Broadcast Independence and Broadcast Packing

by

Kiara A. McDonald
B.Sc., Thompson Rivers University, 2023

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

We acknowledge with respect the Lekwungen peoples on whose traditional territory the university stands, and the Songhees, Esquimalt, and W̱SÁNEĆ peoples whose historical relationships with the land continue to this day.

Broadcast Independence and Broadcast Packing

by

Kiara A. McDonald
B.Sc., Thompson Rivers University, 2023

Supervisory Committee

Dr. Jing Huang, Co-Supervisor
(Department of Mathematics and Statistics, University of Victoria)

Dr. Richard Brewster, Co-Supervisor
(Department of Mathematics and Statistics, Thompson Rivers University)

ABSTRACT

For a simple, connected graph G, a *broadcast* on G is a function $f : V(G) \to \mathbb{N}$, where $f(v)$ is at most the eccentricity of v for every vertex $v \in V(G)$, and the *weight* of f is $f(V) = \sum_{v \in V(G)} f(v)$. A vertex v is broadcasting if $f(v) > 0$ and a vertex u hears v if the distance from u to v at most $f(v)$. An independent broadcast is a broadcast in which no broadcasting vertex hears another vertex. A packing broadcast is a broadcast where no vertex hears more than one vertex. The broadcast independence number $\alpha_b(G)$, respectively the broadcast packing number $P_b(G)$, is the maximum weight over all independent broadcasts, respectively packing broadcasts, on G. In this thesis, we examine these two broadcast parameters in different classes of graphs.

Determining the broadcast independence number is NP-hard for planar graphs of maximum degree 4, and before this thesis, the best known algorithm to compute the broadcast independence number of a tree of order n had time complexity $\mathcal{O}(n^9)$, due to Bessy and Rautenbach. In this thesis, we improve the time complexity of computing the broadcast independence number of a tree to $\mathcal{O}(n^8)$ time. We achieve this through the use of the ball catch graph and maximum weighted independent sets in this graph. This method is also applied to diamond-free interval graphs, paw-free interval graphs, strongly chordal split graphs and several other subclasses of chordal and weakly chordal graphs, showing that the broadcast independence number of such graphs of order n can be found in $\mathcal{O}(n^8)$ time. Further, we establish exact values of $\alpha_b(G)$ of proper interval graphs and subclasses of split graphs.

We also examine the broadcast packing problem in this thesis. By the work of Farber and Lubiw, computing the broadcast packing number of strongly chordal graphs is known to be polynomial time solvable. In contrast, we establish the NP-hardness of the broadcast packing problem in split graphs. We also determine the exact value of the broadcast packing number of proper interval graphs and subclasses of split graphs.

Table of Contents

List of Figures

ACKNOWLEDGEMENTS

First of all, I would like to thank my co-supervisors, Dr. Richard Brewster and Dr. Jing Huang. Rick, thank you for all of your support and mentorship throughout both my undergraduate and graduate programs. This would not have been possible without you, and I will always appreciate the time and effort you have given me. Jing, thank you for all of your guidance and help over the last couple years. Your expertise and creativity has pushed my knowledge of mathematics and made this thesis possible.

I would also like to thank Dr. Saeed Rahmati of Thompson Rivers University. Your encouragement throughout my undergrad is one of the reasons why I completed a graduate degree.

I would like to thank all of the friends I have made throughout my time in Victoria. You have made the past couple years easy and enjoyable. I would also like to thank my friends from Kamloops who, despite not living in the same city as me, helped me throughout my degree.

Finally, and most importantly, I would like to thank my family- Joanna, Art, Brittany and Alba. None of this would have been possible without the endless support you have all given me. I am forever grateful for everything you have done for me.

Chapter 1
Introduction

Graph theory is a well-known and rapidly evolving field of discrete mathematics. The study of graphs is wide-ranging, using tools from algebra to algorithms. Research in this area has many applications, as computational, physical and biological structures can be modeled using graphs. One area of research in graph theory is on *broadcasts* in graphs, which was first introduced by Erwin [20]. Before we formally define what a broadcast in a graph is, let us first introduce some basic graph theory terminology and different types of graphs. In general, we follow the terminology of West [40], and restrict our attention to simple graphs.

A *graph*, G, consists of a set of *vertices* $V(G)$ and a set of *edges* $E(G)$, where each edge is an unordered pair of vertices. The edge $\{u, v\}$ is denoted by $u \sim v$. Let $G = (V, E)$ be a graph and let $v \in V(G)$. The vertex v is called *universal* if v is adjacent to every other vertex of G. We denote the *distance between u and v in G* by $d(u, v)$. The *eccentricity* of v, denoted $ecc_G(v)$, is the maximum distance from v to any other vertex u in the graph. That is, $ecc_G(v) = \max\{d(u, v)|u \in V(G)\}$. When G is clear from context, we simply write $ecc(v)$. The *diameter* of a graph G is the maximum eccentricity over all vertices of G. That is, $diam(G) = \max\{ecc(v)|v \in V(G)\}$. A pair of vertices u and v are *antipodal* if $d(u, v) = diam(G)$. We say that $N_k[v] = \{u|d(u, v) \leq k\}$ is the *ball of radius k centred at v* and $N_k(v) = N_k[v] \setminus \{v\}$ is the *open k-neighbourhood of v*.

A *path graph* on n vertices, denoted P_n, has vertices $v_1, v_2, \ldots, v_n$, and edges $v_{i-1} \sim v_i$ for all $2 \leq i \leq n$. We say P_n is a *path between* v_1 and v_n. The other vertices $v_2, v_3, \ldots v_{n-1}$ are the *internal vertices* of P_n. A *cycle graph* on $n \geq 3$ vertices, denoted C_n, has vertices $v_1, v_2, \ldots, v_n$, and edges $v_1 \sim v_n$ and $v_{i-1} \sim v_i$ for all $2 \leq i \leq n$.

For a subset $S \subseteq V(G)$, we say that *subgraph of G induced by S* is the graph whose vertex set is S and whose edge set contains all edges $u \sim v \in E(G)$ such that $u, v \in S$. Let G and H be graphs. We say G is *H-free* if G does not contain H as an induced subgraph. For a collection $H_1, H_2, \ldots, H_k$ of graphs, we say G is $(H_1, H_2, \ldots, H_k)$-*free* if G does not contain H_i as an induced subgraph, for all $1 \leq i \leq k$.

1.1 Broadcasts on graphs

Formally, a *broadcast* on a graph G is a function $f : V(G) \to \mathbb{N}$, where $f(v) \leq ecc_G(v)$ for every vertex $v \in V(G)$. The *weight* or *cost* of a broadcast f is defined as $f(V) = \sum_{v \in V(G)} f(v)$. A vertex v is *broadcasting*, or equivalently, is a *broadcast vertex*, if $f(v) > 0$. A vertex v is *not broadcasting* if $f(v) = 0$. We say that v has *power* $f(v)$. The set of broadcasting vertices in f is denoted V_f^+; in other words, $V_f^+ = \{v \in V(G) | f(v) > 0\}$. A vertex u *hears the broadcast from* v if $d(u, v) \leq f(v)$ for some $v \in V_f^+$. The set of vertices which u hears is denoted $H(u)$. For a subset $U \subseteq V(G)$, define the weight of U to be $f(U) = \sum_{u \in U} f(u)$. A k-*broadcast* is a broadcast with weight k.

An alternative way to think of broadcasts is as a packing problem, where we place balls of different radii on the vertices of our graph. Given a broadcast f as defined above, we say that we have placed a ball of radius $f(v)$ on the vertex v.

In this chapter, we examine the history of two variations of the broadcasting problem: broadcast independence and broadcast packing. As an analogy and application, consider a city where one wants to place cell towers of different signal strengths on buildings subject to certain conditions. If no tower hears the signal from another tower, the broadcast is independent; if every building in the city hears the signal from at most one tower, then the broadcast is a packing broadcast. The problems of maximizing the weights of independent and packing broadcasts are the focus of this thesis.

Independence is a standard, well-studied problem in graph theory. We say a subset $S \subseteq V(G)$ is an *independent set* if there are no edges between the vertices of S. We can rephrase the previous definition in the following way: a subset $S \subseteq V(G)$ is an independent set if for all distinct $u, v \in S$, $d(u, v) > 1$. It is easy to find an independent set S such that $|S|$ is small, such as $S = \emptyset$, thus it is more interesting to find independent sets with large size. Therefore, in the study of independence, the parameter which we are interested in is $\alpha(G)$, the maximum size of an independent set in a graph G.

Broadcast independence generalizes the notion of independence. In this thesis, we consider broadcast independence as introduced by Erwin [20], but different forms of independent broadcasts have been studied by Neilson and Mynhardt [35, 36, 37]. A broadcast f is *independent* if for every $v \in V_f^+$, $|H(v)| = 1$. In other words, no broadcast vertex hears another vertex. We can rephrase the previous definition as follows:

a broadcast f is independent if for all distinct $u, v \in V_f^+$, $d(u,v) > \max\{f(u), f(v)\}$. If we have an independent broadcast in which $f(v) \leq 1$ for all $v \in V$, the set V_f^+ is an independent set. Much like with usual independence, we are interested in finding the maximum weight of an independent broadcast. The *broadcast independence number* of G, denoted $\alpha_b(G)$, is the maximum weight of an independent broadcast on G. An example of an independent broadcast and a maximum independent broadcast is in Figure 1.1. For each vertex v, the value of $f(v)$ is the label on the vertex.

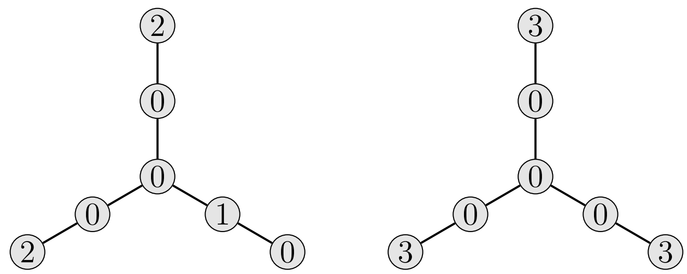


Figure 1.1: An independent broadcast and a maximum independent broadcast, respectively.

As previously mentioned, we can view a broadcast as placing balls of certain radii on the vertices of our graph. Thus, we can say that a broadcast f is independent if for all distinct $u, v \in V_f^+$, $v \notin N_{f(u)}[u]$ and $u \notin N_{f(v)}[v]$. In other words, f is an independent broadcast if for all distinct $u, v \in V_f^+$, v is not contained in the ball of radius $f(u)$ centred at u and u is not contained in the ball of radius $f(v)$ centred at v.

Since the characteristic function of an independent set is an independent broadcast, it follows that $\alpha(G) \leq \alpha_b(G)$. In [4], Bessy and Rautenbach prove that for a graph G of diameter 2, we have $\alpha(G) = \alpha_b(G)$. They also prove that the broadcast independence number is bounded by a function of the independence number.

Theorem 1.1. *[3] For any connected graph G, $\alpha_b(G) \leq 4\alpha(G)$.*

Although the broadcast independence number was introduced in 2001 [20], it has only been determined in some restriced classes of graphs [2, 7, 8, 12, 20, 30]. One reason for this is because of the algorithmic complexity of the problem. Typically, in graph theory, a standard approach is to first solve the problem in paths, cycles and trees, and then consider more general graphs. However, the problem is already challenging for trees. Before this thesis, the best known algorithm for determining the broadcast independence number in trees had a time complexity of $\mathcal{O}(n^9)$, where n is the order of the tree [4]. Additionally, Bessy and Rautenbach also show that the broadcast independence problem is NP-complete for planar graphs with maximum degree 4. Thus, one of the focuses of this thesis was to improve the time complexity of solving the problem in trees, and also determine if other classes of graphs have polynomial time algorithms.

We now present some known results on broadcast independence in several types of graphs.

Erwin provided the first result on the broadcast independence number, focusing on the parameter within paths.

Theorem 1.2. *[20] For $n \geq 3$, $\alpha_b(P_n) = 2(n-2)$.*

The above result could be rephrased as $\alpha_b(P_n) = 2(diam(P_n) - 1)$. We prove an analogous result for proper interval graphs in Section 3.1.

An independent broadcast f on G is a *maximal independent broadcast* if there is no independent broadcast g on G such that $g(v) \geq f(v)$ for all $v \in V(G)$ and $g(u) > f(u)$ for some $u \in V(G)$. In addition to determining $\alpha_b(P_n)$, Erwin [20] also provided a characterization of the structure of maximal independent broadcasts. To understand this characterization, we first define a dominating broadcast. A broadcast f on G is *dominating* if $|H(v)| \geq 1$ for all $v \in V(G)$.

Theorem 1.3. *[20] Let f be an independent broadcast on G. Then f is maximal if and only if the following conditions hold:*

1. *f is a dominating broadcast,*
2. *for all $v \in V_f^+$, $f(v) = \min\{d(u,v) | u \in V_f^+ \text{ and } u \neq v\} - 1$.*

Throughout this thesis, these two conditions are implicitly assumed when we discuss an α_b-broadcast.

In 2014, Bouchemakh and Zemir [7] determined the broadcast independence number in grid graphs. A *grid graph* $G_{m\times n}$ is an $m \times n$ rectangular lattice graph.

Theorem 1.4. *[7] Let m, n be integers such that $5 \leq m \leq n$. Then $\alpha_b(G_{m\times n}) = \lceil \frac{mn}{2} \rceil$.*

In 2020, Bouchika et al. determined the broadcast independence number of cycles.

Theorem 1.5. *[8] For $n \geq 4$, $\alpha_b(C_n) = 2(\lfloor \frac{n}{2} \rfloor - 1)$.*

In [12], Brewster and McDonald determine the broadcast independence number of perfect binary trees and perfect k-ary trees. A *perfect k-ary tree* is a rooted tree in which every vertex has exactly k children and all leaves are at the same height. A perfect k-ary tree of height h is denoted T_h^k. A *perfect binary tree* is a k-ary tree such that $k = 2$. A perfect binary tree of height h is denoted T_h.

Theorem 1.6. *[12] Let T_h be a perfect binary tree of height h. Then*

$$\alpha_b(T_h) = \begin{cases} 24 \cdot \dfrac{16^{\frac{h}{4}} - 1}{15} + 3 & h \equiv 0 \pmod 4,\ h \geq 4, \\ 3 \cdot \dfrac{16^{\lceil \frac{h}{4} \rceil} - 1}{15} + 1 & h \equiv 1 \pmod 4,\ h \geq 5, \\ 6 \cdot \dfrac{16^{\lceil \frac{h}{4} \rceil} - 1}{15} & h \equiv 2 \pmod 4,\ h \geq 6, \\ 12 \cdot \dfrac{16^{\lceil \frac{h}{4} \rceil} - 1}{15} & h \equiv 3 \pmod 4,\ h \geq 3. \end{cases}$$

Theorem 1.7. *[12] Let T_h^k be a perfect k-ary, with $k \geq 3$, tree of height h. Then*

$$\alpha_b(T_h^k) = \begin{cases} 1 + k^2 + \cdots + k^{2m} = \dfrac{(k^2)^{m+1} - 1}{k^2 - 1} & h \equiv 0 \pmod 2, \\ k + k^3 + \cdots + k^{2m+1} = k \cdot \dfrac{(k^2)^{m+1} - 1}{k^2 - 1} & h \equiv 1 \pmod 2. \end{cases}$$

Additionally, Brewster and McDonald provide a linear time algorithm for determining the broadcast independence number in spiders, where a *spider* is a tree obtained by subdividing $K_{1,k}$; that is, there are k leaves, one vertex u such that $d(u) = k$, and the remaining vertices have degree 2 [12].

A further restricted version of broadcast independence is called broadcast packing. A broadcast f is a *packing broadcast* if $|H(u)| \leq 1$ for every $u \in V(G)$. The maximum weight of a packing broadcast on G is the *broadcast packing number*, denoted by $P_b(G)$. An example of a packing broadcast and a maximum packing broadcast is in Figure 1.2.

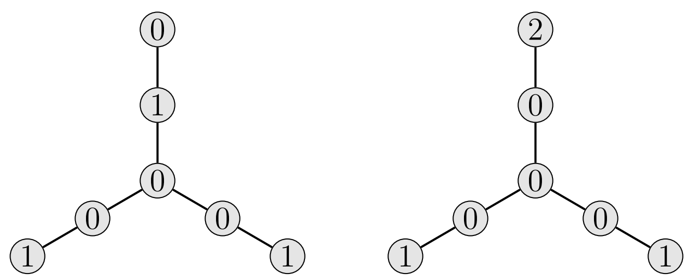


Figure 1.2: A packing broadcast and a maximum packing broadcast, respectively.

The broadcast packing number was first introduced in 2020 [8]. It is easy to see that any packing broadcast is also an independent broadcast. Therefore, we get the following observation:

Proposition 1.8. *[8] For any graph G, $P_b(G) \leq \alpha_b(G)$.*

Much like the broadcast independence number, computing the broadcast packing number is only known to be tractable for a few classes of graphs, either by an exact formula or a polynomial time algorithm.

In this thesis, we use tools from linear programming to solve the broadcast packing problem (see [12, 14, 15]). We first describe the broadcast packing problem as an integer program as in [12]. Given a graph G, for each vertex $v \in V$ and for each k, $0 \leq k \leq ecc_G(v)$, we define a binary variable $x_{v,k}$. If $x_{v,k} = 1$, then the ball of radius k centred at v is in the packing broadcast. The broadcast packing number is the maximum weight broadcast subject to the constraint that each vertex hears at most one broadcast vertex. In symbols,

$$\begin{aligned} P_b(G) \quad &= \quad \max \sum_{v \in V} \sum_{k=1}^{ecc_G(v)} k \cdot x_{v,k} \\ s.t. \quad & \sum_{v \in V} \sum_{k=d(u,v)}^{ecc_G(v)} x_{v,k} \leq 1 \qquad \text{for each } u \in V, \\ & x_{v,k} \in \{0,1\} \text{ and } x_{v,0} = 0 \quad \text{for each } v \in V \end{aligned}$$

We now define a new integer problem called the *multicover* [12]. We introduce a binary variable y_u for each $u \in V$. When $y_u = 1$, we say we have placed a *token* on vertex u. In the multicover, the objective is to minimize the sum of the y_u such that each ball of radius k has at least k tokens on the vertices contained within that ball. In symbols,

$$\begin{aligned} M_c(G) \quad &= \quad \min \sum_{u \in V} y_u \\ s.t. \quad & \sum_{\substack{u \in V \\ d(u,v) \leq k}} y_u \geq k \quad \text{for each } v \in V \text{ and } 1 \leq k \leq ecc_G(v) \\ & y_u \in \{0,1\} \qquad\qquad \text{for each } u \in V \end{aligned}$$

If we allow for the fractional relaxation of the variables $x_{v,k}$ and y_u, the broadcast packing and multicover programs become linear programs instead of integer programs. It is easy to see that the multicover linear program is the dual of the broadcast packing linear program. By the Theorem of Weak Duality in linear programming and by the description of the broadcast packing problem, if the linear programs have integer optima and equal objective functions, we have found the broadcast packing number. For strongly chordal graphs, it is known that these linear programs have integer optima and equal objective functions [21, 32]. The 7-trampoline G (formally defined in Section 1.2) in Figure 1.3 shows that for a non-strongly chordal graph, it is possible for $P_b(G) \neq M_c(G)$ when restricted to integer solutions. The red squares represent the tokens in the multicover.

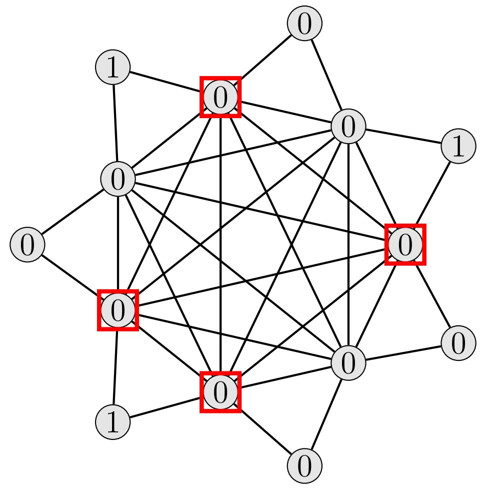


Figure 1.3: The 7-trampoline graph, G, where $P_b(G) = 3 \neq 4 = M_c(G)$.

In 2020, Bouchouika et al. [8] determined the broadcast packing number of paths and cycles, which were the first results on the broadcast packing number.

Theorem 1.9. *[8] For $n \geq 2$, $P_b(P_n) = n - 1$.*

The above result could be rephrased as $P_b(P_n) = diam(P_n)$. We prove an analogous result for proper interval graphs in Section 4.1.

Theorem 1.10. *[8] For $n \geq 2$, $P_b(C_n) = \lfloor \frac{n}{2} \rfloor$.*

In [12], Brewster and McDonald determine the broadcast packing number of perfect binary trees and perfect k-ary trees, where they make use of the dual linear program to prove their results.

Theorem 1.11. *[12] Let T_h be a perfect binary tree of height h. Then*

$$M_c(T_h) = P_b(T_h) = \begin{cases} 2 \cdot \dfrac{8^{\frac{h}{3}} - 1}{7} + 2^{h-1} + 1 & h \equiv 0 \pmod 3, \\ 4 \cdot \dfrac{8^{\frac{h-1}{3}} - 1}{7} + 2^{h-1} + 1 & h \equiv 1 \pmod 3, \\ \dfrac{8^{\frac{h+1}{3}} - 1}{7} + 2^{h-1} + 1 & h \equiv 2 \pmod 3. \end{cases}$$

Theorem 1.12. *[12] Let T_h^k be a perfect k-ary tree, with $k \geq 3$, of height h. Then*

$$M_c(T_h^k) = P_b(T_h^k) = \begin{cases} k \cdot \dfrac{(k^3)^{\frac{h}{3}-1} - 1}{k^3 - 1} + 2 \cdot k^{h-2} + (k-1) \cdot k^{h-2} & h \equiv 0 \pmod 3, \\ k^2 \cdot \dfrac{(k^3)^{\lfloor \frac{h}{3} \rfloor - 1} - 1}{k^3 - 1} + 2 \cdot k^{h-2} + (k-1) \cdot k^{h-2} + 1 & h \equiv 1 \pmod 3, \\ \dfrac{(k^3)^{\lfloor \frac{h}{3} \rfloor} - 1}{k^3 - 1} + 2 \cdot k^{h-2} + (k-1) \cdot k^{h-2} + 1 & h \equiv 2 \pmod 3. \end{cases}$$

1.2 Graph classes

A recurring theme throughout this thesis is to use the structures of different types of graphs, either in terms of vertex orderings or forbidden subgraph characterizations, to prove results about broadcast independence and broadcast packing. In this section, we provide some background on the specific graph classes which we examine in this thesis. We state many well known characterizations of various classes and refer the reader to [11, 25].

A graph G is *chordal* if G does not contain an induced cycle of length 4 or greater. Chordal graphs can also be characterized in terms of a *perfect elimination ordering.* We say a vertex v is *simplicial* in G if $N(v)$ forms a clique in G. A vertex ordering $v_1, v_2, \ldots, v_n$ of the vertices of G is a *perfect elimination ordering* if v_i is simplicial in the subgraph of G induced by the vertices $v_i, v_{i+1}, \ldots, v_n$. A graph is chordal if and only if it has a perfect elimination ordering.

Strongly chordal graphs are a subclass of chordal graphs, which can be characterized in several different ways. An *n-trampoline* ($n \geq 3$) is a graph on $2n$ vertices, say $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$, where $\{u_1, u_2, \ldots, u_n\}$ induces a copy of K_n and each v_i is adjacent to u_i and u_{i+1} with index arithmetic performed modulo n (note that n-trampolines are also commonly referred to as n-suns). Figure 1.3 contains the 7-trampoline. A graph G is *strongly chordal* if G does not contain any induced cycles of length greater than or equal to 4, or any induced n-trampolines for $n \geq 3$. That is, strongly chordal graphs are exactly the chordal graphs which do not contain an n-trampoline, $n \geq 3$, as an induced subgraph. Strongly chordal graphs can also be characterized by a vertex ordering, similar to chordal graphs. A vertex ordering $v_1, v_2, \ldots, v_n$ is a *strong ordering* if it is a perfect elimination ordering and for all $i < j$ and $k < l$, if $v_i \sim v_k$, $v_i \sim v_l$ and $v_j \sim v_k$, then $v_j \sim v_l$. A graph G is strongly chordal if and only if G has a strong ordering.

Weakly chordal graphs were first introduced by Hayward in [27]. A graph G is *weakly chordal* if neither G nor its complement contain an induced cycle of length at least 5. This provides a characterization of weakly chordal graphs in terms of forbidden subgraphs. However, we mainly consider a different characterization in this thesis, which was introduced by Hayward, Hoàng and Maffray [26].

A *two-pair* in a graph H is a pair of non-adjacent vertices x, y such that every induced path in H between x and y has length 2.

Theorem 1.13. *[26] A graph G is weakly chordal if and only if every connected, non-complete, induced subgraph of G contains a two-pair.*

This characterization is used in several theorems throughout this thesis. We provide one more important definition related to weakly chordal graphs. A *k-prism* is the graph on $2k$ vertices $\{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k\}$ where $\{v_1, v_2, \ldots, v_k\}$ and $\{u_1, u_2 \ldots, u_k\}$ each induces a clique and v_iu_i is an edge for all $1 \leq i \leq k$.

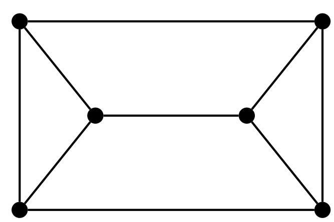


Figure 1.4: The 3-Prism.

The 3-prism is depicted in Figure 1.4. Note that complement of the 3-prism is C_6. Thus, we have the following observation, which is of importance in Chapter 2.

Observation 1.14. *If G is weakly chordal, then G is 3-prism free.*

A graph is an *interval graph* if it is the intersection graph of a family of intervals on the real line. Interval graphs can be characterized in many different ways. First of all, interval graphs are graphs that are both chordal and asteroidal triple-free [31]. Interval graphs are also graphs which are both chordal and co-comparability graphs [24]. Interval graphs can be characterized in terms of *interval orderings*. An interval ordering in a graph G is an ordering $\prec$ of the vertices of G such that for all $x \prec y \prec z$, if $x \sim z$, then $x \sim y$.

Theorem 1.15. *[38] A graph G is an interval graph if and only if it has an interval ordering.*

A subclass of interval graphs is proper interval graphs. A graph is a *proper interval graph* if it is the intersection graph of a family of inclusion-free intervals. Proper interval graphs can also be classified as the interval graphs which do not contain a claw as an induced subgraph (a claw is pictured in Figure 1.5).

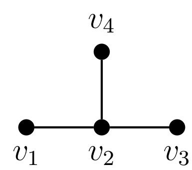


Figure 1.5: The claw graph.

Like interval graphs, proper interval graphs can also be characterized by a vertex ordering, called a *proper interval ordering*. A proper interval ordering in a graph G is an ordering $\prec$ of the vertices of G such that for all $x \prec y \prec z$, if $x \sim z$, then $x \sim y$ and $y \sim z$.

Theorem 1.16. *[11] The graph G is a proper interval graph if and only if it has a proper interal ordering.*

Both the interval order and proper interval order are vital for the proofs in Section 2.3 and Section 4.1, respectively.

A graph G is a *split graph* if its vertex set can be partitioned into an independent set, S, and a clique, C. Such a partition is called a *split parition* of G, denoted by (S, C). Note that a split graph may have different split partitions. For this thesis, in reference to a split graph G, we assume that we consider a split partition (S, C) such that $|S|$ is maximum. Since we chose S such that $|S|$ is maximum, we have for every $v \in C$, there is some $u \in S$ such that $v \sim u$. In this thesis, we thoroughly examine both the broadcast independence and broadcast packing problems in split graphs.

1.3 Ball catch graphs

Let G be a graph. The *ball catch graph* of G, denoted by $B(G)$, is defined as follows: the vertices of $B(G)$ are for all $v \in V(G)$,

$$(v, 1), (v, 2), \ldots, (v, ecc(v))$$

and two vertices (u, i), (v, j) are adjacent in $B(G)$ if and only if $d_G(u, v) \leq \max\{i, j\}$. The ball catch graphs are exactly the underlying undirected graphs of ball catch digraphs defined by Maehara [33]. Figure 1.6 shows the ball catch graph of the claw graph in Figure 1.5.

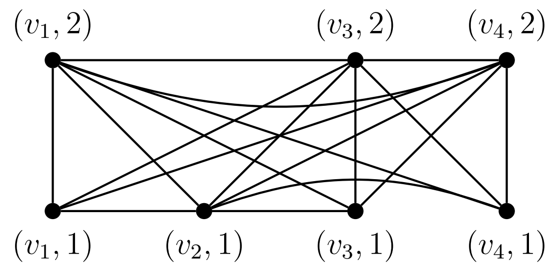


Figure 1.6: The ball catch graph of the claw graph.

The above definition says two vertices $(u, i), (v, j)$ are adjacent in $B(G)$ if either v is contained in the ball of radius i centred at u, or u is contained in the ball of radius j centred at v. Observe that given an independent broadcast f on G and $u, v \in V_f^+$, then $(u, f(u))$ is not adjacent to $(v, f(v))$ in $B(G)$.

We begin with a basic observation about the ball catch graph.

Observation 1.17. *Let G be a graph with n vertices. Then $B(G)$ has $\mathcal{O}(n^2)$ vertices.*

The ball catch graph $B(G)$ can be viewed as a weighted graph, where the weight of a vertex (u, i) is i.

Lemma 1.18. *Let f be a broadcast on G with $V_f^+ = \{v_1, v_2, \ldots, v_k\}$. Then f is independent if and only if $\{(v_1, f(v_1)), (v_2, f(v_2)), \ldots, (v_k, f(v_k))\}$ is a weighted independent set in $B(G)$ of weight $f(V)$.*

Proof. Let f be an independent broadcast on G with $V_f^+ = \{v_1, v_2, \ldots, v_k\}$. Suppose to the contrary that $\{(v_1, f(v_1)), (v_2, f(v_2)), \ldots, (v_k, f(v_k))\}$ is not an independent set in $B(G)$. So $(v_i, f(v_i)) \sim (v_j, f(v_j))$ for $i \neq j$. Thus $d_G(v_i, v_j) \leq \max\{f(v_i)), f(v_j))\}$. But this contradicts that f is independent.

Now suppose $S = \{(v_1, f(v_1)), (v_2, f(v_2)) \ldots, (v_k, f(v_k))\}$ is an independent set in $B(G)$. As S is independent, it is easy to see $v_i \neq v_j$ for all $i \neq j$. We claim that the broadcast

$$f(x) = \begin{cases} f(v_i) & x = v_i \text{ for } 1 \leq i \leq k, \\ 0 & \text{otherwise} \end{cases}$$

is independent. Suppose not. Then $d_G(v_i, v_j) \leq \max\{f(v_i)), f(v_j))\}$ for some $i \neq j$. But then $(v_i, f(v_i)) \sim (v_j, f(v_j))$, contradicting that S is independent. Clearly, the weight of $\{(v_1, f(v_1)), (v_2, f(v_2)), \ldots, (v_k, f(v_k))\}$ in $B(G)$ is equal to $\sum_{i=1}^{k} f(v_i) = f(V)$. □

This yields the following corollary, which is the cornerstone for the results in Chapter 2.

Corollary 1.19. *Solving the broadcast independence problem in G is equivalent to solving the maximum weighted independent set problem in $B(G)$.*

We want to convert the broadcast independence problem into the maximum weighted independent set problem (MWIS) because the following theorem states that MWIS is polynomial time solvable for weakly chordal graphs.

Theorem 1.20. *[39] The maximum weighted independent set problem for weakly chordal graphs can be solved in $\mathcal{O}(n^4)$ time.*

Theorem 1.20 and Observation 1.17 together yield the following.

Corollary 1.21. *The broadcast independence number $\alpha_b(G)$ can be computed in $\mathcal{O}(n^8)$ time for graphs whose ball catch graphs are weakly chordal.*

In Chapter 2, we examine several subclasses of chordal graphs and show that their corresponding ball catch graphs are weakly chordal. By Corollary 1.21, this means that the problem of computing the broadcast independence number for these classes of graphs can be solved in polynomial time.

Chapter 2
Broadcast Independence

In this chapter, we introduce one of the main ideas of this thesis. Our approach is: we start with a graph G for which we want to find the broadcast independence number, $\alpha_b(G)$. The graph G belongs to a certain class of graphs, which has specific structural properties. We then construct the ball catch graph of G, $B(G)$, and show $B(G)$ is weakly chordal. Using this method, we show, by Corollary 1.21, that the broadcast independence number can be computed in $\mathcal{O}(n^8)$ time for trees, diamond-free interval graphs, paw-free interval graphs, strongly chordal split graphs, and several other subclasses of chordal and weakly chordal graphs with small forbidden subgraphs.

2.1 Preliminary results

We begin this section with some basic observations about the ball catch graph. Let G be a graph. For $v \in V(G)$, let $C_v = \{(v,i) | 1 \leq i \leq ecc(v)\}$. Note each C_v induces a clique in $B(G)$ and we call it a *centred clique* of $B(G)$. Let $R_i = \{(v,i) | v \in V(G) \text{ and } i \leq ecc(v)\}$, for each i, $1 \leq i \leq diam(G)$. Observe $C_v \subseteq V(B(G))$ for all $v \in V(G)$, and similarly $R_i \subseteq V(B(G))$ for all $1 \leq i \leq diam(G)$. The *i-th power* of G, where $1 \leq i \leq diam(G)$, is the graph G^i where $V(G^i) = V(G)$ and x, y are adjacent in G^i if and only if $d_G(x,y) \leq i$.

Observation 2.1. *For each $v \in V(G)$, C_v forms a clique in $B(G)$.*

Observation 2.2. *The subgraph of $B(G)$ induced by R_i is isomorphic to G^i.*

As stated in Chapter 1, for a graph G with $diam(G) = 2$, we have $\alpha_b(G) = \alpha(G)$. However, we still consider the ball catch graphs for graphs of diameter 2 in the following sections. The following observation provides a structural result on the ball catch graphs for weakly chordal graphs of diameter 2.

Proposition 2.3. *Let G be a weakly chordal graph such that $diam(G) = 2$. Then $B(G)$ is weakly chordal.*

Proof. Note that any vertex of the form $(v, 2)$ is a universal vertex in $B(G)$. If $B(G)$ contains an induced cycle C_k with $k \geq 5$ or its complement, then the vertices of the cycle must all be of the form $(v, 1)$. In other words, all vertices of this cycle belong to R_1. But by Observation 2.2, this is impossible, as G is weakly chordal. □

Recall that a *two-pair* in a graph is a pair of non-adjacent vertices x, y such that every induced path between x and y has length 2. Theorem 1.13 provides a characterization of weakly chordal graphs in terms of two-pairs. This characterization is vital for the proofs in the subsequent sections of this chapter, where we prove that the ball catch graphs of certain graph classes are weakly chordal.

Throughout the proofs in this chapter, we pick two non-adjacent vertices in a connected, non-complete, induced subgraph of the ball catch graph, subject to certain conditions, and show that they form a two-pair. Therefore, we consider induced paths between vertices of the ball catch graph. We now present results on induced paths within the ball catch graph, and how the vertices of such paths correspond to vertices of the original graph.

Lemma 2.4. *Let G be a graph. Suppose that P is an induced path of length at least 3 in $B(G)$. Then each centred clique of $B(G)$ contains at most one internal vertex of P.*

Proof. Suppose to the contrary that some centred clique C_v contains two internal vertices of P, say $(v, r) \sim (v, r')$. Without loss of generality, $r < r'$. As (v, r) is an internal vertex of P, (v, r) has another neighbour on P, say $(u, t) \neq (v, r')$. Then, $d_G(v, u) \leq \max\{r, t\} \leq \max\{r', t\}$. So $(u, t) \sim (v, r')$, a contradiction. □

Lemma 2.5. *Let G be a graph, and let H be a connected, non-complete, induced subgraph of $B(G)$. Let (v, r) and (u, t) be non-adjacent vertices in H such that $d_G(v, u)$ is minimum. Suppose that P is an induced path of length at least 3 between (v, r) and (u, t) in $B(G)$. Then each centred clique contains at most one vertex of P.*

Proof. Let $P = (v, r) \sim (w_1, s_1) \sim (w_2, s_2) \sim \cdots \sim (w_{k-1}, s_{k-1}) \sim (u, t)$ where $k \geq 3$. By Lemma 2.4, $w_i \neq w_j$ for $i \neq j$. Since P is induced, $v \neq u$, $u \neq w_i$ and $v \neq w_{i+1}$ for $1 \leq i \leq k - 2$. So it suffices to show $v \neq w_1$ and $u \neq w_{k-1}$. Suppose that $v = w_1$. Then $r < s_1$ and $s_2 < s_1$, as otherwise $(v, r) \sim (w_2, s_2)$. If $w_2 = u$, then $k = 3$ as P is an induced path. By the same reasoning, we would have $s_1 < s_2$, a contradiction to $s_2 < s_1$. So $w_2 \neq u$. We have that $d_G(v, w_2) \leq s_1$. Since $k \geq 3$

and $w_2 \neq u$, $d_G(v,u) = d_G(w_1,u) > \max\{s_1,t\} \geq s_1$. But then $d_G(v,w_2) < d_G(v,u)$, a contradiction to the choice of v,u. Hence $v \neq w_1$. A similar argument shows that $u \neq w_{k-1}$. □

Lemma 2.6. *Let G be a graph and let H be a connected, non-complete induced subgraph of $B(G)$. Let (v,r) and (u,t) be non-adjacent vertices in H such that $r+t$ is maximum with respect to all pairs of non-adjacent vertices in H. Supppose P is an induced path of length at least 3 between (v,r) and (u,t) in $B(G)$. Then each centred clique contains at most one vertex of P.*

Proof. Let $P = (v,r) \sim (w_1,s_1) \sim (w_2,s_2) \sim \cdots \sim (w_{k-1},s_{k-1}) \sim (u,t)$, with $k \geq 3$. By Lemma 2.4, $w_i \neq w_j$ for $i \neq j$. Since P is induced, $v \neq u$, $u \neq w_i$ and $v \neq w_{i+1}$ for $1 \leq i \leq k-2$. So it suffices to show $v \neq w_1$ and $u \neq w_{k-1}$. Suppose that $v = w_1$. Then $r < s_1$ and $s_2 < s_1$, as otherwise $(v,r) \sim (w_2,s_2)$. If $w_2 = u$, then $k = 3$ as P is an induced path. By the same reasoning, we would have $s_1 < s_2$, a contradiction to $s_2 < s_1$. So $w_2 \neq u$. But then $(w_1,s_1),(u,t)$ are non-adjacent with $r+t < s_1+t$, a contradiction to our choice of v,u. Hence $v \neq w_1$. A similar argument shows $u \neq w_{k-1}$. □

Lemmas 2.5 and 2.6 are important because they tell us that if we have an induced path between a pair of non-adjacent vertices (subject to certain conditions) in the ball catch graph $B(G)$, then the vertices of such a path correspond to distinct vertices in G. We can prove an analogous result for induced cycles in $B(G)$. (Note that this result is not used in this thesis, but is included for possible future reference.)

Lemma 2.7. *Let G be a graph. Let C_k, $k \geq 5$, be an induced cycle in $B(G)$. Then each centred clique contains at most one vertex of C_k.*

Proof. Suppose to the contrary that C_k contains vertices (v,r) and (v,r') for some $v \in V(G)$, for $1 \leq r < r' < ecc(v)$. Note that $(v,r) \sim (v,r')$ on C. We know (v,r) has another neighbour on C, say (u,t). Then, $d_G(v,u) \leq \max\{r,t\} \leq \max\{r',t\}$. So $(u,t) \sim (v,r')$, a contradiction. □

We conclude this section by providing some necessary conditions on G if $B(G)$ is weakly chordal.

Proposition 2.8. *Let G be a graph. If $B(G)$ is weakly chordal, then G^i is weakly chordal for all i.*

Proof. Follows immediately from Observation 2.2. □

Note the converse of Proposition 2.8 is not true, as demonstrated by Figure 2.1. We refer to the leftmost graph in Figure 2.1 as the *pencil graph*.

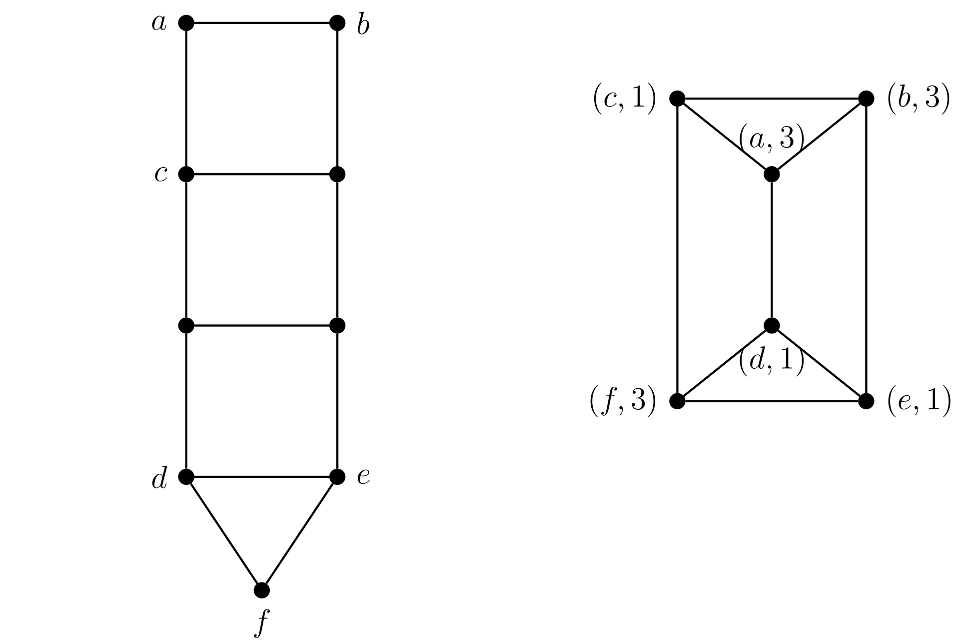


Figure 2.1: A weakly chordal graph (on the left) whose ball catch graph is not weakly chordal, as it contains the 3-prism shown on the right, and whose powers are weakly chordal. The labels of the 3-prism are vertex and radius in $B(G)$.

We now introduce a new graph, which is discussed in Proposition 2.9. A *triangle appended n-trampoline*, with $n \geq 3$, is the graph obtained from the n-trampoline, with vertices $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$, by adding n new vertices $x_1, x_2, \ldots, x_n$ and setting $v_i \sim x_i \sim u_{i+1}$ for $1 \leq i \leq n$, with index arithmetic done modulo n. The triangle appended 3-trampoline and the triangle appended 4-trampoline are in Figure 2.2.

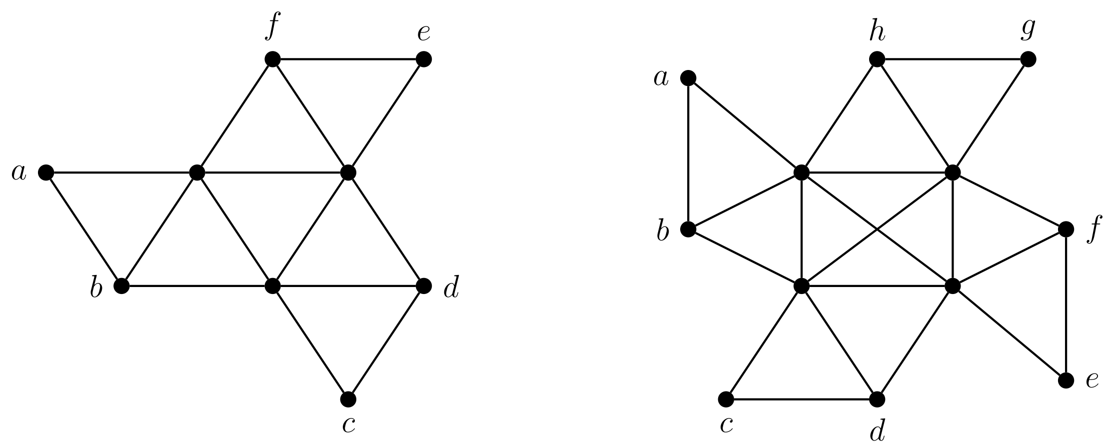


Figure 2.2: The triangle appended 3-trampoline and the triangle appended 4-trampoline, respectively.

Proposition 2.9. *Let G be a graph. If $B(G)$ is weakly chordal, then G does not contain any of the following as induced subgraphs: n-trampoline for $n \geq 5$, triangle appended n-trampoline for $n \geq 3$, or the pencil graph.*

Proof. If G contains an induced n-trampoline for $n \geq 5$, then $(v_1, 2) \sim (v_2, 2) \sim \cdots \sim (v_n, 2) \sim (v_1, 2)$ is an induced cycle of length greater than or equal to 5 in $B(G)$.

If G contains the triangle appended 3-trampoline, then using the labeling of Figure 2.2, $(a, 2) \sim (b, 1) \sim (c, 2) \sim (d, 1) \sim (e, 2) \sim (f, 1) \sim (a, 2)$ is an induced cycle in $B(G)$.

If G contains the triangle appended 4-trampoline, then using the labeling of Figure 2.2, then $(a, 2) \sim (b, 1) \sim (c, 2) \sim (d, 1) \sim (e, 2) \sim (f, 1) \sim (g, 2) \sim (h, 1) \sim (a, 2)$ is an induced cycle in $B(G)$.

If G contains the triangle appended n-trampoline for $n \geq 5$, then G contains an n-trampoline with $n \geq 5$, which we have shown induces a cycle of length n.

If G contains the pencil graph, Figure 2.1 shows an induced copy of the 3-prism in $B(G)$. □

2.2 Ball catch graphs of trees

As previously stated, Bessy and Rautenbach [4] show that the broadcast independence number of a tree of order n can be found in $\mathcal{O}(n^9)$ time. The main theorem of this section uses the ball catch graph of trees to show that the time complexity of the broadcast independence problem in trees can be improved to $\mathcal{O}(n^8)$. We first introduce terminology which is used in Theorem 2.11.

Definition 2.10. *Let T be a tree and let u, v, w be vertices of T. We say that w* branches off *the uv-path in T if some vertex x with $x \notin \{u, v\}$ from the uv-path lies in both the wv-path and wu-path in T. We also say that w* branches off *the uv-path from the vertex x.*

We now state and prove the main result of this section.

Theorem 2.11. *Let T be a tree and let $B(T)$ be the ball catch graph of T. Then $B(T)$ is weakly-chordal.*

Proof. If $diam(T) = 2$, then $B(T)$ is weakly chordal by Proposition 2.3. Thus, we assume $diam(T) \geq 3$. Given a connected, non-complete, induced subgraph of $B(G)$, say H, we show H contains a 2-pair.

Let (v,r) and (u,t) be non-adjacent vertices in H such that $d_T(v,u)$ is minimum over all pairs of non-adjacent vertices of H. We claim $(v,r),(u,t)$ form a 2-pair in H. Suppose to the contrary that there exists an induced path P between (v,r) and (u,t) in H of length $k \geq 3$. Let

$$P = (v,r) \sim (w_1,s_1) \sim (w_2,s_2) \sim \cdots \sim (w_{k-1},s_{k-1}) \sim (u,t).$$

By Lemma 2.5, $v, w_1, w_2, \ldots, w_{k-1}, u$ are distinct in T. We show that there is no possible arrangement these vertices in T. Consider the following cases:

Case 1: u, v and w_1 lie on a path in T.

If w_1 lies on the unique uv-path in T, then $d_T(w_1,u) < d_T(v,u)$, a contradiction with choice of (v,r) and (u,t), as (w_1,s_1) is not adjacent to (u,t) on P. So assume w_1 does not lie on the uv-path in T.

Since $(u,t) \not\sim (v,r)$ and $(u,t) \not\sim (w_1,s_1)$, the vw_1-path in T cannot pass through u. Therefore, the uw_1-path in T must pass through v. Consider the w_1w_2-path in T. If it passes through v, then

$$\max\{s_1,s_2\} \geq d_T(w_1,w_2) > d_T(v,w_2) > \max\{r,s_2\} \geq s_2.$$

So $\max\{s_1,s_2\} = s_1$. Since $(w_1,s_1) \not\sim (u,t)$, $d_T(w_1,u) > \max\{t,s_1\} \geq s_1$. Thus, we have $d_T(w_1,u) > s_1 \geq d_T(w_1,w_2)$. As $d_T(w_1,u) = d_T(w_1,v)+d_T(v,u)$ and $d_T(w_1,w_2) = d_T(w_1,v) + d_T(v,w_2)$, we have $d_T(v,w_2) < d_T(v,u)$, a contradiction to our choice of $(v,r),(u,t)$. Therefore, the w_1w_2-path does not pass through v. Since $(v,r) \not\sim (w_i,s_i)$ and $(v,r) \not\sim (w_{i+1},s_{i+1})$, the w_iw_{i+1}-path in T cannot pass through v for $2 \leq i \leq k-2$. But then the $w_{k-1}u$-path in T must pass through v, which is impossible as $(v,r) \not\sim (u,t)$ and $(v,r) \not\sim (w_{k-1},s_{k-1})$.

Case 2: u, v and w_1 do not lie on a path T.

Suppose that w_1 branches off the uv-path from the vertex a. Note $a \neq w_1$. We claim that w_{k-1} cannot branch off the aw_1-path and w_1 cannot lie on the aw_{k-1}-path. Suppose this is not the case. If $s_{k-1} \geq t$, then $d_T(a,u) < d_T(a,v)$. So $r > s_1$, and hence $d_T(a,w_1) < d_T(a,u)$. Combining these inequalities, $d_T(a,w_1) < d_T(a,v)$. Thus, $d_T(u,w_1) < d_T(u,v)$, a contradiction. On the other hand, if $t > s_{k-1}$, then $d_T(a,w_{k-1}) < d_T(a,v)$ and $d_T(a,w_{k-1}) < d_T(a,w_1)$. So $s_1 > r$. Thus, $d_T(a,v) < d_T(a,u)$. Combining these inequalities, $d_T(a,w_{k-1}) < d_T(a,u)$. So $d_T(v,w_{k-1}) < d_T(v,u)$, a contradiction.

Let i with $2 \leq i \leq k-1$ be the smallest index such that a lies on the w_1w_i-path. Suppose first that the w_1w_i-path does not contain any vertex in the av-path other than a. If $s_i \geq s_{i-1}$, then $d_T(a, w_{i-1}) < d_T(a, v)$. But then $d_T(w_{i-1}, u) < d_T(v, u)$, a contradiction. So $s_{i-1} > s_i$ and $d_T(a, w_i) < d_T(a, u)$. But then $d_T(w_i, v) < d_T(v, u)$, contradiction.

Suppose now that the w_1w_i-path contains a vertex in the av-path. Let a' be the vertex closest to v in T that lies on the w_1w_i-path. By choice of (u, t) and (v, r), $d_T(a, v) \leq d_T(a, w_{i-1})$. If $s_i \geq s_{i-1}$, then $d_T(a', w_{i-1}) < d_T(a', v)$. Combining these inequalities yields

$$d_T(a, v) < d_T(a, w_{i-1}) \leq d_T(a', w_{i-1}) < d_T(a', v)$$

a contradiction to the choice of a and a'. So $s_{i-1} > s_i$. Then $d_T(a, w_i) < d_T(a, u)$. By choice of (v, r) and (u, t), $d_T(a', u) \leq d_T(a', w_i)$. Combining these inequalities, yields

$$d_T(a, w_i) < d_T(a, u) \leq d_T(a', u) < d_T(a', w_i)$$

a contradiction by choice of a and a'. Therefore, (v, r) and (u, t) forms a two-pair as required. □

2.3 Ball catch graphs of interval graphs with small forbidden subgraphs

The diamond graph is K_4 minus an edge. The paw graph is the triangle with a leaf added as a pendant to one of the vertices of the triangle (see Figure 2.3). In this section, we prove that the ball catch graph of any diamond-free interval graph and of any paw-free interval graph is weakly chordal. It is interesting to note that diamond-free interval graphs are graphs which are both interval graphs and block graphs.

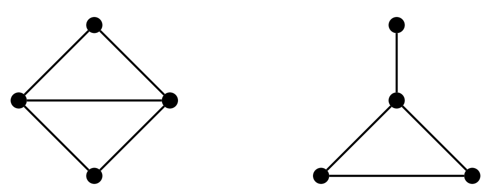


Figure 2.3: The diamond graph and paw graph, respectively.

Theorem 1.15 is of particular interest in this section, as we rely on the interval ordering to prove our results.

Definition 2.12. *Let G be an interval graph with a given interval ordering $\prec$. For distinct vertices $u, v \in V(G)$, v* appears before *u if $v \prec u$.*

Lemma 2.13. *Let G be an interval graph and let $\prec$ be an interval ordering of G. Suppose that $x_1 \sim x_2 \sim \cdots \sim x_k$ is a shortest x_1x_k-path in G. If $x_1 \prec x_k$, then*

1. *$x_1 \prec x_i$ for all $3 \leq i \leq k$,*
2. *$x_i \prec x_{i+1}$ for all $2 \leq i \leq k-1$, and*
3. *$x_i \prec x_k$ for all $1 \leq i \leq k-1$.*

Proof. Suppose to the contrary that $x_i \prec x_1$ for some i with $3 \leq i \leq k-1$. Let i be the largest such index. Then $x_i \prec x_1 \prec x_{i+1}$. Since $x_i \sim x_{i+1}$, the interval ordering implies $x_i \sim x_1$. Thus $x_1 \sim x_i \sim x_{i+1} \sim \cdots \sim x_k$ is a shorter x_1x_k-path in G, a contradiction.

Suppose to the contrary that $x_{i+1} \prec x_i$ for some i with $2 \leq i \leq k-1$. Let i be the smallest such index. If $i = 2$, then $x_3 \prec x_2$. We know from above $x_1 \prec x_3$. Thus $x_1 \prec x_3 \prec x_2$. The interval ordering ensures $x_1 \sim x_3$ and we have a shorter x_1x_k-path, $x_1 \sim x_3 \sim x_4 \sim \cdots \sim x_k$, a contradiction. So assume that $i \geq 3$. By choice of i, $x_{i-1} \prec x_i$. Since $x_{i-1} \sim x_i \sim x_{i+1}$, $x_{i-1} \prec x_i$ and $x_{i+1} \prec x_i$, by the interval ordering, $x_{i-1} \sim x_{i+1}$. But then $x_1 \sim x_2 \sim \cdots \sim x_{i-1} \sim x_{i+1} \sim \cdots \sim x_k$ is a shorter x_1x_k-path, a contradiction.

Clearly, $x_i \prec x_k$ for all $1 \leq i \leq k-1$ follows from the results above. □

Lemma 2.14. *Let G be an interval graph with a given interval order $\prec$. Given three vertices of G, $v \prec w \prec u$, there must be a vertex x on any vu-path such that $x \prec w$ and $x \sim w$. Further, $d_G(v,w) \leq d_G(v,u)$, and if $x \neq v$, then $d_G(w,u) \leq d_G(v,u)$.*

Proof. Consider a vu-path in G, say P. As $v \prec w \prec u$, there must be a vertex x and its successor y on P such that $x \prec w \preceq y$. By the interval order, as $x \sim y$, $x \sim w$. Clearly, $d_G(v,w) \leq d_G(v,u)$, and if $x \neq v$, then $d_G(w,u) \leq d_G(v,u)$. □

We now present the main theorems of this section.

Theorem 2.15. *Let G be a diamond-free interval graph, and let $B(G)$ be the ball catch graph of G. Then $B(G)$ is weakly chordal.*

Proof. If $diam(G) = 2$, then $B(G)$ is weakly chordal by Proposition 2.3. Thus, we assume $diam(G) \geq 3$. Given a connected, non-complete, induced subgraph of $B(G)$, say H, we show H contains a 2-pair. Let (v,r) and (u,t) be non-adjacent vertices in H such that

1. $d_G(v,u)$ is minimum,
2. v, u appear closest in the interval ordering of G, subject to condition 1.

We claim $(v,r),(u,t)$ form a 2-pair in H. Suppose to the contrary that there exists an induced path P between (v,r) and (u,t) in H of length $k \geq 3$. Let

$$P = (v,r) \sim (w_1,s_1) \sim (w_2,s_2) \sim \cdots \sim (w_{k-1},s_{k-1}) \sim (u,t).$$

By Lemma 2.5, $v, w_1, w_2, \ldots, w_{k-1}, u$ are distinct in G. We show that there is no possible arrangement these vertices in the interval ordering of G. Without loss of generality, we assume that $v \prec u$.

Claim 1. *For all* $2 \leq i \leq k-1$, *either* $w_i \prec v$ *or* $u \prec w_i$.

Proof of Claim. Suppose to the contrary that for some i such that $2 \leq i \leq k-1$, we have $v \prec w_i \prec u$. Note that by our choice of (v,r) and (u,t) and i, $d_G(v,u) < d_G(v,w_i)$. As $v \not\sim w_i$, by Lemma 2.14, it follows that $d_G(v,w_i) \leq d_G(v,u)$. This is a contradiction as we would have chosen $(v,r),(w_i,s_i)$ instead of $(v,r),(u,t)$. ◇

Claim 2. *For all* $1 \leq i \leq k-2$, $v \prec w_i$ *or* $w_{i+1} \prec u$.

Proof of Claim. Suppose that for some i with $1 \leq i \leq k-2$, we have $w_i \prec v \prec u \prec w_{i+1}$. We first show that either $i \geq 2$ or $i = 1$ and there is some vertex along a shortest w_1w_2-path which is distinct from w_1 and is adjacent to v.

Suppose to the contrary that $i = 1$ and the only vertex along a shortest w_1w_2-path which is adjacent to v is w_1. Let P' be a shortest w_1w_2-path. By Lemma 2.14, there is some vertex of P' which is adjacent to u, giving $d_G(w_1,u) \leq d_G(w_1,w_2)$. Let P'_u be the w_1u-path containing the P' subpath mentioned above. By our choice of (v,r) and (u,t), $d_G(v,u) \leq d_G(w_1,u)$. Therefore, vP'_u cannot be a shortest vu-path. So consider a shortest vu-path, say P''. Let y be the vertex of P' which is adjacent to w_1 and let x be the vertex of P'' which is adjacent to v. We claim that $x \neq y$, $v \prec y$ and $x \prec y$.

As v is not adjacent to any vertex of P' besides w_1, clearly $x \neq y$. Now suppose to the contrary that $y \prec v$. By Lemma 2.14, the yu-subpath of P'_u must have some vertex which is adjacent to v. But this is a contradiction as the only vertex of P' which is adjacent to v is w_1. As $v \sim x$, $v \prec y$ and $v \not\sim y$, we must have $x \prec y$. Thus $x \neq y$, $x \prec y$ and $v \prec y$ as required. As $d_G(w_1,u) \geq d_G(v,u)$, the only vertex of P'' which w_1 can be adjacent to besides v is x. Since $x \prec w_1 \prec v$ or $w_1 \prec x \prec y$, we have $w_1 \sim x$. Now consider the neighbour of x along P'', say z. By Lemma 2.14, we must have $w_1 \prec z$. Recall that the only vertex of P'' which w_1 can be adjacent to besides v is x. If $w_1 \prec z \preceq y$, then $z \sim w_1$, a contradiction. Thus $y \prec z$. As $x \prec y \prec z$ and $x \sim z$, we have $x \sim y$. But then $\{w_1, v, x, y\}$ induces a diamond.

Thus, either $i \geq 2$ or $i = 1$ and there is some vertex along a shortest w_1w_2-path which is distinct from w_1 and is adjacent to v. Consider a shortest w_iw_{i+1}-path in G, say P'. By Lemma 2.14, $d_G(w_i, u) \leq d_G(w_i, w_{i+1})$. As $(w_i, s_i) \not\sim (u, t)$, $s_i < s_{i+1}$. But by Lemma 2.14, $d_G(w_{i+1}, v) \leq d_G(w_i, w_{i+1})$. As $(w_{i+1}, s_{i+1}) \not\sim (v, r)$, $s_{i+1} < s_i$, a contradiction. ◇

Claim 3. $u \prec w_{k-1}$.

Proof of Claim. Suppose to the contrary that $w_{k-1} \prec v \prec u$. Consider a shortest $w_{k-1}u$-path in G, say P'. By Lemma 2.14, there must be some vertex along this path which is adjacent to v. This gives us that $d_G(v, w_{k-1}) \leq d_G(w_{k-1}, u)$ and $d_G(v, u) \leq d_G(w_{k-1}, u)$. But then we have both $s_{k-1} < t$ and $t < s_{k-1}$, as $(v, r) \not\sim (w_{k-1}, s_{k-1})$ and $(v, r) \not\sim (u, t)$. This is impossible. ◇

Claim 4. $v \prec w_1 \prec u$.

Proof of Claim. First suppose to the contrary that $v \prec u \prec w_1$. Consider a shortest vw_1-path in G. By Lemma 2.14, there must be some vertex along this path which is adjacent to u. This gives us that $d_G(v, u) \leq d_G(v, w_1)$ and $d_G(w_1, u) \leq d_G(v, w_1)$. But then we have both $s_1 < r$ and $r < s_1$, as $(v, r) \not\sim (u, t)$ and $(w_1, s_1) \not\sim (u, t)$. This is impossible.

Now suppose $w_1 \prec v$. By Claim 2, $w_i \prec v \prec u \prec w_{i+1}$ for some $1 \leq i \leq k-2$ is impossible. But as $w_1 \prec v \prec u$ and $v \prec u \prec w_{k-1}$, there must be a some i such that $w_i \prec v \prec u \prec w_{i+1}$ as Claim 1 forbids $v \prec w_i \prec u$. ◇

Therefore, assume that $v \prec w_1 \prec u$. By choice of (v, r) and (u, t), $d_G(v, u) < d_G(w_1, u)$. Consider w_2. By Claims 1, 2 and 3, $u \prec w_2$. Consider a shortest w_1w_2-path in G, say P'. By Lemma 2.14, there must be some vertex along this path which is adjacent to u, say x. Thus $d_G(w_1, u) \leq d_G(w_1, w_2)$, and $d_G(w_1, u) \leq d_G(w_1, x) + 1$. Thus $s_1 < s_2$. Consider a shortest vu-path in G, say P''. By Lemma 2.14, there must be a vertex of P'' which is adjacent to w_1. This vertex must be v, as otherwise $d_G(w_1, u) \leq d_G(v, u)$, a contradiction. Let $y \in V(P'')$ such that y precedes u on P''. That is, $d_G(v, u) = d_G(v, y) + 1$. Observe that $v \neq y$ and $w_1 \neq x$, as $v \not\sim u$ or $w_1 \not\sim u$. First we show that $x \neq y$. Suppose to the contrary that $x = y$. By choice of (v, r) and (u, t), $d_G(v, x) < d_G(w_1, x)$. Further, by choice of x and y, we have that

$$d_G(v, w_2) \leq d_G(v, x) + d_G(x, w_2) < d_G(w_1, x) + d_G(x, w_2) = d_G(w_1, w_2) \leq s_2 \leq \max\{r, s_2\},$$

a contradiction to the fact that $(v, r) \not\sim (w_2, s_2)$. Therefore, $x \neq y$.

As both $x \prec u$ and $y \prec u$, by the interval ordering, $x \sim y$. Further, $d_G(w_1, x) > d_G(v, y)$, as otherwise $d_G(w_1, u) \leq d_G(v, u)$, a contradiction. Thus $d_G(w_1, x) \geq d_G(v, y) + 1 \geq d_G(v, x)$. But then $d_G(v, w_2) \leq d_G(w_1, w_2) \leq \max\{s_1, s_2\} = s_2 \leq \max\{s_2, r\}$, giving $(v, r) \sim (w_2, s_2)$, a contradiction.

Therefore, there is no possible configuration of the vertices of P in the interval ordering of G. Thus (v, r) and (u, t) are a two-pair as required. Thus, $B(G)$ is weakly chordal. □

Theorem 2.16. *Let G be a paw-free interval graph, and let $B(G)$ be the ball catch graph of G. Then $B(G)$ is weakly chordal.*

Proof. If $diam(G) = 2$, then $B(G)$ is weakly chordal by Proposition 2.3. Thus, we assume $diam(G) \geq 3$. Given a connected, non-complete, induced subgraph of $B(G)$, say H, we show H contains a 2-pair. Let (v, r) and (u, t) be non-adjacent vertices in H such that

1. $d_G(v, u)$ is minimum,

2. $\{v, u\}$ is the lexicographically smallest pair with respect to $\prec$, subject to condition 1.

We claim $(v, r), (u, t)$ form a 2-pair in H. Suppose to the contrary that there exists an induced path P between (v, r) and (u, t) in H of length $k \geq 3$. Let

$$P = (v, r) \sim (w_1, s_1) \sim (w_2, s_2) \sim \cdots \sim (w_{k-1}, s_{k-1}) \sim (u, t).$$

By Lemma 2.5, $v, w_1, w_2, \ldots, w_{k-1}, u$ are distinct in G. We now show that there is no possible arrangement these vertices in the interval ordering of G. Without loss of generality, we assume that $v \prec u$. We make the following claims:

Claim 1. *For all $2 \leq i \leq k-1$, either $w_i \prec v$ or $u \prec w_i$.*

Claim 2. *For all $1 \leq i \leq k-2$, $v \prec w_i$ or $w_{i+1} \prec u$.*

Claim 3. *$u \prec w_{k-1}$.*

Claim 4. *$v \prec w_1 \prec u$.*

Observe that these are the same claims as those in the proof of Theorem 2.15. The proof of Claim 1 follows by the choice of (v, r) and (u, t), as in the proof of Claim 1 in Theorem 2.15. The proofs of Claim 3 and 4 are identical to the proofs of the same claims in Theorem 2.15. Thus, we omit the proofs of these claims, and instead provide only a proof of Claim 2.

Proof of Claim 2. Suppose that for some i with $1 \leq i \leq k-2$, we have $w_i \prec v \prec u \prec w_{i+1}$. We first show that either $i \geq 2$ or $i = 1$ and there is some vertex along a shortest w_1w_2-path which is distinct from w_1 and is adjacent to v.

Suppose to the contrary that $i = 1$ and the only vertex along a shortest w_1w_2-path which is adjacent to v is w_1. Let P' be a shortest w_1w_2-path. By the interval order, there is some vertex of P' which is adjacent to u, giving $d_G(w_1, u) \leq d_G(w_1, w_2)$. Let P'_u be the w_1u-path containing the P' subpath mentioned above. By our choice of (v, r) and (u, t), $d_G(v, u) < d_G(w_1, u)$. Therefore, vP'_u cannot be a shortest vu-path. Thus consider a shortest vu-path, say P''. Let y be the vertex of P' which is adjacent to w_1 and let x be the vertex of P'' which is adjacent to v. As $d_G(v, u) < d_G(w_1, u)$, $w_1 \not\sim x$. Therefore, as $w_1 \prec v$, we must have $w_1 \prec x$. We claim that $x \neq y$, $v \prec y$, and $x \prec y$. As v is not adjacent to any vertex of P' besides w_1, clearly $x \neq y$. Now suppose to the contrary that $y \prec v$. By Lemma 2.14, the yu-subpath of P'_u must have some vertex which is adjacent to v. But this is a contradiction as the only vertex of P' which is adjacent to v is w_1. As $v \sim x$, $v \prec y$ and $v \not\sim y$, we must have $x \prec y$. Thus $x \neq y$, $x \prec y$ and $v \prec y$ as required. As $d_G(w_1, u) > d_G(v, u)$, $w_1 \not\sim x$. Therefore, $y \prec x$. But this is a contradiction.

Thus, either $i \geq 2$ or $i = 1$ and there is some vertex along a shortest w_1w_2-path which is distinct from w_1 and is adjacent to v. Consider a shortest w_iw_{i+1}-path in G, say P'. By Lemma 2.14, there must be some vertex along this path which is adjacent to u. So $d_G(w_i, u) \leq d_G(w_i, w_{i+1})$. As $(w_i, s_i) \not\sim (u, t)$, $s_i < s_{i+1}$. By Lemma 2.14, there also must be some vertex along this path which is adjacent to v. So $d_G(w_{i+1}, v) \leq d_G(w_i, w_{i+1})$. As $(w_{i+1}, s_{i+1}) \not\sim (v, r)$, $s_{i+1} < s_i$, a contradiction. ⋄

Suppose that $v \prec w_1 \prec u$. By choice of (v, r) and (u, t), $d_G(v, u) \leq d_G(w_1, u)$. Consider w_2. By Claims 1, 2 and 3, $u \prec w_2$. Consider a shortest w_1w_2-path in G, say P'. By Lemma 2.14, there must be some vertex of P' which is adjacent to u. This gives $d_G(w_1, u) \leq d_G(w_1, w_2)$ and $d_G(w_2, u) \leq d_G(w_1, w_2)$. Therefore, we must have $s_1 < s_2$ and $k - 1 = 2$. As $s_2 > s_1$, the only vertex along P' which v can be adjacent to is w_1.

Consider a shortest vu-path, say P''. Let x be the neighbour of v along P'', and let y be the neighbour of w_1 along P'. As $s_2 > s_1$, $x \neq y$ and $v \not\sim y$. As $d_G(v, u) \leq d_G(w_1, u)$, the only vertices of P'' which w_1 can be adjacent to are v and x. Observe that this implies w_1 precedes all other vertices of P'' in the interval order of G. We claim that $w_1 \sim x$. Suppose to the contrary that $w_1 \not\sim x$. As $v \prec w_1$ and $w_1 \not\sim x$, we must have $w_1 \prec x$ and $v \sim w_1$. Now consider y. If $x \prec y$, then $w_1 \sim x$, a contradiction. But if $y \prec x$, then $v \sim y$, a contradiction. Therefore, $w_1 \sim x$.

Let z be the vertex following x on P''. As $d_G(v, u) \leq d_G(w_1, u)$, $z \neq y$, $w_1 \not\sim z$ and

$w_1 \prec z$. Furthermore, we also must have $y \prec z$, as otherwise, $w_1 \sim z$, a contradiction. As $x \sim z$ and $y \prec z$, by the interval order, $x \sim y$. If $v \not\sim w_1$, $\{v, w_1, x, y\}$ induces the paw graph, a contradiction. If $v \sim w_1$, then $\{v, w_1, x, z\}$ induces the paw graph, a contradiction.

Therefore, there is no possible configuration of the vertices of P in the interval ordering of G. Thus (v, r) and (u, t) are a two-pair in H as required. Thus, $B(G)$ is weakly chordal. □

2.4 Ball catch graphs of strongly chordal split graphs

In this section, we examine the ball catch graph of strongly chordal split graphs. In Chapter 1, we characterized strongly chordal graphs in terms of a strong vertex ordering and a forbidden subgraph characterization. Split graphs are chordal. Thus, when we refer to a strongly chordal split graph, we are talking about the split graphs which do not contain an induced n-trampoline for $n \geq 3$. We now show that the ball catch graph of a strongly chordal split graph is weakly chordal. Recall that for a split graph G, we denote the clique and the independent set of the split partition as C and S, respectively.

Theorem 2.17. *Let G be a strongly chordal split graph. Then $B(G)$ is weakly chordal.*

Proof. If $diam(G) = 2$, then $B(G)$ is weakly chordal by Proposition 2.3. Thus, we assume $diam(G) = 3$. Given a connected, non-complete, induced subgraph of $B(G)$, say H, we show H contains a 2-pair. Let (v, r), (u, t) be a pair of non-adjacent vertices of H such that:

1. $r + t$ is maximum,
2. $d_G(v, u)$ is minimum subject to 1,
3. $|N_t(u) \cap N_r(v)|$ is maximum subject to 2.

As $(v, r), (u, t)$ are non-adjacent and $diam(G) = 3$, $r, t \leq 2$. We claim that (v, r) and (u, t) form a two-pair. Suppose to the contrary that there exists an induced path P between (v, r) and (u, t) in H of length $k \geq 3$. Let

$$P = (v, r) \sim (w_1, s_1) \sim (w_2, s_2) \sim \cdots \sim (w_{k-1}, s_{k-1}) \sim (u, t).$$

By Lemma 2.6, $v, w_1, w_2, \ldots, w_{k-1}, u$ are distinct in G. We cannot have $u, v \in C$, as if that were the case, we would have $(v, r) \sim (u, t)$, a contradiction. Without loss of generality, suppose $v \in S$. We claim $u \in S$. Suppose to the contrary that $u \in C$. As $(v, r), (u, t)$ are non-adjacent and $u \in C$, $d_G(v, u) = 2$. So $r = t = 1$. Now consider w_1. First suppose $w_1 \in C$. As $u \in C$, $(w_1, s_1) \sim (u, t)$, a contradiction. Thus $w_1 \in S$. Then $d_G(v, w_1) = 2$ and $s_1 = 2$, as $r = 1$. But then $s_1 + t > r + t$, a contradiction with our choice of $(v, r), (u, t)$.

Thus $u \in S$. First suppose that $r = t = 1$. By choice of (v, r) and (u, t), $s_i = 1$ for all $1 \leq i \leq k-1$. Now consider w_1. Suppose that $w_1 \in C$. If $w_{k-1} \in C$, $P = (v, 1) \sim (w_1, 1) \sim (w_{k-1}, 1) \sim (u, 1)$. But as $C \subseteq N_{s_{k-1}}(w_{k-1})$ and, as $u \not\sim w_1$, $N_t(u) \subset C$, $|N_r(v) \cap N_{s_{k-1}}(w_{k-1})| > |N_r(v) \cap N_t(u)|$, a contradiction. Therefore, $w_{k-1} \in S$. As $w_{k-1}, u \in S$ and $t = 1$, $s_{k-1} = 2$. But then $(v, r), (w_{k-1}, s_{k-1})$ is a pair of non-adjacent vertices with $r + s_{k-1} > r + t$, a contradiction. Now suppose $w_1 \in S$. As $w_1, v \in S$ and $r = 1$, $s_1 = 2$. But then $(u, t), (w_1, s_1)$ is a pair of non-adjacent vertices with $s_1 + t > r + t$, a contradiction.

Therefore, without loss of generality, we can assume $r = 2$. As $r = 2$, if $w_i \in C$, then $i = 1$. First suppose $w_1 \in C$. As $(w_{k-1}, s_{k-1}) \sim (u, t)$ and $w_{k-1}, u \in S$, $d_G(w_{k-1}, u) = 2$. So either $s_{k-1} = 2$ or $t = 2$. Since $w_1 \in C$, $t \neq 2$. So $s_{k-1} = 2$. But then $(v, r), (w_{k-1}, s_{k-1})$ is a pair of non-adjacent vertices with $r + s_{k-1} > r + t$, a contradiction.

Therefore, $w_1 \in S$. Thus $v, u, w_i \in S$ for all $1 \leq i \leq k-1$. For the remainder of the proof, let $w_0 = v$ and $w_k = u$. As $w_i \in S$ for all $0 \leq i \leq k$, we must have $d_G(w_i, w_{i+1}) = 2$ for all $0 \leq i \leq k-1$. Thus either $s_i = 2$ or $s_{i+1} = 2$ for all $1 \leq i \leq k-2$. Now consider w_{k-1} and w_k. Either $s_{k-1} = 2$ or $t = 2$ as $d_G(w_{k-1}, w_k) = 2$. If $t \neq 2$, then $(w_{k-1}, s_{k-1}), (w_0, r)$ is a pair of non-adjacent vertices with $r + s_{k-1} > r + t$, a contradiction. Thus $t = 2$.

Now consider w_0, w_1, w_2, w_3. If $s_2 = 2$, let x, m, y equal w_0, w_1, w_2 respectively. Otherwise, $s_2 = 1$ and since $d_G(w_1, w_2) = d_G(w_2, w_3) = 2$, we have $s_1 = s_3 = 2$. In this case, let x, m, y equal w_1, w_2, w_3 respectively.

Observe $C \cup \{m\} \subseteq N_2(x) \cap N_2(y)$. Thus, $|N_2(x) \cap N_2(y)| \geq |C| + 1$ and by hypothesis 3, $|N_2(w_0) \cap N_2(w_k)| \geq |N_2(x) \cap N_2(y)|$. Also observe that for $1 \leq i \leq k-1$, $w_i \notin N_2(w_0) \cap N_2(w_k)$ and in particular, $m \notin N_2(w_0) \cap N_2(w_k)$. Thus there is some $z \in (N_2(w_0) \cap N_2(w_k)) \backslash (N_2(x) \cap N_2(y))$. Note $z \in S$. Let $w_{k+1} = z$.

We now show that G must contain an induced n-trampoline for $n \geq 3$. We are going to construct a sequence $z_0, c_0, z_1, c_1, \ldots, z_l, c_l$, where $z_0, z_1, \ldots, z_l$ is a subsequence of $w_{k+1}, w_0, w_1, \ldots, w_k$, and $\{c_0, c_1, \ldots, c_l\} \subseteq C$. Let $z_0 = w_{k+1}$. The subsequence z_0, c_0, z_1, c_1, z_2 will be constructed according to certain cases. We first begin with a claim.

Claim. If $d_G(w_i, w_{i+2}) = 2$, then there exists a vertex $c \in C$ such that

$$\{w_i, w_{i+1}, w_{i+2}\} \subseteq N(c).$$

Proof of Claim. As $d_G(w_i, w_{i+2}) = 2$, $s_i = s_{i+2} = 1$ and $s_{i+1} = 2$. If there is no $c \in C$ such that $\{w_i, w_{i+1}, w_{i+2}\} \subseteq N(c)$, then there exist c_1, c_2, c_3 where $w_i \sim c_1 \sim w_{i+1}$, $w_{i+1} \sim c_2 \sim w_{i+2}$, and $w_i \sim c_3 \sim w_{i+2}$. But then $\{w_i, w_{i+1}, w_{i+2}, c_1, c_2, c_3\}$ forms a 3-trampoline, a contradiction. $\diamond$

The rest of the sequence is defined inductively as follows:

For $2 \leq j \leq k$, (say $z_j = w_i$), we let

$$c_j = \begin{cases} \text{a common neighbour of } w_i, w_{i+1} \text{ and } w_{i+2} & \text{if } d_G(w_i, w_{i+2}) = 2, \\ \text{a common neighbour of } w_i \text{ and } w_{i+1} & \text{if } d_G(w_i, w_{i+2}) = 3 \text{ or } j = k. \end{cases}$$

If $c_j \sim z_0$, we have finished the sequence. If not, we let

$$z_{j+1} = \begin{cases} w_{i+2} & \text{if } d_G(w_i, w_{i+2}) = 2, \\ w_{i+1} & \text{if } d_G(w_i, w_{i+2}) = 3 \text{ or } j = k, \end{cases}$$

and repeat this process.

By construction and as $d_G(w_k, w_{k+1}) = 2$, we will end with a sequence $z_0, c_0, z_1, c_1, \ldots, z_l, c_l$ such that $c_l \sim z_0$. We will show that $z_0, c_0, z_1, c_1, \ldots, z_l, c_l$ is an induced $(l+1)$-trampoline with $l \geq 2$. We now define the subsequence z_0, c_0, z_1, c_1, z_2. Recall $z_0 = z$.

Case 1**.** There is no common neighbour of z, w_0 and w_1.

As $d_G(z, w_0) = 2$, there exists a common neighbour of z and w_0. Let c_0 be a common neighbour of z and w_0 and let $z_1 = w_0$. Let c_1 be a common neighbour of w_0 and w_1, and let $z_2 = w_1$. By assumption, $z_0 \not\sim c_1$. Observe that for all $2 \leq i \leq k$, $w_i \not\sim c_0$ as $r = 2$.

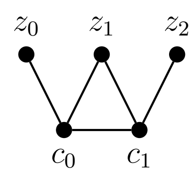
Case 2**.** There is a common neighbour of z, w_0 and w_1 and there is no common neighbour of z, w_1, w_2.

Let c_0 be a common neighbour of z, w_0 and w_1, and let $z_1 = w_1$. If $d_G(w_1, w_3) = 2$, by Claim 1, there exists a common neighbour of w_1, w_2 and w_3. Let c_1 be such a vertex, and let $z_2 = w_3$. Otherwise, if $d_G(w_1, w_3) = 3$, let c_1 be a common neighbour of w_1 and w_2, and let $z_2 = w_2$. As $c_0 \sim w_0$ and $r = 2$, observe that in either case, $z_2 \not\sim c_0$. By assumption, $z \not\sim c_1$.

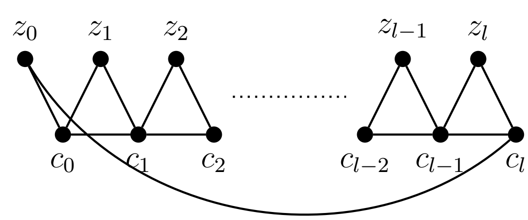
Case 3. There is a common neighbour of z, w_0 and w_1, and there is a common neighbour of z, w_1, w_2.

Let c_0 be a common neighbour of z, w_1 and w_2, and let $z_1 = w_2$. We claim that in this case, $x = w_1$ and $y = w_3$. By assumption $z \in N_2(w_0) \cap N_2(w_2)$. Therefore, x, y cannot be equal to w_0, w_2. So $x = w_1$ and $y = w_3$. By assumption, $z \in N_2(w_1)$. Therefore, as $x = w_1$ and $y = w_3$, we must have $d_G(z, w_3) = 3$. By choice of z, $u \neq w_3$, and $k \geq 4$. If $d_G(w_2, w_4) = 2$, let c_1 be a common neighbour of w_2, w_3 and w_4, which exists by Claim 1, and let $z_2 = w_4$. Otherwise, $d_G(w_2, w_4) = 3$. Then let c_1 be a common neighbour of w_2 and w_3, and let $z_2 = w_3$. As $d_G(z, w_3) = 3$, $z_0 \not\sim c_1$ in either case. If $z_2 = w_3$, then $z_2 \not\sim c_0$ because $d_G(z, w_3) = 3$. If $z_2 = w_4$, then $d_G(w_2, w_4) = 2$. Thus $s_2 = 1$, and therefore, $s_1 = 2$. As $w_1 \sim c_0$ and $s_1 = 2$, then $w_4 \not\sim c_0$, thus $z_2 \not\sim c_0$.

Therefore, in all the cases, we have the following:



By construction, there are no other edges between these vertices. We claim that defining the rest of the sequence as above yields the $(l+1)$-trampoline pictured below (we omit drawing most edges between the vertices of C):



By construction, it is clear that $l \geq 2$, $\{c_0, c_1, \ldots, c_l\}$ is a clique and $\{z_0, z_1, \ldots, z_l\}$ is an independent set. Thus, we need only show that for all $0 \leq j \leq l$, the only non-clique vertices that c_j is adjacent to are z_j and z_{j+1}, with index arithmetic done modulo $l+1$. By construction it is clear that $z_j \sim c_j \sim z_{j+1}$. Suppose to the contrary that $c_j \sim z_t$ for some $t \neq j$ and $t \neq j+1$. Let w_i be such that $z_j = w_i$. By choice of c_j, either c_j is a common neighbour of w_i, w_{i+1} and w_{i+2}, or $d_G(w_i, w_{i+2}) = 3$ and c_j is a common neighbour of w_i and w_{i+1}. In the former case, $s_i = s_{i+2} = 1$ and $s_{i+1} = 2$, and in the latter, either $s_i = 2$ or $s_{i+1} = 2$.

Suppose c_j is a common neighbour of w_i, w_{i+1} and w_{i+2}. Then no other vertex of $\{w_0, w_1, \ldots, w_k\}$ can be adjacent to c_j except w_i, w_{i+1} and w_{i+2} as $s_{i+1} = 2$. Thus, by our process, $z_{j+1} = w_{i+2}$, therefore $t = j$ or $t = j + 1$, a contradiction.

Now suppose $d_G(w_i, w_{i+2}) = 3$ and c_j is a common neighbour of w_i and w_{i+1}. The the only other vertices of $\{w_0, w_1, \ldots, w_k\}$ which can be adjacent to c_j besides w_i and w_{i+1} are w_{i-1} and w_{i+2} as either $s_i = 2$ or $s_{i+1} = 2$. As $d_G(w_i, w_{i+2}) = 3$, $w_{i+2} \not\sim c_j$. If $w_{i-1} \sim c_j$, then c_j is a common neighbour of w_{i-1}, w_i and w_{i+1}. By construction, as $z_j = w_i$, we have that $w_{i-1} \notin \{z_0, z_1, \ldots, z_l\}$.

Therefore, $z_0, c_0, z_1, c_1, \ldots, z_l, c_l$ forms an $(l+1)$-trampoline with $l \geq 2$ as required. This is a contradiction, as G is strongly chordal. Therefore, $(v, r), (u, t)$ forms a two-pair as required. □

2.5 Ball catch graphs of chordal, (P_5, gem)-free graphs

The *gem* is a graph on vertices a, b, c, d, e such that a, b, c, d induces a P_4 and e is adjacent to a, b, c and d. A gem is pictured in Figure 2.4.

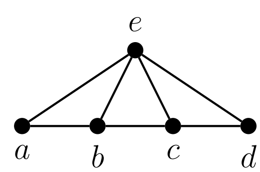


Figure 2.4: The gem graph.

In this section, we prove that the ball catch graph of any chordal, (P_5, gem)-free graph is weakly chordal. Various optimization problems can be solved in linear time for (P_5, gem)-free graphs [5]. We first provide a basic observation about P_5-free graphs, which is used in the proof of Theorem 2.19.

Observation 2.18. *Let G be a P_5-free graph. Then $diam(G) \leq 3$.*

We now prove the main result of this section.

Theorem 2.19. *Let G be a chordal, (P_5, gem)-free graph and let $B(G)$ be the ball catch graph of G. Then $B(G)$ is weakly chordal.*

Proof. If $diam(G) = 2$, then $B(G)$ is weakly chordal by Proposition 2.3. Thus, we assume $diam(G) = 3$. Given a connected, non-complete, induced subgraph of $B(G)$, say H, we show H contains a 2-pair. Let (v, r) and (u, t) be non-adjacent vertices in H such that

1. $r + t$ is maximum,

2. $d_G(u, v)$ is minimum subject to 1.

We claim that (v, r) and (u, t) form a two-pair. Suppose to the contrary that there exists an induced path P between (v, r) and (u, t) in H of length $k \geq 3$. Let

$$P = (v, r) \sim (w_1, s_1) \sim (w_2, s_2) \sim \cdots \sim (w_{k-1}, s_{k-1}) \sim (u, t).$$

By Lemma 2.6, $v, w_1, w_2, \ldots, w_{k-1}, u$ are distinct in G. As $diam(G) = 3$, $r \leq 2$ and $t \leq 2$.

First suppose that $r = t = 1$. As $r + t$ is maximum, for all $1 \leq i \leq k - 1$, $s_i = 1$. Thus, the vertices $v, w_1, w_2, \ldots, w_{k-1}, u$ must induce a path in G. As G is P_5-free, $k = 3$. That is, $v \sim w_1 \sim w_2 \sim u$ is an induced path in G. Further, as $d_G(u, v)$ is minimum and $d_G(v, w_2) = 2$, $d_G(u, v) = 2$. There must be some vertex x such that $v \sim x$ and $u \sim x$. Note that $x \neq w_1$ and $x \neq w_2$. As G is chordal, $x \sim w_1$ and $x \sim w_2$. But then $\{v, w_1, w_2, u, x\}$ induces a gem in G, a contradiction as G is gem-free.

Thus, without loss of generality assume that $r = 2$. Then $d_G(v, w_i) = 3$ for all $2 \leq i \leq k - 1$ and $d_G(v, u) = 3$. Consider w_1 and w_2. At least one of $d_G(v, w_1) = 2$ or $d_G(w_1, w_2) = 2$, as otherwise $d_G(v, w_2) = 2$, a contradiction.

Suppose first that $d_G(v, w_1) = 1$ and $d_G(w_1, w_2) = 2$. In G, we have an induced path $v \sim w_1 \sim x \sim w_2$. Consider (w_3, s_3). We claim that $(w_3, s_3) \neq (u, t)$. If $d_G(w_2, w_3) = 1$, we have a path $v \sim w_1 \sim x \sim w_2 \sim w_3$ in G. As G is P_5-free, this path is not induced. The only possible edge with ends in the path is $x \sim w_3$. Then $d_G(w_1, w_3) = 2$, so $s_1 = s_3 = 1$. As $d_G(w_1, w_2) = 2$ and $s_1 = 1$, $s_2 = 2$. As $r + t$ is maximum, $r = 2$ and $s_2 = 2$, $t = 2$. Thus $(w_3, s_3) \neq (u, t)$ and $k \geq 4$. Now consider (w_4, s_4) (it is possible that $(w_4, s_4) = (u, t)$). As $s_2 = 2$, w_4 cannot be adjacent to any vertex of v, w_1, x, w_2, w_3, in particular w_3. Therefore, $d_G(w_3, w_4) = 2$. As $s_3 = 1$, $s_4 = 2$. Since $s_4 = 2$, a shortest w_3w_4-path in G cannot pass through any of the vertices v, w_1, x, w_2. There must be some vertex $y \notin \{v, w_1, x, w_2\}$ such that $w_3 \sim y$ and $w_4 \sim y$. Since $s_4 = 2$, the only vertex of $\{v, w_1, x, w_2\}$ which can be adjacent to y is x. In fact, $x \sim y$, as otherwise, $v \sim w_1 \sim x \sim w_3 \sim y$ is an induced P_5 in G. But if $x \sim y$, then $v \sim w_1 \sim x \sim y \sim w_4$ is an induced P_5 in G. Therefore, $d_G(w_2, w_3) = 2$.

Suppose there is some shortest w_2w_3-path which passes through x. Then $d_G(w_1, w_3) = 2$, so $s_1 = s_3 = 1$. As $d_G(w_1, w_2) = 2$ and $s_1 = 1$, $s_2 = 2$. As $r + t$ is maximum and $s_2 = 2$, $t = 2$. Thus $(w_3, s_3) \neq (u, t)$, and $k \geq 4$. Now consider (w_4, s_4) (it is possible that $(w_4, s_4) = (u, t)$). Suppose $w_4 \sim w_3$. But then $v \sim w_1 \sim x \sim w_3 \sim w_4$ is an induced P_5. So $d_G(w_3, w_4) = 2$. As $s_3 = 1$, $s_4 = 2$. Note that a shortest w_3w_4-path in G

cannot pass through any of v, w_1, x, w_2. So there must be some vertex $y \notin \{v, w_1, x, w_2\}$ such that $w_3 \sim y$ and $w_4 \sim y$. As $s_4 = 2$, the only vertex in $\{v, w_1, x, w_2\}$ which can be adjacent to y is x. In fact, $x \sim y$, as otherwise, $v \sim w_1 \sim x \sim w_3 \sim y$ is an induced P_5 in G. But if $x \sim y$, then $v \sim w_1 \sim x \sim y \sim w_4$ is an induced P_5 in G.

Therefore, we can assume no shortest w_2w_3-path passes through x. There must be some vertex $y \notin \{v, w_1, x\}$ such that $w_2 \sim y$ and $w_3 \sim y$. Since G is P_5-free and chordal, $w_1 \sim y$ and $x \sim y$. These are the only possible edges we can have with ends in the path $v \sim w_1 \sim x \sim w_2 \sim y \sim w_3$. Because $w_1 \sim y$, $d_G(w_1, w_3) = 2$. Therefore, $s_1 = s_3 = 1$. As $d_G(w_1, w_2) = 2$ and $s_1 = 1$, $s_2 = 2$. As $r + t$ is maximum and $s_2 = 2$, $t = 2$. Thus, $(w_3, s_3) \neq (u, t)$ and $k \geq 4$. Now consider (w_4, s_4) (it is possible that $(w_4, s_4) = (u, t)$). Note that none of v, w_1, x, w_2, y can be adjacent to w_4. Suppose $w_4 \sim w_3$. Then v, w_1, y, w_3, w_4 is an induced P_5 in G, a contradiction. Thus $d_G(w_3, w_4) = 2$. As $s_3 = 1$, $s_4 = 2$. Note that a shortest w_3w_4-path in G cannot pass through any of v, w_1, x, w_2, y. There must be some vertex $z \notin \{v, w_1, x, w_2, y\}$ such that $w_3 \sim z$ and $w_4 \sim z$. None of v, w_1, x, w_2, y can be adjacent to w_4. Furthermore, x and y are the only vertices in $\{v, w_1, x, w_2, y\}$ that can be adjacent to z. In fact, $x \sim z$ and $y \sim z$. If only $x \sim z$, then $x \sim z \sim w_3 \sim y \sim x$ is a C_4. If only $y \sim z$, then $v \sim w_1 \sim x \sim y \sim z$ is an induced P_5. But then $v \sim w_1 \sim x \sim z \sim w_4$ is an induced P_5, a contradiction.

Suppose next that $d_G(v, w_1) = 2$ and $d_G(w_1, w_2) = 1$. Then $v \sim x \sim w_1 \sim w_2$ is an induced path in G. Consider (w_3, s_3) (it is possible that $(w_3, s_3) = (u, t)$). If $d_G(w_2, w_3) = 1$, $v \sim x \sim w_1 \sim w_2 \sim w_3$ is an induced path in G, a contradiction as G is P_5-free. Thus $d_G(w_2, w_3) = 2$. Note that a shortest w_2w_3-path cannot pass through any vertex of v, x, w_1. Thus there must be some vertex $y \notin \{v, x, w_1\}$ such that $w_2 \sim y$ and $w_3 \sim y$. As G is P_5-free and chordal, $w_1 \sim y$ and $x \sim y$. These are the only possible edges with ends in the path $v \sim x \sim w_1 \sim w_2 \sim y \sim w_3$. Because $w_1 \sim y$, $d_G(w_1, w_3) = 2$. This gives $s_1 = s_3 = 1$. As $d_G(w_3, w_2) = 2$ and $s_3 = 1$, $s_2 = 2$. As $r + t$ is maximum and $s_2 = 2$, $t = 2$. Thus $(w_3, s_3) \neq (u, t)$ and $k \geq 4$. Now consider (w_4, s_4) (it is possible that $(w_4, s_4) = (u, t)$).

None of v, x, w_1, w_2, y can be adjacent to w_4. If $w_4 \sim w_3$, then $v \sim x \sim y \sim w_3 \sim w_4$ is an induced P_5 in G, a contradiction. Thus $d_G(w_3, w_4) = 2$. As $s_3 = 1$, $s_4 = 2$. Note that a shortest w_3w_4-path in G cannot pass through any of v, x, w_1, w_2, y. There must be some vertex $z \notin \{v, x, w_1, w_2, y\}$ such that $w_3 \sim z$ and $w_4 \sim z$. As $s_4 = 2$, none of v, x, w_1, w_2, y can be adjacent to w_4. Since G is P_5-free, $x \sim z$ and $y \sim z$. But then $\{y, w_1, x, z, w_3\}$ induces gem, a contradiction.

Suppose now that $d_G(v, w_1) = 2$ and $d_G(w_1, w_2) = 2$. Let x and y be such that

$v \sim x \sim w_1$ and $w_1 \sim y \sim w_2$. As $r = 2$, $x \neq y$. Then $v \sim x \sim w_1 \sim y \sim w_2$ is a path in G. Since G is P_5-free, $x \sim y$, as this is the only possible edge with ends in the path. Consider (w_3, s_3) (it is possible that $(w_3, s_3) = (u, t)$). Assume that $d_G(w_2, w_3) = 1$. None of v, x, w_1 can be adjacent to w_3. As G is P_5-free, $y \sim w_3$. Thus $d_G(w_1, w_3) = 2$, and $s_1 = s_3 = 1$. Since $d_G(w_1, w_2) = 2$, $s_2 = 2$. By choice of (v, r) and (u, t), $t = 2$. So $(w_3, s_3) \neq (u, t)$. Now consider (w_4, s_4) (it is possible that $(w_4, s_4) = (u, t)$). First suppose $w_4 \sim w_3$. As $s_2 = 2$, $w_4 \not\sim y$. But then $v \sim x \sim y \sim w_3 \sim w_4$ is an induced P_5 in G, a contradiction. Therefore, $d_G(w_3, w_4) = 2$. Since $s_3 = 1$, $s_4 = 2$. As $s_4 = 2$, a shortest w_3w_4-path cannot pass through any of v, x, w_1, y, w_2. There must be some vertex $z \notin \{v, x, w_1, y, w_2\}$ such that $w_3 \sim z$ and $w_4 \sim z$. Further, as $s_4 = 2$, v, x, w_1, y, w_2 cannot be adjacent to w_4. As G is P_5-free and chordal, $x \sim z$ and $y \sim z$. But then $\{y, w_1, x, z, w_3\}$ induces gem, a contradiction.

Therefore, $d_G(w_2, w_3) = 2$. Suppose there is some shortest w_2w_3-path which passes through y. Then $d_G(w_1, w_3) = 2$ and $s_1 = s_3 = 1$. As $d_G(w_1, w_2) = 2$ and $s_1 = 1$, $s_2 = 2$. As $r + t$ is maximum and $s_2 = 2$, $t = 2$. Thus $(w_3, s_3) \neq (u, t)$ and $k \geq 4$. Now consider (w_4, s_4) (it is possible that $(w_4, s_4) = (u, t)$). Suppose $w_4 \sim w_3$. As $s_2 = 2$, $w_4 \not\sim y$. But then $v \sim x \sim y \sim w_3 \sim w_4$ is an induced P_5, a contradiction. Therefore, $d_G(w_3, w_4) = 2$. As $s_3 = 1$, $s_4 = 2$. As $s_4 = 2$, a shortest w_3w_4-path in G cannot pass through any of v, x, w_1, y, w_2. There must be some vertex $z \notin \{v, x, w_1, y, w_2\}$ such that $w_3 \sim z$ and $w_4 \sim z$. None of v, x, w_1, y, w_2 can be adjacent to w_4. As G is P_5-free and chordal, $x \sim z$ and $y \sim z$. But then $\{y, w_1, x, z, w_3\}$ induces gem, a contradiction.

Thus, we can assume that no shortest w_2w_3-path passes through y. Then there must be some vertex $z \notin \{v, w_1, x, y\}$ such that $w_2 \sim z$ and $w_3 \sim z$. As G is P_5-free and chordal, $x \sim z$ and $y \sim z$. The only other possible edge in addition to the edges $x \sim z$, $y \sim z$, $x \sim y$ and the path $v \sim x \sim w_1 \sim y \sim w_2 \sim z \sim w_3$ is $w_1 \sim z$. This edge must exist, as otherwise $\{y, w_1, x, z, w_2\}$ induces a gem. Because $w_1 \sim z$, $d_G(w_1, w_3) = 2$. Hence $s_1 = s_3 = 1$. As $d_G(w_1, w_2) = 2$ and $s_1 = 1$, $s_2 = 2$. As $r + t$ is maximum and $s_2 = 2$, $t = 2$. Thus $(w_3, s_3) \neq (u, t)$ and $k \geq 4$. Now consider (w_4, s_4) (it is possible that $(w_4, s_4) = (u, t)$). None of v, x, w_1, y, w_2, z can be adjacent to w_4. First suppose $w_4 \sim w_3$. Observe that $w_4 \not\sim z$ as $s_2 = 2$. But then $v \sim x \sim z \sim w_3 \sim w_4$ is an induced P_5 in G, a contradiction. Therefore, $d_G(w_3, w_4) = 2$. Since $s_3 = 1$, $s_4 = 2$. As $s_4 = 2$, a shortest w_3w_4-path in G cannot pass through any of v, x, w_1, y, w_2, z. There must be some vertex $a \notin \{v, w_1, x, w_2, y\}$ such that $w_3 \sim a$ and $w_4 \sim a$. As $s_4 = 2$, x, y and z are the only vertices of $\{v, x, w_1, y, w_2, z\}$ that can be adjacent to a. In fact, as G is chordal and P_5 free, $x \sim a$, $y \sim a$ and $z \sim a$. But then $\{z, w_2, y, a, w_3\}$ induces a gem.

Therefore, $(v, r), (u, t)$ forms a two-pair as required. □

2.6 Ball catch graphs of ($P_5, P, \overline{P}$, Butterfly, Bull, House)-free weakly chordal graphs

The P graph is a graph on 5 vertices comprised of an induced C_4 and a vertex adjacent to of one of the vertices of the C_4. The graph $\overline{P}$ is the complement graph of P.

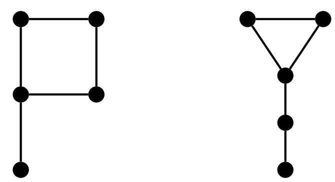


Figure 2.5: The P graph and $\overline{P}$ graph, respectively.

The *butterfly* is the graph on 5 vertices which consists of two triangles sharing a common vertex. The *bull* graph is the graph on 5 vertices consisting of a triangle and two additional vertices, each adjacent to a distinct vertex of the triangle. The *house* graph is the graph on 5 vertices consisting of a triangle and a C_4 sharing an edge.

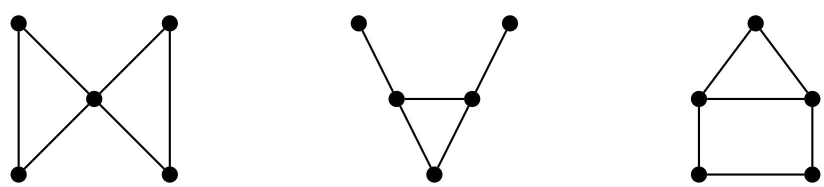


Figure 2.6: The butterfly graph, bull graph and house graph, respectively.

In this section, we examine the ball catch graph of ($P_5, P, \overline{P}$, Butterfly, Bull, House)-free weakly chordal graphs. We now present the main theorem of this section.

Theorem 2.20. *Let G be a ($P_5, P, \overline{P}$, Butterfly, Bull, House)-free weakly chordal graph. Then $B(G)$ is weakly chordal.*

Proof. If $diam(G) = 2$, then $B(G)$ is weakly chordal by Proposition 2.3. Thus, we assume $diam(G) = 3$. Given a connected, non-complete, induced subgraph of $B(G)$, say H, we show H contains a 2-pair. Let (v, r) and (u, t) be non-adjacent vertices in H such that:

1. $d_G(u, v)$ is minimum,

2. $|N(v) \cap N(u)|$ is maximum subject to 1.

We claim that (v, r) and (u, t) form a two-pair. Suppose to the contrary that there exists an induced path P between (v, r) and (u, t) in H of length $k \geq 3$. Let

$$P = (v, r) \sim (w_1, s_1) \sim (w_2, s_2) \sim \cdots \sim (w_{k-1}, s_{k-1}) \sim (u, t).$$

By Lemma 2.5, $v, w_1, w_2, \ldots, w_{k-1}, u$ are distinct in G. As (v, r) and (u, t) are non-adjacent, we have $d_G(v, u) \geq 2$.

First suppose that $d_G(v, u) = 2$. This implies that $r = t = 1$. Let x be such that $v \sim x \sim u$. It is clear that $x \neq w_i$ for all $1 \leq i \leq k-1$. Suppose that $d_G(v, w_1) = 1$. Then $w_1 \sim v \sim x \sim u$ is a path (not necessarily induced) in G. Now consider w_{k-1}. If $d_G(u, w_{k-1}) = 1$, $w_1 \sim v \sim x \sim u \sim w_{k-1}$ is a path in G. This path is not induced as G is P_5-free. The only possible edges among the vertices of this path are $w_1 \sim x$, $w_{k-1} \sim x$ and $w_1 \sim w_{k-1}$. First suppose $w_1 \not\sim w_{k-1}$. If only $w_1 \sim x$, then $\{w_1, v, x, u, w_{k-1}\}$ induces a $\overline{P}$. Similarly, if only $w_{k-1} \sim x$, then $\{w_1, v, x, u, w_{k-1}\}$ induces a $\overline{P}$. But if both $w_1 \sim x$ and $w_{k-1} \sim x$, then $\{w_1, v, x, u, w_{k-1}\}$ induces a butterfly. Therefore, $w_1 \sim w_{k-1}$. As G is weakly chordal, one of the edges $w_1 \sim x$ or $w_{k-1} \sim x$ must exist. Without loss of generality, suppose that $w_1 \sim x$. As $d_G(v, u) = d_G(w_1, u) = 2$, $|N(w_1) \cap N(u)| \geq 2$, and $u \not\sim w_1$, there must exist a vertex y such that $v \sim y \sim u$, but $w_1 \not\sim y$. If $w_{k-1} \not\sim y$, then $y \sim v \sim w_1 \sim w_{k-1} \sim u \sim y$ is an induced C_5 in G, a contradiction. Therefore, $w_{k-1} \sim y$. But then $\{y, u, w_{k-1}, w_1, v\}$ induces a house in G, a contradiction.

Thus, $d_G(v, w_1) = 2$ and $s_1 = 2$. Let z be such that $w_1 \sim z \sim v$. As $s_1 = 2$, $z \neq x$. As G is P_5-free, $w_1 \sim z \sim v \sim x \sim u$ cannot be induced. The only possible edge is $z \sim x$, but then $\{w_1, z, v, x, u\}$ induces a bull.

Thus $d_G(u, v) = 3$. Let $v \sim x \sim y \sim u$ be a shortest uv-path. As $d_G(v, u)$ is minimum, $d_G(v, w_i) = 3$ for all $2 \leq i \leq k-1$, and $d_G(u, w_i) = 3$ for all $1 \leq i \leq k-2$. Therefore, for all $1 \leq i \leq k-1$, $w_i \neq x$ and $w_i \neq y$. Now consider w_1. By choice of (v, r) and (u, t), $d_G(w_1, u) = 3$. Thus $w_1 \not\sim y$. Suppose $d_G(w_1, v) = 1$. As G is P_5-free, $w_1 \sim v \sim x \sim y \sim u$ is not an induced path. The only possible edge with ends in the path is $w_1 \sim x$. But if $w_1 \sim x$, then $\{w_1, v, x, y, u\}$ induces a $\overline{P}$. So $d_G(w_1, v) = 2$. By symmetry, $d_G(w_{k-1}, u) = 2$ as well.

Let z be such that $w_1 \sim z \sim v$. By choice of (v, r) and (u, t), $w_{k-1} \not\sim x$, $w_1 \not\sim y$, and $u \not\sim z$. First suppose $z \neq x$. As G is P_5-free, $w_1 \sim z \sim v \sim x \sim y \sim u$ is not an induced path. The only possible edges with ends in the path are $z \sim x$ or $z \sim y$. But we either get an induced P graph or a bull graph. So $z = x$. By symmetry, $w_{k-1} \sim y$. To avoid P, $w_1 \not\sim w_{k-1}$. If $d_G(w_1, w_{k-1}) = 2$, let a be such that $w_1 \sim a \sim w_{k-1}$. It is easy to check that $a \notin \{v, x, y, u\}$. To avoid C_5, either $x \sim a$ or $y \sim a$. In any case, either $\{v, x, w_1, a, w_{k-1}\}$ or $\{u, y, w_1, a, w_{k-1}\}$ induces a bull graph. Thus $d_G(w_1, w_{k-1}) = 3$.

Therefore, $k \geq 4$. Consider w_2. By choice of (v, r) and (u, t), $w_2 \not\sim x$ and $w_2 \not\sim y$. If $d_G(w_1, w_2) = 1$, then $w_2 \sim w_1 \sim x \sim y \sim u$ induces a P_5, a contradiction. Thus $d_G(w_1, w_2) = 2$. Let a be such that $w_1 \sim a \sim w_2$. It is easy to check that $a \notin \{v, x, y, u, w_{k-1}\}$. So $v \sim x \sim w_1 \sim a \sim w_2$ forms a path in G, which cannot be induced. The only edge with ends in the path can be $x \sim a$. But then $\{v, x, w_1, a, w_2\}$ induces a bull graph, a contradiction.

Therefore, (v, r) and (u, t) form a two-pair as required. $\square$

2.7 Ball catch graphs of (P_6, Triangle, Domino, A, H, X_{18}, X_{172})-free weakly chordal graphs

We follow the naming convention of [18] for the forbidden induced subgraphs. The domino, A, H, X_{18}, and X_{172} are all graphs on six vertices. The domino graph is a graph comprised of two C_4's which share an edge. The A graph is an induced C_4 with two additional vertices each adjacent to a distinct end of an edge of the C_4. The H graph is two P_3's with an edge between the center vertices of each path.

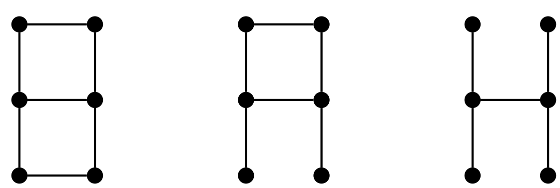


Figure 2.7: The domino, A and H graphs, respectively.

The X_{18} graph is an induced C_4 with a path of length 2 appended to one vertex of the C_4. The X_{172} graph is comprised of a P_5 with an additional vertex adjacent to the fourth vertex of the path.

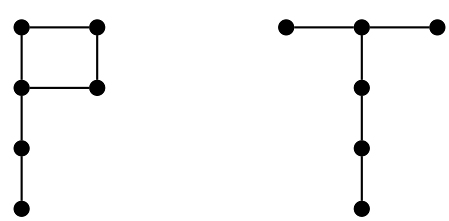


Figure 2.8: The X_{18} and X_{172} graphs, respectively.

In this section, we show that the ball catch graph of a (P_6, Triangle, Domino, A, H, X_{18}, X_{172})-free weakly chordal graph is weakly chordal. For this class of weakly chordal graphs, observe that all forbidden induced subgraphs are on 6 vertices with the exception of the triangle. Forbidding the triangle is important because by Proposition 2.9, we need to ensure that the graph class which we are considering does not contain n-trampolines for $n \geq 5$, and none of the other subgraphs forbid that structure. We now provide a brief observation, and then proceed into the main result of this section.

Observation 2.21. *Let G be a P_6-free graph. Then $diam(G) \leq 4$.*

Theorem 2.22. *Let G be a (P_6, Triangle, Domino, A, H, X_{18}, X_{172})-free weakly chordal graph. Then $B(G)$ is weakly chordal.*

Proof. If $diam(G) = 2$, then $B(G)$ is weakly chordal by Proposition 2.3. Thus, we assume $diam(G) \geq 3$. Given a connected, non-complete, induced subgraph of $B(G)$, say H, we show H contains a 2-pair. Let (v, r) and (u, t) be non-adjacent vertices in H such that $d_G(u, v)$ is minimum. We claim that (v, r) and (u, t) form a two-pair. Suppose to the contrary that there exists an induced path P between (v, r) and (u, t) in H of length $k \geq 3$. Let

$$P = (v, r) \sim (w_1, s_1) \sim (w_2, s_2) \sim \cdots \sim (w_{k-1}, s_{k-1}) \sim (u, t).$$

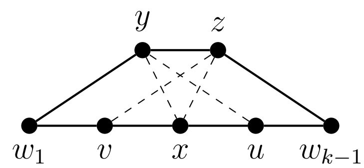
By Lemma 2.6, $v, w_1, w_2, \ldots, w_{k-1}, u$ are distinct in G. By Observation 2.21, we assume $diam(G) \leq 4$. As (v, r) and (u, t) are non-adjacent, we have $d_G(v, u) \geq 2$. By choice of (v, r) and (u, t), no w_i for $1 \leq i \leq k-1$ is on a shortest vu-path in G.

Case 1: $d_G(v, u) = 2$.

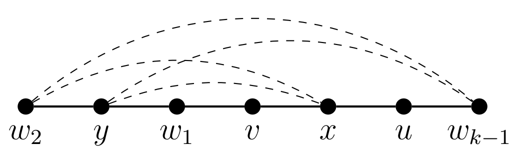
Let x be such that $v \sim x \sim u$. First we claim that $d_G(v, w_1) \leq 2$ and $d_G(u, w_{k-1}) \leq 2$. Without loss of generality suppose $d_G(v, w_1) = 3$. Since $d_G(v, u) = 2$, $s_1 = 3$. So a shortest vw_1-path cannot pass through x. Let y, z be such that $w_1 \sim y \sim z \sim v \sim x \sim u$ is a path in G. As G is P_6-free, this path cannot be induced. But as G is triangle free and $s_1 = 3$, there are no possible edges with ends in this path. Thus, the path $w_1 \sim y \sim z \sim v \sim x \sim u$ is induced, a contradiction. Therefore, $d_G(v, w_1) \leq 2$. An analogous argument shows $d_G(u, w_{k-1}) \leq 2$.

Now suppose $d_G(v, w_1) = 1$ and $d_G(u, w_{k-1}) = 1$. Then we have $w_1 \sim v \sim x \sim u \sim w_{k-1}$. As G is triangle free and weakly chordal, $w_1 \sim v \sim x \sim u \sim w_{k-1}$ is an induced path in G. Thus $d_G(w_1, w_{k-1}) \geq 2$. If $d_G(w_1, w_{k-1}) = 2$, let y be such that $w_1 \sim y \sim w_{k-1}$. Since G is triangle-free, $(v, r) \not\sim (w_{k-1}, s_{k-1})$ and $(u, t) \not\sim (w_1, s_1)$, observe that $y \notin \{v, x, u\}$. Further, as G is triangle-free, $y \not\sim v$ and $y \not\sim u$. As G is weakly chordal, $y \sim x$. But then $\{w_1, v, x, u, w_{k-1}, y\}$ induces a domino, a contradiction. If $d_G(w_1, w_{k-1}) = 3$, let y and z be such that $w_1 \sim y \sim z \sim w_{k-1}$.

By similar reasoning as before, $y, z \notin \{v, x, u\}$. The only possible edges between y and $\{v, x, u\}$ are $y \sim x$ and $y \sim u$. The only possible edges between z and $\{v, x, u\}$ are $z \sim x$ and $z \sim v$. Any subset of these edges either induce a triangle or a cycle of length 5 (see picture below). Therefore, none of these edges can exist. But then $w_1 \sim v \sim x \sim u \sim w_{k-1} \sim z \sim y \sim w_1$ induces a C_7 in G, a contradiction.



Thus $d_G(w_1, w_{k-1}) = 4$, which means that $w_2 \neq w_{k-1}$. Now consider w_2. If $d_G(w_1, w_2) = 1$, $w_2 \sim w_1 \sim v \sim x \sim u \sim w_{k-1}$ is a path in G. It cannot be induced as G is P_6-free. The only possible edges with ends in this path are $w_2 \sim w_{k-1}$ and $w_2 \sim x$. If only $w_2 \sim x$, then $\{w_2, w_1, v, x, u, w_{k-1}\}$ induces an X_{18}, a contradiction. If only $w_2 \sim w_{k-1}$, then $\{w_2, w_1, v, x, u, w_{k-1}\}$ induces a C_6, a contradiction. If both $w_2 \sim x$ and $w_2 \sim w_{k-1}$, then $\{w_2, w_1, v, x, u, w_{k-1}\}$ induces a domino, a contradiction. Now suppose $d_G(w_1, w_2) = 2$. Let y be such that $w_1 \sim y \sim w_2$. Observe that $y \notin \{v, x, u, w_{k-1}\}$ since G is triangle-free, $(v, r) \not\sim (w_2, s_2)$, and $d_G(w_1, w_{k-1}) = 4$. As either $s_1 = 2$ or $s_2 = 2$, and $2 \neq k-1$, $y \not\sim u$. The only possible edges with ends in the path $w_2 \sim y \sim w_1 \sim v \sim x \sim u \sim w_{k-1}$ are $y \sim x$, $y \sim w_{k-1}$, $w_2 \sim x$, or $w_2 \sim w_{k-1}$. Any subset of these edges either induce an A graph, triangle, domino or cycle of length greater than or equal to 5 (see picture below). Therefore, none of these edges can exist. Thus $w_2 \sim y \sim w_1 \sim v \sim x \sim u \sim w_{k-1}$ is an induced path in G, a contradiction.



Therefore, $d_G(w_1, w_2) = 3$. As $w_1 \sim v \sim x \sim u$ is an induced path of length 3 in G and $(w_1, s_1) \not\sim (u, t)$, $s_2 = 3$. Then $d_G(v, w_2) = d_G(u, w_2) = 4$ as $(v, r) \not\sim (w_2, s_2)$ and $2 \neq k-1$. Thus $w_2 \not\sim x$. Let y, z be such that $w_2 \sim z \sim y \sim w_1$. As G is P_6-free, $w_2 \sim z \sim y \sim w_1 \sim v \sim x$ cannot be an induced path. The only possible edge with ends in the path is $y \sim x$, but then $\{w_2, z, y, w_1, v, x\}$ induces an X_{18}.

Now suppose that $d_G(v, w_1) = 1$ and $d_G(u, w_{k-1}) = 2$. Since $d_G(v, u) = 2$, $s_{k-1} = 2$. As $s_{k-1} = 2$, $w_{k-1} \not\sim x$ and $w_{k-1} \not\sim w_1$. Let y be such that $w_1 \sim v \sim x \sim u \sim y \sim w_{k-1}$. This path cannot be induced as G is P_6-free. The only possible edge with ends in the path is $y \sim w_1$. But then $w_1 \sim v \sim x \sim u \sim y \sim w_1$ is an induced C_5, a contradiction.

Therefore, $d_G(v, w_1) = 2$. By symmetry, we also have $d_G(u, w_{k-1}) = 2$. Since $d_G(v, u) = 2$, $s_1 = 2$ and $s_{k-1} = 2$. Therefore, $w_1 \not\sim x$ and $w_{k-1} \not\sim x$. Let y, z be such that $w_1 \sim y \sim v \sim x \sim u \sim z \sim w_{k-1}$ is a path in G. The only possible edges with ends in the path are $w_1 \sim w_{k-1}$ and $y \sim z$. But the edges induce long cycles, a contradiction as G is weakly chordal.

Case 2: $d_G(v, u) = 3$.

By choice of (v, r) and (u, t), for all $2 \leq i \leq k - 1$, $d_G(v, w_i) = 3$, and for all $1 \leq i \leq k-2$, $d_G(u, w_i) = 3$. Let x, y be such that $v \sim x \sim y \sim u$ is a shortest vu-path in G. We claim that if $d_G(v, w_1) \geq 2$, then there is a shortest vw_1-path which passes through x. Let $d_G(v, w_1) = 2$ and suppose to the contrary that there is no shortest vw_1-path which passes through x. Then let $z \neq x$ be such that $w_1 \sim z \sim v \sim x \sim y \sim u$ is a path in G, which cannot be induced. By assumption, $w_1 \not\sim x$. The only possible edge with ends in the path is $z \sim y$. But with this edge, $\{w_1, z, v, x, y, u\}$ induces an A graph, a contradiction. Let $d_G(v, w_1) = 3$ and suppose to the contrary that there is no shortest vw_1-path which passes through x. Then let z, a be such that $w_1 \sim z \sim a \sim v \sim x \sim y \sim u$ is a path in G, which cannot be induced. Note that $z \not\sim x$ by assumption. The only possible edge with ends in the path is $a \sim y$. But if this edge exists $\{z, a, v, x, y, u\}$ induces the A graph, a contradiction. A similar proof shows that if $d_G(u, w_{k-1}) \geq 2$, then there is a shortest uw_{k-1}-path which passes through y.

First suppose that $d_G(v, w_1) = 1$ and $d_G(u, w_{k-1}) = 1$. Then $w_1 \sim v \sim x \sim y \sim u \sim w_{k-1}$ is a path in G, which cannot be induced. As G is triangle free and by choice of (v, r) and (u, t), the only edge with ends in the path is $w_1 \sim w_{k-1}$. But then $w_1 \sim v \sim x \sim y \sim u \sim w_{k-1} \sim w_1$ is an induced C_6 in G, a contradiction as G is weakly chordal. Now suppose that $d_G(v, w_1) = 1$ and $d_G(u, w_{k-1}) = 2$. By the claim above, we have that $w_{k-1} \sim y$. So $\{w_1, v, x, y, u, w_{k-1}\}$ forms an X_{172}, which cannot be induced. By choice of (v, r) and (u, t), the only edge with ends in the X_{172} is $w_1 \sim w_{k-1}$. But with this edge, $w_1 \sim v \sim x \sim y \sim w_{k-1} \sim w_1$ is an induced C_5, a contradiction.

Now suppose that $d_G(v, w_1) = 1$ and $d_G(u, w_{k-1}) = 3$. As $d_G(v, u) = 3$, $s_{k-1} = 3$. Therefore, w_{k-1} cannot be adjacent to any neighbour of v, in particular, $w_{k-1} \not\sim x$. By the claim above, we know that there is a shortest uw_{k-1}-path which contains y. Let z be such that $u \sim y \sim z \sim w_{k-1}$ is a shortest uw_{k-1}-path. Then $w_1 \sim v \sim x \sim y \sim z \sim w_{k-1}$ is a path in G, which cannot be induced. By choice of (v, r) and (u, t), $w_1 \not\sim y$. As G is triangle-free and $s_{k-1} = 3$, there are no possible edges with ends in the path $w_1 \sim v \sim x \sim y \sim z \sim w_{k-1}$, a contradiction.

Now suppose $d_G(v, w_1) = 2$. By symmetry, $d_G(u, w_{k-1}) \geq 2$. First suppose that $d_G(u, w_{k-1}) = 2$. By the claim above, we can assume that $w_1 \sim x$ and $w_{k-1} \sim y$. Then

$\{v, x, y, u, w_1, w_{k-1}\}$ forms an H-graph. As G is H-free and triangle-free, $w_1 \sim w_{k-1}$, as this is the only possible edge with ends in the H. But then v, x, y, u, w_1, w_{k-1} forms an A-graph, a contradiction. Now suppose $d_G(u, w_{k-1}) = 3$. As $d_G(v, u) = 3$, $s_{k-1} = 3$. Since $s_{k-1} = 3$, and by the claim above, we can assume that $w_1 \sim x$ and there is some $z \neq x$ such that $u \sim y \sim z \sim w_{k-1}$ is a shortest uw_{k-1}-path. So $d_G(v, w_{k-1}) = 4$. Then $\{v, x, y, u, w_1, z\}$ forms an H-graph. As G is H-free, $w_1 \sim z$, as this is the only possible edge with ends in the H. But then $\{v, x, y, u, w_1, z\}$ induces an A-graph, a contradiction.

Next suppose $d_G(v, w_1) = 3$. By symmetry, we have that $d_G(u, w_{k-1}) = 3$. By the above claim and choice of $(v, r), (u, t)$, there exist vertices z, a with $z \neq x$ and $a \neq y$ such that $u \sim y \sim z \sim w_{k-1}$ is a shortest uw_{k-1}-path and $v \sim x \sim a \sim w_1$ is a shortest vw_1-path. As $d_G(v, u) = 3$, $s_1 = 3 = s_{k-1}$. Thus $d_G(v, w_{k-1}) = 4$ and $d_G(u, w_1) = 4$. Then $\{v, x, y, u, a, z\}$ forms an H-graph, which cannot be induced. But the only possible edge with ends in the H is $z \sim a$, in which case $\{v, x, y, u, a, z\}$ induces an A-graph, a contradiction.

Case 3: $d_G(v, u) = 4$.

Let x, y, z be such that $v \sim x \sim y \sim z \sim u$ is a shortest vu-path in G. By choice of (v, r) and (u, t), $d_G(w_1, u) = 4$. If $d_G(v, w_1) = 1$, $w_1 \sim v \sim x \sim y \sim z \sim u$. As G is triangle free and $d_G(w_1, u) = 4$, this is an induced path in G, a contradiction as G is P_6-free. Now suppose $d_G(v, w_1) = 2$. If $w_1 \sim x$, $\{v, x, y, z, u, w_1\}$ induces an X_{172}, as $d_G(w_1, u) = 4$ no other edges can exist between these vertices. Thus $w_1 \not\sim x$. So there is some vertex $a \neq x$ such that $w_1 \sim a \sim v$ is a shortest vw_1-path in G. Then $w_1 \sim a \sim v \sim x \sim y \sim z \sim u$ is a path in G, which cannot be induced. As $w_1 \not\sim x$ and $d_G(w_1, u) = 4$, the only edge with ends in the path is $a \sim y$. But then $\{w_1, a, v, x, y, z\}$ induces an A-graph, a contradiction. Now suppose $d_G(v, w_1) = 3$ and there is a shortest vw_1-path which passes through x. As $d_G(w_1, u) = 4$, there is some vertex $a \neq y$ such that $v \sim x \sim a \sim w_1$ is a shortest vw_1-path. Then $w_1 \sim a \sim x \sim y \sim z \sim u$ is a path in G, which cannot be induced. But as $d_G(w_1, u) = 4$ and G is triangle-free, there are no possible edges with ends in the path, a contradiction. Finally, suppose $d_G(v, w_1) = 3$ and no shortest vw_1-path which passes through x. Then there exists vertices a, a' such that $w_1 \sim a \sim a' \sim v$ is a shortest vw_1-path in G. Then $w_1 \sim a \sim a' \sim v \sim x \sim y$ is a path in G, which cannot be induced. Recall, $a \not\sim x$ by assumption. Thus, the only possible edges with ends in path are $a \sim y$ and $a' \sim y$. Since G is triangle free, we cannot have both these edges. If $a \sim y$, then $a \sim a' \sim v \sim x \sim y \sim a$ is an induced C_5 in G. Thus $a' \sim y$. But then $\{w_1, a, a', v, x, y\}$ induces an X_{18}, a contradiction.

Thus (v, r) and (u, t) is a two-pair as required. □

Chapter 3
Exact Values of α_b

In the previous chapter, our results were concerned with the time complexity of determining the broadcast independence number of different classes of graphs. This chapter's focus is determining explicit values for α_b in several different types of graphs: proper interval graphs, triangle-appended trampolines, star split graphs and trampolines. We also provide a thorough examination of the structure of α_b-broadcasts in split graphs, as well as provide a characterization for when $\alpha_b(G) = \alpha(G)$ in split graphs. At the end of this chapter, we provide a polynomial time algorithm for a certain subclass of split graphs, called leaf appended split graphs. This algorithm is different from the one used in the previous chapter.

3.1 Proper interval graphs

In this section, we turn our attention to proper interval graphs. First note that cliques are proper interval graphs with diameter 1. For a clique K_n with $n \geq 2$, we know $\alpha_b(K_n) = 1$. Therefore, in this section, we consider proper interval graphs with diameter 2 or greater.

Recall Theorem 1.16 characterizes proper interval graphs by a vertex ordering, called a *proper interval ordering*. This characterization is crucial for the results of this section. It is important to note that the main result of this section, Theorem 3.7, follows a similar proof method to that of Erwin's proof of $\alpha_b(P_n) = 2(n-2)$. In fact, the results are analogous [20]. This is not necessarily surprisingly, as paths are proper interval graphs. We now present some preliminary lemmas, which are used in the proof of the main result of this section, Theorem 3.7.

Lemma 3.1. *Let G be a connected proper interval graph, and let $\prec$ be a proper interval ordering of G. Suppose that $x_1 \sim x_2 \sim \cdots \sim x_k$ is a shortest x_1x_k-path in G. If*

$x_1 \prec x_k$, *then for all* $1 \leq i \leq k-1$, $x_i \prec x_{i+1}$.

Proof. Suppose to the contrary that there is some i such that $1 \leq i \leq k-1$ and $x_{i+1} \prec x_i$. Let i be the smallest such index. If $i = 1$, let l be the first index such that $x_i \prec x_l$. Then $x_{l-1} \prec x_1 \prec x_l$ and $x_{l-1} \sim x_l$. By the proper interval ordering of G, $x_1 \sim x_l$. But then $x_1 \sim x_l \sim x_{l+1} \sim \cdots \sim x_k$ is a shorter x_1x_k-path, a contradiction.

Now suppose $i \geq 2$. By choice of i, $x_1 \prec x_2 \prec \cdots \prec x_{i-1} \prec x_i$ and $x_{i+1} \prec x_i$. As $x_{i+1} \prec x_i$, $x_{i-1} \prec x_i$, $x_{i+1} \sim x_i$ and $x_{i-1} \sim x_i$, by the proper interval order of G, $x_{i-1} \sim x_{i+1}$. Thus $x_1 \sim x_2 \sim \cdots \sim x_{i-1} \sim x_{i+1} \sim \cdots \sim x_k$ is a shorter x_1x_k-path, a contradiction. □

Lemma 3.2. *Let G be a connected proper interval graph, and let $\prec$ be a proper interval ordering of G. Let $x_i \prec x_j \prec x_k$. Then $d_G(x_j, x_k) \leq d_G(x_i, x_k)$ and $d_G(x_i, x_j) \leq d_G(x_i, x_k)$, with equality only possible in the case where $x_i \sim x_j$ and $x_j \sim x_k$, respectively.*

Proof. Consider a shortest x_ix_k-path in G, say $P = u_1 \sim u_2 \sim \cdots \sim u_m$ with $x_i = u_1$ and $x_k = u_m$. If $x_j = u_l$ for some l, then we are done. Otherwise, let u_l be such that $u_l \prec x_j \prec u_{l+1}$. As $u_l \sim u_{l+1}$, $u_l \sim x_j$ and $u_{l+1} \sim x_j$. If $l = 1$, $x_j \sim u_2 \sim \cdots \sim u_m$ is a path between x_j and x_k of length $d_G(x_i, x_k)$, and $x_i \sim x_j$. If $l \geq 2$, $x_j \sim u_{l+1} \sim \cdots \sim u_m$ is a path between x_j and x_k of length less than $d_G(x_i, x_k)$. Therefore, $d_G(x_j, x_k) \leq d_G(x_i, x_k)$, with equality only possible in the case where $x_i \sim x_j$ as desired. A similar analysis shows $d_G(x_i, x_j) \leq d_G(x_i, x_k)$ with equality only possible in the case where $x_j \sim x_k$. □

Lemma 3.3. *Let G be a connected proper interval graph, and let $\prec$ be a proper interval order of G. Let $x_i \prec x_j \prec x_k$ such that $x_i \not\sim x_j$. Then $d_G(x_i, x_j) + d_G(x_j, x_k) - 1 \leq d_G(x_i, x_k)$.*

Proof. Consider a shortest x_ix_k-path in G, say $P = u_1 \sim u_2 \sim \cdots \sim u_m$, with $x_i = u_1$ and $x_k = u_m$. If $x_j \in V(P)$, then the result holds. So suppose $x_j \notin V(P)$ for any shortest x_ix_k-path P. Let u_l be such that $u_l \prec x_j \prec u_{l+1}$. So $u_l \sim x_j$ and $u_{l+1} \sim x_j$. So the path $u_1 \sim u_2 \sim \cdots \sim u_l \sim x_j \sim u_{l+1} \sim \cdots \sim u_m$ is a x_ix_k-path of length $d_G(x_i, x_k) + 1$, composed of an x_ix_j-subpath, say $P' = u_1 \sim u_2 \sim \cdots \sim u_l \sim x_j$ and a x_jx_k-subpath, say $P'' = x_j \sim u_{l+1} \sim \cdots \sim u_m$. Note that $d_G(x_i, x_j)$, respectively $d_G(x_j, x_k)$, is less than or equal to the length of P', respectively P''. Thus $d_G(x_i, x_j) + d_G(x_j, x_k) \leq d_G(x_i, x_k) + 1$, giving the desired result. □

Lemma 3.4. *Let G be a connected proper interval graph, let $\prec$ be a proper interval order of G, and let A be a maximum size set of pairwise antipodal vertices. Then $|A| = 2$.*

Proof. Suppose $|A| \geq 3$. Consider $x_i, x_j, x_k \in A$ and suppose without loss of generality $x_i \prec x_j \prec x_k$. By Lemma 3.2, $d_G(x_i, x_j) \leq d(x_i, x_k)$. By choice of A, $d_G(x_i, x_j) = d(x_i, x_k)$. By Lemma 3.2, $x_j \sim x_k$. But as $diam(G) \geq 2$, this is not possible. □

Lemma 3.5. *Let G be a connected proper interval graph with order n, and let A be a maximum size set of pairwise antipodal vertices. Suppose $v_1 \prec v_2 \prec \cdots \prec v_n$ is a proper interval order of G. Then we can assume $A = \{v_1, v_n\}$.*

Proof. Suppose $A = \{v_i, v_j\}$ with $v_1 \prec v_i \prec v_j \prec v_n$. By Lemma 3.2, $d_G(v_1, v_j) \geq d_G(v_i, v_j)$. By choice of v_i and v_j, $d_G(v_1, v_j) \leq d_G(v_i, v_j)$. Thus $d_G(v_1, v_j) = d_G(v_i, v_j)$. Thus we can assume $A = \{v_1, v_j\}$. An analogous argument shows we can assume $A = \{v_1, v_n\}$. □

Lemma 3.6. *Let G be a connected proper interval graph, and suppose $v_1 \prec v_2 \prec \cdots \prec v_n$ is a proper interval order of G. Let f be an independent broadcast on G. Then $|H(v_i)| \leq 2$ for all i such that $1 \leq i \leq n$. Further, if $|H(v_i)| = 2$, then $H(v_i) = \{v_j, v_k\}$ where, without loss of generality, $v_j \prec v_i \prec v_k$.*

Proof. If $|H(v_i)| \geq 3$, then there must be $v_j, v_k \in H(v_i)$ such that, without loss of generality, either $v_i \prec v_j \prec v_k$ or $v_j \prec v_k \prec v_i$. Suppose $v_i \prec v_j \prec v_k$. By Lemma 3.2, we have that $d_G(v_j, v_k) \leq d_G(v_i, v_k) \leq f(v_k)$, but this contradicts that f is independent. An analogous argument holds for $v_j \prec v_k \prec v_i$. Therefore, $|H(v_i)| \leq 2$ for all $1 \leq i \leq n$, and by the above argument, if $H(v_i) = \{v_j, v_k\}$, without loss of generality, $v_j \prec v_i \prec v_k$. □

We now state and prove the main result of this section.

Theorem 3.7. *Let G be a connected proper interval graph of order n, and let $v_1 \prec v_2 \prec \cdots \prec v_n$ be a proper interval order of G. Then $\alpha_b(G) = 2(diam(G) - 1)$.*

Proof. First we show $\alpha_b(G) \geq 2(diam(G) - 1)$. By Lemma 3.5, we know that v_1 and v_n are antipodal vertices. Let f be a broadcast on G defined as follows

$$f(x) = \begin{cases} diam(G) - 1 & x = v_1 \text{ or } v_n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is independent. Therefore, $\alpha_b(G) \geq 2(diam(G) - 1)$.

We now show $\alpha_b(G) \leq 2(diam(G)-1)$. We claim that there exists an α_b-broadcast f on G such that $f(v_1) > 0$ and $f(v_n) > 0$.

Suppose not. Let v_i be such that v_1 hears the broadcast f from v_i. By Lemma 3.6, $H(v_1) = \{v_i\}$. Consider the broadcast g defined as follows:

$$g(x) = \begin{cases} f(v_i) & x = v_1, \\ 0 & x = v_i, \\ f(x) & \text{otherwise.} \end{cases}$$

By Lemma 3.2, no v_j such that $v_1 \prec v_j \prec v_i$ can have $f(v_j) > 0$. Furthermore, by Lemma 3.2 for all $v_i \prec v_k$ such that $f(v_k) > 0$, $f(v_i) < d_G(v_i, v_k) \leq d_G(v_1, v_k)$. So g is independent with the same cost as f. A similar argument shows we can assume $f(v_n) > 0$.

Now let f be an α_b-broadcast on G such that $f(v_1) > 0$ and $f(v_n) > 0$. We claim that we can alter the broadcast such that $f(v_i) = 0$ for all i such that $2 \leq i \leq n-1$.

Suppose to the contrary that there exists an i such that $2 \leq i \leq n-1$ and $f(v_i) > 0$. Let i be the smallest such index. By choice of i, for any another broadcast vertex, say v_j, $v_i \prec v_j$, and $d_G(v_1, v_i) < d_G(v_1, v_j)$, by Lemma 3.2. So v_i is the closest broadcast vertex to v_1. Thus $f(v_1) = d_G(v_1, v_i) - 1$. Furthermore, let j be such that, $j \neq i$, $f(v_j) > 0$ and j is minimum. So either v_1 or v_j is the closest broadcast vertex to v_i. So $f(v_i) = \min\{d_G(v_1, v_i), d_G(v_i, v_j)\} - 1$.

We define a broadcast g as follows:

$$g(x) = \begin{cases} f(v_1) + f(v_i) & x = v_1, \\ 0 & x = v_i, \\ f(x) & \text{otherwise.} \end{cases}$$

In g, the closest broadcast vertex to v_1 is v_j. By Lemma 3.3, as $v_1 \prec v_i \prec v_j$, $g(v_1) = f(v_1) + f(v_i) = d_G(v_1, v_i) - 1 + \min\{d_G(v_1, v_i), d_G(v_i, v_j)\} - 1 < d_G(v_1, v_i) + d_G(v_i, v_j) - 1 \leq d_G(v_1, v_j)$ by Lemma 3.3. Thus g is independent and has the same cost as f. We can repeat this argument until we obtain a broadcast with the desired structure.

Thus, we can find an α_b-broadcast in which only v_1 and v_n broadcast. By Lemma 3.5, we know that v_1 and v_n are antipodal vertices, so it follows that $\alpha_b(G) \leq 2(diam(G)-1)$. The result follows. $\square$

3.2 Triangle appended n-trampolines

In this section, we examine broadcast independence in triangle appended n-trampolines with $n \geq 3$, and provide the exact value of $\alpha_b(G)$ for such graphs. Recall, a triangle appended n-trampoline, with $n \geq 3$, is the graph obtained from the n-trampoline, with vertices $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$, by adding n new vertices $x_1, x_2, \ldots, x_n$ and setting $v_i \sim x_i \sim u_{i+1}$ for $1 \leq i \leq n$, with index arithmetic done modulo n. We begin with a basic observation and proposition.

Observation 3.8. *Let G be a triangle appended n-trampoline, with $n \geq 3$. Then $diam(G) = 3$.*

Proposition 3.9. *Let G be a triangle appended n-trampoline, with $n \geq 3$. For an α_b-broadcast f on G, $f(v) \leq 2$ for all $v \in V(G)$ and $|V_f^+| \geq 2$.*

Proof. Suppose to the contrary that there exists an α_b-broadcast f on G where $f(x) = 3$ for some vertex x. As $diam(G) = 3$, x is the only vertex which broadcasts in f. Let u and v be such that $d_G(v, u) = 3$. Clearly, the broadcast g defined by

$$g(x) = \begin{cases} 2 & x = v \text{ or } u, \\ 0 & \text{otherwise} \end{cases}$$

is an independent broadcast with $g(V) > f(V)$, a contradiction. $\square$

Lemma 3.10. *Let G be a triangle appended n-trampoline, with $n \geq 3$. Then there exists an α_b-broadcast f on G such that if $f(v) > 0$, $v \in \{x_1, x_2, \ldots, x_n\}$.*

Proof. Let f be an α_b-broadcast on G. First suppose that $f(v_i) > 0$ for some $i \in \{1, \ldots, n\}$. Consider x_i, where $x_i \sim v_i$. By the structure of G, we have $d_G(y, v_i) \leq d_G(y, x_i)$ for all $y \in V(G)$ distinct from x_i, v_i. Therefore, $N[x_i] \subset N[v_i]$ and $N_2[x_i] \subset N_2[v_i]$. Further, as f is independent, we have $H(x_i) = \{v_i\}$. Define a broadcast g as follows:

$$g(x) = \begin{cases} f(v_i) & x = x_i, \\ 0 & x = v_i, \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, $g(V) = f(V)$, and g is independent.

Now suppose $f(u_{i+1}) > 0$ for some $i \in \{1, \ldots, n\}$. Consider x_i, where $x_i \sim u_{i+1}$. By the structure of G, we have $d_G(y, u_{i+1}) \leq d_G(y, x_i)$ for all $y \in V(G)$ distinct from x_i, u_{i+1}. Therefore, $N[x_i] \subset N[u_{i+1}]$ and $N_2[x_i] \subset N_2[u_{i+1}]$. Further, as f is independent, we have $H(x_i) = \{u_{i+1}\}$. Define a broadcast g as follows:

$$g(x) = \begin{cases} f(u_{i+1}) & x = x_i, \\ 0 & x = u_{i+1}, \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, $g(V) = f(V)$, and g is independent. The result follows. □

Theorem 3.11. *Let G be a triangle appended n-trampoline, with $n \geq 3$. Then $\alpha_b(G) = 2n$.*

Proof. Let f be an α_b-broadcast on G as described in Lemma 3.10. As f is optimal, clearly $f(x_i) = 2$ for all $1 \leq i \leq n$. The result follows. □

3.3 Split graphs

In this section, we study α_b-broadcasts in several types of split graphs. Recall that for a split graph G, we have partitioned $V(G)$ into a clique, C, and an independent set, S, in such a way that the size of the independent set is maximum. Further, for a split graph G, $diam(G) \leq 3$. In this section we restrict our attention to split graphs with diameter 3, as $\alpha_b(G) = \alpha(G)$ if $diam(G) = 2$. We first introduce some preliminary results about the structure of independent broadcasts in split graphs.

Proposition 3.12. *Let f be an α_b-broadcast on a split graph G. Then $f(v) \leq 2$ for all $v \in V(G)$, and $|V_f^+| \geq 2$.*

Proof. Suppose to the contrary that there exists an α_b-broadcast f on G where $f(x) = 3$ for some vertex x. As $diam(G) = 3$, x is the only vertex which broadcasts in f. Let u and v be such that $d_G(v, u) = 3$. Clearly, the broadcast g defined by

$$g(x) = \begin{cases} 2 & x = v \text{ or } u, \\ 0 & \text{otherwise} \end{cases}$$

is an independent broadcast with $g(V) > f(V)$, a contradiction. □

Lemma 3.13. *Let G be a split graph. Then there exists an α_b-broadcast f on G such that if $f(v) > 0$, then $v \in S$.*

Proof. Let f be an α_b-broadcast on G. Suppose there exists a vertex $v \in C$ such that $f(v) > 0$. As C is a clique, v is the only such vertex and $ecc(v) \leq 2$. As $|V_f^+| \geq 2$, $f(v) = 1$. Let $u \in S$ such that $v \sim u$. We claim that $H(u) = \{v\}$.

Proof of Claim. Suppose not. So u hears some other vertex, say x. Thus, $x \in S$. As f is independent and $f(v) = 1$, $d(v, x) = 2$. Thus $f(x) = 1$. But this is impossible as $u, x \in S$. ◇

Furthermore, it is easy to see that for all $x \in V(G)$, $d(x, v) \leq d(x, u)$. Therefore, setting $f(u) = 1$ and $f(v) = 0$ maintains both the weight and independence of f, giving the desired broadcast. □

Throughout the remainder of this section, we assume that when we reference an α_b-broadcast on a split graph, we are referring to one as described in Lemma 3.13. In other words, for a α_b-broadcast f on a split graph G, we assume $V_f^+ \subseteq S$.

Lemma 3.14. *Let G be a split graph and let $v \in S$ such that $deg(v) = 1$. Then there exists an α_b-broadcast f on G such that $f(v) > 0$.*

Proof. Let $v \in S$ be such that $deg(v) = 1$. Consider the following cases:
Case 1. There does not exist a vertex $u \in S$ such that $d_G(u, v) = 2$.

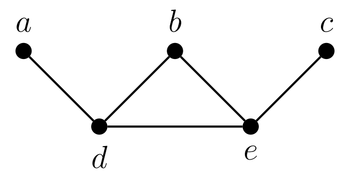
Then clearly as f is an α_b-broadcast, $f(v) = 2$.

Case 2. There exists $u \in S$ such that $d_G(u, v) = 2$.

If $f(v) > 0$, we are done. So suppose that $f(v) = 0$. As f is an α_b-broadcast, there is some u such that $d_G(u, v) = 2$, $u \in H(v)$ and $u \in S$. Thus $f(u) = 2$. We claim that $H(v) = \{u\}$. Suppose not. Then there exists $w \in S$ such that $w \in H(v)$. Therefore, $f(w) = 2$ and $d_G(w, v) = 2$. As $deg(v) = 1$, let $c \in C$ be such that $v \sim c$. As $d_G(v, u) = d_G(w, u) = 2$ and $deg(v) = 1$, $w \sim c$ and $u \sim c$. But then $d_G(w, u) = 2$, contradicting that f is independent. Thus $H(v) = \{u\}$. So we can set $f(v) = f(u) = 1$, and f remains independent with the same weight and we are done. □

Recall that an independent set can be described as an independent broadcast in which all the broadcasting vertices have power 1. Thus, a natural question to ask is "When is the broadcast independence number equal to the independence number?". The following theorem provides a sufficient condition for when this equality holds for split graphs.

Theorem 3.15. *Let G be a split graph such that for every $v \in S$, there exists $u \in S$ such that $d_G(u, v) = 2$, and G does not contain the following as an induced subgraph:*



where $a, b, c \in S$ and $d, e \in C$. Then $\alpha_b(G) = \alpha(G)$.

Proof. Let f be an α_b-broadcast on G. We now show that we can alter f so that $f(v) \leq 1$ for all $v \in S$. Suppose that there exists $v \in S$ such that $f(v) = 2$. Let $v_1, v_2, \ldots, v_k$ be such that $v_i \in S$ and $d_G(v, v_i) = 2$. By assumption, $k \geq 1$. We claim $H(v_i) = \{v\}$ for all $1 \leq i \leq k$. Suppose that for some i, $|H(v_i)| \geq 2$. Consider $u \neq v$ such that $u \in H(v_i)$. As $u \in S$, $d_G(u, v_i) = 2$. Thus $f(u) = 2$. As f is independent, $d_G(v, u) = 3$. Then there exists $c_1, c_2 \in C$ such that $v \sim c_1 \sim v_i$ and $u \sim c_2 \sim v_i$ but $v \not\sim c_2$ and $u \not\sim c_1$. But this is a contradiction as $\{v, v_i, u, c_1, c_2\}$ forms the forbidden subgraph. So $H(v_i) = \{v\}$ for all $1 \leq i \leq k$. Define a broadcast g as follows:

$$g(x) = \begin{cases} 1 & x \in \{v, v_1, v_2, \ldots, v_k\}, \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, g is independent and $g(V) \geq f(V)$, as $k \geq 1$. □

We now provide a characterization of the split graphs for which $\alpha_b(G) = \alpha(G)$.

Theorem 3.16. *Let G be a split graph. Then G has the following property: For all sets of pairwise distance 3 vertices $\{u_1, u_2, \ldots, u_k\}$,*

$$\left| \bigcup_{1 \leq i \leq k} (N_2(u_i) \cap S) \right| \geq k.$$

if and only if $\alpha_b(G) = \alpha(G)$.

Proof. Let f be an α_b-broadcast on G, and suppose G has the property described above. Suppose to the contrary $\alpha_b(G) \neq \alpha(G)$. Let $\{u_1, u_2, \ldots, u_k\} \subseteq V_f^+$ such that $f(u_i) = 2$ for all $1 \leq i \leq k$. As f is independent, $\{u_1, u_2, \ldots, u_k\}$ is a set of pairwise distance 3 vertices. Note that $\sum_{1 \leq i \leq k} f(u_i) = 2k$. By assumption we have that $|\bigcup_{1 \leq i \leq k}(N_2(u_i) \cap S)| \geq k$. We define a broadcast g as follows:

$$g(x) = \begin{cases} 1 & x \in \{u_1, u_2, \ldots, u_k\}, \\ 1 & x \in \bigcup_{1 \leq i \leq k} (N_2(u_i) \cap S), \\ f(x) & \text{otherwise.} \end{cases}$$

It is easy to see that g is independent with $g(V) \geq f(V)$ and for all $v \in V_g^+$, $g(v) = 1$.

Now suppose $\alpha_b(G) = \alpha(G)$. Thus there exists an α_b-broadcast f on G with $f(v) = 1$ for all $v \in V_f^+$, where $V_f^+ = S$. Suppose to the contrary that there exists a set of pairwise distance 3 vertices, say $\{u_1, u_2, \ldots, u_k\}$ such that $|\bigcup_{1\leq i\leq k}(N_2(u_i) \cap S)| < k$. We define a broadcast g as follows:

$$g(x) = \begin{cases} 2 & x \in \{u_1, u_2, \ldots, u_k\}, \\ 0 & x \in \bigcup\limits_{1\leq i\leq k} (N_2(u_i) \cap S), \\ f(x) & \text{otherwise.} \end{cases}$$

It is easy to see that g is independent with $g(V) > f(V)$, a contradiction. □

3.3.1 Star split graphs

In this section, we examine broadcast independence in a specific type of split graphs, which we call *star split graphs*.

Definition 3.17. *Let G be a split graph. We say G is a* star split graph *if for all $v \in S$, $deg(v) = 1$.*

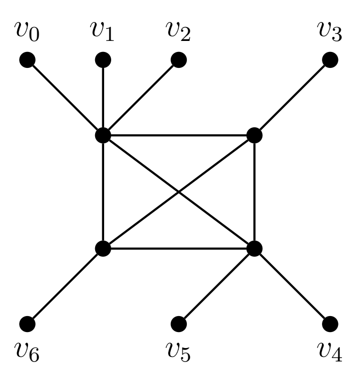


Figure 3.1: A star split graph.

A *star* is the complete bipartite graph $K_{1,k}$ for some positive integer k; that is, a graph consisting of one vertex adjacent to all other vertices. A star is a star split graph, where the clique of the split graph has size 1. In order to prove the main result of this section, we provide a partition of the vertices in the independent set of a star split graph.

Definition 3.18. *Let G be a star split graph. Let $v \in S$. We say v is of* Type A *if there is some $u \in S$ such that $d(u, v) = 2$. We say v is of* Type B *if for all $u \in S$, we* *have $d(u,v) = 3$. We let A, respectively B, denote the set of Type A, respectively Type B, vertices of S.*

For the star split graph in Figure 3.1, we have $A = \{v_0, v_1, v_2, v_4, v_5\}$ and $B = \{v_3, v_6\}$.

We now present results on the structure of an α_b-broadcast on a star split graph.

Observation 3.19. *Let G be a star split graph. If $u, v \in S$ such that $d(u,v) = 2$, then $N(u) = N(v)$, and for any $x \in V(G)$ distinct from v and u, $d(u,x) = d(v,x)$.*

Lemma 3.20. *Let G be a star split graph. Then there is an α_b-broadcast f on G where $f(v) = 1$ for all $v \in A$.*

Proof. Let f be an α_b-broadcast on G, and let $v \in A$. By Lemma 3.14, we know that $f(v) > 0$. Suppose there is some $v \in A$ such that $f(v) = 2$. Since v is of Type A, let $u \in S$ such that $d(v,u) = 2$. So u hears v, thus $f(u) = 0$. We claim that $H(u) = \{v\}$. Suppose to the contrary that u hears another vertex, say x. But by Observation 3.19, $d(u,x) = d(v,x)$. Therefore, v hears x as well, contradicting that f is independent. So setting $f(v) = f(u) = 1$ maintains both the weight and independence of f. □

Lemma 3.21. *Let G be a star split graph. Then for any α_b-broadcast f on G, $f(v) = 2$ for all $v \in B$.*

Proof. Let f be an α_b-broadcast on G. Suppose there is some $v \in B$ such that $f(v) < 2$. By Lemma 3.14, we know $f(v) > 0$. So $f(v) = 1$. But as $d(v,u) = 3$ for all $u \neq v \in S$, we can clearly increase $f(v) = 2$ while maintaining the independence of f, a contradiction. □

We conclude this section with the exact value of α_b of star split graphs.

Theorem 3.22. *Let G be a star split graph. Then*

$$\alpha_b(G) = |A| + 2|B|.$$

Proof. Follows from Lemmas 3.20 and 3.21. □

3.3.2 Trampolines

The next class which we examine are *trampoline graphs.* In this section, we explicitly determine the broadcast independence number of these graphs. We begin by restating the definition of a trampoline graph, along with some additional notation.

Definition 3.23. *An* n-trampoline *is a graph on* $2n$ *vertices, say* $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$*, where* $\{u_1, u_2 \ldots, u_n\}$ *induces a copy of* K_n *and each* v_i *is adjacent to* u_i *and* u_{i+1} *with index arithmetic performed modulo* n*. The vertices* $\{v_1, v_2, \ldots, v_n\}$ *are called the* outer ring*.*

Lemma 3.24. *Let* G *be an* n*-trampoline. Then* $\alpha_b(G) \geq n$*.*

Proof. We define a broadcast f on G as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{v_1, v_2, \ldots, v_n\}, \\ 0 & \text{if } x \in \{u_1, u_2, \ldots, u_n\}. \end{cases}$$

Then $f(V) = n$. It is easy to see that this bound is tight for $n = 3$. □

In the following lemma, we employ a technique known as *discharging*. This method has applications in graph colourings of planar graphs [6, 16].

Lemma 3.25. *Let* G *be an* n*-trampoline for* $n \geq 4$*. Then* $\alpha_b(G) \leq n$*.*

Proof. Let f be an α_b-broadcast on G as described in Lemma 3.13. As $\alpha_b \geq 4$ and $diam(G) = 3$, it is clear that $f(v_i) = 1$ or 2 for each $v_i \in V_f^+$. We assign each vertex on the outer ring a charge equal to its value under f.

We move charge as follows. If a vertex v_i has charge 2, then it gives charge $\frac{1}{2}$ to each of v_{i-1} and v_{i+1}, which must have started with charge zero as $d(v_i, v_{i-1}) = d(v_i, v_{i+1}) = 2$. We do not move charge from vertices with charge 1.

Now vertices that started with 2, have charge 1. Vertices that started with charge 1 still have charge 1. Vertices that started with 0 have charge at most 1. So the total charge is at most n which completes the proof. □

We now state the main result of this section.

Theorem 3.26. *Let* G *be an* n*-trampoline. Then* $\alpha_b(G) = \alpha(G) = n$*.*

3.4 Leaf appended split graphs

In Chapter 2, we proved that α_b can be computed in polynomial time for strongly chordal split graphs using the ball catch graph. In this section, we show that α_b can be computed in polynomial time for another class of split graphs, called *leaf appended split graphs*, using a different algorithm.

Definition 3.27. *Let G be a split graph such that $|C| \geq 3$, and for each $u \in C$, there is some $v \in S$ such that $N(v) = \{u\}$. We say G is a* leaf appended split graph.

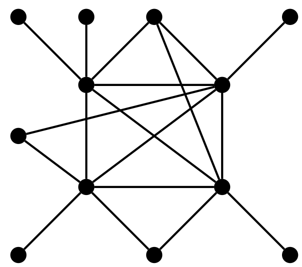


Figure 3.2: A leaf appended split graph.

We now provide an algorithm to find an α_b-broadcast on a leaf appended split graph G, with split partition (S, C). Recall that we assume $|S|$ is maximum over all split partitions of G. We define the following subsets of S: $X = \{v \in S | deg(v) = 1\}$ and $Y = S \setminus X$. Two vertices $x_1 \neq x_2$ are *twins* if $N(x_1) = N(x_2)$. Let $X_1 \subseteq X$ be $X_1 = \{x \in X | x \text{ is a twin}\}$, and let $Y_1 \subseteq Y$ be $Y_1 = \{y \in Y | \text{ for all } x \in X \text{ such that } N_G(x) \subset N_G(y), x \in X_1\}$. Let $X_2 = X \setminus X_1$ and $Y_2 = Y \setminus Y_1$.

Define H as a bipartite graph with bipartition $V(H) = (X_2, Y_2)$ and $x \sim y$ in $E(H)$ if $N(x) \subset N(y)$ (i.e. $x \sim y$ in $E(H)$ if $d_G(x, y) = 2$). We say H is the *auxiliary bipartite graph of* G. Find a maximum matching in H, say M, using the Hopcroft–Karp algorithm [40]. If X_2 is M-covered, define a broadcast g of G as follows:

$$g(v) = \begin{cases} 1 & v \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The following figure shows a split graph G (on the left), with its auxiliary bipartite graph H (on the right) and a maximum matching of H where X_2 is M-covered (the red squiggly edges).

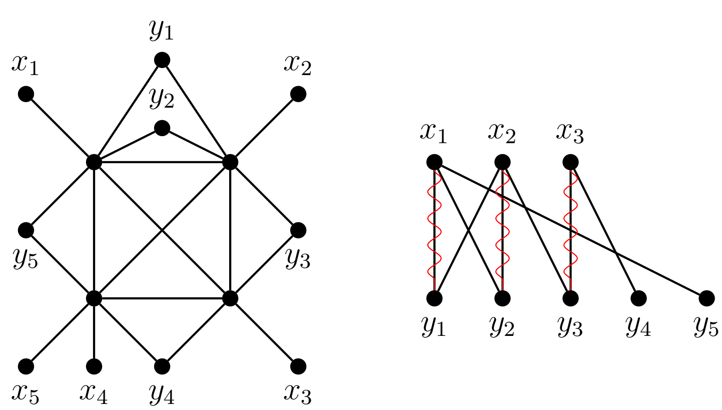


Figure 3.3: A split graph G and its auxiliary bipartite graph H, where X_2 is M-covered.

The example in Figure 3.3 has $X_1 = \{x_4, x_5\}$, $Y_1 = \emptyset$, $X_2 = \{x_1, x_2, x_3\}$, $Y_2 = \{y_1, y_2, y_3, y_4, y_5\}$, and $\alpha_b(G) = 10$, where all vertices of the independent set are broadcasting with power 1.

We now consider the case where X_2 is not M-covered. Let R be the set of vertices reachable by an M-alternating path starting at an M-uncovered vertex of X_2. In this case, we also let an M-alternating path consist of a single uncovered vertex of X_2. Define a broadcast g of G as follows:

$$g(v) = \begin{cases} 2 & v \in R \cap X_2, \\ 0 & v \in R \cap Y_2 \text{ or } v \in C, \\ 1 & \text{otherwise.} \end{cases}$$

The following figure shows a split graph G (on the left), with its auxiliary bipartite graph H (on the right) and a maximum matching M of H (indicated by the red squiggly edges) where X_2 is not M-covered.

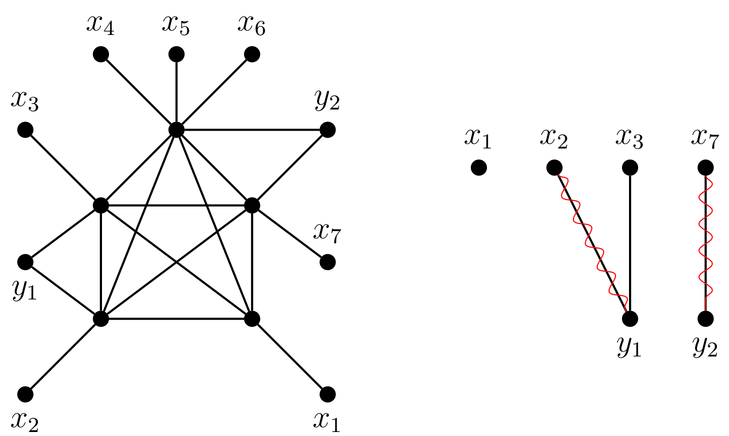


Figure 3.4: A split graph G and its auxiliary bipartite graph H, where X_2 is not M-covered.

Using the graph Figure 3.4, we have $X_1 = \{x_4, x_5, x_6\}$, $Y_1 = \emptyset$, $X_2 = \{x_1, x_2, x_3, x_7\}$, $Y_2 = \{y_1, y_2\}$, and $R = \{x_1, x_2, x_3, y_1\}$. This yields $R \cap (X_2) = \{x_1, x_2, x_3\}$, and $R \cap (Y_2) = \{y_1\}$. Therefore an α_b-broadcast g on G is given by

$$g(v) = \begin{cases} 2 & v \in \{x_1, x_2, x_3\}, \\ 0 & v \in \{y_1\} \text{ or } v \in C, \\ 1 & v \in \{x_4, x_5, x_6, x_7, y_2\}. \end{cases}$$

We now show that this algorithm outputs an α_b-broadcast through a series of lemmas.

Lemma 3.28. *Let G be a leaf-appended split graph. Then there is an α_b-broadcast f on G such that $f(x) = 1$ for all $x \in X_1$ and $f(y) = 1$ for all $y \in Y_1$.*

Proof. To see that $f(x) = 1$ for all $x \in X_1$, the proof follows identically to the proof of Lemma 3.20.

Let f be an α_b-broadcast on G such that $f(x) = 1$ for all $x \in X_1$, and let $y \in Y_1$. Let $X_y = \{x \in X_1 | N(x) \subset N(y)\}$. By choice of f, $f(x) = 1$ for all $x \in X_y$. Therefore, $f(y) \leq 1$. If $f(y) = 1$, we are done. Thus suppose $f(y) = 0$. As $f(x) = 1$ for all $x \in X_y$, y does not hear the broadcast from any vertex of X_y. But as f is an α_b-broadcast, y must hear the broadcast. By choice of y, any other vertex u such that $d_G(y, u) = 2$ must have $deg(u) \geq 2$. As G is leaf-appended, by Lemma 3.14, $f(u) \leq 1$ for all such u. Thus y cannot hear f from any vertex at distance 2. But this is a contradiction as $f(y) = 0$ and f is an α_b-broadcast. Thus, $f(y) = 1$ as required. □

Lemma 3.29. *The broadcast g produced by the above algorithm is an α_b-broadcast on G.*

Proof. Let f be an α_b-broadcast on G. By Lemmas 3.13, 3.14, and 3.28, $f(c) = 0$ for all $c \in C$, $f(x) = 1$ for all $x \in X_1$, $f(y) = 1$ for all $y \in Y_1$, $1 \leq f(x) \leq 2$ for all $x \in X_2$ and $f(y) \leq 1$ for all $y \in Y_2$.

First suppose that X_2 is M-covered. Let $U = \{x \in X_2 | f(x) = 2\}$. Let $U' = \{y \in Y_2 | \exists x \in U \text{ with } xy \in M\}$. In other words, U' is the set of vertices matched to the vertices of U. By definition of U and U', for $v \in U$, $f(v) = 2$ and $g(v) = 1$, and for $v \in U'$, $f(v) = 0$ and $g(v) = 1$. Further, for $v \in (X_2 \cup Y_2) \setminus (U \cup U')$, $g(v) = 1$ and $f(v) \leq 1$. Thus $g(U \cup U') = f(U \cup U') = 2|U|$, and $g((X_2 \cup Y_2) \setminus (U \cup U')) \geq f((X_2 \cup Y_2) \setminus (U \cup U'))$. Furthermore, observe $g(X_1 \cup Y_1) = f(X_1 \cup Y_1)$. Therefore, $g(V) \geq f(V) = \alpha_b(G)$. In fact, by definition of g, $\alpha_b(G) = \alpha(G)$ in this case.

Now suppose that X_2 is not M-covered. Observe that $g(X_1 \cup Y_1) = f(X_1 \cup Y_1)$. Furthermore, by definition, $(X_2 \setminus R)$ is M-covered. Therefore, we can follow the first part of the proof to see that $g((X_2 \setminus R) \cup (Y_2 \setminus R)) \geq f((X_2 \setminus R) \cup (Y_2 \setminus R))$. We claim that $g(X_2 \cap R) \geq f(Y_2 \cap R)$. Suppose not. Then one of the following cases must hold:
Case 1. There exists $y \in R \cap Y_2$ such that $f(y) = 1$ and $g(y) = 0$.

As $y \in Y_2 \cap R$ and M is maximum, there is some $x \in X_2 \cap R$ such that $x \sim y \in M$. As $f(y) = 1$, $f(x) = 1$. In g, we have $g(y) = 0$ and $g(x) = 2$. Thus $f(y) + f(x) = 2 = g(x) + g(y)$.
Case 2. There exists $x \in R \cap X_2$ with $f(x) = 1$ and $g(x) = 2$.

If there is no $y \in Y_2$ such that $x \sim y$ in H, then clearly setting $f(x) = 2$ maintains the independence of f, while increasing the weight of f, a contradiction. Therefore, there is some $y \in Y_2$ such that $x \sim y$. First suppose that there exists $y \in Y_2$ such that $x \sim y \in M$. As $f(x) = 1$, $f(y) \leq 1$. In g, we have $g(x) = 2$ and $g(y) = 0$. Thus $f(x) + f(y) \leq 2 = g(x) + g(y)$. Otherwise there does not exist $y \in Y_2$ such that $x \sim y \in M$. So x is uncovered in M. Recall we have $f(y) \leq 1$ for all $y \in Y \cap R_2$. If there is a $y \in Y \cap R_2$ such that $y \sim x$ and $f(y) = 1$, then any $x' \in X_2 \cap R$ such that $x' \sim y$ must have $f(x') = 1$. In particular, $f(x') = 1$ for $x' \in X_2 \cap R$ such that $y \sim x' \in M$. Continuing this argument and repeating it for all other uncovered vertices of $X_2 \cap R$, we can see that in f, each edge of the matching has been assigned power at most 2, $f(x) = 1$, and all other uncovered vertices $x' \in X_2 \cap R$ have $1 \leq f(x') \leq 2$. In g, each edge of the matching has been assigned power 2, $g(x) = 2$, and all other uncovered vertices $x' \in X_2 \cap R$ have $g(x') = 2$. Therefore, $g(V) > f(V)$, contradicting our choice of f.

In all cases, clearly $g(V) \geq f(V)$, as required. □

Theorem 3.30. *For a leaf appended split graph G, $\alpha_b(G)$ can be computed in polynomial time.*

Chapter 4
Broadcast Packing

In this chapter, we examine the broadcast packing problem in proper interval graphs, triangle-appended trampolines, graphs of diameter 3, and certain split graphs. Similar to broadcast independence, when considering the broadcast packing problem, we want to find either the exact value of the broadcast packing number or a polynomial time algorithm to determine this parameter. In this chapter, we first find the broadcast packing number of proper interval graphs and triangle-appended trampolines. We then discuss graphs of diameter 3 for which we can find the broadcast packing number in linear time. We study broadcast packing in different types of split graphs and show the NP-hardness of the problem in general split graphs. We conclude this chapter by providing an explicit formula for the broadcast packing number for a class of graphs which we call spider appended split graphs.

4.1 Proper interval graphs

In this section, we provide the exact value of the broadcast packing number of proper interval graphs. In Section 3.1, we see that the broadcast independence number of proper interval graphs is analogous to that of paths. The same parallel exists between the broadcast packing number of proper interval graphs and the broadcast packing number of paths [8]. In proof of Theorem 4.1, we use the primal and dual linear programs which are described in Section 1.1.

Theorem 4.1. *Let G be a proper interval graph of order n and let $v_1 \prec v_2 \prec \cdots \prec v_n$ be a proper interval order of G. Then $P_b(G) = M_c(G) = diam(G)$.*

Proof. By Lemma 3.5, $d_G(v_1, v_n) = diam(G)$. Define a broadcast f as follows:

$$f(x) = \begin{cases} diam(G) & x = v_1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is a packing broadcast.

We now find a multicover of size $diam(G)$. Let $P = u_1 \sim u_2 \sim \cdots \sim u_k$ be a shortest v_1v_n-path, with $u_1 = v_1$ and $u_k = v_n$. By Lemma 3.1, we know $u_i \prec u_{i+1}$ for all $1 \leq i \leq k-1$. Let $M = \{u_2, u_3, \ldots, u_k\}$. It is clear that $|M| = diam(G)$. We claim M is a multicover of G.

By Lemma 3.2, we have that for all $v \in V(G)$, $ecc(v) = \max\{d_G(v_1, v), d_G(v_n, v)\}$. Let $v \in V(G)$. Consider the following cases:

Case 1. $ecc(v) = d_G(v_n, v)$.

Let $u_i, u_{i+1} \in V(P)$ be such that $u_i \preceq v \prec u_{i+1}$. If $v = u_i$, cleary $u_i \sim u_{i+1} \sim \cdots \sim u_k$ is a shortest u_iv_n-path, as otherwise P would not be a shortest v_1v_n-path. As $ecc(v) = d_G(v_n, v)$, by choice of M, there is a token at distance d from v for all $1 \leq d \leq ecc(v)$. So $N_r[v]$ contains at least r tokens for all $1 \leq r \leq ecc(v)$.

Suppose $u_i \prec v$. As $u_i \sim u_{i+1}$, we have $v \sim u_{i+1}$. First we claim that for $2 \leq m \leq k-(i+1)$, the only such u_{i+m} such that $v \sim u_{i+m}$ is $m = 2$. Suppose to the contrary that there is some $m > 2$ such that $v \sim u_{i+m}$. As $v \prec u_{i+1}$, $u_{i+1} \sim u_{i+m}$. But then $u_1 \sim u_2 \sim \cdots \sim u_{i+1} \sim u_{i+m} \sim \cdots \sim u_k$ is a shorter v_1v_n-path than P, a contradiction.

We now claim either $v \sim u_{i+1} \sim \cdots \sim u_k$ or $v \sim u_{i+2} \sim \cdots \sim u_k$ is a shortest vv_n-path. Suppose that $v \sim u_{i+1} \sim \cdots \sim u_k$ is not a shortest vv_n-path. Let $P' = w_1 \sim w_2 \sim \cdots \sim w_m$ be a shortest vv_n-path, with $w_1 = v$ and $w_m = v_n$. Consider w_2. It must be that case that $u_{i+1} \prec w_2$, as if $w_2 \prec u_{i+1}$, $u_i \sim w_2 \sim \cdots \sim w_m$ is a shorter u_iv_n-path, contradicting our choice of P. Thus $u_{i+1} \prec w_2$. As $v \prec u_{i+1}$ and $v \sim w_2$, $u_{i+1} \sim w_2$. Therefore, $|\{w_2, w_3, \ldots, w_m\}| \geq |\{u_{i+2}, u_{i+3}, \ldots, u_k\}|$, as otherwise $u_{i+1} \sim w_2 \sim w_3 \sim \cdots \sim w_m$ is a shorter $u_{i+1}v_n$-path, contradicting our choice of P. Further, there must be equality as otherwise $v \sim u_{i+1} \sim \cdots \sim u_k$ is a vv_n-path of lesser or equal length than P'. Without loss of generality, we can assume $\{w_2, w_3, \ldots, w_m\} = \{u_{i+2}, u_{i+3}, \ldots, u_k\}$. The claim follows.

So either $v \sim u_{i+1} \sim \cdots \sim u_k$ or $v \sim u_{i+2} \sim u_{i+3} \sim \cdots \sim u_k$ is a shortest vv_n-path. As $ecc(v) = d_G(v_n, v)$, by choice of M, there is a token at distance d from v for all $1 \leq d \leq ecc(v)$. So $N_r[v]$ contains at least r tokens for all $1 \leq r \leq ecc(v)$.

Case 2. $ecc(v) = d_G(v_1, v)$.

Let $u_i, u_{i+1} \in V(P)$ be such that $u_i \prec v \preceq u_{i+1}$. If $v = u_{i+1}$, cleary $u_1 \sim u_2 \sim$

$\cdots \sim u_{i+1}$ is a shortest v_1u_{i+1}-path, as otherwise P would not be a shortest v_1v_n-path. As $ecc(v) = d_G(v_1, v)$, by choice of M, there is a token at distance d from v for all $1 \leq d \leq ecc(v) - 1$, and v contains a token. Thus $N_r[v]$ contains at least r tokens for all $1 \leq r \leq ecc(v)$.

Suppose $v \prec u_{i+1}$. As $u_i \sim u_{i+1}$, $v \sim u_{i+1}$. By symmetry with the arguments in the previous case, either $u_1 \sim u_2 \sim \cdots \sim u_i \sim v$ or $u_1 \sim u_2 \sim \cdots \sim u_{i-1} \sim v$ is a shortest v_1v-path. As $ecc(v) = d_G(v_1, v)$, by choice of M, there is a token at distance d from v for all $1 \leq d \leq ecc(v) - 1$, and as u_{i+1} contains a token and $v \sim u_{i+1}$, $\{u_{i+1}\} \subseteq N_r[v]$ for all $1 \leq r \leq ecc(v)$. Therefore, $N_r[v]$ contains at least r tokens for all $1 \leq r \leq ecc(v)$.

Therefore, M forms a multicover of size $diam(G)$ as required, giving $P_b(G) = M_c(G) = diam(G)$. □

4.2 Triangle appended n-trampolines

In this section, we determine the exact value of the broadcast packing number of triangle appended n-trampolines with $n \geq 3$. We use the primal and dual linear programs.

Theorem 4.2. *Let G be a triangle appended n-trampoline, with $n \geq 3$. Then $P_b(G) = M_c(G) = n$.*

Proof. Define a broadcast f on G as follows:

$$f(x) = \begin{cases} 1 & x \in \{x_1, x_2, \ldots, x_n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is a packing broadcast with $f(V) = n$.

Let $M = \{u_1, u_2, \ldots, u_n\}$. We claim M is a multicover. For u_i with $1 \leq i \leq n$, as $|M| \geq 3$, clearly $N_1(u_i)$, $N_2(u_i)$ and $N_3(u_i)$ contain the required number of tokens. For v_i, x_i with $1 \leq i \leq n$, by the structure of G, we have $u_i \in N_1(v_i)$ and $u_{i+1} \in N_1(x_i)$. Further, $M \subseteq N_2(v_i)$, $M \subseteq N_3(v_i)$, $M \subseteq N_2(x_i)$, and $M \subseteq N_3(x_i)$. As $|M| \geq 3$, clearly the multicover conditions are satisfied. The result follows. □

4.3 Graphs of diameter 3

In this section, we examine the broadcast packing problem in graphs of diameter 3. Through a series of observations and lemmas, we show that for a graph G with diameter

3, either $P_b(G) = 3$, or solving for the broadcast packing number of G is equivalent to finding a maximum independent set in G^2.

Observation 4.3. *Let f be a P_b-broadcast on G. If $f(v) > 0$ and $f(u) > 0$, then $d_G(u, v) = 3$.*

Proposition 4.4. *Let G be a graph such that $diam(G) = 3$ and the maximum size of a set of pairwise distance 3 vertices of G is 2. Then $P_b(G) = 3$.*

Proof. Let f be a P_b-broadcast on G. We claim that $|V_f^+| = 1$. Suppose not. Let $u, v \in V_f^+$. By Observation 4.3, $d_G(v, u) = 3$. Clearly $1 \leq f(v), f(u) \leq 2$ as f is independent. We claim that $f(v) = f(u) = 1$. Suppose, without loss of generality, $f(v) = 2$. Consider a shortest vu-path in G, say $vxyu$. Then $d_G(v, y) \leq 2$. But then, in either case of $f(u) = 1$ or $f(u) = 2$, $\{v, u\} \subseteq H(y)$, contradicting that $|H(y)| = 1$ as f is a packing broadcast. So $|V_f^+| = 1$. As $diam(G) = 3$, clearly there is some vertex v such that $ecc(v) = 3$. The broadcast g defined by

$$g(x) = \begin{cases} 3 & x = v, \\ 0 & \text{otherwise,} \end{cases}$$

is clearly a packing broadcast with $g(V) \geq f(V)$, as desired. □

Throughout the remainder of this section, we assume that G is a graph of diameter 3 that has at least 3 vertices that are pairwise distance 3 from one another.

Lemma 4.5. *Let f be a P_b-broadcast on G. If $f(v) > 0$, then we can assume $f(v) = 1$ and $ecc(v) = 3$.*

Proof. Note that as $diam(G) = 3$, $f(v) \leq 3$ for all $v \in V(G)$. By assumption, let $v_1, v_2, v_3 \in V(G)$ be such that they are pairwise distance three from one another.

First suppose there exists $v \in V(G)$ such that $f(v) = 3$. As $diam(G) = 3$, note that v is the only vertex which can be broadcasting. Define a broadcast g as follows:

$$g(x) = \begin{cases} 1 & x \in \{v_1, v_2, v_3\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that g is a packing broadcast with the same weight as f.

Now suppose that there exists $v \in V(G)$ such that $f(v) = 2$. We claim that v is the only vertex broadcasting. Suppose not. So there is some $u \in V(G)$ such that $f(u) > 0$. Note that $d_G(v, u) = 3$, as f is a packing broadcast. Consider a shortest vu-path in G, say $vxyu$. Then $d_G(v, y) \leq 2$. But then $\{v, u\} \subseteq H(y)$, contradicting that $|H(y)| = 1$ as f is a packing broadcast. Therefore, v is the only vertex such that $f(v) > 0$. So $f(V) = 2$.

Define a broadcast g as follows:

$$g(x) = \begin{cases} 1 & x \in \{v_1, v_2, v_3\}, \\ 0 & \text{otherwise.} \end{cases}$$

The g is a packing broadcast with greater weight than f, contradicting our choice of f. □

Proposition 4.6. *Solving the broadcast packing problem in G is equivalent to solving for the maximum size of a set of pairwise distance three vertices in G.*

Proof. Follows from Lemma 4.5 and Observation 4.3. □

Observation 4.7. *Solving for the maximum size of a set of pairwise distance three vertices in G is equivalent to finding the size of a maximum independent set in G^2.*

Corollary 4.8. *Solving the broadcast packing problem in G is equivalent to finding the size of a maximum independent set in G^2.*

Thus, we have shown that solving for the broadcast packing number of a graph G such that $diam(G) = 3$ is equivalent to finding a maximum independent set in G^2. The rest of this section collects known results from the literature about structures of graph powers and the time complexity of finding a maximum independent set in different classes of graphs. By collecting these results, we provide a list of different graph classes such that if G belongs to one of these classes and $diam(G) = 3$, then given G^2, we can find $P_b(G)$ in linear time.

Theorem 4.9. *[9, 10] If G is in one of the following graph classes:*

- *Trees,*
- *Interval Graphs,*
- *Strongly Chordal Graphs,*
- *Doubly Chordal Graphs,*
- *Distance Hereditary Graphs,*

then G^2 is chordal.

Theorem 4.10. *[10] If G is either an AT-free graph or a co-comparability graph, then G^2 is a co-comparability graph.*

Theorem 4.11. *[17] If G is a circular arc graph, then G^2 is a circular arc graph.*

Theorem 4.12. *[23, 28, 34] If G is a chordal graph, a co-comparability graph, or a circular arc graph, then the size of a maximum independent set in G can be found in linear time.*

Combining these results together, we obtain the following theorem.

Theorem 4.13. *Let G be a graph of diameter 3 belonging to one of the following graph classes:*

- *Trees,*
- *Interval Graphs,*
- *Strongly Chordal Graphs,*
- *Doubly Chordal Graphs,*
- *Distance Hereditary Graphs,*
- *Co-comparability graphs,*
- *AT-free graphs,*
- *Circular arc graphs.*

Then $P_b(G)$ can be computed in linear time.

Proof. As G^2 can be computed in polynomial time, the result follows from Corollary 4.8 and Theorems 4.9, 4.10 and 4.12. □

4.4 Split graphs

From the results of Section 4.3, in this section, we assume that G is a split graph of diameter 3 that has at least 3 vertices that are pairwise distance 3 from one another. We begin with some preliminary results.

Lemma 4.14. *In a P_b-broadcast f on a split graph G, we can assume that if $f(v) > 0$, then $v \in S$.*

Proof. Let f be a packing broadcast on G and let v be a vertex with $f(v) > 0$. By Lemma 4.5, $ecc(v) = 3$, and hence $v \in S$. $\square$

Observation 4.15. *Solving the broadcast packing problem in a split graph G is equivalent to finding the size of a maximum independent set in G^2 restricted to the vertices of S.*

Proof. Follows from Corollary 4.8 and Lemma 4.14. $\square$

We now provide the exact value of the broadcast packing number for trampolines.

Theorem 4.16. *Let G be an n-trampoline with $n \geq 3$. If $n \leq 5$, $P_b(G) = 3$, and for $n \geq 6$, $P_b(G) = \lfloor \frac{n}{2} \rfloor$.*

Proof. If $n \leq 5$, by Proposition 4.4, $P_b(G) = 3$.

It is easy to see that G^2 restricted to the vertices of S is isomorphic to C_n, and $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$. When $n \geq 6$, G contains at least 3 vertices which are pairwise distance 3. Therefore, the result follows from Observation 4.15. $\square$

As mentioned in Chapter 1, we can find $P_b(G)$ using the primal and dual linear programs. This next theorem uses this technique to find $P_b(G)$, when G is leaf-appended split graph.

Theorem 4.17. *Let G be a leaf appended split graph. Then $P_b(G) = M_c(G) = |C|$.*

Proof. We find a packing broadcast and multicover of the same size.

Let $C = \{u_1, u_2, \ldots, u_k\}$ and $P = \{v_1, v_2, \ldots, v_k\} \subseteq S$ such that $N(v_i) = \{u_i\}$ for all $1 \leq i \leq k$. Let f be the broadcast defined as follows:

$$f(x) = \begin{cases} 1 & x \in P, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that f is a packing broadcast with $f(V) = |C|$.

We claim that C forms a mutlicover of G. Consider a vertex $u \in C$. As $u \in C$, clearly $N[u]$ contains a token. Further, $C \subseteq N_2[u]$ and as $|C| \geq 3$, it is clear that $N_2[u]$ contains at least 2 tokens. Thus the multicover conditions are satisfied for all $u \in C$. Now consider $u \in S$. As all vertices of C contain a token and u must have at least one neighbour in C, $N[u]$ contains at least one token. Now consider $N_2[u]$ and $N_3[u]$. C is a subset of both these neighbourhoods and as $|C| \geq 3$, $N_2[u]$ and $N_3[u]$ both contain at least 3 tokens. Thus the multicover conditions are satisfied for all vertices of G.

Therefore, $P_b(G) = M_c(G) = |C|$. $\square$

4.5 NP-Hardness of broadcast packing in split graphs

In the previous section, we showed $P_b(G) = |C|$ for leaf-appended split graphs. We now show determining $P_b(G)$ for general split graph is NP-hard by reducing the independent set problem in general graphs to broadcast packing in split graphs.

Theorem 4.18. *Determining $P_b(G)$ for general split graphs is NP-hard.*

Proof. Consider an arbitrary graph G. Let $V(G) = \{x_1, x_2, \ldots, x_n\}$ and let $E(G) = \{e_1, e_2, \ldots, e_m\}$. We construct a split graph H as follows.

Let $V(H) = V(G) \cup \{y_1, y_2, \ldots, y_m\}$ and $E(H) = \emptyset$. For each edge e_i of G, $1 \leq i \leq m$, let x_j, x_k be the endpoints of e_i. We add the following edges to $E(H)$: $y_i \sim x_j$ and $y_i \sim x_k$. We then add all possible edges between the vertices $\{y_1, y_2, \ldots, y_m\}$. Clearly, H is a split graph where $\{y_1, y_2, \ldots, y_m\}$ forms a clique and $S = \{x_1, x_2, \ldots, x_n\}$ forms an independent set, and H can be constructed in polynomial time. Note that $H^2[S] \cong G$. By Corollary 4.8, solving the broadcast packing problem for H is equivalent to solving the maximum independent set problem in $H^2[S]$ and hence G. □

4.6 Spider appended split graphs

In this section, we generalize the result of Theorem 4.17 to a class of graphs, which we call *spider appended split graphs*. We first start by defining a *spider*. We use the notation of [12].

Definition 4.19. *[12] A* spider *is tree which is a subdivided star. In other words, we have subdivided $K_{1,k}$ in such a way that there are k leaves, one vertex u with $deg(u) = k$ and all other vertices have degree 2. Let $\{l_1, l_2, \ldots, l_k\}$ be the leaves of the spider, and let $d_i = d(u, l_i)$ for all $1 \leq i \leq k$. We denote the spider by $S(d_1, d_2, \ldots, d_k)$. We say that a path from a leaf to u is a* branch *and u is the* branch vertex *of S. We say that $\sum_{i=1}^{k}(d_i - 1) + 1$ is the* distance sum of *S.*

Definition 4.20. *A graph G is a* spider appended split graph *if G is obtained in the following way:*

- *Let H be a split graph with partition (C, S) with $C = \{c_1, c_2, \ldots, c_m\}$ and $m \geq 3$.*
- *Let $S_i(d_1^i, d_2^i, \ldots, d_{k_i}^i)$ be a spider with leaves $l_1^i, l_2^i, \ldots, l_{k_i}^i$, for $1 \leq i \leq m$.*
- *Let G be the graph obtained from identifying each c_i, with $1 \leq i \leq m$, with the branch vertex of a spider $S_i(d_1^i, d_2^i, \ldots, d_{k_i}^i)$.*

We say the set $\{S_1, S_2, \ldots, S_m\}$ *are* the spiders of G. *Let* D_i *denote the distance sum of* S_i *for all* $1 \leq i \leq m$.

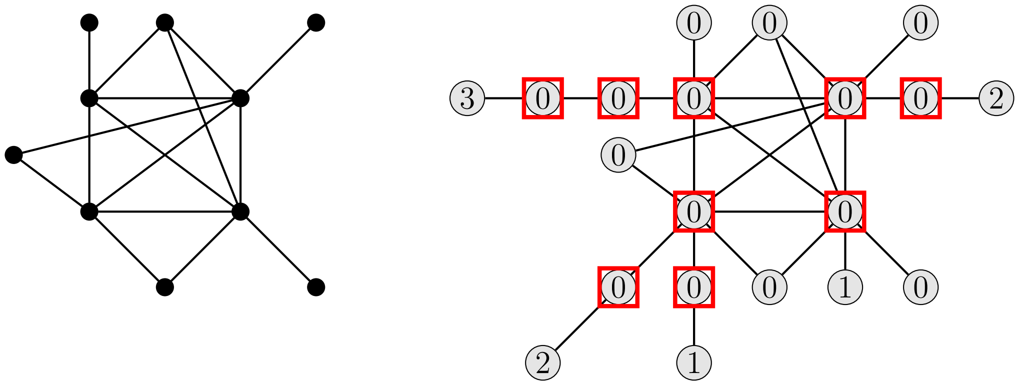


Figure 4.1: A split graph H (on the left) and a spider appended split graph G (on the right) obtained from H with $P_b(G) = M_c(G) = 9$.

It is easy to see that the leaf appended split graphs are exactly the spider appended split graphs where the identified spiders all have branches of length 1. Furthermore, the initial split graph H is not unique in general, as a leaf attached to a vertex in C could belong to H or to a spider.

The next proofs follow in a similar manner to the proofs of Theorem 4.8 and Corollary 4.9 in [12].

Theorem 4.21. *Let* G *be a spider appended split graph, with spiders* $\{S_1, S_2, \ldots, S_m\}$. *Let* $M \subseteq V(G)$ *consist of all internal vertices of the spiders* $\{S_1, S_2, \ldots, S_m\}$. *Then* M *is a mulitcover of* G.

Proof. Consider $v \in V(G)$ and $N_r[v]$ for $r \leq ecc(v)$. It is clear that there is some spider S_i and leaf l_j^i such that $d_G(v, l_j^i) = ecc(v)$. By choice of M, all internal vertices along a shortest v, l_j^i-path contain tokens. Consider a shortest v, l_j^i-path, say P, and let w be the vertex at distance r along this path. If either v or w contain a token, as all internal vertices along P contain tokens, $N_r[v]$ contains at least r tokens. If both v and w do not contain tokens, then w is a leaf and v is either a leaf or a vertex which is only adjacent to vertices of C. In either case, $N_r[v]$ contains $r-1$ tokens from P, and as $C \subseteq N_r[v]$, $|C| \geq 3$, and as $C \subseteq M$, there is also a vertex $u \in C$ such that $u \notin P$ and $u \in N_r[v]$. Thus $N_r[v]$ contains r tokens, and M is a multicover of G.

It is easy to see that

$$|M| = \sum_{j=1}^{m} D_j.$$

□

Corollary 4.22. *Let G be a spider appended split graph, with spiders $\{S_1, S_2, \ldots, S_m\}$. Then*

$$P_b(G) = M_c(G) = \sum_{j=1}^{m} D_j.$$

Proof. Let f be the broadcast defined by for each $1 \leq i \leq m$, $f(l_1^i) = d_1^i$, $f(l_j^i) = d_j^1$ for each $1 \leq j \leq k_i$, and $f(v) = 0$ for all other vertices of G. It is easy to verify that f is a packing broadcast and $f(V) = \sum_{j=1}^{m} D_j$. □

An example of such a packing broadcast and multicover is in Figure 4.1.

Chapter 5
Concluding Remarks and Future Research

In this thesis, we studied the broadcast independence and broadcast packing problems in different classes of graphs. In Chapter 1, we considered the history of both the broadcast independence and broadcast packing problems. In both the literature and this thesis, linear programming is used to prove results about the broadcast packing number of different graphs, as finding a packing broadcast and a multicover of the same size implies an optimal solution. Thus, it is natural to wonder whether a similar approach can be used in the broadcast independence problem. We can set up the broadcast independence problem as an integer program as follows: Given a graph G, for each vertex $v \in V$ and for each k, $1 \leq k \leq ecc_G(v)$, we define a binary variable $x_{v,k}$. If $x_{v,k} = 1$, then the ball of radius k centred at v is in the independent broadcast. The broadcast independence number is the maximum weight broadcast subject to the constraint that each vertex which is broadcast hears only itself. In symbols,

$$
\begin{aligned}
\alpha_b(G) \quad = \quad & \max \sum_{v \in V} \sum_{k=1}^{ecc_G(v)} k \cdot x_{v,k} \\
s.t. \quad & \sum_{1 \leq k \leq ecc_G(v)} x_{v,k} + \sum_{i=d(u,v)}^{ecc_G(u)} x_{u,i} \leq 1 \quad \text{for each } v, u \in V \text{ such that } u \neq v \\
& x_{v,k} \in \{0, 1\}.
\end{aligned}
$$

This program is easy to understand and can be used to computationally find $\alpha_b(G)$ in graphs G with small order. The time it takes to solve this integer program rapidly grows as the order of G increases. (Mixed Integer Programming is NP-hard [29].)

However, this formulation of the integer program depends on the fact that the variables are binary. A relaxation to fractional variables gives infeasible solutions. The challenge is using if-then constraints to require that if a vertex v is broadcasting, then it does not hear any other vertex. The encoding of "if-then" constraints can be done, but again requires binary variables. As a consequence it is hard to state and interpret the dual program to broadcast independence, thus making it impractical to use.

In Chapter 2, we showed that the broadcast independence number of various subclasses of chordal and weakly chordal graphs can be computed in $\mathcal{O}(n^8)$ time by using the ball catch graph. This includes trees, diamond-free interval graphs, paw-free interval graphs, chordal, (P_5, gem)-free graphs, (P_5, P, $\overline{P}$, Butterfly, Bull, House)-free weakly chordal graphs and (P_6, Triangle, Domino, A, H, X_{18}, X_{172})-free weakly chordal graphs. It is easy to see that each of these graph classes are n-trampoline-free for $n \geq 3$. Furthermore, in Chapter 1, all the examples of graphs which do not have a weakly chordal ball catch graph either contain an induced trampoline or an induced C_4. Therefore, one further direction of research is to see if it is possible to show that the ball catch graph of strongly chordal graphs is weakly chordal. This would imply that the broadcast independence number of a strongly chordal graph can be found in polynomial time.

In Chapter 3, we found explicit values for α_b in proper interval graphs, triangle-appended trampolines, star split graphs and trampolines, and also provided a polynomial time algorithm for leaf-appended split graphs. As previously stated, the value of the broadcast independence number of proper interval graphs is analogous to the value of the broadcast independence number of paths. As paths are proper interval graphs, and proper interval graphs have a linear ordering which gives them a structure similar to paths, this result may be unsurprising. Bouchouika et al. proved that $\alpha_b(C_n) = 2(diam(C_n)-1)$ in [8]. We hoped to find the same parallel between broadcast independence in cycles and proper circular arc graphs. However, the graph G in Figure 5.1 shows that this parallel does not exist as $diam(G) = 5$.

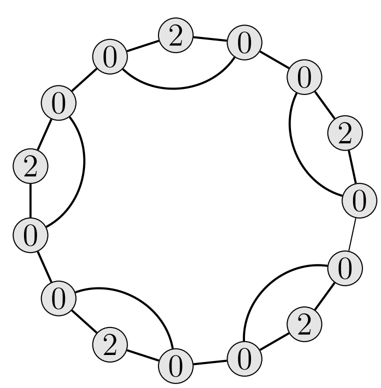


Figure 5.1: A proper circular arc graph G with $\alpha_b(G) \geq 10 > 2(diam(G) - 1) = 8$.

In Chapter 3, we also provided a characterization for when $\alpha_b(G) = \alpha(G)$ for split graphs G. A further direction of research is to determine what other graphs G have $\alpha_b(G) = \alpha(G)$. Furthermore, it is interesting that even though split graphs have diameter less than or equal to 3, the time complexity of the broadcast independence problem in split graphs is still undetermined. Thus, another further direction of research is to determine if the broadcast independence problem in split graphs is polynomial time solvable or NP-complete.

Finally, in Chapter 4, we first determined the broadcast packing number of proper interval graphs. Much like with broadcast independence, we found that the broadcast packing result in proper interval graphs was analogous to the result in paths. Another direction of research would be to determine if there is a similar parallel between proper circular arc graphs and cycles. We also discussed graphs of diameter 3 for which we can find the broadcast packing number in linear time, studied broadcast packing in triangle-appended trampolines and different types of split graphs, showed the NP-hardness of the problem in general split graphs, and found the broadcast packing number of spider appended split graphs. Some further directions of research include determining exact values of the broadcast packing number and establishing the time complexity of the problem in other graph classes.

Bibliography

[1] M. Ahmane, I. Bouchemakh, and E. Sopena. On the broadcast independence number of caterpillars. *Discrete Appl. Math.*, 244:20–35, 2018.

[2] M. Ahmane, I. Bouchemakh, and E. Sopena. On the broadcast independence number of locally uniform 2-lobsters. *Discuss. Math. Graph Theory*, 44(1):199–229, 2024.

[3] S. Bessy and D. Rautenbach. Relating broadcast independence and independence. *Discrete Math.*, 342(12):111589, 7, 2019.

[4] S. Bessy and D. Rautenbach. Algorithmic aspects of broadcast independence. *Discrete Appl. Math.*, 314:142–149, 2022.

[5] H. L. Bodlaender, A. Brandstädt, D. Kratsch, M. Rao, and J. Spinrad. On algorithms for (P_5, gem)-free graphs. *Theoretical Computer Science*, 349(1):2–21, 2005.

[6] O. Borodin. Colorings of plane graphs: A survey. *Discrete Math.*, 313(4):517–539, 2013.

[7] I. Bouchemakh and M. Zemir. On the broadcast independence number of grid graph. *Graphs Combin.*, 30(1):83–100, 2014.

[8] S. Bouchouika, I. Bouchemakh, and E. Sopena. Broadcasts on paths and cycles. *Discrete Appl. Math.*, 283:375–395, 2020.

[9] A. Brandstädt, F. Dragan, V. Chepoi, and V. Voloshin. Dually chordal graphs. *SIAM J. Discrete Math.*, 11(3):437–455, 1998.

[10] A. Brandstädt, F. F. Dragan, Y. Xiang, and C. Yan. Generalized powers of graphs and their algorithmic use. In L. Arge and R. Freivalds, editors, *Algorithm Theory – SWAT 2006*, pages 423–434, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg.

[11] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph classes: a survey.* SIAM Monographs on Discrete Math. and Applications. Society for Industrial and Applied Math. (SIAM), Philadelphia, PA, 1999.

[12] R. C. Brewster and K. A. McDonald. Broadcast independence and packing in certain classes of trees, Submitted 2024.

[13] M. Chudnovsky, M. Pilipczuk, M. Pilipczuk, and S. Thomassé. On the maximum weight independent set problem in graphs without induced cycles of length at least five. *SIAM J. Discrete Math.*, 34(2):1472–1483, 2020.

[14] W. J. Cook, W. H. Cunningham, W. R. Pulleyblank, and A. Schrijver. *Combinatorial optimization.* Wiley-Interscience Series in Discrete Math. and Optimization. John Wiley & Sons, Inc., New York, 1998. A Wiley-Interscience Publication.

[15] G. Cornuéjols. *Combinatorial optimization*, volume 74 of *CBMS-NSF Regional Conference Series in Applied Math.* Society for Industrial and Applied Math. (SIAM), Philadelphia, PA, 2001.

[16] D. W. Cranston and D. B. West. An introduction to the discharging method via graph coloring. *Discrete Math.*, 340(4):766–793, 2017.

[17] A. K. Das and I. Paul. On powers of circular arc graphs, 2023.

[18] H. N. de Ridder et al. Information System on Graph Classes and their Inclusions (ISGCI). `https://www.graphclasses.org`.

[19] J. E. Dunbar, D. J. Erwin, T. W. Haynes, S. M. Hedetniemi, and S. T. Hedetniemi. Broadcasts in graphs. *Discrete Appl. Math.*, 154(1):59–75, 2006.

[20] D. J. Erwin. *Cost domination in graphs.* PhD thesis, Wester Michigan University, 2001.

[21] M. Farber. Domination, independent domination, and duality in strongly chordal graphs. *Discrete Appl. Math.*, 7(2):115–130, 1984.

[22] C. Flotow. Graphs whose powers are chordal and graphs whose powers are interval graphs. *J. Graph Theory*, 24(4):323–330, 1997.

[23] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *J. Comb. Theory, Ser. B*, 16:47–56, 1974.

[24] P. C. Gilmore and A. J. Hoffman. A characterization of comparability graphs and of interval graphs. *Canadian Journal of Math.*, 16:539–548, 1964.

[25] M. C. Golumbic. *Algorithmic graph theory and perfect graphs*, volume 57 of *Annals of Discrete Math.* Elsevier Science B.V., Amsterdam, second edition, 2004. With a foreword by Claude Berge.

[26] R. Hayward, C. T. Hoàng, and F. Maffray. Erratum: "Optimizing weakly triangulated graphs". *Graphs Combin.*, 6(1):33–35, 1990.

[27] R. B. Hayward. Weakly triangulated graphs. *Journal of Combinatorial Theory, Series B*, 39(3):200–208, 1985.

[28] W.-L. Hsu and J. P. Spinrad. Independent sets in circular-arc graphs. *J. Algorithms*, 19(2):145–160, 1995.

[29] R. Karp. Reducibility among combinatorial problems (1972). In *Ideas that created the future—classic papers of computer science*, pages 349–356. MIT Press, Cambridge, MA, 2021. Reprinted from [0378476].

[30] A. Laouar, I. Bouchemakh, and E. Sopena. On the broadcast independence number of circulant graphs. *Discrete Math. Algorithms Appl.*, 16(5):Paper No. 2350053, 2024.

[31] C. G. Lekkerkerker and J. C. Boland. Representation of a finite graph by a set of intervals on the real line. *Fund. Math.*, 51:45–64, 1962/63.

[32] A. Lubiw. Doubly lexical orderings of matrices. *SIAM J. Comput.*, 16(5):854–879, 1987.

[33] H. Maehara. A digraph represented by a family of boxes or spheres. *Journal of Graph Theory*, 8(3):431–439, 1984.

[34] R. M. McConnell and J. P. Spinrad. Modular decomposition and transitive orientation. *Discrete Math.*, 201(1-3):189–241, 1999.

[35] C. M. Mynhardt and L. Neilson. Boundary independent broadcasts in graphs. *J. Combin. Math. Combin. Comput.*, 116:79–100, 2020.

[36] K. Mynhardt and L. Neilson. Comparing upper broadcast domination and boundary independence broadcast numbers of graphs. *Trans. Comb.*, 13(1):105–126, 2024.

[37] L. Neilson. *Broadcast Independence in Graphs*. PhD thesis, University of Victoria, 2019.

[38] S. Olariu. An optimal greedy heuristic to color interval graphs. *Inform. Process. Lett.*, 37(1):21–25, 1991.

[39] J. Spinrad and R. Sritharan. Algorithms for weakly triangulated graphs. *Discrete Appl. Math.*, 59(2):181–191, 1995.

[40] D. B. West. *Introduction to graph theory*. Prentice Hall, Inc., Upper Saddle River, NJ, 1996.