# Mappings On Triangular Algebras

A Dissertation Submitted in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY
in the Department of Mathematics and Statistics.

by
Wai-Shun Cheung

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# MAPPINGS ON TRIANGULAR ALGEBRAS

*by*

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*in the Department of Mathematics and Statistics.*
*We accept this dissertation as conforming*
*to the required standard.*

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# Abstract

In this dissertation, we study certain types of linear mappings on triangular algebras. Triangular algebras are algebras whose elements can be written in the form of 2 × 2 matrices

$$\begin{pmatrix} a & m \\ & b \end{pmatrix}$$

where $a \in A$, $b \in B$, $m \in M$ and where $A$, $B$ are algebras and $M$ is a bimodule. Many widely studied algebras, such as upper triangular matrix algebras and nest algebras, can be viewed as triangular algebras. This dissertation is divided into five chapters. The first chapter is a general account of the basics of triangular algebras, including the unitization of nonunital triangular algebras and the structure of the centre of triangular algebras, as well as a brief introduction to some well-known examples of triangular algebras.

In Chapter 2, we study the general structure of derivations on triangular algebras and obtain some results on the first cohomology groups of triangular algebras. The first cohomology group of an algebra is the quotient space of the space of all derivations over the space of all inner derivations, and it is always a main tool in the research of derivations. In addition, we consider the problem of automatic continuity of derivations in the last section of this chapter.

In Chapter 3, we consider sufficient conditions on a triangular algebra so that every Lie derivation is a sum of a derivation and a linear map whose image lies in the centre of the triangular algebra.

In Chapter 4, we consider sufficient conditions for every commuting map on a triangular algebra to be a sum of a map of the form $x \mapsto ax$ and a map whose image lies in the centre of the triangular algebra.

In the final chapter, we are concerned with the automorphisms of triangular algebras. The study of automorphism is a most important way to understand the underlying structure of an algebra. We deduce some results on the Skolem-Noether groups, or the outer automorphism groups, of triangular algebras and apply those results to generalize some known results about automorphisms on a triangular matrix algebras.

Examiners:

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# Contents

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*To my dear parents. Kai-Leung and Wai-Ming.*
*for their endless love and support.*

# Chapter 1

# Basics of Triangular Algebras

## 1.1 Introduction

In this dissertation, we consider linear operators on a class of algebras of the form

$$\mathfrak{A} = \begin{pmatrix} A & M \\ & B \end{pmatrix},$$

where $A$ and $B$ are algebras and $M$ is a nonzero $(A, B)$-bimodule. Such algebras will be called triangular algebras. Many widely studied algebras, including upper triangular matrix algebras, block triangular matrix algebras, nest algebras, semi-nest algebras, and triangular Banach algebras, may be viewed as triangular algebras.

In this dissertation, we study certain linear operators on triangular algebras. Specifically, derivations, Lie derivations, commuting maps and automorphisms. Our results are then applied to the concrete triangular algebras mentioned in the previous paragraph.

In this chapter, we discuss general properties of triangular algebras and give several useful examples.

## 1.2 Definitions

Throughout the dissertation, $\mathbf{R}$ is always a commutative ring with unity. We shall denote the unity of $\mathbf{R}$ by 1. The unity of an algebra $A$ will be denoted by $1_A$ or simply 1 when no confusion is likely to arise.

Suppose that $A, B$ are $\mathbf{R}$-algebras and $M$ is a nonzero $(A, B)$-bimodule. Consider the set

$$\mathrm{Tri}(A, M, B) = \left\{ \begin{pmatrix} a & m \\ & b \end{pmatrix} : a \in A, b \in B, m \in M \right\}.$$

We can define the matrix-like addition and matrix-like multiplication on $\mathrm{Tri}(A, M, B)$ as below:

$$\begin{pmatrix} a_1 & m_1 \\ & b_1 \end{pmatrix} + \begin{pmatrix} a_2 & m_2 \\ & b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & m_1 + m_2 \\ & b_1 + b_2 \end{pmatrix},$$

and

$$\begin{pmatrix} a_1 & m_1 \\ & b_1 \end{pmatrix} \begin{pmatrix} a_2 & m_2 \\ & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & a_1 m_2 + m_2 b_1 \\ & b_1 b_2 \end{pmatrix}.$$

It is straightforward to verify that $\mathrm{Tri}(A, M, B)$ is an algebra under such addition and multiplication. For convenience, $\mathrm{Tri}(A, M, B)$ may be simply written as $\begin{pmatrix} A & M \\ & B \end{pmatrix}$.

**Definition 1.2.1** An $\mathbf{R}$-algebra $\mathfrak{A}$ is called a *triangular algebra* if there exist $\mathbf{R}$-algebras $A, B$ and nonzero $(A, B)$-bimodule $M$ such that $\mathfrak{A}$ is isomorphic to $\mathrm{Tri}(A, M, B)$ under matrix-like addition and matrix-like multiplication.

**Definition 1.2.2** Let $\mathfrak{A} = \text{Tri}(A, M, B)$ be a triangular algebra. We define three projections $\pi_A : \mathfrak{A} \to A$, $\pi_B : \mathfrak{A} \to B$ and $\pi_M : \mathfrak{A} \to M$ as follows. For any $x = \begin{pmatrix} a & m \\ & b \end{pmatrix}$, we set

$$\pi_A(x) = a, \quad \pi_B(x) = b \quad \text{and} \quad \pi_M(x) = m.$$

Moreover, for any $a \in A$ and $b \in B$, we use $a \oplus b$ to denote $\begin{pmatrix} a & 0 \\ & b \end{pmatrix}$.

**Example 1.2.3** The algebra $T_n(\mathbf{R})$ of $n \times n$ upper triangular matrices over $\mathbf{R}$, may be viewed as a triangular algebra when $n > 1$. In general, if $n > k$, we have $T_n(\mathbf{R}) = \begin{pmatrix} T_{n-k}(\mathbf{R}) & \mathbf{R}^{n-k,k} \\ & T_k(\mathbf{R}) \end{pmatrix}$, where $\mathbf{R}^{n-k,k}$ is the space of $(n-k) \times k$ matrices over $\mathbf{R}$.

The above example demonstrates that the choice of $A$, $B$ and $M$ in the definition of triangular algebras is not unique.

**Example 1.2.4** Let $A$ be any $\mathbf{R}$-algebra. The algebra $T_2(A)$ of $2 \times 2$ upper triangular matrices over $A$ is a triangular algebra, indeed it is naturally isomorphic to $\text{Tri}(A, A, A)$.

**Example 1.2.5** Let $A = B = \left\{ \begin{pmatrix} t & a \\ 0 & t \end{pmatrix} : t, a \in \mathbf{R} \right\}$ and $M = T_2(\mathbf{R})$, then

Tri($A, M, B$) is a triangular algebra. This algebra serves as a counterexample in Chapter 3 and Chapter 4.

We now identify which R-algebras are triangular algebras. For an unital algebra, we have the following result.

**Proposition 1.2.6** *A unital algebra $\mathfrak{A}$ is a triangular algebra if and only if there exists an idempotent $e \in \mathfrak{A}$ such that $(1-e)\mathfrak{A}e = 0$ but $e\mathfrak{A}(1-e) \neq 0$.*

*Proof.* To prove the "if" part, assume that $(1-e)\mathfrak{A}e = 0$ and $e\mathfrak{A}(1-e) \neq 0$ for an idempotent $e \in A$. Then $\mathfrak{A} = \begin{pmatrix} e\mathfrak{A}e & e\mathfrak{A}(1-e) \\ & (1-e)\mathfrak{A}(1-e) \end{pmatrix}$. More precisely, the map

$$x \mapsto \begin{pmatrix} exe & ex(1-e) \\ & (1-e)x(1-e) \end{pmatrix}$$

is an isomorphism from $\mathfrak{A}$ onto Tri($e\mathfrak{A}e, e\mathfrak{A}(1-e), (1-e)\mathfrak{A}(1-e)$).

To prove the converse, assume that $\mathfrak{A}$ is a triangular algebra Tri($A, M, B$) for some $A, M, B$. Write $1 = \begin{pmatrix} a' & m' \\ & b' \end{pmatrix}$. We claim that $e = a' \oplus 0$ is the desired idempotent.

Since $1^2 = 1$, we get $(a')^2 = a'$ and hence $e$ is an idempotent.

That $(1-e)\mathfrak{A}e = 0$ follows from

$$(1-e)\begin{pmatrix} a & m \\ & b \end{pmatrix} e = \begin{pmatrix} 0 & m' \\ & b' \end{pmatrix}\begin{pmatrix} a & m \\ & b \end{pmatrix}\begin{pmatrix} a' & 0 \\ & 0 \end{pmatrix} = 0$$

for arbitrary $a \in A, b \in B, m \in M$.

Finally, for any $m \in M$, we have

$$\begin{aligned} e\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}(1-e) &= \begin{pmatrix} a' & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & m' \\ & b' \end{pmatrix} \\ &= \begin{pmatrix} 0 & a'mb \\ & 0 \end{pmatrix} \\ &= \begin{pmatrix} a' & m' \\ & b' \end{pmatrix}\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\begin{pmatrix} a' & m' \\ & b' \end{pmatrix} \\ &= 1\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}1 \\ &= \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}. \end{aligned}$$

Thus $e\mathfrak{A}(1-e) \neq 0$. ∎

Note that $e\mathfrak{A}(1-e) = 0$ implies $e(1-e) = 0$ or $e^2 = e$. Therefore the condition of $e$ being an idempotent can be omitted.

## 1.3 Unitization

Consider an algebra $\mathcal{A}$. The *unitization* of $\mathcal{A}$, denoted by $\mathcal{A} \vee \mathbf{R}1$, is the smallest unital algebra with $\mathcal{A}$ as a subalgebra. Explicitly,

$$\mathcal{A} \vee \mathbf{R}1 = \{a + \gamma 1 : a \in \mathcal{A}, \gamma \in \mathbf{R}\}.$$

If $\mathcal{A}$ has a unit, then $\mathcal{A} \vee \mathbf{R}1$ is $\mathcal{A}$ itself. Otherwise, the expression $a + \gamma 1$ is treated as a formal expression with the usual addition and multiplication, e.g. $(a_1 + \gamma_1 1)(a_2 + \gamma_2 1) = (a_1 a_2 + \gamma_1 a_2 + \gamma_2 a_1) + \gamma_1\gamma_2 1$.

**Proposition 1.3.1**

$$\begin{pmatrix} \mathcal{A} \vee \mathbf{R}1 & \mathcal{M} \\ & \mathcal{B} \vee \mathbf{R}1 \end{pmatrix} = \begin{pmatrix} \mathcal{A} \vee \mathbf{R}1 & \mathcal{M} \\ & \mathcal{B} \end{pmatrix} \vee \mathbf{R}1 = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \vee \mathbf{R}1 \end{pmatrix} \vee \mathbf{R}1.$$

*In particular, if $\mathcal{A}$ (resp. $\mathcal{B}$) is unital, then the unitization of* $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *is* $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B} \vee \mathbf{R}1)$ *(resp.* $\mathrm{Tri}(\mathcal{A} \vee \mathbf{R}1, \mathcal{M}, \mathcal{B})$*).*

*Proof.* The result follows from

$$\begin{pmatrix} a + \alpha 1 & m \\ & b + \beta 1 \end{pmatrix} = \begin{pmatrix} a + (\alpha - \beta)1 & m \\ & b \end{pmatrix} + \beta 1 = \begin{pmatrix} a & m \\ & b + (\beta - \alpha)1 \end{pmatrix} + \alpha 1.$$

∎

In most cases, we will assume that $\mathcal{A}$ and $\mathcal{B}$ are both unital. The previous proposition allows us to extend some results to the case where either $\mathcal{A}$ or

$B$ is unital. Note that if neither $A$ nor $B$ is unital then the unitization of $\mathrm{Tri}(A, M, B)$ is not of the form in the proposition, indeed we have

$$\begin{pmatrix} A & M \\ & B \end{pmatrix} \vee \mathbf{R}1$$
$$= \left\{ \begin{pmatrix} a+t & m \\ & b+t \end{pmatrix} : a \in A, b \in B, m \in M, t \in \mathbf{R} \right\}.$$

## 1.4 The Centre

Consider an algebra $A$. The *centre* of $A$, denoted by $Z(A)$, is the set

$$\{a \in A : ax = xa \text{ for all } x \in A\}.$$

It is straightforward to verify that $Z(A \vee \mathbf{R}1) = Z(A) \vee \mathbf{R}1$.

The structure of the centre of triangular algebras is given in the next theorem.

**Theorem 1.4.1** *Suppose both $A$ and $B$ are unital. The centre of $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ is given by*

$$Z(\mathfrak{A}) = \{a \oplus b : am = mb \text{ for all } m \in M, a \in Z(A), b \in Z(B)\}.$$

*Proof.* Suppose $a \in Z(A)$, $b \in Z(B)$ and $am = mb$ for all $m \in M$. Then for all $\begin{pmatrix} a' & m \\ & b' \end{pmatrix} \in \mathfrak{A}$, we have

$$\begin{pmatrix} a & 0 \\ & b \end{pmatrix}\begin{pmatrix} a' & m \\ & b' \end{pmatrix} = \begin{pmatrix} aa' & am \\ & bb' \end{pmatrix} = \begin{pmatrix} a'a & mb \\ & b'b \end{pmatrix} = \begin{pmatrix} a' & m \\ & b' \end{pmatrix}\begin{pmatrix} a & 0 \\ & b \end{pmatrix}.$$

Hence $a \oplus b \in Z(\mathfrak{A})$.

Conversely, if $\begin{pmatrix} a & m' \\ & b \end{pmatrix} \in Z(\mathfrak{A})$, we have

$$\begin{pmatrix} a & m' \\ & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} a & m' \\ & b \end{pmatrix} = \begin{pmatrix} a & m' \\ & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}$$

and so $m' = 0$. Since

$$\begin{pmatrix} 0 & am \\ & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ & b \end{pmatrix} \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ & b \end{pmatrix} = \begin{pmatrix} 0 & mb \\ & 0 \end{pmatrix}.$$

then $am = mb$ for any $m \in M$. Now

$$aa' \oplus bb' = (a \oplus b)(a' \oplus b') = (a' \oplus b')(a \oplus b) = a'a \oplus b'b.$$

therefore $aa' = a'a$ and $bb' = b'b$ for any $a' \in A$ and $b' \in B$, and hence $a \in Z(A)$ and $b \in Z(B)$. ■

Next, we recall the definition of faithful modules.

**Definition 1.4.2** [38, p.174] A left (respectively right) $A$-module $M$ is said to be *faithful* if $a = 0$ is the only element in $A$ satisfying $aM = 0$ (respectively $Ma = 0$).

We now define the term faithful for bimodules.

**Definition 1.4.3** An $(A, B)$-bimodule $M$ is called a *faithful* bimodule if it is both a faithful left $A$-module and a faithful right $B$-module.

**Theorem 1.4.4** *If $A, B$ are unital and $M$ is faithful, then the centre of $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ is given by*

$$Z(\mathfrak{A}) = \{a \oplus b : am = mb \text{ for all } m \in M, a \in A, b \in B\}.$$

*Furthermore, $\pi_A(Z(\mathfrak{A})) \subseteq Z(A)$ and $\pi_B(Z(\mathfrak{A})) \subseteq Z(B)$, and there exists a unique algebra isomorphism $\tau$ from $\pi_A(Z(\mathfrak{A}))$ to $\pi_B(Z(\mathfrak{A}))$ such that $am = m\tau(a)$ for every $m \in M$.*

*Proof.* By Theorem 1.4.1, we have

$$Z(\mathfrak{A}) = \{a \oplus b : am = mb \text{ for all } m \in M, a \in Z(A), b \in Z(B)\}.$$

Thus $\pi_A(Z(\mathfrak{A})) \subseteq Z(A)$ and $\pi_B(Z(\mathfrak{A})) \subseteq Z(B)$.

Suppose $am = mb$ for every $m \in M$, then for any $a' \in A$ we have

$$(aa' - a'a)m = a(a'm) - a'(am) = (a'm)b - a'(mb) = 0 \quad \forall m \in M$$

hence $aa' - a'a = 0$ as $M$ is a faithful left $A$-module. Therefore $a \in Z(A)$ and similarly we have $b \in Z(B)$. As a result $a \oplus b \in Z(\mathfrak{A})$.

Next we show that there exists a unique mapping $\tau : \pi_A(Z(\mathfrak{A})) \to \pi_B(Z(\mathfrak{A}))$ satisfying $am = m\tau(a)$ for all $a \in A$ and $m \in M$.

For any $a \in \pi_A(Z(\mathfrak{A}))$, there exists $b \in \pi_A(Z(\mathfrak{A}))$ such that $a \oplus b \in Z(\mathfrak{A})$. Suppose there exists another $b$ satisfying $a \oplus b' \in Z(\mathfrak{A})$, then we have $mb =$

$am = mb'$ for all $m \in M$ and hence $b = b'$ as $M$ is faithful. Therefore the map $\tau$ exists and is unique.

It remains to prove that $\tau$ is an algebra isomorphism.

If $\tau(a) = 0$ then $am = 0$ for every $m \in M$ and thus $a = 0$. Therefore $\tau$ is injective. That $\tau$ is surjective follows from the definition of $\pi_B(Z(\mathfrak{A}))$. For any $a, a' \in \pi_A(Z(\mathfrak{A}))$ and $r \in \mathbf{R}$, we have

$$(ra)m = r(am) = r(m\tau(a)) = m(r\tau(a)),$$

$$(a + a')m = m(\tau(a) + \tau(a')),$$

and

$$(aa')m = a(a'm) = (a'm)\tau(a) = a'(m\tau(a)) = m\tau(a)\tau(a'),$$

thus $\tau(ra) = r\tau(a)$, $\tau(a + a') = \tau(a) + \tau(a')$ and $\tau(aa') = \tau(a)\tau(a')$, proving that $\tau$ is an algebra isomorphism. ■

**Corollary 1.4.5** *Suppose* $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ *with faithful* $M$. *If* $Z(A) = \mathbf{R}1_A$ *(or* $Z(B) = \mathbf{R}1_B$*) then* $Z(\mathfrak{A}) = \mathbf{R}1_{\mathfrak{A}}$.

*Proof.* Suppose $Z(A) = \mathbf{R}1$. By Theorem 1.4.4, if $a \oplus b \in Z(\mathfrak{A})$ then $a = r1_A \in \mathbf{R}1$ and $mb = am = m(r1_B)$, thus $b = r1_B$ as $M$ is faithful. Now $a \oplus b = r1_{\mathfrak{A}}$, proving that $Z(\mathfrak{A}) = \mathbf{R}1_{\mathfrak{A}}$. ■

**Corollary 1.4.6** *If $A$ is unital then* $Z(\text{Tri}(A, A, A)) = \{a \oplus a : a \in Z(A)\}$.

*Proof.* Since $A$ is unital, $\text{Tri}(A, A, A)$ satisfies the condition in Theorem 1.4.4. Indeed if $a \in A$ and $am = 0$ for all $m \in A$, then $a = a \cdot 1 = 0$. Similarly $ma = 0$ for all $m \in A$ implies that $a = 0$. This proves that $A$ is a faithful $(A, A)$-bimodule. The condition that $am = mb$ for all $m \in A$ implies $a = b$ by taking $m = 1$. Thus $Z(\text{Tri}(A, A, A)) = \{a \oplus a : a \in Z(A)\}$. ∎

**Corollary 1.4.7** *Consider the triangular algebra in Example 1.2.5, i.e.*

$$A = B = \left\{ \begin{pmatrix} t & a \\ & t \end{pmatrix} : t, a \in \mathbf{R} \right\} \quad \text{and} \quad M = T_2(\mathbf{R}).$$

*Then* $Z(\text{Tri}(A, M, B)) = \mathbf{R}1$.

*Proof.* As in the proof of the previous corollary, we first establish that every element in the centre is of the form $x \oplus x$ with $x \in A$. The condition that $xm = mx$ for every $m \in T_2(\mathbf{R})$ implies $x \in \mathbf{R}1$. ∎

The simplicity of the structure of the centre for $\text{Tri}(A, M, B)$ with faithful $M$ will be useful in establishing some results in later chapters.

## 1.5 Matrix Algebras

Let $\mathbf{S}$ be a fixed unital $\mathbf{R}$-algebra. Denote by $\mathbf{S}^{k,l}$ the space of $k \times l$ matrices over a unital ring $S$. The standard basis of $\mathbf{S}^{k,l}$ is $\{E_{ij} : 1 \leq i \leq k, 1 \leq j \leq l\}$ where $E_{ij}$ is the matrix with a 1 at the $(i,j)$-entry and 0 elsewhere. By a *matrix algebra* we mean a subalgebra of $M_n(\mathbf{S}) = \mathbf{S}^{n,n}$.

The three most commonly used matrix algebras are the $n \times n$ full matrix algebras $M_n(\mathbf{S})$, the $n \times n$ diagonal matrix algebras $D_n(\mathbf{S})$, and the $n \times n$ upper triangular matrix algebras $T_n(\mathbf{S})$.

A generalization of upper triangular matrix algebras is the upper block triangular matrix algebras.

**Definition 1.5.1** Let $n_1, \ldots, n_k$ be positive integers. The *upper block triangular algebra* $T(n_1, \ldots, n_k)(\mathbf{S})$ is the algebra consisting of elements of the form $(a_{ij})_{1 \leq i,j \leq n}$, where $a_{ij}$ is a $n_i \times n_j$ matrix over $\mathbf{S}$ if $i \leq j$ and $a_{ij} = 0$ if $i > j$.

When $k = 1$, we have $T(n_1)(\mathbf{S}) = \mathbf{S}^{n_1,n_1}$. When $n_1 = \cdots = n_k = 1$, we have $T(1, \ldots, 1)(\mathbf{S})$ is just the upper triangular matrix algebra $T_k(\mathbf{S})$.

In the following result, we will consider when a block triangular matrix algebra is a triangular algebra.

**Theorem 1.5.2** (a) *$M_n(\mathbf{S})$ is a triangular algebra if and only if* $\mathbf{S}$ *is a triangular algebra.*

(b) *Suppose $k > 1$. Let $1 \leq l \leq k-1$, $A = T(n_1, \ldots, n_l)(\mathbf{S})$ and $B = T(n_{l+1}, \ldots, n_k)(\mathbf{S})$ and $M$ be the space of $\left(\sum_{j=1}^{l} n_j\right) \times \left(\sum_{j=l+1}^{k} n_j\right)$ matrices over* $\mathbf{S}$, *then $T(n_1, \ldots, n_k)(\mathbf{S}) = \mathrm{Tri}(A, M, B)$ is a triangular matrix algebra with faithful $M$.*

*Proof.* (a) Assume $\mathbf{S}$ is a triangular algebra. By Proposition 1.2.6, there exists an idempotent $e \in \mathbf{S}$ such that $e\mathbf{S}(1-e) = 0$ and $(1-e)\mathbf{S}e \neq 0$. Therefore $M_n(\mathbf{S})$ is a triangular algebra since $eI_n$ is an idempotent in $M_n(\mathbf{S})$ such that $eI_nM_n(\mathbf{S})(1-e)I_n = 0$ and $(1-e)I_nM_n(\mathbf{S})eI_n \neq 0$. Indeed if $\mathbf{S} = \mathrm{Tri}(\Delta, \Pi, \Sigma)$ then $M_n(\mathbf{S})$ is isomorphic to $\begin{pmatrix} M_n(\Delta) & M_n(\Pi) \\ & M_n(\Sigma) \end{pmatrix}$ under the isomorphism

$$\Phi\left(\left(\begin{pmatrix} d_{ij} & p_{ij} \\ & s_{ij} \end{pmatrix}\right)\right) = \begin{pmatrix} (d_{ij}) & (p_{ij}) \\ & (s_{ij}) \end{pmatrix}.$$

To illustrate, when $n = 2$, we have

$$\Phi\begin{pmatrix} \begin{pmatrix} d_{11} & p_{11} \\ & s_{11} \end{pmatrix} & \begin{pmatrix} d_{12} & p_{12} \\ & s_{12} \end{pmatrix} \\ \begin{pmatrix} d_{21} & p_{21} \\ & s_{21} \end{pmatrix} & \begin{pmatrix} d_{22} & p_{22} \\ & s_{22} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} & \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \\ & \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \end{pmatrix}$$

Conversely we assume that S is not a triangular algebra. Suppose $M_n(\mathrm{S})$ is a triangular algebra, then by Proposition 1.2.6, there exists an idempotent $0 \neq E = (e_{ij}) \in M_n(\mathrm{S})$ satisfying $(I - E)XE = 0$ for every $X \in M_n(\mathrm{S})$ and $EM_n(\mathrm{S})(1 - E) \neq 0$. In particular, if we take $X = sE_{kl}$, then we have

$$e_{ik}se_{jl} = 0 \quad \text{if } i \neq k \tag{1.1}$$

$$e_{kk}se_{jl} = se_{jl}. \tag{1.2}$$

When $i = k = l = j$, we get $e_{kk}se_{kk} = se_{kk}$. Hence $(1 - e_{kk})\mathrm{S}e_{kk} = 0$, and by Proposition 1.2.6, $e_{kk}\mathrm{S}(1 - e_{kk}) = 0$ as S is not a triangular algebra. Therefore $e_{kk}s = e_{kk}se_{kk} = se_{kk}$ and we have $e_{kk} \in Z(\mathrm{S})$.

If $k \neq l$, then by (1.1), we get $e_{lk}e_{ll} = 0$. Interchanging $k$ and $l$, we get $e_{kl}e_{ll} = 0$. By (1.2), we have $e_{kk}e_{kl} = e_{kl}$. Since $e_{kk} \in Z(\mathrm{S})$, we have

$$e_{kl} = e_{kk}e_{kl} = e_{kl}e_{kk} = 0.$$

By (1.2), we have $e_{11}e_{kk} = e_{kk}$ and $e_{kk}e_{11} = e_{11}$. Since $e_{kk} \in Z(\mathrm{S})$, we get

$$e_{kk} = e_{11}e_{kk} = e_{kk}e_{11} = e_{11}.$$

Therefore $E = e_{11}I \in Z(\mathrm{S})I$. As a result $EM_n(\mathrm{S})(1 - E) = 0$, a contradiction.

(b) Straightforward. ■

The following two results are related to the centre of matrix algebras over $\mathbf{S}$.

**Proposition 1.5.3** *Suppose $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ is a matrix algebra with faithful $M$. If $\mathbf{S}I \subseteq \mathfrak{A}$ and $Z(A) = Z(\mathbf{S})I_A$ (or $Z(B) = Z(\mathbf{S})I_B$) then $Z(\mathfrak{A}) = Z(\mathbf{S})I$.*

.

*Proof.* Suppose $Z(A) = Z(\mathbf{S})I$. By Theorem 1.4.4, If $a \oplus b \in Z(\mathfrak{A})$, then (i) $a \in Z(A) = Z(\mathbf{S})I$ and thus $a = sI$ for some $s \in Z(S)$ and (ii) $am = mb$ for all $m \in M$ and thus $mb = am = m(sI)$ for all $m \in M$, and hence $b = sI$ as $M$ is faithful. As a result, $Z(\mathfrak{A}) \subseteq Z(\mathbf{S})I$. The reverse inclusion is trivial as $\mathbf{S}I \subseteq \mathfrak{A}$. ∎

**Theorem 1.5.4** *Consider the block triangular matrix algebra*

$$\mathfrak{A} = T(n_1, \ldots, n_k)(\mathbf{S}).$$

*We have $Z(\mathfrak{A}) = Z(\mathbf{S})I$.*

*Proof.* Take $E$ in the centre of $\mathfrak{A}$. If $k = 1$, then $\mathfrak{A} = M_n(\mathbf{S})$. It is well known [48, Ex 1.9] that the centre of $M_n(\mathbf{S})$ is $Z(\mathbf{S})I$. If $k > 1$, then

$\mathbf{S}I \subseteq \mathfrak{A} = \text{Tri}(M_{n_1}(\mathbf{S}), M, B)$ for some faithful $M$ and $Z(M_{n_1}(\mathbf{S})) = Z(\mathbf{S})I$ and thus $Z(\mathfrak{A}) = Z(\mathbf{S})I$ by Proposition 1.5.3. ■

The next result is about a generating set of the block triangular matrix algebras over $\mathbf{S} = \mathbf{R}$.

**Proposition 1.5.5** *The algebra of $T(n_1, \ldots, n_k)(\mathbf{R})$ is the linear span of all the idempotents it contains. In particular, the same is true for $M_n(\mathbf{R})$ and $T_n(\mathbf{R})$.*

*Proof.* Clearly the set of idempotents

$$\{E_{ii} : 1 \leq i \leq n\} \cup \{E_{ii} + E_{ij} : i \neq j \text{ and } E_{ij} \in T(n_1, \ldots, n_k)(\mathbf{R})\}$$

generates $T(n_1, \ldots, n_k)(\mathbf{R})$. Indeed the subspace generated by these idempotents includes every matrix unit $E_{ij}$ that belongs to the algebra. ■

## 1.6 Nest Algebras

In this section, we give a brief introduction to one of the most well-studied infinite dimensional triangular algebras: nest algebras. We refer the reader to [22] for the general theory of nest algebras.

Consider a complex Hilbert space **H**. A *nest* is a set $\mathcal{N}$ of closed subspaces of **H** satisfying the following four conditions:

(1) $0, \mathbf{H} \in \mathcal{N}$;

(2) If $N_1, N_2 \in \mathcal{N}$ then either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$;

(3) If $\{N_j\}_{j\in J} \subseteq \mathcal{N}$ then $\bigcap_{j\in J} N_j \in \mathcal{N}$;

(4) If $\{N_j\}_{j\in J} \subseteq \mathcal{N}$ then the norm closure of the linear span of $\bigcup_{j\in J} N_j$ also lies in $\mathcal{N}$.

If $\mathcal{N} = \{0, \mathbf{H}\}$ then $\mathcal{N}$ is called a *trivial nest*, otherwise it is called a *non-trivial nest*. A nest $\mathcal{N}$ is said to be *continuous* if, for every $N \in \mathcal{N}$, we have $\inf\{M \in \mathcal{N} : N \subsetneq M\} = N$.

We use $B(\mathbf{H})$ to denote the space of all bounded linear operators over **H**.

**Definition 1.6.1** The *nest algebra associated with* $\mathcal{N}$ is the set

$$\mathcal{T}(\mathcal{N}) = \{T \in B(\mathbf{H}) : T(N) \subseteq N \text{ for all } N \in \mathcal{N}\},$$

i.e. the algebra of all bounded linear operators leaving every subspace in $\mathcal{N}$ invariant.

If $\mathcal{N}$ is trivial then $\mathcal{T}(\mathcal{N}) = B(\mathbf{H})$.

**Example 1.6.2** [22] Consider an orthonormal basis $\{e_j : j = 1, 2, \ldots\}$ of $\mathbf{H}$. Let $N_k = \text{span}\{e_1, \ldots, e_k\}$ and $\mathcal{N} = \{N_k : k = 1, 2, \ldots\} \cup \{0, \mathbf{H}\}$. Then $\mathcal{N}$ is a nest and the associated nest algebra $\mathcal{T}(\mathcal{N})$ is the algebra of operators whose matrix representation with respect to $\{e_j\}$ is upper triangular.

**Example 1.6.3** If $\mathbf{H}$ is of finite dimension, then nest algebras are just algebras of upper block triangular matrices. Indeed if $\mathcal{N}$ is a nest on a finite dimensional space and if $0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_k = \mathbf{H}$ are the subspaces in the nest, then we may choose an orthonormal basis $e_1, \ldots, e_n$ for $\mathbf{H}$ such that $\{e_1, \ldots, e_{n_k}\}$ is a basis for $N_j$. The matrix representation of operators in $\mathcal{T}(\mathcal{N})$ with respect to this basis is the upper block triangular matrix algebra $T(n_1, n_2 - n_1, \ldots, n_k - n_{k-1})(\mathbb{C})$.

**Example 1.6.4** [22, Example 2.4] Let $\mathbf{H} = \mathcal{L}^2[0, 1]$ with Lebesque measure. For each $t \in [0, 1]$, let $H_t = \{f \in \mathcal{L}^2[0, 1] : f(x) = 0 \text{ a.e. for } t \leq x \leq 1\}$, then $\{H_t : 0 \leq t \leq 1\}$ is a continuous nest. This nest is known as the Volterra nest.

We recall the following standard results for further use.

**Lemma 1.6.5** [22, Chapter 2] *If $N \in \mathcal{N} \setminus \{0, \mathbf{H}\}$ and $E$ is the orthonormal projection onto $N$. Then $E\mathcal{N}$ and $(1 - E)\mathcal{N}$ are nests in the Hilbert spaces*

$E$H *and* $(1-E)$H *respectively, and* $\mathcal{T}(E\mathcal{N}) = E\mathcal{T}(\mathcal{N})E$ *and* $\mathcal{T}((1-E)\mathcal{N}) = (1-E)\mathcal{T}(\mathcal{N})$. *Furthermore*

$$\mathcal{T}(\mathcal{N}) = \begin{pmatrix} \mathcal{T}(E\mathcal{N}) & E\mathcal{T}(\mathcal{N})(1-E) \\ & \mathcal{T}((1-E)\mathcal{N}) \end{pmatrix}.$$

**Lemma 1.6.6** [22, Corollary 19.5] $Z(\mathcal{T}(\mathcal{N})) = \mathbb{C}1$.

**Lemma 1.6.7** [53, Proposition 2.6] *Every element of the nest algebra of a continuous nest is a sum of two commutators $xy - yx$.*

## 1.7 Triangular Banach Algebras

By a *triangular Banach algebra*, we mean a Banach algebra $\mathfrak{A}$ which is also a triangular algebra, i.e., there exists an idempotent $\mathfrak{A}$ such that $e\mathfrak{A}(1-e) = 0$ but $(1-e)\mathfrak{A}e \neq 0$. Many of the frequently investigated nonself-adjoint operator algebras, e.g., nest algebras, are indeed triangular Banach algebras. Two more examples are given below.

The first is the join of two operator algebras introduced by Gilfeather and Smith [27]. They investigated the cohomology groups of such algebras.

**Definition 1.7.1** [27] Consider two Hilbert spaces $H$ and $K$. Let $A$ and $B$ be two norm closed unital subalgebras of $B(H)$ and $B(K)$. The *join* of $A$

and $B$ is defined by

$$A\#B = \begin{pmatrix} A & B(K,H) \\ & B \end{pmatrix}$$

where $B(K,H)$ is the space of all bounded linear operator from $K$ to $H$. The norm on $A\#B$ is the operator norm where $A\#B$ is viewed as a subalgebra of $B(H \oplus K)$ in the obvious way.

The next example is a certain finite dimensional perturbation of a nest algebra defined by Deguang [23] and is related to the concept of semi-triangular operators studied by Larson and Wogen [47].

**Definition 1.7.2** [23] Consider a Hilbert space $H$. A subalgebra $\mathfrak{A}$ of $B(H)$ is said to be a *semi-nest algebra* if

(i) $\mathfrak{A}$ is reflexive, i.e.

$$\mathfrak{A} = \{T \in B(H) : PTP = TP \quad \text{for any} \quad P \in Lat(\mathfrak{A})\}$$

where $Lat(\mathfrak{A})$ is the set of projections $P$ satisfying $PAP = AP$ for every $A \in \mathfrak{A}$, i.e. projections on the invariant subspaces of $\mathfrak{A}$; and

(ii) There exists a projection $\hat{P} \in \mathfrak{A} \cap Lat(\mathfrak{A})$ such that $\hat{P}\mathfrak{A}\hat{P}$ is a nest algebra on $PH$ and $dim(I - \hat{P})H < \infty$.

A semi-nest algebra $\mathfrak{A}$ is a triangular Banach algebra if $\hat{P}\mathfrak{A}(1 - \hat{P}) \neq 0$ .

In the remainder of this section, we describe how to construct a triangular Banach algebra $\begin{pmatrix} a & m \\ & b \end{pmatrix}$ from given Banach algebras $\mathcal{A}$, $\mathcal{B}$ and a bimodule $\mathcal{M}$. First consider a triangular Banach $\mathfrak{A}$ algebra with an idempotent $e$ satisfying $e\mathfrak{A}(1-e) = 0$ and $(1-e)\mathfrak{A}e \neq 0$. Let $\mathcal{A} = e\mathfrak{A}e$, $\mathcal{B} = (1-e)\mathfrak{A}(1-e)$ and $\mathcal{M} = (1-e)\mathfrak{A}e$. Then $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras and $\mathcal{M}$ is a Banach space and $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. We observe that $\mathcal{A}$ and $\mathcal{B}$ are closed algebras of $\mathfrak{A}$ and $\mathcal{M}$ may be viewed as an $(\mathcal{A}, \mathcal{B})$-bimodule. The submultiplicity condition of the norm on $\mathfrak{A}$ implies that $\|amb\| \leq \|a\|\|m\|\|b\|$ for every $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $m \in \mathcal{M}$.

Conversely if we start with Banach algebras $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and an $(\mathcal{A}, \mathcal{B})$-bimodule which is also a Banach space with norm $\|\cdot\|_{\mathcal{M}}$ satisfying $\|amb\|_{\mathcal{M}} \leq \|a\|_{\mathcal{A}}\|m\|_{\mathcal{M}}\|b\|_{\mathcal{B}}$, a triangular Banach algebra $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ with norm given by

$$\left\| \begin{pmatrix} a & m \\ & b \end{pmatrix} \right\|_T = \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}}.$$

This is how triangular Banach algebras were defined in [26]. We use the term slightly more generally as the norm may be different than the one given above. However all norms that make $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ a triangular Banach algebra are equivalent as the following proposition shows.

**Proposition 1.7.3** *Let* $(\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}), \|\cdot\|)$ *be a triangular Banach algebra. Then* $\|\cdot\|$ *and* $\|\cdot\|_T$ *are equivalent norms.*

*Proof.* Consider the identity map

$$id : (\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}), \|\cdot\|_T) \to (\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}), \|\cdot\|).$$

We have

$$\begin{aligned}
\left\|\begin{pmatrix} a & m \\ & b \end{pmatrix}\right\| &\leq \left\|\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\right\| + \left\|\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right\| + \left\|\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\right\| \\
&= \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}} \\
&= \left\|\begin{pmatrix} a & m \\ & b \end{pmatrix}\right\|_T .
\end{aligned}$$

The first equality holds as we identify $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ as Banach subspaces of $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Thus $id$ is a bounded linear bijective map between two Banach spaces. By the inverse mapping Theorem [18, Section 12.5], $id^{-1}$ is also bounded. Thus the two norms are equivalent. ∎

Consider a Banach algebra **S**. Then $M_n(\mathbf{S})$ may be identified in the obvious way with $M_n(\mathbb{C}) \otimes \mathbf{S}$. Starting with any algebra norm on $M_n(\mathbb{C})$, we may take one of the tensor norms on $M_n(\mathbb{C})$, for example the projective tensor norm [7, Section 4.2]. Indeed all norms of $M_n(\mathbb{C}) \otimes \mathbf{S}$ that are compatible with the norm on $\mathbf{S}$ (in the sense that $||e_{ij} \otimes s|| = ||s||$ for any $s \in \mathbf{S}$ and every matrix unit $e_{ij}$) are easily seen to be equivalent. The topology induced by any such norm is the product topology on $\mathbf{S}^{n^2}$ when $M_n(\mathbf{S})$ is identified in the obvious way with $\mathbf{S}^{n^2}$. One such norm is

$$||(a_{ij})_{1\leq i,j\leq n}||_1 = \sum_{1\leq i,j\leq n} ||a_{ij}||_S.$$

This norm may also be defined on $A \otimes \mathbf{S}$ for any subalgebra $A$ of $M_n$, e.g. $T_n(\mathbb{C})$ or $D_n(\mathbb{C})$.

# Chapter 2

# Derivations

## 2.1 Introduction

In this chapter, we study derivations on triangular algebras. Derivations have been extensively studied in ring theory and in Banach algebra theory. One may refer to [15, 17, 24, 26, 27, 30] for certain known results about derivations on triangular algebras. First we recall the definitions of derivations and inner derivations. The concept of a derivations may be viewed as a generalization of the differentiation operator on function spaces. The Leibnitz equation $(fg)' = fg' + f'g$ is taken as the defining property.

**Definition 2.1.1** Consider an $\mathbf{R}$-algebra $\mathcal{A}$. An $\mathbf{R}$-linear map $\delta$ on $\mathcal{A}$ is called a *derivation* if it satisfies

$$\delta(aa') = \delta(a)a' + a\delta(a') \quad \text{for every} \quad a, a' \in \mathcal{A}.$$

The $\mathbf{R}$-linear space of all derivations on $\mathcal{A}$ is denoted by $Der(\mathcal{A})$.

**Definition 2.1.2** Consider an algebra $\mathcal{A}$ and a fixed element $a \in \mathcal{A}$, we define a map $\delta_a$ by

$$\delta_a(x) = ax - xa.$$

It is straightforward to verify that $\delta_a$ is indeed a derivation. A derivation which can be written as $\delta_a$ for some $a \in \mathcal{A}$ is said to be *inner*. The $\mathbf{R}$-linear space of all inner derivations on $\mathcal{A}$ is denoted by $Innder(\mathcal{A})$.

We give a general description of derivations on triangular algebras in the next section, and discuss automatic continuity of derivations on triangular Banach algebras in the last section. For the rest of the chapters, our main focus is the first cohomology group, which is defined below.

**Definition 2.1.3** [38, p.373] The *first cohomology group* of an algebra $\mathcal{A}$ is defined to be $H^1(\mathcal{A}) = Der(\mathcal{A})/Innder(\mathcal{A})$. Here $Der(\mathcal{A})$ and $Innder(\mathcal{A})$ are considered groups under addition.

We have the following two propositions for general algebras. The first proposition describes the effect of unitization on the space of derivations.

**Proposition 2.1.4** *$Der(A \vee \mathbf{R}1)$ is isomorphic to $Der(A)$ and $Innder(A \vee \mathbf{R}1)$ is isomorphic to $Innder(A)$. Consequently $H^1(A \vee \mathbf{R}1) = H^1(A)$.*

*Proof.* Define $\phi : Der(A \vee \mathbf{R}1) \to Der(A)$ by $\phi(\delta)(a) = \delta(a)$ for any $a \in A$. We claim that $\phi$ is an isomorphism.

That $\phi$ is a homorphism, i.e.

$$\phi(\delta_1 + \delta_2) = \phi(\delta_1) + \phi(\delta_2)$$

is obvious.

Note that for any $\delta \in Der(A \vee \mathbf{R}1)$, we have $\delta(1) = \delta(1)1 + 1\delta(1)$ and thus $\delta(1) = 0$. Therefore if $\phi(\delta) = 0$ then for any $a + \gamma 1 \in A \vee \mathbf{R}1$,

$$\delta(a + \gamma 1) = \delta(a) = \phi(\delta)(a) = 0.$$

Hence $\delta = 0$ and $\phi$ is injective.

For any $\delta \in Der(A)$, define $\bar{\delta}$ on $A \vee \mathbf{R}1$ by $\bar{\delta}(a + \gamma 1) = \delta(a)$. It is straightforward to verify that $\bar{\delta}$ is a derivation on $A \vee \mathbf{R}1$ and $\phi(\bar{\delta}) = \delta$. Hence $\phi$ is surjective.

It is obvious that $\phi(\delta_{x+\gamma 1}) = \delta_x$ for any $x \in A$, hence $\phi$ is also an isomorphism from $Innder(A \vee \mathbf{R}1)$ onto $Innder(A)$. ■

The next proposition describes the effect of taking direct sums on the space of derivations.

**Proposition 2.1.5** *If $A_1$ and $A_2$ are unital algebras, then $Der(A_1 \oplus A_2) = Der(A_1) \oplus Der(A_2)$ and $Innder(A_1 \oplus A_2) = Innder(A_1) \oplus Innder(A_2)$. Consequently $H^1(A_1 \oplus A_2) = H^1(A_1) \oplus H^1(A_2)$*

*Proof.* If $d_1 \in Der(A_1)$ and $d_2 \in Der(A_2)$, we define a derivation $\Phi(d_1, d_2)$ on $A_1 \oplus A_2$ by the equation $\Phi(d_1, d_2)(x \oplus y) = d_1(x) \oplus d_2(y)$. It is straightforward to verify that this is indeed a derivation. We show that the mapping $d_1 \oplus d_2 \mapsto \Phi(d_1, d_2)$ is an isomorphism from $Der(A_1) \oplus Der(A_2)$ onto $Der(A_1 \oplus A_2)$. It is obvious that $\Phi$ is additive and injective. To prove that it is surjective, suppose that $\delta$ is a derivation on $A_1 \oplus A_2$, let $\delta(a, b) = (f_1(a) + f_2(b), g_1(a) + g_2(b))$. First we prove that $f_2 = 0$ and $g_1 = 0$. Now

$$\begin{aligned} 0 &= \delta((a, 0)(0, b)) \\ &= (\delta(a, 0))(0, b) + (a, 0)(\delta(0, b)) \\ &= (f_1(a), g_1(a))(0, b) + (a, 0)(f_2(b), g_2(b)) \\ &= (af_2(b), g_1(a)b). \end{aligned}$$

and thus $af_2(b) = 0$ and $g_1(a)b = 0$ for all $a \in A_1$ and $b \in A_2$. As a result, $f_2 = 0$ and $g_1 = 0$ as required. Thus $\delta(x \oplus y) = f_1(x) \oplus g_2(y)$. It is now obvious that $f_1$ and $g_2$ are derivations and $\delta = \Phi(f_1, g_2)$. Note that $\delta_{x \oplus y} = \Phi(\delta_x, \delta_y)$, so $Innder(A_1 \oplus A_2) = Innder(A_1) \oplus Innder(A_2)$. ■

## 2.2 Structure of Derivations

In the rest of this chapter, all algebras are unital. The first theorem is a known result (see [26]).

**Theorem 2.2.1** [26] *A linear map $\delta$ over $\mathfrak{A} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a derivation if and only if it can be written as*

$$\delta\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_{\mathcal{A}}(a) & an - nb + f(m) \\ & p_{\mathcal{B}}(b) \end{pmatrix}.$$

*where $n \in \mathcal{M}$ and*

(i) *$p_{\mathcal{A}}$ is a derivation on $\mathcal{A}$, $f(am) = p_{\mathcal{A}}(a)m + af(m)$; and*

(ii) *$p_{\mathcal{B}}$ is a derivation of $\mathcal{B}$, $f(mb) = mp_{\mathcal{B}}(b) + f(m)b$.*

*Proof.* Suppose $\delta$ is a derivation on $\mathfrak{A}$. Write $\delta$ as

$$\delta\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_{\mathcal{A}}(a) + q_{\mathcal{B}}(b) + k_1(m) & r_1(a) - r_2(b) + f(m) \\ & p_{\mathcal{B}}(b) + q_{\mathcal{A}}(a) + k_2(m) \end{pmatrix}.$$

Let $\delta(1 \oplus 0) = \begin{pmatrix} i & n \\ & j \end{pmatrix}$. We have $i = 0$ and $j = 0$ as

$$\begin{aligned}
\begin{pmatrix} i & n \\ & j \end{pmatrix} &= \delta(1 \oplus 0) = \delta((1 \oplus 0)(1 \oplus 0)) \\
&= \delta(1 \oplus 0)(1 \oplus 0) + (1 \oplus 0)\delta(1 \oplus 0) \\
&= \begin{pmatrix} 2i & n \\ & 0 \end{pmatrix}.
\end{aligned}$$

We have $r_1(a) = an$ and $q_A = 0$ as

$$\begin{aligned}
\begin{pmatrix} p_A(a) & r_1(a) \\ & q_A(a) \end{pmatrix} &= \delta(a \oplus 0) = \delta((a \oplus 0)(1 \oplus 0)) \\
&= \delta(a \oplus 0)(1 \oplus 0) + (a \oplus 0)\delta(1 \oplus 0) \\
&= \begin{pmatrix} p_A(a) & r_1(a) \\ & q_A(a) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix} \begin{pmatrix} 0 & n \\ & 0 \end{pmatrix} \\
&= \begin{pmatrix} p_A(a) & an \\ & 0 \end{pmatrix}.
\end{aligned}$$

That $r_2(b) = nb$ and $q_B = 0$ follows

$$\begin{aligned}
0 &= \delta((1\oplus 0)(0\oplus b)) = (1\oplus 0)\delta(0\oplus b) + \delta(1\oplus 0)(0\oplus b) \\
&= \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} q_B(b) & -r_2(b) \\ & p_B(b) \end{pmatrix} + \begin{pmatrix} 0 & n \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix} \\
&= \begin{pmatrix} q_B(b) & nb - r_2(b) \\ & 0 \end{pmatrix}.
\end{aligned}$$

Next we have $k_1 = 0$ and $k_2 = 0$ since

$$\begin{aligned}
0 &= \delta\left(\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\right) \\
&= \delta\left(\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} + \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\delta\left(\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} k_1(m) & f(m) \\ & k_2(m) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} + \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & n \\ & 0 \end{pmatrix} \\
&= \begin{pmatrix} k_1(m) & 0 \\ & 0 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} k_1(m) & f(m) \\ & k_2(m) \end{pmatrix} &= \delta\left(\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right) \\
&= \delta\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\delta\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & n \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} k_1(m) & f(m) \\ & k_2(m) \end{pmatrix} \\
&= \begin{pmatrix} k_1(m) & f(m) \\ & 0 \end{pmatrix}.
\end{aligned}$$

We have just shown that $\delta$ is of the required form. That $p_A$ and $p_B$ are derivations follows from

$$\begin{aligned}
&\begin{pmatrix} p_A(aa') & aa'n - nbb' \\ & p_B(bb') \end{pmatrix} \\
= \; & \delta(aa' \oplus bb') = \delta(a \oplus b)(a' \oplus b') + (a \oplus b)\delta(a' \oplus b') \\
= \; & \begin{pmatrix} p_A(a) & an - nb \\ & p_B(b) \end{pmatrix}\begin{pmatrix} a' & 0 \\ & b' \end{pmatrix} + \begin{pmatrix} a & 0 \\ & b \end{pmatrix}\begin{pmatrix} p_A(a') & a'n - nb' \\ & p_B(b') \end{pmatrix} \\
= \; & \begin{pmatrix} p_A(a)a' + ap_A(a') & aa'n - nbb' \\ & p_B(b)b' + bp_B(b') \end{pmatrix}.
\end{aligned}$$

Finally $f(am) = p_{\mathcal{A}}(a)m + af(m)$ as

$$\begin{aligned}
\begin{pmatrix} 0 & f(am) \\ & 0 \end{pmatrix} &= \delta\left(\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right) \\
&= \delta\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\delta\left(\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} p_{\mathcal{A}}(a) & an \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & f(m) \\ & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & p_{\mathcal{A}}(a)m + af(m) \\ & 0 \end{pmatrix}.
\end{aligned}$$

A similar calculation shows that $f(mb) = mp_B(b) + f(m)b$.

Conversely suppose $\delta$ is of the form

$$\delta\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_{\mathcal{A}}(a) & an - nb + f(m) \\ & p_B(b) \end{pmatrix}$$

with (i) and (ii) satisfied, then

$$
\begin{aligned}
&\delta\left(\begin{pmatrix} a & m \\ & b \end{pmatrix}\begin{pmatrix} a' & m' \\ & b' \end{pmatrix}\right) \\
= \; &\delta\begin{pmatrix} aa' & am' + mb' \\ & bb' \end{pmatrix} \\
= \; &\begin{pmatrix} p_A(aa') & aa'n - nbb' + f(am' + mb') \\ & p_B(bb') \end{pmatrix} \\
= \; &\begin{pmatrix} p_A(a)a' + ap_A(a') & F \\ & p_B(b)b' + bp_B(b') \end{pmatrix} \\
&\text{where } F = aa'n + p_A(a)m' + af(m') - nbb' + mp_B(b') + f(m)b' \\
= \; &\begin{pmatrix} p_A(a) & an - nb + f(m) \\ & p_B(b) \end{pmatrix}\begin{pmatrix} a' & m' \\ & b' \end{pmatrix} \\
&+ \begin{pmatrix} a & m \\ & b \end{pmatrix}\begin{pmatrix} p_A(a') & a'n - nb' + f(m') \\ & p_B(b') \end{pmatrix} \\
= \; &\delta\begin{pmatrix} a & m \\ & b \end{pmatrix}\begin{pmatrix} a' & m' \\ & b' \end{pmatrix} + \begin{pmatrix} a & m \\ & b \end{pmatrix}\delta\begin{pmatrix} a' & m' \\ & b' \end{pmatrix}
\end{aligned}
$$

and so $\delta$ is a derivation. ■

In the case that $M$ is faithful, we have the following two results. The next result differs from Theorem 2.2.1 only in as much as the maps $p_A$ and

$p_B$ are not assumed to be derivations. This becomes part of the conclusion rather than the hypothesis.

**Corollary 2.2.2** *A linear map $\delta$ over $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$, where $\mathcal{M}$ is faithful, is a derivation if and only if it can be written as*

$$\delta\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_{\mathcal{A}}(a) & an - nb + f(m) \\ & p_B(b) \end{pmatrix}.$$

*where $n \in \mathcal{M}$ and*

(i) $f(am) = p_{\mathcal{A}}(a)m + af(m)$*; and*

(ii) $f(mb) = mp_B(b) + f(m)b$.

*Proof.* Suppose (i) is satisfied. Then

$$f(aa'm) = p_{\mathcal{A}}(aa')m + aa'f(m).$$

Also

$$\begin{aligned} f(aa'm) &= p_{\mathcal{A}}(a)a'm + af(a'm) \\ &= p_{\mathcal{A}}(a)a'm + ap_{\mathcal{A}}(a')m + aa'f(m) \end{aligned}$$

and so $p_{\mathcal{A}}(aa')m = p_{\mathcal{A}}(a)a'm + ap_{\mathcal{A}}(a')m$ for any $m \in \mathcal{M}$. As $\mathcal{M}$ is faithful, we get $p_{\mathcal{A}}(aa') = p_{\mathcal{A}}(a)a' + ap_{\mathcal{A}}(a')$ and so $p_{\mathcal{A}}$ is a derivation on $\mathcal{A}$. Similarly $p_B$ is a derivation on $B$. The result now follows from Theorem 2.2.1. ■

**Proposition 2.2.3** *Consider a triangular algebra $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ with faithful $\mathcal{M}$. A derivation $\delta$ on $\mathfrak{A}$, written in the form*

$$\delta\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_{\mathcal{A}}(a) & an - nb + f(m) \\ & p_B(b) \end{pmatrix},$$

*is uniquely determined by $n \in \mathcal{M}$ and $f : \mathcal{M} \to \mathcal{M}$. Furthermore $\delta$ is inner if and only if $f(m) = a_0m - mb_0$ for some fixed $a_0 \in \mathcal{A}$ and $b_0 \in B$.*

*Proof.* Suppose there is another derivation $\delta_1$ of the form

$$\delta_1\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_1(a) & an - nb + f(m) \\ & p_2(b) \end{pmatrix}.$$

By Corollary 2.2.2(i) and (ii), we have

$$\begin{aligned} p_{\mathcal{A}}(a)m &= f(am) - af(m) = p_1(a)m \\ mp_B(b) &= f(mb) - f(m)b = mp_2(b) \end{aligned}$$

which imply that $p_{\mathcal{A}} = p_1$ and $p_B = p_2$ as $\mathcal{M}$ is faithful. Hence $\delta_1 = \delta$.

If $\delta = \delta_z$ for some $z = \begin{pmatrix} a_0 & n \\ & b_0 \end{pmatrix}$, then

$$\delta\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_0m - mb_0 \\ & 0 \end{pmatrix}$$

and $f(m) = a_0 m - m b_0$ as desired.

Conversely, suppose $f(m) = a_0 m - m b_0$. For any $n \in M$, the unique derivation determined by $f$ and $n$ is $\delta_z$, where $z = \begin{pmatrix} a_0 & n \\ & b_0 \end{pmatrix}$. ■

The next lemma is concerned with inner derivations. We first recall the definition of endomorphisms.

**Definition 2.2.4** An **R**-linear map $\phi$ on an $(A, B)$-bimodule $M$ is called an $(A, B)$-*endomorphism* if

$$\phi(amb) = a\phi(m)b$$

for all $m \in M$, $a \in A$, and $b \in B$. The space of all $(A, B)$-endomorphisms of $M$ is denoted by $End(M)$.

**Lemma 2.2.5** *Consider a derivation $\delta$ on* Tri$(A, M, B)$, *written as*

$$\delta \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_A(a) & an - nb + f(m) \\ & p_B(b) \end{pmatrix}.$$

*If $p_A$ and $p_B$ are inner, then $f(m) = \phi(m) + xm - my$ where $\phi$ is an $(A, B)$-endomorphism on $M$, $x \in A$, $y \in B$. If, in addition, $M$ is faithful, then the converse is also true.*

*Proof.* Suppose $p_A = \delta_x$ and $p_B = \delta_y$. Then by Theorem 2.2.1, we have

$$f(am) = \delta_x(a)m + af(m) = (xa - ax)m + af(m)$$

and thus

$$f(am) - x(am) = a(f(m) - xm).$$

Similarly

$$f(mb) + (mb)y = (f(m) + my)b.$$

As a result, we have

$$\begin{aligned} f(amb) - x(amb) + (amb)y &= af(mb) - a(xmb) + (amb)y \\ &= a(f(mb) + (mb)y) - axmb \\ &= a(f(m) + my)b - axmb \\ &= a(f(m) - xm + my)b \end{aligned}$$

Hence the map $\phi(m) = f(m) - xm + my$ is an endomorphism.

Conversely, suppose $M$ is faithful and $f(m) = \phi(m) + xm - my$ for some endomorphism $\phi$. By Theorem 2.2.1 again, we have

$$\begin{aligned} \phi(am) + xam - amy &= f(am) \\ &= p_A(a)m + af(m) \\ &= p_A(a)m + a(\phi(m) + xm - my) \\ &= p_A(a)m + \phi(am) + axm - amy. \end{aligned}$$

Thus

$$xam = p_A(a)m + axm.$$

Since $M$ is faithful, we have $xa = p_A(a) + ax$, or equivalently $p_A = \delta_x$. Similarly $p_B = \delta_y$. ■

## 2.3 The Main Theorem

In this section, we will prove a theorem about the cohomology groups of certain triangular algebras. First we recall certain definitions and facts in group theory, see [29, Chapter 1.1B].

**Definition 2.3.1** Given two groups $G$ and $H$ and a group homomorphism $\theta : G \to Aut(H)$, where $Aut(H)$ is the group of automorphisms of $H$. Then the *semidirect product* of $G$ and $H$, denoted by $G \ltimes_\theta H$, is the group with underlying set $G \times H$ together with the product

$$(g, h)(g_1, h_1) = (gg_1, h\theta(g)(h_1))$$

for $g, g_1 \in G$ and $h, h_1 \in H$.

**Definition 2.3.2** Consider three groups $H, \hat{G}, G$ and group homomorphisms

$j : H \to \hat{G}$, $p : \hat{G} \to G$. The sequence

$$\{1\} \to H \xrightarrow{j} \hat{G} \xrightarrow{p} G \to \{1\}$$

is called a *short exact sequence* if $Ker(j) = \{1\}$, $Im(p) = G$ and $Ker(p) = Im(j)$.

If there exists a group homomorphism $q : G \to \hat{G}$ such that $pq = id_G$, the identity map on $G$, then the sequence is said to be *split*.

**Lemma 2.3.3** [29, Section 1.1.5.] *Consider the split short exact sequence*

$$\{1\} \to H \xrightarrow{j} \hat{G} \xrightarrow{p} G \to \{1\}$$

*with* $q : G \to \hat{G}$ *satisfying* $pq = id_G$. *Then* $j(H)$ *is normal in* $\hat{G}$, $\hat{G} = j(H)q(G) = q(G)j(H)$, $j(H) \cap q(G) = \{1\}$, *and every* $\hat{g} \in \hat{G}$ *has unique factorizations in the form* $j(h)q(g)$ *and* $q(g')j(h')$.

*Furthermore, if we define* $\theta : G \to Aut(H)$ *by*

$$(\theta(g))(h) = j^{-1}(q(g)j(h)q(g^{-1})),$$

*then* $\theta$ *is a group homomorphism and* $\hat{G} = G \ltimes_\theta H$.

*In particular, if* $\hat{G}$ *is an abelian group, then* $\theta$ *is trivial (i.e.* $\theta(g) = id_H$ *for all* $g \in G$*) and so* $\hat{G}$ *is just the Cartesian product group* $G \times H$.

Consider an additive group $K$ with normal subgroup $G$, the equivalence class $k+G$ will be denoted by $[k]$.

Now we are ready to proceed. Note that the first cohomology groups are abelian groups.

**Lemma 2.3.4** *The map $\tilde{\pi}_A : H^1(\mathfrak{A}) \to H^1(A)$ given by*

$$\tilde{\pi}_A([\delta]) = [\pi_A \delta|_A]$$

*is a well-defined group homomorphism.*

*Proof.* Suppose $\delta_1, \delta_2 \in Der(\mathfrak{A})$ satisfy $[\delta_1] = [\delta_2]$. Then $\delta_1 = \delta_z + \delta_2$ for some $z = \begin{pmatrix} a_0 & m_0 \\ & b_0 \end{pmatrix} \in \mathfrak{A}$. Write

$$\delta_i \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p^i_A(a) & an_i - n_i b + f_i(m) \\ & p^i_B(b) \end{pmatrix}$$

for $i = 1, 2$. Then $p^1_A = \delta_{a_0} + p^2_A$ and hence $[\pi_A \delta_1|_A] = [p^1_A] = [p^2_A] = [\pi_A \delta_2|_A]$. Therefore the map $\tilde{\pi}_A$ is well-defined. It is straightforward to verify that $\tilde{\pi}_A$ is a group homomorphism. ∎

We consider a map on $M$ defined by

$$\tau_{a,b}(m) = am - mb \quad \text{for all } m \in M.$$

for fixed $a \in A$ and $b \in B$. It is straightforward to verify that $\tau_{a,b}$ is an $(A, B)$-endomorphism if $a \in Z(A)$ and $b \in Z(B)$.

**Definition 2.3.5** An *inner endomorphism* of an $(A, B)$-bimodule $M$ is an $(A, B)$-endomorphism which can be expressed as $\tau_{a,b}$ for some $a \in Z(A)$ and $b \in Z(B)$. The space of all inner endomorphisms is denoted by $InnEnd(M)$.

We denote by $OutEnd(M)$ the quotient space $End(M)/InnEnd(M)$.

We have our first main theorem.

**Theorem 2.3.6** *Let* $\mathfrak{A} = \text{Tri}(A, M, B)$. *Define a map* $h : End(M) \to Der(\mathfrak{A})$ *by*

$$h(f)\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} 0 & f(m) \\ & 0 \end{pmatrix}.$$

*We have*

(i) *The map* $\Phi : OutEnd(M) \to H^1(\mathfrak{A})$ *defined by*

$$\Phi([f]) = [h(f)]$$

*is injective.*

(ii) *If for every derivation* $\delta$ *on* $\mathfrak{A}$, *we have* $p_B = \pi_B\delta|_B$ *is inner whenever* $p_A = \pi_A\delta|_A$ *is inner, then*

$$0 \to OutEnd(M) \xrightarrow{\Phi} H^1(\mathfrak{A}) \xrightarrow{\bar{\pi}_A} \bar{\pi}_A(H^1(\mathfrak{A})) \to 0 \tag{2.1}$$

*is a short exact sequence, where the map $\tilde{\pi}_{\mathcal{A}}$ is defined by $\tilde{\pi}_{\mathcal{A}}([\delta]) = [\pi_{\mathcal{A}}\delta|_{\mathcal{A}}]$.*

*Proof.* (i) First we show that $\Phi$ is well defined. Suppose $[f] = [g]$ then $f(m) - g(m) = am - mb$ for some $a \in Z(\mathcal{A})$ and $b \in Z(B)$. Then $h(f - g) = \delta_{a \oplus b} \in InnDer(\mathfrak{A})$.

Second we show that $\Phi$ is injective. Suppose $[h(f)] = 0$. Then $h(f)$ is inner, i.e. $h(f) = \delta_z$ for some $z = \begin{pmatrix} a_0 & n \\ & b_0 \end{pmatrix} \in \mathfrak{A}$. Hence

$$\begin{pmatrix} 0 & f(m) \\ & 0 \end{pmatrix} = \begin{pmatrix} \delta_{a_0}(a) & an - nb + a_0 m - m b_0 \\ & \delta_{b_0}(b) \end{pmatrix}$$

and therefore $a_0 \in Z(\mathcal{A})$, $b_0 \in Z(B)$, $n = 0$ (by taking $a = 1$, $b = 0$, $m = 0$) and $f = \tau_{a_0, b_0} \in InnEnd(\mathcal{M})$, i.e. $[f] = [0]$.

(ii) Suppose for every derivation $\delta$, we have $p_B$ is inner whenever $p_{\mathcal{A}}$ is inner. We claim that $Ker(\tilde{\pi}_{\mathcal{A}}) = Im(\Phi)$ and then (ii) follows. To verify the claim, suppose $\tilde{\pi}_{\mathcal{A}}([\delta]) = 0$. Thus $[\pi_{\mathcal{A}}(\delta_{\mathcal{A}})]$ is inner, i.e., $p_{\mathcal{A}} = \pi_{\mathcal{A}}\delta|_{\mathcal{A}}$ is inner. By assumption, $p_B$ is also inner. Write $p_{\mathcal{A}} = \delta_x$ and $p_B = \delta_y$. By Lemma 2.2.5, we have $f(m) = g(m) + xm - my$ for some endomorphism $g$ on $\mathcal{M}$. Hence we have

$$\delta \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} \delta_x(a) & an - nb + g(m) + xm - my \\ & \delta_y(b) \end{pmatrix}$$

and thus $\delta = \delta_z + h(g)$, where $z = \begin{pmatrix} x & n \\ & y \end{pmatrix}$. Therefore $[\delta] = \Phi([g])$ and we have $Ker(\tilde{\pi}_A) \subseteq Im(\Phi)$. The reverse inclusion is obvious since the $(1,1)$-entry of $h(f) = 0$, and so $\tilde{\pi}_A(\Phi(f)) = \tilde{\pi}_A([h(f)]) = 0$. ■

**Remark.** (1) The above Theorem implies that $OutEnd(M)$ can be identified with a subgroup of $H^1(\mathfrak{A})$.

(2) Under the hypothesis of Theorem 2.3.6(ii), we can identify the group $H^1(\mathfrak{A})/OutEnd(M)$ with the subgroup $\tilde{\pi}_A(H^1(\mathfrak{A}))$ of $H^1(A)$ by the fundamental theorem of group isomorphisms [38, p.60]. In particular, if we have that $OutEnd(M) = 0$, then $\tilde{\pi}_A$ is an embedding of $H^1(\mathfrak{A})$ into $H^1(A)$.

**Corollary 2.3.7** [26] *Let* $\mathfrak{A} = \mathrm{Tri}(A, M, B)$. *If* $B$ *has trivial first cohomology group, then* $H^1(\mathfrak{A})/OutEnd(M)$ *can be identified as a subgroup of* $H^1(A)$.

*Proof.* Since every derivation on $B$ is inner, the condition of Theorem 2.3.6(ii) is satisfied. The result follows from Remark (2).

**Corollary 2.3.8** [26] *If both* $A$ *and* $B$ *have trivial first cohomology groups then* $H^1(\mathrm{Tri}(A, M, B)) = OutEnd(M)$.

*Proof.* By Corollary 2.3.7, $H^1(\mathfrak{A})/OutEnd(M)$ is the trivial group and hence $H^1(\mathfrak{A}) = OutEnd(M)$. ■

## 2.4 Matrix Algebras

In this section, we apply the previous results to matrix algebras. We use **S** to denote an algebra over **R**. We will find the first cohomology groups of some triangular algebras in $M_n(\mathbf{S})$.

**Proposition 2.4.1** *Consider a derivation $d$ on* **S**. *it induces a derivation $\hat{d}$ on any matrix algebra $\mathbf{S}I_n \subseteq \mathfrak{A} \subseteq M_n(\mathbf{S})$ by $\hat{d}(a_{ij}) = (d(a_{ij}))$. Furthermore the map $[d] \mapsto [\hat{d}]$ is an embedding of $H^1(\mathbf{S})$ into $H^1(\mathfrak{A})$.*

*Proof.* Let $X = (x_{ij}), Y = (y_{ij}) \in \mathfrak{A}$. Then

$$\begin{aligned}
\hat{d}(XY) &= (d(\sum_{k=1}^{n} x_{ik}y_{kl}))_{i=1,\dots,n;j=1,\dots,n} \\
&= (\sum_{k=1}^{n} x_{ik}d(y_{kl}) + \sum_{k=1}^{n} d(x_{ik})y_{ik})_{i=1,\dots,n;j=1,\dots,n} \\
&= X\hat{d}(Y) + \hat{d}(X)Y
\end{aligned}$$

Hence $\hat{d}$ is a derivation over $\mathfrak{A}$.

The map $D : [d] \mapsto [\hat{d}]$ is well-defined since if $d = \delta_s$, then $\hat{d} = \delta_{sI}$. To prove that $D$ is injective, assume that $[\hat{d}] = 0$, i.e. $\hat{d}$ is inner. So $\hat{d} = \delta_\alpha$

where $\alpha = (a_{ij}) \in \mathfrak{A}$. Then

$$d(s)I = \hat{d}(sI) = \alpha(sI) - s\alpha$$

and $d(s) = a_{11}s - sa_{11}$ by comparing the $(1,1)$-entry. Hence $d$ is inner and the map $D$ is injective. ■

This derivation $\hat{d}$ of $\mathfrak{A}$ is called the *induced derivation by $d$*.

The following result, when restricted to $\mathfrak{A} = T_n(\mathbf{S})$ or $\mathfrak{A} = M_n(\mathbf{S})$, is known [42]. Also see [17, 56].

**Theorem 2.4.2** *Consider a block triangular matrix algebra*

$$\mathfrak{A} = T(n_1, \ldots, n_k)(\mathbf{S}).$$

*Then the map $D : H^1(\mathbf{S}) \to H^1(\mathfrak{A})$ given by $D([d]) = [\hat{d}]$ is a group isomorphism, or equivalently every derivation of $\mathfrak{A}$ is a sum of an inner derivation and an induced derivation.*

*Proof.* We prove the statement by induction on $k$. When $k = 1$, $\mathfrak{A} = M_n(\mathbf{S})$ and the statement is true by [42], and we denote the group isomorphism $D$ by $D_1$ in this case for further use.

Suppose the statement holds for $k < k_0$. Now assume that $k = k_0$. Note that $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ where $A = M_{n_1}(\mathbf{S})$, $M = S^{n_1, n_2 + \cdots + n_k}$ and $B = T(n_2, \ldots, n_{k_0})(\mathbf{S})$.

We claim that, with the same notation as in Theorem 2.3.6,

$$0 \to OutEnd(M) \xrightarrow{\Phi} H^1(\mathfrak{A}) \xrightarrow{\tilde{\pi}_A} \tilde{\pi}_A(H^1(\mathfrak{A})) \to 0$$

is a short exact sequence and $D : H^1(\mathbf{S}) \to H^1(\mathfrak{A})$ is an isomorphism.

We first prove that the sequence is short exact. By Theorem 2.3.6, it suffices to show that $p_B = \pi_B \delta|_B$ is inner whenever $p_A = \pi_A \delta|_A$ is inner. To this end, assume that $p_A = \delta_a$ for some $a = (a_{pq}) \in A$ and let $e_{ij} \in M$ to be the matrix with a 1 at the $(i, j)$-entry and 0 elsewhere. We have, using Corollary 2.2.2,

$$\begin{aligned} e_{11} p_B(sI_l) &= f(e_{11}s) - f(e_{11})s = f(se_{11}) - f(e_{11})s \\ &= p_A(sI_{n_1})e_{11} + sf(e_{11}) - f(e_{11})s \\ &= (sa - as)e_{11} + sf(e_{11}) - f(e_{11})s, \end{aligned}$$

where $l = \sum_{j=1}^{k_0} n_j - n_1$. By the induction hypothesis, $p_B = \delta_\beta + \dot{d}$ for some $\beta \in B$ and an induced derivation $\dot{d}$, and hence, by considering the $(1,1)$-entry of $e_{11} p_B(sI_l)$, we have

$$\beta_{11}s - s\beta_{11} + d(s) = sa_{11} - a_{11}s + sc - cs$$

where $c$ is the $(1,1)$-entry of $f(e_{11})$. Therefore

$$d(s) = \delta_{-c-a_{11}-\beta_{11}}(s).$$

As a result, $p_B$ is inner.

Next we show that $OutEnd(M) = 0$. For any endomorphism $g$ of $M$, we have

$$g(e_{11}) = g(F_{11}e_{11}E_{11}) = F_{11}g(e_{11})E_{11} = \beta e_{11}$$

for some $\beta \in S$ and

$$g(e_{ij}) = g(F_{i1}e_{11}E_{1j}) = F_{i1}g(e_{11})E_{1i} = \beta e_{ij},$$

where $\{F_{uv}\}$ and $\{E_{pq}\}$ are standard matrix units of $A$ and $B$ respectively. Thus $g(m) = m(\beta I)$ and so $g \in InnEnd(M)$.

Finally, by Remark (2) of Theorem 2.3.6, $\tilde{\pi}_A$ is an embedding of $H^1(\mathfrak{A})$ into $H^1(A)$. Recall that the mapping $D_1 : [d] \mapsto [\dot{d}]$ defines a group isomorphism from $H^1(\mathbf{S})$ onto $H^1(A)$. For any derivation $d$ on $\mathbf{S}$, we have

$$\tilde{\pi}_A(D([d])) = \tilde{\pi}_A([\hat{d}]) = [\dot{d}] = D_1([d])$$

and hence $\tilde{\pi}_A D = D_1$. Therefore $D$ is a group isomorphism. The claim is proved and the theorem follows. ■

Before stating the next theorem, we introduce an $\mathbf{R}$-space of upper block triangular matrices over $\mathbf{S}$.

**Definition 2.4.3** $T(n_1, \ldots, n_t; m_1, \ldots, m_t)(\mathbf{S})$ is the $\mathbf{R}$-space consisting of all $\left(\sum_{i=1}^{t} n_i\right) \times \left(\sum_{j=1}^{t} m_j\right)$ matrices of the form $(M_{ij})$ where $M_{ij}$ is a $n_i \times m_j$ matrices over S if $i \leq j$ and $M_{ij} = 0$ if $i > j$.

**Lemma 2.4.4** *Let $M = T(n_1, \ldots, n_t; m_1, \ldots, m_t)(\mathrm{S})$. Suppose that $A \subseteq M_k(\mathrm{S})$ and $B \subseteq M_l(\mathrm{S})$, where $k = n_1 + \cdots + n_t$ and $l = m_1 + \cdots + m_t$, are matrix algebras. If $AM \subseteq M$ and $MB \subseteq M$ under usual matrix multiplications as module multiplications, then $M$ is a faithful bimodule.*

*Proof.* Let $E_{ij}$ be the standard matrix units in $M$. Suppose that $a \in A$ satisfies $aM = 0$, then the $j$-th column of $a$ is the $l$-th column of $aE_{jl} = 0$ and thus $a = 0$. Similarly if $b \in B$ satisfies $Mb = 0$, then the $i$-th row of $b$ is the first column of $E_{1i}b = 0$ and thus $b = 0$. ■

Recall that $D_k(\mathrm{S})$ is algebra of $k \times k$ diagonal matrices over **S**. The next theorem is proved in [26] in the case $k = l$ and $M = M_k(\mathrm{S})$ or $T_k(\mathrm{S})$.

**Theorem 2.4.5** *Let $A = D_k(\mathrm{S})$, $B = D_l(\mathrm{S})$ and $M$ be the space of upper block triangular matrices $T(n_1, \ldots, n_t; m_1, \ldots, m_t)(\mathrm{S})$, where $n_1 + \cdots + n_t = k$ and $m_1 + \cdots + m_t = l$. Then $H^1(\mathrm{Tri}(A, M, B))$ is isomorphic to $(Z(\mathrm{S})^{dimM-k-l+1}) \times H^1(\mathrm{S})$.*

*Proof.* We let $g_{ii}$'s, $h_{jj}$'s and $E_{ij}$'s be the standard bases for $D_k(\mathrm{S})$, $D_l(\mathrm{S})$ and $\mathrm{S}^{k,l}$ respectively. Let $\mathfrak{A} = \mathrm{Tri}(A, M, B)$.

We claim that

$$0 \to OutEnd(M) \xrightarrow{\Phi} H^1(\mathfrak{A}) \xrightarrow{\hat{\pi}_A} \hat{\pi}_A(H^1(\mathfrak{A})) = H^1(\mathbf{S}) \to 0$$

is a split short exact sequence and the map $D : H^1(\mathbf{S}) \to H^1(\mathfrak{A})$ defined by $D([d]) = [\hat{d}]$ satisfies $\hat{\pi}_A D = id_{H^1(\mathbf{S})}$. It would then follow (by Theorem 2.3.3) that $H^1(\mathfrak{A}) = OutEnd(M) \times H^1(\mathbf{S})$.

By Proposition 2.1.5, a derivation $d$ on $D_k(\mathbf{S})$ can be written as

$$d(\sum_{j=1}^{k} a_j h_{jj}) = \sum_{j=1}^{k} d_j(a_j)h_{jj},$$

where $d_i$'s are derivations on $\mathbf{S}$. We write $d = (d_1, \ldots, d_k)$.

Let $\delta$ be a derivation of Tri$(A, M, B)$ of the form

$$\delta\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_A(a) & f(m) + an - nb \\ & p_B(b) \end{pmatrix}.$$

Let $p_A = (\delta_1, \ldots, \delta_k)$ and $p_B = (d_1, \ldots, d_l)$. Take $E_{ij} \in M$ and by Theorem 2.2.1, we have

$$\begin{aligned} d_j(s)E_{ij} &= E_{ij}p_B(sh_{jj}) \\ &= f(E_{ij}sh_{jj}) - f(E_{ij})sh_{jj} \\ &= f(sE_{ij}) - f(E_{ij})sh_{jj} \\ &= p_A(sI_k)E_{ij} + sf(E_{ij}) - f(E_{ij})sh_{jj} \\ &= \delta_i(s)E_{ij} + sf(E_{ij}) - f(E_{ij})sh_{jj}. \end{aligned}$$

Thus, by considering the $(i, j)$-entry, we have

$$d_j(s) = \delta_i(s) + sc - cs$$

where $c$ is the $(i, j)$-entry of $f(E_{ij})$. Hence $[d_j] = [\delta_i]$. Using the fact that the matrices $E_{11}, \ldots, E_{1l}, E_{2l}, \ldots, E_{kl}$ are all in $M$, it follows that

$$[\delta_1] = \cdots = [\delta_k] = [d_1] = \cdots = [d_l].$$

As a result, $[p_A] = [d_1]I_k$ and $[p_B] = [d_1]I_l$. Therefore $\tilde{\pi}_A(H^1(\mathfrak{A})) = H^1(\mathbf{S})$ and $p_A$ is inner if and only if $p_B$ is inner. The claim then follows from Theorem 2.3.6(ii) and Lemma 2.3.3.

It remains to show that $OutEnd(M) = Z(\mathbf{S})^{dim M-k-l+1}$. To this end, we take $f \in End(M)$. Let $E_{ij} \in M$, then $f(E_{ij}) = g_{ii}f(E_{ij})h_{jj} = \alpha_{ij}E_{ij}$ and it is easily decuced that $\alpha_{ij} \in Z(\mathbf{S})$. Set $\alpha_{ij} = 0$ if $E_{ij} \notin M$. It is straightforward to verify that the map $\Theta(f) = (\alpha_{ij})$ defines a group isomorphism from $End(M)$ onto the additive group $T(n_1, \ldots, n_t; m_1, \ldots, m_t)(Z(\mathbf{S}))$, which is isomorphic to $Z(\mathbf{S})^{dim M}$.

Consider an inner endomorphism $\tau_{a,b}$ of $M$, where $a = (a_{ij}) \in Z(A) = D_k(Z(\mathbf{S}))$ and $b = (b_{ij}) \in Z(B) = D_l(Z(\mathbf{S}))$. Since $\tau_{a,b} = \tau_{a-b_{ll}I, b-b_{ll}I}$, we may assume $b_{ll} = 0$. Let $\Theta(\tau_{a,b}) = (c_{ij})$. For $E_{ij} \in M$, we have $c_{ij}E_{ij} =$

$\tau_{a,b}(E_{ij}) = (a_{ii} - b_{jj})E_{ij}$ and hence $c_{ij} = a_{ii} - b_{jj}$. Therefore

$$a_{ii} = c_{il} \quad \text{and} \quad b_{jj} = a_{11} - c_{1j} = c_{1l} - c_{1j}.$$

As a result $\Theta(\tau_{a,b})$ is uniquely determined by the $k + l - 1$ elements at the first row and last column, i.e., $c_{11}, \ldots, c_{1l}, c_{2l}, \ldots, c_{kl}$. Hence

$$\begin{aligned} OutEnd(M) &= T(n_1, \ldots, n_t; m_1, \ldots, m_t)(Z(\mathbf{S}))/\Theta(InnEnd(M)) \\ &= Z(\mathbf{S})^{dim M}/Z(\mathbf{S})^{k+l-1} \\ &= Z(\mathbf{S})^{dim M-k-l+1}. \end{aligned}$$

Now using Theorem 2.3.3, we have $H^1(\mathfrak{A}) = Z(\mathbf{S})^{dim M-k-l+1} \times H^1(\mathbf{S})$. ■

As a corollary to our last theorem, we recover two results from [26]. They were proved in [26] for $\mathbf{S} = \mathbb{C}$.

**Corollary 2.4.6**

$$H^1\begin{pmatrix} D_k(\mathbf{S}) & T_k(\mathbf{S}) \\ & D_k(\mathbf{S}) \end{pmatrix} = T_k(\mathbf{S})/Z(\mathbf{S})^{2k-1} \times H^1(\mathbf{S}).$$

**Corollary 2.4.7**

$$H^1\begin{pmatrix} D_k(\mathbf{S}) & M_k(\mathbf{S}) \\ & D_k(\mathbf{S}) \end{pmatrix} = M_k(\mathbf{S})/Z(\mathbf{S})^{2k-1} \times H^1(\mathbf{S}).$$

The last result in this section is concerned with a finite-dimensional analogue of semi-nest algebras.

**Theorem 2.4.8** *Consider a matrix algebra $\mathfrak{A} = \mathrm{Tri}(A, M, B)$, where $A$ is a matrix algebra containing $D_l(\mathbf{S})$, $B \subseteq M_k(\mathbf{S})$ is a block triangular matrix algebra and $M$ is a faithful left $A$-module generated by the matrix units in it. Then $H^1(\mathfrak{A})$ is a subgroup of $H^1(A)$.*

*Proof.* We claim that

$$0 \to OutEnd(M) \xrightarrow{\Phi} H^1(\mathfrak{A}) \xrightarrow{\tilde{\pi}_A} \tilde{\pi}_A(H^1(\mathfrak{A})) \to 0$$

is a short exact sequence and $OutEnd(M) = 0$. Hence the result follows from Remark 2 of Theorem 2.3.6(ii).

To show that it is a short exact sequence, by Theorem 2.3.6(ii), it suffices to show that for every derivation $\delta$ on $\mathfrak{A}$, if $p_A = \pi_A \delta|_A$ is inner then so is $p_B = \pi_B \delta|_B$. To prove this, assume that $\delta$ on $\mathfrak{A}$ is of the form

$$\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_A(a) & an - nb + f(m) \\ & p_B(b) \end{pmatrix}$$

as in Theorem 2.2.1 and assume that $p_A = \delta_a$ for some $a = (a_{pq}) \in A$. Take $E_{ij} \in M$, by Theorem 2.2.1, we have

$$\begin{aligned} E_{ij} p_B(sI_k) &= f(E_{ij}s) - f(E_{ij})s = f(sE_{ij}) - f(E_{ij})s \\ &= p_A(sI_l)E_{ij} + sf(E_{ij}) - f(E_{ij})s \\ &= (as - sa)E_{ij} + sf(E_{ij}) - f(E_{ij})s. \end{aligned}$$

By Theorem 2.4.2, $p_B = \delta_\beta + d$ for some $\beta \in B$ and induced derivation $d$. Therefore, by considering the $(i, j)$-entry, we have that $d$ is inner and thus $p_B$ is also inner.

It remains to show that $OutEnd(M) = 0$. Consider $g \in End(M)$. Let $\{e_{ij}\}$ and $\{f_{ij}\}$ be the standard bases of $A$ and $B$ respectively. Consider $1 \leq r \leq l$. If there does not exist an $s$ such that $E_{rs} \in M$ then set $\alpha_r = 0$. Otherwise let $s$ be the smallest integer such that $E_{rs} \in M$. We have

$$g(E_{rs}) = g(e_{rr}E_{rs}g_{ss}) = e_{rr}g(E_{rs})f_{ss} = \alpha_r E_{rs}.$$

For $s \leq j$ and $E_{rj} \in M$, we have $g(E_{rj}) = g(E_{rs})f_{sj} = \alpha_r E_{rj}$. Therefore $g(m) = \alpha m$ where $\alpha = \sum \alpha_s e_{ss}$. To prove that $g$ is inner, we shall show that $\alpha \in Z(A)$. Let $a \in A$ and $m \in M$. Then

$$a\alpha m = ag(m) = g(am) = \alpha am.$$

Since this is true for all $m \in M$ and $M$ is faithful, we get $a\alpha = \alpha a$ for every $a \in A$, i.e. $\alpha \in Z(A)$.

■

## 2.5 Triangular Banach Algebras

In this section, we give sufficient conditions on triangular Banach algebras $\mathfrak{A}$ so that every derivation is continuous. Questions similar to this for Banach algebras and operator algebras have attracted quite a bit of attention, see [15, 20, 21, 22, 24, 26, 31, 65]. First we state a classical theorem on nest algebras.

**Theorem 2.5.1** [15] *All derivations of nest algebras are continuous and inner.*

We have the following finite dimensional analogue of Theorem 2.5.1.

**Theorem 2.5.2** *Let $\mathfrak{A}$ be a block triangular matrix algebra over a Banach algebra* $\mathbf{S}$*. Then every derivation of $\mathfrak{A}$ is continuous if and only if every derivation of* $\mathbf{S}$ *is continuous.*

*Proof.* For any derivation $\delta$ on $\mathfrak{A}$, $\delta = \delta_z + d$ for some induced derivation $d$ by Theorem 2.4.2. Since $\delta$ is continuous if and only if $d$ is continuous, we have any derivation of $\mathfrak{A}$ is continuous if and only if any derivation of S is continuous. ∎

The next theorem gives a sufficient conditions that every derivation be continuous.

**Theorem 2.5.3** *Consider a Banach algebra* $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. *Suppose every endomorphism of* $\mathcal{M}$ *is continuous (in particular, this is the case when* $End(\mathcal{M}) = InnEnd(\mathcal{M})$*), and that any one of the following conditions hold:*

(i) *every derivation of* $\mathcal{A}$ *is continuous;*

(ii) *there exists a positive integer* $C$ *such that*

$$||a|| \leq C \sup\{||am|| : m \in \mathcal{M}, ||m|| = 1\}$$

*for every* $a \in \mathcal{A}$.

*and assume that* $\mathcal{B}$ *satisfies one of the conditions (i), (ii) above. Then every derivation of* $\mathfrak{A}$ *is continuous.*

*Proof.* Consider a derivation $\delta$ of $\mathfrak{A}$. By Theorem 2.2.1, $\delta = \delta_z + \hat{\delta}$ for $z = \begin{pmatrix} 0 & n \\ & 0 \end{pmatrix}$ and $\hat{\delta}\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_{\mathcal{A}}(a) & f(m) \\ & p_{\mathcal{B}}(b) \end{pmatrix}$. Since inner derivations are continuous, we have that $\delta$ is continuous if and only if $\hat{\delta}$ is. So we can assume $\delta = \hat{\delta}$. If every derivation of $\mathcal{A}$ is continuous, then $p_{\mathcal{A}}$ is continuous.

If $\|a\| \leq C\sup\{\|am\| : m \in M, \|m\| = 1\}$, then, by Theorem 2.2.1 again,

$$\|p_A(a)m\| = \|f(am) - af(m)\| \leq 2\|f\|\|a\|\|m\|,$$

we have $\|p_A(a)\| \leq 2C\|f\|\|a\|$ and therefore $p_A$ is also continuous. Similarly $p_B$ is continuous and the result follows. ■

We apply the above theorems in the following two corollaries. The first corollary is concerned with the join of two operator algebras. Recall that given two Hilbert spaces $H$ and $K$, the map $x \otimes y^* \in B(K, H)$, where $x \in H$ and $y \in K$, is defined by $(x \otimes y^*)(z) = \langle z, y \rangle x$.

**Corollary 2.5.4** *Suppose that $A \subseteq M_n(\mathbb{C})$ is a matrix algebra and $B$ is a nest algebra associated to a finite nest over a Hilbert space $\mathbf{H}$. Then every derivation on the join $A\#B$ is continuous.*

*Proof.* First note that the inequality in Condition (ii) of Theorem 2.5.3 is an equality with $C = 1$ for both $A$ and $B$. It remains to show that every endomorphism of $M$ is continuous. Suppose that $B = \mathcal{T}(\mathcal{N})$ where $0 = N_1 \subset N_2 \cdots \subset N_k = \mathbf{H}$ are the closed subspaces in $\mathcal{N}$. Let $y$ be a unit vector in $N_k \ominus N_{k-1}$. By [22, Lemma 2.8], the map $x \otimes y^* \in B$ for every $x \in \mathbf{H}$.

Consider $f \in End(M)$. Define $\lambda \in M_n(\mathbb{C})$ by $\lambda u = f(u \otimes y^*)y$ for $u \in \mathbb{C}^n$. We claim that $f(m) = \lambda m$ for all $m \in M$ and thus $f$ is continuous. Take $m \in M$. We have, for every $x \in \mathbf{H}$,

$$f(m)x = f(m)(x \otimes y^*)y = f(mx \otimes y^*)y = \lambda mx.$$

Hence $f(m) = \lambda m$ and the result follows. ∎

The following corollary is concerned with the algebra discussed in Theorem 2.4.5.

**Corollary 2.5.5** *Let $A = D_k(\mathbf{S})$, $B = D_l(\mathbf{S})$ and $M$ be the space of upper block triangular matrices $T(n_1, \ldots, n_t; m_1, \ldots, m_t)(\mathbf{S})$, where $n_1 + \cdots + n_t = k$ and $m_1 + \cdots + m_t = l$. Then every derivation of $\mathfrak{A}$ is continuous if and only if every derivation of* $\mathbf{S}$ *is continuous.*

*Proof.* First assume that every derivation of $\mathbf{S}$ is continuous. By Proposition 2.1.5, a derivation $d$ on $A = D_k(\mathbf{S})$ can be written as

$$d(\sum_{j=1}^{k} a_j h_{jj}) = \sum_{j=1}^{k} d_j(a_j) h_{jj}$$

and hence $d$ is continuous. Similarly every derivation on $B$ is continuous. Therefore by Theorem 2.5.3, it suffices to show that every endomorphism on $M$ is continuous. From the proof of Theorem 2.4.5, every $g \in End(M)$ can be written as $g(m) = \alpha m$ for some $\alpha \in T(n_1, \ldots, n_t; m_1, \ldots, m_t)(Z(\mathbf{S}))$ and thus $g$ is continuous.

Conversely, suppose that every derivation of $\mathfrak{A}$ is continuous. Let $d$ be derivation on $\mathbf{S}$, then the induced derivation $\hat{d}$ on $\mathfrak{A}$ is continuous and thus $d$ is continuous. ■

# Chapter 3

# Lie Derivations

## 3.1 Introduction

Recall that an algebra $\mathcal{A}$ is also a Lie algebra with Lie product defined as $[a, b] = ab - ba$. In this chapter, we will classify those algebras of which every Lie derivation is "almost" a derivation, i.e., a proper derivation.

**Definition 3.1.1** An $R$-linear map $L$ on an algebra $\mathcal{A}$ is said to be a *Lie derivation* if

$$L([a, a']) = [L(a), a'] + [a, L(a')] \quad \text{for all } a, a' \in \mathcal{A}.$$

Lie derivations, as well as other Lie maps (e.g. Lie isomorphisms and commuting maps, have been active research subjects for a long time. One of the earliest results on Lie derivations of associative rings is in Martindale [54], who proved that a Lie derivation of certain primitive rings is always a sum of a derivation and an additive mapping from the ring into the centre of a larger ring. Similar results are in [8, 44, 66]. Miers [60] considered von Neumann algebras and showed that every Lie derivation is the sum of a derivation and a linear map with image lies in the centre of the algebra. Johnson [39] considered the same problem on certain Banach algebras called symmetrically amenable. This motivates the definition of proper Lie derivations, given in [39].

**Definition 3.1.2** A Lie derivation of an algebra $\mathcal{A}$ is said to be *proper* if it is a sum of a derivation of $\mathcal{A}$ and a linear map whose image lies in the centre of $\mathcal{A}$. Otherwise the Lie derivation is said to be *improper*.

We shall start with the following known simple observations.

**Lemma 3.1.3** *The sum of a derivation $\delta$ of an algebra $\mathcal{A}$ and a linear map $h : \mathcal{A} \to Z(\mathcal{A})$ is a Lie derivation if and only if $h([a, a']) = 0$ for every $a, a' \in \mathcal{A}$.*

*Proof.* It is straightforward to verify that every derivation is a Lie derivation and also that any linear combination of Lie derivations is a Lie derivation.

In view of this, it is enough to consider only the map $h$. Since $h(A) \subseteq Z(A)$, we have $[h(a), a'] = 0$ for every $a, a'$. Thus $h$ is a Lie derivation if and only if $h([a, a']) = 0$. ■

**Lemma 3.1.4** *A Lie derivation $L$ on a unital algebra satisfies $L(1) \in Z(A)$.*

*Proof.* This follows from

$$0 = L([1, a]) = [L(1), a] + [1, L(a)] = [L(1), a]$$

for every $a \in A$. ■

The objective of this chapter is to establish sufficient conditions that all Lie derivations of $\mathrm{Tri}(A, M, B)$ are proper. First of all, we have two easy propositions for general algebras. The first proposition is concerned with the unitization of an algebra.

**Proposition 3.1.5** *Let $A$ be an algebra without unity and $A \vee \mathbf{R}1$ be the unitization of $A$. Then $A$ has no improper Lie derivation if and only if $A \vee \mathbf{R}1$ has no improper Lie derivation.*

*Proof.* Let $j : A \to A \vee \mathbf{R}1$ and $\pi : A \vee \mathbf{R}1 \to A$ be the natural inclusion and projection, i.e., $j(a) = a$ and $\pi(a + \gamma 1) = a$.

Assume that $A$ has no improper Lie derivation. Let $L$ be a Lie derivation of $A \vee \mathbf{R}1$. It is easy to verify that $\pi Lj$ is a Lie derivation on $A$. Thus $\pi Lj$ is a proper Lie derivation and can be written as $\delta + h$ where $\delta$ is a derivation of $A$ and $h$ maps into $Z(A)$. Define $\hat{h}, \hat{\delta} : A \vee \mathbf{R}1 \to A \vee \mathbf{R}1$ by $\hat{h}(a+\gamma 1) = L(a) - \delta(a) + \gamma L(1)$ and $\hat{\delta}(a+\gamma 1) = \delta(a)$. We have $L = \hat{\delta} + \hat{h}$. The map $\hat{\delta}$ is easily seen to be a derivation. Also $L(a) - \delta(a) = (\pi Lj)(a) - \delta(a)$ belongs to the centre of $A$ and $L(1)$ belongs to the centre of $A \vee \mathbf{R}1$ by the previous lemma. Therefore $\hat{h}(A) \subseteq Z(A \vee \mathbf{R}1)$.

Conversely, suppose $A \vee \mathbf{R}1$ has no improper Lie derivation. Let $\hat{L}$ be a Lie derivation on $A$. Define a Lie derivation $L$ of $A \vee \mathbf{R}1$ by $L(a+\gamma 1) = \hat{L}(a)$. Then $L = \delta + h$ since $L$ is a proper Lie derivation. Note that $\delta(1) = 0$, thus $\pi\delta j$ is a derivation on $A$ and therefore $\hat{L} = \pi\delta j + \pi h j$ as required. ∎

The next proposition is concerned with the direct product of two algebras.

**Proposition 3.1.6** *$A_1$ and $A_2$ have no improper Lie derivations if and only if $A_1 \oplus A_2$ has no improper Lie derivation.*

*Proof.* Suppose $L$ is a Lie derivation of $A_1 \oplus A_2$. Write $L(a,b) = (L_1(a) + \pi_1(b), L_2(b) + \pi_2(a))$. Since

$$0 = L[(a,0),(0,b)] = ([a, \pi_1(b)], [\pi_2(a), b]),$$

we have $\pi_i$ maps into $Z(A_i)$, and

$$(L_1[a_1, a_2], \pi_2[a_1, a_2]) = L([a_1, a_2], 0) = ([a_1, L_1(a_2)] + [L_1(a_1), a_2], 0),$$

$$(\pi_2[b_1, b_2], L_2[b_1, b_2]) = L(0, [b_1, b_2]) = (0, [b_1, L_2(b_2)] + [L_2(b_1), b_2]),$$

thus we have $L_i$ is a Lie derivation of $A_i$ for $i = 1, 2$.

Assume that $A_1$ and $A_2$ have no improper Lie derivations. Let $L$ be a Lie derivation of $A_1 \oplus A_2$. By the above argument and the fact that every Lie derivation of $A_i$ is a proper Lie derivation, we have $L_i = \delta_i + h_i$ for $i = 1, 2$. Let $\delta(a,b) = (\delta_1(a), \delta_2(b))$ and $h(a,b) = (h_1(a) + \pi_1(b), h_2(b) + \pi_2(a))$, then $L = \delta + h$ and we are done.

Conversely, suppose $A_1 \oplus A_2$ has no improper Lie derivation. We need to show that $A_1$ (and similarly $A_2$) has no improper Lie derivation also. Let $L_1$ be a Lie derivation of $A_1$. Define a Lie derivation of $A_1 \oplus A_2$ by $L(a,b) = (L_1(a), 0)$. As $L$ is a proper Lie derivation, we have $L = \delta + h$ with $\delta(a,0) = (\delta_1(a), h_2(a))$ and $h(a,0) = (h_1(a), -h_2(a))$. It is straightforward to verify that $\delta_1$ is a derivation on $\mathcal{A}_1$ and that $h_1(a) \in Z(\mathcal{A}_1)$. Evidently $L_1 = \delta_1 + h_1$ as required. ■

## 3.2 Structure of Lie Derivations

We will consider only unital algebras in the rest of this chapter.

**Proposition 3.2.1** *A linear map $L$ on $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a Lie derivation $L$ if and only if $L$ is of the form*

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_{\mathcal{A}}(a) + h_B(b) & an - nb + f(m) \\ & h_{\mathcal{A}}(a) + g_B(b) \end{pmatrix}$$

*where $n \in \mathcal{M}$, and $g_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$, $g_B : B \to B$, $h_{\mathcal{A}} : \mathcal{A} \to Z(B)$ with $h_{\mathcal{A}}([a, a']) = 0$, $h_B : B \to Z(\mathcal{A})$ with $h_B([b, b']) = 0$, $f : \mathcal{M} \to \mathcal{M}$ are linear maps satisfying*

(i) *$g_{\mathcal{A}}$ is a Lie derivation on $\mathcal{A}$, $f(am) = g_{\mathcal{A}}(a)m - mh_{\mathcal{A}}(a) + af(m)$ and*

(ii) *$g_B$ is a Lie derivation on $B$, $f(mb) = mg_B(b) - h_B(b)m + f(m)b$.*

*Proof.* Suppose $L$ is a Lie derivation on $\mathfrak{A}$. Write $L$ as

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_{\mathcal{A}}(a) + h_B(b) + k_1(m) & r_1(a) - r_2(b) + f(m) \\ & g_B(b) + h_{\mathcal{A}}(a) + k_2(m) \end{pmatrix}.$$

Let $\delta \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} = \begin{pmatrix} i & n \\ & j \end{pmatrix}$. We have $r_1(a) = an$ and $r_2(b) = nb$ as

$$\begin{aligned}
0 &= L([a \oplus 0, 1 \oplus 0] \\
&= [L(a \oplus 0), 1 \oplus 0] + [a \oplus 0, L(1 \oplus 0)] \\
&= \left[\begin{pmatrix} g_A(a) & r_1(a) \\ & h_A(a) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} i & n \\ & j \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & an - r_1(a) \\ & 0 \end{pmatrix} \quad \text{and} \\
0 &= L([0 \oplus b, 1 \oplus 0] \\
&= [L(0 \oplus b), 1 \oplus 0] + [0 \oplus b, L(1 \oplus 0)] \\
&= \left[\begin{pmatrix} h_B(b) & -r_2(b) \\ & g_B(b) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}, \begin{pmatrix} i & n \\ & j \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & r_2(b) - nb \\ & 0 \end{pmatrix}.
\end{aligned}$$

That $k_1 = 0$ and $k_2 = 0$ follow from

$$\begin{aligned}
&\begin{pmatrix} k_1(m) & f(m) \\ & k_2(m) \end{pmatrix} \\
= \;& L\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \\
= \;& L\left(\left[\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right]\right) \\
= \;& \left[L\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}, L\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right] \\
= \;& \left[\begin{pmatrix} i & n \\ & j \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} k_1(m) & f(m) \\ & k_2(m) \end{pmatrix}\right] \\
= \;& \begin{pmatrix} 0 & im - mj + f(m) \\ & 0 \end{pmatrix}.
\end{aligned}$$

Condition (i) and $h_A([a,a']) = 0$ follow from

$$\begin{aligned}
&\begin{pmatrix} g_A([a,a']) & [a,a']n \\ & h_A([a,a'] \end{pmatrix}) \\
= \;& L([a,a'] \oplus 0) = L([a \oplus 0, a' \oplus 0]) \\
= \;& [L(a \oplus 0), a' \oplus 0] + [a \oplus 0, L(a' \oplus 0)] \\
= \;& \left[\begin{pmatrix} g_A(a) & an \\ & h_A(a) \end{pmatrix}, a' \oplus 0\right] + \left[a \oplus 0, \begin{pmatrix} g_A(a') & a'n \\ & h_A(a') \end{pmatrix}\right] \\
= \;& \begin{pmatrix} [g_A(a), a'] + [a, g_A(a')] & [a,a']n \\ & 0 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\begin{pmatrix} 0 & f(am) \\ & 0 \end{pmatrix} \\
= \;& L\begin{pmatrix} 0 & am \\ & 0 \end{pmatrix} \\
= \;& L\left(\left[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left[L\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, L\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} g_A(a) & an \\ & h_A(a) \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & f(m) \\ & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & g_A(a)m - mh_A(a) + af(m) \\ & 0 \end{pmatrix}.
\end{aligned}
$$

Similarly Condition (ii) and $h_B([b, b']) = 0$ follow from

$$
\begin{aligned}
&\begin{pmatrix} h_B([b, b']) & -n[b, b'] \\ & g_A([b, b'] \end{pmatrix}) \\
&= L(0 \oplus [b, b']) \\
&= L([0 \oplus b, 0 \oplus b']) \\
&= [L(0 \oplus b), 0 \oplus b'] + [0 \oplus b, L(0 \oplus b')] \\
&= \left[\begin{pmatrix} h_B(b) & -nb \\ & g_B(b) \end{pmatrix}, 0 \oplus b'\right] + \left[b \oplus 0, \begin{pmatrix} h_B(b') & -nb' \\ & g_B(b') \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & -n[b, b'] \\ & [g_B(b), b'] + [b, g_B(b')] \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
&\begin{pmatrix} 0 & f(bm) \\ & 0 \end{pmatrix} \\
= \; & L\begin{pmatrix} 0 & bm \\ & 0 \end{pmatrix} \\
= \; & L\left(\left[\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\right]\right) \\
= \; & \left[L\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\right] + \left[\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}, L\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\right] \\
= \; & \left[\begin{pmatrix} 0 & f(m) \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\right] + \left[\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}, \begin{pmatrix} h_B(b) & -nb \\ & g_B(b) \end{pmatrix}\right] \\
= \; & \begin{pmatrix} 0 & mg_B(b) - h_B(b)m + f(m)b \\ & 0 \end{pmatrix}.
\end{aligned}
$$

Finally we have

$$\begin{aligned} 0 &= L([a \oplus 0, 0 \oplus b]) \\ &= [L(a \oplus 0), 0 \oplus b] + [a \oplus 0, L(0 \oplus b)] \\ &= \left[\begin{pmatrix} g_A(a) & an \\ & h_A(a) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\right] + \left[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} h_B(b) & -nb \\ & g_B(b) \end{pmatrix}\right] \\ &= \begin{pmatrix} [a, h_B(b)] & 0 \\ & [h_A(a), b] \end{pmatrix}. \end{aligned}$$

Therefore $[a, h_B(b)] = 0$ and $[h_A(a), b] = 0$. Since $a$ and $b$ are arbitary, we conclude that $h_A(A) \subseteq Z(B)$ and $h_B(B) \subseteq Z(A)$.

Conversely, suppose $L$ is of the form

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) & an - nb + f(m) \\ & h_A(a) + g_B(b) \end{pmatrix}$$

with Conditions (i) and (ii) holding. We have

$$
\begin{aligned}
& L\left(\left[\begin{pmatrix} a & m \\ & b \end{pmatrix}, \begin{pmatrix} a' & m' \\ & b' \end{pmatrix}\right]\right) \\
= \; & L\begin{pmatrix} [a,a'] & am' - m'b - a'm + mb' \\ & [b,b'] \end{pmatrix} \\
= \; & \begin{pmatrix} g_A([a,a']) & [a,a']n - n[b,b'] + f(am' - m'b - a'm + mb') \\ & g_B([b,b']) \end{pmatrix}.
\end{aligned}
$$

and

$$
\begin{aligned}
& \left[L\begin{pmatrix} a & m \\ & b \end{pmatrix}, \begin{pmatrix} a' & m' \\ & b' \end{pmatrix}\right] + \left[\begin{pmatrix} a & m \\ & b \end{pmatrix}, L\begin{pmatrix} a' & m' \\ & b' \end{pmatrix}\right] \\
= \; & \left[\begin{pmatrix} g_A(a) + h_B(b) & an - nb + f(m) \\ & h_A(a) + g_B(b) \end{pmatrix}, \begin{pmatrix} a' & m' \\ & b' \end{pmatrix}\right] \\
& + \left[\begin{pmatrix} a & m \\ & b \end{pmatrix}, \begin{pmatrix} g_A(a') + h_B(b') & a'n - nb' + f(m') \\ & h_A(a') + g_B(b') \end{pmatrix}\right] \\
= \; & \begin{pmatrix} [g_A(a), a'] + [a, g_A(a)] & T \\ & [g_B(b), b'] + [b, g_B(b')] \end{pmatrix} \\
= \; & \begin{pmatrix} g_A([a,a']) & T \\ & g_B([b',b]) \end{pmatrix}.
\end{aligned}
$$

where

$$\begin{aligned} T &= [a, a']n - n[b, b'] \\ &\quad + g_A(a)m' - m'h_A(a) + af(m') - m'g_B(b) + h_B(b)m' - f(m')b \\ &\quad - g_A(a')m + mh_A(a') - a'f(m) + mg_B(b') + h_B(b')m - f(m)b' \\ &= [a, a']n - n[b, b'] + f(am' - m'b - a'm + mb'). \end{aligned}$$

Hence $L$ is a Lie derivation. ∎

In the case that $M$ is faithful, we will show in the next result that the conditions $h_A([a, a']) = 0$ and $h_B([b, b']) = 0$ can be omitted as they are automatically satisfied.

**Corollary 3.2.2** *A linear map $L$ on $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ with faithful $M$ is a Lie derivation if and only if $L$ is of the form*

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) & an - nb + f(m) \\ & h_A(a) + g_B(b) \end{pmatrix}$$

*where $n \in M$, and $g_A : A \to A$, $g_B : B \to B$, $h_A : A \to Z(B)$, $h_B : B \to Z(A)$, $f : M \to M$ are linear maps satisfying*

(i) *$g_A$ is a Lie derivation on $A$, $f(am) = g_A(a)m - mh_A(a) + af(m)$ and*

(ii) *$g_B$ is a Lie derivation on $B$, $f(mb) = mg_B(b) - h_B(b)m + f(m)b$.*

*Proof.* The "only if" part follows from Proposition 3.2.1. To prove the converse, it suffices to show that $h_A([a, a']) = 0$ and $h_B([b, b']) = 0$.

Now, using condition (i), we have

$$\begin{aligned}
f([a, a']m) &= g_A([a, a'])m - mh_A([a, a']) + [a, a']f(m) \\
f([a, a']m) &= f(aa'm) - f(a'am) \\
&= (g_A(a)a'm - a'mh_A(a) + af(a'm)) \\
&\quad -(g_A(a')am - amh_A(a') + a'f(am)) \\
&= (g_A(a)a'm - a'mh_A(a) \\
&\quad +ag_A(a')m - amh_A(a') + aa'f(m)) \\
&\quad -(g_A(a')am - amh_A(a') \\
&\quad +a'g_A(a)m - a'mh_A(a) + a'af(m)) \\
&= ([g_A(a), a'] + [a, g_A(a')])m + [a, a']f(m) \\
&= g_A([a, a'])m + [a, a']f(m)
\end{aligned}$$

as $g_A$ is a Lie derivation. Hence $mh_A([a, a']) = 0$ for all $m \in M$ and then $h_A([a, a']) = 0$ as $M$ is a faithful right $B$ module. Similarly, using condition (ii), we have $h_B([b, b']) = 0$ and the result follows. ■

Recall from Theorem 1.4.4 that when $M$ is faithful, then the centre of $\mathrm{Tri}(A, M, B)$ is

$$Z(\mathrm{Tri}(A, M, B)) = \{a \oplus b : am = mb \text{ for all } m \in M\}.$$

This leads to a relatively simple criteria for a Lie derivation to be proper.

**Theorem 3.2.3** *Given a triangular algebra $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ with faithful $M$. A Lie derivation is proper if and only if $\pi_A L(B) \subseteq \pi_A(Z(\mathfrak{A}))$ and $\pi_B L(A) \subseteq \pi_B(Z(\mathfrak{A}))$ where $A$ and $B$ are identified as subalgebras of $\mathfrak{A}$. That is to say that a Lie derivation $L$ on $\mathfrak{A}$ given by*

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) & an - nb + f(m) \\ & h_A(a) + g_B(b) \end{pmatrix}$$

*is proper if and only if $h_A(A) \subseteq \pi_B(Z(\mathfrak{A}))$ and $h_B(B) \subseteq \pi_A(Z(\mathfrak{A}))$.*

*Proof.* Suppose that $L$ is proper and is written as $\delta + h$, where $\delta$ is a derivation and $h(\mathfrak{A}) \subseteq Z(\mathfrak{A})$. By Corollary 2.2.2,

$$\delta\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_A(a) & an_0 - n_0 b + f_0(m) \\ & p_B(b) \end{pmatrix}$$

with $f_0(am) = p_A(a)m + af_0(m)$ and $f(mb) = mp_B(b) + f_0(m)b$, whereas by the structure of $Z(\mathfrak{A})$,

$$h\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} i_1(a) + i_2(m) + i_3(b) & 0 \\ & j_1(a) + j_2(m) + j_3(b) \end{pmatrix}.$$

Since $L = \delta + h$, we have $n_0 = n$ (by taking $a = 1$, $b = 0$ and $m = 0$), $f_0 = f$, $i_2 = 0$, $j_2 = 0$, $i_3 = h_B$, $j_1 = h_A$, $g_A = p_A + i_1$ and $g_B = p_B + j_3$. Hence

$$h_A(a) = j_1(a) = \pi_B(h\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}) \in \pi_B(Z(\mathfrak{A}))$$

and

$$h_B(b) = i_3(b) = \pi_A(h\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}) \in \pi_A(Z(\mathfrak{A})).$$

Conversely, suppose that $h_A(A) \subseteq \pi_B(Z(\mathfrak{A}))$ and $h_B(B) \subseteq \pi_A(Z(\mathfrak{A}))$. Recall the unique group isomorphism $\tau$ from $\pi_A(Z(\mathfrak{A}))$ to $\pi_B(Z(\mathfrak{A}))$ in Theorem 1.4.4 and define

$$\delta\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_A(a) - \tau^{-1}h_A(a) & an - nb + f(m) \\ & g_B(b) - \tau h_B(b) \end{pmatrix}$$

and

$$h\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} \tau^{-1}h_A(a) + h_B(b) & 0 \\ & h_A(a) + \tau h_B(b) \end{pmatrix}.$$

That $h\begin{pmatrix} a & m \\ & b \end{pmatrix} \in Z(\mathfrak{A})$ is easy to verify. Now by Proposition 3.2.2, we have

$$f(am) = g_A(a)m - mh_A(a) + af(m) = (g_A(a) - \tau^{-1}h_A(a))m + af(m)$$

and similarly

$$f(mb) = mg_B(b) - h_B(b)m + f(m)b = m(g_B(b) - \tau h_B(b)) + f(m)b,$$

thus $\delta$ is a derivation by Corollary 2.2.2. ∎

We now show that the triangular algebra in Example 1.2.5 has an improper Lie derivation.

**Example 3.2.4** Let $A = B = \left\{ \begin{pmatrix} t & a \\ & t \end{pmatrix} : t, a \in \mathbf{R} \right\}$ and $M = \mathcal{T}_2(\mathbf{R})$. The map

$$\begin{pmatrix} t & a & x & y \\ & t & & z \\ & & s & b \\ & & & s \end{pmatrix} \xrightarrow{L} \begin{pmatrix} 0 & b & z & 0 \\ & 0 & & x \\ & & 0 & a \\ & & & 0 \end{pmatrix}$$

is a Lie derivation on $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ but is not a sum of a derivation and a mapping into the centre.

*Proof.* That $L$ is a Lie derivation can be checked by direct verification. By Corollary 1.4.7, $Z(\mathfrak{A}) = \mathbf{R}1$, so $\pi_A(Z(\mathfrak{A})) = \mathbf{R}1$ and $\pi_B(Z(\mathfrak{A})) = \mathbf{R}1$. Since $h_A \begin{pmatrix} t & a \\ & t \end{pmatrix} = \begin{pmatrix} 0 & a \\ & 0 \end{pmatrix} \notin \pi_B(Z(\mathfrak{A}))$, we have $h_A(A) \not\subseteq \pi_B(Z(\mathfrak{A}))$ and thus $L$ is not the sum of a derivation and a mapping into the centre by Theorem 3.2.3.

∎

## 3.3 The Main Theorem

In this section, we will find sufficient conditions on a triangular algebra $\text{Tri}(A, M, B)$ with faithful $M$ so that every Lie derivation is proper. The key idea is that, under those conditions, the maps $h_A$ and $h_B$ in Theorem 3.2.2 always satisfy $A = h_A^{-1}(\pi_B(Z(\mathfrak{A})))$ and $B = h_B^{-1}(\pi_A(Z(\mathfrak{A})))$.

First of all, we have two preliminary results.

**Lemma 3.3.1** *Let $g : A \to A$ be a Lie derivation, $h : A \to Z(B)$ and $f : M \to M$ be linear maps such that $f(am) = g(a)m - mh(a) + af(m)$ for any $a \in A, m \in M$. Define $G : A \times A \to A$ by $G(x, y) = g(xy) - xg(y) - g(x)y$. We have*

(i) $G(x, y) = G(y, x)$;

(ii) $G(x, y)m = mh(xy) - xmh(y) - ymh(x)$, *for any $x, y \in A$, $m \in M$;*

(iii) *Let $F(t) = \sum_{j=1}^{k} r_j t^j$ be a polynomial in $\mathbf{R}[t]$. For any $x \in A$, there exists $w(x) \in \mathfrak{A}$ such that $w(x)m = mh(F(x)) - F'(x)mh(x)$ for every $m \in M$, where $F'(t) = \sum_{j=1}^{k} jr_k t^{j-1}$ is the derivative of $F$.*

*Proof.* As $g$ is a Lie derivation, we have

$$\begin{aligned} G(x, y) - G(y, x) &= g(xy) - xg(y) - g(x)y - g(yx) + yg(x) + g(y)x \\ &= g[x, y] - [x, g(y)] - [g(x), y] = 0. \end{aligned}$$

Thus (i) follows. By the equation $f(am) = g(a)m - mh(a) + af(m)$ from Corollary 2.2.1, we have

$$\begin{aligned} f((xy)m) &= g(xy)m - mh(xy) + xyf(m), \quad \text{and} \\ f(xym) &= f(x(ym)) \\ &= g(x)ym - ymh(x) + xf(ym) \\ &= g(x)ym - ymh(x) + x(g(y)m - mh(y) + yf(m)). \end{aligned}$$

By comparing the two equalities, we have (ii).

To prove (iii), it suffice to show, for $k = 1, 2, \ldots$, that

$$(\sum_{i=1}^{k} x^{k-i}G(x, x^i))m = mh(x^{k+1}) - (k+1)x^k mh(x).$$

Take $y = x$ in (ii) and then $G(x, x)m = mh(x^2) - 2xmh(x)$, thus it is true for $k = 1$. Now suppose it is true for some $k$, therefore

$$\begin{aligned} (\sum_{i=1}^{k+1} x^{k+1-i}G(x, x^i))m &= G(x, x^{k+1})m + x\sum_{i=1}^{k} x^{k-i}G(x, x^i)) \\ &= (mh(x^{k+2}) - xmh(x^{k+1}) - x^{k+1}mh(x)) + \\ &\quad x(mh(x^{k+1}) - (k+1)x^k mh(x)) \\ &= mh(x^{k+2}) - (k+2)x^{k+1}mh(x). \end{aligned}$$

■

**Proposition 3.3.2** *Consider a Lie derivation $L$ on $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ with faithful $M$. Write $L$ in the form*

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) & an - nb + f(m) \\ & h_A(a) + g_B(b) \end{pmatrix}.$$

*Let $V = \{a \in A : h_A(a) \in \pi_B(Z(\mathfrak{A}))\}$. Then $V$ is an $\mathbf{R}$-subalgebra of $\mathfrak{A}$ satisfying the following seven conditions:*

*(i) $V$ contains all the commutators $[x, y]$ for $x, y \in A$;*

*(ii) Suppose $b \in A$ and $F(t)$ is a polynomial in $\mathbf{R}[t]$. If $F'(b) = 0$ then $F(b) \in V$;*

*(iii) Suppose $b \in A$ and $F(t)$ is a polynomial in $\mathbf{R}[t]$. If $F(b) \in V$ and $F'(b)$ is invertible, then $b \in V$;*

*(iv) $V$ contains all the idempotents in $A$ (in particular $1 \in V$);*

*(v) $V$ contains all the elements of the form $x^p$, where $x \in A$ and $p \geq 0$ is the characteristic of $A$;*

*(vi) $\{a \in A : a^{-1} \in V\} \subseteq V$;*

*(vii) If $b \in A$ is invertible with $b^k \in V$ for some positive integer $k$ then $kb \in V$.*

*Proof.* Let $\tau$ be the algebra isomorphism from $\pi_A(Z(\mathfrak{A}))$ to $\pi_B(Z(\mathfrak{A}))$ defined in Lemma 1.4.1. Recall that $x \in V$, i.e. $h_A(x) \in \pi_B(Z(\mathfrak{A}))$, if and only if there exists $a \in A$ such that $mh_A(x) = am$ for all $m \in M$.

Obviously $V$ is a **R**-submodule of $A$. By the Lemma 3.3.1(ii), if $x, y \in V$ then we have

$$\begin{aligned} mh_A(xy) &= G(x,y)m + xmh_A(y) + ymh_A(x) \\ &= (G(x,y) + x\tau^{-1}h_A(y) + y\tau^{-1}(x))m \end{aligned}$$

for all $m \in M$. Thus $xy \in V$ and thus $V$ is a subalgebra.

For any $x, y \in A$, we have $h_A([x,y]) = 0$ by Proposition 3.2.1. Hence $[x,y] \in V$.

Consider $b \in A$ and $F(t) \in \mathbf{R}[t]$. If $F'(b) = 0$, then by Lemma 3.3.1(iii), there exists $w(b) \in A$ such that

$$w(b)m = mh_A(F(b)).$$

Thus $F(b) \in V$.

Consider $b \in A$ and $F(t) \in \mathbf{R}[t]$. If $F(b) \in V$ and $F'(b)$ is invertible, then by Lemma 3.3.1(iii), we have

$$\begin{aligned} F'(b)^{-1}w(b)m &= F'(b)^{-1}mh(F(b)) - mh(b) \\ &= F'(b)^{-1}\tau^{-1}h(F(b))m - mh(b) \end{aligned}$$

and thus $b \in V$ as

$$mh(b) = F'(b)^{-1}(\tau^{-1}h(F(b)) - w(b))m.$$

Take an idempotent $e$ of $A$. Consider $F(t) = 3t^2 - 2t^3 \in \mathbf{R}[t]$. Since $F'(e) = 0$, by (ii), we have $e = F(e) \in V$. In particular $1 \in V$.

Take $b \in A$. Consider $F(t) = t^p \in \mathbf{R}[t]$. Since $F'(t) = 0$, by (ii), we have $b^p = F(b) \in V$.

For any $a \in A$ with $a^{-1} \in V$, we have

$$G(a, a^{-1})m = mh(1) - amh(a^{-1}) - a^{-1}mh(a).$$

As $1 \in V$, that $a \in V$ follows from

$$mh(a) = (G(a, a^{-1}) - \tau^{-1}h(1) - a^2\tau^{-1}(a^{-1}))m.$$

For any invertible element $b \in A$ with $b^k \in V$. Put $F(t) = t^k$ in Lemma 3.3.1(iii), we have

$$w(b)m = mh_A(F(b)) - F'(b)mh_A(b) = mh_A(b^k) - kb^{k-1}mh_A(b),$$

and hence $kb \in V$ as

$$mh_A(kb) = b^{1-k}(w(b) - \tau^{-1}h_A(b^k))m. \qquad \blacksquare$$

Next, we state and prove the main theorem in this section.

**Theorem 3.3.3** *Every Lie derivation of* $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ *with faithful* $M$ *is proper if the following two conditions hold:*

(I) $Z(B) = \pi_B(Z(\mathfrak{A}))$, *or there exists* $a, a' \in A$ *so that* $[a, a']$ *is invertible, or that* $A$ *is generated as an algebra by the commutators and the idempotents (or, more generally, that there is no proper subalgebra of* $A$ *satisfying (i)–(vii) in Lemma 3.3.2);*

(II) $Z(A) = \pi_A(Z(\mathfrak{A}))$, *or there exists* $b, b' \in B$ *so that* $[b, b']$ *is invertible, or that* $B$ *is generated as an algebra by the commutators and the idempotents (or, more generally, that there is no proper subalgebra of* $B$ *satisfying (i)–(vii) in Lemma 3.3.2).*

*Proof.* Consider a Lie derivation on $\mathfrak{A}$ in the form

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) & an - nb + f(m) \\ & h_A(a) + g_B(b) \end{pmatrix}.$$

We have to show that it is a sum of a derivation and a mapping into the centre. By Theorem 3.2.3, this is equivalent to showing that $h_A(A) \subseteq \pi_B(Z(\mathfrak{A}))$ and $h_B(B) \subseteq \pi_A(Z(\mathfrak{A}))$. We claim that if Condition (I) holds, then $h_A(A) \subseteq \pi_B(Z(\mathfrak{A}))$.

Suppose $Z(B) = \pi_B(Z(\mathfrak{A}))$. The result follows since $h_A(A) \subseteq Z(B)$ by Corollary 3.2.2.

Suppose there exists $x, y \in A$ such that $[x, y]^{-1}$ exists. Note that for any

$a \in A$, $m \in M$, we have

$$\begin{aligned}[a, G(x, y)]m &= a(G(x, y))m - G(x, y)(am) \\ &= a(mh_A(xy) - xmh_A(y) - ymh_A(x)) \\ &\quad -(amh_A(xy) - xamh_A(y) - yamh_A(x)) \\ &= [x, a]mh_A(y) + [y, a]mh_A(x).\end{aligned}$$

By taking $a = y$, we get

$$\begin{aligned}mh_A(y) &= [x, y]^{-1}[x, y]mh_A(y) \\ &= [x, y]^{-1}([x, y]mh_A(y) + [y, y]mh_A(x)) \\ &= [x, y]^{-1}[y, G(x, y)]m\end{aligned}$$

thus $y \in h_A^{-1}(\pi_B(Z(\mathfrak{A})))$. For any $a \in A$, we have

$$\begin{aligned}mh_A(a) &= [x, y]^{-1}[x, y]mh_A(a) \\ &= [x, y]^{-1}([a, x]mh_A(y) - [x, G(a, y)]m) \\ &= [x, y]^{-1}([a, x]\tau^{-1}h_A(y) - [x, G(a, y)])m.\end{aligned}$$

Hence $h_A(a) \in \pi_B(Z(\mathfrak{A}))$ by Lemma 3 and the result follows.

Suppose that there is no proper subalgebra of $A$ satisfying (i)–(vii) in Lemma 3.3.2. By Proposition 3.3.2, $V = h_A^{-1}(\pi_B(Z(\mathfrak{A})))$ is a subalgebra of

$A$ satisfying (i)–(vii). Thus we have $A = h_A^{-1}(\pi_B(Z(\mathfrak{A})))$ and the claim is proved.

By symmetry, we have $h_B(B) \subseteq \pi_A(Z(\mathfrak{A}))$ if Condition (II) holds. This ends the proof of the theorem. ■

**Remark.** It is a fact that if $A$ is a simple ring not of characteristic 2, or if $A$ is a von Neumann algebra which has no central abelian summands, then $A$ is generated as an algebra by its commutators (see [34, 61]).

**Corollary 3.3.4** *If* $Z(A) = \mathbf{R}1$, $Z(B) = \mathbf{R}1$ *and* $M$ *is faithful, then every Lie derivation of* $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ *is proper.*

*Proof.* By Corollary 1.4.5, $Z(\mathfrak{A}) = \mathbf{R}1$. Therefore $Z(A) = \pi_A(Z(\mathfrak{A}))$ and $Z(B) = \pi_B(Z(\mathfrak{A}))$. The result follows from Theorem 3.3.3. ■

**Corollary 3.3.5** *If* $Z(A) = \mathbf{R}1$ *and* $A$ *is generated as an algebra by commutators and idempotents, and* $M$ *is faithful, then every Lie derivation of* $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ *is proper.*

*Proof.* As $A$ is generated by commutators and idempotents, Condition (I) in Theorem 3.3.3 holds. By Corollary 1.4.5, $Z(\mathfrak{A}) = \mathbf{R}1$, thus $Z(A) = \pi_A(Z(\mathfrak{A}))$ and Condition (II) holds. Hence the result follows. ■

**Corollary 3.3.6** *If $\mathcal{A} = \mathbf{R}1$ and $\mathcal{M}$ is faithful then every Lie derivation of $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is proper.*

*Proof.* It is a direct consequence of Corollary 3.3.5 as $\mathcal{A}$ is generated by $\{1\}$.

■

**Corollary 3.3.7** *If $Z(\mathcal{A}) = \mathbf{R}1$ and there exists $x, y \in \mathcal{A}$ such that $[x, y]$ is invertible and $\mathcal{M}$ is faithful, then every Lie derivation of $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is proper.*

*Proof.* Condition (I) in Theorem 3.3.3 holds as there exists $x, y \in \mathcal{A}$ such that $[x, y]$ is invertible. Condition (II) holds as $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathfrak{A}))$. ■

**Corollary 3.3.8** *Let $\mathsf{S}$ be an $\mathbf{R}$-algebra. If $\mathcal{A} \subseteq M_k(\mathsf{S})$ and $\mathcal{B} \subseteq M_l(\mathsf{S})$ are matrix algebras such that $Z(\mathcal{A}) = Z(\mathsf{S})I_k$, $Z(\mathcal{B}) = Z(\mathsf{S})I_k$, and $\mathcal{M}$ is subspace of $\mathsf{S}^{k,l}$ which is a faithful bimodule, then every Lie derivation of $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is proper.*

*Proof.* By Theorem 1.5.3, $Z(\mathfrak{A}) = Z(\mathsf{S})I_{k+l}$. Then, by Theorem 3.3.3, every Lie derivation of $\mathfrak{A}$ is proper. ■

## 3.4 Full Matrix Algebras at the corner

It is unknown whether every Lie derivation on $\mathbf{S}^{n,n}$ is proper even when $\mathbf{S} = \mathbf{R}$. However, we have the following result.

**Corollary 3.4.1** *If* $\mathcal{A} = M_n(\mathbf{R})$ *and* $M$ *is faithful, then every Lie derivation of* $\mathrm{Tri}(\mathcal{A}, M, B)$ *is proper.*

*Proof.* This is a consequence of Corollary 3.3.5 since $Z(\mathcal{A}) = \mathbf{R}1$ by Proposition 1.5.4 and $\mathcal{A}$ is generated by commutators and idempotents by Proposition 1.5.5. ■

Next, we give an infinite dimensional generalization of Corollary 3.4.1.

**Corollary 3.4.2** *If* $\mathcal{A} = \mathcal{B}(H)$, *the algebra of bounded operators on a complex Hilbert space* $H$, $B$ *is any algebra, and* $M$ *is a faithful bimodule, then every Lie derivation of* $\mathrm{Tri}(\mathcal{A}, M, B)$ *is proper.*

*Proof.* Note that $H$ is isomorphic to $H \oplus H$ and that $\mathcal{A}$ is isomorphic to $M_2(\mathcal{A})$. As $Z(\mathcal{A}) = \mathbb{C}1$ and $\left[\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is invertible in $\mathcal{A}$, the result follows from Corollary 3.3.7. ■

## 3.5 Triangular Matrix Algebras

**Corollary 3.5.1** *Given a block triangular algebra* $\mathfrak{A} = T(n_1, \ldots, n_k)(\mathbf{S})$ *with* $k > 1$. *Every Lie derivation of* $\mathfrak{A}$ *is of the form* $d + \delta_z + h$, *where* $d$ *is an induced derivation from* $\mathbf{S}$, $\delta_z$ *is the inner derivation defined by* $z$, *and* $h$ *is linear functional which annihilates the commutators.*

*Proof.* It follows from Theorem 1.5.2 and Corollary 3.3.8 that every Lie derivation is proper. By Theorem 2.4.2, every derivation is of the form $d+\delta_z$. Hence we have the result. ■

**Corollary 3.5.2** *If* $A = T(n_1, \ldots, n_k)(\mathbf{R})$, $B$ *is any unital algebra and* $M$ *is faithful, then* $\mathrm{Tri}(A, M, B)$ *has no improper Lie derivation.*

*Proof.* By Proposition 1.5.2, $Z(A) = \mathbf{R}1$ and by Proposition 1.5.4, $A$ is generated by idempotents, so the results follows Corollary 3.3.5. ■

## 3.6 Nest Algebras

The following is an analogue to Corollary 3.4.2.

**Corollary 3.6.1** *Every Lie derivation of nest algebras is of the form* $x \mapsto ax - xa + h(x)1$ *where* $a$ *is a fixed element in the nest algebra and* $h$ *is a linear functional annihilating the commutators.*

*Proof.* First we show that every Lie derivation of a nest algebra is proper. If $\mathcal{N}$ is a trivial nest, i.e. $\mathcal{N} = \{0, H\}$, then $\mathcal{T}(\mathcal{N}) = \mathcal{B}(H)$ is a von Neumann algebra and it has no improper Lie derivation by Mier's result [60]. If $\mathcal{N}$ is not trivial, then we have $\mathcal{T}(\mathcal{N}) = \begin{pmatrix} A & M \\ & B \end{pmatrix}$ where $A$, $B$ are nest algebras as in Lemma 1.6.5, and that $Z(A) = \mathbf{R}1$, $Z(B) = \mathbf{R}1$ by Lemma 1.6.6. Thus every Lie derivation $\mathcal{T}(\mathcal{N})$ is proper by Corollary 3.3.4. Finally, by Proposition 1.6.6, every derivation on a nest algebra is inner. The proof is completed. ∎

**Corollary 3.6.2** *Every Lie derivation on the nest algebra of a continuous nest is an inner derivation.*

*Proof.* By Corollary 3.6.1, every derivation is of the form $\delta_a + h$ where $h$ annihilates commutators. By Lemma 1.6.7, every element of the nest algebra of a continuous nest is a sum of two commutators. Therefore $h = 0$ and every derivation is an inner derivation. ∎

The following result is an analogue to Corollary 3.5.2.

**Corollary 3.6.3** *If $\mathcal{A}$ is nest algebra of a continuous nest, $B$ is any algebra and $\mathcal{M}$ is faithful, then every Lie derivation of* $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, B)$ *is proper.*

*Proof.* Since $Z(\mathcal{A}) = \mathbb{C}1$ by Lemma 1.6.6 and $\mathcal{A}$ is generated by commutators by Lemma 1.6.7, the result follows Corollary 3.3.5. ■

## 3.7 Triangular Banach Algebras

We have the following theorem about continuous Lie derivations of triangular Banach algebras. This differs from Theorem 3.3.3 only in as much as the condition on the algebras being generated by special elements is interpreted in a topological sense. We say that a Banach algebra $\mathfrak{A}$ is the *topologically closed algebra generated* by a subset $S$ if $\mathfrak{A}$ is the smallest closed subalgebra containing $S$.

**Theorem 3.7.1** *Given a triangular Banach algebra* $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, B)$. *Suppose that $\mathcal{M}$ is faithful. If $\mathfrak{A}$ satisfies*

(I) $Z(B) = \pi_B(Z(\mathfrak{A}))$ *or $\mathcal{A}$ is the topologically closed algebra generated by commutators and idempotents: and*

(II) $Z(\mathcal{A}) = \pi_{\mathcal{A}}(Z(\mathfrak{A}))$ *or $B$ is the topologically closed algebra generated by commutators and idempotents,* *then every continuous Lie derivation on $\mathfrak{A}$ is a sum of a derivation and a linear mapping with image in $Z(\mathfrak{A})$.*

*Proof.* Note that $Z(\mathfrak{A})$ is closed, $\pi_A, \pi_B$ are open and continuous. Suppose $L$ is a continuous Lie derivation, then $h_A, h_B$ as in Theorem 3.3.3 are continuous. By Proposition 2.2.5, $V = h_A^{-1}(\pi_B(Z(\mathfrak{A})))$ is a closed subalgebra of $A$ containing commutators and idempotents of $A$. So if $A$ is the closed linear span of commutators and idempotents, then $h_A(A) \subseteq \pi_B(Z(\mathfrak{A}))$. Thus the result follows. ■

We now apply Theorem 3.7.1 to the join of two von Neumann algebras. The class of von Neumann algebras is one of the most widely studied among all the operator algebras. For the definitions and properties of von Neumann algebras, one may refer to [18] or [67]. Recall from [18, Proposition 13.3] that a von Neumann algebra is the norm closed linear span of its projections.

**Corollary 3.7.2** *Suppose $A$ and $B$ are von Neumann algebras on Hilbert spaces $H_1$ and $H_2$ respectively, then every continuous Lie derivation on the join $A\#B = \mathrm{Tri}(A, B(H_2, H_1), B)$ is a sum of a derivation and a linear mapping with image in $Z(\mathfrak{A})$.*

*Proof.* It is known [18, Proposition 13.3] that every von Neumann algebra is the closed linear span of its projections. The result now follows from Theorem 3.7.1. ∎

# Chapter 4

# Commuting Maps

## 4.1 Introduction

In this chapter we will identify those triangular algebras for which every commuting map is "almost" of the form $x \mapsto ax$. Some material in this chapter may appear in [14].

**Definition 4.1.1** Given an $R$-algebra $\mathcal{A}$. A *commuting map* of $\mathcal{A}$ is an $R$-linear map $L$ on $\mathcal{A}$ such that

$$[L(a), a] = 0 \quad \text{for all } a \in \mathcal{A}.$$

As in the previous chapter, we define the term proper commuting maps.

**Definition 4.1.2** A commuting map $L$ of an algebra is said to be *proper* if it can be written as $L(x) = ax + h(x)$, where $a$ lies in the center of the algebra and $h$ is a linear mapping with image in the center of the algebra. A commuting map which is not proper is said to be *improper*.

Commuting maps, like Lie derivations, have been active research subjects for a long time. It is proved in [11] that a commuting map on noncommutative Lie ideals of a prime ring is always of the form $x \mapsto ax + h(x)$, where $x$ lies in the centre of a certain larger ring and $h$ is an additive mapping from the ring into this centre. Results related to commuting maps on prime or semiprime rings are considered in [8, 11, 49, 50]. Brešar [8] considered von Neumann algebras and showed that every commuting map is, according to our definition, proper.

We will determine sufficient conditions for a commuting map of a triangular algebra to be proper. Similar to the previous chapters, we begin with two easy propositions for general algebras. The first one deals with unitization and the second one deals with direct sums.

**Proposition 4.1.3** *Let $\mathcal{A}$ be an algebra without unity and $\mathcal{A} \vee \mathbf{R}1$ be the unitization of $\mathcal{A}$. Then $\mathcal{A}$ has no improper commuting map if and only if $\mathcal{A} \vee \mathbf{R}1$ has no improper commuting map.*

*Proof.* Let $j : A \to A \vee \mathbf{R}1$ and $\pi : A \vee \mathbf{R}1 \to A$ be the natural inclusion and projection.

Assume $A$ has no improper commuting map. Let $L$ be a commuting map of $A \vee \mathbf{R}1$. First observe that for all $a \in A$,

$$\begin{aligned} 0 = [L(1+a), 1+a] &= [L(1), a] + [L(a), a] \\ &= [L(1), a] \\ &= [L(1), a + \gamma 1], \end{aligned}$$

so that $L(1) \in Z(A \vee \mathbf{R}1)$. Then $\pi L j$ is a commuting map on $A$, thus $\pi L j$ is a proper commuting map and can be written as $a \mapsto ax + h(a)$ where $x \in Z(A) \subseteq Z(A \vee \mathbf{R}1)$ and $h$ maps into $Z(A)$. Define $\hat{h}(b) = L(b) - bx$ for all $b \in A \vee \mathbf{R}1$. For $a \in A$ and $b \in A \vee \mathbf{R}1$,

$$\begin{aligned} [\hat{h}(a), b] = [L(a) - ax, b] &= [\pi L(a) - \pi(ax), \pi(b)] \\ &= [ax + h(a) - ax, \pi(b)] \\ &= [h(a), \pi(b)] \\ &= 0. \end{aligned}$$

So $\hat{h}(A) \subseteq Z(A \vee \mathbf{R}1)$. Since $\hat{h}(1) = L(1) - x$ and $L(1), x \in Z(A \vee \mathbf{R}1)$, $\hat{h}(A \vee \mathbf{R}1) \subseteq A \vee \mathbf{R}1$, as needed. Hence $L(b) = bx + \hat{h}(b)$ as required.

Conversely, suppose $A \vee \mathbf{R}1$ has no improper commuting map. Let $\hat{L}$ be a commuting map on $A$. Define a commuting map $L$ of $A \vee \mathbf{R}1$ by $L(a+\gamma 1) = \hat{L}(a)$. Then $L(b) = xb+h(b)$ since $L$ is a proper commuting map. Note that $\pi(x), \pi h(a) \in Z(A)$, thus $\hat{L}(a) = \pi(L(a)) = \pi(x)L(a) + \pi h(a)$ as required. ■

**Proposition 4.1.4** *$A_1$ and $A_2$ have no improper commuting maps if and only if $A_1 \oplus A_2$ has no improper commuting map.*

*Proof.* Suppose $L$ is a commuting map of $A_1 \oplus A_2$. Write $L(a,b) = (L_1(a) + \pi_1(b), L_2(b) + \pi_2(a))$. Since

$$0 = [L(a,0),(a,0)] = ([L_1(a),a],[\pi_2(a),0]),$$

$$0 = [(0,b),L(0,b)] = ([0,\pi_1(b)],[L_2(b),b]),$$

we have $L_i$ is a commuting map of $A_i$ for $i = 1,2$. Furthermore

$$0 = [L(a,b),(a,b)] = ([\pi_1(b),a],[\pi_2(a),b]),$$

hence image of $\pi_i$ lies in $Z(A_i)$ for $i = 1,2$.

Assume that $A_1$ and $A_2$ have no improper commuting maps. Let $L$ be a commuting map of $A_1 \oplus A_2$. By the above argument and the fact that every commuting map of $A_i$ is a proper commuting map, we have $L_1(a) =$

$ax_1+h_1(a)$ for $a \in A_1$ and $L_2(b) = bx_2+h_2(b)$ for $b \in A_2$. Let $x = (x_1, x_2)$ and $h(a_1, a_2) = (h_1(a_1) + \pi_1(a_2), h_2(a_2) + \pi_2(a_1))$, then $L((a,b)) = (a,b)(x_1, x_2) + h((a,b))$ and we are done.

Conversely, suppose $A_1 \oplus A_2$ has no improper commuting map. We need to show that $A_1$ (and similarly $A_2$) has no improper commuting map also. Let $L_1$ be a commuting map of $A_1$. Define a commuting map of $A_1 \oplus A_2$ by $L(a,b) = (L_1(a), 0)$. As $L$ is a proper commuting map, we have $L(c) = c(x_1, x_2) + h(c)$. Evidently $L_1(a) = ax_1 + h_1(a)$ as required. ■

## 4.2 Structure of Commuting Maps

We will consider only unital algebras in the rest of this chapter.

**Proposition 4.2.1** *A linear map $L$ on $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ is a commuting map if and only if $L$ is of the from*

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) + k_A(m) & j_A m - m j_B \\ & h_A(a) + g_B(b) + k_B(m) \end{pmatrix}$$

*where $g_A : A \to A$ with $j_A = g_A(1_A)$, $g_B : B \to B$, $h_A : A \to Z(B)$ with $j_B = h_A(1_A)$, $h_B : B \to Z(A)$, $k_A : M \to Z(A)$, $k_B : M \to Z(B)$ are linear maps satisfying*

(i) $g_{\mathcal{A}}$ *is a commuting map on* $\mathcal{A}$, $a(j_{\mathcal{A}}m - mj_B) = g_{\mathcal{A}}(a)m - mh_{\mathcal{A}}(a)$*;*

(ii) $g_B$ *is a commuting map on* $B$, $(j_{\mathcal{A}}m - mj_B)b = mg_B(b) - h_B(b)m$*;*

(iii) $k_{\mathcal{A}}(m)m = mk_B(m)$.

*Proof.* Suppose $L$ is a commuting map. Write

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_{\mathcal{A}}(a) + h_B(b) + k_{\mathcal{A}}(m) & f_{\mathcal{A}}(a) + f_B(b) + f(m) \\ & h_{\mathcal{A}}(a) + g_B(b) + k_B(m) \end{pmatrix}.$$

Suppose $L(1 \oplus 0) = \begin{pmatrix} j_{\mathcal{A}} & n \\ & j_B \end{pmatrix}$. That $n = 0$ follows from

$$0 = [L(1 \oplus 0), 1 \oplus 0] = \begin{pmatrix} 0 & -n \\ & 0 \end{pmatrix}.$$

Since $[L(x), x] = 0$ for all $x \in \mathfrak{A}$, we have

$$\begin{aligned} 0 &= [L(x + y), (x + y)] \\ &= [L(x), x] + [L(x), y] + [L(y), x] + [L(y), y] \\ &= [L(x), y] - [x, L(y)] \end{aligned}$$

and hence $[L(x), y] = [x, L(y)]$. Therefore

$$\begin{aligned}
\begin{pmatrix} 0 & f_A(a) + f_B(b) \\ & 0 \end{pmatrix} &= [1 \oplus 0, L(a \oplus b)] \\
&= [L(1 \oplus 0), a \oplus b] \\
&= [j_A, a] \oplus [j_B, b],
\end{aligned}$$

and thus $f_A = 0$ and $f_B = 0$.

We have

$$\begin{aligned}
0 \oplus [h_A(a), b] &= [L(a \oplus 0], 0 \oplus b] \\
&= [a \oplus 0, L(0 \oplus b)] \\
&= [a, h_B(b)] \oplus 0
\end{aligned}$$

and hence $[h_A(a), b] = 0$ and $[a, h_B(b)] = 0$. As this is true for all $a \in A$ and $b \in B$, we get that $h_A(A) \in Z(B)$ and $h_B(B) \in Z(A)$.

We also have

$$[g_A(a), a] \oplus [g_B(b), b] = [L(a \oplus b), a \oplus b] = 0$$

and hence $g_A$ and $g_B$ are commuting maps.

Before establishing the results about $Im(k_A)$, $Im(k_B)$ and the form of $f$, we pause to prove condition (iii). For $m \in M$, we have

$$\begin{aligned}
0 &= \left[ L \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} k_A(m) & f(m) \\ & k_B(m) \end{pmatrix} \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} - \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \begin{pmatrix} k_A(m) & f(m) \\ & k_B(m) \end{pmatrix} \\
&= \begin{pmatrix} 0 & k_A(m)m - mk_B(m) \\ & 0 \end{pmatrix}.
\end{aligned}$$

This proves (iii).

Next, we show that $f(m) = j_A m - m j_B$. Recall that $j_A \oplus j_B = L(1_A \oplus 0)$ and hence $j_A = g_A(1_A)$ and $j_B = h_A(1_A)$. We have

$$\begin{aligned}
\begin{pmatrix} 0 & j_A m - m j_B \\ & 0 \end{pmatrix} &= \left[ \begin{pmatrix} j_A & 0 \\ & j_B \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&= \left[ L(1 \oplus 0), \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&= \left[ 1 \oplus 0, L \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & f(m) \\ & 0 \end{pmatrix}.
\end{aligned}$$

Thus $f(m) = j_A m - m j_B$ as required.

Next we prove (i) together with $k_A(m) \in Z(A)$. This follows from

$$\begin{aligned}
\begin{pmatrix} 0 & g_A(a) m - m h_A(a) \\ & 0 \end{pmatrix} &= \left[ L(a \oplus 0), \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&= \left[ a \oplus 0, L \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} [a, k_A(m)] & a(j_A m - m j_B) \\ & 0 \end{pmatrix}.
\end{aligned}$$

Similarly (ii) and $k_B(m) \in Z(B)$ follow from

$$\begin{aligned}
\begin{pmatrix} 0 & h_B(b)m - mg_B(a) \\ & 0 \end{pmatrix} &= \left[ L(0 \oplus b), \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&= \left[ 0 \oplus b, L \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & (mj_B - j_A m)b \\ & [b, k_B(m)] \end{pmatrix}.
\end{aligned}$$

Conversely suppose $L$ is of the given form. Take $x = \begin{pmatrix} a & m \\ & b \end{pmatrix} \in \mathfrak{A}$, we have

$$\begin{aligned}
&\left[ L \begin{pmatrix} a & m \\ & b \end{pmatrix}, \begin{pmatrix} a & m \\ & b \end{pmatrix} \right] \\
&= \left[ \begin{pmatrix} g_A(a) + h_B(b) + k_A(m) & j_A m - m j_B \\ & h_A(a) + g_B(b) + k_B(m) \end{pmatrix}, \begin{pmatrix} a & m \\ & b \end{pmatrix} \right] \\
&= \begin{pmatrix} [g_A(a) + h_B(b) + k_A(m), a] & T \\ & [h_A(a) + g_B(b) + k_B(m), b] \end{pmatrix}
\end{aligned}$$

where $T \in M$ is to be specified presently. First, we note that the diagonal entries in the last expression are zero since $g_A$ and $g_B$ are commuting maps and the images of $h_A, h_B, k_A, k_B$ are in the centre of the corresponding algebras. Now

$$\begin{aligned} T &= (g_A(a) + h_B(b) + k_A(m))m + (j_A m - m j_B)b \\ &\quad -a(j_A m - m j_B) - m(h_A(a) + g_B(b) + k_B(m)). \end{aligned}$$

Using (i) and (ii) we get that

$$\begin{aligned} T &= (g_A(a) + h_B(b) + k_A(m))m + m g_B(b) - h_B(b)m \\ &\quad -(g_A(a)m - m h_A(a)) - m(h_A(a) + g_B(b) + k_B(m)) \\ &= k_A(m)m - m k_B(m) = 0 \text{ by (iii).} \end{aligned}$$

Thus $[L(x), x] = 0$ and so $L$ is a commuting map.

As in the previous chapter, there are appealling results for triangular algebras $\text{Tri}(A, M, B)$ when $M$ is a faithful bimodule. We start with a definition and a lemma.

**Definition 4.2.2** Let $A$ be an algebra, we denote by $[A, A]$ the linear span of all the commutators $xy - yx \in A$.

**Lemma 4.2.3** *Suppose $L$ is a commuting map on $\mathfrak{A} = \text{Tri}(A, M, B)$ with faithful $M$. Then $L$ is of the form*

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) + k_A(m) & j_A m - m j_B \\ & h_A(a) + g_B(b) + k_B(m) \end{pmatrix}$$

*with* $h_{\mathcal{A}}^{-1}(\pi_B(Z(\mathfrak{A})))$ *and* $h_B^{-1}(\pi_{\mathcal{A}}(Z(\mathfrak{A})))$ *being two-sided ideals of* $\mathcal{A}$ *and* $B$ *respectively. Furthermore, we have* $[\mathcal{A}, \mathcal{A}] \subseteq h_{\mathcal{A}}^{-1}(\pi_B(Z(\mathfrak{A})))$ *and* $[B, B] \subseteq h_B^{-1}(\pi_{\mathcal{A}}(Z(\mathfrak{A})))$.

*Proof.* We prove the part of the statement related to $\mathcal{A}$. The part related to $B$ can be proved similarly.

Let $V = \{a \in \mathcal{A} : h_{\mathcal{A}}(a) \in \pi_B(Z(\mathfrak{A}))\} = h_{\mathcal{A}}^{-1}(\pi_B(Z(\mathfrak{A})))$. For any $a, a' \in \mathcal{A}$ and $m \in M$, we have

$$a'a(j_{\mathcal{A}}m - mj_B) = g_{\mathcal{A}}(a'a)m - mh_{\mathcal{A}}(a'a) \tag{4.1}$$

$$a'a(j_{\mathcal{A}}m - mj_B) = a'(g_{\mathcal{A}}(a)m - mh_{\mathcal{A}}(a)) \tag{4.2}$$

$$aa'(j_{\mathcal{A}}m - mj_B) = g_{\mathcal{A}}(aa')m - mh_{\mathcal{A}}(aa') \tag{4.3}$$

$$a(j_{\mathcal{A}}a'm - a'mj_B) = g_{\mathcal{A}}(a)a'm - a'mh_{\mathcal{A}}(a). \tag{4.4}$$

From (4.1) and (4.2), we have

$$0 = g_{\mathcal{A}}(a'a)m - mh_{\mathcal{A}}(a'a) - a'(g_{\mathcal{A}}(a)m - mh_{\mathcal{A}}(a)), \tag{4.5}$$

and from (4.3) and (4.4), we have

$$a[a', j_{\mathcal{A}}]m = g_{\mathcal{A}}(aa')m - mh_{\mathcal{A}}(aa') - g_{\mathcal{A}}(a)a'm + a'mh_{\mathcal{A}}(a). \tag{4.6}$$

Now taking the difference of (4.5) and (4.6), we get

$$(a[a', j_A] - g_A([a, a']) - [a', g_A(a)])m = mh_A([a, a']).$$

Hence $h_A([a, a']) \in \pi_B(Z(\mathfrak{A}))$ and $[a, a'] \in V$.

Suppose that $a \in V$. From (4.5), we have

$$mh_A(a'a) = (g_A(a'a) - a'g_A(a) + a'\tau^{-1}h_A(a))m,$$

and from (4.6), we have

$$mh_A(aa') = (a[a', j_A] + g_A(aa') - g_A(a)a' + a'\tau^{-1}h_A(a))m.$$

Thus $h(a'a), h(aa') \in \pi_B(Z(\mathfrak{A}))$ by Proposition 3. Therefore $a'a, aa' \in V$. As a result, $V$ is an ideal of $A$ containing $[A, A]$. ■

Now we obtain necessary and sufficient conditions for a commuting map of Tri$(A, M, B)$ with faithful $M$ to be proper.

**Theorem 4.2.4** *Consider a commuting map $L$ on $\mathfrak{A} = $ Tri$(A, M, B)$ with faithful $M$. Write*

$$L\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} g_A(a) + h_B(b) + k_A(m) & j_Am - mj_B \\ & h_A(a) + g_B(b) + k_B(m) \end{pmatrix}.$$

*Then the following three conditions are equivalent.*

(I) *L is proper, i.e. L can be written as* $L(c) = cx + h(c)$, *where* $x \in Z(\mathfrak{A})$ *and h maps into* $Z(\mathfrak{A})$,

(II) $h_A(A) \subseteq \pi_B(Z(\mathfrak{A}))$, $h_B(B) \subseteq \pi_A(Z(\mathfrak{A}))$, *and* $k_A(m) \ominus k_B(m) \in Z(\mathfrak{A})$ *for all* $m \in M$.

(III) $j_A = g_A(1_A) \in \pi_A(Z(\mathfrak{A}))$, $j_B = h_A(1_A) \in \pi_B(Z(\mathfrak{A}))$, *and* $k_A(m) \ominus k_B(m) \in Z(\mathfrak{A})$ *for all* $m \in M$.

*Proof.* ((II)⇒(III)) $j_B = h_A(1_A) \in h_A(A) \subseteq \pi_B(Z(\mathfrak{A}))$. By Proposition 4.2.1(ii), we have $j_A m - m j_B = m g_B(1_B) - h_B(1_B) m$ which implies that

$$j_A m = m(j_B + g_B(1_B) - \tau h_B(1_B)),$$

where $\tau$ is the unique map in Theorem 1.4.4 satisfying $am = m\tau(a)$ for every $m \in M$ and $a \in \pi_A(Z(\mathfrak{A}))$. By Theorem 1.4.4, $j_A \in \pi_A(Z(\mathfrak{A}))$.

((III)⇒(I)) Let $h(c) = L(c) - cx$, where $x = (j_A - \tau^{-1}(j_B)) \oplus (\tau(j_A) - j_B) \in Z(\mathfrak{A})$. We claim that $h(\mathfrak{A}) \subseteq Z(\mathfrak{A})$. Note that

$$\begin{aligned} h\begin{pmatrix} a & m \\ & b \end{pmatrix} &= ((g_A(a) - a(j_A + \tau^{-1}(j_B)) \oplus h_A(a)) + \\ &\quad (h_B(b) \ominus (g_B(b) - (\tau(j_A) - j_B)b)) + \\ &\quad k_A(m) \ominus k_B(m). \end{aligned}$$

We have that $(g_A(a) - a(j_A + \tau^{-1}(j_B)) \oplus h_A(a) \in Z(\mathfrak{A})$ follows from

$$\begin{aligned} & (g_A(a) - a(j_A + \tau^{-1}(j_B))m - mh_A(a) \\ = \; & g_A(a)m - mh_A(a) - a(j_Am - mj_B) \\ = \; & 0 \end{aligned}$$

by Proposition 4.2.1(i), and that $h_B(b) \oplus (g_B(b) - (\tau(j_A) - j_B)b) \in Z(\mathfrak{A})$ follows from

$$\begin{aligned} & h_B(b)m - m(g_B(b) - (\tau(j_A) - j_B))b \\ = \; & (h_B(b)m - mg_B(b)) - (j_Am - mj_B)b \\ = \; & 0 \end{aligned}$$

by Proposition 4.2.1(ii), and finally we have $k_A \oplus k_B \in Z(\mathfrak{A})$ by assumption.

((I)⇒(II)) Suppose $L(c) = cx + h(c)$, where $x \in Z(\mathfrak{A})$ and $h$ maps into $Z(\mathfrak{A})$. Then by Theorem 1.4.4, $x = x_A \oplus x_B$ with $x_A \in \pi_A(Z(\mathfrak{A}))$ and $x_B \in \pi_B(Z(\mathfrak{A}))$, and $h(c) = h_1(c) \oplus h_2(c)$ with $h_1(c) \in \pi_A(Z(\mathfrak{A}))$ and $h_2(c) \in \pi_B(Z(\mathfrak{A}))$.

We first show that $h_{\mathcal{A}}(\mathcal{A}) \subseteq \pi_B(Z(\mathfrak{A}))$. For $a \in \mathcal{A}$, we have

$$\begin{aligned}
\begin{pmatrix} g_{\mathcal{A}}(a) & 0 \\ & h_{\mathcal{A}}(a) \end{pmatrix} &= L(a \oplus 0) \\
&= (a \oplus 0)(x_{\mathcal{A}} \oplus x_B) + h_1(a \oplus 0) \oplus h_2(a \oplus 0) \\
&= (ax_{\mathcal{A}} + h_1(a \oplus 0)) \oplus h_2(a \oplus 0).
\end{aligned}$$

Therefore $h_{\mathcal{A}}(a) = h_2(a \oplus 0) \in \pi_B(Z(\mathfrak{A}))$.

Similarly $h_B(B) \subseteq \pi_{\mathcal{A}}(Z(\mathfrak{A}))$.

Finally, let $c = \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}$, we have

$$\begin{pmatrix} k_{\mathcal{A}}(m) & j_{\mathcal{A}}m - mj_B \\ & k_B(m) \end{pmatrix} = L(c) = cx + h(c) = \begin{pmatrix} h_1(c) & mx_B \\ & h_2(c) \end{pmatrix}.$$

and hence

$$k_{\mathcal{A}}(m) \oplus k_B(m) = h_1(c) \oplus h_2(c) \in Z(\mathfrak{A}).$$

The proof of the theorem is now complete. ∎

We now show that the triangular algebra in Example 1.2.5 has improper commuting maps.

**Example 4.2.5** Let $\mathcal{A} = B = \left\{ \begin{pmatrix} t & a \\ & t \end{pmatrix} : t, a \in \mathbf{R} \right\}$ and $\mathcal{M} = T_2(\mathbf{R})$. The maps

$$L_1\begin{pmatrix} t & a & x & y \\ & t & & z \\ & & s & b \\ & & & s \end{pmatrix} = \begin{pmatrix} 0 & t-s & 0 & z-x \\ & 0 & & 0 \\ & & 0 & t-s \\ & & & 0 \end{pmatrix}, \text{ and}$$

$$L_2\begin{pmatrix} t & a & x & y \\ & t & & z \\ & & s & b \\ & & & s \end{pmatrix} = \begin{pmatrix} 0 & x & 0 & 0 \\ & 0 & & 0 \\ & & 0 & z \\ & & & 0 \end{pmatrix}.$$

are improper commuting maps on $\mathfrak{A} = \mathrm{Tri}(A, M, B)$.

*Proof.* That $L_1$ and $L_2$ are commuting maps can be checked by direct verification. By Corollary 1.4.6, $Z(\mathfrak{A}) = \mathbf{R}1$, so $\pi_A(Z(\mathfrak{A})) = \mathbf{R}1$ and $\pi_B(Z(\mathfrak{A})) = \mathbf{R}1$. Since For $L_1$, since $j_A = g_B(1_A) = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \notin \pi_B(Z\mathfrak{A}))$, $L_1$ is therefore improper. For $L_2$, since $k_A(m) \oplus k_B(m) \notin Z(\mathfrak{A}) = \mathbf{R}1$, $L_2$ is also improper. ∎

## 4.3 The Main Theorem

In this section, we establish sufficient conditions on triangular algebras such that every commuting map is proper.

**Theorem 4.3.1** *Every commuting map of* $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ *with faithful* $M$ *is proper if the following three conditions hold:*

(I) $Z(B) = \pi_B(Z(\mathfrak{A}))$, *or* $A$ *is the smallest ideal containing* $[A, A]$*;*

(II) $Z(A) = \pi_A(Z(\mathfrak{A}))$, *or* $B$ *is the smallest ideal containing* $[B, B]$*;*

(III) *there exists* $m_0 \in M$ *such that*

$$Z(\mathfrak{A}) = \{a \oplus b : a \in Z(A), b \in Z(B), am_0 = m_0b\}.$$

*Proof.* By Proposition 4.2.1, $h_A(A) \subseteq Z(B)$. So if $Z(B) = \pi_B(Z(\mathfrak{A}))$, then $h_A(A) \subseteq \pi_B(Z(\mathfrak{A}))$. By Lemma 4.2.3, if $A$ is the smallest ideal containing $[A, A]$, then we have $h_A^{-1}(\pi_B(Z(\mathfrak{A}))) = A$. Hence (I) implies that $h_A(A) \subseteq \pi_B(Z(\mathfrak{A}))$.

Similarly (II) implies that $h_B(B) \subseteq \pi_A(Z(\mathfrak{A}))$.

It remains to show that (III) implies $k_A(m) \oplus k_B(m) \in Z(\mathfrak{A})$. By Proposition 4.2.1(iii), we have $k_A(m) \in Z(A)$, $k_B(m) \in Z(B)$ and $k_A(m)m = mk_B(m)$. As $k_A(m_0)m_0 = m_0k_B(m_0)$, we have $k_A(m_0) \oplus k_B(m_0) \in Z(\mathfrak{A})$ and thus $k_A(m_0)m = mk_B(m_0)$ for all $m \in M$. Now

$$\begin{aligned} & k_A(m_0+m)(m_0+m) \\ = \; & k_A(m_0)m_0 + k_A(m_0)m + k_A(m)m_0 + k_A(m)m \\ = \; & m_0k_B(m_0) + mk_B(m_0) + k_A(m)m_0 + mk_B(m) \end{aligned}$$

and

$$\begin{aligned} & k_A(m_0+m)(m_0+m) \\ = \; & (m_0+m)k_B(m_0+m) \\ = \; & m_0k_B(m_0) + mk_B(m_0) + m_0k_B(m) + mk_B(m) \end{aligned}$$

imply $k_A(m)m_0 = m_0k_B(m)$. By assumption (III), we get that $k_A(m) \oplus k_B(m) \in Z(\mathfrak{A})$ as required. Now Theorem 4.2.4 implies that $L$ is proper. ■

**Corollary 4.3.2** *Suppose* $Z(A) = \mathbf{R}1$, $Z(B) = \mathbf{R}1$ *and* $M$ *is faithful. If Condition* (III) *of Theorem 4.3.1 holds, then every commuting map of* $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ *is proper, i.e. of the form* $a \to ta + h(a)1$, *where* $t \in R$ *and* $h$ *is a linear functional on* $\mathfrak{A}$.

*Proof.* By Corollary 1.4.5, we have $Z(\mathfrak{A}) = \mathbf{R}1$. Hence $Z(A) = \pi_A(Z(\mathfrak{A}))$, $Z(B) = \pi_B(Z(\mathfrak{A}))$. The result then follows Theorem 4.3.1. ■

**Corollary 4.3.3** *Given $Z(A) = \mathbf{R}1$ and $A$ is the smallest ideal containing $[A, A]$. Suppose that Condition* (III) *of Theorem 4.3.1 holds. then every commuting map of $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ is proper. i.e. of the form $a \to ta + h(a)1$. where $t \in R$ and $h$ is a linear functional on $\mathfrak{A}$.*

*Proof.* As $A$ is the smallest ideal containing $[A, A]$ and $Z(A) = \pi_A(Z(\mathfrak{A}))$, Conditions (I) and (II) of Theorem 4.3.1 hold. Hence the result follows. ■

**Corollary 4.3.4** *Let* S *be an* **R**-*algebra. Consider a triangular matrix algebra $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ where $A \subseteq M_k(\mathrm{S})$ and $B \subseteq M_l(\mathrm{S})$ are matrix algebras with $Z(A) = Z(\mathrm{S})I_k$ and $Z(B) = Z(\mathrm{S})I_k$. Assume that $M$ is a faithful bimodule satisfying Condition* (III) *of Theorem 4.3.1. Then every commuting map of* $\mathrm{Tri}(A, M, B)$ *is proper.*

*Proof.* By Theorem 1.5.3, $Z(\mathfrak{A}) = Z(\mathrm{S})I_{k+l}$. Hence Conditions (I) and (II) of Theorem 4.3.1 hold and the result follows. ■

**Corollary 4.3.5** *Let* S *be an* **R**-*algebra. Consider the triangular algebra* $\mathrm{Tri}(A, M, B) = \mathfrak{A}$, *where $\mathrm{S}I_k \subseteq A \subseteq M_k(\mathrm{S})$ and $\mathrm{S}I_l \subseteq B \subseteq M_l(\mathrm{S})$ are matrix algebras and $M \subseteq \mathrm{S}^{k,l}$. Assume that $Z(A) = Z(\mathrm{S})I_k$, $A$ is the smallest ideal containing $[A, A]$ and $M$ is a faithful bimodule satisfying Condition* (III) *of Theorem 4.3.1. Then every commuting map of* $\mathrm{Tri}(A, M, B)$ *is proper.*

*Proof.* The proof is exactly the same as Corollary 4.3.5. ■

The next proposition gives sufficient conditions for Condition (III) of Theorem 4.3.1 to be satisfied. We start with a couple of definitions.

**Definition 4.3.6** Let $M$ be an $(A, B)$-bimodule and $m_0 \in M$. The smallest $(Z(A), Z(B))$-bimodule in $M$ containing $m_0$ is denoted by $\mathcal{M}(m_0)$.

**Definition 4.3.7** Let $M$ be an $(A, B)$-bimodule. If there exists $m_0 \in M$ such that the smallest $(A, B)$-submodule of $M$ containing $m_0$ is $M$ itself, then $M$ is said to be *singly generated* and $m_0$ is a *generator*.

**Proposition 4.3.8** *Condition* (III) *of Theorem 4.3.1 holds if one of the following conditions holds.*

*(i)* $M$ *is a singly generated.*

*(ii)* $Z(A) = \mathbf{R}1$ *and there exists* $m_0 \in M$ *such that* $\mathcal{M}(m_0)$ *is a faithful* $(Z(A), Z(B))$*-bimodule.*

*(iii)* $Z(A) = \mathbf{R}1$ *and there exists* $m_0 \in M$ *such that* $m_0 b = 0$ *for any* $b \in B$ *implies that* $b = 0$.

*Proof.* We note that by Theorem 1.4.1,

$$Z(\mathfrak{A}) \subseteq \{a \oplus b : a \in Z(A), b \in Z(B), am_0 = m_0 b\}.$$

Therefore it suffices to prove the reverse inclusion under each of the conditions (i),(ii),(iii).

(i) Assume that $m_0$ is a generator for $M$. If $a \in Z(A)$ and $b \in Z(B)$ satisfy $am_0 = m_0b$, let

$$M_{a,b} = \{m \in M : am = mb\}.$$

It is straightforward to verify that $M_{a,b}$ is an $(A, B)$-submodule of $M$ containing $m_0$. Since $M$ is singly generated by $m_0$, we have $M = M_{a,b}$. Therefore $am = mb$ for every $m \in M$ and $a \oplus b \in Z(\mathfrak{A})$ by Theorem 1.4.1.

(ii) Let $\mathfrak{A}_1 = \mathrm{Tri}(Z(A), \mathcal{M}(m_0), Z(B))$. By (i), we have

$$Z(\mathfrak{A}_1) \subseteq \{a \oplus b : a \in Z(A), b \in Z(B), am_0 = m_0b\}.$$

Since we assume that $Z(A) = \mathbf{R}1$, then by Corollary 1.4.4, $Z(\mathfrak{A}_1) = \mathbf{R}1$. Thus

$$\mathbf{R}1 \subseteq Z(\mathfrak{A}) \subseteq \{a \oplus b : a \in Z(A), b \in Z(B), am_0 = m_0b\} = \mathbf{R}1$$

and the result follows.

(iii) Suppose $a \in Z(A)$ and $b \in Z(B)$ satisfy $am_0 = m_0b$. Since $Z(A) = \mathbf{R}1$, $a = r1$ for some $r \in \mathbf{R}$ and hence $m_0(r1 - b) = 0$. By the given condition, $r1 - b = 0$ or $b = r1$. Again $a \oplus b = r1 \in Z(\mathfrak{A})$. ∎

## 4.4 Matrix Algebras

We apply the corollaries in the previous section and Proposition 4.3.8 to matrix algebras.

**Corollary 4.4.1** *Given an* $\mathbf{R}$*-algebra* $\mathbf{S}$, *every commuting map of the block triangular matrix algebras* $\mathfrak{A} = T(n_1, \ldots, n_k)(\mathbf{S})$, $k > 1$, *is of the form* $a \mapsto ax + h(a)I$, *where* $x \in Z(\mathfrak{A})$ *and* $h$ *is a linear functional on* $\mathfrak{A}$.

*Proof.* By Theorem 1.5.2(ii), we have $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ where $A = M_{n_1}(\mathbf{S})$ and $B \subseteq M_s(\mathbf{S})$ is a block triangular matrix algebra and $M = \mathbf{S}^{n_1,s}$. Furthermore, by Theorem 1.5.4, $Z(A) = Z(\mathbf{S})I_r$ and $Z(B) = Z(\mathbf{S})I_s$. Next, we prove that $M = \mathbf{S}^{r,s}$ is generated by $E_{s1}$, the matrix with a 1 at the $(s, 1)$-entry and 0 elsewhere. To this end, consider any matrix unit $E_{ij} \in M$, we have $E_{ij} = e_{is}E_{s1}f_{1j}$ for corresponding matrix units $e_{is} \in A$ and $f_{1j} \in B$. Therefore $M$ is generated by $E_{s1}$. By Proposition 4.3.8(i), Condition (III) of Theorem 4.3.1 holds. The result then follows from Corollary 4.3.4. ■

**Corollary 4.4.2** *Suppose the characteristic of* $\mathbf{S}$ *is* $q > 0$. *Consider a triangular algebra* $\mathrm{Tri}(A, M, B)$ *where* $A = T(n_1, \ldots, n_k)(\mathbf{S})$ *is a block triangular algebra satisfying* $q|n_i$ *for* $1 \leq i \leq n$, $\mathbf{S}I \subseteq B \subseteq M_l(\mathbf{S})$ *is a matrix algebra and* $M \subseteq \mathbf{S}^{\sum n_i, l}$ *is singly generated and faithful, then every commuting map* *of* Tri$(A, M, B)$ *is of the form* $a \mapsto ta + h(a)I$, *where* $t \in Z(\mathbf{S})$ *and* $h$ *is a linear functional on* Tri$(A, M, B)$.

*Proof.* Consider matrices $S = S_1 \oplus \cdots \oplus S_k$ and $T = T_1 \oplus \cdots \oplus T_k$ in $A$ determined by

$$S_i(x_1, \ldots, x_{n_i})^t = (0, x_1, \ldots, x_{n_i-1})^t$$

and

$$T_i(x_1, \ldots, x_{n_i})^t = (x_2, 2x_2, \ldots, (n_i - 1)x_{n_i})^t.$$

We have $I = [T, S] \in [A, A]$ and therefore $A$ is the smallest ideal containing $[A, A]$. By Theorem 1.5.4, $Z(A) = Z(\mathbf{S})I$. The result then follows from Proposition 4.3.8(i) and Corollary 4.3.5. ■

**Corollary 4.4.3** *Given an* $\mathbf{R}$*-algebra* $\mathbf{S}$ *with* $1 \in [\mathbf{S}, \mathbf{S}]$. *Suppose that* $\mathbf{S}I_k \subseteq A \subseteq M_k(\mathbf{S})$ *and* $\mathbf{S}I_l \subseteq B \subseteq M_l(\mathbf{S})$ *are matrix algebras over* $\mathbf{S}$ *and* $M \subseteq \mathbf{S}^{k,l}$ *is singly generated and faithful, then every commuting map of* Tri$(A, M, B)$ *is of the form* $a \mapsto ta + h(a)I$, *where* $t \in Z(\mathbf{S})$ *and* $h$ *is a linear functional on* Tri$(A, M, B)$.

*Proof.* Since $1 \in [\mathbf{S}, \mathbf{S}]$, then $I \in [\mathbf{S}, \mathbf{S}]I \subseteq [A, A]$ and therefore $A$ is the smallest ideal containing $[A, A]$. By Theorem 1.5.4, $Z(A) = Z(\mathbf{S})I_k$. The result follows from Corollary 4.3.5. ■

The following corollary is concerned with an algebra discussed in Theorem 2.4.5.

**Corollary 4.4.4** *Given an* $\mathbf{R}$*-algebra* $\mathsf{S}$ *with* $1 \in [\mathsf{S}, \mathsf{S}]$. *Let* $A = D_k(\mathsf{S})$, $B = D_l(\mathsf{S})$ *and* $M = T(n_1, \ldots, n_t; m_1, \ldots, m_t)(\mathsf{S})$, *where* $n_1 + \cdots + n_t = k$ *and* $m_1 + \cdots + m_t = l$. *Then every commuting map of* $\mathrm{Tri}(A, M, B)$ *is of the form* $a \mapsto ta + h(a)I$, *where* $t \in Z(\mathsf{S})$ *and* $h$ *is a linear functional on* $\mathrm{Tri}(A, M, B)$.

*Proof.* Since $I \in [\mathsf{S}, \mathsf{S}]I$, we have $I_k \in [A, A]$ and $I_l \in [B, B]$, and thus Conditions (I) and (II) of Theorem 4.3.1 hold. Take $m_0 = (m_{ij})$ with $m_{ij} = 1$ if $i = 1$ or $j = l$, and 0 elsewhere. Then $am_0 = 0$ for $a \in A$ implies that $a = 0$ and $m_0 b = 0$ for $b \in B$ implies that $b = 0$. Hence $\mathcal{M}(m_0)$ is a faithful $(Z(A), Z(B))$-bimodule. By Proposition 4.3.8(ii), Condition (III) of Theorem 4.3.1 holds and the result follows. ∎

The following corollary is an application of Proposition 4.3.8(iii).

**Corollary 4.4.5** *Given an* $\mathbf{R}$*-algebra* $\mathsf{S}$ *with* $1 \in [\mathsf{S}, \mathsf{S}]$. *Let* $\mathsf{S}I_k \subseteq A \subseteq M_k(\mathsf{S})$ *and* $\mathsf{S}I_{2k} \subseteq B \subseteq M_{2k}(\mathsf{S})$ *be matrix algebras, and* $M = \mathsf{S}^{k,2k}$, *then every commuting map of* $\mathrm{Tri}(A, M, B)$ *is of the form* $a \mapsto ta + h(a)I$, *where* $t \in Z(\mathsf{S})$ *and* $h$ *is a linear functional on* $\mathrm{Tri}(A, M, B)$.

*Proof.* As in the proof of Corollary 4.4.3, we have $\mathcal{A}$ is the smallest ideal containing $[\mathcal{A}, \mathcal{A}]$ and $Z(\mathcal{A}) = Z(\mathrm{S})I$. Taking $[I_k I_k] \in \mathcal{M}$. We have $[I_k I_k]b = 0$ implies $b = 0$. By Proposition 4.3.8(iii), Condition (III) of Theorem 4.3.1 holds. Hence the result follows from Corollary 4.3.5. ∎

## 4.5 Nest Algebras

We extend Corollary 4.4.1 to nest algebras.

**Corollary 4.5.1** *Every commuting map of nest algebras is of the form $a \mapsto ta + h(a)1$, where $t \in \mathbb{C}$ and $h$ is a linear functional.*

*Proof.* If $\mathcal{N}$ is a trivial nest, i.e. $\mathcal{N} = \{0, H\}$, then $\mathcal{T}(\mathcal{N}) = \mathcal{B}(H)$ is a von Neumann algebra which is prime as a ring. Therefore it has no improper commuting map by Brešar's result [9]. If $\mathcal{N}$ is not trivial, then by Lemma 1.6.5, we have $\mathcal{T}(\mathcal{N}) = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ & B \end{pmatrix}$, where $\mathcal{A}$ and $B$ are nest algebras with $Z(\mathcal{A}) = \mathbb{C}1$ and $Z(B) = \mathbb{C}1$. Since $Z(\mathcal{T}(\mathcal{N})) = \mathbb{C}1$ by Lemma 1.6.6, Condition (iii) of Lemma 4.3.8 holds for any nonzero $m \in \mathcal{M}$. Thus every commuting map of $\mathcal{T}(\mathcal{N})$ is of the required form by Corollary 4.3.2. ∎

Recall (Lemma 1.6.7) that every element of the nest algebra of a continuous nest is a sum of two commutators. Applying Lemma 1.5.2 and Corollary 4.3.4, we have the following result:

**Corollary 4.5.2** *If $\mathcal{A}$ is nest algebra of a continuous nest and $\mathcal{M}$ is a faithful bimodule satisfying Condition* (III) *of Theorem 4.3.1, then every commuting map of* $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *is of the form $a \mapsto ta + h(a)1$, where $t \in \mathbb{C}$ and $h$ is a linear functional.*

The last result in this section is concerned with a finite nest over a Hilbert space $\mathbf{H}$.

**Corollary 4.5.3** *Let $\mathcal{N} = \{0 = N_0, N_1, \ldots, N_k\}$ be a nest over a Hilbert space* $\mathbf{H}$ *such that*

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k = \mathbf{H}$$

*and $N_i \ominus N_{i-1}$ is infinite-dimensional for $i = 1, \ldots, k$. If $\mathcal{A} = \mathcal{T}(\mathcal{N})$ and $\mathcal{M}$ is a faithful bimodule satisfying Condition* (III) *of Theorem 4.3.1, then every commuting map of* $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *is of the form $a \mapsto ta + h(a)1$, where $t \in \mathbb{C}$ and $h$ is a linear functional on* $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$.

*Proof.* Let $H_i = N_i \ominus N_{i-1}$. By assumption $H_i$ is an infinite-dimensional Hilbert space and so $H_i$ is isomorphic to $H_i \oplus (H_i \oplus H_i \oplus \ldots)$. We have the shift operator $S_i(x_1, x_2, \dots) = (x_2, x_3, \dots) \in B(H_i)$ and $S^* \in B$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = [S_i, S_i^*] \in [B(H_i), B(H_i)]$$

and similarly $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in [B(H_i), B(H_i)]$. Hence $1_{H_i} \in [B(H_i), B(H_i)]$ and therefore

$$1_{\mathbf{H}} \in [B(H_1), B(H_1)] \oplus \cdots \oplus [B(H_k), B(H_k)] \subseteq [\mathcal{A}, \mathcal{A}]$$

As a result $\mathcal{A}$ is the smallest ideal containing $[\mathcal{A}, \mathcal{A}]$. Since $Z(\mathcal{A}) = \mathbb{C}1$ by Lemma 1.6.6, the result follows from Corollary 4.3.3. ∎

# Chapter 5

# Automorphisms

## 5.1 Introduction

In this chapter, we are concerned with automorphisms on triangular algebras. The study of automorphisms is one of the most important ways to understand the underlying structure of algebras. One may refer to [4, 8, 12, 16, 22, 36, 41, 42, 43] for some known results about automorphisms on triangular algebras. First we recall the definitions of automorphisms and inner automorphisms. All algebras considered in this chapter are unital.

**Definition 5.1.1** Consider an algebra $\mathcal{A}$. A bijective $R$-linear map $\phi$ satisfying

$$\phi(aa') = \phi(a)\phi(a') \quad \text{for all } a, a' \in A$$

is called an *automorphism.* The set of automorphisms of $A$ is denoted by $Aut(A)$.

**Definition 5.1.2** Let $A$ be an $R$-algebra and $x \in A$ is invertible. We define automorphism $\alpha_x$ by

$$\alpha_x(a) = x^{-1}ax.$$

An automorphism which can be written as $\alpha_x$ for some $x \in A$ is said to be *inner.* The set of inner isomorphisms is denoted by $Inn(A)$. An *outer* automorphism is an automorphism which is not inner.

It is common to compare the two sets $Aut(A)$ and $Inn(A)$. Indeed we have the following well-known result.

**Lemma 5.1.3** [46, p.88] *Let $A$ be an algebra. Both $Aut(A)$ and $Inn(A)$ are groups under composition. Furthermore $Inn(A)$ is a normal subgroup of $Aut(A)$.*

The above result leads to the Skolem-Noether group defined in [36] and which may be considered as a measure of the "distance" from $Aut(A)$ to $Inn(A)$. The Skolem-Noether group will be the focus of this chapter.

**Definition 5.1.4** The *Skolem-Noether group* of an algebra $A$, denoted by $SN(A)$, is the quotient group $Aut(A)/Inn(A)$.

The Skolem-Noether group is also called the *outer automorphism group* and denoted by $Out(A)$ in some publications.

In the case of triangular algebras $\mathrm{Tri}(A, M, B)$, it would be convenient if we identify $a \in A$ with $a \oplus 0$, $b \in B$ with $0 \oplus b$ and $m \in M$ as $\begin{pmatrix} 0 & m \\ & b \end{pmatrix}$. The following observation is useful in finding outer automorphisms of triangular algebras.

**Lemma 5.1.5** *An inner automorphism $\phi = \alpha_z$ of $\mathrm{Tri}(A, M, B)$ has the property that $\phi(M) = M$, $\phi(A) \cap B = 0$ and $\phi(B) \cap A = 0$. Furthermore if $z \in A \oplus B$, then $\phi(A) = A$ and $\phi(B) = B$.*

*Proof.* Let $z = \begin{pmatrix} a_0 & m_0 \\ & b_0 \end{pmatrix}$. Then $z^{-1} = \begin{pmatrix} a_0^{-1} & n_0 \\ & b_0^{-1} \end{pmatrix}$ for some $n_0 \in M$. We have

$$\alpha_z(a \oplus 0) = \begin{pmatrix} a_0^{-1}aa_0 & a_0^{-1}am_0 \\ & 0 \end{pmatrix}$$

$$\alpha_z(0 \oplus b) = \begin{pmatrix} 0 & n_0bb_0 \\ & b_0^{-1}bb_0 \end{pmatrix}$$

$$\alpha_z\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_0^{-1}mb_0 \\ & 0 \end{pmatrix}.$$

Therefore $\alpha_z(M) = M$, $\alpha_z(A) \cap B = 0$ and $\alpha_z(B) \cap A = 0$. The last assertion follows from the fact that if $z \in A \oplus B$ then $m_0 = n_0 = 0$. ∎

We now introduce the definition of partible automorphisms.

**Definition 5.1.6** An automorphism $\phi$ of $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ is said to be *partible with respect to* $A, M, B$ if it can be written as $\phi = \alpha_y \bar{\phi}$ where $y \in \mathfrak{A}$ and $\bar{\phi}$ is an automorphism of $\mathfrak{A}$ satisfying $\bar{\phi}(A) = A$, $\bar{\phi}(M) = M$ and $\bar{\phi}(B) = B$. In the case where no confusion is likely to arise, we will simply say that $\phi$ is partible. The set of all partible automorphism of $\mathfrak{A}$ is denoted by $PA(\mathfrak{A})$.

Note that an inner automorphism of an triangular algebra is partible with respect to any possible representation of the triangular algebra.

## 5.2 Structure of Automorphisms

**Lemma 5.2.1** *Every automorphism $\phi$ of* Tri$(A, M, B)$ *can be decomposed as $\phi = \alpha_y \bar{\phi}$ with $\bar{\phi}(1_A) = e_1 \oplus e_2$ where $e_1 \in A$ and $e_2 \in B$ are idempotents.*

*Proof.* Let $\phi \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} = \begin{pmatrix} e_1 & n \\ & e_2 \end{pmatrix}$. Since $1_A = \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}$ is an idempotent, then so is $\begin{pmatrix} e_1 & n \\ & e_2 \end{pmatrix}$. Therefore

$$\begin{pmatrix} e_1 & n \\ & e_2 \end{pmatrix} = \begin{pmatrix} e_1 & n \\ & e_2 \end{pmatrix} \begin{pmatrix} e_1 & n \\ & e_2 \end{pmatrix} = \begin{pmatrix} e_1^2 & e_1 n + n e_2 \\ & e_2^2 \end{pmatrix}$$

and thus $e_1 \in A$ and $e_2 \in B$ are idempotents and $n = e_1 n + n e_2$, which in turn implies that $e_1 n e_2 = 2 e_1 n e_2$, i.e. $e_1 n e_2 = 0$. Let $y = \begin{pmatrix} 1 & ne_2 - e_1 n \\ & 1 \end{pmatrix}$ and $\bar{\phi} = \alpha_y^{-1} \phi$, then

$$\begin{aligned}\bar{\phi}(1 \oplus 0) &= y^{-1}\phi(1 \oplus 0)y \\ &= \begin{pmatrix} 1 & e_1 n - n e_2 \\ & 1 \end{pmatrix} \begin{pmatrix} e_1 & n \\ & e_2 \end{pmatrix} \begin{pmatrix} 1 & n e_2 - e_1 n \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & 2e_1 n e_2 + n - e_1 n - n e_2 \\ & e_2 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & 0 \\ & e_2 \end{pmatrix}\end{aligned}$$

as required. ∎

**Lemma 5.2.2** *Let $\phi \in Aut(\mathfrak{A})$ be such that $\phi(1_A) = e_1 \oplus e_2$. Then $\phi^{-1}(1_A) = f_1 \oplus f_2$ where $e_1, e_2, f_1, f_2$ are idempotents satisfying*

$$\phi(f_1) = e_1, \phi(f_2) = 1_A - e_1, \phi(1_A - f_1) = e_2, \phi(1_B - f_2) = 1_B - e_2.$$

*Moreover, $\mathfrak{A}$ can be partitioned into two different forms*

$$\mathfrak{A}_1 = \begin{pmatrix} f_1 A f_1 & f_1 A(1_A - f_1) & f_1 M f_2 & f_1 M(1_B - f_2) \\ & (1_A - f_1)A(1_A - f_1) & & (1_A - f_1)M(1_B - f_2) \\ & & f_2 B f_2 & f_2 B(1_B - f_2) \\ & & & (1_B - f_2)B(1_B - f_2) \end{pmatrix}$$

*and*

$$\mathfrak{A}_2 = \begin{pmatrix} e_1Ae_1 & e_1A(1_A - e_1) & e_1Me_2 & e_1M(1_B - e_2) \\ & (1_A - e_1)A(1_A - e_1) & & (1_A - e_1)M(1_B - e_2) \\ & & e_2Be_2 & e_2B(1_B - e_2) \\ & & & (1_B - e_2)B(1_B - e_2) \end{pmatrix}.$$

*Furthermore* $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ *can be written as*

$$\phi \begin{pmatrix} a_{11} & a_{12} & m_{11} & m_{12} \\ & a_{22} & & m_{22} \\ & & b_{11} & b_{12} \\ & & & b_{22} \end{pmatrix} = \begin{pmatrix} \phi(a_{11}) & \phi(m_{11}) & \phi(a_{12}) & \phi(m_{12}) \\ & \phi(b_{11}) & & \phi(b_{12}) \\ & & \phi(a_{22}) & \phi(m_{22}) \\ & & & \phi(b_{22}) \end{pmatrix}.$$

*Consequently, we have* $\phi(A) = A$ *and* $\phi(B) = B$ *if and only if* $e_1 = f_1 = 1_A$ *and* $e_2 = f_2 = 0$. *Under these condition, we also have* $\phi(M) = M$.

*Proof.* Note that $\phi(0 \oplus 1_B) = 1_{\mathfrak{A}} - \phi(1_A) = (1_A - e_1) \oplus (1_B - e_2)$. Let $\phi^{-1}(1_A) = \begin{pmatrix} f_1 & n \\ & f_2 \end{pmatrix}$, then

$$\begin{aligned}
\phi(f_1 \oplus 0) &= \phi\left(\begin{pmatrix} f_1 & n \\ & f_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\right) \\
&= \phi\begin{pmatrix} f_1 & n \\ & f_2 \end{pmatrix}\phi\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} \\
&= (1_A \oplus 0)(e_1 \oplus e_2) \\
&= e_1 \oplus 0.
\end{aligned}$$

thus $\phi(f_1) = e_1$ and $\phi(1_A - f_1) = \phi(1_A) - \phi(f_1) = e_2$.

Similarly,

$$\begin{aligned}
\phi(0 \oplus f_2) &= \phi\left(\begin{pmatrix} f_1 & n \\ & f_2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ & 1 \end{pmatrix}\right) \\
&= \phi\begin{pmatrix} f_1 & n \\ & f_2 \end{pmatrix}\phi\begin{pmatrix} 0 & 0 \\ & 1 \end{pmatrix} \\
&= (1_A \oplus 0)((1_A - e_1) \oplus (1_B - e_2)) \\
&= (1_A - e_1) \oplus 0.
\end{aligned}$$

thus $\phi(f_2) = 1_A - e_1$ and $\phi(1_B - f_2) = \phi(1_B) - \phi(f_2) = 1_B - e_2$.

As $\phi(f_1 \oplus f_2) = 1_A = \phi\begin{pmatrix} f_1 & n \\ & f_2 \end{pmatrix}$, we have $n = 0$.

The fact that $e_1, e_2, f_1, f_2$ being idempotents is trivial.

Consider

$$\begin{aligned}
(1_A - e_1)Ae_1 &= (1_A - e_1)\mathfrak{A}e_1 = \phi(f_2)\phi(\mathfrak{A})\phi(f_1) \\
&= \phi(f_2\mathfrak{A}f_1) = 0. \\
(1_A - e_1)Me_2 &= (1_A - e_1)\mathfrak{A}e_2 = \phi(f_2)\phi(\mathfrak{A})\phi(1_A - f_1) \\
&= \phi(f_2\mathfrak{A}(1_A - f_1)) = 0, \quad \text{and} \\
(1_B - e_2)Be_2 &= (1_B - e_2)\mathfrak{A}e_2 = \phi(1_B - f_2)\phi(\mathfrak{A})\phi(1_A - f_1) \\
&= \phi((1_B - f_2)\mathfrak{A}(1_A - f_1) = 0.
\end{aligned}$$

We may therefore write $A$ as $\begin{pmatrix} e_1Ae_1 & e_1A(1-e_1) \\ & (1-e_1)A(1-e_1) \end{pmatrix}$ with similar partitions for $B$ and $M$. Hence $\mathfrak{A}$ have the form $\mathfrak{A}_2$. Applying a similar argument to $\phi^{-1}$, we have the other form $\mathfrak{A}_1$.

Now we want to show that $\phi$ can be written as required.

$$\begin{aligned}
\phi(f_1Af_1) &= \phi(f_1\mathfrak{A}f_1) = \phi(f_1)\phi(\mathfrak{A})\phi(f_1) \\
&= e_1\mathfrak{A}e_1 = e_1Ae_1 \\
\phi(f_1A(1_A - f_1)) &= \phi(f_1\mathfrak{A}(1_A - f_1)) = \phi(f_1)\phi(\mathfrak{A})\phi(1_A - f_1) \\
&= e_1\mathfrak{A}e_2 = e_1Me_2
\end{aligned}$$

$$
\begin{aligned}
\phi((1_A - f_1)A(1_A - f_1)) &= \phi((1_A - f_1)\mathfrak{A}(1_A - f_1)) \\
&= \phi(1_A - f_1)\phi(\mathfrak{A})\phi(1_A - f_1) \\
&= e_2\mathfrak{A}e_2 = e_2 B e_2 \\
\phi(f_1 M f_2) &= \phi(f_1\mathfrak{A}f_2) = \phi(f_1)\phi(\mathfrak{A})\phi(f_2) \\
&= e_1\mathfrak{A}(1_A - e_1) = e_1 A(1_A - e_1) \\
\phi(f_1 M(1_B - f_2)) &= \phi(f_1\mathfrak{A}(1_B - f_2)) = \phi(f_1)\phi(\mathfrak{A})\phi(1_B - f_2) \\
&= e_1\mathfrak{A}(1_B - e_2) = e_1 M(1_B - e_2) \\
\phi((1_A - f_1)M(1_B - f_2)) &= \phi((1_A - f_1)\mathfrak{A}(1_B - f_2)) \\
&= \phi(1_A - f_1)\phi(\mathfrak{A})\phi(1_B - f_2) \\
&= e_2\mathfrak{A}(1_B - e_2) = e_2 B(1_B - e_2) \\
\phi(f_2 B f_2) &= \phi(f_2\mathfrak{A}f_2) = \phi(f_2)\phi(\mathfrak{A})\phi(f_2) \\
&= (1_A - e_1)\mathfrak{A}(1_A - e_1) \\
&= (1_A - e_1)A(1_A - e_1) \\
\phi(f_2 B(1_B - f_2) &= \phi(f_2\mathfrak{A}(1_B - f_2) = \phi(f_2)\phi(\mathfrak{A})\phi(1_B - f_2) \\
&= (1_A - e_1)\mathfrak{A}(1_B - e_2) \\
&= (1_A - e_1)M(1_B - e_2)
\end{aligned}
$$

$$\begin{aligned}\phi((1_B - f_2)B(1_B - f_2)) &= \phi((1_B - f_2)\mathfrak{A}(1_B - f_2)) \\ &= \phi(1_B - f_2)\phi(\mathfrak{A})\phi(1_B - f_2) \\ &= (1_B - e_2)\mathfrak{A}(1_B - e_2) \\ &= (1_B - e_2)B(1_B - e_2).\end{aligned}$$

Finally $\phi(A) = A$ is equivalent to $f_1Af_1 = A = e_1Ae_1$, i.e. $f_1 = e_1 = 1_A$; and $\phi(B) = B$ is equivalent to $(1_B - f_2)B(1_B - f_2) = B = (1_B - e_2)B(1_B - e_2)$, i.e. $f_2 = e_2 = 0$. Under this condition,

$$f_1Mf_2 = (1_A - f_1)M(1_B - f_2) = f_1Mf_2 = (1_A - f_1)M(1_B - f_2) = 0$$

and thus $\phi(M) = M$. ■

We recall that $A$ and $B$ are naturally identified as subalgebras of the triangular algebra $\mathrm{Tri}(A, M, B)$. This is used implicitly in the statement of the next theorem.

**Theorem 5.2.3** *Let $\phi$ be an automorphism of* $\mathrm{Tri}(A, M, B)$. *Then $\phi$ is partible if and only if $\phi(A) \cap B = 0$ and $A \cap \phi(B) = 0$. When this is the case, then we also have $\phi(M) = M$ and so $\phi$ may be written in the form*

$$\phi\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_A(a) & t_A(a) + t_B(b) + f_M(m) \\ & p_B(b) \end{pmatrix}.$$

*The element $y$ implementing the inner automorphism $\alpha_y$ in the decomposition $\phi = \alpha_y\bar{\phi}$ of Definition 5.1.6 may be taken to be $y = \phi(1_A) + 1_B$.*

*Proof.* First assume that $\phi$ is partible, so $\phi = \alpha_y\rho$ for some invertible $y \in \mathfrak{A}$ and automorphism $\rho$ on $\mathfrak{A}$ satisfying $\rho(A) = A$, $\rho(M) = M$, $\rho(B) = B$. By Lemma 5.1.5, we have $\alpha_y(A) \cap B = 0$ and since $\rho(A) = A$, we have $\phi(A) \cap B = 0$. Similarly $A \cap \phi(B) = 0$. We also have $\alpha_y(M) = M$ and hence $\phi(M) = M$.

To prove the converse, assume that $\phi(A) \cap B = 0 = A \cap \phi(B)$. Then we have $\phi(1_A) = \begin{pmatrix} e & n \\ & 0 \end{pmatrix}$ and $\phi(1_B) = \begin{pmatrix} 0 & n' \\ & f \end{pmatrix}$. Since $1 = \phi(1) = \phi(1_A) + \phi(1_B)$, we get that $e = 1_A$, $f = 1_B$ and $n' = -n$, so $\phi(1_A) = \begin{pmatrix} 1 & n \\ & 0 \end{pmatrix}$ and $\phi(1_B) = \begin{pmatrix} 0 & -n \\ & 1 \end{pmatrix}$. Let $y = \phi(1_A) + 1_B = \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}$ and set $\bar{\phi} = \alpha_y^{-1}\phi$, we will show that $\bar{\phi}(A) = A$, $\bar{\phi}(B) = B$ and $\bar{\phi}(M) = M$ proving the converse as well as the assertion in the last sentence of the statement of the theorem.

Now

$$\begin{aligned}\bar{\phi}(1_A) &= y\phi(1_A)y^{-1}\\ &= \begin{pmatrix}1 & n\\ & 1\end{pmatrix}\begin{pmatrix}1 & n\\ & 0\end{pmatrix}\begin{pmatrix}1 & -n\\ & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ & 0\end{pmatrix} = 1_A.\end{aligned}$$

Similarly $\bar{\phi}(1_B) = 1_B$. By the last assertion of Lemma 5.2.2, we get that $\bar{\phi}(A) = A$, $\bar{\phi}(B) = B$ and $\bar{\phi}(M) = M$ as required. ∎

We now give a well-known example of an automorphism which is not partible (and thus is an outer automorphism).

**Example 5.2.4** [42] Let $A = B = M = T_2(\mathbf{R})$. Construct an automorphism $L$ on $\mathrm{Tri}(A, M, B)$ by

$$L\begin{pmatrix}a & b & c & d\\ & e & & f\\ & & g & h\\ & & & i\end{pmatrix} = \begin{pmatrix}a & c & b & d\\ & g & & h\\ & & e & f\\ & & & i\end{pmatrix}.$$

Then $L$ is an outer automorphism.

*Proof.* First we notice that $L$ is an automorphism, indeed $L(C) = Y'^{-1}CY'$ where

$$Y' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that for $0 \oplus e \in A$, we have $L(0 \oplus e) = e \oplus 0 \in B$. Therefore $L(A) \cap B \neq 0$. By Theorem 5.2.3, $L$ is not partible and hence is an outer automorphism.

**Definition 5.2.5** A bijective **R**-linear map $f$ on $M$ is said to be a *bimodule automorphism* if for any $a \in A$, $m \in M$ and $b \in B$,

$$f(amb) = af(m)b.$$

The group of all bimodule automorphisms of $M$ under composition is denoted by $Aut(M)$.

**Lemma 5.2.6** *Suppose that $\phi$ is an automorphism on* Tri$(A, M, B)$ *satisfying $\phi(A) = A$, $\phi(B) = B$ and $\phi(M) = M$. Let $f = \pi_M \phi|_M$. If $\phi|_A = \alpha_x$ and $\phi|_B = \alpha_y$ are inner, then the map $g(m) = xf(m)y$ is a bimodule automorphism. Conversely, if $M$ is a faithful $(A, B)$-bimodule and there exists invertible $x \in A$ and $y \in B$ such that the map $g(m) = xf(m)y$ is a bimodule automorphism, then $\phi|_A = \alpha_x$ and $\phi|_B = \alpha_y$.*

*Proof.* Suppose $\phi\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} x^{-1}ax & f(m) \\ & y^{-1}by \end{pmatrix}$. Then

$$\begin{aligned}
\begin{pmatrix} 0 & f(amb) \\ & 0 \end{pmatrix} &= \phi\begin{pmatrix} 0 & amb \\ & b \end{pmatrix} \\
&= \phi\left(\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\right) \\
&= \begin{pmatrix} x^{-1}ax & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & f(m) \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ & y^{-1}by \end{pmatrix}
\end{aligned}$$

Hence $f(amb) = x^{-1}axf(m)y^{-1}by$, i.e. $xf(amb)y^{-1} = axf(m)y^{-1}b$ and therefore $g(m) = xf(m)y^{-1}$ is a bimodule automorphism of $M$. Suppose $M$ is a faithful bimodule and the map $g(m) = xf(m)y^{-1}$ is a bimodule automorphism. Then setting $\phi_1 = \phi|_A$, we have

$$\begin{aligned}
\begin{pmatrix} 0 & \phi_1(a)m \\ & 0 \end{pmatrix} &= \begin{pmatrix} \phi_1(a) & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \\
&= \phi\left(\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & f^{-1}(m) \\ & 0 \end{pmatrix}\right) \\
&= \phi\begin{pmatrix} 0 & af^{-1}(m) \\ & 0 \end{pmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}\phi_1(a)m &= f(af^{-1}(m)) = x^{-1}g(af^{-1}(m))y \\ &= x^{-1}ag(f^{-1}(m))y = x^{-1}a(xf(f^{-1}(m))y^{-1})y \\ &= x^{-1}axm\end{aligned}$$

and thus $\phi_1(a) = xax^{-1}$ as $M$ is faithful. Similarly $\phi_2(b) = y^{-1}by$ where $\phi_2 = \phi|_B$. ■

## 5.3 The Main Theorems

First of all, we will find sufficient conditions on $A$, $M$ and $B$ so that all automorphisms of $\text{Tri}(A, M, B)$ are partible with respect to $A, M, B$. We start with some definitions.

**Definition 5.3.1** [28] A left $A$-module (or a right $B$-module or an $(A, B)$-bimodule) $M$ is *artinian* if every decreasing chain of submodules of $M$, say

$$M \supseteq M_1 \supseteq M_2 \supseteq \ldots.$$

is eventually constant, i.e. there exists $k$ such that $M_l = M_k$ whenever $k \geq l$.

An algebra $A$ is called *left artinian* (respectively *right artinian*) if $A$ is artinian as a left (respectively right) $A$-module.

**Theorem 5.3.2** *All automorphisms of* Tri$(A, M, B)$ *are partible if any one of the conditions below holds:*

(I) $M$ *is a faithful right* $B$*-module and* $A$ *is not a triangular algebra, i.e.,* $eA(1-e) = 0$ *implies that* $(1-e)Ae = 0$.

(II) $M$ *is a faithful left* $A$*-module and* $B$ *is not a triangular algebra.*

(III) $A$ *is right artinian and* $M$ *is "idempotent faithful", i.e. if* $e \in A, f \in B$ *are idempotents such that* $eMf = 0$, *then* $e = 0$ *or* $f = 0$.

(IV) $B$ *is left artinian and* $M$ *is idempotent faithful.*

(V) $M$ *is idempotent faithful and is an artinian* $(A, B)$*-bimodule.*

*Proof.* By Lemma 5.2.1, we may assume that $\phi(1_A \oplus 0) = e_1 \oplus e_2$. By Lemma 5.2.2, we have $\phi^{-1}(0 \oplus 1_B) = f_1 \oplus f_2$. In view of the last assertion of Lemma 5.2.2, it suffices to prove that $e_1 = f_1 = 1_A$ and $e_2 = f_2 = 0$.

(I) By Lemma 5.2.2, $A = \begin{pmatrix} f_1\mathfrak{A}f_1 & f_1\mathfrak{A}(1_A - f_1) \\ & (1_A - f_1)\mathfrak{A}(1_A - f_1) \end{pmatrix}$. Since $A$ is not a triangular algebra, then we must have $f_1\mathfrak{A}(1_A - f_1) = 0$ and then $e_1\mathfrak{A}e_2 = \phi((1_A - f_1)\mathfrak{A}f_1) = 0$. As $M$ is a faithful right $B$-module and $Me_2 = e_1Me_2 = e_1\mathfrak{A}e_2 = 0$, we have $e_2 = 0$ and then $f_1 = 1_A - \phi^{-1}(e_2) = 1_A$. Similarly $f_2 = 0$ and $e_1 = 1_A$ by considering $\phi^{-1}$.

(II) The argument is similar to (I).

(III) Write $e_1^1 = e_1$. Now $(1 - e_1)Me_2 = 0$, hence $e_1 = 1$ or $e_2 = 0$. Suppose $e_2 = 0$. Then $\phi(1_A) = e_1^1$. Define $e_1^k = \phi(e_1^{k-1}) = \phi^k(1_A)$, we have a descending sequences of right ideals

$$A \supseteq e_1^1 A e_1^1 \supseteq e_1^2 A e_1^2 \supseteq \cdots .$$

Since $A$ is right artinian and $\phi$ is an isomorphism, we conclude that $e_1 = 1_A$. So we always have $e_1 = 1_A$. Similarly, by considering $\phi^{-1}$, we have $f_1 = 1_A$. Finally we get $f_2 = \phi^{-1}(1_A - e_1) = 0$ and $e_2 = \phi(1_A - f_1) = 0$.

(IV) The argument is similar to (III).

(V) Write $e_1^1 = e_1$. Now $(1 - e_1)Me_2 = 0$, hence $e_1 = 1$ or $e_2 = 0$. Suppose $e_2 = 0$. Then $\phi(1_A) = e_1^1$. Define $e_1^k = \phi(e_1^{k-1}) = \bar{\phi}^k(1_A)$, we have a descending sequences of bimodules

$$M \supseteq e_1^1 M \supseteq e_1^2 M \supseteq \cdots .$$

Since $M$ is artinian and $\phi$ is an isomorphism, we conclude that $e_1 = 1$. Following the same argument as (III), we have $e_1 = f_1 = 0$ and $e_2 = f_2 = 0$.

■

Next we will determine the Skolem-Noether groups. We start with a lemma and a definition.

**Lemma 5.3.3** *Suppose every automorphism of* $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ *is partible. Define* $\tilde{\pi}_A : SN(\mathfrak{A}) \to SN(A)$ *by*

$$\tilde{\pi}_A([\phi]) = [\pi_A \bar{\phi}|_A]$$

*where* $\bar{\phi} = \alpha_y \phi$ *satisfies* $\bar{\phi}(A) = A$, $\bar{\phi}(B) = B$ *and* $\bar{\phi}(M) = M$. *Then* $\tilde{\pi}_A$ *is a well-defined homomorphism.*

*Proof.* Let $\phi \in Aut(\mathfrak{A})$. Since $\phi$ is partible, such $\bar{\phi}$ exists. Suppose there exists another $\rho = \alpha_z \phi$ satisfying $\rho(A) = A$, $\rho(B) = B$ and $\rho(M) = M$. Then write $y = \begin{pmatrix} a_1 & m_1 \\ & b_1 \end{pmatrix}$ and $z = \begin{pmatrix} a_2 & m_2 \\ & b_2 \end{pmatrix}$. We have $\bar{\phi}|_A = \alpha_{a_1^{-1}a_2} \rho|_A$ and therefore $[\bar{\phi}|_A] = [\rho|_A]$. Hence $\tilde{\pi}_A$ is well-defined.

Consider two automorphisms $\alpha_{x_1}\phi_1$ and $\alpha_{x_2}\phi_2$ on $\mathfrak{A}$ satisfying $\phi_i(A) = A$, $\phi_i(B) = B$ and $\phi_i(M) = M$ for $i = 1, 2$. Let $x_i = \begin{pmatrix} a_i & m_i \\ & b_i \end{pmatrix}$. We have

$$\alpha_{x_1}\phi_1\alpha_{x_2}\phi_2 = (\alpha_{x_1}\phi_1\alpha_{x_2}\phi_1^{-1})(\phi_1\phi_2) = \alpha_{x_1\phi_1(x_2)}(\phi_1\phi_2)$$

and therefore

$$\begin{aligned} [\pi_A(\alpha_{x_1}\phi_1\alpha_{x_2}\phi_2)|_A] &= [\pi_A(\phi_1\phi_2)|_A] \\ &= [\pi_A\phi_1|_A][\pi_A\phi_2|_A] \end{aligned}$$

as desired. ∎

**Definition 5.3.4** For every pair of invertible elements $a \in Z(A)$ and $b \in Z(B)$, we define a bimodule-automorphism on $M$ by

$$l_{a,b}(m) = amb \quad \text{for all } m \in M.$$

The group of all automorphisms of the form $l_{a,b}$ is denoted by $L_0(M)$.

We also denote by $OutAut(M)$ the quotient group $Aut(M)/L_0(M)$.

The following theorem, together with Theorem 5.3.2, help to determine the Skolem-Noether groups for many triangular algebras. One may refer to Chapter 2.3 for the facts and definitions about short exact sequences.

**Theorem 5.3.5** *Given a triangular algebra $\mathfrak{A} = \text{Tri}(A, M, B)$. Suppose that every automorphism of $\mathfrak{A}$ is partible. Define a map $h : Aut(M) \to Aut(\mathfrak{A})$ by*

$$h(f)\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} a & f(m) \\ & b \end{pmatrix}.$$

*We have*

*(i) The map $\Psi : OutAut(M) \to SN(\mathfrak{A})$ defined by*

$$\Psi([f]) = [h(f)]$$

*is a well-defined injective group homomorphism.*

*(ii) If, for every automorphism $\varphi$ that satisfies $\varphi(A) = A$, $\varphi(B) = B$ and $\varphi(M) = M$, we have $p_B = \pi_B\varphi|_B$ is inner whenever $p_A = \pi_A\varphi|_A$ is inner then*

$$0 \to OutAut(M) \xrightarrow{\Psi} SN(\mathfrak{A}) \xrightarrow{\tilde{\pi}_A} \tilde{\pi}_A(SN(\mathfrak{A})) \to 0 \tag{5.1}$$

*is a short exact sequence, where $\tilde{\pi}_A$ is the map given in Lemma 5.3.3.*

*Proof.* (i) First we show that $\Psi$ is well-defined. Suppose that $[f] = [g]$ then $fg^{-1} = l_{x,y} \in L_0(M)$ and $h(f)h(g)^{-1} = h(fg^{-1}) = \alpha_{x\oplus y^{-1}}$. Therefore $[h(f)] = [h(g)]$.

To prove injectivity, assume that $\Psi([f]) = [id_{\mathfrak{A}}]$. Then $h(f) = \alpha_z$ where $z = \begin{pmatrix} x_1 & m_1 \\ & y_1 \end{pmatrix}$. We have, for every $a \in A$,

$$h(f)(a \oplus 0) = \begin{pmatrix} x^{-1}ax & x^{-1}am_1 \\ & 0 \end{pmatrix}$$

and, by the definition of $h$,

$$h(f)(a \oplus 0) = a \oplus 0.$$

Therefore $x_1 \in Z(A)$ and $m_1 = 0$ by taking $a = x_1^{-1}$. Similarly we have $y_1 \in Z(B)$. Finally

$$h(f)\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_1^{-1}my_1 \\ & 0 \end{pmatrix}$$

and then, by the definition of $h$, we get $f(m) = x_1^{-1}my_1$. Thus $f = l_{x,y} \in L_0(M)$. As a result, $[f] = id_M$ and $\Psi$ is injective.

(ii) Suppose that for every automorphism $\phi$ that satisfies $\phi(A) = A$, $\phi(B) = B$ and $\phi(M) = M$, we have $p_B$ is inner whenever $p_A$ is inner. We claim that $Ker(\tilde{\pi}_A) = Im(\Psi)$ and the result follows. To verify the claim, suppose $\tilde{\pi}_A([\phi]) = [id_A]$. By Lemma 5.3.3, we can assume $\phi(A) = A$, $\phi(M) = M$ and $\phi(B) = B$. Thus $[\pi_A\phi|A] = [id_A]$, i.e., $p_A = \pi_A\phi|_A$ is inner. By assumption $p_B = \pi_B\phi|_B$ is also inner. Write $p_A = \alpha_x$ and $p_B = \alpha_y$. Then $g \in Aut(M)$ where $g(m) = x\phi|_M(m)y^{-1}$ by Lemma 5.2.6. Hence $\phi = \alpha_{x\oplus y}h(g)$ and thus $[\phi] = \Psi([g])$. Therefore we have $Ker(\tilde{\pi}_A) \subseteq Im(\Psi)$. The reverse inclusion is trivial since the $(1,1)$-entry of $h(f) = id_A$, and so $\tilde{\pi}_A(\Psi(f)) = \tilde{\pi}_A([h(f)]) = id_A$. ■

**Corollary 5.3.6** *Let* $\mathfrak{A} = \mathrm{Tri}(A, M, B)$. *Suppose that* $Aut(\mathfrak{A}) = PA(\mathfrak{A})$ *and that for every automorphism* $\phi$ *of the form* $\phi\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} a & f(m) \\ & p_B(b) \end{pmatrix}$, *we* *have $p_B$ is inner. Then*

$$\{1\} \to OutAut(M) \xrightarrow{\Phi} SN(\mathfrak{A}) \xrightarrow{\hat{\pi}_A} \hat{\pi}_A(SN(\mathfrak{A})) \to \{1\} \qquad (5.2)$$

*is a short exact sequence.*

*Proof.* Consider an automorphism $\phi$ on Tri$(A, M, B)$ with $\phi(A) = A$, $\phi(B) = B$ and $\phi(M) = M$. Suppose $\pi_A\phi|_A$ is inner, say $\pi_A\phi|_A = \alpha_x$, then $\alpha_x^{-1}\phi \in Aut(\text{Tri}(A, M, B))$ is of the form

$$\alpha_x^{-1}\phi\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} a & f(m) \\ & \phi|_B(b) \end{pmatrix}.$$

Therefore $\phi|_B$ is inner. Thus by Theorem 5.3.5(ii), the result follows.

**Corollary 5.3.7** *Assume that $Aut(A) = Inn(A)$ and $Aut(B) = Inn(B)$. If $Aut(\text{Tri}(A, M, B)) = PA(\text{Tri}(A, M, B))$, then*

$$SN(\text{Tri}(A, M, B)) = OutAut(M).$$

*Proof.* In this case, the sequence in Theorem 5.3.5(ii) is a short exact sequence. Consider $\rho : SN(A) \to SN(\mathfrak{A})$ defined by $\rho([\alpha_a]) = \alpha_{a\dot{+}1}$, we have $\hat{\phi}_A\rho = id_{SN(A)}$. The result follows from Lemma 2.3.3. ∎

The following corollaries are first obtained by Jøndrup [40] in the case that **R** is a commutative field. Note that **R** is not a triangular algebra as it is commutative.

**Corollary 5.3.8**

$$SN(\mathrm{Tri}(\mathbf{R}, \mathbf{R}^n, \mathbf{R})) = GL_n(\mathbf{R}),$$

*the group of invertible $n \times n$ matrices over* **R**.

*Proof.* By Theorem 5.3.2(ii), every automorphism of $\mathfrak{A} = \mathrm{Tri}(\mathbf{R}, \mathbf{R}^n, \mathbf{R})$ is partible. By Corollary 5.3.7, $SN(\mathfrak{A}) = OutAut(M)$. That $OutAut(M) = GL_n(\mathbf{R})$ follows from the fact that $Aut(M) = GL_n(\mathbf{R})$ and $L_0(M) = \{id_M\}$.

■

In the next corollary, the ring of polynomials in one indeterminate over **R** is denoted as usual by $\mathbf{R}[X]$.

**Corollary 5.3.9**

$$SN(\mathrm{Tri}(\mathbf{R}[X], \mathbf{R}[X], \mathbf{R})) = SN(\mathbf{R}[X]).$$

*Proof.* Every **R**-automorphism $\rho$ of $A = \mathbf{R}[X]$ induces an **R**-automorphism $D(\rho)$ of $\mathrm{Tri}(\mathbf{R}[X], \mathbf{R}[X], \mathbf{R})$ by

$$D(\rho)\begin{pmatrix} F(X) & G(X) \\ & r \end{pmatrix} = \begin{pmatrix} \rho(F(X)) & \rho(G(X)) \\ & r \end{pmatrix}.$$

As the only **R**-automorphism of $B = \mathbf{R}$ is the identity, the condition of Theorem 5.3.5(ii) is satisfied and hence the sequence in Theorem 5.3.5(ii) is a short exact sequence. Moreover the sequence splits since $\tilde{\pi}_A D$ is the identity map on $SN(\mathbf{R}[X])$. For every $(\mathbf{R}[X], \mathbf{R})$-automorphism $f$ on $M = \mathbf{R}[X]$, we have $f(G(X)) = f(G(X)1) = G(X)f(1) = f(1)G(X)$. Since $f$ is bijective, $f(1) \in A$ is invertible and $f \in L_0(M)$. Therefore $OutAut(\mathbf{R}[X]) = [id_M]$ and the result follows from Lemma 2.3.3. ∎

In the next corollary, we use $\mathbf{R}[X, Y] = (\mathbf{R}[X])[Y]$ to denote the ring of polynomials in two indeterminates.

**Corollary 5.3.10**

$$SN(\text{Tri}(\mathbf{R}[X], \mathbf{R}[X, Y], \mathbf{R}[Y])) = SN(\mathbf{R}[X]) \times SN(\mathbf{R}[Y]).$$

*Proof.* Note that $\mathbf{R}[X]$ is commutative and hence is not a triangular algebra, and then every automorphism of $\mathfrak{A} = \text{Tri}(\mathbf{R}[X], \mathbf{R}[X, Y], \mathbf{R}[Y])$ is partible by Theorem 5.3.2(i). Similar to the construction of $\tilde{\pi}_A$, we have $\tilde{\pi}_B : SN(\mathfrak{A}) \to SN(B)$. We claim that the map $\theta : SN(\mathfrak{A}) \to SN(A) \times SN(B)$ defined by $\theta([\phi]) = (\tilde{\pi}_A([\phi]), \tilde{\pi}_B([\phi]))$ is a group automorphism. That $\theta$ is a group homomorphism follows from the fact that $\tilde{\pi}_A$ and $\tilde{\pi}_B$ are group homomorphisms.

Next we prove that $\theta$ is surjective. Let $p_A$ and $p_B$ be $\mathbf{R}$-automorphisms on $A$ and $B$ respectively. Define $f : M \to M$ by $f(rX^sY^t) = rp_A(X^s)p_A(Y^t)$.

It is straightforward to verify that the map $\phi \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_A(a) & f(m) \\ & p_B(b) \end{pmatrix}$ is an **R**-automorphism on $\mathfrak{A}$.

It remains to show that $\theta$ is injective. Suppose $[\phi] \in Ker(\theta)$. Without loss of generality, we can assume that $\phi$ is of the form $\phi \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} \alpha_g(a) & f(m) \\ & \alpha_h(b) \end{pmatrix}$. Since $\mathbf{R}[X]$ and $\mathbf{R}[Y]$ are commutative, $\alpha_g$ and $\alpha_h$ are identity maps. Now for every $rX^sY^t \in M$, we have

$$\begin{aligned}
\begin{pmatrix} 0 & f(rX^sY^t) \\ & 0 \end{pmatrix} &= \phi \begin{pmatrix} 0 & rX^sY^t \\ & 0 \end{pmatrix} \\
&= r\phi \begin{pmatrix} X^s & 0 \\ & 0 \end{pmatrix} \phi \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \phi \begin{pmatrix} 0 & 0 \\ & Y^t \end{pmatrix} \\
&= r \begin{pmatrix} X^s & 0 \\ & 0 \end{pmatrix} \begin{pmatrix} 0 & f(1) \\ & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ & Y^t \end{pmatrix} \\
&= \begin{pmatrix} 0 & f(1)rX^sY^t \\ & 0 \end{pmatrix}.
\end{aligned}$$

$f(rX^sY^t) = f(1)rX^sY^t$. Thus $f(m) = f(1)m$ for every $m \in M$ and that $\phi = \alpha_z$ where $z = \begin{pmatrix} f(1) & 0 \\ & \ddots \\ & & 1 \end{pmatrix}$, i.e. $[\phi] = [id_{\mathfrak{A}}]$ as desired. ■

## 5.4 Matrix Algebras

In this section, we use **S** to denote an algebra over **R**. We want to find the Skolem-Noether groups of some matrix algebras over **S**. First, we consider the **R**-automorphisms of certain matrix algebras induced by **R**-automorphisms of **S**.

**Proposition 5.4.1** *Given an* **R**-*automorphism* $d$ *on* **S**, *it induces an* **R**-*automorphism* $\hat{d}$ *on any unital matrix algebras* $\mathfrak{A} \supseteq D_n(\mathbf{S})$ *by* $\hat{d}(a_{ij}) = (d(a_{ij}))$. *Consequently the map* $[d] \mapsto [\hat{d}]$ *is an embedding of* $SN(\mathbf{S})$ *into* $SN(\mathfrak{A})$.

*Proof.* Let $X = (x_{ij}), Y = (y_{ij}) \in \mathfrak{A}$. Then

$$\begin{aligned}\hat{d}(XY) &= (d(\sum_{k=1}^{n} x_{ik}y_{kl}))_{i=1,\dots,n;j=1,\dots,n} \\ &= (\sum_{k=1}^{n} d(x_{ik})d(y_{kl}))_{i=1,\dots,n;j=1,\dots,n} \\ &= \hat{d}(X)\hat{d}(Y).\end{aligned}$$

Hence $\hat{d}$ is an **R**-automorphism of $\mathfrak{A}$.

The map $D : [d] \mapsto [\hat{d}]$ is well-defined since if $d = \alpha_s$, then $\hat{d} = \alpha_{sI}$. If $\hat{d} = \alpha_z$ where $z = (z_{ij}) \in \mathfrak{A}$, then

$$d(s)E_{11} = \hat{d}(sE_{11}) = z^{-1}(sE_{11})z.$$

Suppose $w = (w_{ij}) = z^{-1}$, we have, by considering the $(1,1)$-entry, $d(s) = w_{11}sz_{11}$. Since $d$ is an $\mathbf{R}$-automorphism, $w_{11} = z_{11}^{-1}$ and $d = \alpha_{z_{11}}$ is inner. Therefore the map $D$ is injective. ∎

The $\mathbf{R}$-automorphism of $\mathfrak{A}$ given in the above proposition will be called the *induced* $\mathbf{R}$*-automorphism by* $d$.

Automorphisms of upper triangular matrix algebras have been studied in [2, 4, 12, 16, 42, 43]. The following theorem is a further generalization of their results.

**Theorem 5.4.2** *Assume that* $\mathbf{S}$ *is not a triangular algebra. Suppose that* $\mathfrak{A} = \mathrm{Tri}(A, M, B)$ *with* $A = M_k(\mathbf{S})$, $M = \mathbf{S}^{k,l}$, *and* $B \supseteq D_l(\mathbf{S})$ *is a matrix algebra. If every* $\mathbf{R}$*-automorphism of* $B$ *is a composition of an inner automorphism and an induced* $\mathbf{R}$*-automorphism from* $\mathbf{S}$, *then we have a short exact sequence*

$$0 \to OutAut(M) \to SN(\mathfrak{A}) \xrightarrow{\hat{\pi}} \hat{\pi}(SN(\mathfrak{A})) \to 0$$

*and that* $OutAut(M) = 0$.

*Proof.* By Theorem 1.5.2, $A$ is not a triangular algebra and thus Condition (I) in Theorem 5.3.2 holds. As a result, every automorphism of Tri$(A, M, B)$ is partible. Take an automorphism of the form $\rho\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} a & f(m) \\ & p_B(b) \end{pmatrix}$. We want to show that $p_B$ is inner and hence we have the short exact sequences in Corollary 5.3.6.

Let $E_{ij}$'s and $F'_{ij}s$ be the standard matrix units of $A$ and $M$ respectively. Note that $f(F_{ij}) = E_{i1}f(F_{1j}) = u_i v_j$ where $u_i$ is a column vector with 1 in the $i-th$ row and 0 elsewhere, and $v_j$ is first row of $f(F_{1j})$. Let $w = [v_1^t, \dots, v_l^t]^t \in M_l(\mathbf{S})$. Then for every $t \in \mathbf{S}$, we have

$$f(tF_{ij}) = tE_{i1}f(F_{ij}) = tu_i v_j = tF_{ij}w.$$

Thus for every $m = \sum t_{ij}F_{ij} \in M$, we have

$$f(m) = f(\sum t_{ij}F_{ij}) = \sum f(t_{ij}F_{ij} = \sum t_{ij}F_{ij}w = mw.$$

As $f$ is bijective, $w$ is invertible. For any $b \in B$,

$$mwp_B(b) = f(m)p_B(b) = f(mb) = mbw$$

which implies that $wp_B(b) = bw$ as $M$ is a faithful right $B$-module. Thus $p_B(b) = w^{-1}bw$. To prove that $p_B$ is inner, we will show that $w \in B$. By assumption, $p_B = \alpha_z\hat{\rho}$ for some $z \in B$ and an $\mathbf{R}$-automorphism $\hat{\rho}$ induced by an $\mathbf{R}$-automorphism $\rho$ of $\mathbf{S}$. As $D_k(\mathbf{S}) \subseteq B$, we have

$$\begin{aligned} E_{ii}wz^{-1} &= wp_B(E_{ii})z^{-1} \\ &= wz^{-1}\rho(E_{ii}) \\ &= wz^{-1}E_{ii}. \end{aligned}$$

Since this is true for every index $i$, we have $wz^{-1} \in D_l(\mathbf{S}) \subseteq B$. As a result $w = (wz^{-1})z \in B$ and thus $p_B$ is inner.

By Lemma 2.3.3, it remains to show that $OutAut(M) = [id_M]$. Take $f \in Aut(M)$. By considering argument similar to the above for the $\mathbf{R}$-automorphism

$$\phi \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} a & f(m) \\ & b \end{pmatrix}$$

on $\mathfrak{A}$, we have $f(m) = mw$ for some invertible $w \in B$. Since

$$mwb = f(m)b = f(mb) = mbw$$

and, as $M$ is a faithful right $B$-module, $wb = bw$ for every $b \in B$. Thus $w \in Z(B)$. Therefore $f \in L_0(M)$ and $OutAut(M) = [id_M]$. ■

**Corollary 5.4.3** *Assume that* $\mathbf{S}$ *is not a triangular algebra. Let* $A = M_k(\mathbf{S})$, $M = \mathbf{S}^{k,l}$, *and let* $B \supseteq D_l(\mathbf{S})$ *be a matrix algerba. If every* $\mathbf{R}$*-automorphisms* *of $A$ and every* $\mathbf{R}$*-automorphisms of $B$ can be written as a composition of an inner automorphism and an induced* $\mathbf{R}$*-automorphism. then every* $\mathbf{R}$*-automorphism on* $\text{Tri}(A, M, B)$ *is a composition of an inner automorphism and an induced* $\mathbf{R}$*-automorphism.*

*Proof.* The sequence in Theorem 5.4.2 is a short exact sequence. Since every $\mathbf{R}$-automorphism of $A$ is a composition of an inner automorphism and an induced $\mathbf{R}$-automorphism. $S\mathcal{N}(\mathbf{S})$ and $S\mathcal{N}(A)$ are isomorphic under the map $D([d]) = [\hat{d}]$. Consider the map $\bar{D} : S\mathcal{N}(\mathbf{S}) \to S\mathcal{N}(\mathfrak{A})$ given by $\bar{D}[d] = [\hat{d}]$. We have $\tilde{\pi}_A(\bar{D}D^{-1}) = id_{S\mathcal{N}(A)}$. Hence the sequence splits, and by Lemma 2.3.3, $D^{-1}\bar{D}$ is an isomorphism from $S\mathcal{N}(A)$ to $S\mathcal{N}(\mathfrak{A})$. Therefore $\bar{D}$ is an isomorphism from $S\mathcal{N}(S)$ to $S\mathcal{N}(\mathfrak{A})$ and the result follows. ∎

**Theorem 5.4.4** *Suppose that* $\mathbf{S}$ *is not a triangular algebra. then every* $\mathbf{R}$*-automorphism of* $T_n(\mathbf{S})$ *is a composition of an inner automorphism and an induced* $\mathbf{R}$*-automorphism.*

*Proof.* We prove by induction. The statement is obviously true for $n = 1$.

Assume the statement is true for $n = k$. By Theorem 1.4.4, $T_{k+1}(\mathbf{S}) = \text{Tri}(\mathbf{S}, \mathbf{S}^k, T_k(\mathbf{S}))$. The result follows from Corollary 5.4.3. ∎

**Corollary 5.4.5** [43] *Every* $\mathbf{R}$*-automorphism of* $T_n(\mathbf{R})$ *is inner.*

*Proof.* Since $\mathbf{R}$ is commutative, it is not a triangular algebra. By Theorem 5.4.4 and that the only $\mathbf{R}$-automorphism of $\mathbf{R}$ is the identity map, every $\mathbf{R}$-automorphism of $T_n(\mathbf{R})$ is inner. ∎

**Corollary 5.4.6** [42] *If every idempotent of* $\mathsf{S}$ *lies in the centre, then every* $\mathbf{R}$-*automorphism of* $T_n(\mathsf{S})$ *is an inner automorphism and an induced automorphism.*

*Proof.* If $\mathsf{S} = \mathrm{Tri}(A, M, B)$, then $1_A \oplus 0$ is an idempotent which does not lie in the centre. Hence $\mathsf{S}$ is not a triangular algebra and the result follows from Theorem 5.4.4. ∎

Recall (see [28, p.2]) that a ring $\mathsf{S}$ is *semiprime* if $a\mathsf{S}a = 0$ implies $a = 0$.

**Corollary 5.4.7** [41] *If* $\mathsf{S}$ *is semiprime, then every* $\mathbf{R}$-*automorphism of* $T_n(\mathsf{S})$ *is a composition of an inner automorphism and an induced automorphism.*

*Proof.* If $\mathsf{S} = \mathrm{Tri}(A, M, B)$, then $\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \mathsf{S} \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} = 0$ which contradicts that $\mathsf{S}$ is semiprime. The result then follows from Theorem 5.4.4. ∎

**Corollary 5.4.8** [4] *Suppose that any nontrivial* $\mathbf{R}$-*endomorphism of* $\mathsf{S}$ *is an* $\mathbf{R}$-*automorphism, then* $SN(T_n(\mathsf{S})) = SN(\mathsf{S})$.

*Proof.* It suffices to show that S is not a triangular algebra. Suppose that $\mathbf{S} = \mathrm{Tri}(A, M, B)$, then the map $\begin{pmatrix} a & m \\ & b \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}$ is a nontrivial **R**-endomorphism but not an **R**-automorphism. ■

**Corollary 5.4.9** *Assume that* S *is not a triangular algebra and that, for every natural number* $k$, $SN(M_k(\mathbf{S}) = SN(\mathbf{S})$, *i.e. every* **R**-*automorphism of* $M_k(\mathbf{S})$ *is a composition of an inner automorphism and an induced* **R**-*automorphism. Then every automorphism of any block triangular algebra* $\mathfrak{A}$ *over* S *is a composition of an inner automorphism and an induced automorphism.*

*Proof.* We prove this by induction on the number of blocks $l$ of $\mathfrak{A} = T(n_1, \ldots, n_l)(\mathbf{S})$. The proof is similar to that of Theorem 5.4.4. ■

The following example is about unique factorization domain. For its definition and properties, one may see [1, Chapter 5]. A factorization domain is a commutative ring and thus is not a triangular algebra.

**Corollary 5.4.10** *If* **R** *is a unique factorization domain, then every* **R**-*automorphism of* $T(n_1, \ldots, n_k)(\mathbf{R})$ *is inner, i.e.* $SN(T(n_1, \ldots, n_k)(\mathbf{R}))$ *is trivial.*

*Proof.* From [36], it is known that $SN(M_k(\mathbf{R}))$ is trivial for all $k$. The result follows from Corollary 5.4.9. ■

The last theorem in this section is concerned with a triangular matrix algebra with $A$ and $B$ being diagonal matrix algebras. The first cohomology group of this algebra is discussed in Theorem 2.4.5. We will only consider the case that $\mathbf{S}$ is *connected*, i.e. the only idempotents of $\mathbf{S}$ are 0 and 1.

We denote by $\mathfrak{S}_n$ the group of $n \times n$ permutation matrices.

**Lemma 5.4.11** *If* $\mathbf{S}$ *is connected, then every* $\mathbf{R}$*-automorphism of* $D_n(\mathbf{S})$ *can be represented by* $(Q; \beta_1, \ldots, \beta_n) \in \mathfrak{S}_n \times (Aut(\mathbf{S})^n)$ *in such a way that*

$$(Q; \beta_1, \ldots, \beta_n)(\sum s_{ii}E_{ii}) = Q^t(\sum \beta_i(s_{ii})E_{ii})Q.$$

*Proof.* Consider $\phi \in End(D_n(\mathbf{S}))$. Since $\mathbf{S}$ is connected and $\phi(E_{ii})$ is an idempotent, we have $\phi(E_{ii}) = \sum_{r \in V_i} E_{rr}$, where $V_i \subseteq \{1, \ldots, n\}$. From

$$0 = \phi(E_{ii}E_{jj}) = \sum_{r \in V_i \cap V_j} E_{rr},$$

we conclude that $V_1, \ldots, V_n$ are pairwise disjoint subsets of $\{1, \ldots, n\}$ and thus $V_i$'s are singletons. Therefore there exists an permutation matrix $Q$ such that $\phi(E_{ii}) = Q^t E_{ii} Q$.

Now for any $i = 1, \ldots, n$ and $b \in \mathbf{S}$, $\phi(sE_{ii}) = \beta_i(s)Q^t E_{ii} Q$ for some automorphism $\beta_i$ of $\mathbf{S}$. Thus the result follows. ■

**Theorem 5.4.12** *Suppose* $\mathbf{S}$ *is connected,* $A = D_k(\mathbf{S})$, $B = D_l(\mathbf{S})$ *and* $M$ *is the space of upper block triangular matrices* $T(n_1, \ldots, n_t; m_1, \ldots, m_t)(\mathbf{S})$, *where* $n_1 + \cdots + n_t = k$ *and* $m_1 + \cdots + m_t = l$. *Then* $SN(\mathrm{Tri}(A, M, B))$ *is a semidirect product of* $G$ *and* $H$, *where*

$$G = \Pi_{i=1}^{t} \mathfrak{S}_{n_i} \times \Pi_{j=1}^{t} \mathfrak{S}_{m_j} \times SN(\mathbf{S})$$

*and* $H = (Z(\mathbf{S})^{*})^{dim(M)-k-l+1}$, *where* $Z(\mathbf{S})^{*}$ *denotes the group of invertible element in* $Z(\mathbf{S})$.

*Proof.* Since $\mathbf{S}$ is connected, it is not a triangular algebra. Otherwise if $\mathbf{S} = \mathrm{Tri}(X, Z, Y)$ then $1_X \oplus 0$ is an idempotent in $\mathbf{S}$, a contradiction.

Let $\mathfrak{A} = \mathrm{Tri}(A, M, B)$. By Lemma 2.3.3, it suffices to show that there exists a split short exact sequence

$$\{1\} \to OutAut(M) \xrightarrow{\Psi} SN(\mathfrak{A}) \xrightarrow{p} G \to \{1\}$$

with $q : G \to SN(\mathfrak{A})$ satisfying $qp = id_G$ and that $OutAut(M) = H$.

Define $\Psi$ in the same way as in Theorem 5.3.5(i). We first define a group homomorphism $p : SN(\mathfrak{A}) \to G$. Since $A$ is not a triangular algebra and $M$ is a faithful $B$-module, every automorphism of $\mathfrak{A}$ is partible by Theorem 5.3.2. Take $\phi \in Aut(\mathrm{Tri}(A, M, B))$. As it is partible, there exists $\bar{\phi} \in [\phi]$ satisfying

$\bar{\varphi}(A) = A$, $\bar{\phi}(B) = B$ and $\bar{\phi}(M) = M$. We write

$$\bar{\phi}\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_A(a) & h(m) \\ & p_B(b) \end{pmatrix}.$$

By the previous lemma, $p_A$ is represented by $(Q_A; \gamma_1, \ldots, \gamma_k)$ and $p_B$ is represented by $(Q_B; \beta_1, \ldots, \beta_l)$.

Let $e_{ii}$'s and $f_{jj}$'s be the standard bases of $A$ and $B$ respectively. Taking a unit matrix $E_{ij} \in M$, we have

$$\begin{aligned} \begin{pmatrix} 0 & h(E_{ij}) \\ & 0 \end{pmatrix} &= \bar{\phi}\begin{pmatrix} 0 & E_{ij} \\ & 0 \end{pmatrix} \\ &= \bar{\phi}\begin{pmatrix} e_{ii} & 0 \\ & 0 \end{pmatrix} \bar{\phi}\begin{pmatrix} 0 & E_{ij} \\ & 0 \end{pmatrix} \bar{\phi}\begin{pmatrix} 0 & 0 \\ & f_{jj} \end{pmatrix} \\ &= \begin{pmatrix} Q_A^t e_{ii} Q_A & 0 \\ & 0 \end{pmatrix} \begin{pmatrix} 0 & h(E_{ij}) \\ & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ & Q_B^t f_{jj} Q_B \end{pmatrix}. \end{aligned}$$

Hence $h(E_{ij}) = Q_A^t e_{ii} Q_A h(E_{ij}) Q_B^t f_{jj} Q_B$. Therefore if $Q_A^t e_{ii} Q_A = e_{\sigma(i)\sigma(i)}$ and $Q_B^t f_{jj} Q_B = e_{\tau(j)\tau(j)}$, then $h(E_{ij}) = m_{ij} E_{\sigma(i)\tau(j)}$ for some $m_{i,j} \in \mathbf{S}$. Since $h(E_{ij}) \in M$ for all $i, j$, we have that $Q_A \in \Pi_{i=t}^{n} \mathfrak{S}_{n_i}$ and $Q_B \in \Pi_{j=t}^{n} \mathfrak{S}_{m_j}$.

Define the map $p$ by $p([\varphi]) = (Q_A, Q_B, [\gamma_1])$. We will show that $p$ is well-defined. Suppose there exists another $\bar{\phi}_1 \in [\phi]$ such that $\bar{\phi}_1(A) = A$,

$\bar{\phi}_1(B) = B$ and $\bar{\phi}_1(M) = M$. Then $\bar{\phi}_1 = \alpha_y\bar{\phi}$ for some $y = \begin{pmatrix} a' & m' \\ & b' \end{pmatrix}$. Write $\bar{\phi}_1$ in the form

$$\bar{\phi}_1 \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_1(a) & f_1(m) \\ & p_1(b) \end{pmatrix}.$$

Then $p_1 = \alpha_{a'}p_A$ is represented by $(Q_A, \alpha_{a'_{11}}\gamma_1, \ldots, \alpha_{a'_{kk}}\gamma_k)$ and $p_2 = \alpha_{b'}p_B$ is represented by $(Q_B, \alpha_{b'_{11}}\beta_1, \ldots, \alpha_{b'_{ll}}\beta_l)$. Therefore the map $p$ is well defined. It is straightforward to verify that $p$ is a group homomorphism.

Define a group homomorphism $q : G \to SN(\mathfrak{A})$ by $q(Q_A, Q_B, [\gamma]) = [\phi]$ where $\phi(X) = \hat{\gamma}((Q_A \oplus Q_B)^t X (Q_A \oplus Q_B)$ for $X \in \mathfrak{A}$ and $\hat{\gamma}$ is the automorphism induced by $\gamma$. Then we have $qp = id_G$ as desired.

Next we prove that $Ker(p) = Im(\Psi)$. That $Im(\Psi) \subseteq Ker(p)$ follows from the definition of $\Psi$ in Theorem 5.3.5(i). To prove the reverse inclusion, let $p([\phi]) = (1, 1, [id_S])$. Without loss of generality, suppose that $\phi\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_A(a) & h(m) \\ & p_B(b) \end{pmatrix}$ and $p_A$ is represented $(1, \gamma_1, \ldots, \gamma_k)$ with $\gamma_1 = id_S$ and $p_B$ is represented $(1, \beta_1, \ldots, \beta_l)$. If $E_{ij} \in M$, then we have

$$h(E_{ij}) = e_{ii}h(E_{ij})f_{jj} = x_{ij}E_{ij}$$

for some $x_{ij} \in \mathbf{S}$. Furthermore, it is not hard to see that

$$h(sE_{ij}) = p_A(se_{ii})h(E_{ij}) = \gamma_i(s)x_{ij}E_{ij}$$

and

$$h(sE_{ij}) = h(E_{ij})p_B(sf_{jj}) = x_{ij}\beta_j(s)E_{ij}.$$

As $\gamma_i$, $\beta_j$ and $h$ are all surjective, we have that $x_{ij}$ is invertible and

$$\gamma_i = \alpha_{x_{ij}^{-1}}\beta_j.$$

Since $E_{11}, \dots, E_{1l}, E_{2l}, \dots, E_{kl}$ are all in $M$, we have $[\gamma_i] = [\beta_j] = [\gamma_1] = [id_{\mathbf{S}}]$ for all $i, j$. Take $z = (x_{1l}x_{1l}^{-1} \oplus \cdots \oplus x_{kl}x_{1l}^{-1}) \oplus (x_{11} \oplus x_{1l})$, then $\alpha_z o \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} a & g(m) \\ & b \end{pmatrix}$ for some $g \in Aut(M)$. Therefore $[o] = \Phi([g]) \in Im\Psi$ and $Ker(p) \subseteq Im(\Psi)$.

Finally we need to show that $OutAut(M) = H$. Using similar technique as in the last part of the proof of Theorem 2.4.5, we define a group isomorphism $\Theta : Aut(M) \to G_1$, where

$$G_1 = T(n_1, \dots, n_t; m_1, \dots, m_t)(Z(\mathbf{S})^*)$$

under coordinate-wise multiplication, and the image of $L_0(M)$ under $\Theta$ is completely determined by the $k+l-1$ entries $x_{11}, \dots, x_{1l}, x_{2l}, \dots, x_{kl}$. Hence the result follows.

■

# Chapter 6

# Conclusion

We have studied four types of linear maps on triangular algebras, i.e. algebras of the form

$$\begin{pmatrix} A & M \\ & B \end{pmatrix}$$

where $A$, $B$ are $\mathbf{R}$-algebras and $M$ is a nonzero $(A, B)$-bimodule. The four types of linear maps are derivations, Lie derivations, commuting maps and automorphisms. In each case we establish sufficient conditions for such maps to be of a special form.

In Chapter 2, we consider the first cohomology group which is the quotient group of the group of all derivations modulo the group of inner derivations. If every derivation of the triangular algebra has the property that its restriction on $B$ is inner whenever its restriction on $A$ is inner, then we are able to construct an exact sequence to determine the first cohomology group of the triangular algebra. With this result at hand, we are able to compute the first cohomology groups of certain matrix algebras. In particular we deduce that every derivation of upper block triangular matrix algebras over an unital algebra $\mathbf{S}$ is the composition of an inner derivation and a derivation induced from a derivation of $\mathbf{S}$.

In Chapter 3, we study Lie derivations of triangular algebras where $M$ is a faithful bimodule. The main result is that every Lie derivations of a triangular algebra with faithful $M$ is a sum of a derivation and a linear map with its image lying in the centre of the algebra, provided that the algebra satisfies certain conditions determined by the centre, commutators and idempotents. For example, those conditions are satisfied if both $A$ and $B$ have trivial centre. In particular nontrivial upper block triangular algebras and nest algebras satisfy these conditions and thus every Lie derivation of these algebras is a sum of derivation and a linear map with its image lying in the centre.

In Chapter 4, we discuss commuting maps of triangular algebras with faithful $M$. A commuting map is a linear map for which every element commutes with its image. We have established sufficient conditions, determined by the centre and commutators, for every commuting map to be of the form $L(x) = ax + h(x)$ where $a$ is a fixed element in the centre and $h$ is a linear map with image lying in the centre. Notably, this result is applicable to the upper block triangular algebras and nest algebras.

In the final Chapter, we study the Skolem-Noether groups of triangular algebras, i.e., the quotient group of the group of all automorphisms modulo the group of all inner automorphisms. An automorphism is partible if it is in the same equivalence class in the Skolem-Noether group as an automorphism which leaves $A$, $M$ and $B$ invariant. We have given five sufficient conditions so that every automorphism is partible. Suppose every automorphism is partible. If every automorphism which leaves $A$, $M$ and $B$ invariant has the property that its restriction on $B$ is inner whenever its restriction on $A$ is inner, then we are able to construct an exact sequence to determine the Skolem-Noether group of the triangular algebra. Applying the results at hand, we are able to show, in particular, that every automorphism of upper triangular matrix algebra over $\mathbf{R}$ is inner.

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