Venn Diagrams with Few Intersections

by

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B Sc , University of Victoria, 1995

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Computer Science

We accept this thesis as conforming
to the required standard

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ABSTRACT

The n-*Venn diagram* is a collection of simple closed curves in the plane, intersecting only at points. The curves divide the plane into 2^n open connected regions. Further, each region must contain a unique set of interiors of the curves. A region's *weight* is the number of curves that contain it. A *monotone* Venn diagram with n curves has the property that every region with weight k, where $1 < k < n$, is adjacent to at least one region with weight $k-1$ and at least one region with weight $k+1$. An n-Venn diagram can be interpreted as a planar graph in which the intersection points of the curves are the vertices. We show that each monotone Venn diagram has at least $\binom{n}{\lfloor n/2 \rfloor}$ vertices and that this bound can be attained for all $n > 1$. For general Venn diagrams, the number of vertices is at least $\lceil \frac{2^n-2}{n-1} \rceil$. Examples are given that demonstrate that this bound can be attained for $1 < n \leq 7$.

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ACKNOWLEDGMENTS

I am very grateful to have had the priviledge and pleasure of researching and writing this thesis. This could not have happened without the helpful guidance of my supervisor, Frank Ruskey. His ability to pick out the elegant simplicity in mathematical truths inspired me to improve my own skill and to see that there truly is beauty in mathematics. Thank-you, Frank.

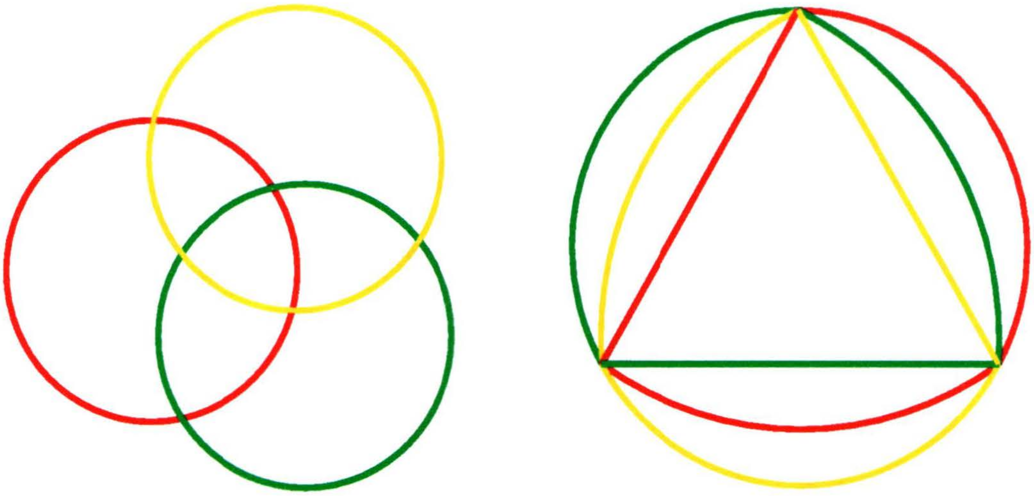
I would also like to thank the National Science and Engineering Research Council of Canada for their financial support during the last two years.

And lastly, I wish to give loving thanks to the Letourneau boys, Gil, Adrian, and Gabe, for helping me keep everything in its proper perspective. Also, thanks to my parents, Pieta and Siewert Bultena, who have always supplied moral support in my continuing pursuit of adulthood.

CHAPTER 1

Introduction

There has been a renewed interest in Venn diagrams in the last few years. Recent surveys have been written by Ruskey [13] and Chilakamarri et al. [3]. In this thesis, we tackle a natural problem that has not received attention: What is the least number of intersections, also called vertices, in a Venn diagram of n curves? The maximum number of vertices on a Venn diagram occurs when the diagram is



Venn Diagram with 3 curves and 6 vertices

Venn Diagram with 3 curves and 3 vertices

Figure 1.1: Example of a simple and a non-simple 3-Venn diagram

simple, meaning that each intersection involves 2 curves. The minimum number of vertices depends on whether the diagram is monotone or non-monotone. We give detailed descriptions of these types of diagrams in Chapter 3.

In Figure 1.1, we see the classic simple Venn diagram of 3 curves and 6 vertices. The Venn diagram next to it is also constructed of 3 curves, but has only 3 vertices. This second diagram has the minimum number of vertices among all Venn diagrams of 3 curves; a complete listing of these may be found in Chilakamarri, Hamburger, and Pippert [3]. We show that this is the minimum value in Theorem 3.1 in Chapter 3.

We give some relevant graph theory definitions, particularly relating to planar graphs, in the remainder of this chapter. In Chapter 2, we discuss operations specific to Venn diagrams that are used in some of our construction techniques. Chapter 3 provides proof for the lower bound of the number of vertices for both monotone and general Venn diagrams. Since the upper bounds are more difficult, they are discussed in two separate chapters.

In Chapter 4, we show that the upper bound of $\binom{n}{\lfloor n/2 \rfloor}$ vertices for monotone Venn diagrams is attainable for all $n > 1$. This is demonstrated, using a specific and recursively constructed set of diagrams. The number of vertices turns out to involve the famous Catalan numbers.

Chapter 5 demonstrates the minimum vertex diagrams for general Venn diagrams when $4 \leq n \leq 7$ and presents a heuristic method that may extend a minimum n-Venn diagram into a minimum $(n+1)$-Venn diagram. Finding a minimum vertex Venn diagram for $n > 7$ remains an open problem. Chapter 6 discusses conclusions.

1.1 Venn Diagrams and Graphs

Venn diagrams were named for and introduced in 1880 by John Venn. We use Grunbaum's definitions for Venn diagrams [9].

1.1.1 Venn Diagrams

A *simple closed curve* in the plane is a non-self-intersecting curve, which, by a *continuous transformation of the plane*, is identical to a circle. This transformation is achieved when we stretch or shrink all or parts of the plane, without tearing, twisting or pasting it to itself [10].

An n-*Venn diagram* in the plane is a collection of simple closed curves $C = C_1, C_2, \ldots, C_n$, such that each of the 2^n sets $X_1 \cap X_2 \cap \ldots \cap X_n$ is a nonempty and connected region, where each X_i is either the bounded interior or the unbounded exterior of C_i. This intersection can be uniquely identified by a subset of $1, 2, \ldots, n$, indicating the subset of the indices of the curves whose interiors are included in the intersection. Pairs of curves are assumed to intersect only at a finite number of points, meaning that intersections occur at points and not curve segments.

We say that two Venn diagrams are *isomorphic* if, by continuous transformation of the plane, one of them can be changed into the other or its mirror image [13].

A simple closed curve is *convex* if any two interior points can be joined by an interior line segment. A Venn diagram is convex if its curves are all convex. A *potentially convex* Venn diagram is isomorphic to a convex Venn diagram. Thus, a potentially convex Venn diagram's curves are not necessarily convex.

1.1.2 Graphs

A *graph* G with r vertices and m edges consists of a *vertex set* $V(G) = \{V_1, \ldots, v_r\}$ and an *edge set* $E(G) = \{e_1, \ldots, e_m\}$, where each edge is an unordered pair of vertices [16]. For any edge $e = \{u, v\}$, it is said to be *incident* with the vertices u and v, and these vertices are *adjacent* to each other [8].

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph *induced* by a set of vertices $V(H) \subseteq V(G)$ contains every edge of G in $V(H)$. A *neighbourhood* of vertex $v \in V(G)$ is the set of all vertices in G that are adjacent to v [16].

1.1.3 The Venn Diagram as a Plane Graph

A *planar graph* can be drawn in the plane with edges, or curves, intersecting only at vertices [8]. A Venn diagram V is a planar graph whose vertices, called *Venn vertices*, are the intersections of the curves, and whose edges are the line segments connecting these vertices. A planar graph embedded in the plane is called a *plane graph*. The actual drawing V of the Venn diagram is a plane graph. The plane graph V is often called the Venn diagram, without causing confusion.

The labelled edges of V are of the form $C_i(v, w)$, where there is a segment on curve C_i with intersection points v and w, and no intersection points between them on C_i. Each face, including the outer infinite face, is called a *region* when referring to V. Each region in the Venn diagram has associated with it a unique subset of $1, 2, \ldots, n$, and a *weight*. The weight is the number of curves that contain the region and is equal to the cardinality of its representative subset.

An x, y *walk* on a graph G is a sequence $x = v_0, e_1, v_1, e_2, \ldots e_t, v_t = y$, of vertices and edges of G, starting at vertex v_0 and ending at v_t involving t edges

$e_i = \{x_{i-1}, x_i\}$, where $1 \leq i \leq t$ [8]. When $x = y$ on the walk, $t > 1$, and no vertices between x and y are repeated, it is called a *cycle*.

A *Hamilton cycle* on a graph G, is a cycle involving all vertices of G. We say G is *Hamiltonian* if there exists such a cycle on G.

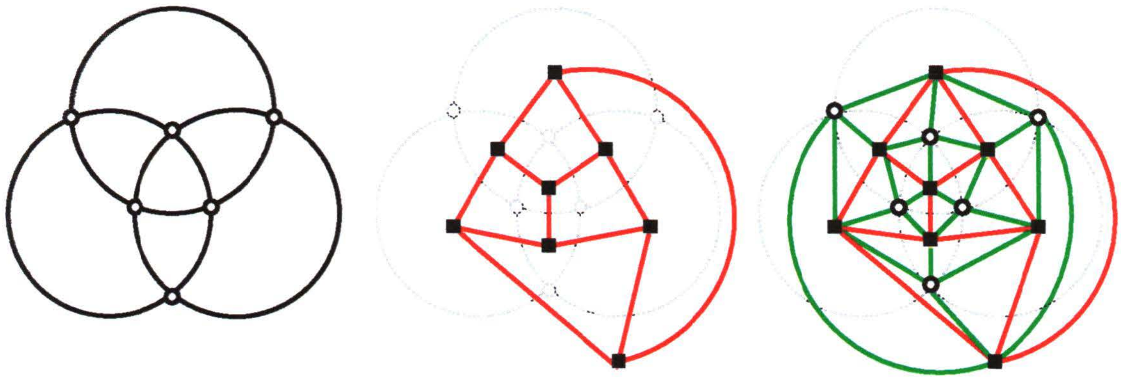
A facial cycle of a region on V is the cycle taken around the region in clockwise order, recording the edges and vertices bordering the region. Note that in a Venn diagram, each edge borders exactly 2 regions whose weights differ by exactly 1. Each of the vertices of this edge are found on facial cycles of both regions.

1.1.4 The Dual Graph

Every plane graph has a unique *dual graph*, which is also a plane graph [8]. We specifically refer to the dual graph $D(V)$, of the Venn diagram V. The dual graph is constructed by placing a vertex within each region of the Venn diagram. For each edge of V, a dual graph edge is drawn which connects the dual vertices within the two adjacent regions. Note that each of the *dual vertices* corresponds to a face in V, and each of the *Venn vertices* corresponds to a face in $D(V)$. We identify each of the dual vertices by the same subset and weight as the associated region on V. We define the *directed* dual graph, $\vec{D}(V)$, by imposing a direction on each edge that moves from the vertex of larger weight to the vertex of smaller weight [13].

1.1.5 The Radual Graph

The vertex set of the *radual graph* $R(V)$ consists of the union of the vertex sets of V and $D(V)$ [12]. The edge set of $R(V)$ consists of all edges in $D(V)$ together with edges between each dual vertex and the following specified Venn vertices. In the radual graph, a dual vertex d is adjacent to a Venn vertex v if v is on the face of V



V
Circle vertices with black edges

D(V)
Square vertices with red edges

R(V)
Square and circle vertices
with red and green edges

Figure 1.2 The plane graph, the dual graph, and the radual graph.

containing d. Note that the edges incident with d in R(V) are alternately incident with Venn vertices and dual vertices as we circle around d in a fixed direction.

Figure 1.2 illustrates the dual and radual graph constructions.

1.2 Monotone and Non-monotone Venn Diagrams

We make a distinction between monotone and non-monotone Venn diagrams. Following [13], we define a diagram to be *monotone* if and only if the directed Venn graph $\vec{D}(V)$ has a unique *sink* (a vertex with no out-going edges) and a unique *source* (a vertex with no incoming edges). An equivalent definition of a monotone Venn diagram is that each vertex with weight $0 < k < n$ in the dual graph is adjacent to a vertex with weight $k-1$ and a vertex with weight $k+1$. Note that a monotone Venn diagram is only isomorphic to another monotone Venn diagram. Of course, all Venn diagrams that do not have this property are *non-monotone*.

It is not always easily apparent whether a Venn diagram is monotone or not. The two Venn diagrams in Figure 1.1 are both monotone. The general constructions of Edwards [6] [7] are monotone. The “necklace property” mentioned in Edwards [5] is a consequence of monotonicity. In contrast, the original general constructions of Venn [15] are non-monotone.

In Chapter 3, we prove that regions with the same weight form *cycles*, connected by shared vertices, within the monotone Venn diagram. These cycles look like strings of connected beads when the regions are coloured according to their weights. On the radual graph, this translates to a cycle of alternating Venn vertices and dual vertices, where all dual vertices have equal weight.

The geometric condition of convexity is equivalent to the purely combinatorial condition of monotonicity. We show, in the following lemma, that every convex Venn diagram is monotone. The proof that every monotone Venn diagram is convex is provided in [1].

Lemma 1.1 *Every convex Venn diagram is a monotone Venn diagram.*

Proof. Observe that a simple closed curve C is convex if and only if every line segment joining an exterior point and an interior point of C intersects C exactly once. On a convex n-Venn diagram V, consider two points v and w, in the regions of weight 0 and n respectively. Because v is exterior to all the curves and w is interior to all the curves, a line segment L joining v and w intersects each of the n curves at exactly one point.

Without loss of generality, we can position L so it does not intersect any of the finite number of vertices of V. Because v can be anywhere in the infinite region around V, L can also be chosen to intersect any region. Consider such a region R with weight k, where $1 < k < n$. As we traverse the segment $v\vec{w}$, we leave a region with weight $k - 1$ just prior to crossing a curve into R. As we cross a curve to exit R, we enter a region with weight $k + 1$. Thus V is monotone. □

CHAPTER 2

Operations on Venn Diagrams

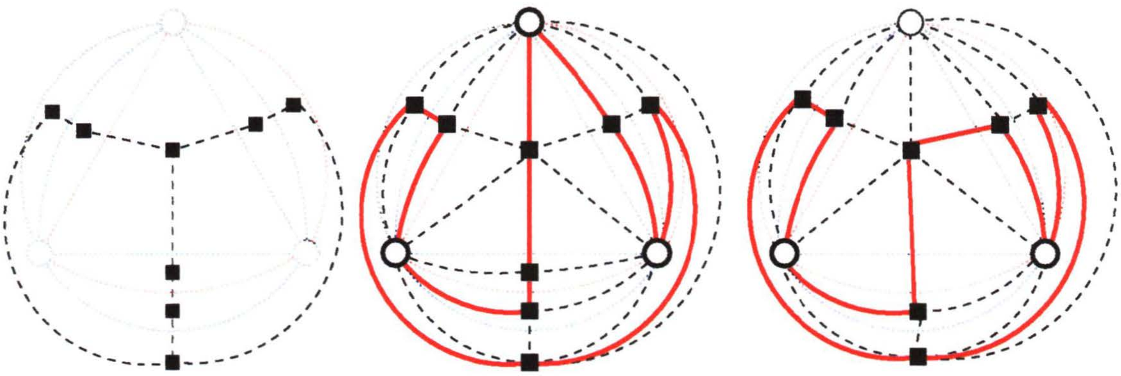
Often we wish to alter an existing plane graph of the Venn diagram while maintaining the Venn diagram properties of the graph. The operations discussed below construct new Venn diagrams from existing Venn diagrams.

2.1 Extending an n-Venn diagram

Both Venn [15] and Edwards [6] [7] demonstrate methods for constructing Venn diagrams for all values of n. However, given *any* Venn diagram V_n of n curves, we can extend it to produce an $(n+1)$-Venn diagram by introducing a *suitable* simple curve which bisects each of the original regions [2]. This curve may cross an edge or vertex of V_n no more than once.

It is easy to see that if the dual graph $D(V_n)$ is Hamiltonian, this cycle may be used as the new curve that extends V_n to V_{n+1}. The dual graph of Figure 1.2 is a 3-cube and thus contains a Hamilton cycle. However, the dual graph is not always Hamiltonian. For example, the dual graph of the minimum vertex 3-Venn diagram shown in Figure 2.1(a) does not contain a Hamilton cycle.

When the dual graph is not Hamiltonian, we look to the radual graph for



(a) No Hamilton cycle on the dual graph

(b) Hamilton cycle on the radual graph used as a suitable curve.

(c) Non-Hamilton cycle on the radual graph used as a suitable curve.

Figure 2.1. Extending the minimum vertex 3-Venn to a 4-Venn diagram.

a suitable curve. Chilakamarri, Hamburger, and Pippert [2] proved that a Venn diagram can be extended if and only if there is a cycle on the radual graph that visits all dual vertices. They also proved that the radual graph of a Venn diagram is Hamiltonian. Figure 2.1(b) demonstrates that this cycle can be used as the suitable curve. However, (c) demonstrates that a suitable curve need not encounter every Venn vertex.

A *reducible* Venn diagram retains Venn diagram properties when one of its curves is removed. A necessary requirement of the curve C that is removed from a reducible n-Venn diagram is that, on a cycle of all its edges, exactly 2^{n-1} vertices are encountered. A number less than this indicates that C does not bisect the 2^{n-1} regions necessary in a Venn diagram with one less curve. When none of the curves of an n-Venn diagram encounter 2^{n-1} vertices on a cycle of its edges, the Venn diagram is *irreducible*.

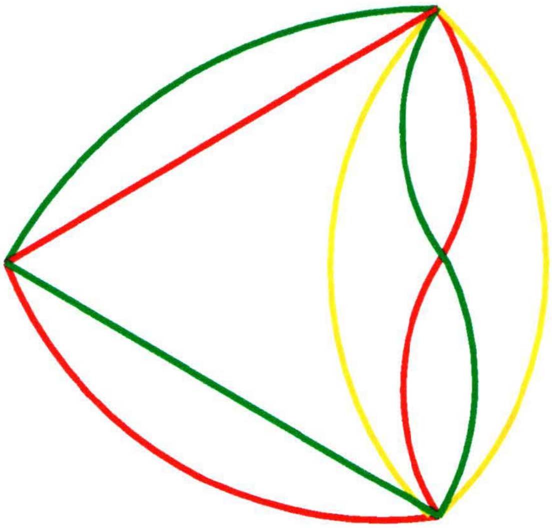
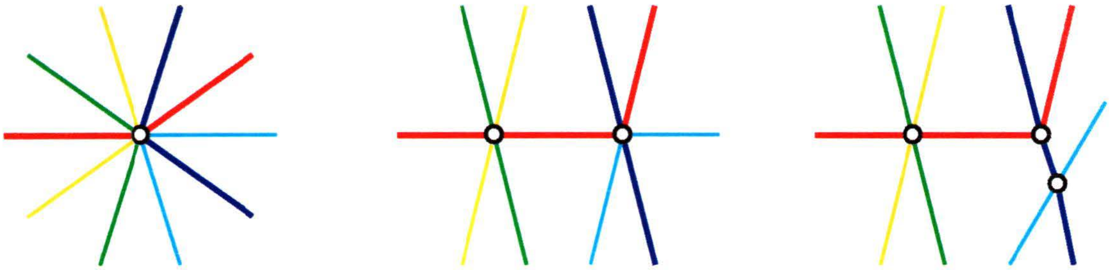


Figure 2.2: An example of a prime 3-Venn diagram.

2.2 Compression and Separation of Vertices

Compression of Venn vertices combines two vertices into one, while *separation* separates a single vertex into two. These are special cases of the graph theory operations called contraction and splitting [16]. Repeatedly applying these operations allows us to alter an existing n-Venn diagram to an n-Venn diagram with a different number of vertices.

It is possible for a Venn diagram to be *prime*. In other words, Venn diagrams do exist whose vertices cannot be separated or compressed. Figure 2.2 illustrates the only prime 3-Venn diagram in Chilakamarri, Hamburger, and Pippert's listing. The other 13 can all be separated and/or compressed to form another diagram in the listing.



Initially: one vertex   Step 1: two vertices   Step 2: three vertices

Figure 2.3: Separating a Venn vertex twice to create 3 vertices.

2.2.1 Compression

We can compress 2 adjacent vertices v and w on a Venn diagram if they share exactly one common curve C_i. This is done by removing the edge $C_i(vw)$ and then mending the curve C_i by merging v and w. The process reduces the number of vertices by one, while maintaining the Venn diagram properties. All curves remain simple and closed and no regions have been altered.

2.2.2 Separation

For a vertex v with $\deg(v) = t$, a *vertex traversal* is the circular listing $T = C_1, C_2, \ldots, C_t$ of the curve segments incident to v, identified in a clockwise rotation around the vertex. Note that this listing contains each incident curve exactly twice. We say that v is separable if and only if there exist integers i and j and sublistings of T, $A = C_i, C_{i+1}, \ldots, C_{j-1}$ and $B = C_j, C_{j+1}, \ldots, C_{i-1}$, that have exactly one curve in common. Moreover, A and B both list more than two curves.

We separate v on this common curve C into 2 vertices whose traversals are A and B, and insert a new segment of C between them. Figure 2.3 shows a vertex which can be separated twice in any order. Compression is demonstrated by reversing the sequence of steps.

An interesting use for vertex separation is to increase the number of vertices on a curve C to 2^{n-1}. This allows us, in some cases, to reduce an irreducible Venn diagram.

CHAPTER 3

The Lower Bound on the Number of Vertices

In this chapter, we prove some properties of general Venn diagrams and monotone Venn diagrams. We also determine the lower bound on the number of vertices in each of these.

3.1 A Lower Bound for General Venn Diagrams

Let $\mathrm{Min}(n)$ be the least number of vertices of a Venn diagram of n curves.

Theorem 3.1 *If* $n > 1$, *then*

$$\mathrm{Min}(n) \geq \left\lceil \frac{2^n - 2}{n-1} \right\rceil$$

Proof. Consider a n-Venn diagram V, with vertex set W. Let f, v, and e denote the number of faces, vertices and edges of V. We denote the degree of vertex w as $\deg(w)$. By definition, for $w \in W$, $\deg(w)$ is no more than $2n$. Since every edge is incident to exactly 2 vertices,

$$2nv \geq \sum_{w \in W} \deg(w) = 2e.$$

Using Euler's theorem [16] for planar graphs $e = f + v - 2$ [8],

$$nv \geq 2^n + v - 2,$$

and

$$v \geq \frac{2^n - 2}{n - 1}.$$

□

3.2 A Lower Bound for Monotone Venn Diagrams

The following lemmas deal with general plane graphs, illustrating that each dual vertex in the radual graph is bordered by a specific cycle. The lemmas are used to prove the lower bound for monotone Venn diagrams.

Lemma 3.2 *The degree of a dual vertex* d *in the radual graph is equal to twice the number of edges on the facial cycle of the region containing* d *in the original plane graph.*

Proof. Consider P, $D(P)$, and $R(P)$, a plane graph, its dual graph, and its radual graph, respectively. Let d be a dual vertex within face F of P. There are an equal number of edges and vertices on the facial cycle of F. Each vertex v_i on this cycle is adjacent to d by definition of $R(P)$. Each edge on the facial cycle of F corresponds to an edge between d and another dual vertex d_i in region S of P. Therefore d is adjacent to the total number of vertices and edges on F's facial cycle. □

Lemma 3 3 *The subgraph of the radual graph* $\mathsf{R(P)}$ *induced by the neighbourhood of a dual vertex* d *is an alternating cycle of dual vertices and vertices of the plane graph* P

Proof Choose any 2 *consecutive* (in a small circle around d) vertices ν and w that are adjacent to d in $\mathsf{R(P)}$ Without loss of generality, let ν be a vertex of P and w a dual vertex in the region S of P Then ν is also contained on the facial cycle of S and therefore is adjacent to w □

An interesting property of monotone Venn diagrams is that they can be *peeled* For an n-Venn diagram V and and integer $k \geq 1$, the k-peeled subgraph V_k of V is obtained by first removing all edges that border two regions in V of weights less than k, and then removing all isolated vertices

Lemma 3 4 *A* k*-peeled subgraph* V_k *of a monotone* n*-Venn* V *contains every original region whose weight is at least* k, *and no bounded regions with a weight less than* k

Proof (by induction on k)

Base Case Note that V_1 is the same as V

Inductive Step For $k \geq 1$, assume the statement is true Consider V_k, the k-peeled graph of a monotone n-Venn diagram V, and its original dual graph $\mathsf{D(V)}$

Each dual vertex with weight k is connected to at least one dual vertex with weight $k-1$, by the definition of a monotone Venn diagram By the induction hypothesis, each dual vertex with weight k is contained in a closed region of V_k, while each weight $k-1$ dual vertex is located in the unbounded region of V_k By definition of the dual graph, there is an edge in the Venn diagram that corresponds to each dual graph edge between two dual vertices with weights of $k-1$ and k. The removal of each of these Venn edges, peels V_k and opens each region with weight k to the outer unbounded region.

None of the regions with weight greater than k are affected. No region with weight k is left bounded in the peeled graph. Therefore the statement is true for V_{k+1}. □

Using the same steps as in the construction of the radual graph of an n-Venn diagram, we construct the radual graph of a k-peeled graph of a monotone n-Venn diagram. Note that if we remove the dual vertex associated with the unbounded region, we have a subgraph of the radual graph associated with the original monotone n-Venn diagram.

Theorem 3.5 *For any radual graph* $R(V)$ *of a monotone* n*-Venn diagram* V*, and any* $0 < k < n$*, there is a cycle of size* $2\binom{n}{k}$ *in* $R(V)$*, consisting of alternating Venn vertices and dual vertices with weight* k.

Proof Consider V_k, the k-peeled graph of an n-Venn diagram V. By Lemma 3.4, there are no regions with weight $k-1$ within V_k. Therefore, all weight $k-1$ regions of V are part of the unbounded region in V_k. Since all the weight k regions in V_k must share an edge with regions with weight $k-1$, there are $\binom{n}{k}$ outer edges on V_k.

Now consider the radual graph, $R(V_k)$. Let d be the dual vertex in $R(V_k)$ of the unbounded region in V_k. By Lemma 3.2, $\deg(d) = 2\binom{n}{k}$, and by Lemma 3.3, the vertices adjacent to d form a cycle which alternates between Venn vertices and dual vertices with weight k. Since neither d nor any of its edges are involved, this cycle is contained in the subgraph of $R(V)$. □

Let M_n be the minimum number of vertices in a monotone Venn diagram. We show that $M_n = \binom{n}{\lfloor n/2 \rfloor}$. We obtain the lower bound of M_n from the number of $(n/2)$-subsets of $\{1, 2, \ldots, n\}$. The upper bound is the same and is proven in the next chapter.

Theorem 3.6 *If* $n > 1$, *then*

$$M_n \geq \binom{n}{\lfloor n/2 \rfloor}$$

Proof. By Theorem 3.5, there exists a cycle on the radual graph of a monotone n-Venn of size $2\binom{n}{k}$, where $k = \lfloor n/2 \rfloor$. Since this cycle alternates between dual vertices and Venn vertices,

$$M_n \geq \binom{n}{\lfloor n/2 \rfloor}.$$

□

CHAPTER 4

The Upper Bound for Monotone Venn Diagrams

In this chapter, we show how to construct a monotone n-Venn diagram with $\binom{n}{\lfloor n/2 \rfloor}$ vertices, thereby proving that $M_n \leq \binom{n}{\lfloor n/2 \rfloor}$

4.1 A Straightened Venn Diagram (SVD)

Suppose V is an n-Venn diagram with a vertex v such that $deg(v) = 2n$. Let v have a vertex traversal such that it is possible to split it into two copies where each copy is adjacent to n distinct curves. Imagine pulling the two copies of v apart, horizontially stretching the rest of the curves so one of the curves C becomes a straight line segment. Each of the curves and the intersections are stretched but not broken, and do not change their original relationships. The resulting diagram represents a Venn diagram with n simple curve segments, beginning and ending at the two copies of v. The exterior region is now represented by the area above the curves and the interior region is represented by the region below the curves. We call this diagram a *straightened representation of* V.

The left Venn diagram in Figure 1.1 cannot be represented in this way because none of the vertices has the required degree. However, the right diagram in the same figure can be straightened by splitting any one of the 3 vertices, and choosing any one of the 3 curves as the straight line segment.

Definition 4.1 *We define an* n-Straightened *Venn Diagram, (n-SVD) as a straightened representation of an n-Venn diagram V_n, with the following properties.*

1. *The curve C_n is a horizontal line segment, beginning and ending on the two copies of the vertex v_1, named v_1^L and v_1^R.*
2. *All vertices of V_n lie on C_n and are numbered $v_1^L, v_2, \ldots, v_m, v_1^R$.*
3. *There are exactly n vertices with degree $2n$, including v_1 and v_2.*
4. *Any vertical line drawn through C_n, which does not pass through a vertex, intersects each curve exactly once.*
5. *All non-adjacent vertices on C_n are the endpoints of exactly 0 or 2 edges.*

Note that this diagram becomes a Venn diagram if we join the two copies of v_1 and make C_n a circle.

Lemma 4.2 *Any SVD represents a monotone Venn diagram.*

Proof. By definition, the SVD represents a Venn diagram. It follows from Property 4 that the vertical line can be seen as a path through the directed dual graph, starting in the lower region, and ending in the upper. □

Lemma 4.3 *The number of curves intersecting at a vertex of an n-SVD has the same parity as n.*

Proof (by induction on v_k)

Let h_k be the number of curves intersecting at a vertex v_k Note that $h_k = deg(v_k)/2$ Also, since an SVD is monotone, h_k is the number of edges from v_i to v_k, for all i where $1 \leq i < k \leq n$

Base Case By Property 3 of Definition 4 1, $h_1 = n$

Inductive Step Assume the statement is true for all v_k, where $k \geq 1$

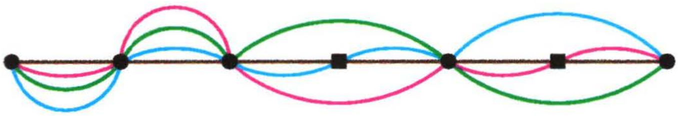
Let the number of edges from v_k to v_{k+1} be c Let the number of edges from v_k to v_l, where $l > k+1$, be d Let the number of edges from v_j to v_{k+1}, where $j < k$, be g Then $h_k = c + d$, and $h_{k+1} = c + g$ By Property 5 of Definition 4 1, d and g are even Then by the induction hypothesis, c must have the same parity as n Therefore, h_{k+1} has the same parity as n □

An illustration of the following proof can be found in Figure 4 1, for the case $n = 4$

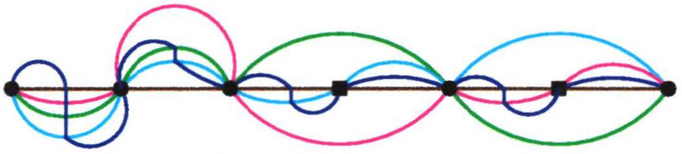
Theorem 4 4 *An* n*-SVD can be extended to an* $(n+1)$*-SVD*

Proof Let sV_n be an SVD, with m vertices Divide the plane into m sections $P_1, P_2, \ldots, P_m$, each section delimited by two vertical lines through two consecutive vertices on sV_n

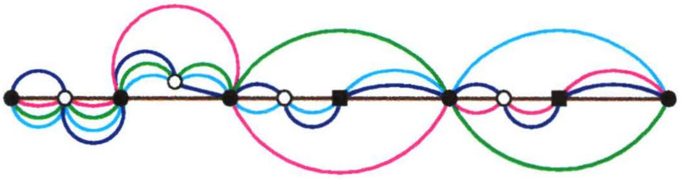
Step 1 We draw a new curve C_{n+1} beginning at v_1^L, and ending on v_1^R In each P_i, we move up to the highest region that has not been previously visited, and sweep downwards as far as possible through all non-visited regions, crossing curves as necessary We exit each P_i through the bordering vertex, and continue in this manner until we reach v_1^R Since we move C_{n+1} through all 2^n regions, at the end of this step, we have a representation of a monotone Venn diagram which has been cut at v_1



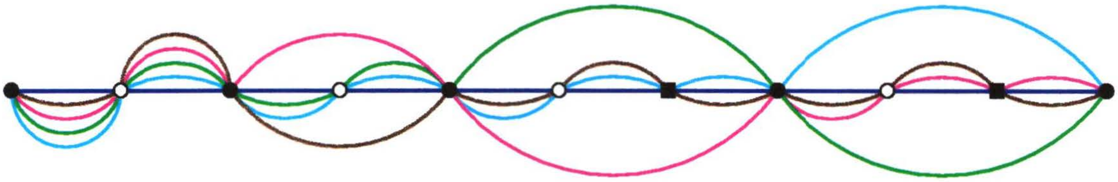
(a) The diagram V_4



(b) The new curve moving through V_4



(c) Compression of vertices along the new curve



(d) Straighten the new curve to get V_5

Figure 4.1: Constructing V_5 from V_4

Step 2: The curve C_{n+1}, while in P_i, intersects $0 \leq r \leq n$ distinct curves on its downward sweep before it exits through the right vertex. For $r \geq 2$, we can apply the compression operation, described in Chapter 2, $r-1$ times and create exactly one vertex from the previous r vertices. This operation, performed similarly in each section, reduces the number of newly created vertices to no more than $2m$.

Step 3: We straighten C_{n+1}.

The new diagram is an $(n+1)$-SVD because of the following:

1. The curve C_{n+1} is the straightened horizontal line segment, beginning and

ending on the two halves of v_1.

2. Since C_{n+1} passes through all existing vertices and creates the only new ones, all vertices lie on C_{n+1}, and are numbered $v_1^L = w_1^L, w_2, \ldots, w_t, w_1^R = v_1^R$, where $t \leq 2m$.

3. The curve C_{n+1} crosses all existing curves in one downward sweep in the first section of V_n between v_1^L and v_2, and these new vertices are compressed to form one vertex of degree $2(n+1)$. After that, C_{n+1} does not venture into the upper or lower regions again, and therefore we cannot produce another compression involving all the curves. C_{n+1} passes through all existing vertices on sV_n, so any vertices that had degree $2n$, now have degree $2(n+1)$. The total number of vertices having degree $2(n+1)$ is $n+1$.

4. Since C_{n+1} is a straight line, it can be intersected by a vertical line exactly once. The curves of sV_n continue to move left to right in the new diagram.

5. If vertices u and w are non-adjacent in sV_n, then there are zero or two edges incident to both. If the number is two, then C_{n+1} has already visited the regions above and below these edges, before passing through u. In either case, C_{n+1} does not alter the number of edges incident to u and w.

   If the vertices are adjacent in sVn, then prior to passing through u, C_{n+1} has either visited the regions above and below the outermost edges, or it has not. If it has not, then C_{n+1} crosses all the edges, and u and w share no edges in the new diagram. If it has, then C_{n+1} does not cross the upper and lower edges, and u and v, if they are no longer adjacent, are the endpoints of exactly two edges.

□

4.1.1 A Specific Set of SVDs

If we use the same construction described in the proof of Theorem 4.4, for all values of n, we create a set of SVDs that has very interesting properties. When discussing SVDs from now on, we specifically refer to this set.

The constructions for the first two diagrams are described below:

1. For $n = 1$, the curve C_1 is a horizontal line segment which divides the plane into an upper and lower region.
2. For $n = 2$, the curve C_2 starts at C_1's leftmost point, moves up to the upper region, crosses C_1 into the lower region and stops at C_2's rightmost point. No compression is necessary and C_2 becomes the new straight line segment.

4.2 Properties of SVDs

4.2.1 Structural Properties

Let sV_n be a straightened n-Venn diagram, constructed as described in the previous section. Let v_i be a vertex on sV_n that has degree $2n$, and let v_j be the next vertex to the right of v_i, which also has degree $2n$. We will call the portion of sV_n which is contained between these 2 vertices a *football*, F^n_k, where k is an index number of the football, as we count from left to right. By Definition 4.1, $1 \leq k \leq n$.

Each football has a *boundary*, which consists of the edges that border the outer and inner region. A boundary edge is sometimes referred to as the upper or lower boundary edge.

Note that due to the method of construction, C_n splits F^{n-1}_1 into F^n_1 and F^n_2. For $k > 1$, C_n does not cross a boundary in F^{n-1}_k. These two facts mean that the

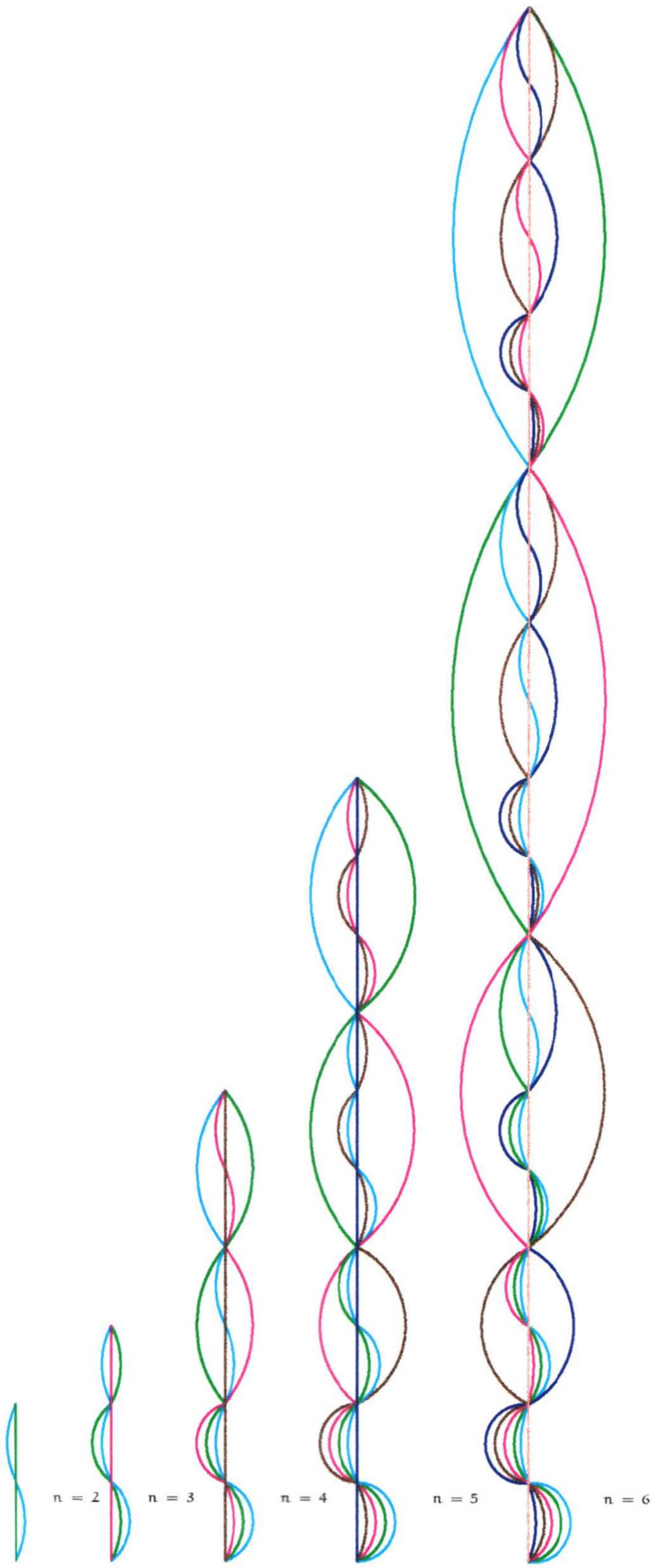


Figure 4 2 The First 6 Straightened Venn Diagrams modified F_k^{n-1} in sV_{n-1} is re-indexed as F_{k+1}^{n} in sV_n.

Lemma 4.5 *The topological structure of* F_k^n, *where* $1 \leq k \leq n$, *is covered by one of the following statements*

1. *When* $1 \leq k \leq 2$, *it is a collection of* n *labelled edges.*
2. *When* $3 \leq k \leq n-1$, *it is a boundary containing* $F_1^{n-2} \ldots F_{k-1}^{n-2}$, *as illustrated in Figure 4.3.*
3. *When* $k = n$ *it is a boundary containing* $F_1^{n-2} \ldots F_{n-2}^{n-2}$

Proof (by induction on n).

See Figure 4.2 for the base cases of $n = 2$ and $n = 3$.

Inductive Step: Assume the lemma is true for F_1^{n-1}.

C_n passes through all curves in F_1^{n-1}, from the upper to lower region, After compression and straightening C_n, we produce F_1^n and F_2^n, divided by the only new vertex. Thus statement 1 is proven.

Assume the lemma is true for F_{k-1}^{n-1} and consider $3 \leq k \leq n-1$. When constructing F_k^n from F_{k-1}^{n-1}, C_n does not cross the boundary, since $k > 2$. For $k = 3$, the action of adding C^n to F_2^{n-1} creates a single vertex compressing $n-3$ labelled edges. When C^n is straightened, the structure within the newly indexed F_3^n's boundary is identical to $F_1^{n-2}F_2^{n-2}$.

For $k > 3$, by the induction hypothesis, F_{k-1}^{n-1}'s boundary contains $F_1^{n-3} \ldots F_{k-2}^{n-3}$. For the special case of $k = n$, the boundary does not contain F_{k-2}^{n-3}, and for simplicity, this is assumed in all future statements concerning F_k^n. When C_n is added to create F_k^n from F_{k-1}^{n-1}, its action inside the boundary follows the same pattern as C_{n-2} does in $F_1^{n-3} \ldots F_{k-2}^{n-3}$, when creating sV_{n-2} from sV_{n-3}. The modified $F_1^{n-3} \ldots F_{k-2}^{n-3}$ in

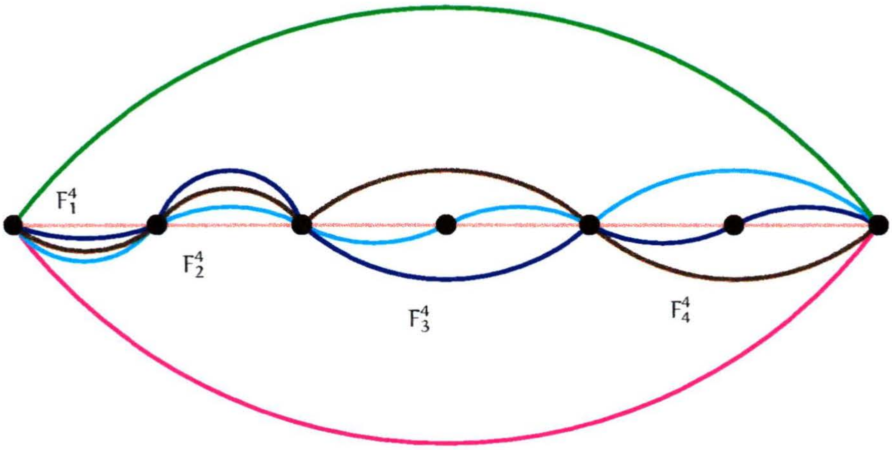


Figure 4.3: The Topological Structure of F_5^6.

sV_{n-3} becomes $F_1^{n-2} \ldots F_{k-1}^{n-2}$, in sV_{n-2}. The modified F_{k-1}^{n-1}, re-indexed as F_k^n in sV_n, contains $F_1^{n-2} \ldots F_{k-1}^{n-2}$. Thus statements 2 and 3 are proven. □

4.2.2 Curve Properties

Another property of these straightened Venn diagrams is that within each football, the *curve segment* of C_i has a predictable placement. It is clear from the construction that the curve segments in F_1^n are the edges ordered $C_n, \ldots, C_1$ and the curve segments in F_2^n are ordered $C_{n-1}, \ldots, C_1, C_n$.

Lemma 4.6 *For $2 \leq k \leq n$, the curve segment of C_j in F_k^n is described by one of the following statements.*

1. *When $1 \leq j < n - k + 1$, the segment is the same as its placement in $F_1^{n-2} \ldots F_{k-1}^{n-2}$.*

2. *When $j = n - k + 1$, the segment is the upper boundary.*

3. *When $j = n - k + 2$, the segment is the lower boundary.*

4. *When $n - k + 2 < j \leq n$, the segment replaces the segment of C_{j-2} in $F_1^{n-2}, \ldots, F_{k-1}^{n-2}$.*

Proof.

Note that for simplicity, when $k = n$, we assume that references to $F_1^{n-2}, \ldots, F_{k-1}^{n-2}$ do not include F_{k-1}^{n-2}.

When $j = n$, C_j is the straight line segment. It is the upper boundary of F_1^n, the lower boundary of F_2^n, and clearly replaces C_{n-2} in all of $F_3^n, \ldots, F_n^n$.

For $k = 2$, the curves are $C_{n-1}, \ldots, C_1, C_n$, from upper to lower boundary, so 1, 2 and 3 are proven, and 4 is not applicable.

For $k > 2$ and $j < n$, we use induction on n. See Figure 4.2 for the base cases of $n = 2$ and $n = 3$.

Inductive Step. Assume the statement is true for any C_j in F_k^{n-1}.

Since $n - k + 1 = n - 1 - (k - 1) + 1$, and $n - k + 2 = n - 1 - (k - 1) + 2$, we use the induction hypothesis to claim that boundaries of F_{k-1}^{n-1} remain the same boundaries when F_k^n is created. Thus statements 2 and 3 are true for all n.

For non-boundary values of j, we invoke the induction hypothesis to claim that C_j in F_{k-1}^{n-1} is either the same as its placement, or is replacing C_{j-2}, in $F_1^{n-3}, \ldots, F_{k-2}^{n-3}$. C_n acts on F_{k-1}^{n-1} in the same manner as C_{n-2} acts on $F_1^{n-3}, \ldots, F_{k-2}^{n-3}$, creating F_k^n or $F_1^{n-2}, \ldots, F_{k-1}^{n-2}$ respectively. So C_j in F_k^n is either still the same as its placement, or is still replacing C_{j-2}, in $F_1^{n-2}, \ldots, F_{k-1}^{n-2}$. Thus statements 1 and 4 are true for all n. □

4.3 Counting the Vertices

We have determined in the proof of Theorem 4.4 that the number of vertices of sV_n is no more than twice the number of vertices of sV_{n-1}. In order to precisely determine the number of vertices, we need to subtract the number of times that C_n passes through 2 existing vertices in sV_{n-1}, without crossing an edge

4.3.1 Singleton Crossings

For $n > 2$, we say sV_n has a *singleton crossing* whenever it has a vertex of degree 4 During the construction of sV_{n+1}, as C_{n+1} exits this vertex, entering section P_i, it confines itself within the 2 curves and does not create a new vertex before exiting P_i See the square vertices of Figure 4 1(a)

Lemma 4 7 *If a singleton crossing occurs on* sV_n, *then* n *is even*

Proof A singleton crossing means that 2 curves cross at one vertex By Lemma 4 3, n must be even □

Let $S(n, k)$ be the number of singleton crossings within the football F^n_k For clarity, we define $S(2, 1) = 0$, and $S(2, 2) = 1$ We define $S(n)$ to be the total number of singleton crossings on an n-SVD

Lemma 4 8 *The number* $S(n, k)$ *is positive if and only if* n *is even and* $n/2 < k \leq n$

Proof (by induction on n)

Obviously $k \leq n$, so we deal specifically with $n/2 < k$

Base Case For $n = 2$, $S(2, 2) = 1$, and $S(2, 1) = 0$

Inductive Step Suppose it is true for $n - 2$ and consider F^n_k

Suppose $S(n, k) > 0$ We know that n is even (by Lemma 4 7) Let F^{n-2}_i be a football contained within the boundary of F^n_k, such that $S(n-2, i) > 0$ Then

$1 \leq i \leq k - 1$ (by Lemma 4 5),

and

$$i > \frac{n-2}{2} \text{ (by the induction hypothesis).}$$

Therefore

$$n/2 - 1 < i \leq k - 1 \Rightarrow n/2 < k.$$

Suppose n is even and $n/2 < k \leq n$. By Lemma 4.5, F_k^n contains $F_1^{n-2}, \ldots, F_{k-1}^{n-2}$ within its boundary. Since $k-1 > n/2 - 1$, by the induction hypothesis, F_{k-1}^{n-2} must have a singleton crossing. Therefore, $S(n,k) > 0$. □

We now present three little corollaries concerning $S(n,k)$.

Corollary 4.9 $S(n,n) = S(n, n-1)$.

Proof. By Lemma 4.5, we know that the unlabelled F_{n-1}^n is identical to the unlabelled F_n^n. □

Corollary 4.10 $S(n, n/2+1) = 1$.

Proof (by induction on n):

Base Case:

$$S(2,2) = 1.$$

Inductive Step: Suppose it is true for $S(n-2, n/2)$. Then

$$\begin{aligned} S(n, n/2+1) &= \sum_{i=1}^{n/2} S(n-2, i) \text{ (by Lemma 4.5)} \\ &= \sum_{i=1}^{n/2-1} S(n-2, i) + S(n-2, n/2) \\ &= 0 + 1 = 1 \text{ (by Lemma 4.8 and the induction hypothesis).} \end{aligned}$$

□

Corollary 4.11 $S(n,k) = S(n,k-1) + S(n-2,k-1)$.

Proof

$$\begin{aligned} S(n,k) &= \sum_{i=1}^{k-1} S(n-2,i) \text{ (by Lemma 4.5)} \\ &= \sum_{i=1}^{k-2} S(n-2,i) + S(n-2,k-1) \\ &= S(n,k-1) + S(n-2,k-1). \end{aligned}$$

□

We define $T(n,k)$ to be the number of well-formed parentheses strings of length $2n$, which begin with exactly k left parentheses. The following recurrence relation for $T(n,k)$ is proven by Ruskey [14]:

$$T(n,k) = \begin{cases} T(n,2) & \text{if } k = 1 \\ T(n,k+1) + T(n-1,k-1) & \text{if } 1 < k < n \\ 1 & \text{if } k = n. \end{cases}$$

An explicit formula for $T(n,k)$ is given below:

$$T(n,k) = \frac{k}{2n-k}\binom{2n-k}{n-k}.$$

We use $T(n,k)$ to demonstrate a relationship between $S(n)$ and $C(n)$, the nth Catalan number. The Catalan numbers are a famous set of numbers, which count many objects. They are named for Eugene Catalan (1814-1894), a Belgian mathematician who used them originally to count the number of ways to parenthesize the expression $x_1x_2x_3 \ldots x_n$ [8]. Some other objects the n^{th} Catalan number counts are:

- the number of binary trees of n nodes,

- the number of ways to triangulate a polygon, and
- the number of ways to permute n consecutive integers, using a stack [8]

The Catalans satisfy the following recurrence relation, with $C(0) = 1$

$$C(n) = \sum_{k=0}^{n-1} C(k)C(n-k)$$

Two explicit values for $C(n)$ are given below

$$\begin{aligned} C(n) &= \frac{1}{n+1}\binom{2n}{n} \\ &= \sum_{i=1}^{n} T(n,i) \end{aligned}$$

Lemma 4 12 *For an even integer* $n > 1$,

$$S(n,k) = T(n/2, n-k+1)$$

Proof (by induction on k)

Base Case

$$S(n, n/2+1) = 1 = T(n/2, n/2)$$

Inductive Step Assume the statement is true for all values less than k

$$\begin{aligned} S(n,k) &= S(n,k-1) + S(n-2,k-1) \text{ (by Corollary 4 11)} \\ &= T(n/2, n-k+2) + T(n/2-1, n-k) \\ &= T(n/2, n-k+1) \end{aligned}$$

And

$$S(n,n) = S(n,n-1) = T(n/2,2) = T(n/2,1)$$

□

Substituting the value into the explicit formula for $T(n,k)$, we have

$$S(n,k) = \frac{n-k+1}{k-1}\binom{k-1}{k-n/2-1}$$

Corollary 4.13 *For $n = 2m$, the number of singleton crossings $S(n)$ on sV_n, is $C(m)$.*

Proof.

$$\begin{aligned} C(m) &= \sum_{i=1}^{m} T(m,i) \text{ (by the explicit formula)} \\ &= \sum_{j=m+1}^{n} S(n,j) \text{ (by Lemma 4.12)} \\ &= S(n). \end{aligned}$$

□

4.3.2 Subtracting the Singleton Crossing from $2M_n$

We know from the proof of Theorem 4.4 and the previous section, that if sVn has M_n vertices, then sV_{n+1} has $2M_n - S(n)$ vertices.

Theorem 4.14 $M_n = \binom{n}{\lfloor n/2 \rfloor}$.

Proof. Theorem 3.6 showed that $M_n \geq \binom{n}{n/2}$. We proceed by induction on n.

Base Cases: Observe that $M_2 = 2$ and $M_3 = 3$, by Figure 4.2.

Inductive Step: Let $n = 2m$, and assume that for all $k < n$,

$$M_k = \binom{k}{\lfloor k/2 \rfloor}$$

Then for n, which is even,

$$\begin{aligned}
M_n &\leq 2M_{n-1} - S(n-1) = 2M_{n-1} \\
&= 2\binom{2m-1}{m-1} \\
&= \frac{2(2m-1)!}{(m-1)!m!} \\
&= \frac{2m(2m-1)!}{m(m-1)!m!} \\
&= \frac{2m!}{m!m!} \\
&= \binom{2m}{m} \\
&= \binom{n}{\lfloor n/2 \rfloor}
\end{aligned}$$

And for $n+1$, which is odd,

$$\begin{aligned}
M_{n+1} &\leq 2M_n - S(n) = 2M_n - C(m) \\
&= 2\binom{2m}{m} - \frac{1}{m+1}\binom{2m}{m} \\
&= \frac{2(m+1)-1}{m+1}\binom{2m}{m} \\
&= \frac{(2m+1)(2m)!}{(m+1)m!m!} \\
&= \frac{(2m+1)!}{(m+1)!m!} \\
&= \binom{2m+1}{m} \\
&= \binom{n}{\lfloor n/2 \rfloor}
\end{aligned}$$

□

CHAPTER 5

The Upper Bound for Non-monotone Venn Diagrams

At present, no proof exists for the minimum number of vertices on general Venn diagrams for $n > 1$. The conjecture is that the value is $\lceil \frac{2^n-2}{n-1} \rceil$. Clearly, if this is true, then the Venn diagram must be a non-monotone Venn diagram.

Venn diagrams with this number of vertices are discovered for 4, 5, 6, and 7 curves. Figure 5.1 shows a minimum 4-Venn discovered by Peter Hamburger and Frank Ruskey [11]. Figures 5.2 and 5.3 are new diagrams which are successively extended from the minimum 4-Venn.

Figure 5.4 is a polar symmetric minimum 7-Venn diagram, discovered in a computer search by Stirling Chow and Ruskey [4]. Note that each vertex in the 7-Venn has the maximum degree; every curve passes through every vertex. The diagram is symmetric in the sense that each curve of the diagram can be obtained by rotating a given curve, see the highlighted one in Figure 5.4, by a multiple of $2\pi/7$ about some point on the plane. This discovery inspires the conjecture that minimum vertex n-Venn diagrams exist for all numbers n. We leave this as an open

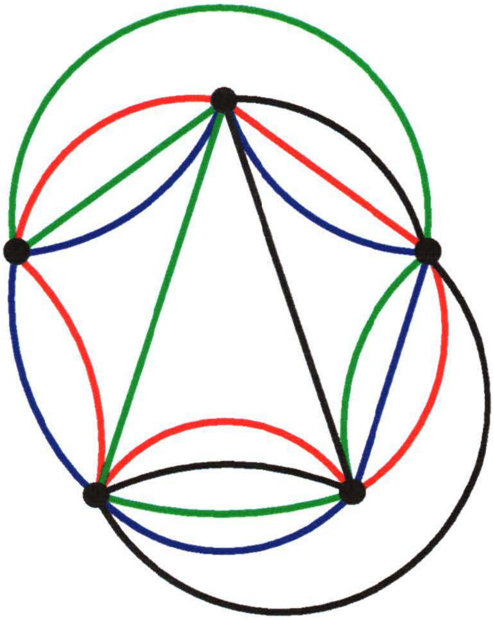


Figure 5.1: A 4-Venn diagram with 5 vertices.

problem.

5.1 A Heuristic Approach to Extending Minimum Vertex Venn Diagrams

Several minimum Venn diagrams for $n = 3, 4, 5,$ and 6 have been constructed by finding a suitable cycle on the radual graph of the smaller minimum Venn diagram. This cycle, after compressions are applied, yields a minimum Venn extension. It was hoped that the methods used would illustrate patterns that predict the formation for larger values of n, but that has not been the case. However, the steps provide a method to discover all minimum vertex Venn diagrams that can be extended from existing Venn diagrams.

Not all minimum vertex Venn diagrams are reducible, even when vertices are

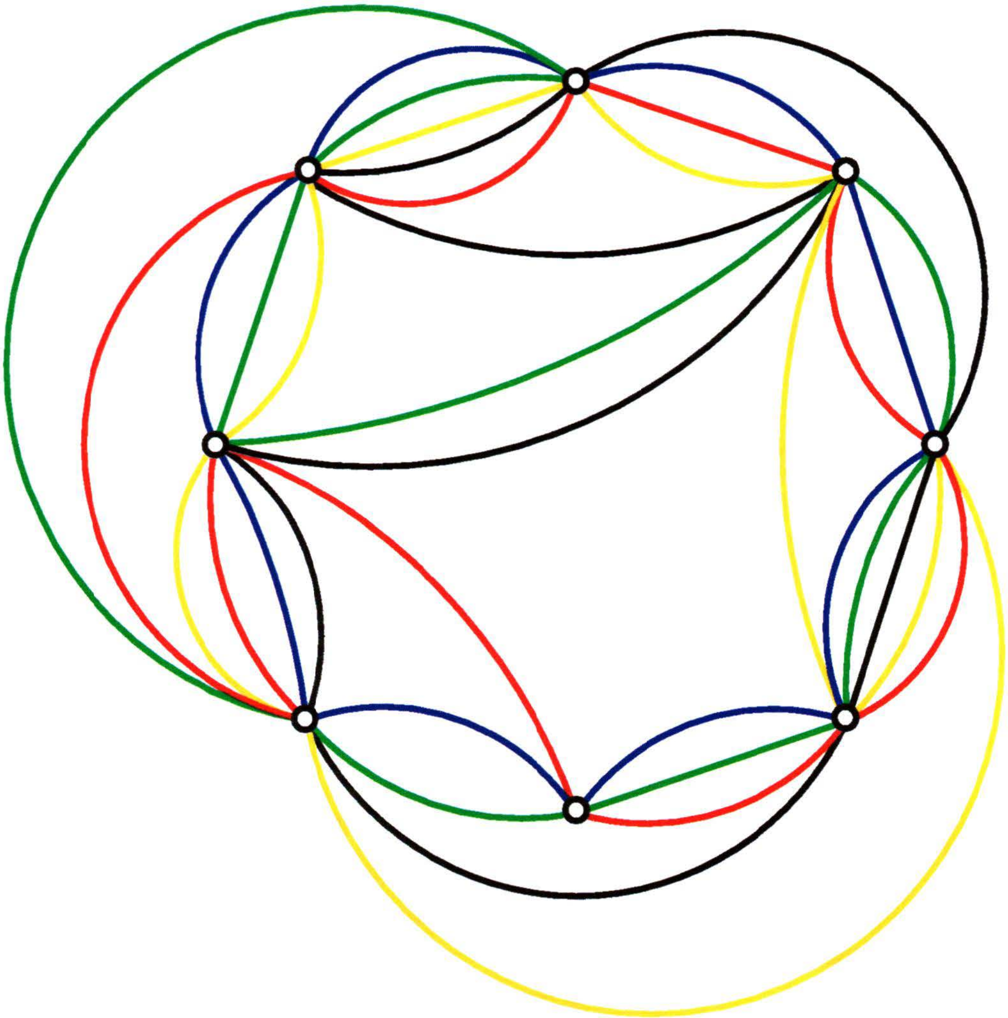


Figure 5.2. A 5-Venn diagram with 8 vertices.

expanded. For example, the 7-Venn in Figure 5.4 is not reducible and therefore cannot be produced using this method. It can be expanded to a diagram with up to 49 vertices, which is much lower than the 64 vertices that one curve must intersect in order to reduce this diagram.

The steps required to produce a minimum n-Venn from an $(n-1)$-Venn are listed below. The goal is to arrange a cycle on the radual graph that includes each dual vertex and allows for maximum compression. After compressions, the number of new vertices does not exceed the difference between $\mathrm{Min}(n)$ and $\mathrm{Min}(n-1)$. Each step must be completed before the next, and the desired diagram is not guaranteed.

1. *Calculate the possible degrees of each vertex on the* n*-Venn*. Except in cases where $(2^n-2)/(n-1)$ is an integer, there can be more than one candidate for the degree sequence. For example, a minimum 4-Venn has 4 vertices whose degrees are all 8 and one whose degree is 6. However, a minimum 5-Venn can have 6 or 7 vertices whose degrees are all 10, depending on whether the 8th vertex has degree 6 or 8.

2. *Find the candidates for compression*. There are 2 types.

   (a) *The new vertex*. If a line segment can cross k consecutive distinct Venn curves as it moves from one dual vertex to another, it is possible to compress these new vertices to a vertex of degree $2(k+1)$. The number k is determined by the required degree of the new vertex. In cases where $(2^n-2)/(n-1)$ is an integer, then $k=n-1$ for each new vertex.

   (b) *The existing vertex that does not have degree* $2(n-1)$. We can assume the new curve will travel through all existing vertices. This was the case in each of the diagrams found using this method. Without compression, an existing vertex that does not have full degree can only increase its degree by 2 on the new diagram. In some cases, this may not be enough. If a line segment can be drawn from an existing vertex v, through t consecutive dual vertices, while crossing distinct curves that are not part of v's traversal, then v can increase its degree by $2t+2$ in the n-Venn diagram.

3. *Find a non-interfering set of paths* We need to find a set of paths in the dual graph which correspond to the candidates for compression described in the previous step A proper set of these paths must not cross each other, and the number of them must satisfy the degree sequence requirement

4. *Find a cycle* We connect all the paths by a cycle on the radual graph that must

    (a) visit each isolated dual vertex exactly once The paths contain the non-isolated dual vertices, so all dual vertices are on the cycle

    (b) connect the isolated dual vertices to Venn vertices only This guarantees that no more new vertices are created

5. *Draw the curve and compress the vertices* If the previous steps were successful, then the cycle is the suitable curve that creates the new n-Venn diagram Vertex compression is applied to sections of the curve that correspond to the chosen set of paths

5.2 An Example: Extending the Minimum 4-Venn Diagram

We demonstrate the heuristic with an example extending the minimum 4-Venn diagram of Figure 5 1 into a minimum 5-Venn diagram Figure 5 5 shows all the paths which allow us to compress 4 vertices into one on the new diagram Depending on the desired vertex degrees in the new diagram, paths of length two or three may also qualify The square vertex can also be joined to a path that crosses the black outer curve as illustrated by the thinner path

A computer program was used to list the possible combinations of non-interfering paths. These lists do not necessarily qualify because a cycle cannot be guaranteed to exist. For example, Figure 5.6 shows a combination of non-interfering paths which do not produce the required cycle. The two square dual vertices are only adjacent to two Venn vertices, yet at least three are required to avoid visiting any vertex more than once in the cycle.

Figure 5.7 shows a cycle that, after compression, yields a desired minimum 5-Venn diagram with 7 vertices of degree 10 and one of degree 6. When we compress along the thick lines of Figure 5.7, we produce a minimum 5-Venn which is isomorphic to the diagram in Figure 5.8.

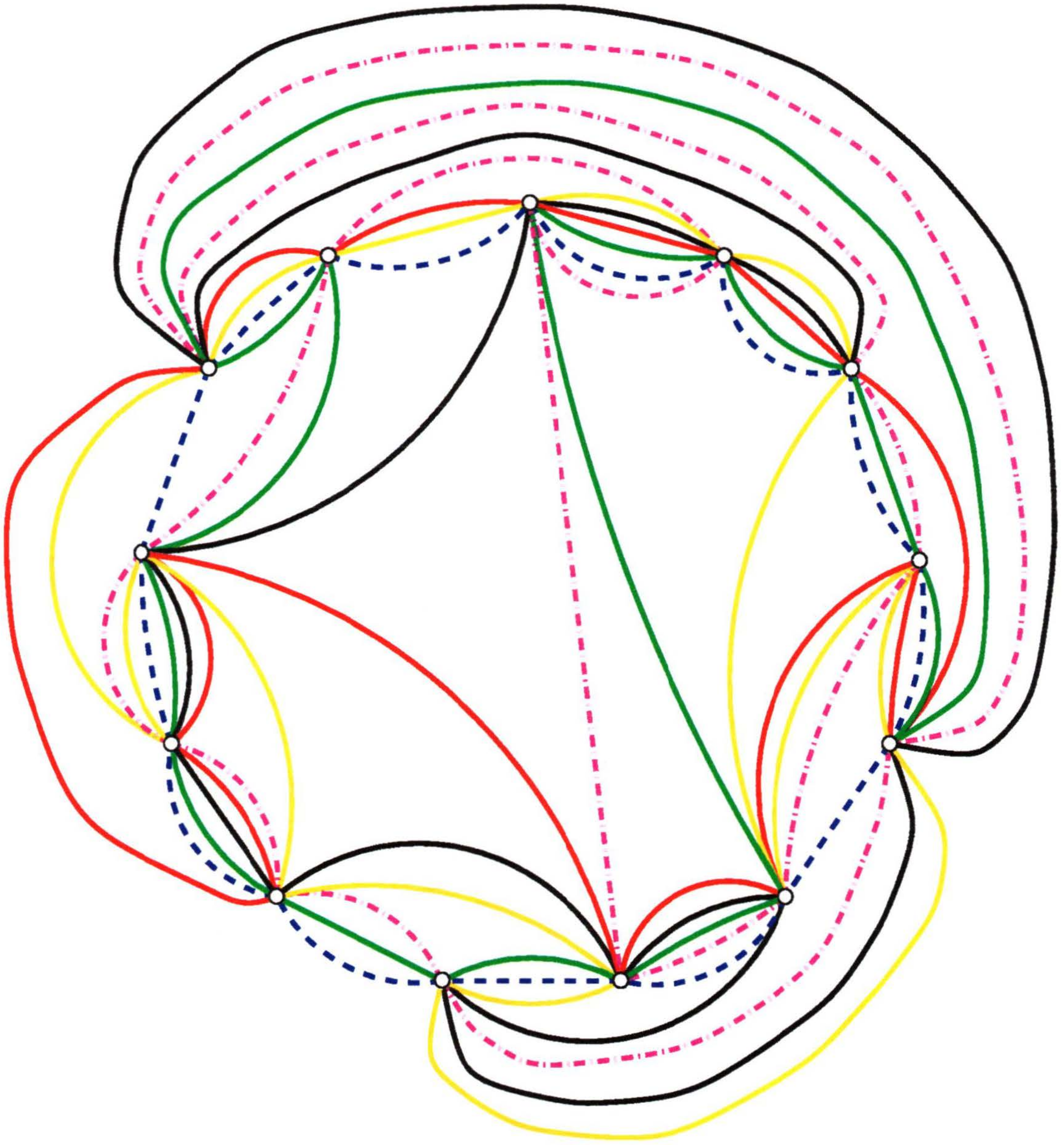


Figure 5.3: A 6-Venn diagram with 13 vertices

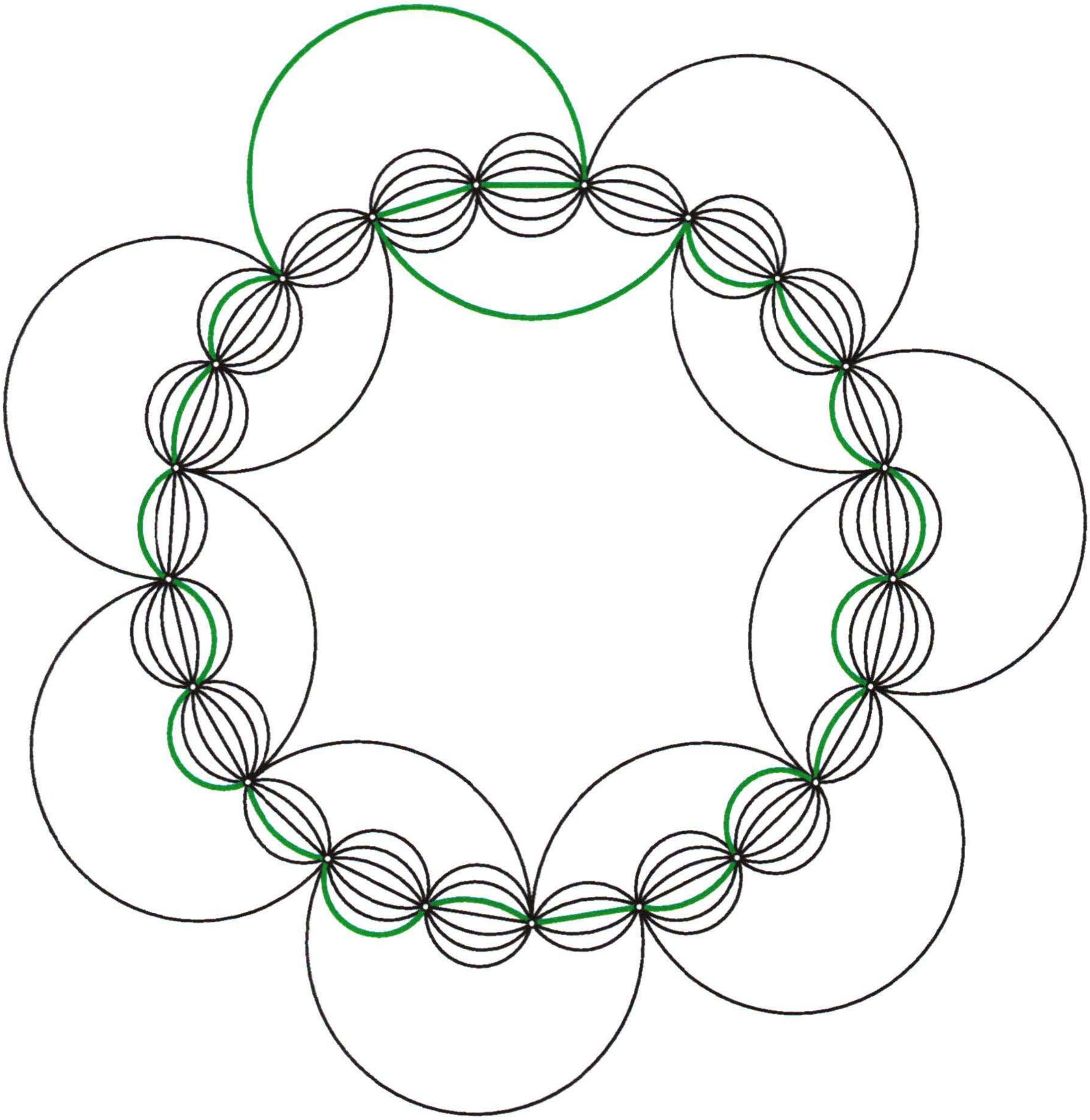


Figure 5.4: A 7-Venn diagram with 21 vertices.

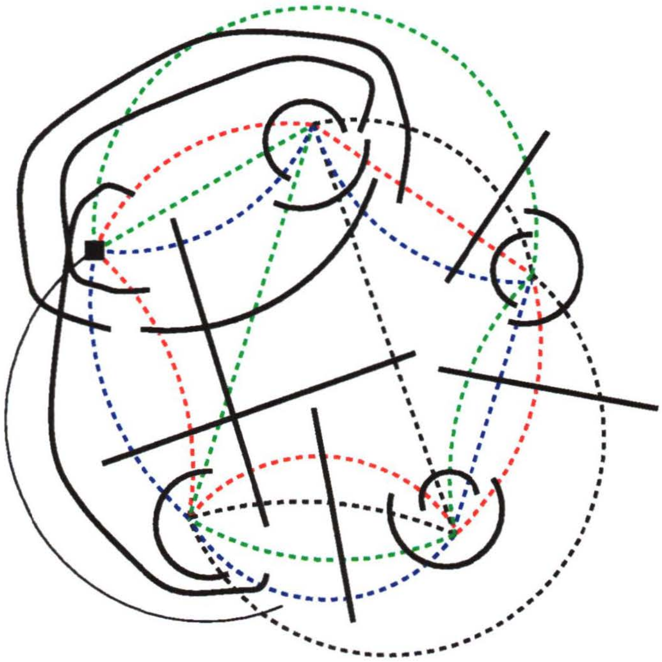


Figure 5 5 All paths that cross 4 distinct curves

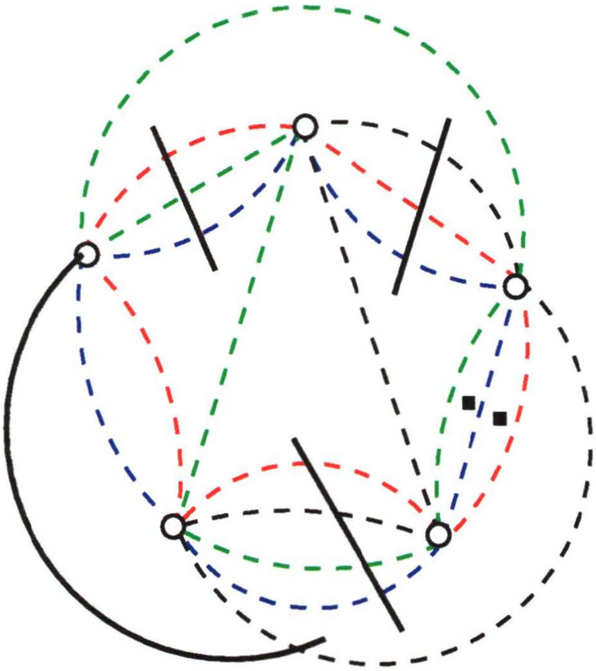


Figure 5 6 A set of paths that cannot be connected to form a cycle

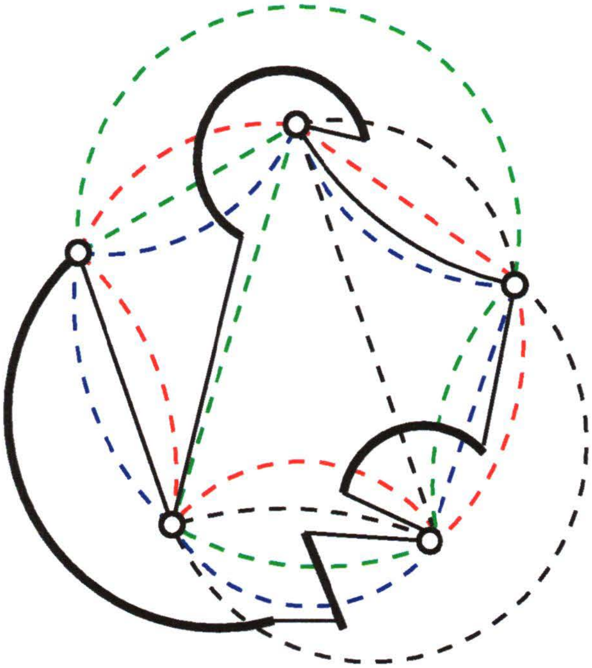


Figure 5.7. A suitable cycle connecting a set of paths.

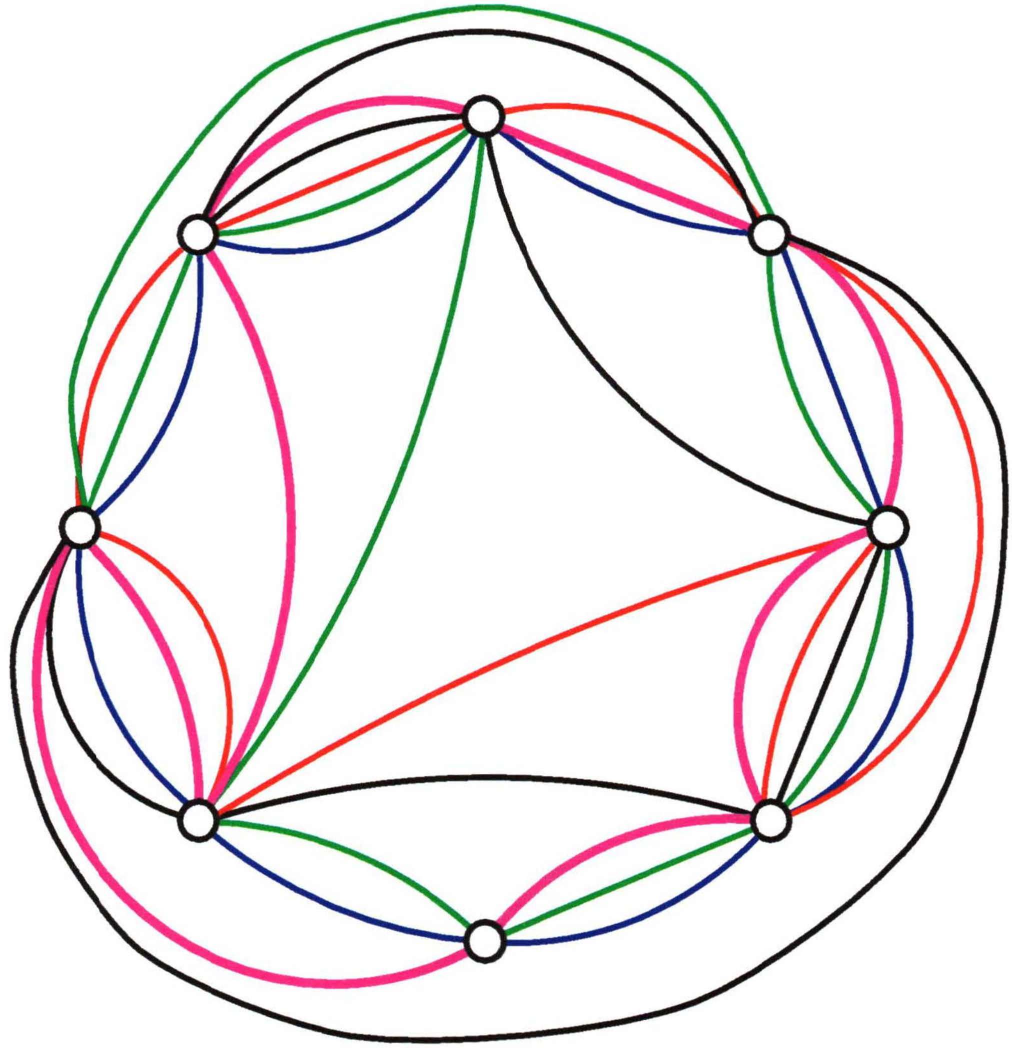


Figure 5 8 The new minimum vertex 5-Venn diagram

CHAPTER 6

Conclusion

Venn diagrams are an interesting set of diagrams, useful in set and combinatorial theory. In this thesis, we have incorporated that study into geometry and graph theory to establish the minimum number of intersections of the Venn curves.

We found it expedient to divide the diagrams into two distinct classifications, those that are monotone and those that are not. The minimum number of intersections depends on this classification. Discussion of the properties of monotone Venn diagrams included the equivalence of potentially convex and monotone Venn diagrams. Any potentially convex diagram can now be easily identified, it will have all its same-weight regions linked together in cycles.

There are many different Venn diagrams and many more ways to draw them. Some constructions exist [6] [7] [15], but most focus on the shapes of the curves. No construction exists that focusses on minimizing the number of intersections. In this dissertation, we successfully developed a recursively constructed set of monotone Venn diagrams, called SVDs. A key element in counting the number of vertices of these diagrams was the number of singleton crossings on an n-SVD. When n is odd, the number of singleton crossings is 0, and when n is even, the number of these crossings is the same as the Catalan number for $n/2$ Subtracting the number of singleton crossings from twice the vertices of an $(n-1)$-SVD produced the desired result of $\binom{n}{\lfloor n/2 \rfloor}$

Constructing non-monotone minimum vertex Venn diagrams is more difficult Presently the only methods involve an heuristic and exhaustive searches Neither method solves this problem efficiently We expect the conjecture that the minimum number of intersections is $\lceil \frac{2^n-2}{n-1} \rceil$ will be solved by the combination of combinatorics, graph theory and geometry

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Title of Thesis:

Venn Diagrams with Few Intersections

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August 21, 1998