

Cops, Robbers, and Pre-Calculus Skills

by

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B.Sc., Dalhousie University, 2017

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MASTER OF SCIENCE

in the Department of Mathematics and Statistics

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University of Victoria

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**ABSTRACT**

This thesis is partitioned into three parts: improving pre-calculus skills of students in an introductory calculus course, the game of Cops and Robber on oriented graphs, and the generalized game of Cops and Robber. In the first part, we study the effect of review modules on student performance using an Educational Action Research framework. In addition to the study, we report what instructors at various institutions say they were doing to improve pre-calculus skills. In the second part, we introduce the game of Cops and Robber on oriented graphs using a two-part article that will appear in the problem-solving journal for high school students and undergraduates, *Crux Mathematicorum*. Next, we present a survey of previous results and provide family of counterexamples to a conjecture that was recently shown to be false. In the last part of the thesis, we consider general Cops and Robber games. We give a survey of previous results and conclude with a new characterization of graphs in which the Cops have a winning strategy.

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# Chapter 1

## Introduction

This thesis is divided into three parts: mathematics education, the game of Cops and Robber on oriented graphs, and the generalized Cops and Robber game. Since it is three parts, we will include relevant background with each part.

In Chapter 2, we investigate the effect of modified just-in-time review modules on student performance in an introductory calculus course for social and biological science students. The three main research questions: (i) Can we remediate student readiness for an introductory calculus course? (ii) Does participating in modified just-in-time review significantly improve student achievement in Calculus? and (iii) What are the current remediation practices of pre-calculus review at post-secondary institutions in Canada? In order to answer the first two questions, we use an Education Action Research methodology: identify the existence of the shortcomings in an educational activity, decide on the problem that is to be improved, formulate a plan, and carry out an intervention. We identified a group of students at risk of failing the course based on previous research, decided on the problem of those students having weak pre-calculus skills, and developed a set of review modules designed to improve pre-calculus skills and, in turn, improve student performance. In order to answer the last question, we asked instructors from post-secondary institutions across Canada to see if and how they were embedding pre-calculus review into introductory calculus courses. We asked

what the current trends are and whether they believed it was successful.

In Chapter 3, we introduce the game of Cop and Robber using a two-part article intended for high school students or undergraduates. It will be published in the problem solving journal, *Crux Mathematicorum*. This two-part article describes the game of Cop and Robber on oriented graphs with minimal notation and terminology. This article explains how to analyze the game, determine who wins, and determine the length of the game assuming both players use an optimal strategy. There are several questions at the end of each part, due to the problem-solving emphasis of the journal. The questions are left without solutions and vary in difficulty. Chapter 3 concludes with a survey of recent results for the game on oriented graphs and introduces some new results.

After readers have seen the game of Cop and Robber on oriented graphs, we describe the generalized Cops and Robber games in Chapter 4. We show that they can be modified to be combinatorial games, so methods from game theory can be used to analyze them. We close the chapter by surveying various characterizations of the games in which the Cops have a winning strategy.

Finally, in Chapter 5, we establish a new characterization of the Cop and Robber games on oriented graphs in which the Cop has a winning strategy. We outline how it can be extended to general Cops and Robber games.

## Chapter 2

# Using Modified Just-in-time Review to Improve Calculus Performance

Student readiness is one of the key factors in success for introductory calculus courses [39]. For students with weaker pre-calculus knowledge, who are either under-prepared or have had time away from mathematics, there was a sixty-six percent rate of failure in introductory Calculus at the University of Victoria, based on data from 2009 [21]. In order to help prevent such failures in an introductory course, research suggests a variety of solutions. These include remedial courses, additional teaching, and summer bridge programs. For more details, see [1, 25, 40]. It is increasingly important to improve student performance as the number of students in introductory mathematics courses continues to grow.

This study focuses on MATH 102, an introductory calculus course for social and biological science students. We developed a set of pre-calculus review modules in order to attempt to improve student performance for those who might be unprepared and therefore be at risk of not successfully completing the course. We conduct a quantitative analysis on the effect of these modules on student's course performance, and a qualitative analysis of the survey given to post-secondary instructors.

## 2.1 Literature Review

The transition from school-level mathematics to university mathematics is often referred to as *secondary-tertiary transition*. This transition has been a long-standing concern and an area of research for mathematics education. As early as 1966, researchers have studied the transition. Stern argued that many students find the transition difficult because students cannot manage the level of autonomy and flexibility in the tertiary environment [52]. Recently, in 2016, Rach and Heinze found that the high percentage of dropout undergraduate students in mathematics in several Western countries represents a big issue for instructors [48]. Di Martino found that the transition causes individual psychological strain for the students involved, even high achievers who did well in high school [23]. Alcock and Simpson said that this strain could be caused due to the fact that certain reasoning strategies are inadequate when applied to university mathematics, although they might have been sufficient in high school mathematics [3].

The European Mathematics Society, EMS, is conducting a survey about the current challenges in the transition from secondary to tertiary mathematics [38]. Questions are asked about: what measures were taken at the institution and how effective they are, which substantial reforms were most important, and which stakeholders are in a position to have substantial positive impact on the transition. For previous relevant information, see [34].

There are many different aspects of the secondary-tertiary transition, including cognitive conflict, conceptual change, and culture shock [19]. In previous work, the EMS Committee on Education found that students required help because they feel unable to even start a problem [25]. The inability to solve a problem and students proposing inadequate reasoning or proof are just two among many possible problematic situations. Some institutions have developed initiatives to mediate these problematic situations. These initiatives can take the form of bridging courses: additional courses, given at the very beginning of the first university year,

that attempt to fill the gap between secondary school and university, often by proposing exercises that require only secondary school knowledge but require more autonomy of the students [25].

When studying the secondary-tertiary transition for students, one component of this transition is *student readiness*, which refers to whether a student is prepared to succeed in a post-secondary level course. Student readiness can be measured through the use of assessment or placement tests, or by achievement in other educational contexts. However, in 2015, Atuahene and Russell examined students' academic readiness in select college level mathematics courses in a United States university using SAT or ACT scores [4]. The study found that approximately 76% of 1315 students were academically ready for university level general education math courses, based on their SAT scores and eligible placement levels, but only 23.19% of 993 students who were academically ready for college-level math courses were academically ready for Calculus I based courses.

Li *et al.* studied the effect of mathematics readiness and student behaviour on knowledge gain and success in mathematics courses [39]. Mathematics readiness was measured via a diagnostic test administered at the beginning of the term, and student course behaviour was taken from instructors' ratings of students' levels of participation, attendance, and completion of homework assignments. They found that mathematics readiness had strong direct effects on math knowledge, which was measured via a posttest, well as indirect effects on course success, as exhibited by student course behaviour.

When it comes to student readiness for Calculus, mathematics education researchers have found that student difficulties in understanding key ideas of calculus are rooted in their weak understanding of the function concept [15, 16, 18, 55, 54, 53, 56, 51, 60]. Students have a strong tendency to view a graph of the function as a picture of an event, rather than a representation of how two variables change together [43]. Furthermore, a common student misconception is viewing a function as a recipe for getting an answer instead of as a mapping

of input values to output values [15, 16].

Researchers know that student readiness has a direct effect on success in an introductory course, but how do we increase the readiness of students who may lack the necessary knowledge to succeed? Sidika Nihan Er surveyed 737 faculty members in the United States and found most instructors had students who needed remediation in mathematics and thought their students' background prevented them from understanding material, but less than 40 percent of instructors thought remedial courses were a sufficient remedy [26].

The effectiveness of remediation courses is inconclusive. There has been positive impact on students' success from remedial mathematics courses [1, 9]. On the other hand, there have also been multiple studies that reported that remedial courses are not a remedy and do not improve students' performance [24, 27, 41, 46, 57, 59]. Most importantly, Bahr conducted a study to see if mathematics remediation worked and found that it does work, but only for some students [5]. This discovery would explain why multiple studies have reported both positive impact and no effect of remediation on student performance. Most students do not benefit from remediation [21].

A possible way to improve student mathematics readiness is through the use of just-in-time teaching methods. Just-in-time teaching, JiTT, is a pedagogy that allows instructors to make adjustments to address student problems using feedback from student work [30]. This technique has been widely used to address issues of remediation and review in undergraduate mathematics. Natarajan and Bennett created "modified" JiTT methods, using online review modules, to make a difference in student achievement on specific calculus topics [44]. The online review modules were administered during the semester, not strictly before a lecture. Natarajan and Bennett found that as long as students worked through the review material at some point during the course, there was a positive impact on course performance. Although completing the review modules ahead of time had more advantages, the results showed gains in learning regardless of timing.

Kay and Kletskin [35] developed online learning objects that consisted of text summary sheets, interactive video-clips demonstrating solution methods to typical problems, and a set of online mastery practice questions. The key advantages for using online learning objects include accessibility, ease of use, reusability, interactivity, flexibility, and adaptability [35]. The majority of students rated all three learning tools as useful or very useful and reported that the tools provided a useful review and helped to improve understanding.

Recently, in 2016, the University of Toronto Scarborough campus created online calculus and pre-calculus learning support modules for mathematical skill development. The modules were designed with the goals of providing students with a strong support for basic Calculus concepts, helping students communicate mathematical ideas, and developing mathematical thinking [49]. There were twelve modules in total with topics including algebraic manipulation, equations and inequalities, analytic geometry, functions, exponential and logarithmic functions, trigonometry, trigonometric functions, limits, continuity, derivatives, integration, and proof techniques. These topics were included because they were believed to be difficult concepts for students. The report did not include any data or analysis for the students who have participated.

## 2.2 Background

We focused on the entry level undergraduate mathematics course for students in the social and biological sciences: MATH 102, Calculus for the Social and Biological Sciences. This course is often the last mathematics course that social science students and biology students need to take.

There has previously been attempted remediations of pre-calculus for students in MATH 102 at UVic. In 2005, Rachel Anderson, a graduate student at UVic, ran pre-calculus tutorials. There was one tutorial per week for the first five weeks of term covering review

material on a review assignment. There was evidence that success on the review assignment was a strong predictor of success in the course. Unfortunately, the pre-calculus tutorials fell off gradually and eventually stopped running in 2009.

Lorraine Dame, in her 2012 PhD thesis, studied student readiness and success in entry level undergraduate mathematics courses, including MATH 102 [21]. Dame found significant differences in median Calculus for the Social and Biological Sciences letter grade for groups of students with differing levels of preparation. These relevant preparations included Pre-Calculus 12 grades and English 12 grades. Two out of three students who completed this course and had entered with a C+ or lower in Pre-Calculus 12, the main prerequisite, failed [21]. Dame made two recommendations for MATH 102: increase the minimum prerequisite in Pre-Calculus to a B or higher, and offer the algebra review assignments in future terms. The grade minimum prerequisite was not implemented.

There were subsequently pre-calculus review assignments online in the course, but no review materials other than those in the appendix of the textbook or freely available online. An internal study that looked at failures of MATH 102 students recommended the Headstart program.

The Headstart program was a pre-arrival review and preparation course. The Headstart program included both pre-calculus objectives and university skill objectives. The program consisted of in-person instruction, written homework assessed by a Teaching Assistant, and access to online study tools through the Pearson product MyMathLab, an online homework system. It was difficult to get students to come to campus early, before the start of term. The program was free of charge, but enrolment was still low, fewer than 50 students attended. Headstart was first held in Summer 2014 with 24 participants. The program was redesigned and held again in Summer 2015.

The attrition and failure rates for MATH 102 since 2014 are shown in Table 2.1. As of Summer Session 2014, grades are now being submitted as a percentage rather than a letter

grade. Therefore, we only included data since the grading change.

<b>Term</b>	<b>Total Head-count</b>	<b>Gradeable Head-count</b>	<b>Attrition Percent</b>	<b>Failure Rate Without N's</b>	<b>Number of N's</b>
Summer 2014	50	39	22.00	17.95	0
Fall 2014	429	390	9.09	26.92	9
Spring 2015	304	279	8.22	13.62	10
Summer 2015	44	36	18.18	22.22	3
Fall 2015	421	389	7.60	18.51	7
Spring 2016	294	272	7.48	26.84	12
Summer 2016	51	51	0	3.92	1
Fall 2016	400	335	16.25	31.34	24
Spring 2017	249	218	12.45	10.10	11
Summer 2017	50	46	8.0	4.35	0
Fall 2017	344	317	7.85	19.87	6
Spring 2018	234	223	4.70	16.14	5
Summer 2018	55	53	3.64	0	0
Fall 2018	289	260	10.03	18.85	4
Spring 2019	189	174	7.94	12.07	3

Table 2.1: Total and Gradeable Headcount for MATH 102 during 2014-2019

In Table 2.1, we show the number of students, attrition rates, and failure rates from Summer 2014 to Spring 2019. Total Headcount represents the number of students after the last day to add the course. Gradeable Headcount represents the number of students on the first day of examinations. A letter grade of N means that the student did not write the final examination or complete course requirements by the end of the term or session. Failure Rates represent the number of students in the Gradeable Headcount who completed the course and received less than 50 percent in the course.

The University of Victoria flags failure rate to be high if the rate is higher than 20 percent. With this in mind, we can infer from Table 2.1 that MATH 102 has high failure rate in most semesters. This regular occurrence of high failure rates is clearly a problem for student achievement and retention. We know, from the previous remediation programs, that

the students in MATH 102 are in need of pre-calculus review. For this study, we will focus on how pre-calculus review can improve student performance.

## 2.3 Statement of the Problem

The goal of this study is to determine if modified just-in-time review modules could improve student readiness for calculus and in turn improve student achievement and retention. The study centres around three specific questions:

1. Can we remediate student readiness for an introductory calculus course?
2. Does participating in modified just-in-time review significantly improve student achievement in Calculus?
3. What are the current remediation practices of pre-calculus review at post-secondary institutions in Canada?

## 2.4 Methodology

Education Action Research, EAR, refers to a wide variety of evaluative, investigative, and analytical research methods designed to diagnose problems or weaknesses [31]. The general goal of EAR is to create a simple, practical, repeatable process of iterative learning, evaluation, and improvement that leads to increasingly better results for schools, teachers, or programs.

A common-sense view of action research provided by McNiff [42] is that we:

- review our current practice,
- identify an aspect we want to improve,

- imagine a way forward,
- try it out, and
- take stock of what happens,
- modify our plan in the light of what we have found, and continue with the ‘action’,
- monitor what we do,
- evaluate the modified action.

This framework is most appropriate for educators who recognize the existence of the shortcomings in their educational activities and who would like to adopt some stance on the problem, formulate a plan, carry out an intervention, evaluate the outcomes and develop further strategies.

### **2.4.1 Learning Object Design**

Our student sample consisted of 177 students enrolled in a first calculus course, MATH 102: Calculus for the Social and Biological Sciences, at the University of Victoria in Semester 2 (January to April) of 2019. Calculus for the Social and Biological Sciences focuses on the calculus of one variable with applications to the social and biological sciences. Topics include: limits, continuity, differentiation, applications of the derivative, exponential and logarithmic growth, and integration.

From Dame’s PhD thesis, there was evidence, based on an assessment test, that students in MATH 102 lacked the pre-calculus knowledge required to be successful in the course. While there have been previous pre-calculus tutorials, assignments, and pre-arrival programs, only the assignments are still offered.

The learning objects consisted of two components: a text-based summary with practice questions, and a set of online questions assessing student progress. We developed four

modules with five topics in each module. The key topics that were included in the modules were numbers and operations, fractions, exponents, polynomials, functions, and solving problems, equations and inequalities. These are topics that Lorraine Dame identified as being key problems in her 2012 thesis [21]. In Figure 2.1, a portion of the text-based summary on inequalities in one of the modules is shown as an example of the structure and style of the text-based summaries in the modules.

**Example 2** Graph the solution set to  $\frac{x}{3} < 0$  on the real number line.

The solution set is  $\{x : \frac{x}{3} < 0\}$ . The expression  $\frac{x}{3}$  is defined for all real numbers  $x$ , and equals zero only if  $x = 0$ . The remaining real numbers are partitioned into two intervals:  $(-\infty, 0)$  and  $(0, \infty)$ . We need to pick a test point in each one and be careful about what happens at  $x = 0$ .

By choosing the test point  $x = -1$  we obtain  $-\frac{1}{3} < 0$ , so every number in  $(-\infty, 0)$  is in the solution set. Since  $\frac{0}{3} = 0$ , the endpoint 0 is not in the solution set. By choosing the test point  $x = 1$  we obtain  $\frac{1}{3} > 0$ , so no number in  $(0, \infty)$  is in the solution set.

Therefore, the solution set to the given inequality is  $(-\infty, 0)$ . Its graph on the real number line is shown below.

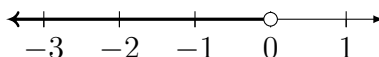


Figure 2.1: Text Summary of Inequalities in Module

The online assessment consists of 20 questions for every module, with roughly four questions for each topic in the module. The assessment was created using a free open-source system, WeBWorK. A sample question from the first module is shown in Figure 2.2.

The modules were released during the beginning of the course, and they were available until the night before the final examination. The completion of the modules was worth two percent of the student's final grade, but it replaced a previously taken assessment test. By completing half of the questions from each WeBWorK module test, students could replace their grade with the full two percent. Anonymized grades of the students were given for the analysis of this study. Information given included: grades of all term work, final examination

## Review Assignment 3: Problem 1

Previous Problem **Problem List** Next Problem This set is **visible to students**.

(1 point) **Library/Rochester/setAlgebra13Inequalities/srw1\_7\_39.pg**  
 Solve the following inequality. Write the answer in [interval notation](#).

$$\frac{x - 9}{x + 20} \geq 0$$

Answer:

[Edit3](#)

Show:  CorrectAnswers  AnswerGroupInfo  Show Auxiliary Resources  AnswerHashInfo  PGInfo

**Preview My Answers** **Check Answers**

You have attempted this problem 0 times.  
 This homework set is closed.

**Show Past Answers**

**Email instructor**

This set is **visible to students**.

Figure 2.2: Example Question in WeBWorK system

grades, and a breakdown of progress for the modules from WeBWorK.

### 2.4.2 Survey

In addition to our modules, we wanted to know what other Canadian post-secondary institutions were doing to improve student readiness for students with weaker pre-calculus knowledge. We specifically wanted to know how instructors embedded pre-calculus review into an introductory calculus course. We emailed lecturers and professors from over 30 Canadian universities to see whether they embedded pre-calculus into a first-semester calculus course at their respective institutions. Fourteen instructors from fourteen different universities responded. We began by asking whether they had any embedding of pre-calculus into a first calculus course. Based on the initial responses, we asked the following questions on current embedding of pre-calculus.

1. Is it done online or in person?
2. Is it integrated into the course lectures or tutorials (if any), or separate review materials?
3. Are students individually encouraged to engage with specific components of the review?
4. What is the participation rate in the review(s)?
5. Is it for marks or not? If so, how many? And how are they earned?
6. Are there specific review questions on pre-calc material? Review assignments?
7. Have there been any changes in the success rate (proportion who pass) by doing this? The attrition and failure rates? Perceptions?
8. What do the students say? That is, the ones who speak up. Is there a perception that it helps, or that someone cares?
9. Is there a difference in outcomes (success, failure, attrition rates) between the ones who participate and the ones who don't?

The list of questions were sent to the instructors who responded to the initial email. They were asked to answer as many questions as they could and with as much detail as possible.

## 2.5 Results on Module Participation

We separated students who used the modules into two groups; students who completed at least 50 percent of all four Modules, and students who completed less than 50 percent, but did a non-zero amount of work on the Modules. In what follows, we will refer to the group of students who completed at least 50 percent as Passed Modules, and refer to the group of students who completed less than 50 percent, but did some amount of work as Participated

Modules. For example, a student who answered 5 questions from each of the four Modules would be in the Participated Modules group.

Out of the 177 students who were registered in the course, twelve students completed at least 50 percent of the modules and were therefore in the Passed Modules group. Two students were in the Participated Modules group. In total, between the two groups, there was a participation rate of 7.91 percent of students.

The mean Final Examination grade for all 177 students was 44.45 out of 70 marks in total. The mean Final Examination grade for the twelve students in Passed Modules was 52.13. The mean Final Examination grade for the Participated Modules was 41.75.

We have included the distribution of final examination grades for all students. The bars in the distribution represent the letter grade of the final examination grades from left to right: F 0% -49%, D 50% -59%, C 60% -69%, B 70% -79%, A 80% -100%.

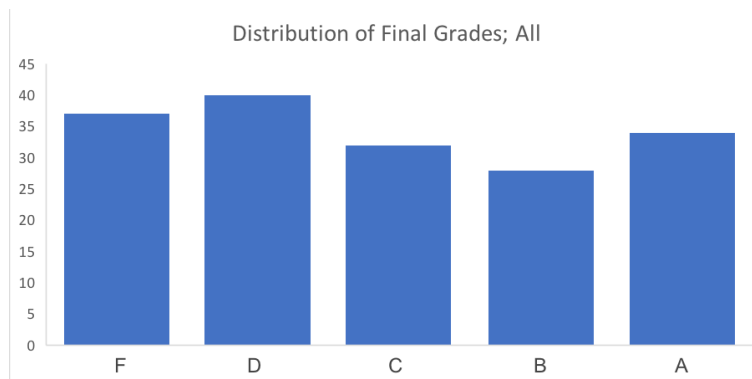


Figure 2.3: Distribution of Final Grades of All Students in MATH 102

The standard deviation of the final grades of MATH 102 was 12.83. Therefore, both Passed Modules and Participated Modules are within one standard deviation of the mean. On the other hand, Passed Modules has mean in the B letter grade, and Participated Modules has mean in the D letter grade.

We took 10 random samples of 12 final examination grades and computed the mean of each random sample. The means were 41.27, 44.7, 44, 39.95, 41.125, 41.5, 44.1, 43.91,

47.27, and 44.91. The samples were roughly between 41 and 47. The standard deviations of each sample were 14.75, 9.12, 9.70, 13.37, 6.95, 12.86, 10.99, 11.70, 8.06, 15.18. The Passed Modules fell within one standard deviation of nine out of the ten samples.

## 2.6 Results from Instructor Survey

We emailed instructors from over 30 universities and colleges and asked if their Mathematics department has any embedding of pre-calculus into a first semester calculus course. We received responses from fourteen universities and colleges stating that they did some sort of pre-calculus embedding and were willing to talk about their experience. From those fourteen instructors who did some pre-calculus embedding, we asked them the nine questions from the Instructor Survey in Subsection 2.4.2. Responses are summarized below.

### **Question 1: Is it done online or in person?**

When we asked the instructors whether they embedded pre-calculus review in person or online, 75 percent of instructors who responded said that they embedded some sort of pre-calculus review in person. This embedding was in various forms: in lecture, tutorial or laboratory, and separate review sessions. On the other hand, 25 percent of instructors said that they embedded pre-calculus review online, either in online quizzes or review materials.

### **Question 2: Is it integrated into the course lectures or tutorials (if any), or separate review materials?**

Instructors gave a wide variety of responses to the integration of review. When asked how the review was integrated, 75 percent said that the review was integrated in lectures, tutorials, or laboratory sessions. Separate review materials accounted for 16.67 percent of instructor responses, and 8.33 percent did not specify how review was integrated. Four instructors stated that their institution offered a two-semester calculus course that covered all the topics of the one-semester calculus course with pre-calculus included.

**Question 3: Are students individually encouraged to engage with specific components of the review?**

One respondent encouraged students to engage with specific components of the review. Nine instructors did not encourage students to engage with specific components, and four instructors did not provide information.

**Question 4: What is the participation rate in the review(s)?**

Participation rates were unknown according to 91.67 percent of instructors. The rest of the instructors did not specify the exact number of students but the review was mandatory, given that it was in the course. It should be noted that Calculus readiness tests were also mandatory because they had to be either completed to get into the course or get credit for the course.

**Question 5: Is it for marks or not? If so, how many? And how are they earned?**

Three reviews were for marks, one review was for bonus marks, three reviews were mandatory tests to enter or get credit for the course, and seven reviews were not for marks.

**Question 6: Are there specific review questions on pre-calc material? Review assignments?**

The answers given for question 6 depended on how instructors embedded pre-calculus into their course. The three institutions that had Calculus readiness tests or online skill testing gave students practice tests to help study the pre-calculus material. If instructors embedded pre-calculus using time outside of scheduled course time, students were provided with a booklet with questions. The two-term Calculus courses had pre-calculus material in assignments. Four instructors said that they did not have specific review questions, and one instructor did not provide any information for question 6.

**Question 7: Have there been any changes in the success rate (proportion who pass) by doing this? The attrition and failure rates? Perceptions?**

Five instructor responded with positive feedback and one instructor reported no change in success rates. One instructor reported high attrition rates due to advising: this instructor said that if a student was not showing progress in the review, the instructor would advise the student out of the course. Seven instructors did not report any information for this question.

**Question 8: What do the students say? That is, the ones who speak up. Is there a perception that it helps, or that someone cares?**

Six instructors said that they had received or heard positive feedback from students, but they did not mention how many students they had heard from. There was no information from eight instructors.

**Question 9: Is there a difference in outcomes (success, failure, attrition rates) between the ones who participate and the ones who don't?**

Lastly on the difference in outcomes, one instructor reported no difference in outcomes, two instructors stated that outcomes might be different due to streaming students into pre-calculus courses, and eleven instructors did not provide any information on outcomes.

## 2.7 Discussion

The research problem that we were investigating is whether modified just-in-time review modules can increase student readiness for calculus and improve final examination marks. We used an Education Action Research methodology to develop the modules and observe the impact on student success.

### 2.7.1 Modules

Although the participation rates for the modules were significantly low, with only 7.9 percent of students participated, there is still useful information from this study. The Passed Modules group had a higher mean grade, but this could be due to students who were al-

ready competent in pre-calculus completing the modules for the marks. Although the Passed Modules had a higher mean than any of the ten random samples of students, the mean was within one standard deviation of nine of the random samples. This analysis suggests that the Passed Modules did not perform significantly higher than students who did not complete the modules.

There are certainly limitations to our study. First, the modules were, in a sense, voluntary. Although the modules could be worth two percent of their final grade, it was replacing a previously taken assessment test. Therefore, the results contain a self-selection bias. Another limitation is that we have no data on whether students actually read and used the text-based summaries. The summaries were posted on a web page, and we did not track who visited the web page, how long they were on a particular page, or whether they completed the practice problems in the summaries. There was also a lack of data for the Passed Module group; the size of the group was a significant obstacle and therefore it was difficult to draw conclusions. This limitation can be remediated by conducting the experiment again with the modules worth two percent, but not replacing a previously taken assessment test. Additionally, participation in the modules could be remediated with better integration of the modules into the course by instructors.

We did not include a student feedback and exit survey to see what students thought of the modules and if they felt it was useful to them. We created a diagnostic test for students to write before and after completing the modules, but that test was not finished in time for the release of the modules. Further research could include this diagnostic test for more accurate results on whether participants gained pre-calculus knowledge from our modules or simply had it to begin with. Statistical analysis would be used on the diagnostic and exiting test to determine if the modules helped improve student readiness. These tests could also include a feedback survey for the students. A different use of the modules could be used in conjunction with the department's calculus readiness test. After a student has written the

test and has found their pre-calculus knowledge gaps, the student would use the modules to improve the knowledge gap and gain entry into their respective calculus course. This process could allow students to avoid a remediation course, which we know are ineffective.

The modules are currently hosted on a faculty member's web page. The modules are in PDF format and therefore have limited features. We would like to convert our TeX files using an XML (eXtensible Markup Language) called PreTeXt. This conversion would allow us to produce the Modules as a HTML e-book, similar to most electronic textbooks. The new Modules would embed examples and practice questions, an index, and the mathematics would be more screen-reader accessible. The features of PreTeXt would allow for smoother navigation and accessibility for students.

### 2.7.2 Survey

Most of the respondents that reported they do embed pre-calculus into an introductory calculus course did so by offering a two-semester calculus course that contains the same material as the typical one semester Calculus I course with added review. Although the students found this two-semester course helpful, instructors noted that those students were at a disadvantage when taking another mathematics course due to the slower pace. Instructors said that the students could not keep up with the faster pace of a typical mathematics course.

Another observation from the survey was that most universities had not done any formal research into success rates of the pre-calculus embedding that was being done at the university. Participation rates, changes in success, student feedback, and outcomes were largely unknown by most of the instructors who responded to the survey. This deficiency shows that there is a need for the study and continued research on pre-calculus reviews in the post-secondary environment.

Calculus Readiness tests are helpful to place students into an appropriate mathematics course depending on their pre-calculus strengths. This placement would stream weak or

under-prepared students into a pre-calculus course. Several of the institutions that we surveyed had a Calculus Readiness test. These were mandatory tests given at the beginning of the semester that were required either to register for a given course, or to get credit for a given course. One institution stated that the results will be used to guide students into an appropriate calculus course or determine which sequence of mathematics courses will be the most beneficial to the student's learning and success.

Institutions with a Calculus Readiness test used the test to place students into the appropriate mathematics course based on their performance. This process is similar to the use of a Calculus Readiness test at Arizona State University, University of Arkansas, and Francis Marion University [17]. They studied the reliability and validity of the instrument as a measure of readiness for success in learning calculus. Correlating their test to American College Testing, ACT, mathematics scores and prerequisite pre-calculus scores showed that the test was useful in deciding a student's readiness for success in the study of calculus.

We got several responses to the survey stating that the department did not have any type of pre-calculus embedding, but would be interested in utilizing other institution's pre-calculus integration techniques and materials. These responses seem to imply that there is a desire from instructors to implement such research at their own institutions. However, the instructors are not accessing the current research focusing on improving pre-calculus skills.

There are also limitations to the instructor survey. First, the survey was voluntary, and initially sent to instructors that we know through various channels. Some of them had an interest in mathematics education and could therefore already be embedding pre-calculus due to their own knowledge of the literature. Additionally, the survey questions were sent after the initial email only to instructors that replied and stated that they had embedded precalculus into an introductory calculus course.

The survey was sent to various colleges and universities with a focus on institutions where we knew someone in the mathematics department. In the future, it would be helpful

to give the survey to a wider audience. The survey could be sent out via a mathematical society such as the Canadian Mathematical Society. Similar to what the EMS is doing with their survey about secondary-tertiary transition, we could reach a broader audience, see the current challenges, and send out a report for all members to see.

## Chapter 3

# The Game of Cop and Robber

The game of Cop and Robber was first introduced by Nowakowski and Winkler [45], and independently, by Quilliot [47]. As introduced, the game was played on undirected graphs. We will play the game on oriented graphs, which we will define in the next section. The rules of the game are the same regardless of whether it is played on undirected or oriented graphs.

We introduce the game of Cop and Robber via the following two-part article which will appear in the problem solving journal *Crux Mathematicorum*, which is aimed at the high school and undergraduate levels. After the article, we present some previous results on the game of Cops and Robber as played on oriented graphs.

Part I of this two-part article introduces the game, and establishes some theoretical results about it. The results come from application of existing theory and definitions. Part II of the article describes a way to analyze the game, determine who wins in a given situation, and how many moves the game will last, assuming both players employ optimal strategies. Due to the problem-solving nature of the journal, each part of the article contains questions for the reader. The questions are left in the thesis because topics are introduced in the questions.

### 3.1 Game of Cop and Robber on Oriented Graphs

The game of Cop and Robber is a two-player game for which the game board is an oriented graph. An example is shown in Figure 3.1. Formally, an *oriented graph* consists of a finite collection of objects called *vertices* and a collection of directed connections between vertices called *arcs*, such that there is at most one arc (in any direction) between any two vertices. The oriented graph shown in Figure 3.1 has four vertices  $a, b, c, d$  and five arcs  $ab, cb, ca, cd, da$ . Each vertex is represented by a dot. The arc  $ab$  is represented by the arrow *from*  $a$  to  $b$  (more precisely, from the dot corresponding to  $a$  to the dot corresponding to  $b$ ), and similarly for the other arcs.

The picture tells you everything about the oriented graph, that is, what the vertices are and what the arcs are. It does not matter how the vertices are placed in the plane or whether the arcs are drawn with straight lines or curves, two pictures represent the same oriented graph when they have the same vertices, and the same collection of ordered pairs of vertices joined by arcs.

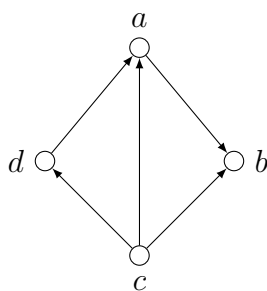


Figure 3.1: An oriented graph with 4 vertices

Given an oriented graph,  $G$ , the game of Cop and Robber is played as follows. There are two players: the Cop and the Robber. First, the Cop chooses a vertex of the graph as their starting position, then the Robber chooses a vertex of the graph as their starting position. The players then move alternately, with the Cop moving first. A move for either player

consists of either remaining on their current vertex, or sliding along an arc in the direction of the arrow to another vertex. The Cop's goal is to *catch* the Robber, which occurs if they are ever on the same vertex. The Robber's goal is to avoid being caught. The Cop wins the game if the Robber is ever caught, and otherwise the Robber wins.

Both the Cop and Robber know everything about the oriented graph,  $G$ , at least in principle, and all of the options available to each other in any situation. That is, the two players have *perfect information* (about the options available to each player).

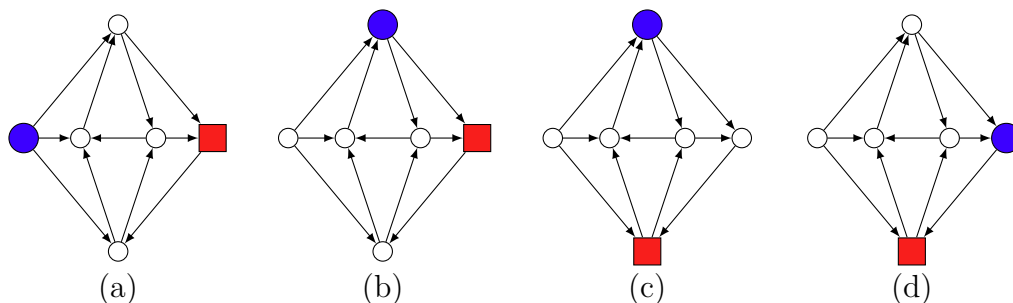


Figure 3.2: The start of a game

The four parts of Figure 3.2 illustrate the start of a game. Since the vertices can be identified by their position in the drawing, the vertex labels don't need to be included in the picture. The Cop's positions are represented by solid round blue vertices and the Robber's positions are represented by square red vertices.

In Figure 3.2 (a) the Cop has chosen an initial position, as has the Robber. Notice that there are no arcs to the vertex chosen by the Cop. If the Cop does not choose this vertex as their initial position, then the Robber can win the game by choosing it. Since there is no arc to this vertex, the Robber could simply stay there and never be caught.

In Figure 3.2 (b) the Cop has slid along an arc to the top vertex in the picture. If the Cop does not make this move, then the Robber has a winning strategy. For example, if the Cop moved to the vertex on the right, then the Robber could respond by moving to the vertex at the bottom and, if it is ever the case that there is an arc from the Robber's vertex to the

Cop's vertex, then the Robber can be behind the Cop on each subsequent move and never be caught (by the condition that there is at most one arc between any two vertices). The same thing would happen if the Cop moved to the bottom vertex and the Robber stayed in place.

Parts (c) and (d) of the figure illustrate two possible next moves. Notice that, in part (d), if the Cop had moved to the next vertex to the left, then he would have guaranteed that the Robber would win: the Robber would be behind. But now the Robber has a move that guarantees a win. What is it?

From the explanation just given it is possible to determine that the Robber can win the game whenever it is played on the oriented graph shown in Figure 3.2.

We observed above that the Cop had to start on a particular vertex, otherwise the Robber could have guaranteed a win. The same observation applies to any oriented graph with a vertex with no arcs to it; such a vertex is called a *source vertex*.

**Proposition 3.1.1.** *If an oriented graph has a source vertex and it is not chosen as the Cop's initial position, then the Robber can win the game.*

**Proof.** Suppose there is a source vertex not chosen as the Cop's initial position. Then the Robber can choose it as their initial position. Since there is no arc to this vertex, the Robber can simply stay there and never be caught.  $\square$

**Corollary 3.1.2.** *If an oriented graph has more than one source vertex, then the Robber can win the game.*

**Proof.** If an oriented graph has more than one source vertex, then one of them is not chosen as the Cop's initial position. Proposition 3.1.1 now tells us that the Robber can win.  $\square$

We also observed above that if Robber can ever get behind the Cop, then the Robber can avoid ever being caught; this observation is also true in any oriented graph.

**Proposition 3.1.3.** *Suppose the game is being played on an oriented graph. If there is an arc from the Robber's vertex to the Cop's vertex, then the Robber can win the game.*

**Proof.** Assume there is an arc from the Robber's vertex to the Cop's vertex.

Suppose it is the Cop's turn to move. Then, since there is at most one arc between any two vertices, the Cop cannot catch the Robber on this turn. If the Cop stays put, then the Robber can avoid capture by also staying put. If the Cop moves to a new vertex, then the Robber can avoid capture by moving to the vertex the Cop just left, as it has an arc to the Cop's new vertex.

Now suppose it is the Robber's turn to move. If the Robber stays put, then there is an arc from the Robber's vertex to the Cop's vertex and it is the Cop's turn to move.  $\square$

**Corollary 3.1.4.** *If an oriented graph has no source vertex, then the Robber can win the game.*

**Proof.** Suppose our oriented graph has no source vertex. Then, no matter which vertex  $c$  the Cop chooses as their initial position, the Robber can choose a vertex with an arc to  $c$ . Proposition 3.1.3 now tells us that the Robber can win.  $\square$

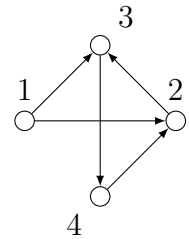
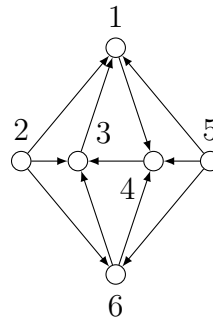
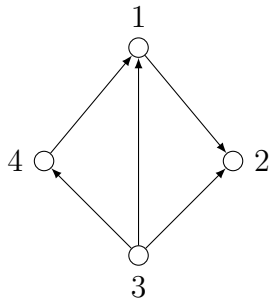
**Corollary 3.1.5.** *If  $G$  is an oriented graph on which the Cop can win the game, then  $G$  has exactly one source vertex and the Cop must choose it as their initial position.*

Notice that the corollary does not say that the Cop can always win the game on an oriented graph with exactly one source vertex. For example, the oriented graph  $G$  in Figure 3.2 has exactly one source vertex and the Robber wins when the game is played on  $G$ .

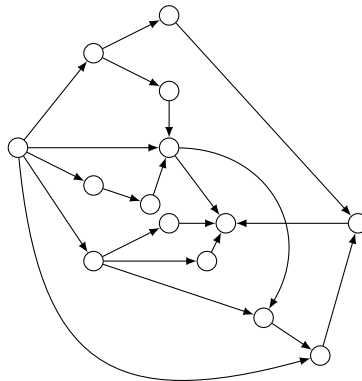
It turns out that, for any given oriented graph  $G$ , it is possible to determine who wins the game when it is played on  $G$ . Furthermore, if the Cop wins, then it is possible to determine how many moves are needed assuming the players always make their best possible move (that is, neither player makes a bad move). We will explain how to do that in Part II of this article.

### Questions

1. Use the results Proposition 3.1.1 through Corollary 3.1.5 to determine which of the Cop and Robber win the game on each of the three given oriented graphs.



2. The Cop has a winning strategy when the game is played on the oriented graph shown below.



- (a) At which vertex should the Cop start in order to win the game?
- (b) Given that the Cop starts at the vertex that you identified in part (a), where should the Robber start in order to make the game last as many moves as possible? And how many moves is that?
3. A *directed cycle* in an oriented graph  $G$  is a sequence of different vertices  $x_1, x_2, \dots, x_k$  such that there are arcs  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_k \rightarrow x_1$ . Explain why an oriented graph

which has exactly one source vertex and is such that the Robber has a winning strategy must have a directed cycle.

4. Suppose that more than one Cop can be placed on the graph at the start of the game, and all Cops can move simultaneously on the Cops' turn. Obviously, if there were a Cop on each vertex of the given oriented graph, then the Cops are guaranteed to catch the Robber. For a given oriented graph  $G$ , the minimum number of Cops needed to guarantee that the Cops can always catch the Robber is called the *Cop number of  $G$* .

- (a) What is the Cop number of each graph in question 1?
- (b) Explain why the Cop number of any oriented graph  $G$  is at least the number of source vertices of  $G$ .
- (c) Show that if the oriented graph  $G$  has no directed cycle, then the Cop number of  $G$  equals the number of source vertices in  $G$ .

## 3.2 Analysis of the game of Cop and Robber on Oriented Graphs

In Part I of this article, we introduced the game of Cop and Robber on oriented graphs, and established some theoretical results that help analyze the game. In this Part, we will describe a way to determine which player wins the game on a given oriented graph, and how many moves the game will last.

Let's quickly recall the rules of the game. The game is played on an oriented graph,  $G$ . To start the game, the two players, the Cop and the Robber, each choose a vertex on the oriented graph. The Cop chooses first. They then alternate taking turns, starting with the Cop. On each turn a player can stay at their current vertex, or slide along the arc in the direction of the arrow to a new vertex. The Cop's goal is to be on the same vertex as the Robber. If this ever happens, the Robber is *caught* and the Cop wins. If the Robber can avoid ever being caught, the Robber wins.

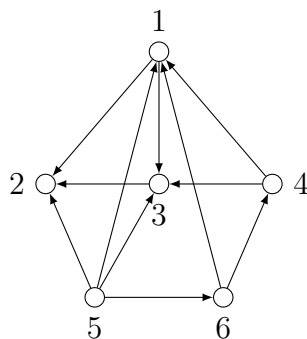


Figure 3.3: Oriented Graph  $G$

To begin our analysis of the game, let's make a list of situations – ordered pairs of the form (Robber's vertex, Cop's vertex) – where if the Robber is at  $x$  and the Cop is at  $y$ , and it is the Robber's turn to move, then the Cop can win in at most  $k$  moves, where  $k \geq 0$ . We

will analyze the game on the oriented graph shown in Figure 3.3.

When  $k = 0$ , this means that the Cop has won, which means the Cop and Robber are on the same vertex, so the list is  $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)$ .

When  $k = 1$ , we are listing situations where the Cop can win in at most 1 move, so all of the situations listed above are included. The list also must include situations where no matter what move the Robber makes, the Cop can move to be on the same vertex. In other words, there is an arc from the Cop's vertex to every vertex to which the Robber could move, including the Robber's current vertex (as staying still is allowed). These are  $(1, 5), (2, 1), (2, 3), (2, 5), (3, 1), (3, 5)$ .

A convenient way to present the lists is using a  $6 \times 6$  array in which the rows correspond to the Robber's positions, and the columns correspond to the Cop's positions. The entry in row  $r$  and column  $c$  corresponds to what is known about the situation where the Robber is on vertex  $r$  and the Cop is on vertex  $c$ , and it is the Robber's turn to move.

The array is filled in iteratively as the lists are constructed. Initially the entry  $(r, c)$  is the ordered pair  $(r, c)$ . If there is an arc from the vertex  $r$  to the vertex  $c$ , i.e. the Robber is behind the Cop, then the  $(r, c)$  entry is permanently changed to  $X$ . (The Robber can win by staying still on the next move, and then moving so as to remain behind the Cop.) If the pair  $(r, c)$  is added to the list when  $k = t$ , the entry is permanently changed to  $t$ .

Since the array has only  $6 \cdot 6 = 36$  entries, eventually there is a value of  $k$  for which no entries are changed. When this happens no entries will ever change since, if an entry could be changed later, it could have been changed for this value of  $k$ . When this happens, the array is completed and can be used to determine the outcome of the game.

The array corresponding to  $k = 1$  is shown in Figure 3.4.

We now continue filling in the array by considering whether any entries are changed when  $k = 2$ . We consider each situation corresponding to an entry which has not been changed and determine whether it can be changed now. The entries that change correspond

$$\begin{bmatrix} 0 & X & X & (1,4) & 1 & (1,6) \\ 1 & 0 & 1 & (2,4) & 1 & (2,6) \\ 1 & X & 0 & (3,4) & 1 & (3,6) \\ X & (4,2) & X & 0 & (4,5) & (4,6) \\ X & X & X & (5,4) & 0 & X \\ X & (6,2) & (6,3) & X & (6,5) & 0 \end{bmatrix}$$

Figure 3.4: The array corresponding to  $k = 1$ 

to situations where, for every vertex to which the Robber could move, there is a vertex to which the Cop can move so that the entry corresponding to the resulting situation is 0 or 1. We will consider the situations corresponding to the entries  $(4,5)$  and  $(4,6)$ , which are bolded in Figure 3.5, and leave consideration of the rest to the reader.

For  $(4,5)$ , the Robber has three options for moves: move to vertex 1, move to vertex 3, or stay on vertex 4. If the Robber moves to vertex 1, the Cop can move to vertex 1 as well, and  $(1,1)$  is on the list when  $k = 0$ . If the Robber moves to vertex 3, the Cop can move to vertex 3 as well, and  $(3,3)$  is on the list when  $k = 0$ . If the Robber stays on vertex 4, then the Cop can stay on vertex 5, or can move to vertex 1, 2, 3, or 6. However,  $(4,1) = X$ , and  $(4,2)$ ,  $(4,3)$ , and  $(4,6)$  are not on the list when  $k = 1$ . Because the Robber can stay on vertex 4 and the Cop does not have a move that results in a position that on the list when  $k = 1$ , the label  $(4,5)$  remains unchanged. For  $(4,6)$ , the Robber's possible moves are to vertex 1, 3, or 4, and the Cop has moves available that result in positions  $(1,1)$ ,  $(3,1)$ , and  $(4,4)$  respectively. Both  $(1,1)$  and  $(4,4)$  are on the list when  $k = 0$  and  $(3,1)$  is on the list when  $k = 1$ . We therefore change the entry in row 4, column 6 from  $(4,6)$  to 2.

When  $k = 3, 4, \dots$  the situation is analyzed exactly as above.

Once we have a completed array, which we denote by  $R$ , we can determine who will win. Because our graph has a source vertex, namely vertex 5, we know from Proposition 3.1.1 that the Cop must choose to start there, otherwise the Robber can win. Remember that the

$$\begin{bmatrix} 0 & X & X & 2 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 & 2 \\ 1 & X & 0 & 2 & 1 & 2 \\ X & (4,2) & X & 0 & \mathbf{(4,5)} & \mathbf{2} \\ X & X & X & (5,4) & 0 & X \\ X & (6,2) & (6,3) & X & (6,5) & 0 \end{bmatrix}$$

Figure 3.5: The array corresponding to  $k = 2$ 

$$R = \begin{bmatrix} 0 & X & X & 2 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 & 2 \\ 1 & X & 0 & 2 & 1 & 2 \\ X & (4,2) & X & 0 & 3 & 2 \\ X & X & X & (5,4) & 0 & X \\ X & (6,2) & (6,3) & X & 4 & 0 \end{bmatrix}$$

Figure 3.6: The completed array,  $R$ , for the oriented graph  $G$  in Figure 3.3

array,  $R$ , in Figure 3.6 is from the perspective that the Cop and Robber occupy vertices and it is the Robber's turn to move. It does not immediately give any information about which vertex the Robber should choose to start the game, or any information about the length of the game, but it can be used to obtain both of those things. We will make an array with 1 row and 6 columns to give information about the Robber's possible starting positions, then the Robber can use it to make a choice. Entry  $j$  of this array will be the number of Cop moves needed for the Cop to win the game if the Robber chooses to start at vertex  $j$ , or  $\infty$  if the Robber can win the game by starting at vertex  $j$ . We will call it the *Cop win time array*.

We need to analyze the situation that arises for each possible choice of vertex where the Robber could start. The possible vertices to which the Cop could then move are 1, 2, 3, 5, and 6. If the Robber starts on one of these vertices, then the Cop can win in at most one

move. Therefore, we only need to consider the case where the Robber starts on vertex 4. We consider the Cop's possible moves to the vertices mentioned above in turn. If the Cop:

- Moves to vertex 1, then because the  $(4, 1)$ -entry of  $R$  is  $X$ , the Robber has a winning strategy by staying behind the Cop.
- Moves to vertex 2, then since there no arcs from vertex 2 to another vertex, the Cop's only choice is to stay there on each subsequent move. The Robber can then win by staying at vertex 4.
- Moves to vertex 3, then because the  $(4, 3)$ -entry of  $R$  is  $X$ , the Robber has a winning strategy by staying behind the Cop.
- Stays at vertex 5, then because the  $(4, 5)$ -entry of  $R$  is 3, the Cop has a winning strategy and can win in at most 3 more Cop moves (exactly 3 more Cop moves if the Robber plays optimally).
- Moves to vertex 6, then because  $(4, 6)$ -entry of  $R$  is 2, the Cop has a winning strategy and can win in at most 2 more Cop moves (exactly 2 more Cop moves if the Robber plays optimally).

Thus, if the Robber starts at vertex 4, then the Cop should move to vertex 6, and will be able to win the game in a total of  $1 + 2 = 3$  Cop moves. This means that entry 4 of the Cop win time array is 3.

The completed Cop win time array,  $C$ , is

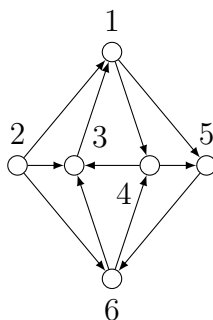
$$C = \begin{bmatrix} 1 & 1 & 1 & 3 & 0 & 1 \end{bmatrix}.$$

The completed array indicates where the Robber should start. If the Robber plays optimally, that is to make the game last as long as possible, then their starting vertex should

be the one corresponding to the largest entry in the Cop win time array. In this case, the Robber would start on vertex 4 and the game will end in 3 Cop moves.

Let's analyze the oriented graph in Figure 3.2 in Part I of this article. The oriented graph on which the game is being played is shown again in Figure 3.7. We also give the completed array and Cop win time array.

Vertex 2 is a source vertex so the Cop must start there by Proposition 3.1.1. We analyze



$$R = \begin{bmatrix} 0 & (1,2) & (1,3) & X & X & (1,6) \\ X & 0 & X & (2,4) & (2,5) & X \\ X & 1 & 0 & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & X & 0 & X & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & 0 & X \\ (6,1) & (6,2) & X & X & (6,5) & 0 \end{bmatrix}$$

$$C = [1 \ 0 \ 1 \ \infty \ \infty \ 1]$$

Figure 3.7: Oriented Graph from Figure 3.2, corresponding completed array,  $R$ , and Cop win time array,  $C$ .

the situation for  $r = 5$  in the Cop win time array. The possible vertices to which the Cop could then move are 1, 2, 3, and 6. We consider these in turn. If the Cop:

- Moves to vertex 1, then because the  $(5,1)$ -entry of  $R$  is unlabelled, the Robber has a move, no matter what move the Cop makes, to a vertex so that the entry in  $R$

corresponding to the players' new positions is unlabelled. Therefore, the Cop does not have a winning strategy.

- Stays on vertex 2 then, similarly, because the  $(5, 2)$ -entry of  $R$  is unlabelled, the Cop does not have a winning strategy.
- Moves to vertex 3 then, similarly again, because the  $(5, 3)$ -entry of  $R$  is unlabelled, the Cop does not have a winning strategy.
- Moves to vertex 6, then because  $(5, 6)$ -entry of  $R$  is  $X$ , the Robber has a winning strategy by staying behind the Cop.

Thus, if the Robber starts at vertex 5, for any move the Cop makes, the Robber has a winning strategy. We label entry 5 of the Cop win time array with  $\infty$ .

To start the game, the Robber chooses a vertex corresponding to a largest entry in the Cop win time array. In this case, since that entry is  $\infty$ , the Robber has a winning strategy that begins by choosing vertex 4 or 5 as the initial position.

We have now seen an oriented graph on which the Cop wins and an oriented graph on which the Robber wins. Our analysis applies in general and leads to the following propositions.

**Proposition 3.2.1.** *In the game of Cop and Robber on a given oriented graph  $G$  with a source vertex  $s$ , the Cop wins if and only if there is a number in every entry of the Cop win time array,  $C$ .*

**Proof.** By Proposition 3.1.3, and Corollaries 3.1.4 and 3.1.5, the Cop must begin the game by choosing vertex  $s$ , otherwise the Robber has a winning strategy. Hence assume that the Cop starts at vertex  $s$ , and that the Cop win time array,  $C$ , has been constructed.

Suppose every entry of  $C$  is a number, and that the Robber starts the game by choosing the vertex  $r$  for which the corresponding entry of  $C$  equals  $k$ . Then, by construction of  $C$ ,

the Cop has a move to a vertex  $c$  so that entry  $(r, c)$  of  $R$  equals  $k$ . Furthermore, it is now the Robber's turn to move. By construction of  $R$ , for any vertex  $r_1$  to which Robber moves, the Cop has a move to a vertex  $c_1$  so that entry  $(r_1, c_1)$  of  $R$  equals  $k_1 < k$ . (In fact, by construction of  $R$ , the Robber can choose  $r_1$  so that  $k_1 = k - 1$ .) From there, for any vertex  $r_2$  to which Robber moves, the Cop has a move to a vertex  $c_2$  so that entry  $(r_2, c_2)$  of  $R$  equals  $k_2 < k_1$ . Continuing in this way, eventually no matter the vertex the Robber chooses, the Cop has a move so that the corresponding entry of  $R$  equals 0. Therefore the Cop wins.

On the other hand, suppose that there is an  $\infty$  in the Cop win time array,  $C$ , corresponding to vertex  $r$ . Then, for any vertex  $c$  where the Cop can move from  $s$ , the entry either entry  $(r, c)$  of the array  $R$  is  $X$ , or unfilled (i.e., it is the ordered pair  $(r, c)$ ). Suppose the Robber chooses  $r$  as the initial vertex.

If entry  $(r, c)$  of  $R$  is  $X$ , then there is an arc from  $r$  to  $c$ . The Robber can maintain the situation where there is an arc to the Cop's current vertex by staying put if the Cop stays put, and by moving to the vertex the Cop just left otherwise. Since there is at most one arc between any two vertices, the Robber cannot be caught on any Cop move. Therefore the Robber wins.

Finally, suppose entry  $(r, c)$  of  $R$  is unfilled. By construction of  $R$ , there is a vertex  $r_1$  where the Robber can move so that, for any vertex  $c_1$  where the Cop moves, entry  $(r_1, c_1)$  of  $R$  is unfilled. Again by construction of  $R$  the Robber can move as above so as to maintain the situation that the entry of  $R$  corresponding to the players' current positions on the Robber's next turn is unfilled. Therefore the Robber wins.

This completes the proof. □

**Corollary 3.2.2.** *In the game of Cop and Robber on a given oriented graph  $G$  with a source vertex  $s$ , the Robber wins if and only if there is an  $\infty$  in the Cop win time array  $C$ .*

The following Proposition applies no matter whether the given oriented graph  $G$  has a source vertex.

**Proposition 3.2.3.** *If every column of the completed array,  $R$ , contains an  $X$ , then the Robber has a winning strategy.*

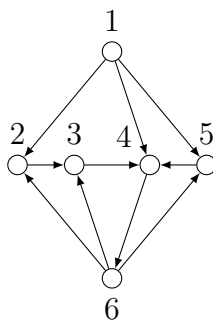
**Proof.** Suppose there is an  $X$  in every column of the completed array. Suppose the Cop starts on vertex  $c$ . Since there is an  $X$  in every column, there is a vertex  $r$  so that entry  $(r, c)$  of  $R$  is  $X$ . The Robber can choose vertex  $r$  to start the game and be behind the Cop. Thus the Robber has a winning strategy.  $\square$

The results and methods of analysis presented in this 2-part article show how to determine which of the Cop and Robber wins the game on a given oriented graph. If the oriented graph does not have exactly one source vertex, then the Robber wins. If it has exactly one source vertex, we know the Cop must start there. To analyze the game in this case, first construct the array  $R$  and, using that, construct the Cop win time array  $C$ . From the Cop win time array, one can determine who wins. The various winning strategies are described in the proof of Proposition 3.2.1

Similar methods can be applied to other types of graphs. These are described in the research paper [11], the contents of which further develop some of the methods and results that can be found in the very readable and interesting book on Cops and Robber by Bonato and Nowakowski [12].

### Questions

1. Who wins the game on the oriented graph shown below?



2. A vertex  $x$  of an oriented graph  $G$  is called a *corner* if there exists a vertex  $w$  such that there is an arc from  $w$  to every vertex where it is possible to move from  $x$  (including  $x$ ), and maybe some other vertices too. Show that if the Cop wins, and the Robber makes the game last as long as possible, then the Cop catches the Robber at a corner.
3. Let  $G$  be an oriented graph with  $n$  vertices. Show that if the Cop has a winning strategy on  $G$ , then the game ends in at most  $n^2$  moves.

### 3.3 Survey of Previous Results for Cops and Robber on Oriented Graphs

Now that we have introduced the game and established preliminary results on the game of Cop and Robber on oriented graphs, we survey earlier results in the literature for this version of the game. We will use terminology and notation from [6]. Note that the game can be played with more than one cop: one player moves a set of  $k > 0$  Cops, and the other side moves the Robber. A move for the Cops consists of each cop staying at their current vertex, or sliding along an arc (or edge) to an adjacent vertex; a move for the Robber is the same as before, as are the goals of each side. The *cop number*,  $c(G)$ , is the number of Cops needed to catch the Robber on a graph  $G$  [2]. If a graph  $G$  has cop number  $k$ , we say that  $G$  is *k-cop-win*.

While there has been very little research done on Cops and Robber when played on digraphs, the first three results are for the game played on digraphs. In 1987, Hamidoune studied upper bounds for the cop number on Cayley digraphs [33]. We will summarize the results below and state the definition of a Cayley graph.

**Definition 3.3.1.** *Let  $H$  be a finite group and  $S \subseteq H - \{e\}$ . Then the Cayley digraph on  $H$ ,  $C(H, S) = (H, E)$ , is defined by  $E = \{(x, y) \mid x^{-1}y \in S\}$ .*

**Theorem 3.3.2.** [33] *Let  $G$  be a strongly connected Cayley digraph on an abelian group with out-degree  $k$ . Then*

$$c(G) \leq k + 1.$$

Next, in 1987, Frankl answered an open question of Hamidoune's on Cayley digraphs on abelian groups [29].

**Theorem 3.3.3.** [29] *Let  $G$  be a connected Cayley digraph on abelian group  $H$ . Suppose the minimum degree of  $G$  is greater than  $d$  and its directed girth is at least  $8t - 3$ . Then*

$$c(G) > d^t.$$

Recent work on the cop number on Cayley digraphs can be found in [14].

Recently, Slívová studied *tournaments*, orientations of complete graphs, in her 2015 Bachelor's Thesis [50]. She showed some lower and upper bounds for the cop number of tournaments.

**Theorem 3.3.4.** [50] *Let  $G$  be a tournament and  $k \in \mathbb{N}, k \geq 2$ . If there exists a vertex  $v \in V(G)$  that  $d^-(v) \geq |V(G)| - 3 \cdot 2^{(k-1)} + 1$  then  $c(G) \leq k$ . Furthermore,  $k$  Cops can win the game in 3 Cop moves.*

**Corollary 3.3.5.** [50] *Let  $k \in \mathbb{N}, k \geq 2$  and  $G$  be a tournament on at most  $3 \cdot 2^k - 2$  vertices. Then  $c(G) \leq k$ .*

Therefore all tournaments on at most 10 vertices have cop number less than three. Geňa Hahn conjectured that for each oriented Steiner triple graph, a special class of graphs defined below, 2 Cops can catch the Robber in at most 2 Cop moves [13]. Slívová presented a counter-example and showed that oriented Steiner triple graphs with arbitrarily large cop number.

**Definition 3.3.6.** *Let  $X$  be a set and  $Y$  be a set of three element subsets of  $X$ . Then we*

say that  $Y$  is a Steiner triple system if each 2-element subset of  $X$  is in exactly one triple from  $Y$ .

**Definition 3.3.7.** A graph  $G$  with vertex set  $V$  and such that for each triple  $\{y_0, y_1, y_2\} \in Y$

$$\text{either } (y_0, y_1), (y_1, y_2), (y_2, y_0) \in E$$

$$\text{or } (y_1, y_0), (y_2, y_1), (y_0, y_2) \in E$$

is an oriented Steiner triple graph for this Steiner triple system.

**Theorem 3.3.8.** [50] For each  $k \in \mathbb{N}$ , there exists an oriented Steiner triple graph  $G$  with cop number  $c(G) \geq k$ .

Darlington *et al.* searched for a characterization of cop-win oriented graphs [22]. They described some necessary conditions for an oriented graph to be cop-win. Most of their results use the idea of a vertex being “reachable” from other vertices in the graph.

**Definition 3.3.9.** [22] A vertex  $x_1$  is reachable from  $x_0$  if there exists a directed path from  $x_0$  to  $x_1$

They have some elementary results similar to those discussed in Section 3.1. They also proposed the idea of cop-dominated cycles. A directed cycle is called *cop-dominated* if the Robber cannot avoid capture by moving to a vertex of the directed cycle and then travelling around the cycle in response to the Cop’s moves. Darlington *et al.* provided necessary conditions for a terminal strong component which is a directed cycle to be cop-dominated.

**Theorem 3.3.10.** [22] Let  $C$  be directed cycle in an oriented graph  $G$  such that every vertex of  $C$  has out-degree 1. The following conditions are necessary for  $C$  to be cop-dominated.

1. Every vertex on  $C$  is an out-neighbour of a vertex not on  $C$ .
2. There exists a vertex of  $G \setminus C$  which corners a vertex on the cycle.

They conjectured that if every directed cycle is cop-dominated, then the graph is cop-win. This conjecture was disproved in 2018 by Khatri *et al.* [36]. Their counterexample is shown in Figure 3.8.

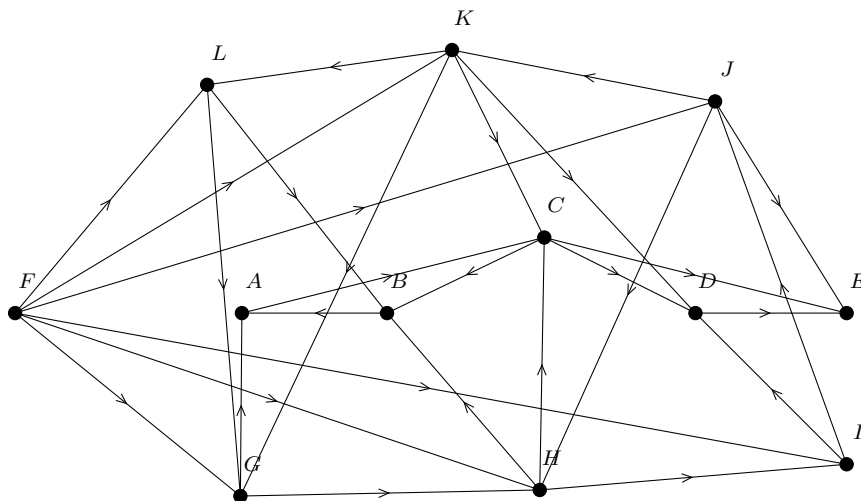


Figure 3.8: Counterexample to oriented cop-win characterization conjecture

The five cycles in the oriented graph in Figure 3.8 are all cop-dominated, but the Robber can avoid capture by moving around on any one of them. We now generalize their counterexample to provide an infinite family of counterexamples.

Let there be two directed cycles,  $C_1$  and  $C_2$ , that share an arc. Suppose  $C_1$  is cop-dominated by a directed path  $P_1$ , and  $C_2$  is cop-dominated by a directed path  $P_2$  that has no vertices in common with  $P_1$ . Construct a graph  $G$  by adding a new vertex,  $s$ , and arcs from  $s$  to every vertex of  $P_1$  and every vertex of  $P_2$ . Let  $\mathcal{F}$  denote the set of all oriented graphs that can be constructed in this way. An oriented graph in this infinite class is shown in Figure 3.9.

**Theorem 3.3.11.** *Let  $G$  be an oriented graph in the class  $\mathcal{F}$ . The Robber can win when the*

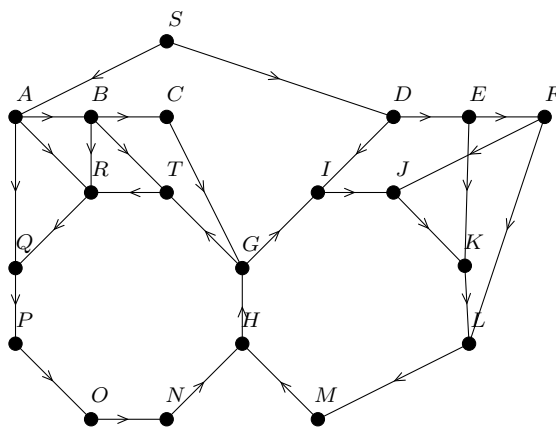


Figure 3.9: Example from the class of counterexamples

game is played on  $G$ .

*Proof.* Let  $G$  be a graph in the class  $\mathcal{F}$ . Since  $s$  is a source vertex, the Cop must start there. Let the arc from vertex  $x$  to vertex  $y$  be the arc that is shared by  $C_1$  and  $C_2$ . Let the Robber choose the vertex  $y$ . The Cop has two options: move to the first vertex of  $P_1$  or move to the first vertex of  $P_2$ . If the Cop moves to  $P_1$ , the Robber will move onto another vertex of  $C_2$ . Since  $C_2$  is not cop-dominated by  $P_1$ , the Robber avoid capture. Similarly, if the Cop moves to  $P_2$ , the Robber will move onto another vertex of  $C_1$ . Since  $C_1$  is not cop-dominated by  $P_2$ , the Robber avoid capture. In both cases, the Robber can avoid being caught from the Cop and wins the game.  $\square$

The  $k$ -capture time is defined as the number of rounds required for  $k$  Cops to capture a Robber in a  $k$ -cop-win graph and the invariant capture time is denoted  $\text{capt}_k(G)$  [10]. When  $k = 1$ , we write  $\text{capt}(G)$  instead of  $\text{capt}_1(G)$ . The capture time of directed graphs has been examined by Kinnersley, who showed that for every  $k \geq 1$  there exists an  $n$ -vertex directed graph  $D$  for which  $\text{capt}_k(D) \in \Theta\left(\binom{n}{k}^{k+1}\right)$  [37].

Next, Khatri *et al.* also examined the capture time of a  $k$ -cop win oriented graph [36]. They also present an infinite family of cop-win oriented graphs with quadratic capture time.

**Theorem 3.3.12.** [36] *There exists an infinite family of oriented graphs on  $n$  vertices with capture time  $\Theta(n^2)$ .*

Lastly, they consider optimal strategies and some situations that occur during optimal play on an oriented graph.

**Theorem 3.3.13.** [36] *There exist cop-win oriented graphs in which, if both the Cop and the Robber play optimally, the distance between the Cop and Robber must strictly increase in two consecutive rounds.*

This result is interesting because when the game is played on oriented graphs, the distance between the Cop and Robber is never increasing. Another result from this paper, that is different than that of the undirected graph case, is that the Cop returns to a previously visited vertex during optimal play.

**Theorem 3.3.14.** [36] *There exists a cop-win oriented graph in which the Cop must revisit a vertex in optimal play.*

While there has been some interesting progress on developing the theory of the game of Cops and Robber on oriented graphs, particularly with tournaments, cop-dominated directed cycles, and quadratic capture times, there is still work that remains to be done in terms of finding new characterizations of cop-win oriented graphs that are different from those derived from the general game that will be discussed in Chapter 4. Additionally, a sufficient condition for oriented graph with directed cycles to be cop-win is still unknown.

## 3.4 Results on the Game of Cops and Robber on Oriented Graphs

Now, we present a result which follows from the work of Nowakowski and Winkler [45], but apparently has never been written down or noted in the graph searching community. This

result applies to both directed and undirected graphs.

**Theorem 3.4.1.** *If  $G$  is a cop-win graph then there exists a strategy where at least one Cop moves in each round.*

*Proof.* Suppose otherwise, that  $G$  is cop-win and the Cop does not move in a round and is on vertex  $c$ . It is now the Robber's turn. Suppose the Robber is on vertex  $r$ . Since  $G$  is cop-win, the position  $(c, r)$  is numbered  $k$ , in the  $k$ -th relation but not the  $(k-1)$ -th relation, as discussed in Section 3.2. By definition of the nested sequence of relations, which will be described in Section 4.4, no matter where the Robber moves from  $r$ , the Cop can win in at most  $k$  moves. If the Robber stays at vertex  $r$ , then the position  $(c, r)$  is still numbered  $k$  and the Cop has a move to a lower numbered position. If the Robber moves to a new vertex  $r'$ , then  $(c, r')$  must be numbered, say  $k_2$ . Since the Robber would not move to a lower numbered position, assume that  $k_2 \geq k$ . Therefore, the Cop should not have stayed on vertex  $c$ , if playing an optimal strategy. So for each case, the Cop has a move to a lower numbered position by moving to a new vertex rather than staying on  $c$ .  $\square$

Next, we look at the structure of a directed path of vertices with arcs to a directed cycle. We know that the Cop should move in each round, and furthermore, the Cop should continue to move along the directed path until the Cop catches the Robber on the cycle. We will need the following definition.

**Definition 3.4.2.** *Let  $G$  be an oriented graph. If there exists a partition  $(P, C)$  of  $V(G)$  such that the subgraph induced by  $P$  is a directed path and the subgraph induced by  $C$  is a directed cycle, then we say that  $G$  is a path-covered cycle.*

Next, we will provide a sufficient condition for cop-win oriented graphs with a directed cycle.

**Theorem 3.4.3.** *Let  $G$  be an oriented graph consisting of a path-covered cycle with directed path  $P = p_1, p_2, \dots, p_n$  and directed cycle  $C = c_1, c_2, \dots, c_l$ . The oriented graph  $G$  is cop-win*

if

- every vertex  $c_i \in \mathcal{C}$  is cornered by at least one vertex  $p_j \in P$ , for all  $i \in \{1, 2, \dots, n\}$  and  $j \in \{1, 2, \dots, l\}$  and
- every vertex  $c_i \in \mathcal{C}$  is cornered by  $p_j$  and  $c_{i+1}$  is cornered by  $p_{j-1}$ , for all  $i \in \{1, 2, \dots, n\}$  and  $j \in \{2, 3, \dots, l\}$ .

*Proof.* Suppose that the Robber starts on a vertex  $c_i$  on the cycle  $\mathcal{C}$ , for some  $i \in \mathbb{N}$ . The Cop must start on the source vertex, which is necessarily on the path  $P$  and move to the next vertex in the path, otherwise the Cop and Robber are both on the cycle and the Robber wins. Let the Cop be on the vertex  $p_1 \in P$ . We will prove that the Cop can catch the Robber if the Robber is on  $c_i$ , which is cornered by  $p_k$ , by induction on  $k$ . If  $k = 1$  then the Cop can win in one move, since  $c_i$  is cornered by  $p_1$ . Assume that the Cop can catch the Robber if the Robber is on  $c_i$ , which is cornered by  $p_k$  for some  $k > 1$ . Suppose that the Cop is on  $p_1$  and it is the Robber's turn to move. The Robber has two options, stay on  $c_i$ , or move to  $c_{i+1}$ . Suppose that the Robber moves to  $c_{i+1}$ . Since  $c_{i+1}$  is cornered by  $p_{k-1}$ , the Cop can catch the Robber, by the induction hypothesis. If the Robber stays on  $c_i$ , then the Cop can move to  $p_2$ . Since  $P$  is a directed path, there is no arc to  $p_1$  so we can remove  $p_1$  from the graph. Let  $G' = G - \{p_1\}$ . In  $G'$ , the Cop is on  $p_1$  and the Robber is on  $c_i$ . However, in  $G'$ ,  $c_i$  is cornered by  $p_{k-1}$ . By the induction hypothesis, the Cop can win on  $G'$ . By induction, the Cop can catch the Robber and  $G$  is cop-win.  $\square$

## Chapter 4

# Generalized Game of Cops and Robber

There have been several characterizations of finite cop-win graphs. When the game was first introduced by Nowakowski and Winkler, and Quilliot independently, they provided a vertex elimination characterization for cop-win undirected graphs. We will see that this characterization does not carry over to oriented graphs. Later in this chapter, we will introduce the generalized game of Cops and Robber [11], and show that any such game is a combinatorial game. In turn, this implies that generalized Cops and Robber games can be analyzed using tools from combinatorial game theory. We then review some characterizations of generalized Cops and Robber games in which the Cops have a winning strategy. Each of these provides a characterization of the cop-win oriented graphs.

### 4.1 Vertex elimination characterization of cop-win graphs

The vertex elimination characterization is based on the idea of a ‘corner’. We call a vertex ‘cornered’ if the Robber’s closed neighbourhood is a subset of the Cop’s closed neighbourhood.

**Definition 4.1.1.** [45] *A vertex  $v$  is a corner of  $G$  if for some  $u \neq v$ ,  $N(v) \subseteq N(u)$ ; we say  $v$  is cornered by  $u$ .*

**Theorem 4.1.2.** [45] *Let  $G$  be a graph, and let  $c$  be a corner of  $G$ . Let  $G' = G - \{c\}$ . Then  $G$  is cop-win if and only if  $G'$  is cop-win.*

Theorem 4.1.2 states that removing a corner from a graph does not change the outcome of the game. We can continue to remove corners from the resulting graphs until no corners exist.

**Definition 4.1.3.** *A graph  $G$  is dismantlable if there is an ordering  $(v_1, v_2, \dots, v_n)$  of the vertices of  $G$  such that for each  $i < n$ ,  $v_i$  is a corner in the subgraph  $G_i$  induced by  $\{v_i, \dots, v_n\}$ .*

**Theorem 4.1.4.** [45] *A finite graph is cop-win if and only if it is dismantlable.*

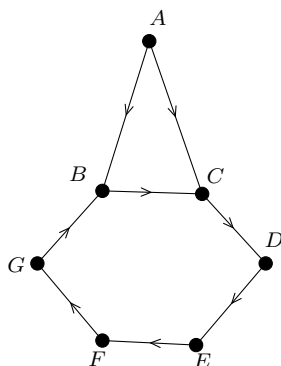


Figure 4.1: Counterexample for the dismantlable characterization on an oriented graph

Note that this characterization does not carry over to oriented graphs or to infinite graphs. For more details on infinite graphs, see [58]. As an example, take Theorem 4.1.2, which states that when a corner is removed from a cop-win graph, the resulting graph is still cop-win. However in the oriented graph case, this is not true. In the graph pictured in Figure 4.1, the vertex  $B$  is a corner; in fact, it is the only corner in the graph. The graph is robber-win; the robber can simply stay on a vertex on the cycle that does not have an arc

from  $A$ . The Cop has to either stay on  $A$  or move onto the directed cycle. In both cases, the robber has a winning strategy. The graph  $G'$  that results from removing the corner  $B$  has no directed cycles and therefore is cop-win.

## 4.2 The Generalized Game of Cops and Robber

The many variations of the game of Cops and Robber, whether it is played on undirected or oriented graphs, admit essentially the same analysis. The analysis is the same with one Cop or more than one Cop (for example, see Question 4 in Section 3.1). Therefore an appropriate definition of a general Cops and Robber game would permit the analysis of all Cops and Robber games. A *generalized Cops and Robber game* is any discrete-time process  $\mathcal{G}$  that satisfies the following rules [11]:

1. There are two players named the Pursuer and the Evader.
2. There is perfect information.
3. There is a set  $\mathcal{P}_P$  of allowed positions for the Pursuer and a set  $\mathcal{P}_E$  of allowed positions for the Evader. The set of positions of the game is a subset  $\mathcal{P} \subseteq \mathcal{P}_P \times \mathcal{P}_E$ . Similarly, the set of states of the game is the subset  $\mathcal{S} \subseteq \mathcal{P} \times \{P, E\}$  such that  $((p_P, q_E), X) \in \mathcal{S}$  if the position  $(p_P, q_E)$ , with  $X$  next to move.
4. For each state of the game and each player, there is a non-empty set of allowed moves. Each allowed move leaves the position of the other player unchanged.
5. The rules of the game determine the set  $\mathcal{I} \subseteq \mathcal{P}_P \times \mathcal{P}_E$  of allowed start positions. We define the set  $\mathcal{I}_P = \{p_P : (p_P, q_E) \in \mathcal{I} \text{ for some } q_E \in \mathcal{P}_E\}$  and, for  $p_P \in \mathcal{P}_P$ , We define the set  $\mathcal{I}_E(p_P) = \{q_E \in \mathcal{P}_E : (p_P, q_E) \in \mathcal{I}\}$ . The game  $\mathcal{G}$  begins by the Pursuer choosing a position  $p_P \in \mathcal{I}_P$ , and then the Evader choosing a position  $q_E \in \mathcal{I}_E(p_P)$ .

6. After each player has chosen their initial position, the sides move alternately with the Pursuer moving first. Each player, on their turn, must choose an allowed move given the current state of the game.
7. The rules of the game specify when the Pursuer has caught the Evader. That is, there is a subset of allowed positions  $\mathcal{F}$  that are final positions. The Pursuer wins the game  $\mathcal{G}$  if, at any time-step, the current position of the game belongs to  $\mathcal{F}$ . The Evader wins if their current position never belongs to  $\mathcal{F}$ .

We define a generalized Cops and Robber game  $\mathcal{G}$  to be *position independent* if, for all  $p_P \in \mathcal{P}_P$  and  $q_E \in \mathcal{P}_E$ , whenever  $(p_P, q_E) \notin \mathcal{F}$  the set of allowed moves for the Pursuer at the position  $(p_P, q_E)$  depends only on  $p_P$  and the set of allowed moves for the Evader at the position  $(p_P, q_E)$  depends only on  $q_E$ . That is, if the game is not over, then from any position of the game, the set of available moves for a player does not depend on the position of the other player. This collection of moves includes optimal moves and other moves. In what follows, we will assume that the game of Cops and Robber is position independent unless we say otherwise.

### 4.3 Cop and Robber as a Combinatorial Game

For this section, the game of Cop and Robber refers to the game when played with one Cop and one Robber. In order to view the game of Cop and Robber as a combinatorial game, we must define what it means to be such a game.

**Definition 4.3.1.** [8] *A normal combinatorial game is a game that satisfies the following conditions:*

1. *There are just two players.*
2. *Both players know what is going on; there is perfect information.*

3. *There are finitely many positions, and particular starting positions.*
4. *There are clearly defined rules that specify the moves that either player can make from a given position to their options. There are no elements of chance.*
5. *The two players move alternately.*
6. *The rules are such that play will always come to an end because some player will be unable to move.*
7. *The player unable to move loses.*

We will describe several ways to alter the game of Cop and Robber so that the game is finite with the same outcome and therefore can be analyzed as a combinatorial game. The following methods will be described: adding a position array and marking positions as they occur, having a fixed number of rounds for the Cop to catch the Robber, and playing on a related graph.

The first way to alter the game of Cop and Robber is to play the game as normal and have an additional game board: an array. The array is indexed by Cop positions and Robber positions. As the game is played, the positions are marked in the array after the Robber's move is played. Under optimal play, no position is repeated, or else the Robber could repeat the same sequence of moves forever. A legal move for the Cop, where the Cop is at a vertex  $c$  and the Robber is at a vertex  $r$ , is a move from vertex  $c$  to a vertex  $c'$  if and only if  $(c', r)$  is an option from  $(c, r)$  and  $(c', r)$  is not marked in the array. A legal move for the Robber, where the Cop is at a vertex  $c$  and the Robber is at a vertex  $r$ , is a move from vertex  $r$  to a vertex  $r'$  if and only if  $(c, r')$  is an option from  $(c, r)$  for the Robber and  $(c, r)$  is not a final position. The combinatorial game continues until the Cop catches the Robber (they are on the same vertex), and the Robber cannot move, or the Robber moves so that all of the Cop's allowed positions are already marked in the array and the Cop cannot move.

The second way to change the game of Cop and Robber is to play the game as normal and fix the number of moves, and require the Cop to catch the Robber within that number. If the Cop does not catch the Robber after the fixed number of moves, then the Cop cannot move. A legal move for the Robber, when the Cop is at vertex  $c$  and the Robber is at vertex  $r$ , is from vertex  $r$  to a vertex  $r'$  if and only if  $(c, r')$  is an option from  $(c, r)$  for the Robber and  $(c, r)$  is not a final position. We fix the number of rounds to be the total number of positions, denoted  $N$ . Under optimal play, if all positions have been played and the Robber has not been caught, then the Robber will repeat the sequence of moves again. The combinatorial game is played until the Cop catches the Robber and the Robber cannot move, or until the Cop has made  $N$  moves, and then the Cop cannot move. A similar approach was used for considering the eternal domination set problem as a combinatorial game by Finbow *et al.* [28].

Finally, the last alteration is to play an equivalent game on a modified graph. This combinatorial game is played on one game board; i.e. there is no separate array to mark or round count to change. We modify the graph  $G$  as follows: take an array of all of the positions in the game of Cop and Robber, and index the rows by the Cop positions and the columns by the Robber positions. We add vertical arcs from a position  $(c, r)$  to  $(c', r)$  if that is an option from that position for the Cop. Similarly, we add horizontal arcs from a position  $(c, r)$  to  $(c, r')$  if that is an option from that position for the Robber. Next, we remove the arcs from the final positions, including loops. The Cop plays on the vertical edges, and the Robber plays on the horizontal edges. After each Robber move, the vertex that represents the current position of the Cop and Robber is coloured. A legal move for the Cop must be an allowed option and the vertex must not be coloured. A legal move for the Robber must be an allowed option. The combinatorial game is played until the Robber is in a final position and the Robber cannot move, or until the Cop cannot move.

## 4.4 Survey of Previous Results on Cops and Robber Game

Recall that we will only focus on position independent games. There are several characterizations of cop-win graphs for a generalized game of Cops and Robber, which we will describe in this section. Note that some of the cop-win characterizations were first constructed for undirected graphs but can be adapted for a generalized game.

The first characterization comes from Nowakowski and Winkler [45]. This characterization is based on a recursive relation.

Let  $G$  be a graph; we define a relation  $\preceq$  on the vertices of the graph. The relation is defined recursively by setting  $x \preceq_0 x$  for each vertex  $x$  of the graph. For each ordinal  $\alpha$ , define  $\preceq_\alpha$  by  $x \preceq_\alpha y$  if and only if, for each  $u \in N(x)$ , there exists  $v \in N(y)$  such that  $u \preceq_\beta v$  for some  $\beta < \alpha$ . Let  $\rho$  be the least ordinal such that  $\preceq_\rho = \preceq_{\rho+1}$  and define  $\preceq = \preceq_\rho$ . We will call this relation the *Nowakowski-Winkler relation*.

**Theorem 4.4.1.** [45] *The graph  $G$  is cop-win if and only if  $\preceq = V(G) \times V(G)$ .*

The second characterization comes from Clarke and MacGillivray [20]. Let  $G$  be a finite graph and let  $\mathcal{P} = \mathcal{P}(G)$  be the graph whose vertices are the possible positions of the  $k$  Cops on the graph  $G$ , with  $pq \in E(\mathcal{P})$  if and only if it is possible for the Cops to move from position  $p$  to position  $q$ , or vice versa. For  $y \in V(G)$ , let  $J_y = \{(q, y) \in V(\mathcal{P} \times G)\}$ .

**Definition 4.4.2.** *A vertex  $(p, x)$  of  $\mathcal{P} \times G$  is called removable with respect to  $S \subseteq V(\mathcal{P} \times G)$  if either*

1. *in position  $p$ , one of the Cops is located at vertex  $x$ ; or*
2. *for every  $y \in N_G(x)$ ,  $N_{\mathcal{P} \times G}((p, x)) \cap J_y \cap S \neq \emptyset$ .*

**Theorem 4.4.3.** [20] *A finite graph  $G$  is  $k$ -cop-win if and only if there is a sequence  $S = (p_1, x_1), (p_2, x_2), \dots, (p_t, x_t)$  of vertices of  $P \times G$  such that*

1.  $t \leq |V(\mathcal{P} \times G)|$ ;
2. for  $1 \leq i \leq t$ , the vertex  $(p_i, x_i)$  is removable with respect to  $\{(p_j, x_j) : j < i\}$ ; and
3. for every  $x \in V(G)$ ,  $(p_t, x)$  belongs to  $S$ .

The next characterization also comes from Clarke and MacGillivray [20]. For a finite graph  $G$ , define a bipartite graph  $B = B(G)$  with bipartition  $(V(G), V(\mathcal{P}))$  and  $x \in V(G)$  is adjacent to  $p \in V(\mathcal{P})$  when  $x \preceq p$ .

**Theorem 4.4.4.** [20] *A connected finite graph  $G$  is  $k$ -cop-win if and only if there is a vertex  $p \in V(\mathcal{P})$  that is adjacent in  $B$  to every vertex of  $V(G)$ .*

The fourth characterization is from Hahn and MacGillivray [32] and independently from Berarducci and Intrigila [7]. This characterization creates a directed bipartite graph whose vertices and edges indicate possible moves and states of the game. This graph is labelled in such a way as to indicate the winner of the game. More precisely, consider a digraph  $D$  on which a game of Cop and Robber is played. After each move, the situation of the game can be described by stating where each player is and whose turn it is to move. Consider  $V(D) \times V(D) \times \{c, r\}$  and use  $c_{xy}$  and  $r_{xy}$  instead of  $(x, y, c)$  and  $(x, y, r)$ , respectively. Thus, we can read  $c_{xy}$  as: on the Cop's move, the Cop is at  $x$  and the Robber at  $y$ , and read  $r_{xy}$  analogously. We construct the digraph's move digraph  $M = M_D$  on vertex set  $V_M = C_M \cup R_M$ , where  $C_M = \{c_{xy} : x, y \in V(D)\}$  and  $R_M = \{r_{xy} : x, y \in V(D)\}$ . There is an arc from  $c_{xy}$  to  $r_{wy}$  if  $x \neq y$  and  $xw \in E(D)$ , and there is an arc from  $r_{xy}$  to  $c_{xz}$  if  $x \neq y$  and  $yz \in E(D)$ . The move digraph  $M$  is bipartite and arcs from a vertex  $c_{xy}$  and arcs from a vertex  $r_{xy}$  represent the possible moves for the Cop and for the Robber, respectively.

Using a relation similar to [45], we can label each vertex  $v$  of  $M_D$  with a non-negative integer  $l(v)$  which indicates the number of Cop's moves until the Cop catches the Robber. If vertex  $v$  is a vertex of the form  $r_{wy}$  and  $l(v) = k$ , then the Cop will capture the Robber in  $k$  moves after the Robber has moved. To begin, all vertices of  $M_D$  are labelled  $\infty$  and then label positions where the Cop has caught the Robber with zero. The positions labelled zero correspond to the Cop and Robber being on the same vertex, regardless of whose turn it is, so  $c_{xx}$  and  $r_{xx}$  are labelled 0 for all  $x \in V(D)$ . Now, for each pair of different vertices  $x$  and  $y$  in  $V(D)$ , we change the label of  $c_{xy}$  to one more than the minimum label of all possible vertices that the Cop can move to. We change the label of  $r_{xy}$  to the maximum label of all possible vertices that the Robber can move to.

**Theorem 4.4.5.** [32] *All vertices of  $M_D$  can be labelled with an integer if and only if  $D$  is cop-win.*

Let  $D = (V, A)$  be a digraph with minimum out-degree at least one. Define  $C_k(D) = (V^k, A_k)$  by setting  $A_k = \{(u, v) : u, v \in V^k, v_i \in N_D^+(u_i), i = 1, 2, \dots, k\}$ . Define  $R(D) = (V, A_1)$ , where  $A_1$  is defined above. Thus, a move by either the Cops or the Robber on  $D$  corresponds to moves by one Cop on  $C_k(D)$  or by one Robber on  $R(D)$ . The  $(k, 1, D)$  augmented game digraph  $\mathcal{D}^*$  is constructed from  $\mathcal{D}$  by first adding two new control vertices  $\alpha$ , from which arcs go to all the vertices of  $C_k(D)$ , and  $\beta$ , from which arcs go to all the vertices of  $R(D)$ , and by adding the arcs as defined:  $A(\mathcal{D}^*) = A(\mathcal{D}) \cup \{(\alpha, x) : x \in V^k \times \{c\}\} \cup \{(\beta, y) : y \in V \times \{r\}\} \cup \{(u, v) : u = ((x_1, x_2, \dots, x_k), c), v = (y, r), y \in \prod_{i=1}^k N^+(x_i)\}$ .

**Theorem 4.4.6.** [32] *A digraph  $D$  is  $k$ -cop-win if and only if the augmented game digraph  $\mathcal{D}^*$  is 1-cop-win.*

The theorems in this section can be combined into the following result.

**Theorem 4.4.7.** *Let  $G$  be a graph. For a generalized game of Cops and Robber, the following statements are equivalent.*

- The graph  $G$  is cop-win.
- $\preceq = V(G) \times V(G)$  [45].
- The graph  $G$  has a removable vertex ordering [20].
- There is a vertex  $p \in V(\mathcal{P})$  that is adjacent in  $B(G)$  to every vertex of  $V(G)$ .
- The graph  $G$  has a move digraph  $M_D$  that can be labelled with integers [32].

Another important concept from the game of Cops and Robber is the length of the game, that is, how many rounds it takes the Cop to catch the Robber. The result comes from Bonato and MacGillivray [11], which is for a generalized Cops and Robber game. The sequence of relations,  $\preceq$ , can be used to label the allowed start positions, where  $c$  is the position of the Cop,  $\mathcal{I}_C$  is the set of allowed start positions of the Cop,  $\mathcal{I}_R(c)$  is the set of allowed start positions of the Robber, and  $r$  is the position of the Robber. The label of the position  $(c, r)$  is  $l(c, r)$ , which is labelled  $k$  such that  $r \preceq_k c$ , where  $\preceq_k$  is the Nowakowski-Winkler relation defined earlier.

**Theorem 4.4.8.** [11] *Suppose the Cop has a winning strategy in the Cops and Robber game  $\mathcal{G}$ . Assuming optimal play, the length of the game is*

$$\min_{c \in \mathcal{I}_C} \{ \max_{r \in \mathcal{I}_R(c)} \{ l(c, r) \} \}.$$

Recall the Cop win time array,  $C$ , from Section 3.2. In the oriented case,  $C$  is a vector and we can determine the length of the game from its largest entry. By Theorem 3.2.1, there is an infinity in the Cop win time array if and only if the Robber wins when the game is played on  $G$ . We can expand this result to all graphs, not just oriented graphs. When constructing the Cop win time array,  $C$ , we will use the completed array,  $R$  as before. Instead of a vector, this Cop win time array will be an  $n \times n$  array. Note that for an undirected graph, there will be no entries labelled  $X$  in  $R$  because there is an arc from  $r$  to  $c$  and an arc from  $c$  to

$r$ . However, for a directed graph, there might be entries labelled  $X$  in  $R$ .

**Proposition 4.4.9.** *In the game of Cop and Robber on a graph  $G$ , the Cop wins if and only if there exists a row of the Cop win time array  $C$  such that every entry is an integer.*

*Proof.* Suppose every entry in a row of  $C$  is a number, and that the Robber starts the game by choosing the vertex  $r$  for which the corresponding entry of  $C$  is labelled  $k$  for some  $k > 0$ . Then, by construction of  $C$ , the Cop has a move to a vertex  $c$  so that entry  $(r, c)$  of  $R$  is labelled  $k$ . It is now the Robber's turn to move. By construction of  $R$ , for any vertex  $r_1$  to which Robber moves, the Cop has a move to a vertex  $c_1$  so that entry  $(r_1, c_1)$  of  $R$  is labelled  $k_1 < k$ . (The Robber can choose  $r_1$  so that  $k_1 = k - 1$ .) From there, for any vertex  $r_2$  to which Robber moves, the Cop has a move to a vertex  $c_2$  so that entry  $(r_2, c_2)$  of  $R$  is labelled  $k_2 < k_1$ . Continuing in this way, the Robber will move to a vertex so that the Cop has a move with the corresponding entry of  $R$  is labelled 0. Therefore the Cop wins.

On the other hand, suppose that there is an  $\infty$  in the Cop win time array,  $C$ , corresponding to a vertex  $r$ . We claim that if the Robber chooses  $r$  as the initial vertex then there is a winning strategy for the Robber regardless of the Cop's starting position. Then, for any vertex  $c$  where the Cop can move from their starting position, either the entry  $(r, c)$  of the array  $R$  is labelled  $X$ , or is unfilled (i.e., it is the ordered pair  $(r, c)$ ).

If entry  $(r, c)$  of  $R$  is labelled  $X$ , then there is an arc from  $r$  to  $c$ . The Robber can maintain the situation where there is an arc to the Cop's current vertex by staying in place if the Cop stays on their current vertex, and by moving to the vertex the Cop just left otherwise. Since there is at most one arc between any two vertices, the Robber cannot be caught on any Cop move. Hence the Robber has a winning strategy.

Finally, suppose entry  $(r, c)$  of  $R$  is unfilled. By construction of  $R$ , there is a vertex  $r_1$  where the Robber can move so that, for any vertex  $c_1$  where the Cop moves, entry  $(r_1, c_1)$  of  $R$  is unfilled. Again by construction of  $R$ , the Robber can move as above so as to maintain

the situation that the entry of  $R$  corresponding to the players' current positions on the Robber's next turn is unfilled. Hence the Robber has a winning strategy.  $\square$

## Chapter 5

# Cops and Robber on Oriented Graphs

### 5.1 New Results for Oriented Graphs

Recall that Theorem 4.4.5 uses an auxiliary move digraph which is then labelled with integers to characterize cop-win games. The same result holds for oriented graphs. In the case where the game as described in Chapter 3 is played on an oriented graph, the move digraph has  $2n^2$  vertices. We know from Khatri *et al.* [36] that there exist oriented graphs with quadratic capture time. This fact suggests that a vertex elimination ordering will not suffice to characterize cop-win oriented graphs.

We will give a characterization of cop-win oriented graphs by defining an arc elimination ordering of the Cartesian product of an oriented graph  $G$  with itself. The vertices in the product represent Cop and Robber positions. The arcs of the Cartesian product will represent the allowed moves for each player; the horizontal arcs represent the Robber's moves, and the vertical arcs represent the Cop's moves.

**Definition 5.1.1.** *The Cartesian product of graphs  $G$  and  $H$  is the graph  $G \square H$  such that*

- *the vertex set,  $V(G \square H) = V(G) \times V(H)$ ; and*
- *two vertices  $(u, u')$  and  $(v, v')$  are adjacent in  $G \square H$  if and only if either*
  - $u = v$  and  $u'$  is adjacent to  $v'$  in  $H$ , or*

$u' = v'$  and  $u$  is adjacent to  $v$  in  $G$ .

To begin, take the Cartesian product of the graph  $G$  with itself. We will consider arcs of the form  $((u, v), (u, w))$  to be *vertical* and arcs of the form  $((u, v), (w, v))$  to be *horizontal*. We call the vertical arcs Cop arcs, and the horizontal arcs Robber arcs. At each vertex of the product, we include loops, one corresponding to a move where the Cop stays in place (thus it is a Cop arc), and one corresponding to a move where the Robber stays in place (thus it is a Robber arc).

**Definition 5.1.2.** *Let  $G$  be an oriented graph. We define the Cop out-degree of a vertex  $(x, y)$  in  $G \square G$  to be the number of Cop arcs from  $(x, y)$  and the Cop in-degree of a vertex  $(x, y)$  in  $G \square G$  to be the number of Cop arcs to a vertex  $(x, y)$ .*

**Definition 5.1.3.** *Let  $G$  be an oriented graph. We define the Robber out-degree of a vertex  $(x, y)$  in  $G \square G$  to be the number of Robber arcs from  $(x, y)$  and the Robber in-degree of a vertex  $(x, y)$  in  $G \square G$  to be the number of Robber arcs to a vertex  $(x, y)$ .*

**Algorithm 1.** *Construct a sequence,  $S$ , of arcs of  $G \square G$  and a sequence  $P_0, P_1, \dots, P_N$  of graphs as follows:*

Set  $P_0 = G \square G$ .

1. For each  $x \in G$ , add all arcs pointing from  $(x, x)$  to the sequence  $S$ , and delete those arcs from  $P_0$  to obtain  $P_1$ .
2. For  $i \geq 1$ ,
  - (a) Suppose that the vertex  $(x, y) \in V(P_i)$  has a Cop arc to a vertex  $(x, r)$  that has Robber out-degree zero. Add all Cop arcs pointing from  $(x, y)$  to the sequence  $S$ , and delete them from  $P_i$  to obtain  $P_{i+1}$ .

- (b) Suppose that  $(u, v) \in V(P_i)$  has a Robber arc to a vertex  $(c, v)$  that has Cop out-degree zero. Add the arc  $((u, v)(c, v))$  to the sequence  $S$ , and delete it from  $P_i$  to obtain  $P_{i+1}$ .

Repeat until there is no such vertex in  $P_i$ , in which case you move on to 3.

3. If there exists a vertex  $c \in V(G)$  such that, for all  $r \in V(G)$ , the vertex  $(c, r)$  that has Robber out-degree zero in  $P_i$ , then add all arcs of  $P_i$  to the sequence  $S$  and delete those arcs to obtain  $P_{i+1} = P_N$ . Otherwise, set  $P_N = P_i$ .

**Theorem 5.1.4.** *Let  $G$  be an oriented graph. Then  $G$  is cop-win if and only if the sequence  $S$  constructed by Algorithm 1 contains all arcs of  $P_0$ .*

*Proof.* For the forward direction, we will prove the contrapositive: If the sequence  $S$  constructed by Algorithm 1 does not contain all arcs of  $P_0$ , then  $G$  is robber-win. Suppose that there is at least one arc of  $P_0$  not in the sequence  $S$ . Then, for every  $c \in V(G)$  there exists a vertex  $r \in V(G)$  such that the vertex  $(c, r)$  has Robber out-degree greater than zero.

Claim: *If, for some  $i$ ,  $(c, r)$  has Robber out-degree greater than zero in  $P_i$  then,  $r \not\prec_i c$ .*

We will prove the claim by induction on  $i$ . The statement is true for  $i = 0$ . Consider  $P_{k+1}$ . Suppose that  $(c, r)$  has Robber out-degree greater than zero in  $P_{k+1}$ . Since  $(c, r)$  has Robber out-degree greater than zero, there is at least one neighbour  $(c, r')$  that has Cop out-degree greater than zero in  $P_k$ . Since  $(c, r')$  has Cop out-degree greater than zero, then any move the Cop can make to is a position  $(c', r')$  such that  $(c', r')$  has Robber out-degree greater than zero in  $P_j$ , where  $j < k$ . By the induction hypothesis,  $r' \not\prec_k c'$ . Therefore, there exists  $r' \in N[r]$  such that  $r' \not\prec_k c'$  for any  $c' \in N[c]$ . By the Nowakowski-Winkler relation,  $r \not\prec_{k+1} c$  in  $P_{k+1}$ . If there exists  $r \in V(G)$  such that, for all  $c \in V(G)$ ,  $(c, r)$  has Robber out-degree greater than zero in  $P_N$ , then  $r \not\prec c$ . By Theorem 4.4.1,  $G$  is not cop-win.

For the reverse direction, suppose that the sequence  $S$  contains all arcs of  $P_0$ . Then by Step 4 of Algorithm 1, there exists  $c \in V(G)$  such that for all  $r \in V(G)$  the vertex  $(c, r)$  has

Robber out-degree zero in  $P_{N-1}$ .

Claim: *If, for some  $i$ ,  $(c, r)$  has Robber out-degree zero in  $P_i$  then,  $r \preceq c$ .*

We will prove the claim by induction on  $i$ . The statement is true for  $i = 1$ . All arcs pointing from  $(x, x)$  can be added to the sequence  $S$  and  $(x, x)$  has Robber out-degree zero in  $P_1$ . By definition of the Nowakowski-Winkler relation,  $x \preceq_0 x$  for any  $x \in V(G)$ . Suppose that  $(c, r)$  has Robber out-degree zero, then  $r \preceq c$  in  $P_k$ , for some  $k \geq 1$ . Consider  $P_{k+1}$ . Suppose that  $(c, r)$  has Robber out-degree zero in  $P_{k+1}$ . If  $(c, r)$  has Robber out-degree zero in  $P_k$  then by induction hypothesis,  $r \preceq c$ . Otherwise, either Step 2(a) or 2(b) was applied. If Step 2(a) was applied, then  $(c, r)$  would have Robber out-degree zero in  $P_k$  because only Cop arcs are removed in Step 2(a). Therefore, there is some  $(x, y)$  such that  $((c, r)(x, y))$  is removed using Step 2(b) in  $P_k$ . All other Robber arcs pointing from  $(c, r)$  were previously removed by Step 2(b) in some  $P_j$  where  $j < k$ . Therefore, any such arc was from  $(c, r)$  is to a vertex that has Cop out-degree zero. Therefore,  $(x, y)$  had a Cop arc to a vertex that has Robber out-degree zero in  $P_j$ , where  $j < k$ . From  $(c, r)$ , any move the Robber could make corresponds to a position  $(c, r')$  such that  $(c, r')$  has Cop out-degree zero in  $P_k$ . In turn,  $(c, r')$  has Cop out-degree zero because there is an arc  $(c, r')(c', r')$  in  $P_j$ , where  $(c', r')$  has Robber out-degree zero in  $P_j$ , where  $j < k$ . By induction hypothesis,  $r' \preceq_t c'$  for some  $t$ . Therefore, for any  $r' \in N[r]$  there exists  $c' \in N[c]$  such that  $r' \preceq_t c'$  for some  $t$ . Let  $T = \{t : r' \preceq_t c' \text{ for } r' \in N[r] \text{ and } c' \in N[c]\}$ . Then  $T$  is finite and it has a greatest element  $t^*$ . Therefore, by the definition of the Nowakowski-Winkler relation,  $r \preceq_{t^*} c$ . Thus  $r \preceq c$ .

Therefore, if there exists  $c \in V(G)$  such that  $(c, r)$  has Robber out-degree zero in  $P_i$  for all  $r \in V(G)$ , then there exists  $c \in V(G)$  such that  $r \preceq c$  for all  $r \in V(G)$ . Therefore,  $G$  is cop-win.

□

We can use the arc elimination ordering for the generalized game of Cops and Robber when the graph and elimination are slightly modified. We will now describe how to construct

the product for the generalized game of Cops and Robber using a graph  $G$ . The modified graph  $\mathcal{G}$  has vertex set  $V(\mathcal{G}) = V(G) \times V(G)$ . There is a Cop arc in  $\mathcal{G}$  from  $(x, y)$  to  $(x', y)$  if and only if the Cop is allowed to move from the state  $(x, y)$  to the state  $(x', y)$ . There is a Robber arc in  $\mathcal{G}$  from  $(x, y)$  to  $(x, y')$  if and only if the Robber is allowed to move from the state  $(x, y)$  to the state  $(x, y')$ .

The elimination of arcs in  $\mathcal{G}$  occurs as follows:

**Algorithm 2.** *Construct a sequence,  $S$ , of arcs of  $\mathcal{G}$  and a sequence  $P_0, P_1, \dots, P_N$  of graphs as follows: Set  $P_0 = \mathcal{G}$ .*

1. *For each  $x, y \in V(G)$ , if  $(x, y)$  is a final position, then add all arcs pointing from  $(x, y)$  to the sequence  $S$ , and delete those arcs from  $P_0$  to obtain  $P_1$ .*
2. *For  $i \geq 1$ ,*
  - (a) *Suppose that the vertex  $(x, y) \in V(P_i)$  has a Cop arc to a vertex  $(x, r)$  that has Robber out-degree zero. Add all Cop arcs pointing from  $(x, y)$  to the sequence  $S$ , and delete them from  $P_i$  to obtain  $P_{i+1}$ .*
  - (b) *Suppose that  $(u, v) \in V(P_i)$  has a Robber arc to a vertex  $(c, v)$  that has Cop out-degree zero. Add the arc  $((u, v)(c, v))$  to the sequence  $S$ , and delete it from  $P_i$  to obtain  $P_{i+1}$ .*

*Repeat until there is no such vertex in  $P_i$ , in which case you move on to 3.*

3. *If there exists a vertex  $c \in V(G)$  such that, for all  $r \in V(G)$ , the vertex  $(c, r)$  that has Robber out-degree zero in  $P_i$ , then add all arcs of  $P_i$  to the sequence  $S$  and delete those arcs to obtain  $P_{i+1} = P_N$ . Otherwise, set  $P_N = P_i$ .*

Due to the construction of  $\mathcal{G}$ , the Algorithm 2 is the same as Algorithm 1 after Step 1. The proof of Theorem 5.1.4 can therefore be replicated to prove the following more general theorem.

**Theorem 5.1.5.** *Let  $G$  be a graph. Then  $G$  is cop-win if and only if the sequence  $S$  constructed by Algorithm 2 contains all arcs of  $\mathcal{G}$ .*

We can use Theorem 4.4.7 for the generalized game of Cops and Robber, and add our new edge elimination characterization to form the next theorem.

**Theorem 5.1.6.** *Let  $G$  be a graph. For a generalized game of Cops and Robber, the following statements are equivalent.*

- *The graph  $G$  is cop-win.*
- $\preceq = V(G) \times V(G)$  [45].
- *The graph  $G$  has a removable vertex ordering [20].*
- *There exists a vertex  $p \in V(\mathcal{P})$  that is adjacent in  $B(G)$  to every vertex of  $V(G)$ .*
- *The graph  $G$  has a move digraph  $M_D$  that can be labelled with integers [32].*
- *The sequence  $S$  constructed by Algorithm 2 contains all arcs of  $\mathcal{G}$ .*

## 5.2 Conclusion

We will conclude the chapter with a summary of the new results given in this part of the thesis, and present some directions for future research. We started this part of the thesis with an introduction to the game of Cops and Robber on oriented graphs through an article intended for high school students and undergraduates. The article provided some results based on existing theory, and introduced new topics which will be mentioned in the thesis through the questions at the end of each part.

After the article, we presented a survey of results on the game when played on oriented graphs. Particularly, we presented a conjecture for an oriented graph with directed cycles to be cop-win. We generalized a counterexample to this conjecture to find an infinite class of counterexamples to the conjecture. Furthermore, we provided sufficient conditions for oriented graphs with directed cycles to be cop-win.

Next, we looked at the generalized game of Cops and Robber and how the game of Cops and Robber can be viewed as a combinatorial game. We modified the game of Cops and Robber to be finite with the same player winning, which validates why it can be analyzed with combinatorial game theory methods. Finally, we presented a survey of previous results on the generalized game of Cops and Robber.

Finally, we constructed a related graph to represent the Cop and Robber positions in the original graph and their respective moves from a given position. We presented an arc elimination ordering in a related graph in order to characterize cop-win oriented graphs, and described how the algorithm can be adapted to the generalized game of Cops and Robber.

Although there has been research on the game of Cops and Robber on Cayley digraphs, there has been little research done on other classes of digraphs. Future research could explore other digraph classes and their cop numbers. Additionally, there is still no structural characterization for cop-win oriented graphs. Future research could also determine another characterization of cop-win oriented graphs. First, one could look for a characterization for cop-win oriented graphs with a directed cycle and that result might be able to be generalized for cop-win oriented graphs with multiple cycles.

# Bibliography

- [1] Clifford Adelman. “The Kiss of Death? An Alternative View of College Remediation.” In: *National Crosstalk* 6.3 (1998).
- [2] M. Aigner and M. Fromme. “A game of cops and robbers”. In: *Discrete Applied Mathematics* 8.1 (1984), pp. 1–12.
- [3] Lara Alcock and Adrian Simpson. “Definitions: Dealing with Categories Mathematically”. In: *For the Learning of Mathematics* 22.2 (2002), pp. 28–34.
- [4] Francis Atuahene and Tammy A. Russell. “Mathematics Readiness of First-Year University Students.” In: *Journal of Developmental Education* 39.3 (2016), pp. 12–32.
- [5] Peter Riley Bahr. “Does Mathematics Remediation Work?: A Comparative Analysis of Academic Attainment among Community College Students”. In: *Research in Higher Education* 49.5 (2008), pp. 420–450.
- [6] Jørgen Bang-Jensen and Gregory Gutin. *Digraphs: theory, algorithms and applications*. London;New York; Springer, 2001.
- [7] A. Berarducci and B. Intrigila. “On the Cop Number of a Graph”. In: *Advances in Applied Mathematics* 14.4 (1993), pp. 389–403.
- [8] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning ways for your mathematical plays*. 2nd ed. Natick, MA: A.K. Peters, 2001.

- [9] Eric P. Bettinger and Bridget Terry Long. “Addressing the Needs of Underprepared Students in Higher Education: Does College Remediation Work?” In: *The Journal of Human Resources* 44.3 (2009), pp. 736–771.
- [10] A. Bonato et al. “The capture time of a graph”. In: *Discrete Mathematics* 309.18 (2009). Combinatorics 2006, A Meeting in Celebration of Pavol Hell’s 60th Birthday (May 1–5, 2006), pp. 5588 –5595.
- [11] Anthony Bonato and Gary MacGillivray. “Characterizations and algorithms for generalized Cops and Robbers games”. In: *Contributions to Discrete Mathematics* (Apr. 2017), pp. 110–122.
- [12] Anthony Bonato and Richard Nowakowski. *The Game of Cops and Robbers on Graphs*. Amer. Mathematical Society, Sept. 2011.
- [13] Michel Boyer et al. “Cops-and-robbers: remarks and problems”. In: 2013.
- [14] Peter Bradshaw, Seyyed Aliasghar Hosseini, and Jérémie Turcotte. “Cops and robbers on directed and undirected abelian Cayley graphs”. In: *arXiv* (2019).
- [15] Daniel Breidenbach et al. “Development of the process conception of function”. In: *Educational Studies in Mathematics* 23.3 (1992), pp. 247–285.
- [16] Marilyn Carlson. “A cross-sectional investigation of the development of the function concept”. In: *Research in Collegiate Mathematics Education* 7 (Feb. 1998).
- [17] Marilyn Carlson, Bernard Madison, and Richard West. “A Study of Students’ Readiness to Learn Calculus”. In: *International Journal of Research in Undergraduate Mathematics Education* 1 (Sept. 2015).
- [18] Marilyn Carlson, Michael Oehrtman, and Nicole Engelke Infante. “The Precalculus Concept Assessment: A Tool for Assessing Students’ Reasoning Abilities and Understandings”. In: *Cognition and Instruction* 28 (Apr. 2010), pp. 113–145.

- [19] Megan Clark and Miroslav Lovric. “Suggestion for a theoretical model for secondary-tertiary transition in mathematics”. In: *Mathematics Education Research Journal* 20.2 (2008), pp. 25–37.
- [20] Nancy E. Clarke and Gary MacGillivray. “Characterizations of k-copwin graphs”. In: *Discrete Mathematics* 312.8 (2012), pp. 1421–1425.
- [21] Lorraine F. Dame. “Student readiness, engagement and success in entry level undergraduate mathematics courses.” PhD thesis. University of Victoria (B.C.), 2012.
- [22] Elise Darlington et al. “Cops and Robbers on Oriented Graphs”. In: *Rose-Hulman Undergraduate Mathematics Journal* 17.1 (2016).
- [23] Pietro Di Martino and Francesca Gregorio. “The Mathematical Crisis in Secondary–Tertiary Transition”. In: *International Journal of Science and Mathematics Education* 17.4 (2019), pp. 825–843.
- [24] Giorgio Di Pietro. “The Short-term Effectiveness of a Remedial Mathematics Course: Evidence from a UK University”. In: *The Manchester School* 82 (2014), pp. 363–384.
- [25] Education Committee of the EMS. “Why is University Mathematics difficult for students? Solid findings about the secondary-tertiary transition.” In: *Newsletter of the European Mathematical Society* 91 (2013), pp. 46–48.
- [26] Sidika N. Er. “Mathematics readiness of first-year college students and missing necessary skills: perspectives of mathematics faculty”. In: *Journal of Further and Higher Education* 42.7 (2018), pp. 937–952.
- [27] Ebenezer F. Kolajo. “From developmental education to graduation: A community college experience”. In: *Community College Journal of Research and Practice* 28 (Apr. 2004), pp. 365–371.

- [28] Stephen Finbow et al. “A note on the eternal dominating set problem”. English. In: *International Journal of Game Theory* 47.2 (2018), pp. 543–555.
- [29] Peter Frankl. “On a pursuit game on Cayley graphs”. In: *Combinatorica* 7.1 (1987), pp. 67–70.
- [30] Andrew Gavrin. “Just-in-Time Teaching”. In: *Published in Metropolitan Universities* 17 (Jan. 2006), pp. 9–18.
- [31] *Glossary of Education Reform*. URL: <https://www.edglossary.org/action-research/>.
- [32] Geňa Hahn and Gary MacGillivray. “A note on k-cop, l-robber games on graphs”. In: *Discrete Mathematics* 306.19 (2006). Creation and Recreation: A Tribute to the Memory of Claude Berge, pp. 2492–2497.
- [33] Yahya Ould Hamidoune. “On a Pursuit Game on Cayley Digraphs”. In: *European Journal of Combinatorics* 8.3 (1987), pp. 289–295.
- [34] Elena Nardi John Monaghan and Tommy Dreyfus. “Calculus in upper secondary and beginning university mathematics”. In: MatRIC. Kristiansand, Norway, 2019.
- [35] Robin Kay and Ilona Kletskin. “Evaluating the use of learning objects for improving calculus readiness”. In: *Journal of Computers in Mathematics and Science Teaching* 29(1) (2010), pp. 87–104.
- [36] Devvrit Khatri et al. “A study of cops and robbers in oriented graphs”. In: *arXiv e-prints* (2018), arXiv:1811.06155.
- [37] William B. Kinnersley. “Cops and Robbers is EXPTIME-complete”. In: *Journal of Combinatorial Theory, Series B* 111 (2015), pp. 201–220.
- [38] Boris Koichu and Alon Pinto. *The secondary-tertiary transition in mathematics: What are our current challenges and what can we do about them?*

- [39] Kevin Li et al. “Readiness, Behavior, and Foundational Mathematics Course Success”. In: *Journal of Developmental Education* 37.1 (2013), pp. 14–36.
- [40] Shiyu Liu. “Entering the STEM Pipeline: Exploring the Impacts of a Summer Bridge Program on Students’ Readiness”. In: *Journal of College Student Development* 59.5 (2018), pp. 635–640.
- [41] Paco Martorell and Isaac McFarlin. “Help or Hindrance? The Effects of College Remediation on Academic and Labor Market Outcomes”. In: *The Review of Economics and Statistics* 93.2 (2011), pp. 436–454.
- [42] J McNiff. “Action research: Principles and practice, 3rd edition”. In: *Action Research: Principles and Practice, 3rd Edition* (Jan. 2013), pp. 1–226.
- [43] Steve Monk. “Students’ understanding of a function given by a physical model”. In: *The concept of function: Aspects of epistemology and pedagogy*. Ed. by G. Harel and E. Dubinsky. MAA Notes 25, 1992, pp. 175–193.
- [44] Rekha Natarajan and Andrew Bennett. “Improving Student Learning of Calculus Topics via Modified Just-in-Time Teaching Methods”. In: *Primus* 24(2) (2014), pp. 149–159.
- [45] Richard Nowakowski and Peter Winkler. “Vertex-to-vertex pursuit in a graph”. In: *Discrete Mathematics* 43.2 (1983), pp. 235–239.
- [46] Wendy E. O’Connor and Todd G. Morrison. “Do remedial mathematics programmes improve students’ mathematical ability”. In: *Studies in Educational Evaluation* 23.3 (1997), pp. 201–207.
- [47] Alain Quilliot. “Jeux et pointes fixes sur les graphes”. PhD thesis. Ph. D. Dissertation, Université de Paris VI, 1978.

- [48] Stefanie Rach and Aiso Heinze. “The Transition from School to University in Mathematics: Which Influence Do School-Related Variables Have?” In: *International Journal of Science and Mathematics Education* 15.7 (2016), pp. 1343–1363.
- [49] Zohreh Shahbazi and Adon Irani. “Online Calculus and Pre-Calculus Modules”. In: EdMedia. June 2016.
- [50] Veronika Slivova. “Cops and robber game on directed complete graphs”. MA thesis. Charles University in Prague, 2015.
- [51] N. Smith. “Students’ emergent conceptions of the fundamental theorem of calculus”. PhD thesis. Arizona State University, 2008.
- [52] George Stern. “Myth and Reality in the American College”. In: *AAUP Bulletin* 52.4 (1966), pp. 408–414.
- [53] David Tall. “Functions and calculus”. In: *International handbook of mathematics education, part 1*. Ed. by A. Bishop et al. Dordrecht: Kluwer Academic Publishers, 1996, pp. 289–325.
- [54] David Tall. “The transition to advanced mathematical thinking: Functions, infinity, and proof”. In: *Handbook of research on mathematics teaching and learning*. Ed. by D.A. Grouws. New York: Macmillan, 1992, pp. 495–511.
- [55] David Tall and Shlomo Vinner. “Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity”. In: *Educational Studies in Mathematics* 12 (May 1981).
- [56] Patrick Thompson. “Images of rate and operational understanding of the Fundamental theorem of calculus”. In: *Educational Studies in Mathematics* 26 (Jan. 1994), pp. 229–274.

- [57] Pansy Waycaster. “Tracking Developmental Students into Their First College Level Mathematics Course”. In: *Inquiry* 16.1 (2011), pp. 53–66.
- [58] Mackenzie Wheeler. *Cops and Robbers on Infinite Graphs*. 2017.
- [59] Robin M. Wucherer. “Beliefs and Instructional Practices of Four Community College Remedial Mathematics Instructors”. PhD thesis. Illinois State University, 2011.
- [60] M. Zandieh. “A theoretical framework for analyzing student understanding of the concept of derivative”. In: *Research in collegiate mathematics education IV*. Ed. by E. Dubinsky, A. H. Schoenfeld, and J. Kaput. Providence: American Mathematical Society, 2000, pp. 103–127.