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Article

Multiplicity of Solutions for Fractional-Order Differential Equations via the $\kappa(x)$ -Laplacian Operator and the Genus Theory

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Abstract: In this paper, we investigate the existence and multiplicity of solutions for a class of quasi-linear problems involving fractional differential equations in the χ -fractional space $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$. Using the Genus Theory, the Concentration-Compactness Principle, and the Mountain Pass Theorem, we show that under certain suitable assumptions the considered problem has at least k pairs of non-trivial solutions.

Keywords: fractional differential equations; $\kappa(x)$ -Laplacian; χ -Hilfer fractional derivative; existence; multiplicity of solutions; genus theory; Concentration-Compactness Principle; Mountain Pass Theorem; variable exponents; variational methods

MSC: Primary 26A33, 34A08; Secondary 35A15, 35J20, 35J66, 35J92



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1. Introduction

Let $I = [a, b]$ ($-\infty < a < b < \infty$) denote a finite interval on the real axis \mathbb{R} . In the theory and application of fractional integrals and fractional derivatives, it is known that the right-sided Hilfer fractional derivative ${}^H\mathcal{D}_{a+}^{\gamma,\beta}$ and the left-sided Hilfer fractional derivative ${}^H\mathcal{D}_{b-}^{\gamma,\beta}$ of order γ ($0 < \gamma < 1$) and type β ($0 \leq \beta \leq 1$) reduce, when $\beta = 0$ and $\beta = 1$, to the corresponding relatively more familiar Riemann-Liouville fractional derivatives and Liouville-Caputo fractional derivatives, respectively (see [1–4] for details, along with several other related recent works [5–9]). For $n - 1 < \gamma < n$ ($n \in \mathbb{N}$), let $f, \chi \in C^n(I \subset \mathbb{R})$, where the function χ is increasing and $\chi'(x) \neq 0$ in the interval I . Then, we have the right-sided χ -H ${}^H\mathcal{D}_{a+}^{\gamma,\beta;\chi}$ and the left-sided χ -Hilfer fractional derivative ${}^H\mathcal{D}_{b-}^{\gamma,\beta;\chi}$ of order γ ($0 < \gamma < 1$) and type β ($0 \leq \beta \leq 1$).

Let

$$\theta = (\theta_1, \theta_2, \dots, \theta_N), \quad d = (d_1, d_2, \dots, d_N) \quad \text{and} \quad \gamma = (\gamma_1, \gamma_2, \dots, \gamma_N),$$

where $0 < \gamma_1, \gamma_2, \dots, \gamma_N < 1$ with $\theta_j < d_j$ for all $j \in \{1, 2, \dots, N\}$ and $N \in \mathbb{N}$, and use

$$\Delta = I_1 \times I_2 \times \dots \times I_N = [\theta_1, d_1] \times [\theta_2, d_2] \times \dots \times [\theta_N, d_N],$$

where T_1, d_2, \dots, T_N and $\theta_1, \theta_2, \dots, \theta_N$ are positive constants. We consider $\chi(\cdot)$ to be an increasing and positive monotone function on $(\theta_1, d_1), (\theta_2, d_2), \dots, (\theta_N, d_N)$ having

a continuous derivative $\chi'(\cdot)$ on $(\theta_1, d_1], (\theta_2, d_2], \dots, (\theta_N, d_N]$. The χ -Riemann-Liouville fractional integral of order γ and of N variables $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_N) \in L^1(\Delta)$, denoted by $\mathcal{I}_{\theta, x_j}^{\gamma; \chi}(\cdot)$, is defined as follows (see [10–12]):

$$\mathcal{I}_{\theta, x_j}^{\gamma; \chi} \varphi(x_j) := \frac{1}{\Gamma(\gamma_j)} \int \int \cdots \int_{\Delta} \chi'(s_j) [\chi(x_j) - \chi(s_j)]^{\gamma_j - 1} \varphi(s_j) ds_j$$

with

$$\begin{aligned} \chi'(s_j) [\chi(x_j) - \chi(s_j)]^{\gamma_j - 1} &= \chi'(s_1) [\chi(x_1) - \chi(s_1)]^{\gamma_1 - 1} \chi'(s_2) [\chi(x_2) - \chi(s_2)]^{\gamma_2 - 1} \\ &\quad \cdots \chi'(s_N) [\chi(x_N) - \chi(s_N)]^{\gamma_N - 1}, \end{aligned}$$

where

$$\begin{aligned} \Gamma(\gamma_j) &:= \Gamma(\gamma_1) \Gamma(\gamma_2) \cdots \Gamma(\gamma_N), \\ \varphi(s_j) &:= \varphi(s_1) \varphi(s_2) \cdots \varphi(s_N) \end{aligned}$$

and

$$ds_j := ds_1 ds_2 \cdots ds_N$$

for all $j \in \{1, 2, \dots, N\}$. We can then define $\mathcal{I}_{d, x_j}^{\gamma; \chi}(\cdot)$ by analogy.

Furthermore, we let $\varphi, \chi \in C^n(\Delta)$ be two functions such that χ is increasing, $\chi'(x_j) \neq 0$ ($j \in \{1, 2, \dots, N\}$), and $x_j \in \Delta$. The χ -Hilfer fractional partial derivative (χ -H) of functions of N variables, denoted by ${}^H\mathcal{D}_{\theta, x_j}^{\gamma, \beta; \chi}(\cdot)$, of order γ ($n - 1 < \gamma < n$) and type β ($0 \leq \beta \leq 1$), is defined as follows (see [10–12]):

$${}^H\mathcal{D}_{\theta, x_j}^{\gamma, \beta; \chi} \varphi(x_j) := \mathcal{I}_{\theta, x_j}^{\beta(1-\gamma); \chi} \left(\frac{1}{\chi'(x_j)} \left(\frac{\partial^N}{\partial x_j^N} \right) \mathcal{I}_{\theta, x_j}^{(1-\beta)(1-\gamma); \chi} \varphi(x_j) \right) \tag{1}$$

with

$$\partial x_j = \partial x_1 \partial x_2 \cdots \partial x_N \quad \text{and} \quad \chi'(x_j) = \chi'(x_1) \chi'(x_2) \cdots \chi'(x_N)$$

for all $j \in \{1, 2, \dots, N\}$. We can then define ${}^H\mathcal{D}_{T, x_j}^{\gamma, \beta; \chi}(\cdot)$ by analogy.

Throughout this work, we use the following notations:

$$\mathcal{I}_{-}^{\gamma; \chi}(\cdot) := \mathcal{I}_{T, x_j}^{\gamma; \chi}(\cdot),$$

$$\mathcal{I}_{+}^{\gamma; \chi}(\cdot) := \mathcal{I}_{\theta, x_j}^{\gamma; \chi}(\cdot),$$

$${}^H\mathcal{D}_{+}^{\gamma, \beta; \chi}(\cdot) := {}^H\mathcal{D}_{\theta, x_j}^{\gamma, \beta; \chi}(\cdot)$$

and

$${}^H\mathcal{D}_{-}^{\gamma, \beta; \chi}(\cdot) := {}^H\mathcal{D}_{T, x_j}^{\gamma, \beta; \chi}(\cdot).$$

In the present paper, we consider the following class of quasi-linear fractional-order problems with variable exponents:

$$\begin{cases} {}^H\mathcal{D}_{-}^{\gamma, \beta; \chi} \left(\left| {}^H\mathcal{D}_{+}^{\gamma, \beta; \chi} \varphi \right|^{\kappa(x) - 2} {}^H\mathcal{D}_{+}^{\gamma, \beta; \chi} \varphi \right) = \lambda |\varphi|^{n(x) - 2} \varphi + \mathbf{F}(x, \varphi) \\ \varphi = 0, \end{cases} \tag{2}$$

where $\Delta := [0, T] \times [0, T] \times [0, T]$ is a bounded domain with a smooth boundary, and (for simplicity) ${}^H\mathcal{D}_{+}^{\gamma, \beta; \chi}(\cdot)$ and ${}^H\mathcal{D}_{-}^{\gamma, \beta; \chi}(\cdot)$ are the χ -H of order γ ($\frac{1}{\kappa(x)} < \gamma < 1$) and type β ($0 \leq \beta \leq 1$), $\lambda > 0$; $\kappa, n : \bar{\Delta} \rightarrow \mathbb{R}$ are Lipschitz functions such that:

- (p1) $1 < \kappa_- \leq \kappa(x) \leq \kappa_+ < 3, \kappa_+ < n_- \leq n(x) \leq \kappa_+^*(x)$ for all $x \in \bar{\Delta}$;
- (p2) The set $A = \{x \in \bar{\Delta} : n(x) = \kappa_+^*(x)\}$ is not empty.

We now make several assumptions which are detailed below.

Let $f : \bar{\Delta} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function provided by

$$f(x, t) = \zeta(x)|t|^{\kappa(x)-2}t + \psi(x, t)$$

with $\zeta \in L^\infty(\Delta)$ and the function $\psi : \bar{\Delta} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following conditions:

- (g1) The function g is odd with respect to t , that is, $\psi(x, -t) = -\psi(x, t)$ for all $(x, t) \in \bar{\Delta} \times \mathbb{R}$;
 $\psi(x, t) = o(|t|^{\kappa(x)-1})$ when $|t| \rightarrow 0$ uniformly in x ;
 $\psi(x, t) = o(|t|^{n(x)-1})$ when $|t| \rightarrow \infty$ uniformly in x .
- (g2) $\psi(x, t) \leq \frac{1}{\kappa_+}\psi(x, t)t$ for all $t \in \mathbb{R}$, and at almost every point $x \in \Delta$, where

$$\psi(x, t) = \int_0^t \psi(x, s)ds.$$

In addition, consider the following:

- (H1) There exists $\bar{\gamma} > 0$ such that

$$\int_{\Delta} \left(\frac{1}{\kappa(x)} \left| {}^H\mathcal{D}_+^{\gamma, \beta; \chi} \varphi \right|^{\kappa(x)-2} - \zeta(x)|\varphi|^{n(x)} \right) dx \geq \bar{\gamma} \int_{\Delta} \frac{1}{\kappa(x)} |\varphi|^{\kappa(x)} dx.$$

- (H2) $\kappa(x) = \kappa_+$ for all $x \in \Gamma = \{x \in \Delta : \zeta(x) > 0\}$.

The derivative operator

$${}^H\mathcal{D}_-^{\gamma, \beta; \chi} \left(\left| {}^H\mathcal{D}_+^{\gamma, \beta; \chi} \varphi \right|^{\kappa(x)-2} {}^H\mathcal{D}_+^{\gamma, \beta; \chi} \varphi \right)$$

is a natural generalization of the operator

$${}^H\mathcal{D}_-^{\gamma, \beta; \chi} \left(\left| {}^H\mathcal{D}_+^{\gamma, \beta; \chi} \varphi \right|^{\kappa-2} {}^H\mathcal{D}_+^{\gamma, \beta; \chi} \varphi \right),$$

with $\kappa(x) = \kappa > 1$ being a real constant.

In recent years, there has been growing interest in the study of equations with growth conditions involving variable exponents. The study of such problems has been stimulated by their applications in elasticity [13], electro-rheological fluids [14,15] and image restoration [16]. Lebesgue spaces with variable exponents appeared for the first time as early as 1931 in the work of Orlicz [17]. Applications of this work include clutches, damper and rehabilitation equipment, and more [18–21].

Zhikov [13] was the first to work with the Lavrentiev phenomenon involving variational problems with variable exponents. His work motivated a great deal of research worldwide into variational and differential equations with variable exponent problems, including the works of, among others, Acerbi and Mingione [14], Alves [22,23], Alves and Ferreira [24,25], Antontsev and Shmarev [26], Bonder and Silva [27], Fu [28], Kovacic and Rakosnik [29], and Fan et al. [30,31].

In 2005, Chabrowski and Fu [32] considered the following $\kappa(x)$ -Laplacian problem:

$$\begin{cases} -\operatorname{div}(\zeta(x)|\nabla\varphi|^{\kappa(x)-2}\nabla\varphi) + b(x)|\varphi|^{\kappa(x)-2}\varphi = \mathbf{F}(x, \varphi) & \text{in } \Delta \\ \varphi = 0 & \text{on } \partial\Delta, \end{cases}$$

where $1 < \kappa_1 \leq b(x) \leq \kappa_2 < n, \Delta \subset \mathbb{R}^n$ is bounded, $0 < \zeta_0 \leq \zeta(x) \in L^\infty(\Delta)$ and $0 \leq b_0 \leq b(x) \in L^\infty(\Delta)$. In fact, Chabrowski and Fu [32] investigated the existence of

solutions in $W_0^{1,\kappa(x)}(\Delta)$ in the superlinear and sublinear cases using the Mountain Pass Theorem (MPT). Subsequently, in 2016, Alves and Ferreira [25] discussed the existence of solutions for a class of quasi-linear problems involving variable exponents by applying the Ekeland variational principle and the Mountain Pass Theorem (MPT).

In 2022, Taarabti [33] investigated the existence of positive solutions of the following equation:

$$\begin{cases} \Delta_{\kappa(x)}\varphi + v|\varphi|^{\kappa(x)-2}\varphi = \lambda k(x)|\varphi|^{\gamma(x)-2} + s(x)|\varphi|^{\beta(x)-2}\varphi & \text{in } \Delta \\ \varphi = 0 & \text{on } \partial\Delta. \end{cases} \quad (3)$$

For further details about the parameters and functions of problem (3), see [33].

Over the years, interest in fractional differential equations involving variational techniques has been gaining remarkable popularity and attention from researchers. However, works in this direction remain very limited, especially those involving the χ -Hilfer fractional derivative operators (see [34–36]). The pioneering work involving the χ -Hilfer fractional derivative operator, the m -Laplacian, and the Nehari manifold was conducted by da Costa Sousa et al. [37] in 2018. We mention here that classical variational techniques have been applied in partial differential equations involving fractional derivatives; see, for example, [38–41].

Recently, Zhang and Zhang [42] investigated the properties of the following problem:

$$\begin{cases} (-\Delta_{\kappa(x)})^s \varphi = f(x) & \text{in } \Delta \\ \varphi = 0 & \text{on } \partial\Delta, \end{cases}$$

where $0 < s < 1$, $\kappa : \bar{\Delta} \times \bar{\Delta} \rightarrow (1, \infty)$ is a continuous function with $s\kappa(x, y) < N$ for any $(x, y) \in \bar{\Delta} \times \bar{\Delta}$ and $0 \leq f \in L^1(\Delta)$.

Research on fractional Laplacian operators has been fairly productive in recent years. For example, in 2019 Xiang et al. [41] carried out interesting work on a multiplicity of solutions for the variable-order fractional Laplacian equation with variable growth using the MPT and Ekeland's variational principle. For other interesting results on multiplicity of solutions, see the works by Ayazoglu et al. [43], Xiang et al. [44], Colasuono et al. [45], and Mihailescu and Radulescu [46], as well as the references cited in each of these publications.

In 2021, Rahmoune and Biccari [47] investigated a multiplicity of solutions for the fractional Laplacian operator involving variable exponent nonlinearities of the type

$$(-\Delta)_{q(\cdot)}^{s(\cdot)}\varphi + \lambda V\varphi = \gamma|\varphi|^{p(\cdot)-2}\varphi + \beta|\varphi|^{p(\cdot)-2}u, \quad \in \Delta,$$

with $\varphi = 0$ in \mathbb{R}^n/Δ . Their results were obtained using the MPT and Ekeland's variational principle. Finally, several open and interesting problems about the existence and multiplicity of solutions were highlighted at the end of their paper [47] (see of the aforementioned recent related works [41,43–46] on the same subject).

In the year 2020, da Costa Sousa et al. [48] considered a mean curvature type problem involving a χ -H operator and variable exponents provided by

$$\begin{cases} {}^H\mathcal{L}_T^{\gamma,\beta;\chi}\zeta(x) = \lambda f(x)|\zeta(x)|^{n(x)-2}\zeta(x) + g(x)|\zeta(x)|^{s(x)-2}\zeta(x) & \text{in } \Delta \\ \mathcal{I}_{0+}^{\beta(\beta-1);\chi}\zeta(0) = \mathcal{I}_T^{\beta(\beta-1);\chi}\zeta(T) = 0 & \text{on } \partial\Delta, \end{cases} \quad (4)$$

where

$$\begin{aligned}
 {}^H\mathcal{L}_T^{\gamma,\beta;\chi} \zeta(x) := & {}^H\mathcal{D}_T^{\gamma,\beta;\chi} \left[\left(1 + \frac{|{}^H\mathcal{D}_{0+}^{\gamma,\beta;\chi} \zeta(x)|^{\kappa(x)}}{\sqrt{1 + |{}^H\mathcal{D}_{0+}^{\gamma,\beta;\chi} \zeta(x)|^{2\kappa(x)}}} \right) \right. \\
 & \left. \cdot |{}^H\mathcal{D}_{0+}^{\gamma,\beta;\chi} \zeta(x)|^{\kappa(x)-2} {}^H\mathcal{D}_{0+}^{\gamma,\beta;\chi} \zeta(x) \right]. \tag{5}
 \end{aligned}$$

For further details about the parameters and functions of problem (5), see [48].

Motivated by the above-mentioned works, our main object in this paper is to investigate the multiplicity of solutions to problem (2) by applying the Concentration-Compactness Principle (CCP), the Mountain Pass Theorem (MPT) for paired functionals, and the genus theory. More precisely, we present the following theorem.

Theorem 1. *Suppose that a function g satisfies the conditions (\mathbf{g}_1) and (\mathbf{g}_2) and that the conditions (\mathbf{p}_1) , (\mathbf{p}_2) , (\mathbf{H}_1) , and (\mathbf{H}_2) are satisfied. Then there exists a sequence $\{\lambda_k\} \subset (0, +\infty)$ with $\lambda_k > \lambda_{k+1}$ for all $k \in \mathbb{N}$ such that, for $\lambda \in (\lambda_{k+1}, \lambda_k)$, the problem (2) has at least k pairs of non-trivial solutions.*

The rest of this paper is organized as follows. In Section 2, we present important definitions results needed for the further development of the paper. In Section 3, we present technical lemmas and discuss the main result of the paper, that is, the multiplicity of solutions for a class of quasi-linear fractional-order problems in (2) via the Genus Theory, the Concentration-Compactness Principle, and the Mountain Pass Theorem (MPT). Finally, in Section 4, we conclude the paper by presenting several closing remarks and observations.

2. Mathematical Background and Auxiliary Results

Consider the space $L_+^\infty(\Delta)$ provided by

$$L_+^\infty(\Delta) := \left\{ \varphi \in L^\infty(\Delta) : \operatorname{ess\,inf}_{x \in \Delta} \varphi \geq 1 \right\}$$

and we assume that $s \in L_+^\infty(\Delta)$. For each $s \in L_+^\infty(\Delta)$, consider the numbers s_- and s_+ by

$$s_- = \operatorname{ess\,inf}_\Delta h \quad \text{and} \quad s_+ = \operatorname{ess\,sup}_\Delta h.$$

The Lebesgue space with the variable exponent $L^{s(x)}(\Delta)$ is defined as follows (see [27]):

$$L^{s(x)}(\Delta) = \left\{ \varphi : \Delta \rightarrow \mathbb{R} \text{ is measurable: } \int_\Delta |\varphi|^{s(x)} dx < \infty \right\},$$

which the norm

$$\|\varphi\|_{L^{s(x)}(\Delta)} = \inf \left\{ \lambda > 0 : \int_\Delta \left| \frac{\varphi}{\lambda} \right|^{s(x)} dx \leq 1 \right\}. \tag{6}$$

On the space $L^{s(x)}(\Delta)$, we consider the modular function $\rho : L^{s(x)}(\Delta) \rightarrow \mathbb{R}$ defined by

$$\rho(\varphi) = \int_\Delta |\varphi|^{s(x)} dx.$$

Definition 1. *Let $0 < \gamma \leq 1, 0 \leq \beta \leq 1$ and $p \in C^+(\bar{\Delta})$. The right-sided χ -fractional derivative space provided by $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi} := \mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$ is the closure of $C_0^\infty(\Delta)$ with the following norm:*

$$\|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}} = \inf \left\{ \lambda > 0 : \int_\Delta \left| \frac{\varphi}{\lambda} \right|^{\kappa(x)} + \left| \frac{{}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi}{\lambda} \right|^{\kappa(x)} dx \leq 1 \right\}$$

where ${}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\cdot)$ is the right-sided χ -H with $0 < \gamma \leq 1$ and type $0 \leq \beta \leq 1$ as in (1), which is provided by

$$\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi} := \left\{ \varphi \in L^{\kappa(x)}(\Delta) : {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \in L^{\kappa(x)}(\Delta) \text{ and } \varphi(\Delta) = 0 \right\},$$

where the space $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$ is the closure of $C_0^\infty(\Delta)$.

Proposition 1 (see [37,49]). Let $0 < \gamma \leq 1, 0 \leq \beta \leq 1$ and $1 < \kappa(x) < \infty$. Then, for all $\varphi \in \mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta, \mathbb{R})$,

$$\|\varphi\|_{L^{\kappa(x)}} \leq \frac{(\chi(T) - \chi(0))^\gamma}{\Gamma(\gamma + 1)} \left\| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right\|_{L^{\kappa(x)}}. \tag{7}$$

Remark 1. In view of Inequality (7), we can consider the space $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta, \mathbb{R})$ with respect to the following equivalent norm:

$$\|\varphi\| = \left\| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right\|_{L^{\kappa(x)}}. \tag{8}$$

Proposition 2 (see [37,49]). Let $0 < \gamma \leq 1, 0 \leq \beta \leq 1$ and $1 < \kappa(x) < \infty$. Then, the space $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta; \mathbb{R})$ is a separable Banach space and is reflexive.

Proposition 3 (see [27]). Let $\varphi \in L^{s(x)}(\Delta)$. Then, each of the following assertions holds true:

- (1) If $u \neq 0$, then $\|\varphi\|_{L^{s(x)}(\Delta)} = \lambda$ if and only if $\rho(\frac{\varphi}{\lambda}) = 1$;
- (2) $\|\varphi\|_{L^{s(x)}(\Delta)} < 1$ ($= 1, > 1$) if and only if $\rho(\varphi) < 1$ ($= 1, > 1$);
- (3) If $\|\varphi\|_{L^{s(x)}(\Delta)} > 1$, then $\|\varphi\|_{L^{s(x)}(\Delta)}^{s-} \leq \rho(\varphi) \leq \|\varphi\|_{L^{s(x)}(\Delta)}^{s+}$;
- (4) If $\|\varphi\|_{L^{s(x)}(\Delta)} < 1$, then $\|\varphi\|_{L^{s(x)}(\Delta)}^{s+} \leq \rho(\varphi) \leq \|\varphi\|_{L^{s(x)}(\Delta)}^{s-}$.

Proposition 4 (see [37,49]). Let $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$ and $\{\varphi_n\} \subset \mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$. Then, the same conclusion as in Proposition 3 occurs when considering $\|\cdot\|$ and ρ_0 .

Corollary 1 (see [27]). Let $\{\varphi_n\} \subset L^{s(x)}(\Delta)$. Then,

- (1) $\lim_{n \rightarrow \infty} \|\varphi\|_{L^{s(x)}(\Delta)} = 0$ if and only if $\lim_{n \rightarrow \infty} \rho(\varphi_n) = 0$.
- (2) $\lim_{n \rightarrow \infty} \|(\varphi_n)\|_{L^{s(x)}(\Delta)} = \infty$ if and only if $\lim_{n \rightarrow +\infty} \rho(\varphi_n) = \infty$.

Proposition 5 (see [25,28], Hölder-Type Inequality). Let $f \in L^{\kappa(x)}(\Delta)$ and $g \in L^{p'(x)}(\Delta)$. Then, the following holds true:

$$\int_{\Delta} |f(x)g(x)| \, dx \leq C_p \|f\|_{L^{\kappa(x)}(\Delta)} \|g\|_{L^{p'(x)}(\Delta)}.$$

Lemma 1 (see [25,28]). Let $h, r \in L_+^\infty(\Delta)$, with $s(x) \leq r(x)$ at almost every point $x \in \Delta$ and $\varphi \in L^{r(x)}(\Delta)$. Then, $|\varphi|^{s(x)} \in L^{\frac{r(x)}{s(x)}}(\Delta)$ and

$$\left\| |\varphi|^{s(x)} \right\|_{L^{\frac{r(x)}{s(x)}}(\Delta)} \leq \|\varphi\|_{L^{r(x)}(\Delta)}^{s+} + \|\varphi\|_{L^{r(x)}(\Delta)}^{s-}$$

or equivalently,

$$\left\| |\varphi|^{s(x)} \right\|_{L^{\frac{r(x)}{s(x)}}(\Delta)} \leq \max \left\{ \|\varphi\|_{L^{r(x)}(\Delta)}^{s+}, \|\varphi\|_{L^{r(x)}(\Delta)}^{s-} \right\}.$$

Lemma 2 (see [25,28], Brezis-Lieb Lemma). Let $\{\Psi_n\} \subset L^{s(x)}(\Delta, \mathbb{R}^m)$ with $m \in \mathbb{N}$. Then,

- (1) $\Psi_n(x) \rightarrow \Psi(x)$ at almost every point in $x \in \Delta$;
- (2) $\sup_{n \in \mathbb{N}} |\Psi_n|_{L^{s(x)}(\Delta, \mathbb{R}^m)} < \infty$.

Furthermore, $\Psi \in L^{s(x)}(\Delta, \mathbb{R}^m)$ and

$$\int_{\Delta} (|\Psi_n|^{s(x)} - |\Psi_n - \Psi|^{s(x)} - |\Psi|^{s(x)}) dx = o_n(1).$$

Remark 2. In the space $\mathcal{H}_{s(x)}^{\gamma, \beta; \chi}(\Delta)$, we consider $\rho_1 : \mathcal{H}_{s(x)}^{\gamma, \beta; \chi}(\Delta) \rightarrow \mathbb{R}$ (modular function) provided by

$$\rho_1(\varphi) = \int_{\Delta} (|\mathcal{H}\mathcal{D}_+^{\gamma, \beta; \chi} \varphi|^{s(x)} + |\varphi|^{s(x)}) dx.$$

Remark 3. If we define

$$\|\varphi\|_1 := \inf \left\{ \int_{\Delta} \frac{(|\mathcal{H}\mathcal{D}_+^{\gamma, \beta; \chi} \varphi|^{s(x)} + |\varphi|^{s(x)})}{t^{s(x)}} dx \leq 1 \right\}$$

then $\|\cdot\|_{\mathcal{H}_{s(x)}^{\gamma, \beta; \chi}(\Delta)}$ and $\|\cdot\|_1$ are equivalent norms in the space $\mathcal{H}_{s(x)}^{\gamma, \beta; \chi}(\Delta)$.

We now present several definitions and a version of the Lions’ Compactness-Concentration Principle in the setting of the χ -H operator.

Definition 2 (see [50]). A finite measure μ on Δ is a continuous linear functional over $C_0(\Delta)$, and the respective norm is defined by

$$\|\mu\| := \sup_{\substack{\Phi \in C_0(\Delta) \\ \|\Phi\|_{\infty} = 1}} |\langle \mu, \Phi \rangle|,$$

with

$$\langle \mu, \Phi \rangle = \int_{\Delta} \Phi d\mu.$$

We denote by $\mathcal{M}(\Delta)$ and $\mathcal{M}^+(\Delta)$ the spaces of finite measures and positive finite measures over Δ , respectively. There are two important convergence properties in $\mathcal{M}(\Delta)$, as detailed below.

Definition 3 (see [50]). A sequence $\mu_n \rightarrow \mu$ in $\mathcal{M}(\Delta)$ (strongly converges), if $\|\mu_n - \mu\| \rightarrow 0$.

Definition 4 (see [50]). A sequence $\mu_n \rightarrow \mu$ in $\mathcal{M}(\Delta)$ (weakly converges), if $\langle \mu_n, \Phi \rangle \rightarrow \langle \mu, \Phi \rangle$ for all $\Phi \in C_0(\Delta)$.

Lemma 3 (see [20], Simon Inequality). Let $x, y \in \mathbb{R}^N$. Then, there is a constant $C = C(\kappa)$ such that

$$\langle |x|^{\kappa-2}x - |y|^{\kappa-2}y, x - y \rangle \geq \begin{cases} C \frac{|x-y|^2}{(|x|+|y|)^{2-\kappa}} & \text{if } 1 < \kappa < 2 \\ C|x-y|^{\kappa} & \text{if } \kappa \geq 2, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in \mathbb{R}^N .

Lemma 4 (see [51], Strauss Compactness Lemma). Let $\mathcal{P} : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ and $\mathcal{Q} : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions such that

$$\sup_{x \in \mathbb{R}^N, |t| \leq a} |\mathcal{P}(x, t)| < \infty \text{ and } \sup_{x \in \mathbb{R}^N, |t| \leq a} |\mathcal{Q}(x, t)| < \infty$$

for each $a > 0$. In addition, let $\lim_{|s| \rightarrow \infty} \frac{\mathcal{P}(x, s)}{\mathcal{Q}(x, s)} = 0$ uniformly in $x \in \mathbb{R}^N$. Suppose that $\{(\varphi_n)\}$ and v are measurable functions on \mathbb{R}^N such that

$$\sup_n \int_{\mathbb{R}^N} |\mathcal{Q}(x, (\varphi_n))| dx < \infty$$

and

$$\lim_n \mathcal{P}(x, (\varphi_n)) = v \text{ at almost every point in } \mathbb{R}^N.$$

Then, for every bounded Borel set $\mathcal{B} \subset \mathbb{R}^N$,

$$\lim_n \int_{\mathcal{B}} |\mathcal{P}(x, (\varphi_n)) - v| dx = 0.$$

Moreover, if

$$\lim_{|s| \rightarrow \infty} \frac{\mathcal{P}(x, s)}{\mathcal{Q}(x, s)} = 0$$

uniformly in $x \in \mathbb{R}^N$ and

$$\lim_{|x| \rightarrow \infty} (\varphi_n) = 0$$

uniformly in n , then $\mathcal{P}(x, (\varphi_n)) \rightarrow v$ in $L^1(\mathbb{R}^N)$.

Lemma 5 (see [27]). Let μ and ν be two non-negative and bounded measures on $\bar{\Delta}$ such that for $1 \leq \kappa(x) < r(x) < \infty$ there exists a constant $c > 0$ satisfying the following inequality:

$$\|\Xi\|_{L_\nu^{r(x)}(\Delta)} \leq c \|\Xi\|_{L_\mu^{\kappa(x)}(\Delta)}.$$

Then, there exist $\{x_j\}_{j \in J} \subset \bar{\Delta}$ and $\{v_j\}_{j \in J} \subset (0, \infty)$ such that

$$\nu = \sum_i v_i \delta_{x_i}.$$

Proposition 6. Let $m, n \in C(\bar{\Delta})$ such that $1 \leq n(x) \leq \kappa_\gamma^*(x)$ for all $x \in \bar{\Delta}$, and let the functions κ and n be log-Hölder continuous. Then, there is a continuous embedding $\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta) \hookrightarrow L^{\kappa(x)}(\Delta)$.

Lemma 6 (see [27]). Let $\Xi_n \rightarrow \Xi$ a.e. and $\Xi_n \rightarrow \Xi$ in $L^{\kappa(x)}(\Delta)$. Then,

$$\lim_{n \rightarrow \infty} \left(\int_{\Delta} |\Xi(x)|^{\kappa(x)} dx - \int_{\Delta} |\Xi(x) - \Xi_n(x)|^{\kappa(x)} dx \right) = \int_{\Delta} |\Xi(x)|^{\kappa(x)} dx.$$

Lemma 7 (see [27]). For the sequence $\{v_j\}_{j \in \mathbb{N}}$, let ν be a non-negative and finite Radon measure in Δ such that $v_j \rightarrow \nu$ weakly $*$ in the sense of measurement. Then,

$$\|\Xi\|_{L_{v_j}^{n(x)}(\Delta)} \rightarrow \|\Xi\|_{L_\nu^{n(x)}(\Delta)} \text{ as } j \rightarrow \infty$$

for all $\Xi \in C^\infty(\bar{\Delta})$.

Considering $\Xi \in C^\infty(\bar{\Delta})$, from the Poincaré inequality for variable exponents we obtain

$$\|\Xi(\varphi_j)\|_{L^{n(x)}(\Delta)} S \leq \left\| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\Xi(\varphi_j)) \right\|_{L^{\kappa(x)}(\Delta)}. \tag{9}$$

If we take the limit as $j \rightarrow \infty$ in (9), from Lemma 7 we have

$$\|\Xi\|_{L_v^{n(x)}(\Delta)} S \leq \|\Xi\|_{L_\mu^{\kappa(x)}(\Delta)}. \tag{10}$$

Theorem 2. Let $q, r \in C(\Delta)$ be such that

$$1 < n_- \leq n_+ < N \quad \text{and} \quad 1 \leq n(x) \leq r^*(x)$$

in Δ with bounded domain of \mathbb{R}^N with a smooth border. Also let $\{(\varphi_n)\}$ be a weakly convergent sequence in the space $\mathcal{H}_{s(x)}^{\gamma,\beta;\chi}(\Delta)$ with a weak limit φ such that

- (1) $\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{r(x)} \rightharpoonup \mu$ in $\mathcal{M}(\Delta)$;
- (2) $|(\varphi_n)|^{n(x)} \rightharpoonup \nu$ in $\mathcal{M}(\Delta)$.

Suppose further that $\varphi = \{x \in \Delta : n(x) = r^*(x)\}$ is non-empty. Then, for some countable set Λ ,

$$\begin{aligned} \nu &= |\varphi|^{n(x)} + \sum_{j \in \Lambda} \nu_j \delta_{x_j}, \quad \nu_j > 0, \\ \mu &\geq |{}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi|^{r(x)} + \sum_{j \in \Lambda} \mu_j \delta_{x_j}, \quad \mu_j > 0 \end{aligned}$$

and

$$S \nu_j^{\frac{1}{r^*(x_j)}} \leq S \mu_j^{\frac{1}{r(x_j)}} \quad \forall j \in \Lambda,$$

where $\{x_j\}_{j \in \Lambda} \subset \varphi$ and S is the best constant provided by

$$S = \inf_{\Xi \in C_0^\infty(\Delta)} \frac{\left\| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi \right\|_{L^{r(x)}(\Delta)}}{\|\Xi\|_{L^{n(x)}(\Delta)}}.$$

Proof. Let $\Xi \in C^\infty(\Delta)$ and let $v_j = (\varphi_j) - \varphi$. Then, by using Lemma 6, we have

$$\lim_{j \rightarrow \infty} \left(\int_{\Delta} |\Xi|^{n(x)} |(\varphi_j)|^{n(x)} dx - \int_{\Delta} |\Xi|^{n(x)} |v_j|^{n(x)} dx \right) = \int_{\Delta} |\Xi|^{n(x)} |\varphi|^{n(x)} dx.$$

Moreover, by using the Hölder measure inequality (10) and Lemma 5 and after taking limits, we obtain the following representation:

$$\nu = |\varphi|^{n(x)} + \sum_{j \in I} \nu_j \delta_{x_j}.$$

Suppose that $x_1 \in \Delta/A$. Let $B = B(x_1, r) \subset\subset \Delta - A$. Then,

$$n(x) < \kappa_\gamma^*(x) - \delta$$

for some $\delta > 0$ in \bar{B} . Using Proposition 6, the embedding $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(B) \hookrightarrow L^{n(x)}(B)$ is compact. Therefore, $(\varphi_j) \rightarrow \varphi$ strongly in $L^{n(x)}(B)$, and thus $|(\varphi_j)|^{n(x)} \hookrightarrow |x|^{n(x)}$ strongly in $L^1(B)$. This is a contradiction to our assumption that $x_1 \in B$.

Next, by applying (9) to $\Xi(\varphi_j)$ and taking into account the fact that $(\varphi_j) \rightarrow \varphi$ in $L^{\kappa(x)}(\Delta)$, we find that

$$S \|\Xi\|_{L_v^{n(x)}(\Delta)} \leq \|\Xi\|_{L_\mu^{n(x)}(\Delta)} + \left\| \left({}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi \right) \varphi \right\|_{L^{\kappa(x)}(\Delta)}.$$

We consider $\Xi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \Xi \leq 1$ and assume that it is supported in the unit ball of \mathbb{R}^N . For a fixed $j \in I$, we let $\varepsilon > 0$ be arbitrary. We set $\Xi_{i_0,\varepsilon}(x) := \varepsilon^{-n} \Xi((x - x_j)\varepsilon)$. Then, decomposition of v yields

$$\rho_v(\Xi_{i_0,\varepsilon}) := \int_\Delta |\Xi_{i_0,\varepsilon}|^{n(x)} dx = \int_\Delta |\Xi_{i_0,\varepsilon}|^{n(x)} |\varphi|^{n(x)} dx + \sum_{i \in I} v_i \Xi_{i_0,\varepsilon}(x_i)^{q(x_i)} \geq v_{i_0}.$$

We use

$$q_{i,\varepsilon}^+ := \sup_{B_\varepsilon(x_i)} n(x), \quad q_{i,\varepsilon}^- := \inf_{B_\varepsilon(x_i)} n(x), \quad p_{i,\varepsilon}^+ := \sup_{B_\varepsilon(x_i)} \kappa(x) \quad \text{and} \quad p_{i,\varepsilon}^- := \inf_{B_\varepsilon(x_i)} \kappa(x).$$

Then, if

$$\rho_v(\Xi_{i_0,\varepsilon}) < 1 \quad \text{and} \quad \rho_v(\Xi_{i_0,\varepsilon}) > 1,$$

it follows that

$$\|\Xi_{i_0,\varepsilon}\|_{L_v^{n(x)}(\Delta)} \geq \rho_v(\Xi_{i_0,\varepsilon})^{\frac{1}{q_{i_0,\varepsilon}^-}} \geq v_{i_0}^{\frac{1}{q_{i_0,\varepsilon}^-}}$$

and

$$\|\Xi_{i_0,\varepsilon}\|_{L_v^{n(x)}(\Delta)} \geq \rho_v(\Xi_{i_0,\varepsilon})^{\frac{1}{q_{i_0,\varepsilon}^+}} \geq v_{i_0}^{\frac{1}{q_{i_0,\varepsilon}^+}},$$

respectively. Consequently, we have

$$\max \left\{ v_{i_0}^{\frac{1}{q_{i_0,\varepsilon}^-}}, v_{i_0}^{\frac{1}{q_{i_0,\varepsilon}^+}} \right\} S \leq \|\Xi_{i,\varepsilon}\|_{L_\mu^{n(x)}(\Delta)} + \left\| \left({}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right) \varphi \right\|_{L^{\kappa(x)}(\Delta)}.$$

Thus, by means of Proposition 3 and Corollary 1, we have

$$\begin{aligned} & \left\| \left({}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right) \varphi \right\|_{L^{\kappa(x)}(\Delta)} \\ & \leq \max \left\{ \rho \left(\left({}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right) \varphi \right)^{\frac{1}{\kappa^-}}, \rho \left(\left({}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right) \varphi \right)^{\frac{1}{\kappa^+}} \right\}. \end{aligned}$$

meaning that by using the Hölder inequality (see Proposition 5), it follows that

$$\begin{aligned} \rho \left(\left({}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right) \varphi \right) & = \int_\Delta \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right|^{\kappa(x)} \|\varphi\|^{\kappa(x)} dx \\ & \leq \left\| \|\varphi\|^{\kappa(x)} \right\|_{L^{\gamma(x)}(B_\varepsilon(x_i))} \left\| \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right|^{\kappa(x)} \right\|_{L^{\gamma'(x)}(B_\varepsilon(x_i))}, \end{aligned}$$

where

$$\gamma(x) = \frac{n}{n - \kappa(x)} \quad \text{and} \quad \gamma'(x) = \frac{n}{\kappa(x)}.$$

Next, by applying the relation

$${}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\Xi_{i,\varepsilon}) = {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \left(\frac{x - x_i}{\varepsilon} \right) \frac{1}{\varepsilon},$$

it follows that

$$\begin{aligned} & \left\| \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right|^{\kappa(x)} \right\|_{L^{\gamma'(x)}(B_\varepsilon(x_i))} \\ & \leq \max \left\{ \rho \left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right|^{\kappa(x)} \right)^{\frac{p^+}{n}}, \rho \left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right|^{\kappa(x)} \right)^{\frac{p^-}{n}} \right\} \end{aligned}$$

and

$$\begin{aligned} \rho \left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \right|^{\kappa(x)} \right) &= \int_{B_\varepsilon(x_i)} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \left(\frac{x-x_i}{\varepsilon} \right) \right|^n \frac{1}{\varepsilon^n} dx \\ &= \int_{B_1(0)} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon}(y) \right|^n dy. \end{aligned}$$

Thus, clearly, we have

$${}^H\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_{i,\varepsilon} \varphi \rightarrow 0$$

in $L^{\kappa(x)}(\Delta)$ (strongly). We note here that

$$\int_\Delta |\Xi_{i,\varepsilon}|^{\kappa(x)} d\mu \leq \mu(B_\varepsilon(x_i)).$$

Therefore, we obtain

$$\|\Xi_{i,\varepsilon}\|_{L^{\kappa(x)}(\Delta)} \leq \max \left\{ \mu(B_\varepsilon(x_i))^{\frac{1}{p_{i,\varepsilon}^+}}, \mu(B_\varepsilon(x_i))^{\frac{1}{p_{i,\varepsilon}^-}} \right\},$$

meaning that

$$S \min \left\{ v_i^{\frac{1}{q_{i,\varepsilon}^+}}, v_i^{\frac{1}{q_{i,\varepsilon}^-}} \right\} \leq \max \left\{ \mu(B_\varepsilon(x_i))^{\frac{1}{p_{i,\varepsilon}^+}}, \mu(B_\varepsilon(x_i))^{\frac{1}{p_{i,\varepsilon}^-}} \right\}.$$

As κ and n are continuous functions and $n(x_i) = \kappa_\gamma^*(x_i)$, upon letting $\varepsilon \rightarrow \infty$, we obtain

$$S v_i^{\frac{1}{\kappa_\gamma^*(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}},$$

where

$$\mu_i := \lim_{\varepsilon \rightarrow \infty} \mu(B_\varepsilon(x_i)).$$

Finally, we prove that

$$\mu \geq \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} + \sum \mu_i \delta_{x_i}.$$

In fact, we have $\mu \geq \sum \mu_i \delta_{x_i}$. As $(\varphi_j) \rightharpoonup \varphi$ in $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$ (weakly), ${}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_j) \rightharpoonup {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi$ weakly in $L^{\kappa(x)}(\varphi)$ for all $U \subset \Delta$. From the weakly lower semi-continuity of the norm, we find that

$$d\mu \geq \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)}.$$

As $\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)}$ is orthogonal to μ , we arrive at the desired result. This completes the proof of Theorem 2. \square

Definition 5. When E is an abstract Banach space and $I \in C^1(E, \mathbb{R})$, we say that a sequence $\{v_n\}$ in E is a Palais-Smale (PS) sequence for \mathcal{I} at level c . We denote this by $(PS)_c$ when $\mathcal{I}(v_n) \rightarrow c$

and $\mathcal{I}'(v_n) \rightarrow c$ in E^* as $n \rightarrow \infty$. We say that \mathcal{I} satisfies the PS condition at level c when every sequence $(PS)_c$ has a subsequence convergent in E .

Theorem 3 (see [52]). Let \mathbf{U} and \mathbf{V} be an infinite-dimensional space and a finite-dimensional space (\mathbf{U} being a Banach space), respectively, with

$$\mathbf{U} = \mathbf{V} \oplus \mathbf{W} \quad \text{and} \quad \mathcal{I} \in C^1(\mathbf{U}, \mathbb{R})$$

being an even functional with $\mathcal{I}(0) = 0$ satisfying the following conditions:

- (I₁) There are constants $\delta, \sigma > 0$ such that $\mathcal{I}(\varphi) \geq \delta > 0$ for each $\varphi \in \partial B_\sigma \cap \mathbf{W}$;
- (I₂) There exists $\varphi > 0$ such that \mathcal{I} satisfies the condition $(PS)_c$ for $0 < c < \varphi$;
- (I₃) For each subspace, $\tilde{\mathbf{U}} \subset \mathbf{U}$ exists with

$$\mathbf{R} = \mathbf{R}(\tilde{\mathbf{U}}) > 0$$

such that

$$\mathcal{I}(\varphi) \leq 0 \quad (\forall \varphi \in \tilde{\mathbf{U}} \setminus B_{\mathbf{R}}(0)).$$

Suppose that $\{e_1, \dots, e_k\}$ is a basis for the vector space \mathbf{V} . For $m \geq k$, choose inductively $e_{m+1} \notin \mathbf{U}_m := \text{span}\{e_1, \dots, e_m\}$. Let $\mathbf{R}_m = \mathbf{R}(\mathbf{U}_m)$ and $D_m = B_{\mathbf{R}_m}(0) \cap \mathbf{U}_m$. Define the following sets:

$$G_m := \{s \in C(D_m, \mathbf{U}) : h \text{ is odd and } s(\varphi) = \varphi, \forall \varphi \in \partial B_{\mathbf{R}_m}(0) \cap \mathbf{U}_m\}$$

and

$$\Gamma_j := \left\{s(\overline{D_m \setminus \Xi}) : s \in G_m, m \geq j, \Xi \in \Sigma \text{ and } q(\Xi) \leq m - j\right\},$$

where Σ is the family of the sets $\Xi \subset \mathbf{U} \setminus \{0\}$ such that Ξ is closed in \mathbf{U} and symmetric with respect to 0; that is,

$$\Sigma = \{\Xi \subset \mathbf{U} \setminus \{0\} : \Xi \text{ is closed in } \mathbf{U} \text{ and } \Xi = -\Xi\}$$

and $q(\Xi)$ is the gender of $\Xi \in \Sigma$. For each $j \in \mathbb{N}$, define

$$\epsilon_j := \inf_{K \in \Gamma_j} \max_{\varphi \in K} \mathcal{I}(\varphi).$$

If $0 < \beta \leq \epsilon_j \leq \epsilon_{j+1}$ for $j > k$ and, if $j > k$ and $\epsilon_j < \varphi$, then ϵ_j is a critical value for \mathcal{I} . Furthermore, if

$$\epsilon_j = \epsilon_{j+1} = \dots = \epsilon_{j+l} = \epsilon < \varphi \quad (j > k),$$

then $q(K_\epsilon) \geq l + 1$, where

$$K_\epsilon := \{\varphi \in \mathbf{U} : \mathcal{I}(\varphi) = \epsilon \text{ and } \mathcal{I}'(\varphi) = 0\}.$$

3. Main Results

Consider the following energy functional of (2) provided by

$$\mathcal{E}_\lambda : \mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta) \rightarrow \mathbb{R},$$

which is defined by

$$\mathcal{E}_\lambda(\varphi) = \int_\Delta \frac{1}{\kappa(x)} \left| {}^H\mathcal{D}_+^{\gamma, \beta; \chi} \varphi \right|^{\kappa(x)} dx - \lambda \int_\Delta \frac{1}{n(x)} |\varphi|^{n(x)} dx - \int_\Delta \frac{\zeta(x)}{\kappa(x)} |\varphi|^{\kappa(x)} dx - \int_\Delta \psi(x, \varphi) dx.$$

Thus, using condition (g₁), it is shown that

$$\mathcal{E}_\lambda \in C^1\left(\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta), \mathbb{R}\right)$$

with

$$\begin{aligned} \mathcal{E}'_\lambda(\varphi)v &= \int_\Delta \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)-2} {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi {}^H\mathcal{D}_+^{\gamma,\beta;\chi} v \, dx \\ &\quad - \lambda \int_\Delta |\varphi|^{n(x)-2} \varphi v \, dx - \int_\Delta \zeta(x) |\varphi|^{\kappa(x)-2} \varphi v \, dx \\ &\quad - \int_\Delta \psi(x, \varphi) v \, dx \end{aligned}$$

for all $\varphi, v \in \mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$. Therefore, the critical points of the energy functional $\mathcal{E}_\lambda(\cdot)$ are solutions to problem (2).

In our first result in this section (Lemma 8 below), we prove that the functional $\mathcal{E}_\lambda(\cdot)$ satisfies the first geometry of the MPT for even functionals.

Lemma 8. *Under conditions (\mathbf{H}_1) and (\mathbf{g}_1) , $\mathcal{E}_\lambda(\cdot)$ satisfies hypothesis (\mathbf{I}_1) of Theorem 3.*

Proof. Given $\delta > 0$, we obtain

$$\begin{aligned} &\int_\Delta \frac{1}{\kappa(x)} \left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} - \zeta(x) |\varphi|^{\kappa(x)} \right) dx \\ &= \frac{1}{1+\delta} \int_\Delta \frac{1}{\kappa(x)} \left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} - \zeta(x) |\varphi|^{\kappa(x)} \right) dx \\ &\quad + \frac{\delta}{1+\delta} \int_\Delta \frac{1}{\kappa(x)} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} dx - \frac{\delta}{1+\delta} \int_\Delta \frac{1}{\kappa(x)} \zeta(x) |\varphi|^{\kappa(x)} dx. \end{aligned}$$

From condition (\mathbf{H}_1) , it follows that

$$\begin{aligned} &\int_\Delta \frac{1}{\kappa(x)} \left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} - \zeta(x) |\varphi|^{\kappa(x)} \right) dx \\ &\geq \frac{\bar{\gamma}}{1+\delta} \int_\Delta \frac{|\varphi|^{\kappa(x)}}{\kappa(x)} + \frac{\delta}{1+\delta} \int_\Delta \frac{1}{\kappa(x)} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} dx \\ &\quad - \frac{\delta}{1+\delta} \int_\Delta \frac{1}{\kappa(x)} \zeta(x) |\varphi|^{\kappa(x)} dx \\ &\geq \frac{\delta}{1+\delta} \int_\Delta \frac{1}{\kappa(x)} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} dx + \frac{1}{1+\delta} \int_\Delta \left(\frac{\bar{\gamma}}{\kappa_+} - \frac{\delta \zeta(x)}{\kappa_-} \right) |\varphi|^{\kappa(x)} dx. \end{aligned}$$

For a sufficiently small δ , provided that $\zeta(x) \in L^\infty(\Delta)$, we can assume that

$$\frac{1}{1+\delta} \left(\frac{\bar{\gamma}}{\kappa_+} - \frac{\delta \zeta(x)}{\kappa_-} \right) \geq \frac{1}{1+\delta} \left(\frac{\bar{\gamma}}{\kappa_+} - \frac{\delta \|\zeta\|_{L^\infty(\Delta)}}{\kappa_-} \right) = \bar{\gamma}_0 > 0.$$

Therefore, for all $\varphi \in \mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$, we find that

$$\begin{aligned} &\int_\Delta \frac{1}{\kappa(x)} \left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} - \zeta(x) |\varphi|^{\kappa(x)} \right) dx \\ &\geq \frac{\delta}{1+\delta} \int_\Delta \frac{1}{\kappa(x)} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} dx + \bar{\gamma}_0 \int_\Delta |\varphi|^{\kappa(x)} dx. \end{aligned} \tag{11}$$

For the sake of verifying the above developments, given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|\psi(x, t)| \leq \frac{\varepsilon}{\kappa(x)} |t|^{\kappa(x)} + \frac{C_\varepsilon}{n(x)} |t|^{n(x)} \quad \forall (x, t) \in \bar{\Delta} \times \mathbb{R}. \tag{12}$$

Indeed, from hypothesis **(g₁)** we know that

$$\psi(x, t) = o(|t|^{n(x)-1}) \quad \text{when } |t| \rightarrow \infty$$

uniformly in x . Thus, given $\varepsilon > 0$, there is a number $R = R(\varepsilon) > 0$ such that $|\psi(x, t)| \leq \varepsilon |t|^{n(x)-1}$ for all $x \in \bar{\Delta}$ and $|t| \geq R$.

By continuity and from the inequalities above, there exists $\mathbf{M} > 0$ such that

$$|\psi(x, t)| \leq \mathbf{M} + \varepsilon |t|^{n(x)-1} \quad \forall (x, t) \in \bar{\Delta} \times \mathbb{R}.$$

Again, it follows from condition **(g₁)** that, given $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$, satisfying

$$|\psi(x, t)| \leq \varepsilon |t|^{n(x)-1} \quad \forall (x, t) \in \bar{\Delta} \times [-\delta, \delta]. \tag{13}$$

We can now assume that $\delta < 1$. Therefore, for $|t| \geq \delta$, we have $|t|^{n(x)-1} \geq \delta^{n_+-1}$, such that

$$\frac{|\psi(x, t)|}{|t|^{n(x)-1}} \leq \frac{\mathbf{M}}{|t|^{n(x)-1}} + \varepsilon \leq \frac{\mathbf{M}}{\delta^{n_+-1}} + \varepsilon = C_\varepsilon \quad \forall x \in \bar{\Delta} \quad \text{and } |t| \geq \delta. \tag{14}$$

Thus, from Inequalities (13) and (14) we obtain

$$|\psi(x, t)| \leq \varepsilon |t|^{k(x)-1} + C_\varepsilon |t|^{n(x)-1} \quad \forall (x, t) \in \bar{\Delta} \times \mathbb{R},$$

which leads to the statement to be verified.

Next, using the definition of $\mathcal{E}_\lambda(\cdot)$ and the Equations (11) and (12), we obtain

$$\begin{aligned} \mathcal{E}_\lambda(\varphi) &\geq \frac{\delta}{(1+\delta)\kappa_+} \int_\Delta |{}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi|^{k(x)} dx + \left(\gamma_0 - \frac{\varepsilon}{\kappa_-}\right) \int_\Delta |\varphi|^{k(x)} dx \\ &\quad - \frac{(\lambda + C_\varepsilon)}{n_-} \int_\Delta |\varphi|^{n(x)} dx. \end{aligned}$$

Consequently, if ε is small enough and $\|\varphi\| \leq 1$, from Proposition 4 we find that

$$\mathcal{E}_\lambda(\varphi) \geq \frac{\delta}{(1+\delta)\kappa_+} \|{}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi\|_{L^{k(x)}(\Delta)}^{k_+} - \frac{(\lambda + C_\varepsilon)}{n_-} \int_\Delta |\varphi|^{n(x)} dx.$$

Using Sobolev embeddings, there exists $\epsilon_1 > 0$ such that

$$\|\varphi\|_{L^{n(x)}(\Delta)} \leq \epsilon_1 \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)}, \quad \forall \varphi \in \mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta).$$

Thus, if we apply Proposition 3, we obtain

$$\mathcal{E}_\lambda(\varphi) \geq c_3 \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)}^{k_+} - c_4 \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)}^{n_-}$$

for the positive constants ϵ_2, c_3 , and c_4 . Because $\kappa_+ < n_-$ if $\|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)} = \sigma > 0$ is sufficiently small, $\exists \tilde{\beta} > 0$ such that

$$\mathcal{E}_\lambda(\varphi) \geq \tilde{\beta} > 0, \quad \forall \varphi \in \partial B_\sigma(0).$$

We have thus completed the proof of Lemma 8. \square

Lemma 9. Under the conditions **(H₁)** and **(g₁)**, $\mathcal{E}_\lambda(\cdot)$ satisfies the condition **(I₃)**.

Proof. Let \tilde{E} be a sub-space of $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$ of a finite dimension.

For verification of the above statement, given $\varepsilon > 0$, there is a constant $\mathbf{M} > 0$ satisfying

$$\mathbf{F}(x, t) \geq -\mathbf{M} - \varepsilon|t|^{n(x)} \quad \forall (x, t) \in \bar{\Delta} \times \mathbb{R}. \tag{15}$$

In fact, we have

$$\frac{f(x, t)}{|t|^{n(x)-1}} \leq |\zeta(x)| \frac{|t|^{\kappa(x)-1}}{|t|^{n(x)-1}} + \frac{|\psi(x, t)|}{|t|^{n(x)-1}} \rightarrow 0$$

when $|t| \rightarrow \infty$. Hence, given $\varepsilon > 0$, $\exists R_0 > 0$ such that $|f(x, t)| \leq \varepsilon|t|^{n(x)-1}$ for all $x \in \bar{\Delta}$ and $|t| \geq R_0$. Moreover, because f is continuous, it follows that

$$|f(x, t)| \leq \mathbf{M} + \varepsilon|t|^{n(x)-1} \quad \forall x \in \bar{\Delta} \times \mathbb{R}$$

for some positive constant \mathbf{M}_0 . Thus, we have

$$\frac{|\mathbf{F}(x, t)|}{|t|^{n(x)}} \leq \frac{\mathbf{M}_0}{|t|^{n(x)-1}} + \frac{\varepsilon}{n_-} = o(1) \text{ with } |t| \rightarrow \infty.$$

Furthermore, given $\varepsilon > 0$, $\exists R > 0$ such that

$$|\mathbf{F}(x, t)| \leq \varepsilon|t|^{n(x)} \quad \forall x \in \bar{\Delta} \quad \text{and} \quad |t| \geq R.$$

By continuity, there is a constant $\mathbf{M} > 0$ such that $\mathbf{F}(x, t) \geq -\mathbf{M}$ for all $x \in \bar{\Delta}$ and $|t| \leq R$. Therefore, we obtain

$$\mathbf{F}(x, t) \geq -\mathbf{M} - \varepsilon|t|^{n(x)} \quad \forall (x, t) \in \bar{\Delta} \times \mathbb{R},$$

thereby proving the claim.

Using Inequality (15) and $\mathcal{E}_\lambda(\cdot)$, we have

$$\mathcal{E}_\lambda(\varphi) \leq \frac{1}{\kappa_-} \int_{\Delta} |{}^H\mathcal{D}_+^{\gamma, \beta; \chi} u|^{\kappa(x)} dx - \frac{\lambda}{n_+} \int_{\Delta} |\varphi|^{n(x)} dx + \varepsilon \int_{\Delta} |\varphi|^{n(x)} dx + \mathbf{M}|\Delta|.$$

Now, upon setting

$$\varepsilon = \frac{\lambda}{2n_+},$$

we can conclude that

$$\mathcal{E}_\lambda(\varphi) \leq \frac{1}{\kappa_-} \int_{\Delta} |{}^H\mathcal{D}_+^{\gamma, \beta; \chi} \varphi|^{\kappa(x)} dx - \frac{\lambda}{2n_+} \int_{\Delta} |\varphi|^{n(x)} dx + \mathbf{M}|\Delta|.$$

By applying Proposition 3, it follows that

$$\mathcal{E}_\lambda(\varphi) \leq \frac{1}{\kappa_-} \max \left\{ \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta)}^{\kappa_-}, \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta)}^{\kappa_+} \right\} - \frac{\lambda}{2n_+} \min \left\{ \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta)}^{n_-}, \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta)}^{n_+} \right\} + \mathbf{M}|\Delta|.$$

Because $\dim \tilde{E} < \infty$, any two norms in \tilde{E} are equivalent, and thus $\exists c > 0$ (constant) such that

$$\mathcal{E}_\lambda(\varphi) \leq \frac{1}{\kappa_-} \max \left\{ \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta)}^{\kappa_-}, \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta)}^{\kappa_+} \right\} - \frac{\lambda c}{2n_+} \min \left\{ \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta)}^{n_-}, \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta)}^{n_+} \right\} + \mathbf{M}|\Delta|.$$

Moreover, as $\kappa_+ < n_-$, we have

$$\Psi(s) = \frac{s^{\kappa_+}}{\kappa_-} - \frac{\lambda c s^{q_-}}{2n_+} \rightarrow -\infty$$

when $s \rightarrow \infty$. Consequently, for a sufficiently large $R > 0$, the last inequality implies that

$$\mathcal{E}_\lambda(\varphi) \leq \frac{\|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)}^{m_+}}{\kappa_-} - \frac{\lambda c \|\varphi\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)}^{n_-}}{2n_+} + \mathbf{M}|\Delta| < 0$$

for all $\varphi \in \tilde{E}$ with $\|\varphi\| \geq R$. Hence, $\mathcal{E}_\lambda(\cdot) < 0$ over $\tilde{E} \setminus B_R(0)$. \square

We now establish a compactness condition for the functional $\mathcal{E}_\lambda(\cdot)$. We prove that the (PS) condition holds true below a certain level, provided that the parameter λ is less than 1.

Lemma 10. *Let the conditions (\mathbf{H}_1) , (\mathbf{g}_1) , and (\mathbf{g}_2) be satisfied. Then, every sequence (PS) for the functional $\mathcal{E}_\lambda(\cdot)$ is bounded in $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$.*

Proof. Let $\{(\varphi_n)\}$ be a sequence (PS)_c for the functional $\mathcal{E}_\lambda(\cdot)$. Then,

$$\mathcal{E}_\lambda(\varphi_n) \rightarrow c \text{ and } \mathcal{E}'_\lambda(\varphi_n) \rightarrow 0 \text{ when } n \rightarrow \infty. \tag{16}$$

We note that

$$\begin{aligned} & \lambda \int_\Delta \left(\frac{1}{\kappa_+} - \frac{1}{n(x)} \right) |(\varphi_n)|^{n(x)} dx \\ &= \mathcal{E}_\lambda(\varphi_n) - \frac{1}{\kappa_+} \mathcal{E}'_\lambda(\varphi_n)(\varphi_n) + \int_\Delta \left(\frac{1}{\kappa_+} - \frac{1}{\kappa(x)} \right) \left| {}^{\text{H}}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)} dx \\ &+ \int_\Delta \left(\frac{1}{\kappa(x)} - \frac{1}{\kappa_+} \right) \zeta(x) |(\varphi_n)|^{\kappa(x)} dx \\ &- \int_\Delta \left(G(x, (\varphi_n)) - \frac{1}{\kappa_+} g(x, (\varphi_n))(\varphi_n) \right) dx. \end{aligned}$$

In the above equality, using the hypotheses (\mathbf{g}_2) and (16), we obtain

$$\begin{aligned} & \lambda \int_\Delta \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) |(\varphi_n)|^{n(x)} dx \\ & \leq \lambda \int_\Delta \left(\frac{1}{\kappa_+} - \frac{1}{n(x)} \right) |(\varphi_n)|^{n(x)} dx \\ & \leq c + 1 + \|(\varphi_n)\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)} + \|a\|_\infty \int_\Delta \left(\frac{1}{\kappa_-} - \frac{1}{\kappa_+} \right) |(\varphi_n)|^{\kappa(x)} dx \tag{17} \end{aligned}$$

for sufficiently large n . Provided $\varepsilon > 0$, note that $\exists C_\varepsilon > 0$ such that

$$|t|^{\kappa(x)} \leq \varepsilon |t|^{n(x)} + C_\varepsilon \forall (x, t) \in \Delta \times \mathbb{R}.$$

Upon combining the last inequality with (17), we obtain

$$\begin{aligned} & \lambda \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) \int_\Delta |(\varphi_n)|^{n(x)} dx \\ & \leq c + 1 + \|(\varphi_n)\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)} + \varepsilon \|a\|_\infty \left(\frac{1}{\kappa_-} - \frac{1}{\kappa_+} \right) \int_\Delta |(\varphi_n)|^{n(x)} dx \\ & \quad + \|a\|_\infty \left(\frac{1}{\kappa_-} - \frac{1}{\kappa_+} \right) C_\varepsilon |\Delta|, \end{aligned}$$

which implies that

$$\left[\lambda \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) - \|a\|_\infty \left(\frac{1}{\kappa_-} - \frac{1}{\kappa_+} \right) \varepsilon \right] \int_\Delta |(\varphi_n)|^{n(x)} dx \leq c + 1 + \|(\varphi_n)\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)} + c_5,$$

where c_5 is a positive constant. If we set

$$\varepsilon = \frac{\lambda}{2} \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) \left[\|a\|_\infty \left(\frac{1}{\kappa_-} - \frac{1}{\kappa_+} \right) \right]^{-1},$$

we obtain

$$\frac{\lambda}{2} \left(\frac{1}{\kappa_+} - \frac{1}{q_-} \right) \int_\Delta |(\varphi_n)|^{n(x)} dx \leq c + 1 + \|(\varphi_n)\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)} + c_5,$$

which yields

$$\int_\Delta |(\varphi_n)|^{n(x)} dx \leq c_6 \left(1 + \|(\varphi_n)\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)} \right).$$

Now, using the definition of $\mathcal{E}_\lambda(\cdot)$ together with (11), we obtain

$$\frac{\delta}{(1 + \delta)\kappa_+} \int_\Delta |{}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n)|^{\kappa(x)} dx \leq \mathcal{E}_\lambda(\varphi_n) + \lambda \int_\Delta \frac{|(\varphi_n)|^{n(x)}}{n(x)} dx + \int_\Delta G(x, (\varphi_n)) dx.$$

Of the growth conditions over g , given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|\psi(x, t)| \leq \varepsilon |t|^{n(x)} + C_\varepsilon$$

for all $x \in \bar{\Delta}$ and $t \in \mathbb{R}$, meaning that

$$\begin{aligned} & \frac{\delta}{(1 + \delta)\kappa_+} \int_\Delta |{}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n)|^{\kappa(x)} dx \\ & \leq c + o_n(1) + \frac{\lambda}{n_-} \int_\Delta |(\varphi_n)|^{n(x)} dx + \varepsilon \int_\Delta |(\varphi_n)|^{n(x)} dx + C_\varepsilon |\Delta| \\ & \leq c + o_n(1) + \left(\frac{\lambda}{n_-} + \varepsilon \right) c_6 \left(1 + \|(\varphi_n)\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)} \right) + C_\varepsilon |\Delta|. \end{aligned}$$

Therefore, for sufficiently large n , we have

$$\int_\Delta |{}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n)|^{\kappa(x)} dx \leq c_7 \left(1 + \|(\varphi_n)\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)} \right),$$

where c_7 is a positive constant. If

$$\|(\varphi_n)\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)} > 1,$$

then it follows from Proposition 4 that

$$\|(\varphi_n)\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)}^{\kappa_-} \leq c_7 \left(1 + \|(\varphi_n)\|_{\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)} \right).$$

Finally, as $\kappa_- > 1$, the above inequality implies that $\{(\varphi_n)\}$ is bounded in $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$. We have thus completed the proof of Lemma 10. \square

From the reflexivity of $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$, if $\{(\varphi_n)\}$ is a sequence $(PS)_c$ for $\mathcal{E}_\lambda(\cdot)$, then up to subsequence $(\varphi_n) \rightharpoonup \varphi$ in $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$. As the immersion of $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$ in $L^{n(x)}(\Delta)$ is continuous, $(\varphi_n) \rightharpoonup \varphi$ in $L^{n(x)}(\Delta)$. On the other hand, the immersion $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$ in $L^{r(x)}(\Delta)$ is compact for $1 < r_- \leq r \ll \kappa_\gamma^*$.

Consequently, $(\varphi_n) \rightarrow \varphi$ in $L^{\kappa(x)}(\Delta)$. From the CCP, for Lebesgue spaces with variable exponents (see Theorem 2) there are two non-negative measures $\mu, \nu \in \mathcal{M}(\Delta)$, a countable set Λ , points $\{x_j\}_{j \in J}$ in A , and sequences $\{\mu_j\}_{j \in J}$ and $\{\nu_j\}_{j \in J} \subset [0, \infty)$, and thus we have

$$|\mathbb{H}\mathcal{D}_+^{\gamma, \beta; \chi}(\varphi_n)|^{\kappa(x)} \rightarrow \mu \geq |\mathbb{H}\mathcal{D}_+^{\gamma, \beta; \chi} \varphi|^{r(x)} + \sum_{j \in \Lambda} \mu_j \delta_{x_j} \text{ in } \mathcal{M}(\Delta),$$

$$|(\varphi_n)|^{n(x)} \rightarrow \nu = |\varphi|^{n(x)} + \sum_{j \in \Lambda} \nu_j \delta_{x_j} \text{ in } \mathcal{M}(\Delta)$$

and

$$S\nu_j^{\frac{1}{q(x_j)}} \leq \mu_j^{\frac{1}{m(x_j)}} \quad \forall j \in \Lambda.$$

Our objective is now to establish a lower estimate for $\{v_i\}$. For this purpose, we need to prove the following lemma.

Lemma 11. Let $\Xi \in C_0^\infty(\Delta)$, satisfying the following conditions:

$$\Xi(x) = 1 \text{ in } B_1(0), \sup p \Xi \subset B_2(0) \text{ and } 0 \leq \Xi(x) \leq 1 \quad \forall x \in \Delta.$$

Then, for $\varepsilon > 0, z \in \bar{\Delta}$ and $\varphi \in L^{\kappa(x)}(\Delta)$, it is asserted that

$$\int_{\Delta} \left| \varphi \mathbb{H}\mathcal{D}_+^{\gamma, \beta; \chi} \Xi_\varepsilon(x-z) \right|^{\kappa(x)} dx \leq C \left(\|\varphi\|_{L^{\kappa_\gamma^*(x)}(B_{2\varepsilon}(z))}^{\kappa_+} + \|\varphi\|_{L^{\kappa_\gamma^*(x)}(B_{2\varepsilon}(z))}^{\kappa_-} \right), \quad (18)$$

where

$$\Xi_\varepsilon(x) = \Xi\left(\frac{x}{\varepsilon}\right)$$

for all $x \in \Delta$ and for a constant C independent of ε and z .

Proof. We note that

$$\begin{aligned} & \int_{\Delta} \left| \varphi \mathbb{H}\mathcal{D}_+^{\gamma, \beta; \chi} \Xi_\varepsilon(x-z) \right|^{\kappa(x)} dx \\ &= \int_{B_{2\varepsilon}(z)} \left| \varphi \right|^{\kappa(x)} \left| \frac{1}{\varepsilon} \mathbb{H}\mathcal{D}_+^{\gamma, \beta; \chi} \Xi\left(\frac{x-z}{\varepsilon}\right) \right|^{\kappa(x)} dx \\ &\leq c_p \left\| \left| \varphi \right|^{\kappa(x)} \right\|_{L^{\frac{p_\gamma^*(x)}{\kappa(x)}}(B_{2\varepsilon}(z))} \left\| \left| \frac{1}{\varepsilon} \mathbb{H}\mathcal{D}_+^{\gamma, \beta; \chi} \Xi\left(\frac{\cdot-z}{\varepsilon}\right) \right|^{\kappa(x)} \right\|_{L^{\frac{\kappa_\gamma^*(x)}{\kappa_\gamma^*(x)-\kappa(x)}}(B_{2\varepsilon}(z))}, \end{aligned}$$

where c_p is the constant provided by the Hölder inequality (see Proposition 5). Thus, upon changing the variable, we have

$$\begin{aligned} & \int_{B_{2\varepsilon}(z)} \left| \frac{1}{\varepsilon} \mathbb{H}\mathcal{D}_+^{\gamma, \beta; \chi} \Xi\left(\frac{x-z}{\varepsilon}\right) \right|^{\frac{\kappa(x)\kappa_\gamma^*(x)}{\kappa_\gamma^*(x)-\kappa(x)}} dx \\ &= \int_{B_2(0)} \left| \frac{1}{\varepsilon} \mathbb{H}\mathcal{D}_+^{\gamma, \beta; \chi} \Xi(y) \right|^N \varepsilon^N dx \\ &= \int_{B_2(0)} \left| \mathbb{H}\mathcal{D}_+^{\gamma, \beta; \chi} \Xi(y) \right|^N dx. \end{aligned}$$

Now, the result follows from Proposition 3 and Lemma 1. We have thus completed the proof of Lemma 11. \square

Lemma 12. Under the conditions of Lemma 10, let $\{(\varphi_n)\}$ be a sequence (PS) for the functional $\mathcal{E}_\lambda(\cdot)$ and $\{v_j\}$. Then, for each $j \in \Lambda$,

$$v_j \geq \frac{S^N}{\lambda^{\frac{N}{\kappa(x_j)}}} \quad \text{or} \quad v_j = 0.$$

Proof. First, for each $\varepsilon > 0$ let $\Xi_\varepsilon \in C_0^\infty(\Delta)$, as in Lemma 11. Therefore, we have

$$\{\Xi_\varepsilon(\cdot - x_j)(\varphi_n)\} \subset \mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$$

for any $j \in \Lambda$. In addition, by direct calculation we can see that $\{\Xi_\varepsilon(\cdot - x_j)(\varphi_n)\}$ is bounded in $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$. Thus, we obtain

$$\mathcal{E}'_\lambda(\varphi_n)(\Xi_\varepsilon(\cdot - x_j)(\varphi_n)) = o_n(1)$$

or equivalently,

$$\begin{aligned} & \int_\Delta \left| \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)} \Xi_\varepsilon(x - x_j) \, dx \\ & + \int_\Delta \left| \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)-2} (\varphi_n) \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \text{H}\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_\varepsilon(x - x_j) \, dx + o_n(1) \\ & = \lambda \int_\Delta |(\varphi_n)|^{n(x)} \Xi_\varepsilon(x - x_j) \, dx + \int_\Delta \zeta(x) |(\varphi_n)|^{\kappa(x)} \Xi_\varepsilon(x - x_j) \, dx \\ & + \int_\Delta g(x, (\varphi_n)) (\varphi_n) \Xi_\varepsilon(x - x_j) \, dx. \end{aligned} \tag{19}$$

Now, for each $\delta > 0$, by applying the Cauchy-Schwartz and Young inequalities, we obtain

$$\begin{aligned} & \int_\Delta \left| \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)-2} (\varphi_n) \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \text{H}\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_\varepsilon(x - x_j) \, dx \\ & \leq \delta \int_\Delta \left| \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)} \, dx + C_\delta \int_\Delta \left| (\varphi_n) \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)} \, dx. \end{aligned} \tag{20}$$

Thus, by the Lebesgue Dominated Convergence Theorem and the limit of $\{(\varphi_n)\}$, we can conclude that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_\Delta \left| \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)-2} (\varphi_n) \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \text{H}\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_\varepsilon(x - x_j) \, dx \\ & \leq \delta \varepsilon_1 + C_\delta \int_\Delta \left| \varphi \text{H}\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_\varepsilon(x - x_j) \right|^{\kappa(x)} \, dx. \end{aligned} \tag{21}$$

Therefore, by applying Lemma 11, it follows that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_\Delta \left| \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)-2} (\varphi_n) \text{H}\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \text{H}\mathcal{D}_+^{\gamma,\beta;\chi} \Xi_\varepsilon(x - x_j) \, dx \\ & \leq \delta \varepsilon_1 + CC_\delta \left(\|\varphi\|_{L^{p_\gamma^*(x)}(\mathcal{B}_{2\varepsilon}(x_j))}^{\kappa_+} + \|\varphi\|_{L^{p_\gamma^*(x)}(\mathcal{B}_{2\varepsilon}(x_j))}^{\kappa_-} \right). \end{aligned} \tag{22}$$

Applying the Strauss Lemma (see Lemma 4) with

$$\mathcal{P}(x, t) = \psi(x, t)t \quad \text{and} \quad \mathcal{Q}(x, t) = |t|^{\kappa(x)} + |t|^{n(x)},$$

and using the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n \rightarrow \infty} \int_{\Delta} \zeta(x) |(\varphi_n)|^{\kappa(x)} \Xi_{\varepsilon}(x - x_j) dx = \int_{\Delta} \zeta(x) \|\varphi\|^{\kappa(x)} \Xi_{\varepsilon}(x - x_j) dx. \tag{23}$$

Next, using Equation (19) and Equations (21) to (23), we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Delta} \left| {}^H\mathcal{D}_+^{\gamma, \beta; \chi} (\varphi_n) \right|^{\kappa(x)} \Xi_{\varepsilon}(x - x_j) dx \\ & \leq \lambda \lim_{n \rightarrow \infty} \int_{\Delta} |(\varphi_n)|^{n(x)} \Xi_{\varepsilon}(x - x_j) dx + \int_{\Delta} \zeta(x) \|\varphi\|^{\kappa(x)} \Xi_{\varepsilon}(x - x_j) dx \\ & \quad + \int_{\Delta} \psi(x, \varphi) \varphi \Xi_{\varepsilon}(x - x_j) dx + \delta \varepsilon_1 + CC_{\delta} \left[\|\varphi\|_{L^{\kappa_+^*(x)}(\mathcal{B}_{2\varepsilon}(x_j))}^{\kappa_+} + \|\varphi\|_{L^{\kappa_-^*(x)}(\mathcal{B}_{2\varepsilon}(x_j))}^{\kappa_-} \right], \end{aligned}$$

where C is a constant independent of ε and j . Because

$$\left| {}^H\mathcal{D}_+^{\gamma, \beta; \chi} (\varphi_n) \right|^{\kappa(x)} \rightarrow \mu \quad \text{and} \quad |(\varphi_n)|^{n(x)} \rightarrow v \quad \text{in } \mathcal{M}(\Delta),$$

we have

$$\lim_{n \rightarrow \infty} \int_{\Delta} \left| {}^H\mathcal{D}_+^{\gamma, \beta; \chi} (\varphi_n) \right|^{\kappa(x)} \Xi_{\varepsilon}(x - x_j) dx \geq \int_{\mathcal{B}_{\varepsilon}(x_j)} d\mu \geq \mu_j(\{x_j\}) = \mu_j$$

and

$$\lim_{n \rightarrow \infty} \int_{\Delta} |(\varphi_n)|^{n(x)} \Xi_{\varepsilon}(x - x_j) dx = \int_{\mathcal{B}_{2\varepsilon}(x_j)} \Xi_{\varepsilon}(x - x_j) dv \leq \int_{\mathcal{B}_{2\varepsilon}(x_j)} dv \leq v_j.$$

Consequently, we have

$$\begin{aligned} \mu_j & \leq \lim_{n \rightarrow \infty} \int_{\Delta} \left| {}^H\mathcal{D}_+^{\gamma, \beta; \chi} (\varphi_n) \right|^{\kappa(x)} \Xi_{\varepsilon}(x - x_j) dx \\ & \leq \lambda v_j + \int_{\mathcal{B}_{2\varepsilon}(x_j)} \zeta(x) |\varphi|^{\kappa(x)} \Xi_{\varepsilon}(x - x_j) dx + \int_{\mathcal{B}_{2\varepsilon}(x_j)} \psi(x, \varphi) \varphi \Xi_{\varepsilon}(x - x_j) dx \\ & \quad + \delta \varepsilon_1 + CC_{\delta} \left(\|\varphi\|_{L^{\kappa_+^*(x)}(\mathcal{B}_{2\varepsilon}(x_j))}^{\kappa_+} + \|\varphi\|_{L^{\kappa_-^*(x)}(\mathcal{B}_{2\varepsilon}(x_j))}^{\kappa_-} \right). \end{aligned} \tag{24}$$

Upon first letting $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$, we obtain $\mu_j \leq \lambda v_j$. Hence,

$$Sv_j^{\frac{1}{\kappa_+^*(x_j)}} \leq \mu_j^{\frac{1}{\kappa(x_j)}} \leq (\lambda v_j)^{\frac{1}{\kappa(x_j)}},$$

and thus

$$v_j \geq \frac{S^N}{\lambda^{\frac{N}{\kappa(x_j)}}} \quad \text{or} \quad v_j = 0.$$

This evidently concludes our proof of Lemma 12. \square

We are now able to demonstrate that the (PS) condition for the functional $\mathcal{E}_{\lambda}(\cdot)$ holds true below a certain level. More precisely, we can prove the following lemma.

Lemma 13. *Let the conditions (\mathbf{H}_1) , (\mathbf{H}_2) , (\mathbf{g}_1) , and (\mathbf{g}_2) be satisfied. If $\lambda < 1$, then $\mathcal{E}_{\lambda}(\cdot)$ satisfies the condition $(\text{PS})_d$ for*

$$d < \lambda^{1 - \frac{N}{\kappa_+}} \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) S^N.$$

Proof. Let the sequence $\{(\varphi_n)\}$ be $(PS)_d$ for the energy functional $\mathcal{E}_\lambda(\cdot)$ with

$$d < \lambda^{1-\frac{N}{\kappa_+}} \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) S^N,$$

that is,

$$\mathcal{E}_\lambda(\varphi_n) = d + o_n(1) \quad \text{and} \quad \mathcal{E}'_\lambda(\varphi_n) = o_n(1).$$

We observe that

$$d = \lim_{n \rightarrow \infty} \left[\int_\Delta \left(\frac{1}{\kappa(x)} - \frac{1}{\kappa_+} \right) \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)} dx + \lambda \int_\Delta \left(\frac{1}{\kappa_+} - \frac{1}{\kappa_\gamma^*(x)} \right) |(\varphi_n)|^{n(x)} dx \right] + \int_\Delta \left(\frac{1}{\kappa_+} - \frac{1}{\kappa(x)} \right) \zeta(x) |(\varphi_n)|^{\kappa(x)} dx - \int_\Delta \left(G(x, (\varphi_n)) - \frac{1}{\kappa_+} g(x, (\varphi_n))(\varphi_n) \right). \tag{25}$$

Then, it follows from conditions (H_1) and (H_2) that

$$d \geq \lambda \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) \lim_{n \rightarrow \infty} \int_\Delta |(\varphi_n)|^{n(x)} dx. \tag{26}$$

Now, recalling that

$$\lim_{n \rightarrow \infty} \int_\Delta |(\varphi_n)|^{n(x)} dx = \int_\Delta |\varphi|^{n(x)} dx + \sum_{j \in \Lambda} v_j \geq v_j,$$

it follows that if $v_j > 0$ for some $j \in \Lambda$, then

$$d \geq \lambda \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) v_j \geq \lambda \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) \frac{S^N}{\lambda^{\frac{N}{\kappa(x_j)}}}.$$

Thus, for $\lambda < 1$, we find that

$$d \geq \lambda \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) \left(\frac{S}{\lambda^{\frac{1}{\kappa_+}}} \right)^N = \lambda^{1-\frac{N}{\kappa_+}} \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) S^N, \tag{27}$$

which is absurd. Therefore, we must have $v_j = 0$ for all $j \in \Lambda$, implying that

$$\int_\Delta |(\varphi_n)|^{n(x)} dx \rightarrow \int_\Delta |\varphi|^{n(x)} dx. \tag{28}$$

Combining the above limit with Lemma 2, we have

$$\int_\Delta |(\varphi_n) - \varphi|^{n(x)} dx \rightarrow 0 \text{ when } n \rightarrow \infty.$$

Thus, by Proposition 3, $(\varphi_n) \rightarrow \varphi$ in $L^{n(x)}(\Delta)$.

Now let us denote by $\{\mathcal{P}_n\}$ the sequence provided by

$$\mathcal{P}_n(x) := \left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)-2} {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) - \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)-2} {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right) {}^H\mathcal{D}_+^{\gamma,\beta;\chi}((\varphi_n) - \varphi). \tag{29}$$

From the above definition of \mathcal{P}_n we find that

$$\begin{aligned} \int_{\Delta} \mathcal{P}_n(x) dx &= \int_{\Delta} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)} dx \\ &\quad - \int_{\Delta} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)-2} {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi dx \\ &\quad - \int_{\Delta} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)-2} {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi {}^H\mathcal{D}_+^{\gamma,\beta;\chi}((\varphi_n) - \varphi) dx. \end{aligned}$$

Because $(\varphi_n) \rightarrow \varphi$ in $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$, we have

$$\int_{\Delta} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)-2} {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi {}^H\mathcal{D}_+^{\gamma,\beta;\chi}((\varphi_n) - \varphi) dx \rightarrow 0 \tag{30}$$

when $n \rightarrow \infty$. This implies that

$$\begin{aligned} \int_{\Delta} \mathcal{P}_n(x) dx &= \int_{\Delta} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)} dx - \int_{\Delta} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right|^{\kappa(x)-2} \\ &\quad \times {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi dx + o(1). \end{aligned}$$

On the other hand, because

$$\mathcal{E}'_{\lambda}(\varphi_n)(\varphi_n) = o_n(1) \quad \text{and} \quad \mathcal{E}'_{\lambda}(\varphi_n)\varphi = o_n(1),$$

we have

$$\begin{aligned} \int_{\Delta} \mathcal{P}_n(x) dx &= o_n(1) + \lambda \int_{\Delta} |(\varphi_n)|^{n(x)} dx + \int_{\Delta} \zeta(x)|(\varphi_n)|^{\kappa(x)} dx + \int_{\Delta} g(x, (\varphi_n)) dx \\ &\quad - \lambda \int_{\Delta} |(\varphi_n)|^{n(x)-2}(\varphi_n)\varphi dx - \int_{\Delta} \zeta(x)|(\varphi_n)|^{\kappa(x)-2} dx - \int_{\Delta} g(x, (\varphi_n)) dx. \end{aligned}$$

Combining (28) with the Strauss lemma (see Lemma 4), we can conclude that

$$\int_{\Delta} \mathcal{P}_n dx \rightarrow 0 \text{ when } n \rightarrow \infty. \tag{31}$$

Let us now consider the following sets:

$$\Delta_+ = \{x \in \Delta : \kappa(x) \geq 2\} \text{ and } \Delta_- = \{x \in \Delta : 1 < \kappa(x) < 2\}. \tag{32}$$

It follows from Lemma 3 that

$$\mathcal{P}_n(x) \geq \begin{cases} \frac{2^{3-\kappa_+}}{\kappa_+} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) - {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} & \text{if } \kappa(x) \geq 2 \\ (\kappa_- - 1) \frac{\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) - {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^2}{\left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right| + \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right| \right)^{2-\kappa(x)}} & \text{if } 1 < \kappa(x) < 2. \end{cases} \tag{33}$$

Consequently, we obtain

$$\int_{\Delta_+} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) - {}^H\mathcal{D}_+^{\gamma,\beta;\chi} \varphi \right|^{\kappa(x)} dx = o_n(1). \tag{34}$$

Now, by applying the Hölder inequality (see Proposition 5), we have

$$\int_{\Delta_-} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) - {}^H\mathcal{D}_+^{\gamma,\beta;\chi}\varphi \right|^{\kappa(x)} dx \leq C \|g_n\|_{L^{\frac{2}{\kappa(x)}(\Delta_-)}} \|s_n\|_{L^{\frac{2}{2-\kappa(x)}(\Delta_-)'}}$$

where

$$g_n(x) = \frac{\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) - {}^H\mathcal{D}_+^{\gamma,\beta;\chi}\varphi \right|^{\kappa(x)}}{\left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right| + \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}\varphi \right| \right)^{\frac{\kappa(x)(2-\kappa(x))}{2}}}$$

and

$$s_n(x) = \left(\left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) \right| + \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}\varphi \right| \right)^{\frac{\kappa(x)(2-\kappa(x))}{2}},$$

$C > 0$ (constant). In addition, by direct calculation we can see that

$$\left\{ \|s_n\|_{L^{\frac{2}{2-\kappa(x)}(\Delta_-)}} \right\}$$

is a bounded sequence and

$$\int_{\Delta_-} |g_n|^{\frac{2}{\kappa(x)}} dx \leq C \int_{\Delta_-} \mathcal{P}_n(x) dx.$$

Therefore, we have

$$\int_{\Delta} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}(\varphi_n) - {}^H\mathcal{D}_+^{\gamma,\beta;\chi}\varphi \right|^{\kappa(x)} dx \rightarrow 0 \text{ when } n \rightarrow \infty. \tag{35}$$

From Equations (31), (34), and (35), we deduce that $(\varphi_n) \rightarrow \varphi$ in $\mathcal{H}_{\kappa(x)}^{\gamma,\beta;\chi}(\Delta)$. We thus conclude the proof of Lemma 13. \square

Lemma 14. Under conditions (\mathbf{g}_1) and (\mathbf{H}_1) , there is a sequence $\{\mathbf{M}_m\} \subset (0, \infty)$ independent of λ with $\mathbf{M}_\lambda \leq \mathbf{M}_{m+1}$ such that, for all $\lambda > 0$,

$$c_m^\lambda = \inf_{K \in \Gamma_m} \max_{\varphi \in K} \mathcal{E}_\lambda(\varphi) < \mathbf{M}_m.$$

Proof. First, we observe that

$$\begin{aligned} c_m^\lambda &= \inf_{K \in \Gamma_m} \max_{\varphi \in K} \left\{ \int_{\Delta} \frac{1}{\kappa(x)} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}\varphi \right|^{\kappa(x)} dx - \int_{\Delta} \frac{\lambda}{n(x)} \|\varphi\|^{\kappa(x)} dx - \int_{\Delta} \mathbf{F}(x, \varphi) dx \right\} \\ &\leq \inf_{K \in \Gamma_m} \max_{\varphi \in K} \left\{ \int_{\Delta} \frac{1}{\kappa(x)} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}\varphi \right|^{\kappa(x)} dx - \int_{\Delta} \mathbf{F}(x, \varphi) dx \right\}. \end{aligned}$$

Let

$$\mathbf{M}_m = \inf_{K \in \Gamma_m} \max_{\varphi \in K} \left\{ \int_{\Delta} \frac{1}{\kappa(x)} \left| {}^H\mathcal{D}_+^{\gamma,\beta;\chi}\varphi \right|^{\kappa(x)} dx - \int_{\Delta} \mathbf{F}(x, \varphi) dx \right\} + 1.$$

Then, by the definition of the set Γ_m and by the properties of the infimum of a set, it follows that $\mathbf{M}_m \leq \mathbf{M}_{m+1}$. Therefore, as $\mathbf{F}(x, t) \leq c_1 + \epsilon_2|t|^{n(x)}$, we can conclude that $\mathbf{M}_m < \infty$, proving the result asserted by Lemma 14. \square

Finally, we prove the main result (Theorem 1) of this paper.

Proof. Proof of Theorem 1: First, λ_k for each $k \in \mathbb{N}$ such that

$$\mathbf{M}_k < \lambda_k^{1-\frac{N}{\kappa_+}} \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) S^N.$$

Thus, for $\lambda \in (\lambda_k, \lambda_{k+1}]$ we have

$$0 < \epsilon_1^\lambda \leq \epsilon_2^\lambda \leq \dots \leq \epsilon_k^\lambda < \mathbf{M}_k \leq \lambda_k^{1-\frac{N}{\kappa_+}} \left(\frac{1}{\kappa_+} - \frac{1}{n_-} \right) S^N.$$

Now, by Theorem 3, the levels provided by

$$\epsilon_1^\lambda \leq \epsilon_2^\lambda \leq \dots \leq \epsilon_k^\lambda$$

are the critical values of the functional $\mathcal{E}_\lambda(\cdot)$. Thus, if

$$\epsilon_1^\lambda < \epsilon_2^\lambda < \dots < \epsilon_k^\lambda,$$

the functional $\mathcal{E}_\lambda(\cdot)$ has at least k critical points, meaning that if $\epsilon_j^k = \epsilon_{j+1}^\lambda$ for some $j = 1, 2, \dots, k$, it follows from Theorem 3 that $K_{\epsilon_j^k}$ is an infinite set. Consequently, problem (2) has infinite solutions in this case. In either case, therefore, we can see that the problem (2) has at least k pairs of non-trivial solutions. Our proof of Theorem 1 is thus completed. \square

4. Application

In this section, we present an application of the investigated result.

First, let us consider $\kappa(x) = 2$, $n(x) = 2$, $\zeta(x) = 2$, $\chi(x) = x$ and $\beta = 1$ in Equation (2). Then, we have the following class of quasi-linear fractional-order problems:

$$\begin{cases} {}^c\mathcal{D}_-^\gamma ({}^c\mathcal{D}_+^\gamma \varphi) = (2 + \lambda)\varphi + \psi(x, \varphi) \\ \varphi = 0, \end{cases} \tag{36}$$

where $\Delta := [0, T] \times [0, T] \times [0, T]$ is a bounded domain with a smooth boundary and, for simplicity, ${}^c\mathcal{D}_+^\gamma(\cdot)$, ${}^c\mathcal{D}_-^\gamma(\cdot)$ are the Liouville-Caputo fractional derivatives of order γ ($\frac{1}{2} < \gamma < 1$) and $p, q : \bar{\Delta} \rightarrow \mathbb{R}$ are Lipschitz functions such that:

- (p1) $1 < \kappa_- \leq 2 \leq \kappa_+ < 3, \kappa_+ < n_- \leq 2 \leq \kappa_\gamma^*(x)$ for all $x \in \bar{\Delta}$ and
- (p2) The set $A = \{x \in \bar{\Delta} : 2 = p_\gamma^*(x)\}$ is not empty.

We now make several assumptions, which are detailed below.

Let $f : \bar{\Delta} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function provided by

$$f(x, t) = 2t + \psi(x, t)$$

with $a \in L^\infty(\Delta)$ and the function $g : \bar{\Delta} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following conditions:

- (g1) The function g is odd with respect to t , that is, $\psi(x, -t) = -\psi(x, t)$ for all $(x, t) \in \bar{\Delta} \times \mathbb{R}$, $\psi(x, t) = o(|t|^{\kappa(x)-1})$ when $|t| \rightarrow 0$ uniformly in x , and $\psi(x, t) = o(|t|^{n(x)-1})$ when $|t| \rightarrow \infty$ uniformly in x ;
- (g2) $\psi(x, t) \leq \frac{1}{\kappa_+} \psi(x, t)t$ for all $t \in \mathbb{R}$ and at almost every point $x \in \Delta$, where

$$\psi(x, t) = \int_0^t \psi(x, s) ds.$$

Furthermore, we make the following assumptions:

- **(H₁)** There exists $\bar{\gamma} > 0$ such that

$$\int_{\Delta} \left(\frac{1}{2} - 2|\varphi|^2 \right) dx \geq \bar{\gamma} \int_{\Delta} \frac{1}{2} |\varphi|^2 dx.$$

- **(H₂)** $2 = \kappa_+$ for all $x \in \Gamma = \{x \in \Delta : 2 > 0\}$.

Theorem 4. Suppose that a function g satisfies the conditions **(g₁)** and **(g₂)** and that the conditions **(p₁)**, **(p₂)**, **(H₁)** and **(H₂)** are also satisfied. Then there exists a sequence $\{\lambda_k\} \subset (0, +\infty)$ with $\lambda_k > \lambda_{k+1}$ for all $k \in \mathbb{N}$ such that, for $\lambda \in (\lambda_{k+1}, \lambda_k)$, the problem (36) has at least k pairs of non-trivial solutions.

Remark 4. We have presented an application of problem (2) in a particular case in the sense of the Liouville–Caputo fractional derivative. However, it is possible for particular choices of β and χ to obtain a new class of particular cases, especially when the limit $\gamma \rightarrow 1$.

5. Concluding Remarks and Observations

In the investigations presented in this paper, we have successfully addressed a problem involving the multiplicity of solutions for a class of fractional-order differential equations via the $\kappa(x)$ -Laplacian operator and the Genus Theory. We have first presented several definitions, lemmas and other preliminaries related to the problem. Applying these lemmas and other preliminaries, we have then studied the existence and multiplicity of solutions for a class of quasi-linear problems involving fractional differential equations in the χ -fractional space $\mathcal{H}_{\kappa(x)}^{\gamma, \beta; \chi}(\Delta)$ via the Genus Theory, the Concentration-Compactness Principle (CCP) and the Mountain Pass Theorem (MPT). We have considered a number of corollaries and consequences of the main results in this paper. On the other hand, although we have obtained several results in this paper, many open questions remain about the theory involving the χ -Hilfer fractional derivative. As presented in the introduction, the first work with m -Laplacian via the χ -Hilfer derivative was developed in 2019. It should be noted that there have been few further developments thus far. In this sense, several future questions need to be answered, in particular, those that are itemized below:

- Is it possible to discuss the results in Orlicz Spaces and Generalized Orlicz Spaces?
- Would it be possible to obtain the existence and multiplicity of solutions of Equation (2) unified with Kirchhoff's problem?
- Is it possible to extend these results to more general and global fractional-calculus operators?

Yet another possibility is to further extend this work to the distributed-order Hilfer fractional derivative and the ψ -Hilfer fractional derivative operators with variable exponents. For additional details, see [53,54], and (for recent developments) see [55], which is based upon the Riemann–Liouville, the Liouville–Caputo, and the Hilfer fractional derivatives).

As can be seen, it is a new area and there are many questions yet to be answered. Surely, this calls for the attention of researchers toward the discussing of new and complex problems.

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