SOME FORMULAS FOR THE BERNOULLI AND EULER POLYNOMIALS AT RATIONAL ARGUMENTS

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1. INTRODUCTION

In the usual notations, let $B_n(x)$ and $E_n(x)$ denote, respectively, the classical Bernoulli and Euler polynomials of degree n in x, defined by the generating functions:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi) \tag{1.1}$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (|t| < \pi). \tag{1.2}$$

Also let

$$B_n := B_n(0) \quad \text{and} \quad E_n := E_n(0), \tag{1.3}$$

where B_n and E_n are, respectively, the Bernoulli and Euler numbers of order n.

The Riemann Zeta function $\zeta(s)$ and the Hurwitz Zeta function $\zeta(s,a)$ are defined (for $\text{Re}(s) > 1$) by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{and} \quad \zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (a \neq 0, -1, -2, \cdots) \tag{1.4}$$

and (for $\text{Re}(s) \leqq 1; s \neq 1$) by their meromorphic continuations (see, for details, Titchmarsh [11]).

By employing a number of properties and characteristics of certain Dirichlet and trigonometric series, Cvijović and Klinowski [5] evaluated the Bernoulli polynomials $B_n(x)$ with $n = 2, 3, 4, \cdots$, and the Euler polynomials $E_n(x)$ with $n = 1, 2, 3, \cdots$, for $0 \leqq x \leqq 1$ when x is a rational number. For the sake of ready reference, we recall here the *main* results of Cvijović and Klinowski [5] as Theorem A and Theorem B below:

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Theorem A. *In terms of the Hurwitz Zeta function $\zeta(s,a)$ defined by (1.4), the Bernoulli polynomials $B_n(x)$ at rational arguments are given by*

$$B_{2n-1}\left(\frac{p}{q}\right) = (-1)^n \frac{2(2n-1)!}{(2q\pi)^{2n-1}} \sum_{j=1}^{q} \zeta\left(2n-1, \frac{j}{q}\right) \sin\left(\frac{2jp\pi}{q}\right) \tag{1.5}$$

$$(n \in \mathbb{N}\backslash\{1\}; \mathbb{N} := \{1,2,3,\cdots\}; p \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}; q \in \mathbb{N}; 0 \leqq p \leqq q)$$

and

$$B_{2n}\left(\frac{p}{q}\right) = (-1)^{n-1} \frac{2(2n)!}{(2q\pi)^{2n}} \sum_{j=1}^{q} \zeta\left(2n, \frac{j}{q}\right) \cos\left(\frac{2jp\pi}{q}\right) \tag{1.6}$$

$$(n \in \mathbb{N}; p \in \mathbb{N}_0; q \in \mathbb{N}; 0 \leqq p \leqq q).$$

Theorem B. *In terms of the Hurwitz Zeta function $\zeta(s,a)$ defined by (1.4), the Euler polynomials $E_n(x)$ at rational arguments are given by*

$$E_{2n-1}\left(\frac{p}{q}\right) = (-1)^n \frac{4(2n-1)!}{(2q\pi)^{2n}} \sum_{j=1}^{q} \zeta\left(2n, \frac{2j-1}{q}\right) \cos\left(\frac{(2j-1)p\pi}{q}\right) \tag{1.7}$$

$$(n \in \mathbb{N}; p \in \mathbb{N}_0; q \in \mathbb{N}; 0 \leqq p \leqq q)$$

and

$$E_{2n}\left(\frac{p}{q}\right) = (-1)^n \frac{4(2n)!}{(2q\pi)^{2n+1}} \sum_{j=1}^{q} \zeta\left(2n+1, \frac{2j-1}{2q}\right) \sin\left(\frac{(2j-1)p\pi}{q}\right) \tag{1.8}$$

$$(n \in \mathbb{N}; p \in \mathbb{N}_0; q \in \mathbb{N}; 0 \leqq p \leqq q).$$

As pointed out by Cvijović and Klinowski [5, p. 1535], the formula (1.5) was derived in a completely different way by Almkvist and Meurman [1, p. 107, Proposition 10], who applied this formula to show *eventually* that

$$q^{2n-1} B_{2n-1}\left(\frac{p}{q}\right) \quad (n \in \mathbb{N}\backslash\{1\})$$

is an integer for *all* positive integers p and q. The main object of this paper is to present some remarkably shorter proofs of each of Theorem A and Theorem B. Indeed we show that Theorem B is actually a simple consequence of Theorem A. We also consider a number of other results which are relevant to our present investigation.

2. Shorter Proofs of Theorem A and Theorem B

First Proof of Theorem A. We begin by recalling the following known result (*cf.*, *e.g.*, Magnus *et al.* [10, p. 27]):

$$B_n(x) = -\frac{2\cdot n!}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{1}{k^n}\cos\left(2k\pi x - \frac{n\pi}{2}\right) \tag{2.1}$$

$$\left(n \in \mathbb{N}\backslash\{1\} \quad \text{and} \quad 0 \leqq x \leqq 1; n = 1 \quad \text{and} \quad 0 < x < 1\right),$$

which, in view of the definition in (1.4) *and* the elementary series identity:

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{q}\sum_{k=0}^{\infty} f(qk+j) \qquad (q \in \mathbb{N}), \tag{2.2}$$

immediately yields

$$B_n\left(\frac{p}{q}\right) = -\frac{2\cdot n!}{(2q\pi)}\sum_{j=1}^{q}\zeta\left(n, \frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q} - \frac{n\pi}{2}\right) \tag{2.3}$$

$$\left(n \in \mathbb{N}\backslash\{1\} p \in \mathbb{N}_0; q \in \mathbb{N}; 0 \leqq p \leqq q\right),$$

where we have also set

$$x = \frac{p}{q} \qquad \left(p \in \mathbb{N}_0; q \in \mathbb{N}; 0 \leqq p \leqq q\right).$$

Upon replacing n in (2.3) by $2n-1$ and $2n$, respectively, we obtain the assertions (1.5) and (1.6) of Theorem A.

Second Proof of Theorem A. The series identity (2.2), when applied to the classical Hurwitz formula (*cf.*, *e.g.*, Whittaker and Watson [12, p. 269]):

$$\begin{aligned}\zeta(s,a) &= \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left\{\sin\left(\frac{s\pi}{2}\right)\sum_{k=1}^{\infty}\frac{\cos(2ka\pi)}{k^{1-s}}\right.\\ &\qquad \left. + \cos\left(\frac{s\pi}{2}\right)\sum_{k=1}^{\infty}\frac{\sin(2ka\pi)}{k^{1-s}}\right\}\end{aligned} \tag{2.4}$$

with

$$s \longmapsto 1-s \quad \text{and} \quad a = \frac{p}{q} \qquad (p, q \in \mathbb{N}),$$

gives us Rademacher's formula (*cf.* Magnus *et al.* [10, p. 23]; see also Apostol [3, p. 261]):

$$\zeta\left(1-s, \frac{p}{q}\right) = \frac{2\Gamma(s)}{(2q\pi)^s}\sum_{j=1}^{q}\zeta\left(s, \frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q} - \frac{s\pi}{2}\right) \tag{2.5}$$

$$(p, q \in \mathbb{N}).$$

Upon setting $s = n$ $(n \in \mathbb{N}\backslash\{1\})$ in (2.5), if we apply the familiar relationship [10, p. 23]:

$$\zeta(1-n, a) = -\frac{B_n(a)}{n} \qquad (n \in \mathbb{N}), \tag{2.6}$$

we arrive once again at the formula (2.3) which unifies the assertions (1.5) and (1.6) of Theorem A, the missing case $p = 0$ for the Bernoulli numbers B_n being easily verified *directly* by means of the well-known identities:

$$B_{2n} = (-1)^{n-1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n) \qquad (n \in \mathbb{N}) \tag{2.7}$$

and

$$\zeta(s) = q^{-s}\sum_{j=1}^{q}\zeta\left(s, \frac{j}{q}\right) \qquad (q \in \mathbb{N}). \tag{2.8}$$

First Proof of Theorem B. We now show that Theorem B is actually a simple consequence of Theorem A. Indeed, by appealing to the known relationship [10, p. 29]:

$$E_n(x) = \frac{2}{n+1}\left[B_{n+1}(x) - 2^{n-1}B_{n+1}\left(\frac{x}{2}\right)\right] \qquad (n \in \mathbb{N}_0), \tag{2.9}$$

we find from (2.3) [with n replaced by $n+1$] that

$$\begin{aligned} E_n\left(\frac{p}{q}\right) &= \frac{2}{n+1}\left[-\frac{2(n+1)!}{(2q\pi)^{n+1}}\sum_{j=1}^{q}\zeta\left(n+1, \frac{j}{q}\right)\sin\left(\frac{2jp\pi}{q} - \frac{n\pi}{2}\right)\right. \\ &\left. +2^{n+1}\frac{2(n+1)!}{(4q\pi)^{n+1}}\sum_{j=1}^{2q}\zeta\left(n+1, \frac{j}{2q}\right)\sin\left(\frac{jp\pi}{q} - \frac{n\pi}{2}\right)\right] \\ &= \frac{4 \cdot n!}{(2q\pi)^{n+1}}\left[\sum_{j=1}^{2q}\zeta\left(n+1, \frac{j}{2q}\right)\sin\left(\frac{jp\pi}{q} - \frac{n\pi}{2}\right)\right. \\ &\left. -\sum_{j=1}^{q}\zeta\left(n+1, \frac{j}{p}\right)\sin\left(\frac{2jp\pi}{q} - \frac{n\pi}{2}\right)\right] \end{aligned} \tag{2.10}$$

$$(n \in \mathbb{N}; p \in \mathbb{N}_0; q \in \mathbb{N}; 0 \leqq p \leqq q).$$

Obviously, since

$$\begin{aligned} &\sum_{j=1}^{2q}\zeta\left(n+1, \frac{j}{2q}\right)\sin\left(\frac{jp\pi}{q} - \frac{n\pi}{2}\right) \\ &= \sum_{j=1}^{q}\zeta\left(n+1, \frac{2j-1}{2q}\right)\sin\left(\frac{(2j-1)p\pi}{q} - \frac{n\pi}{2}\right) \\ &+ \sum_{j=1}^{q}\zeta\left(n+1, \frac{j}{q}\right)\sin\left(\frac{2jp\pi}{q} - \frac{n\pi}{2}\right), \end{aligned} \tag{2.11}$$

upon separating the even and odd terms, (2.10) leads us immediately to the formula:

$$E_n\left(\frac{p}{q}\right) = \frac{4 \cdot n!}{(2q\pi)^{n+1}} \sum_{j=1}^{q} \zeta\left(n+1, \frac{2j-1}{2q}\right) \sin\left(\frac{(2j-1)p\pi}{q} - \frac{n\pi}{2}\right) \tag{2.12}$$

$$\left(n \in \mathbb{N}; p \in \mathbb{N}_0; q \in \mathbb{N}; 0 \leqq p \leqq q\right),$$

which yields the assertions (1.7) and (1.8) of Theorem B when n is replaced by $2n-1$ and $2n$, respectively.

Second Proof of Theorem B. Just as in our first proof of Theorem A, the unification (2.12) of the assertions (1.7) and (1.8) of Theorem B can be proven *directly* by merely applying the series identity (2.2) in the following known result (*cf.*, *e.g.*, Magnus *et al.* [10, p. 30]):

$$E_n(x) = \frac{4 \cdot n!}{\pi^{n+1}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{n+1}} \sin\left((2k-1)\pi x - \frac{n\pi}{2}\right) \tag{2.13}$$

$$(n \in \mathbb{N} \quad \text{and} \quad 0 \leqq x \leqq 1; n = 0 \quad \text{and} \quad 0 < x < 1),$$

with, of course,

$$x = \frac{p}{q} \qquad \left(p \in \mathbb{N}; q \in \mathbb{N}; 0 \leqq p \leqq q\right).$$

3. Formulas Involving the Hurwitz-Lerch Zeta Function

Since

$$\zeta(n+1, z) = \frac{(-1)^{n+1}}{n!} \psi^{(n)}(z) \qquad (n \in \mathbb{N}; z \neq 0, -1, -2, \cdots), \tag{3.1}$$

where $\psi^{(n)}(z)$ $(n \in \mathbb{N}_0)$ denotes the Polygamma function defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \{\log \Gamma(z)\} \qquad (n \in \mathbb{N}_0), \tag{3.2}$$

each of the formulas (2.3) and (2.12), and their even and odd integer versions given by Theorem A and Theorem B, can easily be restated in terms of the Polygamma function.

A general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (*cf.*, *e.g.*, Erdélyi *et al.* [7, p. 27 *et seq.*])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{3.3}$$

$$(a \neq 0, -1, -2, \cdots; s \in \mathbb{C} \text{ when } |z| < 1; \operatorname{Re}(s) > 1 \text{ when } |z| = 1)$$

contains, as its special cases, not only the Riemann and Hurwitz Zeta functions:

$$\zeta(s) = \Phi(1, s, 1) \qquad \text{and} \qquad \zeta(s, a) = \Phi(1, s, a) \tag{3.4}$$

and the Lerch Zeta function:

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi} \Phi\left(e^{2\pi i\xi}, s, 1\right) \tag{3.5}$$

$$(\xi \in \mathbb{R}; \mathrm{Re}(s) > 1),$$

but also other functions such as the Polylogarithmic function:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1) \tag{3.6}$$

$$(s \in \mathbb{C} \quad \text{when} \quad |z| < 1; \mathrm{Re}(s) > 1 \quad \text{when} \quad |z| = 1)$$

and the the generalized Zeta function (*cf.* Whittaker and Watson [12, p. 280, Example 8]);

$$\phi(\xi, a, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(e^{2\pi i\xi}, s, a\right) \tag{3.7}$$

$$(a \neq 0, -1, -2, \cdots; \mathrm{Re}(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z}; \mathrm{Re}(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}),$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions.

For the general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (3.3), it is easily seen by using the series identity (2.2) that

$$\Phi(z, s, a) = q^{-s} \sum_{j=1}^{q} \Phi\left(z^q, s, \frac{a+j-1}{q}\right) z^{j-1}, \tag{3.8}$$

which, in the special case when

$$z = \exp\left(\frac{2p\pi i}{q}\right) \qquad (p \in \mathbb{Z}; q \in \mathbb{N}),$$

yields the summation formula:

$$\phi\left(\frac{p}{q}, a, s\right) = q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{a+j-1}{q}\right) \exp\left(\frac{2(j-1)p\pi i}{q}\right) \tag{3.9}$$

for the generalized Zeta function $\phi(\xi, a, s)$ defined by (3.7) in terms of the Hurwitz Zeta function $\zeta(s, a)$.

For $z = 1$, (3.8) reduces at once to the identity [8, p. 360, Entry (54.13.1)]:

$$\zeta(s, a) = q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{a+j-1}{q}\right), \tag{3.10}$$

which, for $a = 1$, yields the well-known result (2.8). On the other hand, by setting $a = \frac{1}{2}$ in (3.8) and (3.9), we have

$$\sum_{n=1}^{\infty} \frac{z^n}{(2n-1)^s} = (2q)^{-s} \sum_{j=1}^{q} \Phi\left(z^q, s, \frac{2j-1}{2q}\right) z^{j-1} \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} \frac{e^{(2n+1)p\pi i/q}}{(2n+1)^s} = (2q)^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{2j-1}{2q}\right) \exp\left(\frac{(2j-1)p\pi i}{q}\right), \tag{3.12}$$

respectively. Lastly, in their special cases when $a = 1$, (3.8) and (3.9) yield the following companions of the summation formulas (3.11) and (3.12), respectively.

$$\sum_{n=1}^{\infty} \frac{z^n}{n^s} =: \mathrm{Li}_s(z) = q^{-s} \sum_{j=1}^{q} \Phi\left(z^q, s, \frac{j}{q}\right) z^j \tag{3.13}$$

and

$$\sum_{n=1}^{\infty} \frac{e^{2np\pi i/q}}{n^s} =: \ell_s\left(\frac{p}{q}\right) = q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{j}{q}\right) \exp\left(\frac{2jp\pi i}{q}\right). \tag{3.14}$$

In particular, when $s = \nu \quad (\nu > 1)$, by simply equating the real and imaginary parts in (3.12) and (3.14), we immediately obtain the following summation formulas involving trigonometric series:

$$\sum_{n=0}^{\infty} \frac{\cos\left[(2n+1)p\pi/q\right]}{(2n+1)^{\nu}} = (2q)^{-\nu} \sum_{j=1}^{q} \zeta\left(\nu, \frac{2j-1}{2q}\right) \cos\left(\frac{(2j-1)p\pi}{q}\right); \tag{3.15}$$

$$\sum_{n=0}^{\infty} \frac{\sin\left[(2n+1)p\pi/q\right]}{(2n+1)^{\nu}} = (2q)^{-\nu} \sum_{j=1}^{q} \zeta\left(\nu, \frac{2j-1}{2q}\right) \sin\left(\frac{(2j-1)p\pi}{q}\right); \tag{3.16}$$

$$\sum_{n=1}^{\infty} \frac{\cos(2np\pi/q)}{n^{\nu}} = q^{-\nu} \sum_{j=1}^{q} \zeta\left(\nu, \frac{j}{p}\right) \cos\left(\frac{2jp\pi}{q}\right); \tag{3.17}$$

$$\sum_{n=1}^{\infty} \frac{\sin(2np\pi/q)}{n^{\nu}} = q^{-\nu} \sum_{j=1}^{q} \zeta\left(\nu, \frac{j}{p}\right) \sin\left(\frac{2jp\pi}{q}\right). \tag{3.18}$$

The assertions made by the Lemma of Cvijović and Klinowski [5, p. 1530] are *precisely* the special summation formulas (3.12) and (3.18) *with* $s = \nu$ $(\nu > 1)$. Formula (3.14) with $s = \nu$ $(\nu > 1)$ is, in fact, also one of the three *main* results in another paper by Cvijović and Klinowski [6, p. 207, Equation (7)]; the other two *main* results of Cvijović and Klinowski [6, p. 207, Equations (8a) and (8b)] are essentially the same as (3.16) and (3.15), respectively, *with* p replaced by $2p$. Cvijović and Klinowski [6, p. 208, Equations (9a) and (9b)] also gave the special summation formulas (3.16) and (3.15), respectively, *with* $q = Q$.

Each of the trigonometric sums (3.15) to (3.18), and indeed also various *further* special cases of many of the summation formulas considered here, can be found to be derived in the work of Cvijović and Klinowski [5].

4. An Application of Lerch's Functional Equation

For the generalized Zeta function $\phi(\xi, a, s)$ defined by (3.7), the functional equation:

$$\begin{aligned}\phi(\xi, a, 1-s) &= \frac{\Gamma(s)}{(2\pi)^s}\left\{\exp\left[\left(\frac{1}{2}s - 2a\xi\right)\pi i\right]\phi(-a, \xi, s)\right.\\ &\quad \left. + \exp\left[\left\{-\frac{1}{2}s + 2a(1-\xi)\right\}\pi i\right]\phi(a, 1-\xi, s)\right\} \qquad (4.1)\end{aligned}$$

$$(s \in \mathbb{C}; 0 < \xi < 1)$$

was first given in 1887 by Lerch [9], whose proof of (4.1) follows the lines of the first Riemann proof of the functional equation for $\zeta(s)$ [Equation (2.4) with $a = 1$] and makes use of a certain contour integral (which provides the analytic continuation of $\phi(\xi, a, s)$ as an entire function of s for $\xi \in \mathbb{R} \setminus \mathbb{Z}$). Several other interesting proofs of Lerch's functional equation (4.1) can be found in the works of (for example) Apostol [2] and Berndt [4]. Moreover, in terms of the general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (3.3), the functional equation (4.1) was derived in (among other places) Erdélyi *et al.* [7, p. 29, Equation 1.11(7)].

In view of the summation formula (3.9), Lerch's functional equation (4.1) with

$$a = \frac{p}{q} \qquad (p \in \mathbb{Z}; q \in \mathbb{N})$$

yields

$$\begin{aligned}\phi\left(\xi, \frac{p}{q}, 1-s\right) &= \frac{\Gamma(s)}{(2p\pi)^s}\left\{\sum_{j=1}^{q}\zeta\left(s, \frac{\xi+j-q}{q}\right)\right.\\ &\quad \cdot\exp\left[\left(\frac{s}{2} - \frac{2(\xi+j-1)p}{q}\right)\pi i\right]\\ &\quad + \sum_{j=1}^{q}\zeta\left(s, \frac{j-\xi}{q}\right)\\ &\quad \left.\cdot\exp\left[\left(-\frac{s}{2} + \frac{2(j-\xi)p}{q}\right)\pi i\right]\right\}. \qquad (4.2)\end{aligned}$$

Finally, in terms of a *mild* generalization of the classical Bernoulli polynomials $B_n(x)$, which was defined by Apostol [2] by means of the generating function [2, p. 165, Equation (3.1)]:

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda)\ \frac{t^n}{n!}\ \left(|t + \log\lambda| < 2\pi\right), \qquad (4.3)$$

so that [*cf.* Equation (1.1)]

$$\mathcal{B}_n(x; 1) = B_n(x) \qquad (n \in \mathbb{N}_0), \qquad (4.4)$$

it is known that [2, p. 164]

$$\phi(\xi, a, 1-n) = -\frac{\mathcal{B}_n\left(a; e^{2\pi i\xi}\right)}{n} \qquad (n \in \mathbb{N}_0). \qquad (4.5)$$

Thus, setting $s = n \quad (n \in \mathbb{N})$ in (4.2) and using (4.5), we obtain the following (presumably new) analogue of (2.3) for the generalized Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ defined by (4.3):

$$\begin{aligned}\mathcal{B}_n\left(\frac{p}{q}; e^{2\pi i\xi}\right) &= -\frac{n!}{(2q\pi)^n}\left\{\sum_{j=1}^{q}\zeta\left(n, \frac{\xi+j-1}{q}\right)\right. \\ &\quad \cdot \exp\left[\left(\frac{n}{2} - \frac{2(\xi+j-1)p}{q}\right)\pi i\right] \\ &\quad + \sum_{j=1}^{q}\zeta\left(n, \frac{j-\xi}{q}\right) \\ &\quad \left. \cdot \exp\left[\left(-\frac{n}{2} + \frac{2(j-\xi)p}{q}\right)\pi i\right]\right\}, \end{aligned} \tag{4.6}$$

which holds true whenever each side exists. Indeed, in its *special* case when $\xi \in \mathbb{Z}$, the summation formula (4.6) can easily be shown to reduce to the known result (2.3).

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$$\mathcal{R}(w,x,s) = \sum_{k=0}^{\infty}\frac{e^{2k\pi ix}}{(w+k)^s},$$

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