CERTAIN CLASSES OF MULTIVALENT FUNCTIONS OF ORDER α AND TYPE β

by

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ABSTRACT

Let $\mathscr{S}_p^{\lambda}(\alpha,\beta)$ denote the class of p–valent λ–spiral–like functions of order α and type β. Also let $\mathscr{C}_p^{\lambda}(\alpha,\beta)$ denote the class of p–valent λ–Robertson functions of order α and type β. In the present paper, the authors give several representation formulas, distortion theorems, and coefficient bounds for functions belonging to these subclasses of analytic and p–valent functions in the open unit disk. They also obtain the sharp radius of starlikeness for the class $\mathscr{S}_p^{\lambda}(\alpha,\beta)$ and the sharp radius of convexity for the class $\mathscr{C}_p^{\lambda}(\alpha,\beta)$. Finally, some important open problems involving the various function classes considered in this paper are pointed out.

1. INTRODUCTION

For a fixed positive integer p, let $\mathscr{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n , \tag{1.1}$$

which are analytic and p–valent in the open unit disk

$$\mathscr{U} = \{z : |z| < 1\}.$$

For $|\lambda| < \pi/2$ and p a fixed positive integer, let $\mathscr{S}_p^{\lambda}(\alpha)$ denote the class of functions $f(z) \in \mathcal{A}_p$ which satisfy the inequality:

$$\mathrm{Re}\left\{e^{i\lambda}\,\frac{zf'(z)}{f(z)}\right\} > \alpha\cos\lambda \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some α $(0 \leqq \alpha < p)$. Following Patil and Thakare [20], we say that the functions belonging to the class $\mathcal{S}_p^{\lambda}(\alpha)$ are p–valent λ–spiral–like functions of order α.

Furthermore, for $|\lambda| < \pi/2$ and p a fixed positive integer, let $\mathcal{C}_p^{\lambda}(\alpha)$ denote the class of functions $f(z) \in \mathcal{A}_p$ which satisfy the inequality:

$$\mathrm{Re}\left\{e^{i\lambda}\left[1 + \frac{zf''(z)}{f'(z)}\right]\right\} > \alpha\cos\lambda \qquad (z \in \mathcal{U}) \tag{1.3}$$

for some α $(0 \leqq \alpha < p)$. We say that the functions belonging to the class $\mathcal{C}_p^{\lambda}(\alpha)$ are p–valent λ–Robertson functions of order α.

It follows from (1.2) and (1.3) that

$$f(z) \in \mathcal{C}_p^{\lambda}(\alpha) \iff \frac{zf'(z)}{p} \in \mathcal{S}_p^{\lambda}(\alpha). \tag{1.4}$$

Motivated by a number of recent works ([11],[17],[1]), we introduce here the concept of *type* for the classes $\mathcal{S}_p^{\lambda}(\alpha)$ and $\mathcal{C}_p^{\lambda}(\alpha)$.

Definition 1. A function $f(z) \in \mathcal{A}_p$ is said to be a p–valent λ–spiral–like function of order α and type β, that is, $f(z) \in \mathcal{S}_p^{\lambda}(\alpha,\beta)$, if and only if the inequality

$$\left|\frac{\frac{zf'(z)}{f(z)} - p}{2\beta\left[\frac{zf'(z)}{f(z)} - p + (p-\alpha)\,e^{-i\lambda}\cos\lambda\right] - \left[\frac{zf'(z)}{f(z)} - p\right]}\right| < 1 \tag{1.5}$$

holds true for some λ, α, and β $(|\lambda| < \pi/2; 0 \leqq \alpha < p; 0 < \beta \leqq 1)$, and for all $z \in \mathcal{U}$.

Definition 2. A function $f(z) \in \mathcal{A}_p$ is said to be a p–valent λ–Robertson function of order α and type β, that is, $f(z) \in \mathcal{C}_p^\lambda(\alpha,\beta)$, if and only if the inequality

$$\left| \frac{\left[1 + \frac{z f''(z)}{f'(z)} - p\right]}{2\beta\left[1 + \frac{zf''(z)}{f'(z)} - p + (p-\alpha)\, e^{-i\lambda} \cos \lambda\right] - \left[1 + \frac{zf''(z)}{f'(z)} - p\right]} \right| < 1 \qquad (1.6)$$

holds true for some λ, α, and β $(|\lambda| < \pi/2;\ 0 \leqq \alpha < p;\ 0 < \beta \leqq 1)$, and for all $z \in \mathcal{U}$.

It follows immediately from Definition 1 and Definition 2 that

$$f(z) \in \mathcal{C}_p^\lambda(\alpha,\beta) \quad \Leftrightarrow \quad \frac{z f'(z)}{p} \in \mathcal{S}_p^\lambda(\alpha,\beta)\, . \qquad (1.7)$$

We note that $\mathcal{S}_1^\lambda(\alpha,\beta) = \mathcal{S}^\lambda(\alpha,\beta)$ is the class of λ–spiral–like functions of order α and type β $(|\lambda| < \pi/2;\ 0 \leqq \alpha < 1;\ 0 < \beta \leqq 1)$ which was studied earlier by Mogra and Ahuja [12]. On the other hand, $\mathcal{C}_1^\lambda(\alpha,\beta) = \mathcal{C}^\lambda(\alpha,\beta)$ is the class of λ–Robertson functions of order α and type β $(|\lambda| < \pi/2;\ 0 \leqq \alpha < 1;\ 0 < \beta \leqq 1)$ which was studied earlier by Ahuja [1].

Furthermore, by specializing the parameters λ, α, β, and p, we obtain the following subclasses studied by various earlier authors. We should like to refer, in this connection, to the excellent treatises on the subject of univalent functions by Pommerenke [21], Goodman [10], and Duren [6].

(i) $\mathcal{S}_p^\lambda(\alpha,1) = \mathcal{S}_p^\lambda(\alpha)$, $\mathcal{C}_p^\lambda(\alpha,1) = \mathcal{C}_p^\lambda(\alpha)$; $\mathcal{S}_p^\lambda\left[\alpha, \frac{2M-1}{2M}\right] = \mathcal{F}_M(\lambda,\alpha,p)$

and $\mathscr{C}_p^\lambda\left[\alpha, \frac{2M-1}{2M}\right] = \mathscr{G}_M(\lambda,\alpha,p)$ $\left[M > \frac{1}{2}\right]$ (Aouf [2]);

(ii) $\mathscr{S}_1^\lambda(\alpha,1) = \mathscr{S}^\lambda(\alpha)$ (Libera [14]);

(iii) $\mathscr{S}_1^\lambda\left[0, 1 - \frac{1}{2}\cos\lambda\right] = \mathscr{H}(\lambda)$ (Goel [7]);

(iv) $\mathscr{S}_1^0(\alpha,\beta) = \mathscr{S}^*(\alpha,\beta)$ (Juneja and Mogra [11]);

(v) $\mathscr{C}_1^\lambda(\alpha,1) = \mathscr{C}^\lambda(\alpha)$ (Chichra [5] and Sizuk [24]);

(vi) $\mathscr{C}_1^\lambda(1,1) = \mathscr{C}^\lambda$ (Robertson [23], Libera and Ziegler [15], and Bajpai and Mehrok [3]);

(vii) $\mathscr{S}_1^\lambda\left[0, \frac{2M-1}{2M}\right] = \mathscr{F}_{\lambda,M}$ and $\mathscr{C}_1^\lambda\left[0, \frac{2M-1}{2M}\right] = \mathscr{G}_{\lambda,M}$ $\left[M > \frac{1}{2}\right]$ (Kulshrestha [13]);

(viii) $\mathscr{S}_1^\lambda\left[\frac{1-\beta+2\alpha\beta}{1+\beta}, \frac{1+\beta}{2}\right] = \mathscr{S}_{\alpha,\beta}^\lambda$ (Makówka [16]);

(ix) $\mathscr{S}_1^0\left[\frac{1-\beta}{1+\beta}, \frac{1+\beta}{2}\right] = \mathscr{S}(\beta)$ (Padmanabhan [19]);

(x) $\mathscr{S}_1^\lambda(0,1) = \mathscr{S}^\lambda$ (Špaček [25] and Zamorski [27]),

and

(xi) $\mathscr{S}_1^0\left[\alpha, \frac{1}{2}\right] = \overline{\mathscr{S}}_\alpha$ (Wright [26]).

Since our classes $\mathscr{S}_p^\lambda(\alpha,\beta)$ and $\mathscr{C}_p^\lambda(\alpha,\beta)$ include various known subclasses, as observed above, a systematic study of their properties will lead to a unified study of these subclasses. In the present paper, we first give a representation formula for each of the classes $\mathcal{S}_p^\lambda(\alpha,\beta)$ and $\mathcal{C}_p^\lambda(\alpha,\beta)$. We deal next with the distortion properties and coefficient bounds for functions belonging to these classes. Finally, we obtain the sharp radius of starlikeness for $\mathcal{S}_p^\lambda(\alpha,\beta)$ and the sharp radius of convexity for $\mathcal{C}_p^\lambda(\alpha,\beta)$. Our results for the general classes $\mathcal{S}_p^\lambda(\alpha,\beta)$ and $\mathcal{C}_p^\lambda(\alpha,\beta)$ can indeed be applied to derive the corresponding known as well as new properties and characteristics for many of their aforementioned subclasses.

It may be of interest to remark in passing that Goodman's work [8] happens to be the forerunner of all of the aforementioned papers. In fact, it was the basis for much of the later developments where more parameters were added.

2. REPRESENTATION FORMULAS

Lemma 1. *Let the function* $f(z)$ *be defined by* (1.1). *Then* $f(z) \in \mathcal{S}_p^\lambda(\alpha,\beta)$ *if and only if it can be represented as follows:*

(i)
$$f(z) = z^p\left[\frac{f_1(z)}{z}\right]^p \qquad (z \in \mathcal{U}) \tag{2.1}$$

for

$$f_1(z) \in \mathcal{S}^\lambda\left[\frac{\alpha}{p},\beta\right];$$

(ii)
$$f(z) = z^p\left[\frac{f_2(z)}{z}\right]^{p\,e^{-i\lambda}\cos\lambda} \qquad (z \in \mathcal{U}) \tag{2.2}$$

for

$$f_2(z) \in \mathscr{S}^*\left[\frac{\alpha}{p}, \beta\right].$$

Proof. (i) By direct computation, we find from the representation (2.1) that

$$\frac{\frac{z f'(z)}{f(z)} - p}{2\beta\left[\frac{z f'(z)}{f(z)} - p + (p-\alpha)e^{-i\lambda}\cos\lambda\right] - \left[\frac{z f'(z)}{f(z)} - p\right]}$$

$$= \frac{\frac{z f_1'(z)}{f_1(z)} - 1}{2\beta\left[\frac{z f_1'(z)}{f_1(z)} - 1 + \left[1 - \frac{\alpha}{p}\right]e^{-i\lambda}\cos\lambda\right] - \left[\frac{z f_1'(z)}{f_1(z)} - 1\right]}. \tag{2.3}$$

Now the assertion (i) follows from (2.3) in view of Definition 1 and of its special case when $p = 1$.

(ii) By direct computation, we find from the representation (2.2) that

$$\frac{\frac{z f'(z)}{f(z)} - p}{2\beta\left[\frac{z f'(z)}{f(z)} - p + (p-\alpha)e^{-i\lambda}\cos\lambda\right] - \left[\frac{z f'(z)}{f(z)} - p\right]}$$

$$= \frac{\frac{z f_2'(z)}{f_2(z)} - 1}{2\beta\left[\frac{z f_2'(z)}{f_2(z)} - \frac{\alpha}{p}\right] - \left[\frac{z f_2'(z)}{f_2(z)} - 1\right]}.$$

Again, making use of Definition 1 and its special case when $p - 1 = \lambda = 0$, we arrive at the assertion (ii) of Lemma 1.

With a view to stating some known results required in our investigation, let $\mathcal{Q}$ denote the class of functions $\varphi(z)$ which are analytic in $\mathcal{U}$ and satisfy the inequality:

$$|\varphi(z)| \leqq 1 \qquad (\forall\, z \in \mathcal{U}). \tag{2.4}$$

We first state

Lemma 2 (Mogra and Ahuja [17, p. 146, Theorem 1]). *A function* $f(z)$ *defined by* (2.3) *is in the class* $\mathcal{S}^{\lambda}(\alpha,\beta)$ *if and only if*

$$\frac{zf'(z)}{f(z)} = \frac{1 + \{(2\beta-1) - 2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}z\,\varphi(z)}{1 + (2\beta-1)z\,\varphi(z)} \qquad (z \in \mathcal{U}) \tag{2.5}$$

or, equivalently,

$$f(z) = z\exp\left[-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\int_0^z \frac{\varphi(t)}{1 + (2\beta-1)t\,\varphi(t)}\,dt\right] \quad (z \in \mathcal{U}) \tag{2.6}$$

for some function $\varphi(z) \in \mathcal{Q}$.

An immediate consequence of Lemma 1(i) and the assertion (2.5) of Lemma 2 is

Lemma 3. *A function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{S}_p^{\lambda}(\alpha,\beta)$ *if and only if*

$$\frac{zf'(z)}{f(z)} = \frac{p + \{p(2\beta-1) - 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda\}z\,\varphi(z)}{1 + (2\beta-1)z\,\varphi(z)} \qquad (z \in \mathcal{U}) \tag{2.7}$$

for some function $\varphi(z) \in \mathcal{Q}$.

Next we prove

Lemma 4. *If a function*

$$g(z) = p + \sum_{n=1}^{\infty} b_n z^n , \tag{2.8}$$

analytic in $\mathcal{U}$, *satisfies the inequality:*

$$\left| \frac{g(z) - p}{2\beta\{g(z) - p + (p-\alpha)e^{-i\lambda} \cos \lambda\} - \{g(z) - p\}} \right| < 1, \tag{2.9}$$

for $|\lambda| < \pi/2$, $0 \leqq \alpha < p$, $0 < \beta \leqq 1$, *and for all* $z \in \mathcal{U}$, *then*

$$g(z) = \frac{p + \{p(2\beta-1) - 2\beta(p-\alpha)e^{-i\lambda} \cos \lambda\}z\ \varphi(z)}{1 + (2\beta-1)z\ \varphi(z)} \tag{2.10}$$

for some $\varphi(z) \in \mathcal{Q}$. *Conversely, a function* $g(z)$ *given by* (2.10) *for some* $\varphi(z) \in \mathcal{Q}$ *is analytic in* $\mathcal{U}$ *and satisfies* (2.4) *for all* $z \in \mathcal{U}$.

Proof. The first part of Lemma 4 is obtained immediately by an application of Schwarz's lemma [18]. The second part follows from the observation that the function

$$w = \frac{p + \{p(2\beta-1) - 2\beta(p-\alpha)e^{-i\lambda} \cos \lambda\}z}{1 + (2\beta-1)z} \tag{2.11}$$

maps $\mathscr{U}$ onto the disk

$$\left| \frac{p - w}{2\beta\{w - p + (p-\alpha)e^{-i\lambda} \cos \lambda\} - (w-p)} \right| < 1 \tag{2.12}$$

in the w–plane.

Yet another representation formula for functions in the class $\mathscr{S}_p^\lambda(\alpha,\beta)$ is given by

Theorem 1. *A function* $f(z)$, *defined by* (1.1) *and analytic in* $\mathscr{U}$, *is in the class* $\mathscr{S}_p^\lambda(\alpha,\beta)$ *if and only if*

$$f(z) = z^p \exp\left[-2\beta(p-\alpha)e^{-i\lambda} \cos \lambda \int_0^z \frac{\varphi(t)}{1 + (2\beta-1)t\,\varphi(t)}\,dt\right] \quad (z \in \mathscr{U}) \tag{2.13}$$

for some $\varphi(z) \in \mathscr{Q}$.

Proof. First suppose that $f(z) \in \mathscr{S}_p^\lambda(\alpha,\beta)$. Noting that $zf'(z)/f(z)$ satisfies the hypothesis of the first part of Lemma 4, we see that

$$\frac{z f'(z)}{f(z)} = \frac{p + \{p(2\beta-1) - 2\beta(p-\alpha)e^{-i\lambda} \cos \lambda\}z\,\varphi(z)}{1 + (2\beta-1)z\,\varphi(z)} \tag{2.14}$$

for some function $\varphi(z) \in \mathscr{Q}$. It is easily observed from (2.14) that

$$\frac{f'(z)}{f(z)} - \frac{p}{z} = -\frac{2\beta(p-\alpha)e^{-i\lambda}\,\varphi(z) \cos \lambda}{1 + (2\beta-1)z\,\varphi(z)}. \tag{2.15}$$

Upon integrating both sides of (2.15) from 0 to z, if we exponentiate the resulting equation, we obtain the representation formula (2.13).

Conversely, if (2.13) holds true, then

$$\frac{z\,f'(z)}{f(z)} = \frac{p + \{p(2\beta-1) - 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda\}z\,\varphi(z)}{1 + (2\beta-1)z\,\varphi(z)} \quad (z \in \mathcal{U};\ \varphi(z) \in \mathcal{Q}). \quad (2.16)$$

Now Theorem 1 follows by appealing to the second part of Lemma 4.

Alternatively, Theorem 1 can be proven by applying Lemma 1(i) to the integral representation (2.6) asserted by Lemma 2.

In view of the relationship (1.7), it is not difficult to deduce from the above results the following representation formulas for functions belonging to the class $\mathcal{C}_p^{\lambda}(\alpha,\beta)$:

COROLLARY 1. *A function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{C}_p^{\lambda}(\alpha,\beta)$ *if and only if its derivative* $f'(z)$ *can be represented as follows*:

(i)
$$f'(z) = p\,z^{p-1}[f_1'(z)]^p \quad (2.17)$$

for

$$f_1'(z) \in \mathcal{C}^{\lambda}\left[\frac{\alpha}{p}, \beta\right];$$

(ii)
$$f'(z) = p\,z^{p-1}[f_2'(z)]^{p\,e^{-i\lambda}\cos\lambda} \quad (2.18)$$

for

$$f_2(z) \in \mathcal{C}_1^{0}\left[\frac{\alpha}{p}, \beta\right] = \mathcal{C}\left[\frac{\alpha}{p}, \beta\right];$$

(iii)
$$f'(z) = p\,z^{p-1}\exp\left[-2\beta(p-\alpha)e^{-i\lambda}\cos\lambda\int_0^z \frac{\varphi(t)}{1 + (2\beta-1)t\,\varphi(t)}\,dt\right]$$

for some function $\varphi(z) \in \mathcal{Q}$.

For various choices of the parameters involved in Theorem 1 and Corollary 1, we can obtain the corresponding representation formulas for functions in the simpler classes described above.

3. A SUFFICIENT CONDITION

We now establish a sufficient condition for a function to be in each of the classes $\mathcal{S}_p^\lambda(\alpha,\beta)$ and $\mathcal{C}_p^\lambda(\alpha,\beta)$.

Theorem 2. *Let the function* $f(z)$ *defined by* (1.1) *be analytic in* $\mathcal{U}$. *Then* $f(z) \in \mathcal{S}_p^\lambda(\alpha,\beta)$ *if, for some* λ *and* α $(|\lambda| < \pi/2;\ 0 \leqq \alpha < p)$,

$$\sum_{n=p+1}^{\infty} \left\{2n(1-\beta) - p + |p(1-2\beta) + 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda|\right\} |a_n|$$

$$\leqq 2\beta(p-\alpha)\cos\lambda\,, \tag{3.1}$$

whenever $0 < \beta \leqq \frac{1}{2}$, *and*

$$\sum_{n=p+1}^{\infty} \left\{(n-p) + |(2\beta-1)(n-p) + 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda|\right\} |a_n|$$

$$\leqq 2\beta(p-\alpha)\cos\lambda, \tag{3.2}$$

whenever $\frac{1}{2} \leqq \beta \leqq 1$.

Proof. Let $|z| = r < 1$. Noting that

$$|zf'(z) - pf(z)| < \sum_{n=p+1}^{\infty} (n-p)|a_n| rp, \tag{3.3}$$

and

$$\Big|2\beta[zf'(z) - pf(z) + (p-\alpha)e^{-i\lambda}f(z)\cos\lambda] - [zf'(z) - pf(z)]\Big|$$

$$> \Big\{2\beta(p-\alpha)\cos\lambda - \sum_{n=p+1}^{\infty} (1-2\beta)n\,|a_n| - \sum_{n=p+1}^{\infty} \Big|p(1-2\beta) + 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda\Big|\,|a_n|\Big\} rp, \tag{3.4}$$

we see that

$$\Big|zf'(z) - pf(z)\Big| - \Big|2\beta[zf'(z) - pf(z) + (p-\alpha)e^{-i\lambda}f(z)\cos\lambda] - [zf'(z) - pf(z)]\Big|$$

$$< \left[\sum_{n=p+1}^{\infty} \Big\{2n(1-\beta)-p + \Big|p(1-2\beta)+2\beta(p-\alpha)e^{-i\lambda}\cos\lambda\Big|\Big\}|a_n| - 2\beta(p-\alpha)\cos\lambda\right] rp, \tag{3.5}$$

provided that $0 < \beta \leqq \frac{1}{2}$. The right–hand side of (3.5) is non–positive by (3.1), so that $f(z) \in \mathscr{S}_p^{\lambda}(\alpha,\beta)$ by Definition 1.

For the second part, we assume that (3.2) holds true for $\frac{1}{2} \leqq \beta \leqq 1$. In this case, we observe that

$$\left|2\beta[zf'(z) - pf(z) + (p-\alpha)e^{-i\lambda}f(z)\cos\lambda] - [zf'(z) - pf(z)]\right|$$

$$> \left\{2\beta(p-\alpha)\cos\lambda - \sum_{n=p+1}^{\infty} \left|(2\beta-1)(n-p) + 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda\right| |a_n|\right\} rp. \tag{3.6}$$

Making use of (3.3), (3.6), and (3.2), we complete the proof of Theorem 2.

Corollary 2. *Let the function* $f(z)$ *defined by* (1.1) *be analytic in* $\mathscr{U}$. *Then* $f(z)$ *is in the class* $\mathscr{C}_p^{\lambda}(\alpha,\beta)$ *if, for some* λ *and* α ($|\lambda| < \pi/2$; $0 \leqq \alpha < p$),

$$\sum_{n=p+1}^{\infty} \frac{n}{p}\left\{2n(1-\beta) - p + |p(1-2\beta) + 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda|\right\} |a_n|$$

$$\leqq 2\beta(p-\alpha)\cos\lambda, \tag{3.7}$$

whenever $0 < \beta \leqq \frac{1}{2}$, *and*

$$\sum_{n=p+1}^{\infty} \frac{n}{p}\left\{(n-p) + |(2\beta-1)(n-p) + 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda|\right\} |a_n|$$

$$\leqq 2\beta(p-\alpha)\cos\lambda, \tag{3.8}$$

whenever $\frac{1}{2} \leqq \beta \leqq 1$.

Proof. Since

$$\frac{z\,f'(z)}{p} = z^p + \sum_{n=p+1}^{\infty} \frac{n}{p}\, a_n\, z^n, \tag{3.9}$$

by replacing a_n by $(n/p)a_n$ in Theorem 2, we immediately have Corollary 2 in view of the equivalence relation (1.7).

For various choices of the parameters involved in Theorem 2 and Corollary 2, we can obtain the corresponding results for functions belonging to the numerous simpler classes described in Section 1.

4. DISTORTION THEOREMS

Theorem 3. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $\mathscr{S}_p^{\lambda}(\alpha,\beta)$, *then*

$$|f(z)| \leqq r^p \left[\frac{\{1 + (2\beta - 1)r\}^{1-\cos\lambda}}{\{1 - (2\beta - 1)r\}^{1+\cos\lambda}} \right]^{\{\beta(p-\alpha)/(2\beta-1)\}\cos\lambda} \tag{4.1}$$

and

$$|f(z)| \geqq r^p \left[\frac{\{1 - (2\beta - 1)r\}^{1-\cos\lambda}}{\{1 + (2\beta - 1)r\}^{1+\cos\lambda}} \right]^{\{\beta(p-\alpha)/(2\beta-1)\}\cos\lambda} \tag{4.2}$$

for $|z| = r$ $(0 < r < 1)$, $|\lambda| < \pi/2$, $0 \leqq \alpha < p$, *and* $\beta \in (0,\frac{1}{2}) \cup (\frac{1}{2},1]$; *and*

$$|f(z)| \leqq r^p \exp[(p-\alpha) r \cos \lambda] \tag{4.3}$$

and

$$|f(z)| \geqq r^p \exp[-(p-\alpha) r \cos \lambda] \tag{4.4}$$

for $|z| = r \ (0 < r < 1)$, $|\lambda| < \pi/2$, $0 \leqq \alpha < p$, *and* $\beta = \frac{1}{2}$. *The estimates* (4.1) *through* (4.4) *are sharp for all admissible values of* λ, α, β, *and* p.

Proof. Since $f(z) \in \mathscr{S}_p^{\lambda}(\alpha,\beta)$, the condition (1.5) in conjunction with the Schwarz lemma [18] implies that

$$\left| \frac{zf'(z)}{f(z)} - \xi \right| < R,$$

where

$$\xi = \frac{p - (2\beta-1)[p(2\beta-1) - 2\beta(p-\alpha) \cos^2\lambda] r^2 - i\beta(2\beta-1)(p-\alpha)r^2 \sin 2\lambda}{1 - (2\beta-1)^2 r^2}$$

and

$$R = \frac{2\beta(p-\alpha) r \cos \lambda}{1 - (2\beta-1)^2 r^2} \qquad (|z| = r).$$

Hence we have

$$\frac{p - 2\beta(p-\alpha) r \cos \lambda + (2\beta-1)[2\beta(p-\alpha) \cos^2\lambda - p(2\beta-1)] r^2}{1 - (2\beta-1)^2 r^2} \leqq \operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\}$$

$$\leqq \frac{p + 2\beta(p-\alpha) r \cos \lambda + (2\beta-1)[2\beta(p-\alpha) \cos^2\lambda - p(2\beta-1)]r^2}{1 - (2\beta-1)^2\, r^2} . \tag{4.5}$$

Observing that

$$\log\left[\left|\frac{f(z)}{z^p}\right|\right] = \mathrm{Re}\left[\log\frac{f(z)}{z^p}\right] = \mathrm{Re}\left[\int_0^z \left[\frac{f'(s)}{f(s)} - \frac{p}{s}\right] ds\right]$$

$$= \int_0^r t^{-1}\, \mathrm{Re}\left[t\, e^{i\theta}\, \frac{f'(te^{i\theta})}{f(te^{i\theta})} - p\right] dt ,$$

and applying (4.5), we find that

$$\log\left[\left|\frac{f(z)}{z^p}\right|\right] \leqq [2\beta(p-\alpha) \cos \lambda] \int_0^r \frac{1 + (2\beta-1)t \cos \lambda}{1 - (2\beta-1)^2\, t^2}\, dt . \tag{4.6}$$

Now suppose that $|\lambda| < \pi/2$, $0 \leqq \alpha < p$, and $\beta \in (0,\frac{1}{2}) \cup (\frac{1}{2},1]$. Then (4.6) yields

$$\log\left[\left|\frac{f(z)}{z^p}\right|\right] \leqq \frac{\beta(p-\alpha) \cos \lambda}{2\beta - 1} \log \left\{\frac{\{1 + (2\beta-1)r\}^{1-\cos \lambda}}{\{1 - (2\beta-1)r\}^{1+\cos \lambda}}\right\} ,$$

which leads us to (4.1). For the case when $|\lambda| < \pi/2$, $0 \leqq \alpha < p$, and $\beta = \frac{1}{2}$, (4.6) immediately gives (4.3).

In view of the fact that

$$\log\left[\left|\frac{f(z)}{z^p}\right|\right] = \mathrm{Re}\left\{\log\frac{f(z)}{z^p}\right\} = \int_0^r \mathrm{Re}\left\{\frac{\partial}{\partial t}\left[\log\frac{f(t)}{t^p}\right]\right\} dt = \int_0^r t^{-1}\, \mathrm{Re}\left\{\frac{t f'(t)}{f(t)} - p\right\} dt,$$

and with the aid of (4.5), we may write

$$\log\left[\left|\frac{f(z)}{z^p}\right|\right] \geqq -[2\beta(p-\alpha)\cos\lambda]\int_0^r \frac{1-(2\beta-1)t\cos\lambda}{1-(2\beta-1)^2t^2}\,dt\,. \tag{4.7}$$

If $\beta \neq \frac{1}{2}$, then (4.2) follows upon evaluating the integral in (4.7). If, on the other hand, $\beta = \frac{1}{2}$, then we immediately get (4.4) from (4.7).

The extremal function for all of the inequalities is given by

$$f(z) = \begin{cases} z^p\left[1-(2\beta-1)z\,e^{i\theta}\right]^{-\{2\beta(p-\alpha)/(2\beta-1)\}e^{-i\lambda}\cos\lambda} & \left[\beta \neq \frac{1}{2}\right] \\ \\ z^p \exp](p-\alpha)z\,e^{i(\theta-\lambda)}\cos\lambda] & \left[\beta = \frac{1}{2}\right] \end{cases} \tag{4.8}$$

where $|\lambda| < \pi/2$ and $0 \leqq \alpha < p$; and $\theta(0 \leqq \theta \leqq 2\pi)$ is determined by the equation:

$$\tan\left[\frac{\theta}{2}\right] = \left\{\frac{1-(2\beta-1)r}{1+(2\beta-1)r}\right\}\cot\left[\frac{\pi}{2}-\frac{\lambda}{2}\right] \tag{4.9}$$

for the equality in (4.1) and (4.3), and by the equation:

$$\tan\left[\frac{\theta}{2}\right] = \left\{\frac{1-(2\beta-1)r}{1+(2\beta-1)r}\right\}\cot\left[-\frac{\lambda}{2}\right] \tag{4.10}$$

for the equality in (4.2) and (4.4).

Corollary 3. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $\mathscr{C}_p^{\lambda}(\alpha,\beta)$, *then*

$$|f'(z)| \leqq pr^{p-1}\left[\frac{\{1+(2\beta-1)r\}^{1-\cos\lambda}}{\{1-(2\beta-1)r\}^{1+\cos\lambda}}\right]^{\{\beta(p-\alpha)/(2\beta-1)\}\cos\lambda} \tag{4.11}$$

and

$$|f'(z)| \geqq pr^{p-1}\left[\frac{\{1-(2\beta-1)r\}^{1-\cos\lambda}}{\{1+(2\beta-1)r\}^{1+\cos\lambda}}\right]^{\{\beta(p-\alpha)/(2\beta-1)\}\cos\lambda} \tag{4.12}$$

for $|z| = r < 1,\ |\lambda| < \pi/2,\ 0 \leqq \alpha < p,$ *and* $\beta \in (0,\tfrac{1}{2}) \cup (\tfrac{1}{2},1];$ *and*

$$|f'(z) \leqq pr^{p-1}\exp[(p-\alpha)r\cos\lambda] \tag{4.13}$$

and

$$|f'(z)| \geqq pr^{p-1}\exp[-(p-\alpha)r\cos\lambda] \tag{4.14}$$

for $|z| = r < 1,\ |\lambda| < \pi/2,\ 0 \leqq \alpha < p,$ *and* $\beta = \frac{1}{2}$.

The function $f(z)$ *given by*

$$f'(z) = \begin{cases} pz^{p-1}[1-(2\beta-1)z\,e^{i\theta}]^{-\{2\beta(p-\alpha)/(2\beta-1)\}e^{-i\lambda}\cos\lambda} & (\beta \neq \frac{1}{2}) \\ pz^{p-1}\exp[(p-\alpha)z\,e^{i(\theta-\lambda)}\cos\lambda] & (\beta = \frac{1}{2}) \end{cases} \tag{4.15}$$

provides equality in (4.11) *and* (4.13) *when* θ *is given by Equation* (4.9). *The function* $f(z)$

given by (4.15) *also provides equality in* (4.12) *and* (4.14) *when* θ *is given by Equation* (4.10).

For various choices of the parameters in Theorem 3 and Corollary 3, the corresponding known or new results can be deduced for the function classes studied earlier in the literature.

5. COEFFICIENT BOUNDS

Let Ω denote the class of bounded analytic functions $w(z)$ in $\mathcal{U}$ which satisfy the conditions $w(0) = 0$ and $|w(z)| \leqq |z|$ for $z \in \mathcal{U}$. We shall require the following result in our investigation:

Lemma 4 (Keogh and Merkes [12, p. 10, Equation (7)]). *Let the function* $w(z)$ *defined by*

$$w(z) = \sum_{n=1}^{\infty} c_n z^n \tag{5.1}$$

be in the class Ω. *Then*

$$|c_2 - \nu c_1^2| \leqq \max\{1, |\nu|\} . \tag{5.2}$$

for any complex number ν. *Equality in* (5.2) *may be attained with the functions* $w(z) = z^2$ *and* $w(z) = z$ *for* $|\nu| < 1$ *and* $|\nu| \geqq 1$, *respectively.*

Theorem 4. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $\mathscr{S}_p^{\lambda}(\alpha,\beta)$, $\beta \neq \frac{1}{2}$, *and if* μ *is any complex number, then*

$$|a_{p+2} - \mu a_{p+1}^2| \leq [\beta(p-\alpha)\cos\lambda] \max\left\{1, \, |2\beta(p-\alpha)(2\mu-1)\cos\lambda - (2\beta-1)e^{i\lambda}|\right\}. \tag{5.3}$$

The result (5.3) *is sharp for each* μ.

Proof. Since $f(z) \in \mathscr{S}_p^{\lambda}(\alpha,\beta)$, setting $z\,\varphi(z) = w(z)$ in the representation (2.14), we have

$$\frac{z\,f'(z)}{f(z)} = \frac{p + \{p(2\beta-1) - 2\beta(p-\alpha)\,e^{-i\lambda}\cos\lambda\}w(z)}{1 + (2\beta-1)\,w(z)} \tag{5.4}$$

where $w(z)$ defined by (5.1) is in the class Ω. Rewriting (5.4) in the form:

$$w(z) = \frac{z f'(z) - p\,f(z)}{\{p(2\beta-1) - 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda\}f(z) - (2\beta-1)z\,f'(z)}, \tag{5.5}$$

and applying the definition (1.1), it can be shown that

$$w(z) = -\frac{e^{i\lambda}\sec\lambda}{2\beta(p-\alpha)}\,a_{p+1}\,z - \frac{e^{i\lambda}\sec\lambda}{2\beta(p-\alpha)}\left[2a_{p+2} - \frac{2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + 2\beta-1}{2\beta(p-\alpha)\,e^{-i\lambda}\cos\lambda}\,a_{p+1}^2\right]z^2 + \cdots . \tag{5.6}$$

Now compare the coefficients of z and z^2 on both sides of (5.6), using the definition (5.1). We thus obtain

$$c_1 = -\frac{e^{i\lambda}\ \sec\ \lambda}{2\beta(p-\alpha)}\ a_{p+1} \tag{5.7}$$

and

$$c_2 = -\frac{e^{i\lambda}\ \sec\ \lambda}{\beta(p-\alpha)}\ a_{p+2} + \frac{e^{i\lambda}\ \sec\ \lambda}{4\beta^2(p-\alpha)^2}\left\{2\beta(p-\alpha) + (2\beta-1)e^{i\lambda}\ \sec\ \lambda\right\} a_{p+1}^2 . \tag{5.8}$$

Consequently, we have

$$a_{p+1} = -\frac{2\beta(p-\alpha)}{\sec\ \lambda}\ c_1\ e^{-i\lambda} \tag{5.9}$$

and

$$a_{p+2} = -\frac{\beta(p-\alpha)}{\sec\ \lambda}\ c_2\ e^{-i\lambda} + \left[\frac{2\beta(p-\alpha)\ +\ (2\beta-1)e^{i\lambda}\ \sec\ \lambda}{4\beta(p-\alpha)}\right] a_{p+1}^2\ . \tag{5.10}$$

Using (5.2), (5.9) and (5.10), we readily obtain (5.3).

Finally, the assertion (5.3) of Theorem 4 is sharp in view of the fact that the assertion (5.2) of Lemma 4 is sharp.

Corollary 4. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{S}_p^{\lambda}(\alpha,\beta)$, $\beta \neq \frac{1}{2}$, *then*

$$|a_{p+1}| \leqq 2\beta(p-\alpha)\cos\ \lambda, \tag{5.11}$$

and

$$|a_{p+2}| \leqq [\beta(p-\alpha)\cos\lambda]\max\left\{1,\ |2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (2\beta-1)|\right\}. \tag{5.12}$$

The bounds in (5.11) *and* (5.12) *are attained by the function* $f(z)$ *defined by*

$$f(z) = z^p[1-(2\beta-1)z]^{-\{2\beta(p-\alpha)/(2\beta-1)\}e^{-i\lambda}\cos\lambda} \quad \left[\beta \neq \frac{1}{2}\right]. \tag{5.13}$$

Proof. The assertions (5.11) and (5.12) of Corollary 4 follow directly from (5.9) and (5.3), respectively.

Corollary 5. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathscr{C}_p^{\lambda}(\alpha,\beta)$, $\beta \neq \frac{1}{2}$, *and if* μ *is any complex number, then*

$$|a_{p+2} - \mu a_{p+1}^2| \leqq \left[\frac{p(p-\alpha)\beta\cos\lambda}{p+2}\right] \cdot \max\left\{1,\ \left|(2\beta-1) - 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda\left[\frac{2p(p+2)\mu - (p+1)^2}{(p+1)^2}\right]\right|\right\}. \tag{5.14}$$

The result (5.14) *is sharp for each* μ.

Proof. In view of the equivalence relationship (1.7), Corollary 5 follows easily from Theorem 4.

Corollary 6. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathscr{C}_p^{\lambda}(\alpha,\beta)$, $\beta \neq \frac{1}{2}$, *then*

$$|a_{p+1}| \leqq \frac{2\beta\ p(p-\alpha)}{p+1} \cos\lambda \tag{5.15}$$

and

$$|a_{p+2}| \leqq \frac{p(p-\alpha)\,\beta\ \cos\lambda}{p+2} \max\left\{1,\ |(2\beta-1) + 2\beta(p-\alpha)e^{-i\lambda} \cos\lambda|\right\}. \tag{5.16}$$

The bounds in (5.15) *and* (5.16) *are attained by the function* $f(z)$ *defined by*

$$f'(z) = p\, z^{p-1}[1-(2\beta-1)z]^{-\{2\beta(p-\alpha)/(2\beta-1)\}e^{-i\lambda}\cos\lambda} \left[\beta \neq \frac{1}{2}\right]. \tag{5.17}$$

We now state and prove

Lemma 5. *If* p *and* m *are positive integers,* $|\lambda| < \pi/2$, $0 \leqq \alpha < p$, *and* $0 < \beta \leqq 1$, *then*

$$\prod_{j=0}^{m-1} \frac{|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + j(2\beta-1)|^2}{(j+1)^2} = \frac{1}{m^2}\left\{4\beta^2(p-\alpha)^2\cos^2\lambda\right.$$

$$+ \sum_{k=1}^{m-1}\left[|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + k(2\beta-1)|^2 - k^2\right]$$

$$\left.\cdot \prod_{j=0}^{k-1} \frac{|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + j(2\beta-1)|^2}{(j+1)^2}\right\}. \tag{5.18}$$

Proof. For $m = 1$, the assertion (5.18) is obvious. Thus, in order to prove Lemma 5 by induction on m, suppose that the result (5.18) holds true for $m = q-1$ $(q = 1,2,3,\cdots)$. Then, observing that

$$\frac{1}{q^2}\left\{4\beta^2(p-\alpha)^2\cos^2\lambda + \sum_{k=1}^{q-1}\left[|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + k(2\beta-1)|^2 - k^2\right]\right.$$

$$\left.\cdot\prod_{j=0}^{k-1}\frac{|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + j(2\beta-1)|^2}{(j+1)^2}\right\}$$

$$= \frac{1}{q^2}\left\{4\beta^2(p-\alpha)^2\cos^2\lambda + \sum_{k=1}^{q-2}\left[|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + k(2\beta-1)|^2 - k^2\right]\right.$$

$$\cdot\prod_{j=0}^{k-1}\frac{|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + j(2\beta-1)|^2}{(j+1)^2}$$

$$+\left[|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (q-1)(2\beta-1)|^2 - (q-1)^2\right]$$

$$\left.\cdot\prod_{j=0}^{q-2}\frac{|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + j(2\beta-1)|^2}{(j+1)^2}\right\}$$

$$= \left[\frac{q-1}{q}\right]^2\prod_{j=0}^{q-2}\frac{|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + j(2\beta-1)|^2}{(j+1)^2}$$

$$+\frac{\left[|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (q-1)(2\beta-1)|^2 - (q-1)^2\right]}{q^2}$$

$$\cdot \prod_{j=0}^{q-2} \frac{|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + j(2\beta-1)|^2}{(j+1)^2}$$

$$= \prod_{j=0}^{q-2} \frac{|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + j(2\beta-1)|^2}{(j+1)^2}$$

$$\cdot \frac{|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (q-1)(2\beta-1)|^2}{q^2}$$

$$= \prod_{j=0}^{q-2} \frac{|2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + j(2\beta-1)|^2}{(j+1)^2}, \tag{5.19}$$

we conclude that the result (5.18) is valid also for $m = q$. This evidently completes the proof of Lemma 5.

THEOREM 5. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathscr{S}_p^{\lambda}(\alpha,\beta)$. *If*

$$\beta(p-\alpha)(k+p-\alpha)\cos^2\lambda > (1-\beta)k[k+(p-\alpha)\cos^2\lambda] \tag{5.20}$$

and

$$N = \left[\frac{\beta(p-\alpha)(k+p-\alpha)\cos^2\lambda}{(1-\beta)\,k[k+(p-\alpha)\cos^2\lambda]}\right] \quad (k = 1,2,\cdots,n-p-1), \tag{5.21}$$

then

$$|a_n| \leqq \frac{1}{(n-p)!} \prod_{k=0}^{n-p-1} |2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + k(2\beta-1)| \tag{5.22}$$

$$(n = p+1,\ p+2,\ \cdots,\ p+N)$$

and

$$|a_n| \leqq \frac{1}{(p+N)!\,(n-p)} \prod_{k=0}^{p+N} |2\beta(p-\alpha)e^{-i\lambda} \cos \lambda + k(2\beta-1)| \quad (n > p+N). \tag{5.23}$$

If, on the other hand,

$$\beta(p-\alpha)(k+p-\alpha) \cos^2\lambda \leqq (1-\beta)\, k[k + (p-\alpha) \cos^2\lambda], \tag{5.24}$$

then

$$|a_n| \leqq \frac{2\beta(p-\alpha) \cos \lambda}{n - p} \quad (n = p+1, p+2, p+3, \cdots). \tag{5.25}$$

The bounds in (5.22) *and* (5.25) *are sharp for all admissible* λ, α, β, *and* p, *and for each integer* $n \geqq p + 1$.

Proof. Since $f(z) \in \mathscr{S}_p^{\lambda}(\alpha,\beta)$, (5.4) gives

$$\left\{(2\beta-1)zf'(z) - [(2\beta-1)p - 2\beta(p-\alpha)e^{-i\lambda} \cos \lambda]f(z)\right\} w(z)$$

$$= pf(z) - zf'(z), \qquad w(z) \in \Omega\ . \tag{5.26}$$

Rewriting (5.26) in the form:

$$\left\{(2\beta-1)\left[p\,z^p + \sum_{n=1}^{\infty} (p+n)a_{p+n}\,z^{p+n}\right] - \left[(2\beta-1)p - 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda\right]\right.$$

$$\left.\cdot\left[z^p + \sum_{n=1}^{\infty} a_{p+n}\,z^{p+n}\right]\right\} w(z)$$

$$= p\left[z^p + \sum_{n=1}^{\infty} a_{p+n}\,z^{p+n}\right] - \left[p\,z^p + \sum_{n=1}^{\infty} (p+n)a_{p+n}\,z^{p+n}\right] \tag{5.27}$$

or, equivalently,

$$\left\{2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + \sum_{n=1}^{\infty}\left[2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (2\beta-1)n\right]a_{p+n}\,z^n\right\} w(z)$$

$$= -\sum_{n=1}^{\infty} n\,a_{p+n}\,z^n, \tag{5.28}$$

we find that

$$\sum_{n=0}^{\infty}\left\{\left[2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (2\beta-1)n\;a_{p+n}\,z^n\right\} w(z) = -\sum_{n=0}^{\infty} n\,a_{p+n}\,z^n, \tag{5.29}$$

where $a_p = 1$ and $w(z)$ is given, as before, by (5.1).

Equating the coefficients of z^m on both sides of (5.29), we obtain

$$\sum_{n=0}^{m-1}\left[2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (2\beta-1)n\right]a_{p+n}\,c_{m-n} = -m\,a_{p+m}, \tag{5.30}$$

which shows that a_{p+m} on the right–hand side depends only on the coefficients

$$a_p,\ a_{p+1},\ \cdots,\ a_{p+m-1}$$

occurring on the left–hand side. Hence we can write

$$\sum_{n=0}^{m-1} \left\{\left[2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (2\beta-1)n\right] a_{p+n} z^n\right\} w(z)$$

$$= -\sum_{n=0}^{m} n\, a_{p+n} z^n + \sum_{n=m+1}^{\infty} A_n z^n \tag{5.31}$$

for $m = 1,2,3,\cdots,$ and for a proper choice of A_n $(n \geqq 0)$.

Let $z = re^{i\theta}$, $0 < r < 1$, $0 \leqq \theta \leqq 2\pi$. Then

$$\sum_{n=0}^{m-1} |2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (2\beta-1)n|^2\, |a_{p+n}|^2\, r^{2n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{n=0}^{m-1} \left\{2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (2\beta-1)n\right\} a_{p+n}\, r^n e^{in\theta}\right|^2 d\theta$$

$$\geqq \frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{n=0}^{m-1} \left\{2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + (2\beta-1)n\right\} a_{p+n}\, r^n e^{in\theta}\right|^2 |w(r\,e^{i\theta})|^2\, d\theta$$

$$\geqq \frac{1}{2\pi}\int_0^{2\pi} \left| -\sum_{n=0}^{m} n\, a_{p+n}\, r^n e^{in\theta} + \sum_{n=m+1}^{\infty} A_n r^n e^{in\theta}\right|^2 d\theta$$

$$\geqq \sum_{n=0}^{m} n^2 \, |a_{p+n}|^2 \, r^{2n} + \sum_{n=m+1}^{\infty} |A_n|^2 \, r^{2n}$$

$$\geqq \sum_{n=0}^{m} n^2 \, |a_{p+n}|^2 \, r^{2n} . \tag{5.32}$$

Letting $r \to 1$ in (5.32), we obtain

$$\sum_{n=0}^{m-1} \left[\left| 2\beta(p-\alpha) e^{-i\lambda} \cos \lambda + (2\beta-1) n \right|^2 - n^2 \right] |a_{p+n}|^2 \geqq m^2 \, |a_{p+m}|^2 . \tag{5.33}$$

Setting $m = k - p$ in (5.33), we are led finally to the inequality:

$$(k-p)^2 \, |a_k|^2 \leqq 4\beta^2 (p-\alpha)^2 \cos^2 \lambda$$

$$+ \sum_{n=1}^{k-p-1} \left[\left| 2\beta(p-\alpha) e^{-i\lambda} \cos \lambda + (2\beta-1) n \right|^2 - n^2 \right] |a_{p+n}|^2 \tag{5.34}$$

$$(k = p+1, \ p+2, \ p+3, \cdots).$$

The following two cases will now arise:

Case 1. Let the inequality (5.20) hold true. Suppose also that $k \leqq p + N$ in (5.34), where N is given by (5.21). Then, for $k = p + 1$, (5.34) immediately yields

$$|a_{p+1}| \leqq 2\beta(p-\alpha) \cos \lambda , \tag{5.35}$$

which proves (5.22) for $n = p + 1$.

Setting $k = p + 2$ in (5.34), and making use of (5.35), it is not difficult to verify that (5.22) holds true also for $n = p + 2$.

The general result (5.22) for

$$n = p+1,\ p+2,\ \cdots,\ p+N \tag{5.36}$$

can be established by applying the above technique successively. For a fixed integer $q \geqq 2$, suppose that (5.22) holds true for

$$n = p+1,\ p+2,\ \cdots,\ p+q-1 \qquad (q = 2,3,\cdots,N-1).$$

Then, applying (5.22) (with $k = p+q$) and Lemma 5 suitably, we can easily show that (5.22) holds true also for $n = p + q$, thus completing the proof of the first part of Theorem 5.

With a view to proving the assertion (5.23) of Theorem 5, suppose that $n > p + N$, where N is given by (5.21). Then, retaining only the terms of the series on the right–hand side of (5.34) from $n = 1$ to $n = N$, we have

$$\begin{aligned}(k-p)^2 |a_k|^2 \leqq{} & 4\beta^2(p-\alpha)^2 \cos^2\lambda \\ & + \sum_{n=1}^{N} \left[|2\beta(p-\alpha)e^{-i\lambda} \cos\lambda + (2\beta-1)n|^2 - n^2 \right] |a_{p+n}|^2 ,\end{aligned} \tag{5.37}$$

where we have obviously dropped the non–negative terms from $n = N + 1$ to $n = k - p - 1$.

Now we substitute in (5.37) the upper bounds for

$$|a_{p+1}|,\ |a_{p+2}|,\ \cdots,\ |a_{p+N}|,$$

given by (5.22), and the assertion (5.23) follows upon simplifying the resulting equation.

Case 2. Let the inequality (5.24) hold true. Then it follows from (5.34) that

$$(k-p)^2\ |a_k|^2 \leqq 4\beta^2(p-\alpha)^2 \cos^2\lambda \quad (k = p+1, p+2, p+3, \cdots),$$

which evidently proves the assertion (5.25) of Theorem 5.

The bound in (5.22) is sharp for the function $f(z)$ given by (5.13), and the bounds in (5.25) are sharp for the functions $f_n(z)$ given by

$$f_n(z) = \begin{cases} z^p[1-(2\beta-1)\,z^{n-p}]^{-\{2\beta(p-\alpha)/(2\beta-1)(n-p)\}e^{-i\lambda}\cos\lambda} & (\beta \neq \frac{1}{2};\ n \geqq p+1) \\ \\ z^p \exp\left[\{(p-\alpha)/(n-p)\}z^{n-p}e^{-i\lambda}\cos\lambda\right] & (\beta = \frac{1}{2};\ n \geqq p+1). \end{cases} \tag{5.38}$$

An immediate consequence of Theorem 5 may be stated as

Corollary 6. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathscr{C}_p^{\lambda}(\alpha,\beta)$. *Then, under the hypotheses* (5.20) *and* (5.21),

$$|a_n| \leqq \frac{p}{(n-p)!\,n} \prod_{k=0}^{n-p-1} |2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + k(2\beta-1)| \tag{5.39}$$

$$(n = p+1,\ p+2, \cdots,\ p+N)$$

and

$$|a_n| \leqq \frac{p}{(p+N)!\,n(n-p)} \prod_{k=0}^{p+N} |2\beta(p-\alpha)e^{-i\lambda}\cos\lambda + k(2\beta-1)| \quad (n > p+N). \tag{5.40}$$

If, on the other hand, the condition (5.24) *holds true, then*

$$|a_n| \leqq \frac{2\beta\, p(p-\alpha)\cos\lambda}{n(n-p)} \qquad (n = p+1, p+2, p+3, \cdots). \tag{5.41}$$

The estimates in (5.39) *are sharp for the function* $f(z)$ *given by*

$$1 + \frac{zf''(z)}{f'(z)} = \frac{p - [p(2\beta-1) - 2\beta(p-\alpha)e^{-i\lambda}\cos\lambda]z}{1 - (2\beta-1)z} \quad (\beta \neq \tfrac{1}{2}), \tag{5.42}$$

while the estimates in (5.41) *are sharp for the functions* $f_n(z)$ *given by*

$$f_n'(z) = \begin{cases} pz^{p-1}[1-(2\beta-1)z^{n-p}]^{-\{2\beta(p-\alpha)/(2\beta-1)(n-p)\}e^{-i\lambda}\cos\lambda} & (\beta \neq \frac{1}{2};\ n \geqq p+1) \\ pz^{p-1}\exp\left[\{(p-\alpha)/(n-p)\}z^{n-p}\, e^{-i\lambda}\cos\lambda\right] & (\beta = \frac{1}{2};\ n \geqq p+1). \end{cases} \tag{5.43}$$

6. RADIUS OF STARLIKENESS AND CONVEXITY

We first state and prove

Theorem 6. *The sharp radius of starlikeness of the class* $\mathscr{S}_p^{\lambda}(\alpha,\beta)$ *is given by*

$$r_s = p\left\{\beta(p-\alpha)\cos\lambda + \left[\beta^2(p-\alpha)^2\cos^2\lambda - p(2\beta-1)^2\left[\frac{2\beta}{2\beta-1}(p-\alpha)\cos^2\lambda - p\right]\right]^{\frac{1}{2}}\right\}^{-1}, \tag{6.1}$$

provided that $\beta \neq \frac{1}{2}$. *The expression in* (6.1) *is real and finite only when* $\beta \neq \frac{1}{2}$ *and such that*

$$\beta^2(p-\alpha)^2 \cos^2\lambda \geqq p(2\beta-1)^2 \left[\frac{2\beta}{2\beta-1}(p-\alpha)\cos^2\lambda - p\right]. \tag{6.2}$$

Proof. From the first part of the inequalities in (4.5), it follows readily that

$$\mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > 0 \quad \text{for} \quad |z| < r_s, \tag{6.3}$$

where r_s is given by (6.1), provided that $\beta \neq \frac{1}{2}$ and the condition (6.2) is satisfied.

To show that the result (6.1) is sharp, we let $f(z)$ be given by (5.13), and put

$$\zeta = \frac{r[(2\beta-1)r - e^{i\lambda}]}{1-(2\beta-1)r\, e^{i\lambda}}. \tag{6.4}$$

We thus obtain

$$\frac{\zeta f'(\zeta)}{f(\zeta)} = \frac{p - 2\beta(p-\alpha)r\cos\lambda + (2\beta-1)^2\left[\frac{2\beta}{2\beta-1}(p-\alpha)\cos^2\lambda - p\right]r^2}{1-(2\beta-1)^2 r^2}, \tag{6.5}$$

which obviously has a zero real part when r is given by (6.1). This completes the proof of Theorem 6.

For $\beta = 1$, Theorem 6 readily yields

COROLLARY 7. *The sharp radius of starlikeness of the class* $\mathcal{S}_p^\lambda(\alpha)$ *is given by*

$$r_s = p\left\{(p-\alpha)\cos\lambda + \sqrt{p^2\sin^2\lambda + \alpha^2\cos^2\lambda}\right\}^{-1}. \tag{6.6}$$

The result (6.6) *is sharp.*

Making use of the relationship (1.7) between the classes $\mathcal{C}_p^\lambda(\alpha,\beta)$ and $\mathcal{S}_p^\lambda(\alpha,\beta)$, we can deduce the following consequence of Theorem 6.

COROLLARY 8. *The sharp radius of convexity of the class* $\mathcal{C}_p^\lambda(\alpha,\beta)$ *is given by* (6.1). *The result is sharp for the function* $f(z)$ *given by* (5.17), ζ *being defined* (*as before*) *by* (6.4).

7. CONCLUDING REMARKS AND OPEN PROBLEMS

The representation formulas, distortion theorems, coefficient bounds, and the sharp radii of starlikeness and convexity for the general classes $\mathcal{S}_p^\lambda(\alpha,\beta)$ and $\mathcal{C}_p^\lambda(\alpha,\beta)$, obtained in this paper, are indeed applicable in the derivation of the corresponding known or new properties and characteristics for their various subclasses including, for example, those that are enumerated in Section 1.

The present paper would normally be classified under *Geometric Function Theory* which is the study of the relationship between the analytic properties of $f(z)$ and the geometric properties of the image domain

$$\mathcal{D} = f(\mathcal{U}). \tag{7.1}$$

An excellent (and useful) example of such an important interplay is provided by Louis de Branges' theorem on the coefficients of a function $f(z) \in \mathcal{S}$, where

$$\mathcal{S} = \{f(z) : f(z) \in \mathcal{A}_1 \text{ and } f(z) \text{ is univalent in } \mathcal{U}\}. \tag{7.2}$$

For the general classes $\mathcal{S}_p^\lambda(\alpha,\beta)$ and $\mathcal{C}_p^\lambda(\alpha,\beta)$ studied in this paper, we have not given any geometric properties. Thus, to the lists of various interesting problems discussed, for example, by Goodman [9] (and, more recently, by Brannan and Hayman [4]), we should like to add a set of important open problems concerning the determination of some useful geometric properties of the general classes $\mathcal{S}_p^\lambda(\alpha,\beta)$ and $\mathcal{C}_p^\lambda(\alpha,\beta)$, and indeed also of many of their known subclasses mentioned in Section 1.

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