple Lie algebra over C. Suppose first that $T^{(2)} \neq 0$. Then the chain of C-spaces

$$TC \supseteq T^{(2)}C \supseteq \cdots \supseteq T^{(m)}C \supseteq \cdots$$

levels off, say, $0 \neq T^{(m)}C = T^{(m+1)}C$. From $[T^{(m)}C,KC] = [T^{(m+1)}C,KC] \subseteq T^{(m)}C$ we see that $T^{(m)}C$ is a Lie ideal of KC, and so $T^{(m)}C = KC$, whence $TC = KC$. Then $[VC,KC] = [VC,[TC,KC]] \subseteq VC$ and so VC is also a Lie ideal of KC, whence $VC = KC$. Next suppose that char. $R = 0$ and $T^{(2)} = 0$. By Theorem 5.2(b) $V^q = 0$ for appropriate q. But the special nature of RC is such that the only possible nilpotent subring of RC is of the form Cx, $x^2 = 0$. Therefore we may write $VC = Cv$, $v \in V$, $v^2 = 0$. We have thus shown

Remark 5.3. Let $\tilde{R}$ be of class $C(2)$ and let $V \neq 0$ be a GLI of K.

(a) If char. $R \neq 2$ and $T^{(2)} \neq 0$, then $VC = KC$.

(b) If char. $R = 0$ and $T^{(2)} = 0$, then $VC = Cv$, $v^2 = 0$.

Now suppose $\tilde{R}$ is of class $BD(4)$ and $V \neq 0$ is a GLI of K. The skew elements $L = KC$ of RC can be written as a Lie theoretic direct sum $L_1 \oplus L_2$, where L_i is a simple 3-dimensional Lie algebra over C. In fact the associative subring $Q_i = L_i + C1$ is such that $\tilde{Q}_i$ is of class $C(2)$. For ρ_i the projection of L on L_i and for $X \subseteq K$ we let $X_i = \rho_i XC$.

Suppose first that $T_i^{(2)} \neq 0$, $i = 1,2$. Then for all m $T_i^{(m)} \neq 0$, $i = 1,2$. As before we know that for some m $T^{(m)}C$ is a Lie ideal of L and the only possibility here is that $T^{(m)}C = L$, whence $TC = L$. Therefore the $\star$-ideal $I = I_T \neq 0$ and so $0 \neq [I \cap K,K] \subseteq T$ (see Lemma 2.8). Setting

$U = [[I \cap K, K], K]$ we then have $0 \neq [V,U] \subseteq V$ where $[V,U]$ is a Lie subideal of K.

Suppose next that char. $R = 0$ and for some j $T_j^{(2)} = 0$. Then Remark 5.3 can be applied to each of the rings Q_i, $i = 1,2$ in order to conclude that for each i, $i = 1,2$, either $V_i = L_i$ or $V_i = Cv_i, v_i^2 = 0$. Summarizing these considerations we have

Remark 5.4. Let $\tilde{R}$ be of class BD(4) and let $V \neq 0$ be a GLI of K.

(a) If char. $R \neq 2$ and $T_i^{(2)} \neq 0$, $i = 1,2$, then V contains a nonzero Lie subideal of K.

(b) If char. $R = 0$ and some $T_j^{(2)} = 0$ then for each $i = 1,2$ either $V_i = L_i$ or $V_i = Cv_i$, $v_i^2 = 0$.

6. INVOLUTIONS OF THE SECOND KIND

Throughout this section R will denote a $\star$-prime ring with involution of the second kind. We fix a skew element $\beta \neq 0$ in C and set $\alpha = \beta^2 \in C_\star$. We also fix a $\star$-ideal $I \neq 0$ of R such that $\beta I \subseteq R$ and $\beta^{-1} I \subseteq R$. We note that $I = I \cap K + I \cap S = I \cap K + \beta\, \beta^{-1}(I \cap S) \subseteq K + \beta K$. For U an additive subgroup of R and $\lambda \in C$ we shall use the compact notation U_λ to denote the polynomials in λ over U, *i.e.*, a typical element of U_λ is of the form $\sum_{i=0}^{m} u_i \lambda^i$, $u_i \in U$. specifically in this section we will have occasion to work in the chain of coverings $R \subseteq R_\alpha \subseteq R_\beta$, noting in particular that the skew (resp. symmetric) elements of R_α are K_α (resp. S_α).

Now let V be a generalized Lie ideal of K of index $\leq n$. Since

$I \subseteq K + \beta K$ we see that $V^{(n)}(I) \subseteq V_\beta$. Analogous to the set T_V used in the previous section we will find here as a useful took the set

$$T_0 = \{t \in R \mid [V,[t,I]] \subseteq V_\beta\}.$$

For any additive subgroup W of R we define W to be a *Lie I-inner ideal of R* if $W^{(2)}(I) \subseteq W$. We begin with

Lemma 6.1. T_0 *is* (a) *a subring of* R, (b) *a Lie* I*–inner ideal of* R, *and* (c) *invariant under* $\star$.

Proof. (a) Let $u,t \in T_0$, $x \in I$, and $v \in V$. We first remark that $[tu,x] = [t,ux] + [u,xt]$, noting that ux, $xt \in I$. Then

$$[v,[tu,x]] = [v,[t,ux]] + [v,[u,xt]] \in V_\beta$$

since $t,u \in T_0$, and so $tu \in T_0$.

(b) Let $t,u \in T_0$, $r,x \in I$, and $v \in V$. Using the Jacobi identity we first have

$$[[t,[u,r]],x] = [[t,x],[u,r]] + [t,[[u,r],x]]. \tag{12}$$

Commuting (12) with v and using the Jacobi identity again we have

$$[v,[[t,[u,r]],x]] = [[v,[t,x]],[u,r]] + [[t,x],[v,[u,r]]]$$

$$+ [v,[t,[[u,r],x]]] \in [V_\beta,[u,r]] + [[t,x],V_\beta]$$

$$+ V_\beta \subseteq V_\beta.$$

(c) If $t \in T$, $x \in I$, and $v \in V$ then

$$[v,[t^\star,x]] = -[v^\star,[x^\star,t]^\star] = -[[x^\star,t],v]^\star \in V_\beta^\star = V_\beta.$$

LEMMA 6.2. *Either* T_0 *contains a nonzero* $\star$*-ideal of* R *or* $T_0^{(2)} = 0$.

Proof. Let $x \in T_0^{(2)}$, $y \in T_0$, and $r \in I$. Since $[T_0^{(2)},I] \subseteq T_0^{(2)}(I) \subseteq T_0$ we note that $[x,I] \subseteq T_0$. Using this observation in conjunction with T_0 being a subring we see from $[x,yr] = [x,y]r + y[x,r]$ that $[x,y]I \subseteq T_0$. Now let $r_0,r_1,r_2 \in I$. Then

$$(13) \qquad [[r_0,[[x,y],r_1]],\ [[x,y],r_2]] \in T_0.$$

Expansion of (13) yields $r_0[x,y]r_1[x,y]r_2 + [x,y]w \in T_0$ for some $w \in I$, and so we conclude that $r_0[x,y]r_1[x,y]r_2 \in T_0$. Thus the ideal $J = I[x,y]I[x,y]I \subseteq T_0$. If $J \neq 0$ then $J^\star \subseteq T_0$ by Lemma 6.1(c), whence the $\star$-ideal $J + J^\star$ lies in T_0 and we are finished. We may therefore assume that $J = 0$, which implies that $[x,y]I[x,y] = 0$ (since I is a $\star$-ideal) and in turn that $[x,y] = 0$ (since R is semiprime and I is a $\star$-ideal). We have thus established that $T_0^{(3)} = [T_0^{(2)},T_0] = 0$, which implies $T_0^{(2)}(T_0^{(2)}(I)) = 0$. In particular for $a \in T_0^{(2)}$ we have $[a,[a,I]] = 0$. By Lemma 2.3 we conclude that $a = 0$, and the proof is complete. ∎

Before proceeding to the main result of this section we wish to show that even an exceptional Lie subideal of K contains a nearly standard Lie ideal of K. To this end if W is a Lie subideal of K we define P_W to be the $\star$-ideal $IW^{[4]}I$.

Lemma 6.3. $[P_W \cap K, I \cap K] \subseteq W_\alpha$.

Proof. Since W is a Lie subideal of K it is easy to see that $[W^{[1]},K] \subseteq W$. From this, using the Jacobi identity, we obtain $[W^{[2]},K] \subseteq W^{[1]}$, and it then follows that

$$[W^{[2]},I] = [W^{[2]},I \cap K + \beta\, \beta^{-1}(I \cap S)] \subseteq W^{[1]} + \beta\, W^{[1]} \subseteq W_\beta^{[1]}.$$

More generally an easy induction establishes $[W^{[m+1]},I] \subseteq W^{[m]}$. Now let $p,q \in W^{[2]}$ and $x \in I$. Then

$$[p,q]x = [p,qx] - q[p,x] \in [W^{[2]},I] + W^{[2]}[W^{[2]},I]$$

$$\subseteq W_\beta^{[1]} + [W^{[1]},W^{[1]}]W_\beta^{[1]} \subseteq \overline{W_\beta^{[1]}}$$

(the subring generated by $W_\beta^{[1]}$). Therefore $W^{[3]}I \subseteq \overline{W_\beta^{[1]}}$. Next let $p,q \in W^{[3]}$ and $x,y \in I$. We write

$$x[p,q]y = [x,[p,q]]y + [p,q]xy = [[x,p],q]y + [p,[x,q]]y$$

$$+ [p,q]xy \in [W_\beta^{[2]}, W^{[3]}]I + [W^{[3]}, W_\beta^{[2]}]I + W^{[4]}I$$

$$\subseteq W_\beta^{[3]}I \subseteq \overline{W_\beta^{[1]}}.$$

Thus $P = P_W \subseteq \overline{W_\beta^{[1]}}$. With the aid of Lemma 2.4 we note that

$$[P\cap K, I\cap K] \subseteq [\overline{W_\beta^{[1]}}, I] = [W_\beta^{[1]}, I] = [W_\alpha^{[1]} + \beta\, W_\alpha^{[1]},\ I\cap K + I\cap S]$$

$$\subseteq [W_\alpha^{[1]}, I\cap K] + [W_\alpha^{[1]}, \beta(I\cap S)] + S_0,$$

where S_0 is symmetric. Therefore

$$[P\cap K, I\cap K] \subseteq [W_2^{[1]}, I\cap K] + [W_\alpha^{[1]}, \beta(I\cap S)] \subseteq [W_\alpha^{[1]}, K] \subseteq W_\alpha.$$

Theorem 6.4. *Let* R *be a* $\star$*-prime ring with involution of the second kind. Let* β *be a skew element of* C, *let* I *be a nonzero* $\star$*-ideal such that* $\beta I \subseteq R$ *and* $\beta^{-1}I \subseteq R$, *and set* $\alpha = \beta^3 \in C_\star$. *Let* V *be a noncentral generalized Lie ideal of* K *of index* $\leq n$ *and let*
$T_0 = \{t \in R \mid [V,[t,I]] \subseteq V_\beta\}$.

(a) *Suppose* char. $R \neq 2$ *and* $T_0^{(2)} \neq 0$. *Then* $[J\cap K, K] \subseteq V_\alpha$ *for some nonzero* $\star$*-ideal* J *of* R *unless* $\tilde{R}$ *is of class* $C(2)$ *in which case* $[J\cap K, I\cap K] \subseteq V_\alpha$ *for some nonzero* $\star$*-ideal* J *of* R.

(b) *Suppose* char. $R = 0$ *and* $T_0^{(2)} = 0$. *Then* $V \subseteq C + B$, B *a subset of* RC, $B^{3n-3} = 0$.

Proof. We first remark that it suffices to prove the theorem in the case where α lies in the centroid of R and $\alpha V \subseteq V$, *i.e.*, $R = R_\alpha$ and $V = V_\alpha$.

(a) By Lemma 6.2 T_0 contains a nonzero $\star$-ideal J of R and without loss of generality we may assume that $J \subseteq I$. By the definition of T_0 we then have $[V,J^{(2)}] \subseteq V_\beta$. In particular $[V,(J\cap K)^{(2)}] \subseteq V_\beta$ whence $[V,(J\cap K)^{(2)}]$ must lie in $V = V_\alpha$ (the skew component of V_β). Setting $U = (J\cap K)^{(2)}$ we write $[V,U] \subseteq V$, noting that $U \not\subseteq 0$. It follows that $[V,U]$ is a Lie ideal of U and therefore $[V,U]$ is a Lie subideal of K. By [9, Lemma 5.2(b)] $[V,U] \not\subseteq 0$ and by [9, Lemma 5.2(c)] $[V,U]^{[4]} \not\subseteq 0$. If $\tilde{R}$ is not of class $C(2)$ then $[V,U]$ is not exceptional. In this case we know there is a nonzero $\star$-ideal $J_0 = J_{[V,U]}$ such that $[J_0\cap K,K] \subseteq [V,U] \subseteq V$ and we are done. If $\tilde{R}$ is of class $C(2)$ we still have the nonzero $\star$-ideal $P = I[V,U]^{[4]}I$, and by Lemma 6.3 $[P\cap K,I\cap K] \subseteq [V,U] \subseteq V$.

(b) We now assume $T_0^{(2)} = 0$. Making use of Lemma 2.5 we see from $[V,[V^{(n-1)},I]] \subseteq [V,V^{(n-1)}(I)] = V^{(n)}(I) \subseteq V_\beta$ that $V^{(n-1)} \subseteq T_0$, whence $[V^{(n-1)},V^{(n-1)}] = 0$. We also note that $V^{(2n-2)}(I) = V^{(n-2)}(V^{(n)}(I)) \subseteq V^{(n-2)}(V_\beta) = V_\beta^{(n-1)}$. We set $D = \{\mathrm{ad}\ v \mid v \in V\} \subseteq \mathrm{Der}\, R$ and let Δ and Ω be products of elements of D such that $|\Delta| \geq 2n-2$ and $|\Omega| \geq 2n-2$. We have just shown that $[x^\Delta,y^\Omega] = 0$ for all $x,y \in I$. By Theorem 3.3 $x^\Gamma = 0$ for all $x \in R$ and all Γ for which $|\Gamma| \geq 6n-7$, *i.e.*, $V^{(6n-7)}(R) = 0$. Then by Theorem 4.3 $V \subseteq C + B$, B a subset of RC, $B^{3n-3} = 0$, and the proof is complete. ∎

Remark 6.5. Suppose R is $\star$-prime but not prime. Then the skew element β in C may be chosen so that $\beta^2 = 1$ (as mentioned earlier in Remark 2.1). Accordingly the conclusion in part (a) of Theorem 6.4 may be strengthened to read: " $[J\cap K,K] \subseteq V$ " (if $\tilde{R}$ is not of class $C(2)$) and " $[J\cap K,I\cap K] \subseteq V$ " (if $\tilde{R}$ is of class $C(2)$).

Remark 6.6. Suppose $R = Q \oplus Q^o$, Q a prime ring, Q^o the opposite ring, $\star$ the exchange involution $(x,y) \longrightarrow (y,x)$. Choosing $\beta = (1,-1)$ as a central skew element we not only have $\beta^2 = (1,1)$ but also the $\star$-ideal I such that $\beta I \subseteq R$ and $\beta^{-1} I \subseteq R$ may be taken to be R itself. Therefore the entire conclusion in part (a) of Theorem 6.4 may be further strengthened to simply read " $[J\cap K,K\} \subseteq V$."

Taking advantage of the Lie isomorphism $x \longrightarrow (x,-x)$ between the prime ring Q (considered as a Lie ring under $[x,y]$) and the skew elements K of $R = Q \oplus Q^o$, we are able to recapture Theorem 1.2 (the main result of [10]) as a corollary to Theorem 6.4 and Remark 6.6.

Theorem 6.7. *Let* Q *be a prime ring, let* V *be a noncentral generalized Lie ideal of* Q *of index* $\leq n$, *and let* $T = \{t \in Q \mid [V,[t,Q]] \subseteq V\}$.

(a) *If* char. $Q \neq 2$ *and* $T^{(2)} \neq 0$ *then* $[J,Q] \subseteq V$ *for some nonzero ideal* J *of* Q.

(b) *If* char. $Q = 0$ *and* $T^{(2)} = 0$ *then* $V \subseteq C + B$, C *the extended centroid of* Q, B *a subset of* RC, $B^{3n-3} = 0$.

REFERENCES

[1] W.E. Baxter and W.S. Martindale, The extended centroid in $\star$-prime rings, *Comm. Algebra* 10(1982), 847–874.

[2] G.M. Benkart, The Lie inner structure of associative rings, *J. Algebra* **43**(1976), 561–584.

[3] C.-L. Chuang, Differential identities with automorphisms and antiautomorphisms II, *J. Algebra*, to appear.

[4] I.N. Herstein, On the Lie structure of an associative ring, *J. Algebra* **14**(1970), 561–571.

[5] V.K. Kharchenko, Differential identities of prime rings, *Algebra and Logic* **17**(1978), 155–168.

[6] W.S. Martindale, Prime rings satisfying a generalized polynomial identity, *J. Algebra* **12**(1969), 576–584.

[7] W.S. Martindale, Prime rings with involution and generalized polynomial identities, *J. Algebra* **22**(1972), 520–616.

[8] W.S. Martindale and C.R. Miers, On the iterates of derivations of prime rings, *Pac. J. Math.* **104**(1983), 179–190.

[9] W.S. Martindale and C.R. Miers, Herstein's Lie theory revisted, *J. Algebra* **98**(1986), 14–37.

[10] W.S. Martindale and C.R. Miers, Nilpotency and generalized Lie ideals, *J. Algebra* **127**(1989), 244–254.

[11] W.S. Martindale and C.R. Miers, Nilpotent inner derivations of the skew elements of prime rings with involution, *Canad. J. Math.*, to appear.