TOPOLOGICAL ORBIT EQUIVALENCE

by

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DMS-622-IR November 1992

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This talk describes joint work with Thierry Giordano and Christian Skau. I will focus on the main questions and results and I will not give any proofs.

Let us consider a compact metrizable space, X, and a single homeomorphism φ of X. The system (X,φ) is said to be minimal if the only closed φ-invariant subsets of X are X itself and the empty set. This is equivalent to the condition that the φ-orbit of any point x in X, $\{\varphi^n x \mid n \in \mathbf{Z}\}$, is dense in X. Two such systems, (X_1,φ_1) and (X_2,φ_2), are (topologically) orbit equivalent if there is a homeomorphism $h : X_1 \to X_2$ which, for any x in X_1, maps the φ_1-orbit of x onto the φ_2-orbit of $h(x)$. We wish to consider the following.

Question. For minimal systems (X_1,φ_1) and (X_2,φ_2), is it true that these are orbit equivalent if and only if the crossed-product C^*-algebras [7] $C(X_1) \times_{\varphi_1} \mathbf{Z}$ and $C(X_2) \times_{\varphi_2} \mathbf{Z}$ are $*$-isomorphic?

What we are asking for would be a topological analogue of Krieger's celebrated theorem which deals with non-singular transformations of measure spaces and their von Neumann crossed products [4]. We have replaced the ergodicity hypothesis with minimality.

The question seems rather hopeless in this generality and we add as a hypothesis (throughout the rest of the paper) that X_1 and X_2 be totally disconnected; *i.e.* $\dim X_1 = \dim X_2 = 0$. With the hypotheses of metrizability and φ_1 and φ_2 being minimal, this implies that X_1 and X_2 are Cantor sets. The hypothesis is not a bad one from a dynamical point of view -- such systems have a certain universal property [3] -- and this is usually a

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[†] **From a talk given at the NATO advanced research workshop, Istanbul, July 1991.**

[††] **Supported in part by NSERC.**

reasonable first case to consider.

From a more practical viewpoint, this hypothesis has two important consequences. The first is that it allows us to use George Elliott's recent remarkable classification theorem and the second is the use of Vershik transformations as models for the dynamics.

Elliott's Theorem. Combining the results of [10] and [2], the C^*-algebra $C(X) \times_\varphi Z$ (with (X,φ) as above) is the inductive limit

$$A_1 \to A_2 \to A_3 \to \cdots$$

where each A_n is a tensor product of the continuous functions on the circle with a finite dimensional C^*-algebra. Moreover, this C^*-algebra satisfies the additional condition of having real rank zero [2]. Elliott's theorem [2] classifies such C^*-algebras up to *-isomorphism and the complete invariant is certain K-theory data. Now in our case, $K_1(C(X) \times_\varphi \mathbf{Z}) \cong \mathbf{Z}$ (from the Pimsner-Voiculescu exact sequence). For the C^*-algebras we are considering, Elliott's theorem easily provides the following.

THEOREM. *With* (X_i,φ_i), $i = 1,2$, *as above,* $C(X_1) \times_{\varphi_1} \mathbf{Z}$ and $C(X_2) \times_{\varphi_2} \mathbf{Z}$ *are *-isomorphic if and only if* $K_0(C(X_1) \times_{\varphi_1} \mathbf{Z})$ and $K_0(C(X_2) \times_{\varphi_2} \mathbf{Z})$ *are isomorphic as ordered groups with distinguished order units.*

The K_0-group of such a crossed product may be computed via the Pinsner-Voiculescu exact sequence [8]. Let $C(X,Z)$ denote the continuous functions $f : X \to Z$ considered as a group with pointwise addition. Then $K_0(C(X) \times_\varphi \mathbf{Z})$ is isomorphic to the quotient of $C(X,Z)$ by the subgroup $\{f - f \circ \varphi \mid f \in C(X,Z)\}$, the so-called coboundaries. The order structure may be computed from the results of [10]; specifically, $K_0(C(X) \times_\varphi \mathbf{Z})^+$ is the image in the quotient group of the non-negative functions in $C(X,Z)$. The order unit is the image of the constant function 1. We denote this group by $K^0(X,\varphi)$. Let us remark that this group is a familiar object in dynamical systems -- see Parry and Tuncel [6]. The point here is that it should be viewed as an ordered group.

Vershik Transformations. This is a brief account of recent work of Richard Herman, Christian Skau and myself [3].

Let us begin with a Bratteli diagram (drawn vertically) with a single vertex, v_0, at the top level. Suppose that we have an order on the edge set E so that two edges are comparable if and only if they end at the same lower vertex. This is called an ordered Bratteli diagram. We let X denote the space of infinite paths in the diagram (beginning at v_0). There is a standard topology on X in which it is totally disconnected compact and metrizable. We make the hypothesis that our diagram is *essentially simple*: there is a unique infinite path of maximal edges and a unique infinite path of minimal edges. We are now able to define an order relation on X as follows. Two paths, given by edge lists $(e_1,e_2,e_3,\cdots)$ and $(e_1',e_2',e_3',\cdots)$ are comparable if, for some N, $e_n = e_n'$ for all $n \geq N$; *i.e.* they are *cofinal*. Choose the least such N and order the paths by comparing the edges $e_{N\text{-}1}$ and $e_{N\text{-}1}'$. Now we are able to define a map φ on X as simply taking the successor of any path. Our hypothesis of essentially simple means that there is one path with no successor and one path with no predecessor. Then φ may be extended to a homeomorphism of X by mapping the former to the latter.

This kind of transformation was first introduced by Vershik (see, for example, [12]) who called them "adic transformations". We have chosen the term "Vershik transformations". The main point of difference is that Vershik regards them as models in a measure theory setting while we work entirely in a topological one.

We want to use these as models for our dynamical systems.

THEOREM [3]. *Let* (X,φ) *be any minimal zero dimensional system. Then* (X,φ) *is topologically conjugate to a Vershik transformation.*

Specifically, this means there is a Vershik map (X',φ') and a homeomorphism $h : X \to X'$ so that $h\varphi = \varphi' h$. (This means that the systems are isomorphic and is easily seen to imply $K^0(X,\varphi) \cong K^0(X',\varphi')$).

In view of Elliott's theorem, we must compute $K^0(X,\varphi)$ for a Vershik map (X,φ).

THEOREM [3]. *Let* (X,φ) *be a minimal Vershik transformation arising from an essentially simple ordered Bratteli diagram. Then* $K^0(X,\varphi)$ *is order isomorphic to the dimension group associated with the underlying Bratteli diagram.*

Finally, it is important to understand the orbits of a Vershik map. Let $x_{\max}$ and $x_{\min}$ denote the infinite paths of maximal and minimal edges, respectively, in our ordered Bratteli diagram. Consider the equivalence relation on X, $x \sim y$ if and only if x and y have the same φ-orbit; *i.e.* the equivalence classes are just the orbits.

THEOREM. *Let* (X,φ) *be the minimal Vershik map arising from an essentially simple ordered Bratteli diagram. Then the equivalence relation defined by* φ *as above is equal to the equivalence relation* $\approx$ *on* X *generated by* $x \approx y$ *if* x *and* y *are cofinal and* $x_{\max} \approx x_{\min}$.

Results. Let us restate the original question in an equivalent way, making use of the results we have so far.

Let (X_1,φ_1) and (X_1,φ_2) be two Vershik maps. Are they orbit equivalent if and only if $K^0(X_1,\varphi_1) \cong K^0(X_2,\varphi_2)$? It turns out, as we will see, that the "if" implication is valid while the "only if" is not. However, what seems to emerge as a theme is that there is a "hierarchy" of types of orbit equivalence. Let us set out some ideas and notation.

If $h : X_1 \to X_2$ is an orbit equivalence, consider $\varphi = \varphi_1$ and $\psi = h^{-1}\varphi_2 h$ as two homeomorphisms of $X = X_1$ with precisely the same orbits. Given x in X, φx is in the φ-orbit of x which equals the ψ-orbit of x and so there is a (unique) integer, $m(x)$, so that

$$\varphi x = \psi^{m(x)} x$$

Similarly, we have $n : X \to Z$ so that

$$\psi x = \varphi^{n(x)} x.$$

(Note that reversing the rôles of φ_1 and φ_2 above simply interchanges $m(\cdot)$ and $n(\cdot)$).

Without going into detail, let us mention that the m and n have special properties. For example if m is bounded, then it is continuous as a map from X to Z. Also, they are clearly related. Again as an example, if m is continuous on X (or on the orbit of a point x) then n is continuous on X (or on the orbit of x, respectively).

In this setting, we would describe a topological conjugacy as an orbit equivalence where the associated m and n are identically one.

The following result is due to Mike Boyle [1].

THEOREM. *If two minimal systems are orbit equivalent so that the associated m and n are continuous, then the systems are topologically conjugate or one is conjugate to the inverse of the other ("flip conjugate").*

The next level below conjugacy for an orbit equivalence is the case when n and m each have at most one point of discontinuity. We refer to this as strong orbit equivalence.

THEOREM. *Two minimal systems (X_1,φ_1) and (X_2,φ_2) are strong orbit equivalent if and only if $K^0(X_1,\varphi_1) \cong K^0(X_2,\varphi_2)$.*

COROLLARY. *If $K^0(X_1,\varphi_1) \cong K^0(X_2,\varphi_2)$ then (X_1,φ_1) and (X_2,φ_2) are orbit equivalent.*

To proceed further, we must deal with the subgroup of infinitesimal elements of $K^0(X,\varphi)$, which we denote by $\mathrm{Inf}\, K^0(X,\varphi)$. Recall that an element g in an ordered group G is called infinitesimal if every multiple of it is less than the order unit. For simple dimension groups -- including $K^0(X,\varphi)$ when (X,φ) is minimal [3] -- this is equivalent to saying that for every state ω on G (*i.e.* a positive group homomorphism to $\mathbb{R}$), $\omega(g) = 0$. In our situation, the states on $K^0(X,\varphi)$ may be identified with finite invariant measures on X [9]. In this way, we see that $\mathrm{Inf}\, K^0(X,\varphi)$ is isomorphic to the image of

$$\left\{ f \in C(X,Z) \mid \int f\,d\mu = 0, \text{ for every } \varphi\text{-invariant probability measure } \mu \text{ on } X \right\}$$

in the quotient (dividing out by the coboundaries). The question "given a function which integrates to zero under every invariant measure, is it a coboundary?" is a familiar one in dynamical systems and $\mathrm{Inf}\ K^0(X,\varphi)$ is an algebraic measure of (the failure of) this property.

Note that $K^0(X,\varphi)/\mathrm{Inf}\ K^0(X,\varphi)$ is also an ordered group with an order unit in a natural way. This turns out to be a complete invariant for orbit equivalence.

THEOREM. *Two systems* (X_1,φ_1) *and* (X_2,φ_2) *are orbit equivalent if and only if*

$$K^0(X_1,\varphi_1)/\mathrm{Inf}\ K^0(X_1,\varphi_1) \cong K^0(X_2,\varphi_2)/\mathrm{Inf}\ K^0(X_2,\varphi_2)$$

as ordered groups with order unit.

The following is a straightforward consequence of the theorem and the earlier discussion.

COROLLARY. *Two systems* (X_1,φ_1) *and* (X_2,φ_2) *are orbit equivalent if and only if there is a homeomorphism* $h : X_1 \to X_2$ *which induces a bijection between the* φ*-invariant probability measure on* X_1 *and the* φ_2*-invariant probability measures on* X_2.

If h is an orbit equivalence, then the associated m and n, while not continuous in general, are Borel and it follows easily that h induces a bijection as above between the invariant probability measures. However, in the "if" direction, the homeomorphism of the hypothesis need not be an orbit equivalence itself.

In addition to the ideas of using Vershik transformations which have been sketched, the proof involves careful analysis of the *AF*-algebras constructed in [9]. It is especially important to understand the short exact sequence appearing in section 4 of [9]. It is also interesting that we require certain results from homological algebra in dealing with the group extensions. Another fact which emerges from the proof is that the orbit equivalence is always possible so that the associated m and n have discontinuity sets which are a convergent sequence in X (with its limit point).

We are also able to describe orbit equivalence for the type of group action considered by

Krieger in [5] -- a locally finite, ample group acting on X with open fix-point sets -- with the added hypothesis that the orbits are dense. We also have results for orbit equivalence for which the associated m and n have finitely many discontinuities. These results, with proofs, will appear in the final paper.

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