

SOME FORMULAS INVOLVING q -JACOBI
AND RELATED POLYNOMIALS

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DM-443-IR

JUNE 1987

ABSTRACT

A number of linear and bilinear generating functions, and connection formulas, are proved for q -Jacobi polynomials and for various q -orthogonal polynomials associated with them.

Relationships between different q -extensions of the classical Gegenbauer (or ultraspherical) polynomials are also studied systematically.

1. INTRODUCTION AND NOTATIONS

For real or complex q , $|q| < 1$, put

$$(1.1) \quad (\lambda; q)_{\infty} = \prod_{j=0}^{\infty} (1 - \lambda q^j)$$

and let $(\lambda; q)_{\mu}$ be defined by

$$(1.2) \quad (\lambda; q)_{\mu} = \frac{(\lambda; q)_{\infty}}{(\lambda q^{\mu}; q)_{\infty}}$$

for arbitrary parameters λ and μ , so that

$$(1.3) \quad (\lambda; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1 - \lambda)(1 - \lambda q) \cdots (1 - \lambda q^{n-1}), & \forall n \in \{1, 2, 3, \dots\}. \end{cases}$$

Then a generalized basic (or q -) hypergeometric function is defined by (cf. [19, Chapter 3]; see also [8])

$$(1.4) \quad {}_r\phi_s \left[\begin{matrix} \alpha_1, \dots, \alpha_r; \\ \beta_1, \dots, \beta_s; \end{matrix} \middle| q, z \right] \\ = \sum_{n=0}^{\infty} (-1)^{(1-r+s)n} q^{(1-r+s)n(n-1)/2} \frac{(\alpha_1, \dots, \alpha_r; q)_n}{(\beta_1, \dots, \beta_s; q)_n} \frac{z^n}{(q; q)_n},$$

where (and throughout this paper) we find it to be convenient to write

$$(1.5) \quad (\lambda_1, \dots, \lambda_k; q)_\mu \stackrel{\text{def}}{=} (\lambda_1; q)_\mu \cdots (\lambda_k; q)_\mu,$$

and, for convergence of the infinite series in (1.4),

$$|q| < 1 \quad \text{and} \quad |z| < \infty \quad \text{when} \quad r \leq s,$$

or

$$\max\{|q|, |z|\} < 1 \quad \text{when} \quad r = s + 1,$$

provided that no zeros appear in the denominator.

In a series of memoirs on expansions of certain infinite products, L.J. Rogers ([15], [16], [17]) introduced the continuous q -Hermite polynomials $H_n(x|q)$ and the continuous q -ultraspherical polynomials $C_n(x; \nu|q)$, which are given explicitly by

$$(1.6) \quad H_n(\cos \theta |q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \cos(n-2k)\theta = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} e^{i(n-2k)\theta},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}$$

denotes the q -binomial coefficient, and

$$(1.7) \quad C_n(\cos \theta; \nu | q) = \sum_{k=0}^n \frac{(\nu; q)_k (\nu; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} \cos(n-2k)\theta$$

$$= \sum_{k=0}^n \frac{(\nu; q)_k (\nu; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} e^{i(n-2k)\theta}.$$

In fact, Rogers applied his results involving these polynomials to prove the celebrated Rogers–Ramanujan identities. The definition (1.7) provides a q -extension of the following well-known representation for the classical Gegenbauer (or ultraspherical) polynomials (see, e.g., [14, p. 283, Equation (37)]):

$$(1.8) \quad C_n^\nu(\cos \theta) = \sum_{k=0}^n \frac{(\nu)_k (\nu)_{n-k}}{k! (n-k)!} \cos(n-2k)\theta,$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$ in terms of Gamma functions.

The polynomials $C_n^\nu(x)$ are a subclass of the classical Jacobi polynomials and have the following important hypergeometric representations [14, p. 279, Equation (15); p. 280, Equation (19)]:

$$(1.9) \quad C_n^\nu(x) = \frac{(2\nu)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 2\nu+n; \\ \nu+\frac{1}{2}; \end{matrix} \frac{1-x}{2} \right]$$

and

$$(1.10) \quad C_n^\nu(x) = \frac{(\nu)_n}{n!} (2x)^n {}_2F_1 \left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}; \\ 1-\nu-n; \end{matrix} \frac{1}{x^2} \right]$$

(see also Gasper [11, p. 1066]).

In terms of the little q -Jacobi polynomials defined by (cf. [12, p. 29] and [3, p. 11, Equation (3.1)])

$$(1.11) \quad P_n^{(\alpha, \beta)}(x; q) = \frac{(\alpha q; q)_n}{(q; q)_n} {}_2\phi_1 \left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}; \\ \alpha q; \end{matrix} q, xq \right],$$

a q -extension of the definition (1.9) may be provided by

$$\frac{(\nu^2; q)_n}{(\nu\sqrt{q}; q)_n} P_n^{(\nu/\sqrt{q}, \nu/\sqrt{q})}(x; q).$$

On the other hand, using the big q -Jacobi polynomials defined by (cf. [4, p. 47, Equation (3.28)])

$$(1.12) \quad P_n^{(\alpha, \beta)}(x; \gamma, \delta; q) = \frac{(\alpha q, -\delta\alpha q/\gamma; q)_n}{(q, -q; q)_n} \left[\frac{\gamma}{\alpha q} \right]^n \cdot {}_3\phi_2 \left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, \alpha x q/\gamma; \\ \alpha q, -\delta\alpha q/\gamma; \end{matrix} q, q \right],$$

a q -extension of the definition (1.9) is given by

$$\begin{aligned}
(1.13) \quad C_n^\nu(x; q) &= \frac{(q, \nu^2; q)_n}{(\nu\sqrt{q}, -\nu\sqrt{q}; q)_n} P_n^{(\nu/\sqrt{q}, \nu/\sqrt{q})}(x; \nu/\sqrt{q}, \nu/\sqrt{q}; q) \\
&= \frac{(\nu^2; q)_n}{(q; q)_n} q^{-n} {}_3\phi_2 \left[\begin{matrix} q^{-n}, \nu^2 q^n, xq; \\ \nu\sqrt{q}, -\nu\sqrt{q}; \end{matrix} q, q \right],
\end{aligned}$$

which is, in fact, a q -extension of (1.10) as well (see Section 2).

The orthogonality relation for the q -ultraspherical polynomials (1.7) was proven by Askey and Ismail [5]. The orthogonality relations for the little as well as big q -Jacobi polynomials were proven by Andrews and Askey ([3, p. 13, Theorem 9] and [4, p. 48, Equation (3.30)], respectively), and their results can be suitably specialized to deduce the orthogonality relations for the aforementioned q -Gegenbauer polynomials stemming from (1.9) and (1.10). The object of the present paper is to derive several further properties (such as linear and bilinear generating functions, connection formulas, et cetera) for the big q -Jacobi polynomials (1.12) and for various q -orthogonal polynomials associated with them.

It may be of interest to remark in passing that, since [4, p. 49, Equation (3.36)]

$$\begin{aligned}
(1.14) \quad P_n^{(\alpha, \beta)}(x; 1, 0; q) &= (-1)^n P_n^{(\beta, \alpha)}(-x; 0, 1; q) \\
&= \frac{(-1)^n q^{n(n-1)/2}}{(-q; q)_n} P_n^{(\beta, \alpha)}(x; q),
\end{aligned}$$

and since [cf. Equation (1.13)]

$$(1.15) \quad f_n(x|q) = (q; q)_n \lim_{\nu \rightarrow 0} \left\{ (\sqrt{q}/\nu)^n C_n^\nu(\nu x/\sqrt{q}; q) \right\},$$

where ([1, p. 60, Equation (6.5)]; see also [8, p. 196, Equation (5.6.3.3)])

$$(1.16) \quad f_n(x|q) = \sum_{k=0}^{[n/2]} (-1)^k q^{k(k-1)} \frac{(q; q)_n}{(q^2; q^2)_k (q; q)_{n-2k}} x^{n-2k},$$

our results will apply also to the little q -Jacobi polynomials (1.11), and to the q -Hermite polynomials (1.16) considered, for example, by Al-Salam and Carlitz [1].

2. GENERATING FUNCTIONS FOR q -GEGENBAUER POLYNOMIALS

We begin by showing that the definition (1.13) is equivalent to the following q -extension of (1.10) considered in the literature (cf. [8, p. 208, Equation (5.9.1.9)]):

$$(2.1) \quad C_n^\nu(x; q) = \frac{(\nu^2; q^2)_n}{(q; q)_n} x^n {}_2\phi_1 \left[\begin{matrix} q^{-n}, q^{1-n}; \\ q^2, \frac{1}{x^2} \end{matrix} \middle| q^{2-2n}/\nu^2 \right].$$

Indeed, on writing the ${}_3\phi_2$ series in (1.13) in the reverse order, we find that

$$(2.2) \quad C_n^\nu(x; q) = \frac{(\nu^2; q^2)_n (xq; q)_n}{(q; q)_n} (-1)^n q^{-n(n+1)/2} \cdot {}_3\phi_2 \left[\begin{matrix} q^{-n}, \nu^{-1} q^{(1-2n)/2}, -\nu^{-1} q^{(1-2n)/2}; \\ \nu^{-2} q^{1-2n}, q^{-n}/x; \end{matrix} \middle| q, -\frac{1}{x} \right].$$

Now transform this last ${}_3\phi_2$ series by means of a known result [13, p. 957, Equation (3.1)]

$$(2.3) \quad {}_3\phi_2 \left[\begin{matrix} a, b, -b; \\ b^2, az; \end{matrix} \middle| q, -z \right] = \frac{(z; q)_\infty}{(az; q)_\infty} {}_2\phi_1 \left[\begin{matrix} a, aq; \\ b^2 q; \end{matrix} \middle| q^2, z^2 \right],$$

and (2.2) leads us to the representation (2.1).

Another q -hypergeometric representation for $C_n^\nu(x; q)$ would follow if we transform the ${}_2\phi_1$ series occurring in (2.1) by appealing to the formula [22, p. 370, Equation (3)]

$$(2.4) \quad {}_2\phi_1 \left[\begin{matrix} a, b; \\ e; \end{matrix} \middle| q, \frac{ec}{ab} \right] = \frac{(e/a, e/b; q)_\infty}{(e, e/ab; q)_\infty} {}_3\phi_2 \left[\begin{matrix} a, b, c; \\ abq/e, 0; \end{matrix} \middle| q, q \right],$$

$a \quad \text{or} \quad b \quad \text{or} \quad c = q^{-N} \quad (N = 0, 1, 2, \dots),$

and we thus have

$$(2.5) \quad C_n^\nu(x; q) = \frac{(\nu^2; q)_n}{(q; q)_n} q^{n(n-1)/2} {}_3\phi_2 \left[\begin{matrix} q^{-n}, q^{1-n}, \nu^2/(x^2 q); \\ \nu^2 q, 0; \end{matrix} \middle| q^2, q^2 \right]$$

or, equivalently,

$$(2.6) \quad C_n^\nu(x; q) = \frac{(\nu^2; q)_n}{(q; q)_n} x^n {}_3\phi_1 \left[\begin{matrix} q^{-n}, q^{1-n}, x^2 q^2; \\ \nu^2 q; \end{matrix} q^2, \left[\frac{\nu q^{n-1}}{x} \right]^2 \right],$$

which follows readily from (2.5) when we apply a particular case ($a=g=0$) of Sears's transformation [18, p. 167, Equation (8.3)]

$$(2.7) \quad {}_4\phi_3 \left[\begin{matrix} a, b, c, d; \\ e, g, h; \end{matrix} q, q \right] = \frac{(g/c, g/d, eg/ab, eg/abcd; q)_\infty}{(g, g/cd, eg/abc, eg/abd; q)_\infty} \cdot {}_4\phi_3 \left[\begin{matrix} e/a, e/b, c, d; \\ e, cdq/g, cdq/h; \end{matrix} q, q \right],$$

$$c \quad \text{or} \quad d = q^{-N} \quad (N = 0, 1, 2, \dots) \quad \text{and} \quad egh = abcdq.$$

These last representations (2.5) and (2.6) are q -extensions of a familiar expression for the classical Gegenbauer polynomials [14, p. 280, Equation (20)].

Next we turn to the derivation of the following generating functions for q -Gegenbauer polynomials:

$$(2.8) \quad \sum_{n=0}^{\infty} \frac{(\lambda; q)_n}{(\nu^2; q)_n} C_n^\nu(x; q) t^n = \frac{(\lambda xt; q)_\infty}{(xt; q)_\infty} {}_3\phi_3 \left[\begin{matrix} \lambda, \lambda q, x^2 q^2; \\ \nu^2 q, \lambda xt, \lambda xtq; \end{matrix} q^2, \frac{\nu^2 t^2}{q} \right],$$

which, for $\lambda = \nu^2$, immediately yields

$$(2.9) \quad \sum_{n=0}^{\infty} c_n^{\nu}(x; q) t^n = \frac{(\nu^2_{xt}; q)_{\infty}}{(xt; q)_{\infty}} {}_2\phi_2 \left[\begin{matrix} \nu^2, x^2 q^2; \\ \nu^2_{xt}, \nu^2_{xtq}; \end{matrix} q^2, \frac{\nu^2 t^2}{q} \right];$$

$$(2.10) \quad \sum_{n=0}^{\infty} c_n^{\nu}(x; q) \frac{t^n}{(\nu^2; q)_n} = \frac{1}{(xt; q)_{\infty}} {}_1\phi_1 \left[\begin{matrix} x^2 q^2; \\ \nu^2 q; \end{matrix} q^2, \frac{\nu^2 t^2}{q} \right],$$

which is an obvious special case of (2.8) when $\lambda = 0$;

$$(2.11) \quad \sum_{n=0}^{\infty} \frac{(-\nu\sqrt{q}; q)_n}{(\nu^2, \nu\sqrt{q}; q)_n} c_n^{\nu}(x; q) t^n \\ = {}_2\phi_1 \left[\begin{matrix} -\nu/x\sqrt{q}, 0; \\ \nu\sqrt{q}; \end{matrix} q, xt \right] {}_1\phi_1 \left[\begin{matrix} xq; \\ \nu\sqrt{q}; \end{matrix} q, \nu t\sqrt{q} \right],$$

which is independent of the generating function (2.8).

To prove the generating function (2.8), we make use of the representation (2.6) and change the order of summation. On evaluating the inner sum by means of the q -binomial theorem:

$$(2.12) \quad \sum_{n=0}^{\infty} \frac{(\lambda; q)_n}{(q; q)_n} z^n = \frac{(\lambda z; q)_{\infty}}{(z; q)_{\infty}} \quad (|q| < 1, |z| < 1),$$

we arrive at the right-hand side of (2.8). Formula (2.11), on the other hand, can be proven by applying the ${}_3\phi_2$ representation (1.13). Indeed, on transforming the ${}_3\phi_2$ series by means of (2.7) with $a = 0$, the

left-hand side of (2.11) becomes

$$\sum_{n=0}^{\infty} \frac{(-\nu/x\sqrt{q}; q)_n}{(q, \nu\sqrt{q}; q)_n} (xt)^n {}_3\phi_2 \left[\begin{matrix} q^{-n}, \nu^{-1}q^{(1-2n)/2}, xq; \\ \nu\sqrt{q}, -\nu^{-1}xq^{(1-2n)/2}; \end{matrix} q, -\nu q^{n+\frac{1}{2}} \right],$$

which, on simplification, yields the right-hand side of (2.11).

Formulas (2.8) to (2.11) provide q -extensions of various known generating functions for the classical Gegenbauer polynomials (see, e.g., [14, Chapter 17]; see also [6, p. 20, Equation (4.3); p. 21, Equation (4.6)]). Formula (2.11), in particular, is a q -extension of Bateman's generating function [14, p. 280, Equation (18)].

Lastly, we prove the following q -extension of a formula due to Feldheim [9, p. 128, Equation (33')]:

$$(2.13) \quad C_n^\nu(x; q) = \frac{(\nu^2; q)_n}{(\nu\sqrt{q}; q)_n} \sum_{k=0}^n \frac{(\nu/\mu\sqrt{q}; q)_k (-q; q)_{n-k}}{(q; q)_k (-\nu\sqrt{q}; q)_{n-k}} \cdot (\alpha q)^k P_{n-k}^{(\mu, \nu^2 q^{k-1}/\mu)}(x; \mu, \nu/\sqrt{q}; q).$$

Denoting, for convenience, the right-hand side of (2.13) by S , and making use of the definition (1.12), we have

$$\begin{aligned}
S &= \frac{(\nu^2; q)_n}{(\nu\sqrt{q}; q)_n} \sum_{k=0}^n \frac{(\nu/\mu\sqrt{q}; q)_k (\mu q; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} \mu^k q^{-n+k} \\
&\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{-n+k}, \nu^2 q^n, xq; \\ \mu q, -\nu\sqrt{q}; \end{matrix} \right]_{q, q} \\
&= \frac{(\nu^2; q)_n}{(\nu\sqrt{q}; q)_n} \sum_{k=0}^n \frac{(\nu/\mu\sqrt{q}; q)_{n-k} (\mu q; q)_k}{(q; q)_{n-k} (q; q)_k} \mu^{n-k} q^{-k} \\
&\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{-k}, \nu^2 q^n, xq; \\ \mu q, -\nu\sqrt{q}; \end{matrix} \right]_{q, q} \\
&= \frac{(\nu^2; q)_n}{(\nu\sqrt{q}; q)_n} \sum_{j=0}^n \frac{(\nu^2 q^n, xq; q)_j (\nu/\mu\sqrt{q}; q)_{n-j}}{(q, -\nu\sqrt{q}; q)_j (q; q)_{n-j}} (-1)^j \mu^{n-j} q^{-j(j+1)/2} \\
&\quad \cdot {}_2\phi_1 \left[\begin{matrix} \mu q^{1+j}, q^{-n+j}; \\ \mu\nu^{-1} q^{(3-2n+2j)/2}; \end{matrix} \right]_{q, \mu\nu^{-1} q^{(1-2j)/2}}.
\end{aligned}$$

Now sum the ${}_2\phi_1$ series by the q -Gauss theorem [19, p. 97, Equation (3.3.2.6)] and simplify the resulting expression to get the left-hand side of (2.13) via the representation (1.13).

3. A BILINEAR GENERATING FUNCTION FOR THE BIG q -JACOBI POLYNOMIALS

Associated with a bounded complex sequence $\{\Omega_n\}_{n=0}^{\infty}$, we prove the following bilinear generating function for the big q -Jacobi polynomials defined by (1.12):

$$\begin{aligned}
 (3.1) \quad & \sum_{\ell, m=0}^{\infty} \frac{(\gamma/x, \gamma/y; q)_{\ell} (-x/\delta, -y/\delta; q)_{m}}{(-\gamma/\delta, -\gamma/\delta; q)_{\ell+m} (q, \alpha q; q)_{\ell} (q, \beta q; q)_{m}} \Omega_{\ell+m} \left[\frac{xyt}{\beta \delta^2 q} \right]^{\ell} t^m \\
 &= \sum_{n=0}^{\infty} \frac{(\alpha \beta q, -\alpha \beta \delta q^2/\gamma, -\alpha \beta \delta q^2/\gamma, q, -q, -q; q)_{n}}{(\alpha q, \beta q, -\gamma/\delta, -\gamma/\delta, -\delta \alpha q/\gamma, -\delta \alpha q/\gamma; q)_{n} (\alpha \beta q; q)_{2n}} \\
 & \quad \cdot P_n^{(\alpha, \beta)}(x; \gamma, \delta; q) P_n^{(\alpha, \beta)}(y; \gamma, \delta; q) \left[\frac{t}{\beta \delta^2 q} \right]^n \sum_{\ell=0}^{\infty} \frac{\Omega_{n+\ell} t^{\ell}}{(q, \alpha \beta q^{2n+2}; q)_{\ell}},
 \end{aligned}$$

which, for $\Omega_n = (\lambda, \mu; q)_n$, would yield a q -extension of Feldheim's result for the classical Jacobi polynomials (cf. [10, p. 464, Equation (3.3)]; see also [21, p. 168, Problem 14(iii)]).

To prove the bilinear generating function (3.1), we set

$$a = \alpha \beta q, \quad b = \gamma/y, \quad c = \alpha q, \quad d = -\gamma/\delta, \quad e = \gamma/x, \quad \text{and} \quad f = -\gamma/\delta$$

in a known result [23, p. 945, Equation (3.1)], and make use of (2.7) with $a \rightarrow \infty$. In view of the definition (1.12), we thus find that

$$\begin{aligned}
(3.2) \quad & P_n^{(\alpha, \beta)}(x; \gamma, \delta; q) P_n^{(\alpha, \beta)}(y; \gamma, \delta; q) \\
&= \frac{(-\gamma/\delta, -\gamma/\delta, \beta q; q)_n}{(\alpha q, -\alpha\beta\delta q^2/\gamma, -\alpha\beta\delta q^2/\gamma; q)_n} q^{n(n+5)/2} \left[-\frac{\alpha^2 \beta \delta^2}{\gamma^2} \right]^n \\
&\cdot \sum_{\ell, m \geq 0} \frac{(q^{-n}, \alpha\beta q^{n+1}; q)_{\ell+m} (\gamma/x, \gamma/y; q)_\ell (-x/\delta, -y/\delta; q)_m}{(-\gamma/\delta, -\gamma/\delta; q)_{\ell+m} (q, \alpha q; q)_\ell (q, \beta q; q)_m} \left[\frac{xy}{\beta\delta^2} \right]^\ell q^m.
\end{aligned}$$

Now employing the formula (3.2) in the case $r = 2$ of another known result [20, p. 316, Equation (1.10)], we get (3.1).

Some consequences of the bilinear generating function (3.1) are worthy of note. First of all, since [cf. Equation (1.14)]

$$(3.3) \quad \lim_{\gamma \rightarrow 0} \left\{ P_n^{(\alpha, \beta)}(x; \gamma, -1; q) \right\} = \frac{(-1)^n q^{n(n-1)/2}}{(-q; q)_n} P_n^{(\alpha, \beta)}(x; q),$$

our result (3.1) for $\gamma \rightarrow 0$, $\delta = -1$, and

$$\Omega_n = (\lambda, \mu; q)_n \quad (n = 0, 1, 2, \dots),$$

would correspond to a known bilinear formula for the little q -Jacobi polynomials [24, p. 99, Equation (1.1)]. Furthermore, in view of (1.13), (3.1) for

$$\alpha = \beta = \gamma = \delta = \nu/\sqrt{q}$$

yields the following bilinear generating function for the q -Gegenbauer polynomials $C_n^\nu(x; q)$:

$$\begin{aligned}
 (3.4) \quad & \sum_{\ell, m=0}^{\infty} \frac{(\nu/x\sqrt{q}, \nu/y\sqrt{q}; q)_\ell (-x\sqrt{q}/\nu, -y\sqrt{q}/\nu; q)_m}{(-1, -1; q)_{\ell+m} (q, \nu\sqrt{q}; q)_\ell (q, \nu\sqrt{q}; q)_m} \Omega_{\ell+m} \left[\frac{xyt\sqrt{q}}{\nu^3} \right]^\ell t^m \\
 &= \sum_{n=0}^{\infty} \frac{(q, -\nu^2 q, -\nu^2 q; q)_n}{(-1, -1, \nu^2, \nu^2, \nu^2 q^n; q)_n} C_n^\nu(x; q) C_n^\nu(y; q) \left[\frac{t\sqrt{q}}{\nu^3} \right]^n \\
 & \quad \cdot \sum_{\ell=0}^{\infty} \frac{\Omega_{n+\ell} t^\ell}{(q, \nu^2 q^{n+1}; q)_\ell},
 \end{aligned}$$

which, for $\Omega_n = (-1, -1; q)_n$, reduces to the elegant form (cf. [7, p. 168, Corollary 6]):

$$\begin{aligned}
 (3.5) \quad & 2^{\Phi_1} \left[\begin{array}{c} \nu/x\sqrt{q}, \nu/y\sqrt{q}; \\ q, \frac{xyt\sqrt{q}}{\nu^3} \\ \nu\sqrt{q}; \end{array} \right] 2^{\Phi_1} \left[\begin{array}{c} -x\sqrt{q}/\nu, -y\sqrt{q}/\nu; \\ q, t \\ \nu\sqrt{q}; \end{array} \right] \\
 &= \sum_{n=0}^{\infty} \frac{(q, -\nu^2 q, -\nu^2 q; q)_n}{(\nu^2, \nu^2, \nu^2 q^n; q)_n} 2^{\Phi_1} \left[\begin{array}{c} -q^n, -q^n; \\ q, t \\ \nu^2 q^{n+1}; \end{array} \right] C_n^\nu(x; q) C_n^\nu(y; q) \left[\frac{t\sqrt{q}}{\nu^3} \right]^n.
 \end{aligned}$$

Lastly, in view of the limit relationship (1.15), we replace x , y , and t in (3.5) by

$$\nu x/\sqrt{q}, \quad \nu y/\sqrt{q}, \quad \text{and} \quad \nu t\sqrt{q},$$

respectively, and let $\nu \rightarrow 0$. We thus obtain the bilinear generating function:

$$(3.6) \quad \sum_{n=0}^{\infty} f_n(x|q) f_n(y|q) \frac{t^n}{(q; q)_n} = {}_2\phi_1 \left[\begin{matrix} 1/x, 1/y; \\ 0; \end{matrix} q, xyt \right],$$

which may be looked upon as a q -analogue of Mehler's formula for the classical Hermite polynomials.

4. CONNECTION FORMULAS

In this section we first prove the following connection formula for the big q -Jacobi polynomials:

$$(4.1) \quad P_n^{(\lambda, \mu)}(x; \nu, \delta \mu / \beta; q) = \sum_{k=0}^{[n/2]} A_{n,k} P_{n-2k}^{(\alpha, \beta)}(x; \gamma, \delta; q),$$

where the (connection) coefficients $A_{n,k}$ are given by

$$(4.2) \quad A_{n,k} = \frac{(\mu q, -\beta \nu q / \delta; q)_n (-q, \alpha \beta q; q)_{n-2k} (\lambda \mu q; q)_{2n-2k}}{(-q, \lambda \mu q; q)_n (q; q)_{2k} (\mu q, -\beta \nu q / \delta; q)_{n-2k} (\alpha \beta q; q)_{2n-4k}} \\ \cdot \left[\frac{\delta}{\beta} \right]_q^{2k} q^{2k(2k-n-1)} {}_4\phi_3 \left[\begin{matrix} -2k, \lambda \mu q^{2n-2k+1}, \beta q^{n-2k+1}, -\beta \gamma q^{n-2k+1} / \delta; \\ \alpha \beta q^{2n-4k+2}, \mu q^{n-2k+1}, -\beta \nu q^{n-2k+1} / \delta; \end{matrix} q, q \right].$$

Multiplying each member of (4.1) by

$$\frac{(xq/\gamma, -xq/\delta; q)_{\infty}}{(\alpha xq/\gamma, -\beta xq/\delta; q)_{\infty}} P_m^{(\alpha, \beta)}(x; \gamma, \delta; q) d_q x,$$

and integrating both sides of the resulting equation from $-\delta$ to γ , if we apply the known orthogonality relation [4, p. 48, Equation (3.30)], we readily find that

$$(4.3) \quad A_{n,k} h_{n-2k} = \int_{-\delta}^{\gamma} \frac{(xq/\gamma, -xq/\delta; q)_{\infty}}{(\alpha xq/\gamma, -\beta xq/\delta; q)_{\infty}} P_n^{(\lambda, \mu)}(x; \nu, \delta\mu/\beta; q) \cdot P_{n-2k}^{(\alpha, \beta)}(x; \gamma, \delta; q) d_q x,$$

where

$$(4.4) \quad h_n = \Delta(\gamma\delta)^n q^{n(n-1)/2} \frac{(1-\alpha\beta q)(\alpha q, \beta q, -\alpha\delta q/\gamma, -\beta\gamma q/\delta; q)_n}{(1-\alpha\beta q^{2n+1})(q, -q, -q, \alpha\beta q; q)_n}$$

and

$$(4.5) \quad \Delta = \frac{\gamma\delta(1-q)(q, \alpha\beta q^2, -\gamma/\delta, -\delta/\gamma; q)_{\infty}}{(\gamma+\delta)(\alpha q, \beta q, -\alpha\delta q/\gamma, -\beta\gamma q/\delta; q)_{\infty}}.$$

Now transform the ${}_3\phi_2$ series representing the q -Jacobi polynomials

$$P_n^{(\lambda, \mu)}(x; \nu, \delta\mu/\beta; q)$$

by appealing to (2.7) [with $a=0$] twice, and we thus find from (4.3) that

$$(4.6) \quad A_{n,k} = \frac{(\mu q, -\beta \nu q / \delta; q)_n (\alpha q, -\alpha \delta q / \gamma; q)_{n-2k}}{(q, -q; q)_n (q, -q; q)_{n-2k}} \left[-\frac{\delta}{\beta q} \right]^n \left[\frac{\gamma}{\alpha q} \right]^{n-2k} \\ \cdot \sum_{\ell=0}^n \frac{(q^{-n}, \lambda \mu q^{n+1}; q)_\ell}{(\mu q, -\beta \nu q / \delta; q)_\ell} \frac{q^\ell}{(q; q)_\ell} \sum_{j=0}^{n-2k} \frac{(q^{-n+2k}, \alpha \beta q^{n-2k+1}; q)_j}{(\alpha q, -\alpha \delta q / \gamma; q)_j} \frac{q^j}{(q; q)_j} I(\alpha q^j, \beta q^\ell),$$

where, for convenience,

$$(4.7) \quad I(\alpha, \beta) = \int_{-\delta}^{\gamma} \frac{(-xq/\gamma, -xq/\delta; q)_\infty}{(\alpha xq/\gamma, -\beta qx/\delta; q)_\infty} d_q x.$$

The value of this last integral in (4.7) is known to be the constant Δ given by (4.5), as already observed by Andrews and Askey [4, p. 47, Equation (3.27)]. Consequently, (4.6) becomes

$$(4.8) \quad A_{n,k} = \frac{(\mu q, -\beta \nu q / \delta; q)_n (-q, \alpha \beta q; q)_{n-2k} (\alpha \beta q^2; q)_{2n-4k}}{(q, -q; q)_n (\beta q, -\beta \gamma q / \delta; q)_{n-2k} (\alpha \beta q; q)_{2n-4k}} \left[-\frac{1}{\alpha \beta} \right]^n (\alpha \delta)^{2k} \\ \cdot q^{-n-(n-2k)(n-2k+1)/2} \sum_{\ell=0}^n \frac{(q^{-n}, \beta q, -\beta \gamma q / \delta, \lambda \mu q^{n+1}; q)_\ell}{(\alpha \beta q^2, -\beta \nu q / \delta, \mu q; q)_\ell} \frac{q^\ell}{(q; q)_\ell} \\ \cdot {}_2\phi_1 \left[\begin{matrix} q^{-n+2k}, \alpha \beta q^{n-2k+1}; \\ \alpha \beta q^{\ell+2}; \end{matrix} \middle| q, q \right].$$

Summing this ${}_2\phi_1$ series by a q -extension of the familiar Chu-Vandermonde theorem [19, p. 97, Equation (3.3.2.7)], and simplifying the resulting expression, we are led eventually to the second member of (4.2).

We now consider some special or limiting cases of the connection formula (4.1). Indeed, in view of the limit relationship (3.3), (4.1) with $\gamma \rightarrow 0$, $\nu \rightarrow 0$, and $\delta = -1$ yields the following connection formula for the little q -Jacobi polynomials:

$$(4.9) \quad P_n^{(\lambda, \mu)}(\beta x / \mu; q) = \sum_{k=0}^{[n/2]} B_{n,k} P_{n-2k}^{(\alpha, \beta)}(x; q),$$

where

$$(4.10) \quad B_{n,k} = \frac{(\mu q; q)_n (\alpha \beta q; q)_{n-2k} (\lambda \mu q; q)_{2n-2k}}{(\lambda \mu q; q)_n (\mu q; q)_{n-2k} (\alpha \beta q; q)_{2n-4k} (q; q)_{2k}} \\ \cdot \beta^{-2k} q^{k(6k-6n-1)} {}_3\phi_2 \left[\begin{matrix} -2k, \lambda \mu q^{2n-2k+1}, \beta q^{n-2k+1}; \\ \alpha \beta q^{2n-4k+2}, \mu q^{n-2k+1}; \end{matrix} ; q, q \right].$$

Formula (4.9) differs markedly from a connection formula given earlier by Andrews and Askey [3, p. 14, Theorem 10].

Next we replace each of the parameters λ , μ , and ν in (4.1) and (4.2) by μ/\sqrt{q} , set

$$\alpha = \beta = \gamma = \delta = \nu/\sqrt{q},$$

and make use of the relationship (1.13) between q -Gegenbauer polynomials and the big q -Jacobi polynomials. Summing the resulting ${}_4\phi_3$ series in (4.2) by a q -extension of Watson's theorem [2, p. 333, Theorem 1], we thus obtain

$$(4.11) \quad C_n^\nu(x; q) = \sum_{k=0}^{[n/2]} \frac{(1-\nu^2 q^{2n-4k}) (\mu^2; q^2)_{n-k} (\mu^2/\nu^2; q^2)_k}{(1-\nu^2) (\nu^2 q^2; q^2)_{n-k} (q^2; q^2)_k} \left[\frac{\nu^2}{q} \right]^k C_{n-2k}^\nu(x; q),$$

which is, of course, equivalent to the following well-known result due to Rogers (cf. [16]; see also [5] and [6]):

$$(4.12) \quad C_n(x; \mu|q) = \sum_{k=0}^{[n/2]} \frac{(1-\nu q^{n-2k}) (\mu; q)_{n-k} (\mu/\nu; q)_k}{(1-\nu) (\nu q; q)_{n-k} (q; q)_k} \nu^k C_{n-2k}(x; \nu|q).$$

Acknowledgements

The present investigation was carried out at the University of Victoria while the second author was on academic leave from Bareilly College (Rohilkhand University). This work was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant A-7353. The second author wishes to thank the University Grants Commission of India for providing financial assistance.

REFERENCES

- [1] W.A. Al-Salam and L. Carlitz, Some orthogonal q -polynomials, *Math. Nachr.* 30(1965), 47-61.
- [2] G.E. Andrews, On q -analogues of the Watson and Whipple summations, *SIAM J. Math. Anal.* 7(1976), 332-336.
- [3] G.E. Andrews and R. Askey, Enumeration of partitions: The role of Eulerian series and q -orthogonal polynomials, in Higher Combinatorics (M. Aigner, Ed.), pp. 3-26, D. Reidel Publishing Company, Dordrecht and Boston, 1977.
- [4] G.E. Andrews and R. Askey, Classical orthogonal polynomials, in Polynômes Orthogonaux et Applications (C. Brezinski, A. Draux, A.P. Magnus, P. Maroni et A. Ronveaux, Eds.), pp. 36-62, Springer-Verlag, Berlin, Heidelberg and New York, 1985.
- [5] R. Askey and M. Ismail, A generalization of ultraspherical polynomials, in Studies in Pure Mathematics (P. Erdős, Ed.), pp. 55-78, Birkhäuser, Basel, 1983.
- [6] R. Askey and J. Wilson, Some Basic Hypergeometric Orthogonal Polynomials that Generalize Jacobi Polynomials, *Mem. Amer. Math. Soc.*, Vol. 54(No. 319), Amer. Math. Soc., Providence, Rhode Island, 1985.
- [7] D.M. Bressoud, Linearization and related formulas for q -ultraspherical polynomials, *SIAM J. Math. Anal.* 12(1981), 161-168.
- [8] H. Exton, q -Hypergeometric Functions and Applications, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Brisbane, Chichester and Toronto, 1983.
- [9] E. Feldheim, Relations entre les polynômes de Jacobi, Laguerre et Hermite, *Acta Math.* 74(1941), 117-138.
- [10] E. Feldheim, Contributions à la theorie des polynômes de Jacobi, *Mat. Fiz. Lapok* 48(1941), 453-504.
- [11] G. Gasper, Rogers' linearization formula for the continuous q -ultraspherical polynomials and quadratic transformation formulas, *SIAM J. Math. Anal.* 16(1985), 1061-1071.
- [12] W. Hahn, Über Orthogonalpolynome, die q -Differenzgleichungen, *Math. Nachr.* 2(1949), 4-34.
- [13] V.K. Jain, Some transformations of basic hypergeometric functions. Part II, *SIAM J. Math. Anal.* 12(1981), 957-961.

- [14] E.D. Rainville, Special Functions, Macmillan, New York, 1960.
- [15] L.J. Rogers, On the expansion of some infinite products, Proc. London Math. Soc. 24(1893), 337-352.
- [16] L.J. Rogers, Second memoir on the expansion of certain infinite products, Proc. London Math. Soc. 25(1894), 318-343.
- [17] L.J. Rogers, Third memoir on the expansion of certain infinite products, Proc. London Math. Soc. 26(1895), 15-32.
- [18] D.B. Sears, On the transformation theory of basic hypergeometric functions, Proc. London Math. Soc. (2) 53(1951), 158-180.
- [19] L.J. Slater, Generalized Hypergeometric Functions, Cambridge Univ. Press, Cambridge, London and New York, 1966.
- [20] H.M. Srivastava, Certain q-polynomial expansions for functions of several variables, IMA J. Appl. Math. 30(1983), 315-323.
- [21] H.M. Srivastava and H.L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.
- [22] A. Verma, Certain summation formulae for basic hypergeometric series, Canad. Math. Bull. 20(1977), 369-375.
- [23] A. Verma and V.K. Jain, Some transformations of basic hypergeometric functions. Part I, SIAM J. Math. Anal. 12(1981), 943-956.
- [24] A. Verma and V.K. Jain, Self-reproducing kernels and bilinear formulas for q-orthogonal polynomials, Rocky Mountain J. Math. 13(1983), 99-116.

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