Cross-Sperner Systems

by

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We acknowledge and respect the lək̓ʷəŋən peoples on whose traditional territory the
university stands and the Songhees, Esquimalt, and W̱SÁNEĆ peoples whose
historical relationships with the land continue to this day.

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ABSTRACT

Two sets A and B are *comparable* if $A \subseteq B$ or $B \subseteq A$. A collection of families $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \in \mathcal{P}([n])^k$ is *cross-Sperner* if there is no pair $i \neq j$ for which some $F_i \in \mathcal{F}_i$ is comparable to some $F_j \in \mathcal{F}_j$. Two natural measures of the 'size' of such systems are the sum $\sum_{i=1}^{k} |\mathcal{F}_i|$ and the product $\prod_{i=1}^{k} |\mathcal{F}_i|$. Let (n, k) be the maximum size of the sum measure for a cross-Sperner system $(\mathcal{F}_1, \cdots, \mathcal{F}_k) \in \mathcal{P}([n])^k$, and let $\pi(n, k)$ be the maximum size of the product measure for a cross-Sperner system $(\mathcal{F}_1, \cdots, \mathcal{F}_k) \in \mathcal{P}([n])^k$. We prove new upper and lower bounds on (n, k) and $\pi(n, k)$ for general n and $k \geq 2$ which improve considerably on the previous best bounds. In this thesis we prove that

$$\left(\frac{2^n}{ek}\right)^k \leq \pi(n, k) \leq \left(1 + \frac{1}{k}\right)\left(\frac{2^n}{2k}\right)^k,$$

and

$$2^n - \frac{3}{\sqrt{2}}\sqrt{2^n k} + 2(k-1) \ \leq \ \sigma(n, k) \ \leq \ 2^n - 2\sqrt{2^n(k-1)} + 2(k-1).$$

In particular, we construct a rich family of counterexamples to a conjecture of Gerbner, Lemons, Palmer, Patkós, and Szécsi from 2011.

To prove these bounds, we exploit a connection between cross-Sperner systems and the *comparability number* of a family of sets. Define the comparability number of a family $\mathcal{F} \subseteq \mathcal{P}([n])$ to be the number of sets comparable to $\mathcal{F}$. Then define $c(n, m)$ to be the minimum comparability number of a family $\mathcal{F} \subseteq \mathcal{P}([n])$ where $|\mathcal{F}| = m$. We prove that for $1 \leq m \leq 2^n$,

$$c(n, m) \geq 2^{n/2+1}\sqrt{m} - m.$$

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Chapter 1
Introduction and History

The focus of this thesis is extremal combinatorics. This area of mathematics concerns maximizing or minimizing the size of discrete structures that satisfy certain conditions. Many of these discrete structures are sets or families of sets. Here we consider problems at the intersection of two main themes in extremal set theory: containment and multi-family theorems. At the intersection of these two themes we find interesting and exciting results, which we will explore in this thesis.

The results we focus on pertain to set systems, or subsets of the power set. We will standardize our notation regarding sets and set systems here. The canonical way to organize the sets comprised of numbers 1 up to n is the *hypercube*. Let $[n]$ denote the set of integers $\{1, 2, \cdots, n\}$ and let $\mathcal{P}([n])$ be the power set of $[n]$.

Definition 1.0.1. The *n-dimensional hypercube* is a graph, denoted $\mathcal{Q}_n$, where each vertex is a set in $\mathcal{P}([n])$ and there is an edge if and only if two sets differ in exactly one element.

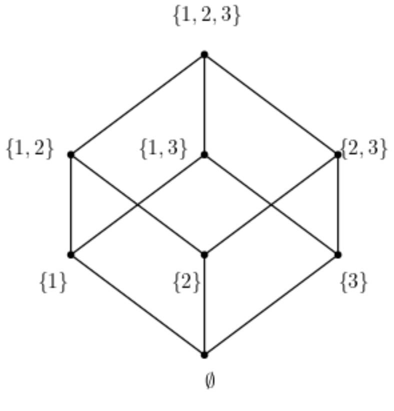

Figure 1.1: 3-dimensional hypercube

It is helpful to think of the hypercube as being divided into levels. The vertices are organized into $n+1$ levels labeled 0 to n where level i has all the sets of size i, of which there are $\binom{n}{i}$. The largest level of the hypercube has size $\binom{n}{\lfloor n/2 \rfloor}$ (see Figure 1.2). Let $\binom{[n]}{k}$ be the family of sets $X \subseteq [n]$ such that $|X| = k$.

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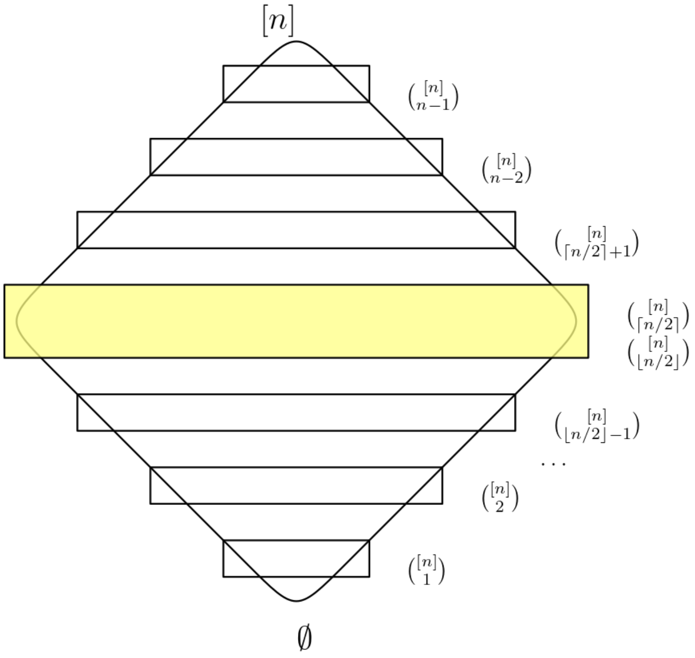

Figure 1.2: An n-dimensional hypercube without the edges. Notice that the largest row is the middle row.

In order to place our results in context, we begin with an overview of the two themes to which our results relate. In Section 1.1 we discuss containment and Sperner's Theorem. In Section 1.2 we discuss multi-family theorems. In Section 1.3 we state the main results of this thesis.

1.1 Antichains and Sperner's Theorem

We begin with Sperner's Theorem which focuses on subsets of the hypercube where no two sets are *comparable*.

Definition 1.1.1. Two sets A and B are *comparable* if $A \subseteq B$ or $B \subseteq A$. If A and B are not comparable, they are *incomparable*.

Example 1.1.2. The sets $\{1, 3, 5\}$ and $\{1, 3\}$ are comparable since $\{1, 3\} \subseteq \{1, 3, 5\}$. The sets $\{1, 2\}$ and $\{2, 3\}$ are incomparable since $\{1, 2\} \not\subseteq \{2, 3\}$ and $\{2, 3\} \not\subseteq \{1, 2\}$.

Comparability is a relation that we can define on the sets of the power set. In particular, the power set $\mathcal{P}([n])$ is a partially ordered set, also known as a *poset*, with the relation of comparability.

Definition 1.1.3. A *partially ordered set* is a pair $P = (X, \leq)$ where X is a ground set and $\leq$ is a partial ordering on X that satisfies the following three conditions.

1. For all $x \in P$, $x \leq x$ (reflexive).
2. If $x, y \in P$ such that $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric).
3. If $x, y, z \in P$ such that $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive).

This ordering is called 'partial' because not every pair of elements needs a relation between them. It is easy to see that $(\mathcal{P}([n]), \subseteq)$ is a poset. There are many other examples of posets, including $(\mathbb{R}, \leq)$ and $\mathbb{N}$ equipped with the relation of divisibility.

It is natural to ask how large a family $\mathcal{F} \subseteq \mathcal{P}([n])$ can be so that no two sets are comparable. We call a family with this property an *antichain*.

Definition 1.1.4. A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is an *antichain* if no pairs of distinct sets $A, B \in \mathcal{F}$ are comparable. This is also known as a *Sperner family*.

Example 1.1.5. The family $\{\{1, 2\}, \{1, 3\}, \{4, 5, 6\}\}$ is an antichain.

Notice that if $\mathcal{F} \subseteq \binom{[n]}{k}$, then $\mathcal{F}$ is an antichain since all the sets are the same size. One of the first and central theorems in extremal set theory is Sperner's Theorem which answers the question of how large an antichain can be.

Theorem 1.1.6 (Sperner's Theorem [39])**.** *If* $\mathcal{F} \subseteq \mathcal{P}([n])$ *is an antichain, then* $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ *with equality when* $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ *or* $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$.

As Sperner's Theorem is so fundamental to extremal combinatorics, we present a probabilistic proof here, inspired by [15]. In order to do so, we need to introduce a dual notion to antichains – *chains*.

Definition 1.1.7. A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is a *chain* if for every pair of sets $A, B \in \mathcal{F}$, A and B are comparable. If $\mathcal{F} \subseteq \mathcal{P}([n])$ is a chain where $|\mathcal{F}| = k$, then we call $\mathcal{F}$ a k-chain.

Example 1.1.8. The family $\{\{1\}, \{1, 2\}, \{1, 2, 5, 6\}\}$ is a 3-chain.

We can now prove Theorem 1.1.6 (Sperner's Theorem).

Proof of Theorem 1.1.6. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be an antichain. Let σ be a permutation of $[n]$ chosen uniformly at random. Define

$$\mathcal{C}_\sigma = \{\emptyset, \{\sigma(1)\}, \{\sigma(1), \sigma(2)\}, \{\sigma(1), \sigma(2), \sigma(3)\}, \cdots, [n]\}.$$

Notice that

$$\emptyset \subseteq \{\sigma(1)\} \subseteq \{\sigma(1), \sigma(2)\} \subseteq \{\sigma(1), \sigma(2), \sigma(3)\} \subseteq \cdots \subseteq [n],$$

so $\mathcal{C}_\sigma$ is a chain. Define a random variable

$$X := |\mathcal{F} \cap \mathcal{C}_\sigma|.$$

For each $A \in \mathcal{F}$, let

$$X_A = \begin{cases} 1 \text{ if } A \in \mathcal{C}_\sigma \\ 0 \text{ otherwise.} \end{cases}$$

Thus $X = \sum_{A \in \mathcal{F}} X_A$. Since $\mathcal{C}_\sigma$ is a chain, it contains exactly one set of size $|A|$. Since the permutation was chosen uniformly at random, the probability that a given set of size $|A|$ is in $\mathcal{C}_\sigma$ is $\frac{1}{\binom{n}{|A|}}$. So,

$$\mathbf{E}[X_A] = \mathbf{P}(A \in \mathcal{C}_\sigma) = \frac{1}{\binom{n}{|A|}}.$$

By linearity of expectation,

$$\mathbf{E}[X] = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}.$$

Since $\mathcal{F}$ is an antichain, we must have that $X = |\mathcal{F} \cap \mathcal{C}_\sigma| \leq 1$. To see this, suppose $X \geq 2$. Then there are two distinct sets, say $A, B \in \mathcal{F}$, such that $A, B \in \mathcal{C}_\sigma$. But $\mathcal{C}_\sigma$ is a chain, so A and B are comparable, contradicting that $\mathcal{F}$ is an antichain. Thus,

$$1 \geq \mathbf{E}[X] = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}.$$

The binomial coefficient $\binom{n}{k}$ is maximized when $k = \lfloor n/2 \rfloor$, so

$$1 \geq \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \geq \frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}},$$

proving that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. □

Kleitman [25] and Katona [24] (independently) proved a generalization of Sperner's theorem.

Theorem 1.1.9 (Kleitman [25], Katona [24]). *Let S_1, S_2 be a partition of a set S and let $\mathcal{A} = \{A_1, A_2, \cdots, A_N\}$ be a family of subsets of S. If no two distinct A_i, A_j satisfy*

$$A_i \cap S_1 = A_j \cap S_1 \text{ and } A_i \cap S_2 \subseteq A_j \cap S_2$$

or,

$$A_i \cap S_1 \subseteq A_j \cap S_1 \text{ and } A_i \cap S_2 = A_j \cap S_2,$$

then $N \leq \binom{n}{\lfloor n/2 \rfloor}$.

Sperner's theorem can also be generalized to bound the the size of a family with no chain of length r, as done by Erdős in [13].

Theorem 1.1.10 (Erdős [13]). *Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a family with no chain of length r. Then $|\mathcal{F}|$ is at most the sum of the $r-1$ largest binomial coefficients.*

That is to say, the extremal family contains sets from as close to the middle layer as possible. The case $r = 2$ is Sperner's Theorem. Theorem 1.1.10 falls into the general category of *saturation results.* In general, a family $\mathcal{F} \subseteq \mathcal{P}([n])$ is saturated with respect to a certain property if adding any set $X \subseteq [n]$ to $\mathcal{F}$ will destroy that property. In this context, we consider families with no chain of a certain length.

Definition 1.1.11. A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is a *saturated k-Sperner system* if $\mathcal{F}$ does not contain a chain of length $k+1$ and for every $X \in \mathcal{P}([n]) \setminus \mathcal{F}$, $\mathcal{F} \cup \{X\}$ contains a $(k+1)$-chain. A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is an *oversaturated k-Sperner system* if, for every $X \in \mathcal{P}([n]) \setminus \mathcal{F}$, the family $\mathcal{F} \cup \{X\}$ has more $(k+1)$-chains than the original family $\mathcal{F}$.

Minimizing the size of a saturated and overstaturated k-Sperner system was first studied by Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi, and Patkós in [17] where they found upper and lower bounds on the minimum size of a saturated and oversaturated k-Sperner system for certain values of n and k. In [33], Morrison, Noel and Scott improved the upper bound for all values of n and k. Turán problems have also been widely studied. These problems focus on maximizing the number of edges in a graph G that do not contain a copy of some graph H.

Given that we know the size of the extremal families with no k-chains, it is natural to ask what happens when the size of the family is slightly bigger than the maximum size of a family with no k-chains. Results of this kind are known as supersaturation theorems. Kleitman [27] proved the following theorem.

Theorem 1.1.12 (Kleitman [27]). *Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be such that $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor} + t$ for some $t > 0$. Then $\mathcal{F}$ contains at least $t\lceil \frac{n+1}{2} \rceil$ 2-chains.*

Moreover, he showed that the minimum number of 2-chains in a family is obtained by choosing sets of size as close to $n/2$ as possible. Kleitman went on to conjecture that the same families minimize the number of k-chains. This conjecture has remained open for over 50 years. Independently, Das, Gan, and Sudakov [10] and Dove, Griggs, Kang, and Sereni [12] showed that the conjecture is true if the size of the family is at most the size of the k middle layers of the hypercube. Balogh and Wagner [4] proved that the result holds asymptotically. Recently, the conjecture of Kleitman has been resolved fully by Samotij [37].

Maximizing the number of 2-chains has also been studied. For a family $\mathcal{F}$ of size m, the number of 2-chains is at most $\binom{m}{2}$, with equality when $\mathcal{F}$ is an m-chain and

$m \leq n+1$. In [11], Daykin and Frankl showed the the maximum number of comparable pairs in a family of size $m = 2^{o(n)}$ is $(1+o(1))\binom{m}{2}$. That is, there are large families where almost all pairs are comparable. The extremal families are 'towers of cubes', which can be visualized as subcubes stacked tip-to-tip in the hypercube. Alon, Das, and Glebov [1], considered the two family version of counting the number of 2-chains. Let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}([n])$ be two families. They proved that if $|\mathcal{F}||\mathcal{G}| = n^d 2^n$, then the number of comparable pairs $(F, G) \in \mathcal{F} \times \mathcal{G}$ is at most $2^{-d/300}|\mathcal{F}||\mathcal{G}|$.

Another well-known problem in extremal combinatorics is *Dedekind's Problem*. This is the problem of counting the antichains in $\mathcal{P}([n])$. There are at least $2^{\binom{n}{\lfloor n/2 \rfloor}}$ antichains, as every subchain of $\binom{[n]}{\lfloor n/2 \rfloor}$ is an antichain. An upper bound of $3^{\binom{n}{\lfloor n/2 \rfloor}}$ was proved by Hansel in [21]. Kleitman in [29] improved the upper bound to $2^{(1+o(1))\binom{n}{\lfloor n/2 \rfloor}}$ by partitioning $\mathcal{P}([n])$ into particular chains.

In combinatorics, it is often interesting to incorporate randomness into discrete structures. The random version of Sperner's theorem has inspired many results. Let $\mathcal{P}([n], p)$ be a random family obtained by including sets of $\mathcal{P}([n])$ independently with probability p. Rényi [36] introduced this model and gave a threshold for when $\mathcal{P}([n], p)$ is not an antichain. Osthus [34] considered the size of the largest antichain in $\mathcal{P}([n], p)$. He proved that as $pn/\log n \to \infty$, asymptotically almost surely, the size of the maximal antichain in $\mathcal{P}([n], p)$ is $(1+o(1))p\binom{n}{\lfloor n/2 \rfloor}$. That is, the largest antichain is asymptotically the same as the expected number of sets of size $\lfloor n/2 \rfloor$ in $\mathcal{P}([n], p)$. Balogh, Mycroft, and Treglown in [3] used the hypergraph container method [2] to give the exact values of p for which the largest antichain is $(1+o(1))p\binom{n}{\lfloor n/2 \rfloor}$. A random generalization of the largest family with no chain of length r has also been studied. Collares and Morris [9] showed as $pn \to \infty$, asymptotically almost surely the largest family in $\mathcal{P}([n], p)$ with no r chain has size $(k-1+o(1))p\binom{n}{\lfloor n/2 \rfloor}$.

1.2 Multi-Family Theorems

Two family theorems generalize many of the classical results in extremal combinatorics by extending them to multiple families. Given two families $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}([n])$, it is natural to study when these two families satisfy some relationship between them. There are many ways to measure the 'size' of multiple families. One is to measure the sum $|\mathcal{F}|+|\mathcal{G}|$. Another common measurement of size is the product $|\mathcal{F}| \times |\mathcal{G}|$. In this section we discuss some of the multi-family theorems that have been studied. Generalizing results from single families to multiple families originated from the study of intersecting families. In order to provide some context, we give a brief overview of intersecting family theorems.

1.2.1 Intersecting Families

The property of intersection is related to the property of containment. Intersection requires that only part of a set be contained in the other, where as containment requires that the full set be contained in the other.

Definition 1.2.1. A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is *intersecting* if for every pair $X, Y \in \mathcal{F}$, $X \cap Y \neq \emptyset$.

Example 1.2.2. The family

$$\{\{1,2,3\},\{2,3,4\},\{1,2\},\{1,3\}\}$$

is an intersecting family, as every pairwise intersection is non-empty.

An intersecting family $\mathcal{F} \subseteq \mathcal{P}([n])$ always satisfies $|\mathcal{F}| \leq 2^{n-1}$. To see this, suppose $|\mathcal{F}| > 2^{n-1}$. Then, by the pigeonhole principle, there must exist $X \in \mathcal{P}([n])$ such that X and $[n] \setminus X \in \mathcal{F}$. But $X \cap ([n] \setminus X) = \emptyset$, contradicting the fact that $\mathcal{F}$ is intersecting. This bound can be achieved by letting

$$\mathcal{F} = \{X \subseteq [n] : 1 \in X\}.$$

It is perhaps more interesting to consider intersecting families $\mathcal{F} \subseteq \binom{[n]}{k}$. The Erdős-Ko-Rado Theorem [14] is the first major theorem regarding intersecting families $\mathcal{F} \subseteq \binom{[n]}{k}$, and gives a bound on their size.

Theorem 1.2.3 (Erdős-Ko-Rado [14])**.** *If n and k are integers satisfying $n \geq 2k$, then every intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ satisfies $|\mathcal{F}| \leq \binom{n-1}{k-1}$.*

This is a tight bound. If we let $\mathcal{F} = \{A \subseteq [n] : 1 \in A\}$, then $\mathcal{F}$ is an intersecting family with size $\binom{n-1}{k-1}$. Notice that if $n < 2k$, then any two sets in $\binom{[n]}{k}$ have nonempty intersection, so the intersecting family can be larger then $\binom{n-1}{k-1}$. We can also consider families where the size of the intersection is bounded.

Definition 1.2.4. A family $\mathcal{F} \subseteq \binom{[n]}{k}$ is *t-intersecting* if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$.

Wilson [40], building on the work of Frankl [16], gave the exact bound on the size of t-intersecting families.

Theorem 1.2.5 (Wilson [40])**.** *If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a t-intersecting family and $n \geq (k-t+1)(t+1)$, then $|\mathcal{F}| \leq \binom{n-t}{k-t}$.*

This bound is tight. Fix a set $X \subseteq [n]$ such that $|X| = t$. Then let

$$\mathcal{F} = \{Y \cup X : Y \subseteq [n] \setminus X, |Y| = n - t\},$$

so $|\mathcal{F}| = \binom{n-t}{k-t}$.

A different generalization of intersecting families is *almost intersecting* families.

Definition 1.2.6. A family $\mathcal{F}$ is *ℓ-almost intersecting* if for every $F \in \mathcal{F}$, the number of $G \in \mathcal{F}$ whose intersection with F is empty is ℓ.

Notice that if $\ell = 0$, $\mathcal{F}$ is an intersecting family. In [18] Gerbner, Lemons, Palmer, Patkós, and Szécsi give an upper bound on the size of an ℓ-almost interesecting family.

Theorem 1.2.7 (Gerbner, Lemons, Palmer, Patkós, Szécsi [18]). *If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a ℓ-almost intersecting family where $\ell > 0$, then $|\mathcal{F}| \leq \ell\binom{2k\ell}{k\ell}$.*

1.2.2 Cross-Intersecting Families

We can now continue our discussion of the multi-family version of intersecting families. Kleitman made the generalization to *cross-intersecting* families in [28].

Definition 1.2.8. If $\mathcal{F} = \{F_1, F_2, \cdots, F_k\}$ and $\mathcal{G} = \{G_1, G_2, \cdots, G_\ell\}$ satisfy the condition that for every $1 \leq i \leq k$ and $1 \leq j \leq \ell$, $F_i \cap G_j \neq \emptyset$, and $F_i \neq G_j$ then $\mathcal{F}$ and $\mathcal{G}$ are *cross-intersecting.*

Example 1.2.9. If

$$\mathcal{F} = \{\{1,2\}, \{1,2,3\}, \{1,2,4\}\}$$

and

$$\mathcal{G} = \{\{1\}, \{1,3\}, \{1,4\}\},$$

then every set in $\mathcal{F}$ has non-empty intersection with every set in $\mathcal{G}$. Thus $\mathcal{F}$ and $\mathcal{G}$ are cross-intersecting.

Kleitman [28] gave the first bound on maximizing the sum of cross-intersecting families.

Theorem 1.2.10 (Kleitman, [28]). *If $\mathcal{F} \subseteq \binom{[n]}{k}$ and $\mathcal{G} \subseteq \binom{[n]}{\ell}$ are cross intersecting where $k + \ell \leq n$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ or $|\mathcal{G}| \leq \binom{n-1}{\ell-1}$.*

It is also natural to consider maximizing the product of cross-intersecting families, as done by Pyber [35].

Theorem 1.2.11 (Pyber, [35]). *If $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}([n])$ are cross-intersecting families satisfying the conditions*

$$F_i \cap G_j \neq \emptyset, \quad |F_i| \leq k, \quad |G_j| \leq \ell$$

for all $F_i \in \mathcal{F}$ and $G_j \in \mathcal{G}$ then *i.* $|\mathcal{F}||\mathcal{G}| \leq \binom{n-1}{k-1}\binom{n-1}{\ell-1}$ *when* $k > \ell$ *and* $2k + \ell - 2 \geq n$*;*

ii. $|\mathcal{F}||\mathcal{G}| \leq \binom{n-1}{k-1}^2$ *when* $k = \ell$ *and* $2k \leq n$*.*

This bound was improved a few years later by Matsumoto and Tokushige [32] by making the restriction on the size of k and l sharp. They proved the following theorem.

Theorem 1.2.12 (Matsumoto, Tokushige [32])**.** *If* $\mathcal{F} \subseteq \binom{[n]}{k}$ *and* $\mathcal{G} \subseteq \binom{[n]}{\ell}$ *are cross-intersecting with* $n \geq 2k$ *and* $n \geq 2\ell$*, then* $|\mathcal{F}||\mathcal{G}| \leq \binom{n-1}{k-1}\binom{n-1}{\ell-1}$*.*

1.2.3 Other Multi-Family Theorems

Many other two family theorems exist, including the Bollobás and Lovász two families theorems. The Bollobás two families theorem [8] was originally used to prove a result on saturated graphs. The theorem is as follows.

Theorem 1.2.13 (Bollobás [8])**.** *Let* $A_1, A_2, \cdots, A_m \in \binom{[n]}{r}$ *and* $B_1, B_2, \cdots, B_m \in \binom{[n]}{s}$ *satisfy the following two properties:*

i. $A_i \cap B_i = \emptyset$ *for all* i*,*

ii. $A_i \cap B_j \neq \emptyset$ *for all* $i \neq j$*.*

Then $m \leq \binom{r+s}{r}$*.*

This bound is tight. Let $n = r + s$. Take $A_1, A_2, \cdots, A_m$ to be all the sets in $\binom{[r+s]}{r}$ and $B_i = [n] \setminus A_i$. Then $m = \binom{[r+s]}{r}$, $A_i \cap B_i = A_i \cap ([n] \setminus A_i) = \emptyset$, and $A_i \cap B_j \neq \emptyset$. The Lovász two families theorem is a skewed version of Theroem 1.2.13.

Theorem 1.2.14 (Lovász [31])**.** *Let* $A_1, A_2, \cdots, A_m \in \binom{[n]}{r}$ *and* $B_1, B_2, \cdots, B_m \in \binom{[n]}{s}$ *satisfy the following two properties:*

i. $A_i \cap B_i = \emptyset$ *for all* i*,*

ii. $A_i \cap B_j \neq \emptyset$ *for all* $j < i$*.*

Then $m \leq \binom{r+s}{r}$*.*

Theorem 1.2.14 is useful in proving results on weak saturation of graphs.

1.3 Main Results

The main results in this thesis all relate to *cross-Sperner* families. The study of cross-Sperner families lies at the intersection of the study of containment and multi-family theorems. While they were first introduced in the 1970's, there has been recent interest in these structures.

Definition 1.3.1. A pair of non-empty families $(\mathcal{F}, \mathcal{G}) \subseteq \mathcal{P}([n])^2$ are *cross-Sperner* if for all pairs $F \in \mathcal{F}$ and $G \in \mathcal{G}$, F and G are incomparable.

Example 1.3.2. Let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}([5])$ where

$$\mathcal{F} = \{\{1\}, \{1,2\}, \{1,3\}, \{2,3\}\}$$

and

$$\mathcal{G} = \{\{5\}, \{3,4\}, \{2,5\}, \{3,5\}\}.$$

Then $\mathcal{F}, \mathcal{G}$ are cross-Sperner.

Seymour gave the first result on cross-Sperner families.

Theorem 1.3.3. (Seymour's Theorem [38]) *Let* $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}([n])$ *be cross-Sperner families. Then,* $|\mathcal{F}|^{1/2} + |\mathcal{G}|^{1/2} \leq 2^{n/2}$.

We can extend the notion of cross-Sperner to k families.

Definition 1.3.4. For $k \geq 2$, a collection of non-empty families $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \subseteq \mathcal{P}([n])^k$ is *cross-Sperner* if, for all $i \neq j$, the sets F_i and F_j are incomparable for any $F_i \in \mathcal{F}_i$ and $F_j \in \mathcal{F}_j$. We may also write that $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is *cross-Sperner in* $\mathcal{P}([n])$.

Example 1.3.5. The system

$$\begin{aligned} F_1 &= \{\{1,3\}, \{3,4\}, \{1,3,4\}\}, \\ F_2 &= \{\{1,2\}, \{2,4\}, \{1,2,4\}\}, \\ F_3 &= \{\{2,3\}\} \end{aligned}$$

is a cross-Sperner family in $\mathcal{P}([4])$.

Let $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ be cross-Sperner in $\mathcal{P}([n])$. There are several natural measures of the 'size' of such a family. These include the sum $\sum_{i=1}^{k} |\mathcal{F}_i|$, the product $\prod_{i=1}^{k} |\mathcal{F}_i|$, or the sizes of the individual families. The general study of these quantities was initiated by Gerbner, Lemons, Palmer, Patkós, and Szécsi [19], who essentially proved best possible bounds on cross-Sperner *pairs* of families. Two of the main goals of this thesis are to prove bounds on the size of the product and the sum of a cross-Sperner system with $k \geq 3$ families. For the product, define

$$\pi(n,k) := \max\left\{\prod_{i=1}^{k} |\mathcal{F}_i| : (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \text{ is cross-Sperner in } \mathcal{P}([n])\right\}.$$

For the sum, define

$$\sigma(n,k) := \max\left\{\sum_{i=1}^{k} |\mathcal{F}_i| : (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \text{ is cross-Sperner in } \mathcal{P}([n])\right\}.$$

Note that if we allowed empty families in a cross-Sperner system, then $\sigma(n,k) = 2^n$ by taking one family to be $\mathcal{P}([n])$ and the remaining families to be empty. Any cross-Sperner system with an empty family would have a product measure of zero. For this reason, we define cross-Sperner systems to have non-empty families.

We will prove the following upper and lower bound on the maximum product, $\pi(n,k)$, of a cross-Sperner system.

Theorem 1.3.6. *Let n and $k \geq 2$ be integers. For n sufficiently large,*

$$\left(\frac{2^n}{ek}\right)^k \leq \pi(n,k).$$

Theorem 1.3.7. *Let n and $k \geq 2$ be integers. Then*

$$\pi(n,k) \leq \left(\frac{2^n}{k^2}\right)^k \left\lfloor\frac{k}{2}\right\rfloor^{\lfloor k/2\rfloor} \left\lceil\frac{k}{2}\right\rceil^{\lceil k/2\rceil}.$$

And finally, we will prove a lower and upper bound on the maximum sum, (n,k), of a cross-Sperner system.

Theorem 1.3.8. *Let n, k be integers with $n \geq 2k$. Then*

$$2^n - \frac{3}{\sqrt{2}}\sqrt{2^n k} + 2(k-1) \;\leq\; \sigma(n,k) \;\leq\; 2^n - 2\sqrt{2^n(k-1)} + 2(k-1).$$

In order to prove Theorems 1.3.7 and 1.3.8, we exploit the connection between cross-Sperner systems and comparability. Notice that if $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}([n])$ are cross-Sperner, then every set in $\mathcal{G}$ is incomparable to every set in $\mathcal{F}$. Suppose $|\mathcal{F}| = m$. Then

$$|\mathcal{F}| + |\mathcal{G}| \leq m + 2^n - (\#\text{ of sets comparable to } \mathcal{F}).$$

If we knew the minimum number of sets comparable to a family $\mathcal{F} \subseteq \mathcal{P}([n])$ of size m, this would give an upper bound on the sum of cross-Sperner pairs. We define $c(n,m)$ to be the minimum number of sets comparable to a family $\mathcal{F} \subseteq \mathcal{P}([n])$ where $|\mathcal{F}| = m$. We prove the following lower bound on $c(n,m)$.

Theorem 1.3.9. *For $1 \leq m \leq 2^n$,*

$$c(n,m) \geq 2^{n/2+1}\sqrt{m} - m.$$

In Chapter 2, we discuss minimizing the comparability number and prove Theorem 1.3.9. In Chapter 3, we prove Theorem 1.3.6 and 1.3.7 on the maximum product of a cross-Sperner system. In Chapter 4, we consider the sum of a cross-Sperner system and prove Theorem 1.3.8. In Chapter 5 we give some concluding remarks and open questions. This thesis contains joint work with Natalie Behague, Natasha Morrison, and Ashna Wright (see [7]).

Chapter 2
Minimizing Comparability

2.1 Introduction

2.1.1 Intuition

In this chapter we consider a natural question regarding comparability. Recall the definition of comparability.

Definition 1.1.1. Two sets A and B are *comparable* if $A \subseteq B$ or $B \subseteq A$. If A and B are not comparable, they are *incomparable*.

Our question is as follows.

Question 2.1.1. *Suppose* $\mathcal{F} \subseteq \mathcal{P}([n])$ *such that* $|\mathcal{F}| = m$. *What is the minimum number of sets that* $\mathcal{F}$ *is comparable to?*

Not only is this a natural question that is interesting in its own right, but understanding the minimum number of comparable sets will be useful to prove Theorems 1.3.7 and 1.3.8. For a given family $\mathcal{F} \subseteq \mathcal{P}([n])$, we call the number of sets comparable to $\mathcal{F}$ the *comparability number*.

Definition 2.1.2. Given a family $\mathcal{F} \subseteq \mathcal{P}([n])$ define the *comparability number of* $\mathcal{F}$ to be

$$c(n, \mathcal{F}) := |\{X \subseteq [n] : X \text{ is comparable to some } A \in \mathcal{F}\}|.$$

When the setting is clear from context, we may write $c(\mathcal{F})$ for $c(n, \mathcal{F})$. If $\mathcal{F} = \{F\}$, for some $F \subseteq [n]$, we may write $c(F)$ for $c(\mathcal{F})$.

Example 2.1.3. Consider the set $\{1, 2\} \in \mathcal{P}([4])$. The comparability number of $\{1, 2\}$ is 7. The sets comparable to it are

$$\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}.$$

Our goal is to minimize $c(n, \mathcal{F})$ over all families of size m. For this we need the following definition.

Definition 2.1.4. Let

$$c(n, m) := \min_{\substack{\mathcal{F} \subseteq \mathcal{P}([n]) \\ |\mathcal{F}| = m}} c(\mathcal{F}).$$

If $\mathcal{F} \subseteq \mathcal{P}([n])$ is a family of size m such that $c(\mathcal{F}) = c(n, m)$, then we call $\mathcal{F}$ an (n, m)-*best family.*

The main result of this chapter is the following theorem.

Theorem 1.3.9. *For* $1 \leq m \leq 2^n$,

$$c(n, m) \geq 2^{n/2+1}\sqrt{m} - m.$$

When determining the comparability number of a family, it is helpful to think of the sets that are *contained in* or *contain* sets in the family. For this, we have the following definitions.

Definition 2.1.5. Let $X \in \mathcal{P}([n])$. Then the *downset* of X is

$$\{Y \subseteq [n] : Y \subseteq X\}.$$

If $\mathcal{F} \subseteq \mathcal{P}([n])$ is a family, then *downset* of $\mathcal{F}$, denoted $d(\mathcal{F})$, is

$$d(\mathcal{F}) = \{X \subseteq [n] : X \subseteq F \text{ for some } F \in \mathcal{F}\}.$$

Definition 2.1.6. Let $X \in \mathcal{P}([n])$. Then the *upset* of X is

$$\{Y \subseteq [n] : X \subseteq Y\}.$$

If $\mathcal{F} \subseteq \mathcal{P}([n])$ is a family, then *upset* of $\mathcal{F}$, denoted $u(\mathcal{F})$, is

$$u(\mathcal{F}) = \{X \subseteq [n] : F \subseteq X \text{ for some } F \in \mathcal{F}\}.$$

If $\mathcal{F} = \{F\}$ for some $F \subseteq [n]$, we may write $d(F)$ for $d(\mathcal{F})$ and $u(F)$ for $u(\mathcal{F})$.

Example 2.1.7. The downset of $\{\{1, 2\}\} \subseteq \mathcal{P}([4])$ is

$$d(\{1, 2\}) = \{\{1, 2\}, \{1\}, \{2\}, \emptyset\}.$$

The upset of $\{\{1, 2\}\} \subseteq \mathcal{P}([4])$ is

$$u(\{1, 2\}) = \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}.$$

We can describe the comparability number using the upset and downset of a family.

Observation 2.1.8. *Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a family of sets. Then*

$$c(\mathcal{F}) = |d(\mathcal{F}) \cup u(\mathcal{F})| = |d(\mathcal{F})| + |u(\mathcal{F})| - |d(\mathcal{F}) \cap u(\mathcal{F})|.$$

There are certain families $\mathcal{F}$ for which counting the upset, downset and their intersection is straightforward. From Example 2.1.7, we can calculate that $c(\{1,2\}) = 4+4-1 = 7$. For more complex families, if there are two sets $X, Y \in \mathcal{F}$ such that $X \subseteq Y$, then $u(Y) \subseteq u(X)$ and $d(X) \subseteq d(Y)$. In this way, we only need to consider the *minimal* and *maximal* sets in a family when counting the upset and downset.

Definition 2.1.9. A set $X \in \mathcal{F}$ is *minimal* if there does not exist a $Y \neq X \in \mathcal{F}$ such that $Y \subseteq X$. A set $Y \in \mathcal{F}$ is *maximal* if there does not exist a set $X \neq Y \in \mathcal{F}$ such that $Y \subseteq X$.

Example 2.1.10. Suppose

$$\mathcal{F} = \{\{1\}, \{2\}, \{2,3\}, \{1,2,3\}\} \subseteq \mathcal{P}([4]).$$

Then the minimal sets in $\mathcal{F}$ are $\{1\}$ and $\{2\}$ and the maximal set is $\{1,2,3\}$.

It will be useful to consider sets we call *convex*.

Definition 2.1.11. A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is *convex* if for all $X, Y \in \mathcal{F}$ such that $X \subseteq Y$, $\{Z : X \subseteq Z \subseteq Y\} \subseteq \mathcal{F}$.

If $\mathcal{F}$ is a convex family, every set contained in the upset of the minimal sets and the downset of the maximal elements is in $\mathcal{F}$. This makes it simple to determine $|u(\mathcal{F}) \cap d(\mathcal{F})|$.

Observation 2.1.12. *If a family $\mathcal{F}$ is convex, $u(\mathcal{F}) \cap d(\mathcal{F}) = \mathcal{F}$.*

Recall from Observation 2.1.8, one of the terms in computing the comparability number is $|u(\mathcal{F}) \cap d(\mathcal{F})|$ which, from Observation 2.1.12, is simple to determine when $\mathcal{F}$ is convex. Lemma 2.1.13 below gives us the tool to assume there exists a convex family that minimizes comparability.

Lemma 2.1.13. *Let $\mathcal{F} \subseteq \mathcal{P}([n])$. Let $Z \subseteq [n]$ be such that $Z \notin \mathcal{F}$ and there exist $X, Y \in \mathcal{F}$ such that $X \subseteq Z \subseteq Y$. Then $c(\mathcal{F} \cup Z) = c(\mathcal{F})$.*

Proof. Suppose $X, Y \in \mathcal{F}$ and $Z \notin \mathcal{F}$ such that $X \subseteq Z \subseteq Y$. Notice that $d(Z) \subseteq d(Y)$ and $u(Z) \subseteq u(X)$. By Observation 2.1.8, $c(\mathcal{F}) = c(\mathcal{F} \cup Z)$. □

Lemma 2.1.13 proves that a convex (n,m)-best family always exists.

Corollary 2.1.14. *There exists an (n, m)-best family that is convex.*

Proof. Suppose an (n, m)-best family $\mathcal{F}$ is not convex. Let

$$\mathcal{G} = \{Z \in \mathcal{P}([n]) : X \subseteq Z \subseteq Y, \text{ where } X, Y \in \mathcal{F}, Z \notin \mathcal{F}\}.$$

By Lemma 2.1.13, we may add a set from $\mathcal{G}$ to $\mathcal{F}$ without increasing $c(\mathcal{F})$. Then we can remove a minimal or maximal set from $\mathcal{F}$ so $|\mathcal{F}| = m$. Removing a minimal or maximal set will not increase the comparability number (see Observation 2.3.15). Continue this process until $\mathcal{G}$ is empty. Since $\mathcal{F}$ was already an (n, m)-best family, and we have not changed the comparability number in this process, our altered family after this process is (n, m)-best. □

In Example 2.1.15, we show that a family can be made convex without increasing the comparability number.

Example 2.1.15. Suppose

$$\mathcal{F} = \{\{1\}, \{2\}, \{2, 3\}, \{1, 2, 3\}\} \subseteq \mathcal{P}([4]).$$

Then adding the set $\{1, 2\}$ will not change $c(\mathcal{F})$ as $u(\{1, 2\}) \subseteq u(\{1\})$ and $d(\{1, 2\}) \subseteq d(\{1, 2, 3\})$. Let $\mathcal{F}' = \{1, 2\} \cup \mathcal{F}$. Then $c(\mathcal{F}') = c(\mathcal{F})$. We can remove $\{2\}$ from $\mathcal{F}'$ to obtain a family of the original size. By removing $\{2\}$ we reduce the comparability number since $\{2, 4\} \in u(\mathcal{F})$, but is not comparable to any other set in $\mathcal{F}$.

Observation 2.1.8 and Corollary 2.1.14 will be very useful in determining the comparability number and proving Theorem 1.3.9, as well as other results in this chapter. Before proving these results, we give some context as to where these results lie in combinatorics.

2.1.2 Context

Minimizing comparability has connections to other classical results in extremal combinatorics, namely the Kruskal-Katona Theorem and isoperimetric problems. We give a brief description of these results to highlight the interest in these types of problems.

Minimizing comparability can be viewed as a generalization of the Kruskal-Katona Theorem. This theorem determines the families $\mathcal{F} \subseteq \binom{[n]}{k}$ that minimizes the lower shadow which is defined as

$$\triangle\mathcal{F} = \{X \in \binom{[n]}{k-1} : X \subseteq F \text{ for some } F \in \mathcal{F}\}.$$

The Kruskal-Katona theorem says that the families with the smallest lower shadow are 'initial segments of colex'. Before stating the theorem, we will define the *colexicographic ordering.*

Definition 2.1.16. Let $A = \{a_1, a_2, \cdots, a_k\} \subseteq [n]$ and $B = \{b_1, b_2, \cdots, b_k\} \subseteq [n]$ be distinct sets where $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k$. We say that $A \prec_{colex} B$ if and only if $a_j < b_j$ where $j = \max\{i : a_i \neq b_i\}$. This is the *colexicographic ordering* or *colex ordering* on $\binom{[n]}{k}$.

Example 2.1.17. The first 7 terms of the colex ordering on $\binom{[6]}{3}$ are

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 5\}, \{1, 3, 5\}, \{2, 3, 5\}.$$

Now we can state the Kruskal-Katona Theorem.

Theorem 2.1.18 (Kruskal-Katona Theorem [23, 30])**.** *Let $k, m \geq 1$ and let $\{X_1, X_2, \cdots, X_m\}$ be the initial segment of colex of $\binom{[n]}{k}$. Then every $\mathcal{F} \subseteq \binom{[n]}{k}$ where $|\mathcal{F}| = m$ satisfies*

$$|\triangle\{X_1, X_2, \cdots, X_m\}| \leq |\triangle\mathcal{F}|.$$

This result also implies that there is a nesting of families that minimize the lower shadow. That is, there is a sequence of families $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k$ such that $|\mathcal{F}_i| = i$, $|\triangle\mathcal{F}_i|$ is minimal, and $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$. One may expect that a similar result would be true when minimizing the union of the lower shadow and upper shadow, which are the sets $X \in \binom{[n]}{k+1}$ such that $F \subseteq X$ for some $F \in \mathcal{F}$ of a family $\mathcal{F}$. In [5], Bashov proved that the k-uniform families that minimize the lower and upper shadow are the intersection between the Hamming ball and $\binom{[n]}{k}$. In [6] he went on to show that there is no nesting of families that minimizes the upper and lower shadow, as in the Kruskal-Katona Theorem. In this chapter we consider non-uniform families of sets that minimize the downset and upset, rather than just the upper and lower shadow.

It is interesting to note that the problem of minimizing comparability can be interpreted as a *vertex isoperimetric problem*, as noted in [19]. Isoperimetric problems ask how small the ‘boundary’ of a set of a given size can be. Suppose we are given a graph G with vertex set $V(G)$ and edge set $E(G)$. Let $U \subseteq V(G)$ where $|U| = m$. The vertex boundary is the set

$$N(U) := \{v \in V(G) \setminus U : \{u, v\} \in E(G) \text{ for some } u \in U\}.$$

To translate the problem of minimizing comparability into a vertex isoperimetric problem, we must define a graph. Let Γ be a graph with vertex set $\mathcal{P}([n])$ where two distinct sets $X \in \mathcal{P}([n])$ and $Y \in \mathcal{P}([n])$ are adjacent if and only if they are comparable. Now, for $U \subseteq \mathcal{P}([n])$, $N(U)$ consists of the sets that are comparable to U. The goal is to find the set of vertices of a given size that minimize $|N(U)| = c(U) - |U|$.

Isoperimetric questions have been widely studied. One of the first results on the isoperimetric problem in discrete math is for the hypercube. Recall that the n-dimensional hypercube, Q_n, has vertex set $\mathcal{P}([n])$ and an edge $\{X, Y\}$ if and only if X and Y differ in exactly one element. Harper's Theorem [22] resolves the vertex isoperimetric problem for Q_n. To state Harper's Theorem, we first need to define the *lexicographic ordering* on $\mathcal{P}([n])$.

Definition 2.1.19. Let $A = \{a_1, a_2, \cdots, a_k\} \subseteq [n]$ and $B = \{b_1, b_2, \cdots, b_k\} \subseteq [n]$ be distinct sets where $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k$. We say that $A \prec_{lex} B$ if and only if $a_j < b_j$ where $j = \min\{i : a_i \neq b_i\}$. This is the *lexicographic ordering* or *lex ordering* on $\binom{[n]}{k}$.

Now we can define the ordering for Harper's Theorem.

Definition 2.1.20. Given two sets $X, Y \in \mathcal{P}([n])$, $X \prec Y$ in the *simplicial ordering* if $|X| < |Y|$ or $|X| = |Y|$ and $X \prec_{lex} Y$.

Now we can state Harper's Theorem.

Theorem 2.1.21 (Harper's Theorem [22]). *Let $U \subseteq V(Q_n)$. Let C be an initial segment of the simplicial ordering of $\mathcal{P}([n])$ with $|C| = |U|$. Then $|N(C)| \leq |N(U)|$.*

Harper's Theorem also implies that there is a sequence of nested families that minimize the vertex boundary of the hypercube. Theorem 1.3.9, as well as the other results in this chapter, has the same flavour as the Kruskal-Katona Theorem and Harper's Theorem. A recent paper of Gowty, Horsley, and Mammoliti [20] concerning the comparability number was published independently of Behague, Kuperus, Morrison, and Wright [7]. They give a very different proof of Theorem 1.3.9 (see Corollary 1.2 of [20]) and use it as we do to deduce Theorem 4.4.1. They also provide some very interesting further analysis of the comparability number and families that minimize $c(n, m)$. Most notably, they show that there is a sequence of nested families that minimize comparability.

This chapter is structured as follows. In Section 2.2 we prove Theorem 1.3.9. In Section 2.3, we illustrate a large family of extremal examples. In Section 2.4 we determine the comparability number for small families.

2.2 General Lower Bound on $c(n, m)$

In this section, we will prove the following theorem.

Theorem 1.3.9. *For $1 \leq m \leq 2^n$,*

$$c(n, m) \geq 2^{n/2+1}\sqrt{m} - m.$$

This result is best possible when $m = 2^d$ and n and d have the same parity. This will be discussed more in Section 2.3. To prove Theorem 1.3.9, we need the following lemma of Kleitman. For completeness, we give a proof of this lemma inspired by [26].

Lemma 2.2.1. (Harris-Kleitman Inequality [26]). *Let* $\mathcal{U} \subseteq \mathcal{P}([n])$ *be an upset and* $\mathcal{D} \subseteq \mathcal{P}([n])$ *be a downset. Then*

$$\frac{|\mathcal{U} \cap \mathcal{D}|}{2^n} \leq \frac{|\mathcal{U}|}{2^n} \cdot \frac{|\mathcal{D}|}{2^n}.$$

Proof. We will prove that $|\mathcal{U} \cap \mathcal{D}| 2^n \leq |\mathcal{U}||\mathcal{D}|$ by induction on n. For $n = 1$, the result is trivial. Suppose the theorem holds for $n-1$. Let $\mathcal{U} \subseteq \mathcal{P}([n])$ be an upset, and $\mathcal{D} \subseteq \mathcal{P}([n])$ be a downset. Define

$$\begin{aligned} \mathcal{U}_n &= \{A \in \mathcal{U} : n \in A\} \\ \mathcal{U}_{\overline{n}} &= \{A \in \mathcal{U} : n \notin A\}. \end{aligned}$$

Define $\mathcal{D}_n$ and $\mathcal{D}_{\overline{n}}$ similarly. For every $A \in \mathcal{U}_{\overline{n}}$, $A \cup \{n\} \in \mathcal{U}_n$ since $\mathcal{U}$ is an upset. Thus $|\mathcal{U}_n| \geq |\mathcal{U}_{\overline{n}}|$. For every $B \in \mathcal{D}_n$, $B \setminus \{n\} \in \mathcal{D}_{\overline{n}}$ since $\mathcal{D}$ is a downset. Thus $|\mathcal{D}_n| \leq |\mathcal{D}_{\overline{n}}|$. Notice that $\mathcal{U}_{\overline{n}}$ is an upset in $\mathcal{P}([n-1])$ and $\mathcal{D}_{\overline{n}}$ is a downset in $\mathcal{P}([n-1])$. By the induction hypothesis,

$$|\mathcal{U}_{\overline{n}} \cap \mathcal{D}_{\overline{n}}| 2^{n-1} \leq |\mathcal{U}_{\overline{n}}||\mathcal{D}_{\overline{n}}|. \tag{2.1}$$

Now define

$$\begin{aligned} \mathcal{U}'_n &= \{A \setminus \{n\} : A \in \mathcal{U}_n\} \\ \mathcal{D}'_n &= \{B \setminus \{n\} : B \in \mathcal{D}_n\}. \end{aligned}$$

As $\mathcal{U}'_n \in \mathcal{P}([n-1])$ is an upset and $\mathcal{D}'_n \in \mathcal{P}([n-1])$ is a downset, the inductive hypothesis holds, giving

$$|\mathcal{U}_n \cap \mathcal{D}_n| 2^{n-1} = |\mathcal{U}'_n \cap \mathcal{D}'_n| 2^{n-1} \leq |\mathcal{U}'_n||\mathcal{D}'_n| = |\mathcal{U}_n||\mathcal{D}_n|. \tag{2.2}$$

Combining (2.1) and (2.2) gives

$$2^n |\mathcal{U} \cap \mathcal{D}| = 2^n (|\mathcal{U}_n \cap \mathcal{D}_n| + |\mathcal{U}_{\overline{n}} \cap \mathcal{D}_{\overline{n}}|) \leq 2(|\mathcal{U}_n||\mathcal{D}_n| + |\mathcal{U}_{\overline{n}}||\mathcal{D}_{\overline{n}}|).$$

Since $|\mathcal{U}_{\overline{n}}| \leq |\mathcal{U}_n|$ and $|\mathcal{D}_{\overline{n}}| \geq |\mathcal{D}_n|$,

$$|\mathcal{U}_n||\mathcal{D}_n| + |\mathcal{U}_{\overline{n}}||\mathcal{D}_{\overline{n}}| \leq |\mathcal{U}_n||\mathcal{D}_{\overline{n}}| + |\mathcal{U}_{\overline{n}}||\mathcal{D}_n|.$$

This gives

$$2^n |\mathcal{U} \cap \mathcal{D}| \leq |\mathcal{U}_n||\mathcal{D}_n| + |\mathcal{U}_{\overline{n}}||\mathcal{D}_{\overline{n}}| + |\mathcal{U}_n||\mathcal{D}_{\overline{n}}| + |\mathcal{U}_{\overline{n}}||\mathcal{D}_n| = (|\mathcal{U}_n| + |\mathcal{U}_{\overline{n}}|)(|\mathcal{D}_n| + |\mathcal{D}_{\overline{n}}|) = |\mathcal{U}||\mathcal{D}|,$$

as required. ☐

We also require the arithmetic mean - geometric mean (AM-GM) inequality.

Lemma 2.2.2. (AM-GM Inequality). *For any n nonnegative real numbers $x_1, x_2, \cdots, x_n$ we have*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{\frac{1}{n}}.$$

Now we can prove Theorem 1.3.9.

Proof. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be such that $|\mathcal{F}| = m$ and $c(\mathcal{F}) = c(n,m)$. We may assume that $\mathcal{F}$ is convex by Lemma 2.1.13. Define

$$u(\mathcal{F}) = \{X \in \mathcal{P}([n]) : F \subseteq X \text{ for some } F \in \mathcal{F}\}$$

and

$$d(\mathcal{F}) = \{X \in \mathcal{P}([n]) : X \subseteq F \text{ for some } F \in \mathcal{F}\}.$$

By Observation 2.1.8, $c(\mathcal{F}) = |u(\mathcal{F})| + |d(\mathcal{F})| - |u(\mathcal{F}) \cap d(\mathcal{F})|$. Since $\mathcal{F}$ is convex, by Observation 2.1.12, $|u(\mathcal{F}) \cap d(\mathcal{F})| = m$. Applying the AM-GM inequality (Lemma 2.2.2),

$$c(\mathcal{F}) \geq 2\sqrt{|u(\mathcal{F})||d(\mathcal{F})|} - m.$$

Since $u(\mathcal{F})$ is an upset and $d(\mathcal{F})$ is a downset, we apply the Harris-Kleitman inequality (Lemma 2.2.1) to get

$$c(\mathcal{F}) \geq 2\sqrt{2^n m} - m = 2^{\frac{n}{2}+1}\sqrt{m} - m.$$

Thus $c(n,m) \geq 2^{\frac{n}{2}+1}\sqrt{m} - m$. □

This result is interesting in its own right, as well as being useful for proving Theorem 1.3.7 in Chapter 3 and Theorem 1.3.8 in Chapter 4. Having determined a lower bound on $c(n,m)$, we show that this result is tight for certain values of n and m.

2.3 Structure of Some Extremal Families

In this section we will describe a large class of (n,m)-best families for infinitely many values of n and m. Not only is it interesting to understand some of the families that achieve equality in Theorem 1.3.9, these constructions will also inspire the proof of Theorem 1.3.8. We begin by defining two family structures, the subcube and the star.

Definition 2.3.1. A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is a *subcube of dimension d* if $\mathcal{F}$ is convex with a unique minimal and unique maximal set, and $|\mathcal{F}| = 2^d$. Let $\mathcal{C}(d,\ell)$ be the class of subcubes of dimension d whose minimal set has size ℓ and maximal set has size $\ell + d$. If $\mathcal{F} \in \mathcal{C}(d,\ell)$ we say $\mathcal{F}$ is a $\mathcal{C}(d,\ell)$ subcube.

Example 2.3.2. The family

$$\mathcal{F} = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$$

is a subcube of dimension 2. Specifically, $\mathcal{F}$ is a $\mathcal{C}(2,1)$ subcube.

Definition 2.3.3. A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is a *star* if it is of the form

$$\mathcal{F} = \{X, X \cup \{a_1\}, X \cup \{a_2\}, \cdots, X \cup \{a_{m-1}\}\}$$

for some $\{a_1, a_2, \cdots, a_{m-1}\} \subseteq [n] \setminus X$. Let $\mathcal{S}(m, \ell)$ be the class of stars of size m whose minimal set has size ℓ. If $\mathcal{F} \in \mathcal{S}(m, \ell)$ we say $\mathcal{F}$ is an $\mathcal{S}(m, \ell)$ star. A family $\mathcal{F}^* \subseteq \mathcal{P}([n])$ is an *upside-down star* if it is of the form

$$\mathcal{F}^* = \{X, X \setminus \{a_1\}, X \setminus \{a_2\}, \cdots, X \setminus \{a_{m-1}\}\}$$

for some $\{a_1, a_2, \cdots, a_{m-1}\} \subseteq X$. Let $\mathcal{S}^*(m, \ell)$ be the class of stars of size m whose maximal set has size ℓ. If $\mathcal{F}^* \in \mathcal{S}^*(m, \ell)$ we say $\mathcal{F}^*$ is an $\mathcal{S}^*(m, \ell)$ upside-down star.

Example 2.3.4. The family

$$\mathcal{F} = \{\{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}\}$$

is a star. Specifically, $\mathcal{F}$ is an $\mathcal{S}(4,2)$ star. The family

$$\mathcal{F}^* = \{\{1,2,3,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}\}$$

is an upside-down star. In particular, $\mathcal{F}^*$ is an $\mathcal{S}^*(4,4)$ star.

It is relatively straightforward to find the comparability number of the subcube.

Lemma 2.3.5. *Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a $\mathcal{C}(d, \ell)$ subcube. Then*

$$c(\mathcal{F}) = 2^{\ell+d} + 2^{n-\ell} - 2^d.$$

Proof. Let $X \in \mathcal{F}$ be the minimal set in $\mathcal{F}$ of size ℓ. Let $Y \in \mathcal{F}$ be the maximal set in $\mathcal{F}$ of size $\ell + d$. By Observation 2.1.8, we have that

$$c(\mathcal{F}) = |u(\mathcal{F})| + |d(\mathcal{F})| - |u(\mathcal{F}) \cap d(\mathcal{F})|.$$

Since $\mathcal{F}$ is a subcube, it is convex. By Observation 2.1.12, $|u(\mathcal{F}) \cap d(\mathcal{F})| = |\mathcal{F}| = 2^d$. Notice that $u(\mathcal{F}) = u(X)$ since X is the unique minimal set of $\mathcal{F}$, so $X \subseteq Z$ for all $Z \in \mathcal{F}$. Similarly, $d(\mathcal{F}) = d(Y)$ since Y is the unique maximal element. So $Z \subseteq Y$ for all $Z \in \mathcal{F}$. Thus

$$c(\mathcal{F}) = 2^{|Y|} + 2^{n-|X|} - 2^d,$$

as desired. ☐

The comparability number of the star is slightly more tedious to determine.

Lemma 2.3.6. *Let $\mathcal{F}$ be a $\mathcal{S}(m,\ell)$ star. Then $c(\mathcal{F}) = m2^\ell + 2^{n-\ell} - m$.*

Proof. Let $X \subseteq [n]$ be such that $|X| = \ell$. Let $\mathcal{F} = \{X, X \cup \{a_1\}, X \cup \{a_2\}, \cdots, X \cup \{a_{m-1}\}\}$ for $\{a_1, a_2, \cdots, a_{m-1}\} \subseteq [n] \setminus X$, so $\mathcal{F}$ is a $\mathcal{S}(m,\ell)$ star. To begin, observe that

$$|u(X) \cup d(X)| = 2^\ell + 2^{n-\ell} - 1.$$

Notice that $u(X \cup \{a_i\}) \subseteq u(X)$ for all $1 \leq i \leq m-1$. So we only need to consider the downsets of the remaining sets in $\mathcal{F}$. As

$$d(X \cup \{a_1\}) \setminus d(X) = \{Y \cup \{a_1\} : Y \subseteq X\},$$

we have

$$|d(X \cup \{a_1\}) \setminus d(X)| = 2^\ell.$$

In general,

$$\left| d(X \cup \{a_i\}) \setminus \left(d(X) \cup \bigcup_{j<i} d(X \cup \{a_j\}) \right) \right| = 2^\ell.$$

Putting it all together,

$$c(\mathcal{F}) = |u(X) \cup d(X)| + |d(X \cup \{a_1\}) \setminus d(X)| + \sum_{i=2}^{m-1} \left| d(X \cup \{a_i\}) \setminus \left(d(X) \cup \bigcup_{j<i} d(X \cup \{a_j\}) \right) \right|.$$

Thus $c(\mathcal{F}) = m2^\ell + 2^{n-\ell} - m$. ☐

Stars and upside-down stars are symmetric with respect to taking complements.

Observation 2.3.7. *For every family $\mathcal{F} \subseteq \mathcal{P}([n])$ in $\mathcal{S}(m,\ell)$, there exists a family $\mathcal{F}^* \subseteq \mathcal{P}([n])$ in $\mathcal{S}^*(m, n-\ell)$ such that $\mathcal{F}^* = \{[n] \setminus F : F \in \mathcal{F}\}$.*

It will be helpful to notice that taking the complement of a family does not change the comparability number. Define $\mathcal{F}^c := \{[n] \setminus F : F \in \mathcal{F}\}$.

Lemma 2.3.8. *Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a family. Define $\mathcal{F}^c := \{[n] \setminus F : F \in \mathcal{F}\}$. Then $c(\mathcal{F}) = c(\mathcal{F}^c)$.*

Proof. Let $X \in d(\mathcal{F})$. Then $[n] \setminus X \in u(\mathcal{F}^c)$. Moreover, for $Y \in u(\mathcal{F}^c)$, $[n] \setminus Y \in d(\mathcal{F})$. Thus, $|d(\mathcal{F})| = |u(\mathcal{F}^c)|$ Similarly, there is a 1-1 correspondence between $u(\mathcal{F}^c)$ and $d(\mathcal{F})$. Thus, by Observation 2.1.8,

$$c(\mathcal{F}^c) = |\{[n] \setminus X : X \text{ is comparable to } \mathcal{F}\}| = c(\mathcal{F}).$$

□

Using Observation 2.3.7 and Lemma 2.3.8, we can also prove that upside-down stars have the same comparability number as stars.

Corollary 2.3.9. *Let $\mathcal{F}^*$ be a $\mathcal{S}^*(m,\ell)$ star. Then $c(\mathcal{F}^*) = m2^\ell + 2^{n-\ell} - m$.*

Proof. By Observation 2.3.7, there exists a family $\mathcal{F} \subseteq \mathcal{P}([n])$ such that $\mathcal{F}^c = \mathcal{F}^*$ and $\mathcal{F}$ is a $\mathcal{S}(m,\ell)$ star. By Lemma 2.3.6, $c(\mathcal{F}) = m2^\ell + 2^{n-\ell} - m$. By Lemma 2.3.8, $c(\mathcal{F}) = c(\mathcal{F}^c) = c(\mathcal{F}^*)$, as desired. □

From now on we will only consider the star construction as we describe the extremal families. One should note, however, that whenever the star is an extremal construction, the upside-down star is also an extremal construction.

Now we can prove that for certain values of n and m, the star and the cube are extremal constructions that minimize comparability.

Theorem 2.3.10. *Let $n, d \in \mathbb{N}$ have the same parity and $\frac{n+d}{2} \geq 2^d$. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a $\mathcal{S}(2^d, \frac{n-d}{2})$ star. Then $\mathcal{F}$ is an $(n, 2^d)$-best family.*

Proof. Apply Lemma 2.3.6 with $\ell = \frac{n-d}{2}$. Notice that we need $n - \frac{n-d}{2} \geq 2^d$ in order to guarantee there are enough sets to build a star of this form. □

Theorem 2.3.11. *Let $n, d \in \mathbb{N}$ have the same parity. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a $\mathcal{C}(d, \frac{n-d}{2})$ subcube. Then $\mathcal{F}$ is an $(n, 2^d)$-best family.*

Proof. Apply Lemma 2.3.5 with $\ell = \frac{n-d}{2}$. □

Using these constructions we can define an operation that 'moves' between the star and the cube and keeps the comparability number the same. In this way, we describe a large family of constructions that achieve the comparability number $c(n, 2^d)$, when n and d have the same parity.

Lemma 2.3.12. *Let $X \in [n]$ and $\mathcal{G} \subseteq \mathcal{P}([n] \setminus X)$ be a downset. Then the family $\mathcal{F} = \{X \cup Y : Y \in \mathcal{G}\}$ has $c(\mathcal{F}) = 2^{n-|X|} + (|\mathcal{F}|)2^{|X|} - |\mathcal{F}|$.*

Proof. Let $X \in [n]$ and $\mathcal{G} \subseteq \mathcal{P}([n] \setminus X)$ be a downset. Define $\mathcal{F} = \{X \cup Y : Y \in \mathcal{G}\}$. We will define an operation ϕ that we apply to $\mathcal{F}$ that will maintain its comparability number.

The operation ϕ takes in a family $\mathcal{F}$ and a set $Y_1 \in \mathcal{G}$ such that $X \cup Y_1$ is the maximal element of a chain in $\mathcal{F}$ with length at least 3. Then $\phi(\mathcal{F}, Y_1)$ is a new family $\mathcal{F}'$ obtained from $\mathcal{F}$ by replacing $X \cup Y_1$ with $X \cup Y_2$ where $Y_2 \subseteq [n] \setminus X$, $|Y_2| = 1$ and $X \cup Y_2 \notin \mathcal{F}$.

Now we will show that $c(\mathcal{F}) = c(\mathcal{F}')$. For any set $S \in \mathcal{F}$, define

$$c_{\mathcal{F}}(S) := |\{Y \subseteq [n] : Y \text{ is comparable to } S, Y \text{ is incomparable to all } Z \neq S \in \mathcal{F}\}|.$$

So $c_{\mathcal{F}}(S)$ is the number of sets that are only comparable to S and not comparable to any other set in $\mathcal{F}$. Since ϕ only changes one set in $\mathcal{F}$, it suffices to show that $c_{\mathcal{F}}(X \cup Y_1) = c_{\mathcal{F}'}(X \cup Y_2)$.

First we consider $c_{\mathcal{F}}(X \cup Y_1)$. Notice that the sets that are only comparable to $X \cup Y_1$ and incomparable to all other sets in $\mathcal{F}$ are of the form $\{Z \cup Y_1 : Z \subseteq X\}$, Thus $c_{\mathcal{F}}(X \cup Y_1) = 2^{|X|}$. Now we consider $c_{\mathcal{F}'}(X \cup Y_2)$. Again, the sets that are only comparable to $X \cup Y_2$ and incomparable to all other sets in $\mathcal{F}$ are of the form $\{Z \cup Y_2 : Z \subseteq X\}$. Thus $c_{\mathcal{F}}(X \cup Y_2) = 2^{|X|}$. Applying the operation ϕ to $\mathcal{F}$ decreases the comparability number by $2^{|X|}$ and increases the comparability number by $2^{|X|}$. Thus $c(\mathcal{F}) = c(\mathcal{F}')$. Also notice that $|\mathcal{F}| = |\mathcal{F}'|$.

Now we apply ϕ to $\mathcal{F}$ repeatedly until all the chains have length 2. Call this family $\mathcal{H}$. Now, $\mathcal{H}$ is a star and by Lemma 2.3.6, $c(\mathcal{H}) = 2^{n-|X|} + (|\mathcal{H}|)2^{|X|} - |\mathcal{H}|$. Since the comparability number of the family does not change after each application of ϕ, $c(\mathcal{F}) = 2^{n-|X|} + (|\mathcal{F}|)2^{|X|} - |\mathcal{F}|$. □

Lemma 2.3.12 allows us to describe a large family of extremal families $\mathcal{F} \in \mathcal{P}([n])$ that minimize comparability when $|\mathcal{F}| = 2^d$ and n and d have the same parity.

Theorem 2.3.13. *Let n and d have the same parity. Fix $X \subseteq [n]$ such that $|X| = \frac{n-d}{2}$. Let $\mathcal{G} \subseteq \mathcal{P}([n] \setminus X)$ be a downset such that $|\mathcal{G}| = 2^d$. Then the family $\mathcal{F} = \{X \cup Y : Y \in \mathcal{G}\}$ is a $(n, 2^d)$-best family. In particular, the class of $\mathcal{C}(d, \frac{n-d}{2})$ cubes and $\mathcal{S}(2^d, \frac{n-d}{2})$ stars are $(n, 2^d)$-best families.*

Proof. By Lemma 2.3.12, $c(\mathcal{F}) = 2^{\frac{n+d}{2}+1} - 2^d = c(n, 2^d)$. Again by Lemma 2.3.12, $\mathcal{C}(d, \frac{n-d}{2})$ cubes and $\mathcal{S}(2^d, \frac{n-d}{2})$ stars are all families of this form. □

Example 2.3.14 will illustrate the operation ϕ and Theorem 2.3.13. See Figure 2.1 for a visual representation of this process.

Example 2.3.14. Let $n = 11$, $d = 3$ and $X = \{1, 2, 3, 4\}$. Let

$$\mathcal{G} = \{\emptyset, \{5\}, \{6\}, \{7\}, \{5, 6\}, \{5, 7\}, \{6, 7\}, \{5, 6, 7\}\}.$$

Then

$$\mathcal{F} = \{\{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 4, 7\}, \{1, 2, 3, 4, 5, 6\}, \\ \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 6, 7\}, \{1, 2, 3, 4, 5, 6, 7\}\}.$$

We first let $Y_2 = \{8\}$, so

$$\mathcal{F}_1 := \phi(\mathcal{F}, \{5, 6, 7\}) = \{\{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 4, 7\}, \\ \{1, 2, 3, 4, 8\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 6, 7\}\}.$$

Applying the operation ϕ again with $Y_2 = \{9\}$ we have

$$\begin{aligned}\mathcal{F}_2 := \phi(\mathcal{F}, \{6,7\}) = \{&\{1,2,3,4\}, \{1,2,3,4,5\}, \{1,2,3,4,6\}, \{1,2,3,4,7\},\\ &\{1,2,3,4,8\}, \{1,2,3,4,9\}, \{1,2,3,4,5,6\}, \{1,2,3,4,5,7\}\}.\end{aligned}$$

Another application of the operation ϕ gives

$$\begin{aligned}\mathcal{F}_3 := \phi(\mathcal{F}, \{5,7\}) = \{&\{1,2,3,4\}, \{1,2,3,4,5\}, \{1,2,3,4,6\}, \{1,2,3,4,7\},\\ &\{1,2,3,4,8\}, \{1,2,3,4,9\}, \{1,2,3,4,10\}, \{1,2,3,4,5,6\}\}.\end{aligned}$$

One final application of ϕ gives

$$\begin{aligned}\mathcal{F}_4 := \phi(\mathcal{F}, \{5,6\}) = \{&\{1,2,3,4\}, \{1,2,3,4,5\}, \{1,2,3,4,6\}, \{1,2,3,4,7\},\\ &\{1,2,3,4,8\}, \{1,2,3,4,9\}, \{1,2,3,4,10\}, \{1,2,3,4,11\}\}.\end{aligned}$$

Notice that every chain in $\mathcal{F}_4$ has length 2, so we can no longer apply the operation ϕ, and $\mathcal{F}_4$ is a $\mathcal{S}(8,4)$ star. We also have that $c(\mathcal{F}) = 248$ and $c(\mathcal{F}_4) = 248$. By Theorem 1.3.9, $c(11,8) \geq 248$. Thus, $\mathcal{F}$ and $\mathcal{F}_4$, as well as $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$, are $(11,8)$-best families.

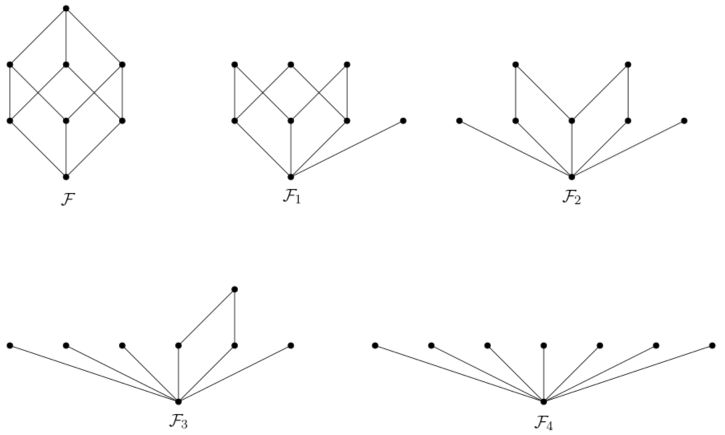

Figure 2.1: Illustration of Example 2.3.14. Each vertex represents a set of $\mathcal{F}$. The vertices in the bottom layer represent sets of size four. The vertices in the next layer up represent sets of size five and the vertices in the third layer up (if any) represent sets of size six. There is an edge when one set is contained in the other, and the sets differ in size by one.

While we cannot determine the exact class of (n,m)-best families for values of m that are not powers of two, we can say something about the size of the sets in an extremal family. We first observe that $c(n,m)$ is non-decreasing. This will be used in

Lemmas 2.3.16 and 2.3.17 to compare the comparability number of families of size m to families of size $2^{\lceil \log_2 m \rceil}$.

Observation 2.3.15. *$c(n, m) \leq c(n, m+1)$.*

To see this, note that if $c(n, m) > c(n, m+1)$ then we could remove a set from an $(n, m+1)$-best family and get a family of size m with comparability number less than $c(n, m)$.

The next two lemmas tell us that an (n, m)-best family only has sets of size 'close' to $n/2$, where 'close' is determined by the size of the family. Note that the 'middle' levels of the hypercube are the sets of size $n/2$ if n is even and the sets of size $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ if n is odd. Lemmas 2.3.16 and 2.3.17 will be useful in proving Lemma 2.4.3.

Lemma 2.3.16. *Let $\mathcal{F}$ be an (n, m)-best family. If n and $\lceil \log_2(m) \rceil$ have the same parity then*

$$\mathcal{F} \subseteq \bigcup_{i=\frac{n-\lceil \log_2 m \rceil}{2}}^{\frac{n+\lceil \log_2 m \rceil}{2}} \binom{[n]}{i}.$$

Proof. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be an (n, m)-best family and suppose n and $\lceil \log_2(m) \rceil$ have the same parity. Suppose that there exists $X \in \mathcal{F}$ such that $|X| \leq \frac{n-\lceil \log_2(m) \rceil}{2} - 1$. Then

$$|u(X)| \geq 2^{\frac{n+\lceil \log_2(m) \rceil}{2}+1} \text{ and } |d(X)| \geq 1.$$

Then, by Observation 2.1.8

$$c(X) \geq 2^{\frac{n+\lceil \log_2(m) \rceil}{2}+1} + 1.$$

Let $\mathcal{C} \subseteq \mathcal{P}([n])$ be a $\mathcal{C}(\lceil \log_2(m) \rceil, \frac{n-\lceil \log_2(m) \rceil}{2})$ subcube. By Observation 2.3.15, $c(n, m) \leq c(\mathcal{C})$ since $|\mathcal{C}| = 2^{\lceil \log_2(m) \rceil}$. By Lemma 2.3.5,

$$c(\mathcal{C}) = 2^{\frac{n+\lceil \log_2(m) \rceil}{2}+1} - 2^{\lceil \log_2(m) \rceil}.$$

So $c(\mathcal{F}) \geq c(X) > c(\mathcal{C})$, contradicting that $\mathcal{F}$ is an (n, m)-best family.

By Lemma 2.3.8 if we assume there exists a $Y \in \mathcal{F}$ such that $|Y| \geq \frac{n+\lceil \log_2(m) \rceil}{2} + 1$, then $c(\mathcal{F}) > c(\mathcal{C})$. Thus $\mathcal{F}$ only contains sets of size m such that $\frac{n-\lceil \log_2 m \rceil}{2} \leq m \leq \frac{n+\lceil \log_2 m \rceil}{2}$. □

When the parity of n and $\lceil \log_2 m \rceil$ are not the same, the result is slightly weaker.

Lemma 2.3.17. *Let $\mathcal{F}$ be an (n, m)-best family. If n and $\lceil \log_2(m) \rceil$ have different parity then*

$$\mathcal{F} \subseteq \bigcup_{i=\frac{n-\lceil \log_2 m-1 \rceil}{2}}^{\frac{n+\lceil \log_2 m+1 \rceil}{2}} \binom{[n]}{i}.$$

Proof. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be an (n, m)-best family and suppose n and $\lceil \log_2(m) \rceil$ have different parities. Suppose that there exists $X \in \mathcal{F}$ such that $|X| \leq \frac{n - \lceil \log_2(m) \rceil - 1}{2} - 1$. Then $u(X) \geq 2^{\frac{n + \lceil \log_2(m) \rceil + 1}{2} + 1}$. By Observation 2.1.8,

$$c(X) \geq 2^{\frac{n + \lceil \log_2(m) \rceil + 1}{2} + 1}.$$

Let $\mathcal{C} \subseteq \mathcal{P}([n])$ be a $\mathcal{C}(\lceil \log_2(m) \rceil, \frac{n - \lceil \log_2(m) \rceil - 1}{2})$ subcube. Then by Lemma 2.3.5,

$$c(\mathcal{C}) = 2^{\frac{n + \lceil \log_2(m) \rceil + 1}{2} + 1} - 2^{\lceil \log_2(m) \rceil},$$

so $c(\mathcal{F}) \geq c(X) > c(\mathcal{C})$. This contradicts Observation 2.3.15, so $|X| \geq \frac{n - \lceil \log_2(m) \rceil - 1}{2}$. By Lemma 2.3.8, if there exists a $Y \in \mathcal{F}$ such that $|Y| \geq \frac{n + \lceil \log_2(m) \rceil + 1}{2} + 1$, then $c(\mathcal{F}) > c(\mathcal{C})$. Thus $\mathcal{F}$ only has sets of size m where $\frac{n - \lceil \log_2 m - 1 \rceil}{2} \leq m \leq \frac{n + \lceil \log_2 m + 1 \rceil}{2}$. □

2.4 Comparability Number for Small Values of m

In this section we determine the comparability number $c(n, m)$ for small values of m. These results on small comparability numbers will be useful in improving the bounds for the sum of cross-Sperner systems in Section 4.4. Using simple counting arguments, we can determine the values of $c(n, 1), c(n, 2)$, and $c(n, 3)$.

Lemma 2.4.1. $c(n,1) = \begin{cases} 2^{\frac{n}{2}+1} - 1, & \textit{if } n \textit{ is even} \\ 2^{\lfloor n/2 \rfloor} + 2^{\lceil n/2 \rceil} - 1, & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. We can explicitly compute the comparability number of a family $\mathcal{F} \in \mathcal{P}([n])$ where $\mathcal{F} = \{X\}$. Notice that $d(X)$ is all the subsets of X. There are $2^{|X|}$ subsets of X. The $u(X)$ is all the sets that contain X. There are $2^{n-|X|}$ such sets. Since the set X is counted in both the upset and the downset, $|d(X) \cap u(X)| = 1$. By Observation 2.1.8,

$$c(\mathcal{F}) = 2^{|X|} + 2^{n-|X|} - 1. \tag{2.3}$$

To find the minimum of (2.3), we take the derivative and find when it equals 0. The derivative of (2.3) is

$$\frac{d}{d|X|}(2^{|X|} + 2^{n-|X|} - 1) = \ln(2)(2^{|X|} - 2^{n-|X|}).$$

So (2.3) is minimized when $|X| = n/2$. Thus, when n is even, $c(\mathcal{F}) = 2(2^{n/2}) - 1$. When n is odd, (2.3) is minimized when $|X| = \lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$. Then $c(\mathcal{F}) = 2^{\lfloor n/2 \rfloor} + 2^{\lceil n/2 \rceil} - 1$. □

To determine the minimum number of sets comparable to a family of size two, we do a similar case analysis, expect this time we must consider the size of the intersection of the two sets in the family.

Lemma 2.4.2. *For $n \geq 3$,*

$$c(n,2) = \begin{cases} 3(2^{\frac{n}{2}}) - 2, & \text{if } n \text{ is even} \\ 2^{\lceil \frac{n}{2} \rceil + 1} - 2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $\mathcal{F} = \{X, Y\} \subseteq \mathcal{P}([n])$ where $X \neq Y$. Then

$$c(\mathcal{F}) = 2^{|X|} + 2^{n-|X|} + 2^{|Y|} + 2^{n-|Y|} - 2^{|X\cap Y|} - 2^{n-|X\cup Y|} - 2. \tag{2.4}$$

The first four terms in (2.4) count the upsets and downsets of X and Y. By the principle of inclusion and exclusion, we have double counted the downset of $X \cap Y$, the upset of $X \cup Y$, and X and Y themselves.

Let $|X| = k$ and $|Y| = \ell$. We now have the following two cases.

Case 1: Suppose $k = \ell$. Then $|X \cap Y| \leq k - 1$ and $|X \cup Y| \geq k + 1$. So

$$c(\mathcal{F}) \geq 2^k + 2^{n-k} + 2^k + 2^{n-k} - 2^{k-1} - 2^{n-k-1} - 2 = \frac{3}{2}2^k + \frac{3}{2}2^{n-k} - 2. \tag{2.5}$$

To find the minimum of (2.5), we take the derivative and find when it is equal to zero. The derivative of (2.5) is $\frac{3\ln 2}{2}(2^k - 2^{n-k})$, which is zero when $k = \frac{n}{2}$. Thus, when n is even, $k = \frac{n}{2}$ is an integer and (2.5) gives $c(\mathcal{F}) \geq 3(2^{\frac{n}{2}}) - 2$. This bound can be achieved. Take $X \in \binom{[n]}{\frac{n}{2}}$. Pick some $x \in X$ and $y \in [n] \setminus X$. Define $Y := (X \setminus \{x\}) \cup \{y\}$. Then $|X \cap Y| = k - 1$ and $|X \cup Y| = k + 1$, so $c(\{X, Y\}) = 3(2^{\frac{n}{2}}) - 2$ when n is even.

When n is odd, $\frac{n}{2}$ is not an integer, so we consider when $k = \lfloor \frac{n}{2} \rfloor$ and when $k = \lceil \frac{n}{2} \rceil$, as these are the closest integers to $\frac{n}{2}$. For both values of k, (2.5) is $2^{\lceil \frac{n}{2} \rceil + 1} - 2$. This bound can also be achieved. Take $X \in \binom{[n]}{\lfloor n/2 \rfloor}$. Pick $x \in X$ and $y \in [n] \setminus X$ and define $Y := (X \setminus \{x\}) \cup \{y\}$. Then $|X \cap Y| = \lfloor n/2 \rfloor - 1$ and $|X \cup Y| = \lceil \frac{n}{2} \rceil$, so $c(\{X, Y\}) = 2^{\lceil \frac{n}{2} \rceil + 1} - 2$, when n is odd. The bound can also be achieved by taking $X \in \binom{[n]}{\lceil \frac{n}{2} \rceil}$ and choosing a set $Y \in \binom{[n]}{\lceil \frac{n}{2} \rceil}$ such that $|X \cap Y| = \lfloor \frac{n}{2} \rfloor$ and $|X \cup Y| = \lceil \frac{n}{2} \rceil + 1$.

In the next case we will show that taking sets of different sizes will produce a family with comparability number strictly greater than that of the families considered in Case 1.

Case 2: Suppose $k \neq \ell$. First we will show that we can assume $\ell = k - 1$. Suppose $\ell < k - 1$ and $\ell \leq \lfloor \frac{n}{2} \rfloor$. Take $x \in X \setminus Y$ and define $Y' := Y \cup \{x\}$. Then $|Y'| = \ell + 1$. Notice that $|X \cup Y'| = |X \cup Y|$ but $|X \cap Y'| = |X \cap Y| + 1$. Define $\mathcal{F}' := \{X, Y'\}$.

Then

$$\begin{aligned}
c(\mathcal{F}) - c(\mathcal{F}') &= (2^{\ell} + 2^{n-\ell} - 2^{|X\cap Y|}) - (2^{\ell+1} + 2^{n-\ell-1} - 2^{|X\cap Y'|}) \\
&\geq 2^{\ell} - 2^{\ell+1} + 2^{n-\ell} - 2^{n-\ell-1} - 2^{|X\cap Y|} + 2^{|X\cap Y|+1} \\
&= -2^{\ell} + 2^{n-\ell-1} + 2^{|X\cap Y|} \\
&\geq -2^{\ell} + 2^{n-\ell-1} + 1 > 0 \quad \text{since } \ell \leq \left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

If $\ell \geq \lceil \frac{n}{2} \rceil$, then $|[n] \setminus Y| \leq \lfloor \frac{n}{2} \rfloor$ and $|[n] \setminus X| < \lfloor \frac{n}{2} \rfloor - 1$. Let $\ell' = |[n] \setminus X|$ and $k' = |[n] \setminus Y|$. Then $\ell' < \lfloor \frac{n}{2} \rfloor$. By Lemma 2.3.8, $c(\mathcal{F}) = c(\{[n] \setminus X, [n] \setminus Y\})$, so the argument also holds when $\ell \geq \lceil \frac{n}{2} \rceil$. Thus $c(\mathcal{F}) > c(\mathcal{F}')$. So, we can assume that $\ell = k - 1$, as otherwise we can create a family $\mathcal{F}'$ from $\mathcal{F}$ where $c(\mathcal{F}) > c(\mathcal{F}')$.

Now suppose $\ell = k - 1$. Then $|X \cap Y| \leq k - 1$ and $|X \cup Y| \geq k$. This implies

$$c(\mathcal{F}) \geq 2^{k} + 2^{n-k} + 2^{k-1} + 2^{n-k+1} - 2^{k-1} - 2^{n-k} - 2. \tag{2.6}$$

To minimize (2.6) we take the derivative and find when it equals zero. The derivative of (2.6) is $\ln 2 \left(2^{k} - 2^{n-k+1}\right)$ which is zero when $k = \frac{n-1}{2}$. Thus,

$$c(\mathcal{F}) \geq 2^{\frac{n-1}{2}} + 2^{\frac{n+1}{2}+1} - 2 > 2^{\lceil \frac{n}{2} \rceil + 1} - 2 > 3(2^{\frac{n}{2}}) - 2.$$

Thus, the case where $k \neq \ell$ has a larger comparability number than the case where $k = \ell$ (Case 1). We conclude that $c(n, 2) = 2^{\lceil \frac{n}{2} \rceil + 1} - 2$ when n is odd and $c(n, 2) = 3(2^{\frac{n}{2}}) - 2$, as in Case 1.

□

The main result of the remainder of this section determines $c(n, 3)$.

Lemma 2.4.3. *For $n \geq 4$,*

$$c(n,3) = \begin{cases} \frac{7}{2}(2^{\frac{n}{2}}) - 3, & \textit{if } n \textit{ is even} \\ \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3, & \textit{if } n \textit{ is odd.} \end{cases}$$

Claim 2.4.4 proves the upper bound on $c(n, 3)$.

Claim 2.4.4. *For $n \geq 4$,*

$$c(n,3) \leq \begin{cases} \frac{7}{2}(2^{\frac{n}{2}}) - 3, & \textit{if } n \textit{ is even} \\ \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3, & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof of Claim 2.4.4. We will give a construction of a family $\mathcal{F} = \{X, Y, Z\} \subseteq \mathcal{P}([n])$ to prove an upper bound on $c(n, 3)$. Suppose $|X| = k - 1$ and $|Y| = |Z| = k$. Moreover, suppose $X \subseteq Y$ and $X \subseteq Z$ and $Y \neq Z$. Then $|Y \cap Z| = k-1$. Let $\mathcal{F} = \{X, Y, Z\}$. Notice that $u(Y) \subseteq u(X)$ since $X \subseteq Y$ and $u(Z) \subseteq u(Y)$ since $X \subseteq Z$. Thus, by the principle of inclusion and exclusion,

$$c(\mathcal{F}) = |u(Y)| + |d(Y)| + |d(Z)| - |d(Y \cap Z)| - 3 = 2^{n-k+1} + 2^k + 2^k - 2^{k-1} - 3.$$

If n is even and $k = \frac{n}{2}$, then $c(\mathcal{F}) = \frac{7}{2}(2^{\frac{n}{2}}) - 3$. If n is odd and $k = \frac{n+1}{2}$, then $c(\mathcal{F}) = \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3$. $\square$

The purpose of Claim 2.4.5 is to show that an $(n,3)$-best family is contained within two middle layers of the hypercube.

Claim 2.4.5. *Suppose $\mathcal{F} = \{X, Y, Z\} \subseteq \mathcal{P}([n])$ is an $(n,3)$-best family. If n is even, then $\frac{n}{2} \leq |X|, |Y|, |Z| \leq \frac{n}{2} + 1$ or $\frac{n}{2} - 1 \leq |X|, |Y|, |Z| \leq \frac{n}{2}$. If n is odd, $\frac{n-1}{2} \leq |X|, |Y|, |Z| \leq \frac{n+1}{2}$.*

Proof of Claim 2.4.5.

Case 1: Suppose n is even. Without loss of generality, suppose $|X| = \frac{n}{2} - 1$ by Lemma 2.3.16. For contradiction, suppose $|Y| = \frac{n}{2}+1$. Then $|X \cap Y| \leq \frac{n}{2}-1$ and $|X \cup Y| \geq \frac{n}{2}+1$. Thus

$$\begin{aligned} c(\mathcal{F}) \geq c(\{X, Y\}) &\geq 2^{\frac{n}{2}-1} + 2^{\frac{n}{2}+1} + 2^{\frac{n}{2}+1} + 2^{\frac{n}{2}-1} - 2^{\frac{n}{2}-1} - 2^{\frac{n}{2}-1} - 2 \\ &= 4(2^{\frac{n}{2}}) - 2. \end{aligned}$$

Since $4(2^{\frac{n}{2}}) - 2 > \frac{7}{2}(2^{\frac{n}{2}}) - 3$, by Claim 2.4.4, this cannot be an $(n,3)$-best family. Thus, we must have that $\frac{n}{2} \leq |X|, |Y|, |Z| \leq \frac{n}{2} + 1$ or $\frac{n}{2} - 1 \leq |X|, |Y|, |Z| \leq \frac{n}{2}$.

Case 2: Suppose n is odd. By Lemma 2.3.17, an $(n,3)$-best family when n is odd will only have sets of size $\frac{n-3}{2}$, $\frac{n-1}{2}$, $\frac{n+1}{2}$, and $\frac{n+3}{2}$.

We show first that we cannot have a set of size $\frac{n-3}{2}$ and a set of size $\frac{n+3}{2}$. Suppose $|X| = \frac{n-3}{2}$ and $|Z| = \frac{n+3}{2}$. Then $|X \cap Z| \leq \frac{n-3}{2}$ and $|X \cup Z| \geq \frac{n+3}{2}$. Then,

$$\begin{aligned} c(\mathcal{F}) \geq c(\{X, Z\}) &\geq 2^{\frac{n-3}{2}} + 2^{\frac{n+3}{2}} + 2^{\frac{n+3}{2}} + 2^{\frac{n-3}{2}} - 2^{\frac{n-3}{2}} - 2^{\frac{n-3}{2}} - 2 \\ &= 2(2^{\frac{n+3}{2}}) - 2 \\ &> \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3. \end{aligned}$$

By Claim 2.4.4, this cannot be an $(n,3)$-best family. Thus, we cannot have a set of size $\frac{n-3}{2}$ and a set of size $\frac{n+3}{2}$.

Now we show that we cannot have a set of size $\frac{n-3}{2}$ and a set of size $\frac{n+1}{2}$, or, by Lemma 2.3.8, a set of size $\frac{n+3}{2}$ and a set of size $\frac{n-1}{2}$. Suppose that $|X| = \frac{n-3}{2}$ and

$|Z| = \frac{n+1}{2}$. Then, $|X \cap Z| \leq \frac{n-3}{2}$ and $|X \cup Z| \geq \frac{n+1}{2}$. Now,

$$\begin{aligned} c(\mathcal{F}) \geq c(\{X, Z\}) &\geq 2^{\frac{n-3}{2}} + 2^{\frac{n+3}{2}} + 2^{\frac{n+1}{2}} + 2^{\frac{n-1}{2}} - 2^{\frac{n-3}{2}} - 2^{\frac{n-1}{2}} - 2 \\ &= (2^{\frac{3}{2}} + 2^{\frac{1}{2}})2^{\frac{n}{2}} - 2 \\ &> \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3. \end{aligned}$$

By Claim 2.4.4, this cannot be an $(n, 3)$-best family. Thus, we cannot have a set of size $\frac{n-3}{2}$ and a set of size $\frac{n+1}{2}$ (or a set of size $\frac{n+3}{2}$ and a set of size $\frac{n-1}{2}$ by Lemma 2.3.8).

Thus, it must be the case that if n is odd and $\mathcal{F}$ is an $(n, 3)$-best family, $\frac{n-1}{2} \leq |X|, |Y|, |Z| \leq \frac{n+1}{2}$.

□

Claim 2.4.6 says that if an $(n, 3)$-best family has two sets of the same size, then the size of the intersection of those two sets is as large as possible.

Claim 2.4.6. *Suppose $\mathcal{F} = \{X, Y, Z\} \subseteq \mathcal{P}([n])$ is an $(n, 3)$-best family such that $|X| = |Y| = k$ and $|Z| = \ell > k$. Then $|X \cap Y| = k - 1$.*

Proof of Claim 2.4.6. Suppose $\mathcal{F} = \{X, Y, Z\} \subseteq \mathcal{P}([n])$ is an $(n, 3)$-best family such that $|X| = |Y| = k$ and $|Z| = \ell > k$. Moreover, suppose $|X \cap Y| \leq k - 2$. Then $|X \cup Y| \geq k + 2$. Thus,

$$c(\mathcal{F}) > c(\{X, Y\}) \geq 2^{k+1} + 2^{n-k+1} - 2^{k-2} - 2^{k-2} - 2.$$

By Claim 2.4.5, when n is even, $k = \frac{n}{2}$ without loss of generality. In this case,

$$c(\mathcal{F}) > 2^{\frac{n}{2}+1} + 2^{\frac{n}{2}+1} - 2^{\frac{n}{2}-2} - 2^{\frac{n}{2}-2} - 2 = \frac{7}{2}(2^{\frac{n}{2}}) - 2.$$

By Claim 2.4.4, $\mathcal{F}$ cannot be an $(n, 3)$-best family.

By Claim 2.4.5, when n is odd $k = \frac{n-1}{2}$ without loss of generality. In this case,

$$c(\mathcal{F}) > 2^{\frac{n+1}{2}} + 2^{\frac{n+3}{2}} - 2^{\frac{n-5}{2}} - 2^{\frac{n-3}{2}} - 2 = \frac{21}{4\sqrt{2}}(2^{\frac{n}{2}}) - 2 > \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3.$$

Thus, by Claim 2.4.4, this is not an $(n, 3)$-best family. □

Claim 2.4.7 says that if a family has two sets of different sizes, then there is a 2-chain.

Claim 2.4.7. *If $\mathcal{F} = \{X, Y, Z\} \subseteq \mathcal{P}([n])$ is an $(n, 3)$-best family such that $|X| = |Y| = k$ and $|Z| = \ell > k$, then $X \subseteq Z$ or $Y \subseteq Z$.*

Proof of Claim 2.4.7. Suppose $\mathcal{F} = \{X, Y, Z\} \subseteq \mathcal{P}([n])$ is an $(n, 3)$-best family such that $|X| = |Y| = k$ and $|Z| = \ell > k$. Moreover, suppose X and Z are incomparable and Y and Z are incomparable. By the principle of inclusion and exclusion,

$$\begin{aligned} c(\mathcal{F}) = 2^{|X|}+2^{n-|X|}+2^{|Y|}+2^{n-|Y|}+2^{|Z|}+2^{n-|Z|}-2^{|X\cap Y|}-2^{n-|X\cup Y|}-2^{|X\cap Z|}-2^{n-|X\cup Z|} \\ - 2^{|Y\cap Z|} - 2^{n-|Y\cup Z|} + 2^{|X\cap Y\cap Z|} + 2^{n-|X\cup Y\cup Z|} - 3. \quad (2.7) \end{aligned}$$

By construction, $|X \cap Y| \leq k-1$, $|X \cup Y| \geq k+1$, $|X \cap Z| \leq k-1$, $|X \cup Z| \geq \ell+1$, $|Y \cap Z| \leq k-1$ and $|Y \cup Z| \geq \ell+1$. So

$$\begin{aligned} c(\mathcal{F}) &\geq 2^{k+1} + 2^{n-k+1} + 2^{\ell} + 2^{n-\ell} - 2^{k-1} - 2^{n-k-1} - 2^{k-1} - 2^{n-\ell-1} - 2^{k-1} - 2^{n-\ell-1} - 3 \\ &= 2^{k-1} + \frac{3}{2}(2^{n-k}) - 2^{\ell} - 3. \end{aligned}$$

Now we construct a new family $\mathcal{F}'$ from $\mathcal{F}$ by replacing Z with Z' such that $|Z'| = \ell$ and $X \subseteq Z$. Then $|X \cap Z'| = k$ and $|X \cup Z'| = \ell$. By Claim 2.4.6, $|X \cap Y| = k-1$ and $|X \cup Y| = k+1$. Thus $|Y \cap Z| \geq k-1$ and $|Y \cup Z| \leq \ell+1$. Then $|X \cap Y \cap Z'| = k-1$ and $|X \cup Y \cup Z'| = \ell+1$. Thus

$$c(\mathcal{F}') \leq 2^{k+1}+2^{n-k-1}+2^{\ell}+2^{n-\ell}-2^{k-1}-2^{n-k-1}-2^{k}-2^{\ell}-2^{k-1}-2^{n-\ell-1}+2^{k-1}+2^{n-\ell-1}-3.$$

If $c(\mathcal{F}) - c(\mathcal{F}') > 0$, then $\mathcal{F}$ is not an $(n, 3)$-best family and we can decrease the comparability number of $\mathcal{F}$ by replacing Z with a set Z' that is comparable to X. When n is even, by Claim 2.4.5, $k = \frac{n}{2}$ and $\ell = \frac{n}{2} + 1$ without loss of generality. Then

$$c(\mathcal{F}) - c(\mathcal{F}') \geq \frac{3}{2}(2^{\frac{n}{2}}) + 2^{\frac{n}{2}+1} - 2^{\frac{n}{2}-1} > 0,$$

as desired. When n is odd, by Claim 2.4.5, $k = \frac{n-1}{2}$ and $\ell = \frac{n+1}{2}$ without loss of generality. Then

$$c(\mathcal{F}) - c(\mathcal{F}') \geq \frac{3}{2}(2^{\frac{n+1}{2}}) + 2^{\frac{n+1}{2}} - 2^{\frac{n-1}{2}} > 0,$$

as desired. □

With these claims, we can prove Lemma 2.4.3.

Proof of Lemma 2.4.3. Claims 2.4.5, 2.4.6, and 2.4.7 and Lemma 2.3.8 narrow down the structure of an $(n, 3)$-best family. Claim 2.4.5 says that an $(n, 3)$-best family is contained in the two middle layers of the hypercube. If the family has sets of different sizes, then by Claim 2.4.6, we know that the two sets of the same size intersect as much as possible. By Claim 2.4.7 we know that when there are two sets of different sizes, there is a 2-chain. We have narrowed down the constructions to the following three cases. Let $\mathcal{F} = \{X, Y, Z\} \subseteq \mathcal{P}([n])$. Fix $k = \frac{n}{2}$ when n is even and $k = \frac{n-1}{2}$ when n is odd.

1. $|X| = |Y| = k$, $|Z| = k+1$ and $X \subseteq Z, Y \subseteq Z$.
2. $|X| = |Z| = k$, $|Y| = k+1$ and $X \subseteq Y, Z \not\subseteq Y$.
3. $|X| = |Y| = |Z|$ where $|X| = k$ when n is odd and $|X| \in \{k, k+1\}$ when n is even.

We proceed by cases.

Case 1: Suppose $|X| = |Y| = k$ and $|Z| = k+1$. Moreover, suppose $X \subseteq Z$ and $Y \subseteq Z$ and $Y \neq Z$. Then $|Y \cap Z| = k-1$ by Claim 2.4.6. By Observation 2.1.8 we have that

$$c(\mathcal{F}) = |d(Z)| + |u(X)| + |u(Y)| - |u(X \cap Y)| - 3 = 2^{k+1} + 2^k + 2^k - 2^{n-k-1} - 3. \quad (2.8)$$

First suppose n is even. Then $k = \frac{n}{2}$ and we have the construction described in Claim 2.4.4. Now suppose n is odd. Then $k = \frac{n-1}{2}$ and we again have the construction described in Claim 2.4.4.

Case 2: Suppose $|X| = |Z| = k, |Y| = k+1$ and $X \subseteq Y, Z \not\subseteq Y$. By Claim 2.4.6, $|X \cap Z| = k-1$, so $|X \cup Z| = k+1$. By construction, $|X \cap Y| = k$, so $|X \cup Y| = k+1$, and $|Y \cap Z| = k-1$, so $|Y \cup Z| = k+2$. Thus, $|X \cap Y \cap Z| = k+2$ and $|X \cup Y \cup Z| = k+2$. Now

$$\begin{aligned} c(\mathcal{F}) = 2^{k+1} + 2^{n-k+1} + 2^{k+1} + 2^{n-k-1} - 2^{k-1} - 2^{n-k-1} - 2^k - 2^{n-k-1} - 2^{k-1} \\ - 2^{n-k-2} + 2^{k+2} + 2^{n-k-2} - 3. \end{aligned}$$

When n is even and $k = \frac{n}{2}$, $c(\mathcal{F}) = 6(2^{\frac{n}{2}}) + 2^{\frac{n}{2}-1} - 3 > \frac{7}{2}(2^{\frac{n}{2}}) - 3$. Thus, by Claim 2.4.4, this is not an $(n,3)$-best family when n is even. When n is odd and $k = \frac{n-1}{2}$, $c(\mathcal{F}) = 6(2^{\frac{n-1}{2}}) + 2^{\frac{n-1}{2}} - 3 > \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3$. Thus by Claim 2.4.4, this is not an $(n,3)$-best family when n is odd.

Case 3: Suppose $|X| = |Y| = |Z| = k$.

First suppose n is even and $k = \frac{n}{2} + 1$. Then $|X \cap Y|, |X \cap Z|, |Y \cap Z| \leq \frac{n}{2}$. Then $|X \cup Y|, |X \cup Z|, |Y \cup Z| \geq \frac{n}{2} + 2$. By the principle of inclusion and exclusion,

$$\begin{aligned} c(\mathcal{F}) &\geq 3(2^{\frac{n}{2}+1}) + 3(2^{\frac{n}{2}-1}) - 3(2^{\frac{n}{2}}) - 3(2^{\frac{n}{2}-2}) - 3 \\ &> \frac{7}{2}(2^{\frac{n}{2}}) - 3. \end{aligned}$$

Thus, by Claim 2.4.4, this is not an $(n,3)$-best family, so $k = \frac{n}{2}$ when n is even. Now assume $c(\{X,Y\}) \leq c(\{X,Z\}) \leq c(\{Y,Z\})$. Suppose n is even and $k = \frac{n}{2}$. By Lemma 2.4.2, $c(\{X,Y\}) \geq 3(2^{\frac{n}{2}}) - 2$. First suppose there exists $i \in [n]$ such that $i \in Z$

but $i \notin X \cup Y$. We now consider the number of sets comparable to Z that are not comparable to X or Y. Let $Z = A \cup \{i\}$. Then the number of sets in the downset of Z that are not comparable to X or Y is at least

$$|\{B \cup \{i\} : B \subseteq A\}| = 2^{\frac{n}{2}-1}.$$

The number of sets that are in the upset of Z and are not comparable to X or Y is at least

$$|\{Z \cup B : X \setminus Z \not\subseteq B, Y \setminus Z \not\subseteq B\}| \geq 2^{n-(\frac{n}{2}-2)}.$$

So

$$c(\mathcal{F}) \geq 3(2^{\frac{n}{2}}) + 2^{\frac{n}{2}-1} + 2^{\frac{n}{2}+2} - 3 > \frac{7}{2}(2^{\frac{n}{2}}) - 3.$$

Thus, by Claim 2.4.4, this is not an $(n, 3)$-best family.

Now suppose $Z \subseteq X \cup Y$. Then $|X \cap Y| \leq \frac{n}{2} - 2$ and $|X \cup Y| \geq \frac{n}{2} + 2$. So

$$c(\mathcal{F}) \geq 4(2^{\frac{n}{2}}) - 2^{\frac{n}{2}-1} - 2^{\frac{n}{2}-2} - 2 = \frac{7}{2}(2^{\frac{n}{2}}) - 2.$$

Adding the set Z will increase the comparability number by at least one, since Z is comparable to itself, so this is not an $(n, 3)$-best family by Claim 2.4.4. Thus, when n is even and all the sets are the same size, we do not have an $(n, 3)$-best family.

We will do the same thing for n odd. Assume $k = \frac{n-1}{2}$. By Lemma 2.4.2, $c(\{X, Y\}) \geq 2^{\frac{n+1}{2}+1} - 2$. First suppose there exists $i \in [n]$ such that $i \in Z$ but $i \notin X \cup Y$. We now consider the number of sets comparable to Z that are not comparable to X or Y. Let $Z = A \cup \{i\}$. Then the number of sets in the downset of Z that are not comparable to X or Y is at least

$$|\{B \cup \{i\} : B \subseteq A\}| = 2^{\frac{n-1}{2}-1}.$$

The number of sets that are in the upset of Z and are not comparable to X or Y is at least

$$|\{Z \cup B : X \setminus Z \not\subseteq B, Y \setminus Z \not\subseteq B\}| \geq 2^{n-(\frac{n-1}{2}-2)}.$$

So

$$c(\mathcal{F}) \geq 2^{\frac{n+1}{2}+1} + 2^{\frac{n-1}{2}-1} + 2^{\frac{n+1}{2}+2} - 3 > \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3.$$

Thus, by Claim 2.4.4, this is not an $(n, 3)$-best family. Now suppose $Z \subseteq X \cup Y$. Then $|X \cap Y| \leq \frac{n-1}{2} - 2$ and $|X \cup Y| \geq \frac{n-1}{2} + 2$. So

$$c(\mathcal{F}) \geq 2(2^{\frac{n-1}{2}}) + 2(2^{\frac{n+1}{2}}) - 2^{\frac{n-1}{2}-1} - 2^{\frac{n+1}{2}-2} - 2 > \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3.$$

Adding the set Z will increase the comparability number by at least one, since Z is comparable to itself, so by Claim 2.4.4 this is not an $(n, 3)$-best family. Thus, when n is odd and all the sets are the same size, we do not have an $(n, 3)$-best family.

This concludes all of the cases for an $(n, 3)$-best family. So for $n \geq 4$,

$$c(n,3) \geq \begin{cases} \frac{7}{2}(2^{\frac{n}{2}}) - 3, & \text{if } n \text{ is even} \\ \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3, & \text{if } n \text{ is odd.} \end{cases}$$

Together with Claim 2.4.4, we have our result. □

Chapter 3
Bounding the Product Measure of Cross-Sperner Systems

3.1 Introduction and Intuition

We begin by building some intuition and understanding of cross-Sperner systems. Recall the definition of a cross-Sperner system, restated here.

Definition 1.3.4. For $k \geq 2$, a collection of non-empty families $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \subseteq \mathcal{P}([n])^k$ is *cross-Sperner* if, for all $i \neq j$, the sets F_i and F_j are incomparable for any $F_i \in \mathcal{F}_i$ and $F_j \in \mathcal{F}_j$. We may also write that $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is *cross-Sperner in* $\mathcal{P}([n])$.

We begin with the simplest case of cross-Sperner *pairs*. As mentioned earlier, Seymour's Theorem, restated here, is the first result on cross-Sperner pairs.

Theorem 1.3.3. (Seymour's Theorem [38]) *Let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}([n])$ be cross-Sperner families. Then, $|\mathcal{F}|^{1/2} + |\mathcal{G}|^{1/2} \leq 2^{n/2}$.*

For completeness, we present a proof of Seymour's Theorem, following the argument from [38]. To prove Seymour's Theorem, we use the arithmetic mean-geometric mean (AM-GM) inequality (Lemma 2.2.2) and the Harris-Kleitman inequality (Lemma 2.2.1).

Proof of Seymour's Theorem. Since $\mathcal{F}$ and $\mathcal{G}$ are cross-Sperner families, we can partition $\mathcal{P}([n])$ into the following sets:

$$\begin{aligned}
\mathcal{A} &= \{Z \in [n] : \exists X \in \mathcal{F}, X \subseteq Z \text{ and } \exists Y \in \mathcal{G}, Y \subseteq Z\}, \\
\mathcal{B} &= \{Z \in [n] : \exists X \in \mathcal{F}, X \subseteq Z \text{ and } \nexists Y \in \mathcal{G}, Y \subseteq Z\}, \\
\mathcal{C} &= \{Z \in [n] : \nexists X \in \mathcal{F}, X \subseteq Z \text{ and } \exists Y \in \mathcal{G}, Y \subseteq Z\}, \\
\mathcal{D} &= \{Z \in [n] : \nexists X \in \mathcal{F}, X \subseteq Z \text{ and } \nexists Y \in \mathcal{G}, Y \subseteq Z\}.
\end{aligned}$$

So, $\mathcal{A} = u(\mathcal{F}) \cap u(\mathcal{G})$, $\mathcal{B} = u(\mathcal{F}) \setminus u(\mathcal{G})$, $\mathcal{C} = u(\mathcal{G}) \setminus u(\mathcal{F})$, and $\mathcal{D} = \mathcal{P}([n]) \setminus (u(\mathcal{F}) \cup u(\mathcal{G}))$. Let $\mathcal{U} = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{V} = \mathcal{B} \cup \mathcal{D}$. Then $\mathcal{U} = u(\mathcal{F})$, so $\mathcal{U}$ is an upset. Moreover, $\mathcal{V} = \mathcal{P}([n]) \setminus u(\mathcal{G})$ so $\mathcal{V}$ is a downset and $d(G) \subseteq \mathcal{V}$. Thus, we can apply the Harris-Kleitman inequality (Lemma 2.2.1), so

$$|\mathcal{U} \cap \mathcal{V}|2^n = |\mathcal{B}|2^n \leq (|\mathcal{A}| + |\mathcal{B}|)(|\mathcal{B}| + |\mathcal{D}|).$$

Since $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| + |\mathcal{D}| = 2^n$, we can substitute it in to get

$$|\mathcal{A}||\mathcal{B}| + |\mathcal{B}|^2 + |\mathcal{C}||\mathcal{B}| + |\mathcal{D}||\mathcal{B}| \leq |\mathcal{A}||\mathcal{B}| + |\mathcal{B}|^2 + |\mathcal{A}||\mathcal{D}| + |\mathcal{D}||\mathcal{B}|.$$

Thus,

$$|\mathcal{B}||\mathcal{C}| \leq |\mathcal{A}||\mathcal{D}| \leq \left(\frac{|\mathcal{A}| + |\mathcal{D}|}{2}\right)^2 \leq \left(\frac{2^n - |\mathcal{B}| - |\mathcal{C}|}{2}\right)^2,$$

where the second inequality comes from the AM-GM inequality. This implies,

$$2^n \geq |\mathcal{B}| + |\mathcal{C}| + 2|\mathcal{B}|^{1/2}|\mathcal{C}|^{1/2} = (|\mathcal{B}|^{1/2} + |\mathcal{C}|^{1/2})^2.$$

Simplifying gives $|\mathcal{B}|^{1/2} + |\mathcal{C}|^{1/2} \leq 2^{n/2}$. Since $\mathcal{F} \subseteq \mathcal{B}$ and $\mathcal{G} \subseteq \mathcal{C}$, this gives the result. □

Gerbner, Lemons, Palmer, Patkós, and Szécsi [19] proved a tight bound on the product of cross-Sperner pairs.

Theorem 3.1.1 (Theorem 1.2 of [19]). *Let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}([n])$ be cross-Sperner families. Then $|\mathcal{F}||\mathcal{G}| \leq 2^{2n-4}$.*

This bound is tight. To see this, define $\mathcal{F} := \{F \subseteq [n] : 1 \in F, n \notin F\}$ and $\mathcal{G} := \{G \subseteq [n] : 1 \notin G, n \in G\}$. First notice $\mathcal{F}$ and $\mathcal{G}$ are cross-Sperner. For every $F \in \mathcal{F}$ and $G \in \mathcal{G}$, $1 \in F$ and $1 \notin G$, so $F \not\subseteq G$. Similarly, since $n \in G$ and $n \notin F$, $G \not\subseteq F$. As $|\mathcal{F}| = |\mathcal{G}| = 2^{n-2}$, the product is as desired. This bound on the product is also a consequence of Seymour's Theorem (Theorem 1.3.3) via an application of the AM-GM inequality.

Proof of Theorem 3.1.1. Let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}([n])$ be cross-Sperner families. By Seymour's Theorem, $|\mathcal{F}|^{1/2} + |\mathcal{G}|^{1/2} \leq 2^{n/2}$. Applying the AM-GM inequality, we get

$$(|\mathcal{F}|^{1/2} \cdot |\mathcal{G}|^{1/2})^{1/2} \leq \frac{|\mathcal{F}|^{1/2} + |\mathcal{G}|^{1/2}}{2} \leq 2^{\frac{n}{2}-1}.$$

□

It should be noted that this is a much simpler proof than the one presented in [19].

We now extend to cross-Sperner systems with 3 or more families. In this chapter we bound $\pi(n,k)$. Recall that for $k \geq 3$,

$$\pi(n,k) := \max\left\{\prod_{i=1}^{k}|\mathcal{F}_i| : (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \text{ is cross-Sperner in } \mathcal{P}([n])\right\}.$$

In [19], it was observed that Theorem 3.1.1 can also be used to show $\pi(n,k) \leq 2^{k(n-2)}$. To see this, let $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ be cross-Sperner in $\mathcal{P}([n])$. Then

$$\prod_{i=1}^{k}|\mathcal{F}_i| = \left(\prod_{1\leq i<j\leq k}|\mathcal{F}_i||\mathcal{F}_j|\right)^{\frac{1}{k-1}}$$

since, for each $1 \leq i \leq k$, $|\mathcal{F}_i|$ occurs $k-1$ times in the product on the right hand side. By Theorem 3.1.1, $|\mathcal{F}_i||\mathcal{F}_j| \leq 2^{2(n-2)}$. Thus,

$$\prod_{i=1}^{k}|\mathcal{F}_i| \leq \left(\left(2^{2(n-2)}\right)^{\frac{k(k-1)}{2}}\right)^{\frac{1}{k-1}} = 2^{k(n-2)}.$$

In [19] they conjectured the following bound on $\pi(n,k)$ that exploits the connection between antichains and cross-Sperner system.

Conjecture 3.1.2 (Gerbner, Lemons, Palmer, Patkós, Szécsi, [19])**.** *Let $\ell = \ell(k)$ be the least positive integer such that $\binom{\ell}{\lfloor \ell/2 \rfloor} \geq k$. Then $\pi(n,k) \leq 2^{k(n-\ell)}$.*

Conjecture 3.1.2 is based on the following construction. Since $\binom{\ell}{\lfloor \ell/2 \rfloor} \geq k$, by Sperner's Theorem there exists an antichain $\{S_1, S_2, \cdots, S_k\} \subseteq \mathcal{P}([\ell])$ of size k. Let $\mathcal{F}_i = \{F \subseteq [n] : F \cap [\ell] = S_i\}$. Then $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ form a cross-Sperner system in $\mathcal{P}([n])$. To see this, suppose $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is not cross-Sperner. Then there exist some $F \in \mathcal{F}_i$ and $F' \in \mathcal{F}_j$ such that $F \subseteq F'$. Since $F \cap [\ell] = S_i$ and $F' \cap [\ell] = S_j$, then $S_i \subseteq S_j$, contradicting the fact that $\{S_1, S_2, \cdots S_k\}$ is an antichain. Notice that $\prod_{i=1}^{k}|\mathcal{F}_i| = 2^{k(n-\ell)}$.

In this chapter we disprove this conjecture with the following theorem.

Theorem 1.3.6. *Let n and $k \geq 2$ be integers. For n sufficiently large,*

$$\left(\frac{2^n}{ek}\right)^k \leq \pi(n,k).$$

A crude application of Stirling's approximation yields that $\ell(k) = \omega(\log k)$. So, in particular, there is a function $g(k)$ tending to infinity with k such that $2^{k(n-\ell)} = O\left(2^{kn}(k \cdot g(k))^{-k}\right)$. Therefore our lower bound is exponentially larger than the conjectured upper bound of $2^{k(n-\ell)}$.

We also prove an upper bound on $\pi(n,k)$.

Theorem 1.3.7. *Let n and $k \geq 2$ be integers. Then*

$$\pi(n,k) \leq \left(\frac{2^n}{k^2}\right)^k \left\lfloor \frac{k}{2} \right\rfloor^{\lfloor k/2 \rfloor} \left\lceil \frac{k}{2} \right\rceil^{\lceil k/2 \rceil}.$$

To prove the upper bound on $\pi(n,k)$, we exploit a connection between the cross-Sperner property and the notion of minimizing comparability. Notice that if $(\mathcal{F}_1, \mathcal{F}_2)$ is cross-Sperner, then every set in $\mathcal{F}_1$ is incomparable to every set in $\mathcal{F}_2$. Furthermore, $2^n - c(n,m)$ is the maximum number of sets incomparable to a family of size m. The following key idea allows us to use Theorem 1.3.9 to say something about the size of cross-Sperner systems.

Key Idea 3.1.3. Let $(\mathcal{F}_1, \mathcal{F}_2) \subseteq \mathcal{P}([n])^2$ be cross-Sperner where $|\mathcal{F}_1| = m$. Then $|\mathcal{F}_2| \leq 2^n - c(n,m)$.

This chapter is organized as follows. In Section 3.2, we prove the lower bound in Theorem 1.3.6. In Section 3.3, we prove Theorem 1.3.7 using Key Idea 3.1.3. In Section 3.4, we give the exact value of $\pi(3,3)$, $\pi(4,4)$, and $\pi(4,3)$. c

3.2 Lower Bound on $\pi(n,k)$

Theorem 1.3.6 follows directly from the following (slightly stronger) statement.

Lemma 3.2.1. *Let n, k be integers with $k \geq 2$ and $n > k \log_2 k + k$. Then*

$$\pi(n,k) \geq \left(\left(\frac{1}{k} - \frac{1}{2^{\lfloor n/k \rfloor}}\right)\left(1 - \frac{1}{k}\right)^{k-1}\right)^k 2^{kn}.$$

Proof. Partition $[n]$ into k parts $A_1, A_2, \cdots, A_k$, each of size $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$. For each $1 \leq i \leq k$, take $\mathcal{X}_i$ to be an initial segment of colex in $\mathcal{P}(A_i)$ such that $|\mathcal{X}_i| = \lambda_i 2^{|A_i|}$ for some $0 < \lambda_i < 1$. Set $\mathcal{Y}_i := \mathcal{P}(A_i) \setminus \mathcal{X}_i$. Now we construct a cross-Sperner system $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$. Define

$$\mathcal{F}_i := \{F \in \mathcal{P}([n]) : F \cap A_i \in \mathcal{X}_i, F \cap A_j \in \mathcal{Y}_j \text{ for all } j \neq i\}.$$

Refer to Example 3.2.2 for an example of this construction.

To see that $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is cross-Sperner in $\mathcal{P}([n])$, consider $S \in \mathcal{F}_i$ and $T \in \mathcal{F}_j$. We must show that S and T are incomparable. If $S \subseteq T$, then $S \cap A_j \subseteq T \cap A_j$. By the definition of our cross-Sperner system, there is some $Y \in \mathcal{Y}_j$ and $X \in \mathcal{X}_j$ such that $Y \subseteq X$. This contradicts that $\mathcal{X}_j$ is an initial segment of colex in $\mathcal{P}(A_j)$ and $\mathcal{Y}_j := \mathcal{P}(A_j) \setminus \mathcal{X}_j$. Analogously, we see that T cannot be a subset of S. Hence

$(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is cross-Sperner as required. We can think of each $F \in \mathcal{F}_i$ as being of the form

$$F = X \cup \left(\bigcup_{\substack{j \neq i \\ Y_j \in \mathcal{Y}_j}} Y_j \right).$$

for some $X \in \mathcal{X}_i$. Thus,

$$|\mathcal{F}_i| = |\mathcal{X}_i| \prod_{j \neq i} |\mathcal{Y}_j|.$$

So,

$$\pi(n,k) \geq \prod_{i=1}^{k} |\mathcal{F}_i| = \prod_{i=1}^{k} \left(|\mathcal{X}_i| \prod_{j \neq i} |\mathcal{Y}_j| \right).$$

To complete the proof, it remains to optimize the value of the λ_i. We have

$$|\mathcal{F}_i| = \lambda_i 2^{|A_i|} \prod_{j \neq i} (1-\lambda_j) 2^{|A_j|} = \lambda_i 2^{|A_1|+|A_2|+\cdots+|A_k|} \prod_{j \neq i} (1-\lambda_j) = \lambda_i 2^n \prod_{j \neq i} (1-\lambda_j).$$

So

$$\prod_{i=1}^{k} |\mathcal{F}_i| = \left(\prod_{i=1}^{k} \lambda_i (1-\lambda_i)^{k-1} \right) 2^{kn}. \tag{3.1}$$

For each $1 \leq k \leq i$, set $\lambda_i = \frac{1}{2^{|A_i|}} \lfloor \frac{2^{|A_i|}}{k} \rfloor$. Then

$$\frac{1}{k} - \frac{1}{2^{\lfloor n/k \rfloor}} \leq \frac{1}{k} - \frac{1}{2^{|A_i|}} \leq \lambda_i \leq \frac{1}{k}.$$

For $n > k \log_2 k + k$ we have $2^{-\lfloor n/k \rfloor} \leq 2^{-(n/k-1)} < \frac{1}{k}$, so λ_i is not zero. Therefore, with this choice of λ_i we get

$$\prod_{i=1}^{k} |\mathcal{F}_i| \geq \left(\left(\frac{1}{k} - \frac{1}{2^{\lfloor n/k \rfloor}} \right) \left(1 - \frac{1}{k} \right)^{k-1} \right)^{k} 2^{kn},$$

as required. □

Note that if k is a power of 2 in the proof of Lemma 3.2.1, $\lambda_i = \frac{1}{k}$ for all $1 \leq i \leq k$. Therefore in this case we can eliminate the $-\frac{1}{2^{\lfloor n/k \rfloor}}$ term.

For clarity, Example 3.2.2 demonstrates the construction given in Lemma 3.2.1.

Example 3.2.2. Let $n = 6$ and $k = 3$. Partition $[6]$ into

$$A_1 = \{1,2\}, A_2 = \{3,4\}, A_3 = \{5,6\}.$$

Then let

$$\mathcal{X}_1 = \{\emptyset, \{1\}\},$$
$$\mathcal{X}_2 = \{\emptyset, \{3\}\},$$
$$\mathcal{X}_3 = \{\emptyset, \{5\}\}.$$

So

$$\mathcal{Y}_1 = \{\{2\}, \{1,2\}\},$$
$$\mathcal{Y}_2 = \{\{4\}, \{3,4\}\},$$
$$\mathcal{Y}_3 = \{\{6\}, \{5,6\}\}.$$

By the definition of $\mathcal{F}_i$ given in Lemma 4.2.2, the cross-Sperner system is

$$\mathcal{F}_1 = \{\{4,6\}, \{4,5,6\}, \{3,4,6\}, \{3,4,5,6\}, \{1,4,6\}, \{1,4,5,6\}, \{1,3,4,6\}, \{1,3,4,5,6\}\},$$
$$\mathcal{F}_2 = \{\{2,6\}, \{2,5,6\}, \{2,3,6\}, \{2,3,5,6\}, \{1,2,6\}, \{1,2,5,6\}, \{1,2,3,6\}, \{1,2,3,5,6\}\},$$
$$\mathcal{F}_3 = \{\{2,4\}, \{2,4,5\}, \{2,3,4\}, \{2,3,4,5\}, \{1,2,4\}, \{1,2,4,5\}, \{1,2,3,4\}, \{1,2,3,4,5\}\}.$$

We now deduce Theorem 1.3.6, restated below for convenience, from Lemma 3.2.1.

Theorem 1.3.6. *Let n and $k \geq 2$ be integers. For n sufficiently large,*

$$\left(\frac{2^n}{ek}\right)^k \leq \pi(n,k).$$

Proof. Take n sufficiently large so that

$$\frac{1}{2^{\lfloor n/k \rfloor}} \leq \frac{1}{k} - \frac{1}{ek}\left(1 + \frac{1}{k-1}\right)^{k-1} = \frac{1}{ek}\left(e - \left(1 + \frac{1}{k-1}\right)^{k-1}\right).$$

This is possible as $\left(1 + \frac{1}{k-1}\right)^{k-1}$ tends to e from below. Substituting this into Lemma 3.2.1, we see that

$$\pi(n,k) \geq \left(\frac{1}{ek}\left(1 + \frac{1}{k-1}\right)^{k-1}\left(1 - \frac{1}{k}\right)^{k-1}\right)^k 2^{kn} = \left(\frac{1}{ek}\right)^k 2^{kn}.$$

□

3.3 Upper Bound on $\pi(n, k)$

The goal of this subsection is to prove Theorem 1.3.7, restated below for convenience.

Theorem 1.3.7. *Let n and $k \geq 2$ be integers. Then*

$$\pi(n,k) \leq \left(\frac{2^n}{k^2}\right)^k \left\lfloor \frac{k}{2} \right\rfloor^{\lfloor k/2 \rfloor} \left\lceil \frac{k}{2} \right\rceil^{\lceil k/2 \rceil}.$$

We will use the following lemma.

Lemma 3.3.1. *Let $1 \leq j < k$ and let $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k) \subseteq \mathcal{P}([n])^k$ be cross-Sperner. Then $\left(\bigcup_{i=1}^j \mathcal{F}_i, \bigcup_{i=j+1}^k \mathcal{F}_i\right)$ is cross-Sperner in $\mathcal{P}([n])$.*

Proof. Suppose for contradiction that $\left(\bigcup_{i=1}^j \mathcal{F}_i, \bigcup_{i=j+1}^k \mathcal{F}_i\right)$ is not cross-Sperner. Then there exist some $X \in \bigcup_{i=1}^j \mathcal{F}_i$ and $Y \in \bigcup_{i=j+1}^k \mathcal{F}_i$ such that $X \subseteq Y$ or $Y \subseteq X$. Since $X \in \mathcal{F}_i$ for some $1 \leq i \leq j$, and $Y \in \mathcal{F}_t$ for some $j+1 \leq t \leq k$ we deduce that $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k)$ is not cross-Sperner, a contradiction. □

We now use Lemma 3.3.1, along with Theorem 1.3.9, to prove an upper bound on $\pi(n,k)$.

Proof of Theorem 1.3.7. Suppose $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is cross-Sperner in $\mathcal{P}([n])$. Let $a = \lfloor k/2 \rfloor, b = \lceil k/2 \rceil$, and observe that $a + b = k$. Let $\mathcal{G} = \bigcup_{i=1}^a \mathcal{F}_i$ and $\mathcal{H} = \bigcup_{a+1}^k \mathcal{F}_i$. Notice that $(\mathcal{G}, \mathcal{H}) \subseteq \mathcal{P}([n])^2$ is cross-Sperner by Lemma 3.3.1. If $|\mathcal{G}| = m$, by Key Idea 3.1.3, $|\mathcal{H}| \leq 2^n - c(n,m)$. Moreover, since each product is maximized when the families are of equal sizes,

$$\prod_{i=1}^a |\mathcal{F}_i| \leq \left(\frac{m}{a}\right)^a \quad \text{and} \quad \prod_{j=a+1}^k |\mathcal{F}_j| \leq \left(\frac{2^n - 2^{n/2+1}\sqrt{m} + m}{b}\right)^b.$$

Thus,

$$\prod_{i=1}^k |\mathcal{F}_i| = \prod_{i=1}^a |\mathcal{F}_i| \prod_{j=a+1}^k |\mathcal{F}_j| \leq \left(\frac{m}{a}\right)^a \left(\frac{2^n - 2^{n/2+1}\sqrt{m} + m}{b}\right)^b := h(m). \tag{3.2}$$

To find an upper bound on the left hand side of (3.2), we differentiate with respect to m to find the value of m that maximizes the right hand side:

$$\frac{d}{dm}h(m) = \left(\frac{m}{a}\right)^a \left(\frac{(2^{n/2} - \sqrt{m})^2}{b}\right)^b (a(\sqrt{m} - 2^{n/2}) + b\sqrt{m})(m^{3/2} - m2^{n/2})^{-1}.$$

Setting this equal to zero yields $m \in \{0, 2^n, \frac{a^2 2^n}{k^2}\}$. A simple calculation shows that (3.2) is maximized when $m = \frac{a^2 2^n}{k^2}$. Thus

$$\prod_{i=1}^k |\mathcal{F}_i| \leq \left(\frac{2^n}{k^2}\right)^k a^a b^b = \left(\frac{2^n}{k^2}\right)^k \left\lfloor \frac{k}{2} \right\rfloor^{\lfloor k/2 \rfloor} \left\lceil \frac{k}{2} \right\rceil^{\lceil k/2 \rceil},$$

as required. □

The next corollary will make it easier to compare the upper and lower bound for $\pi(n,k)$.

Corollary 3.3.2. *When k is even, $\pi(n,k) \leq \left(\frac{2^n}{2k}\right)^k$. When k is odd, $\pi(n,k) \leq \left(1+\frac{1}{k}\right)\left(\frac{2^n}{2k}\right)^k$.*

Proof. When k is even, the upper bound given by Theorem 1.3.7 is $\left(\frac{2^n}{2k}\right)^k$. When k is odd, let $k = 2r+1$. Then the upper bound given by Theorem 1.3.7 is

$$\left(\frac{2^n}{k^2}\right)^k r^r(r+1)^{(r+1)} = \left(\frac{2^n}{k^2}\right)^k \left(\frac{k-1}{2}\right)^{\frac{k-1}{2}} \left(\frac{k+1}{2}\right)^{\frac{k+1}{2}}.$$

Simplifying we get

$$\left(\frac{2^n}{2k^2}\right)^k (k-1)^{\frac{k-1}{2}}(k+1)^{\frac{k+1}{2}} = k^k \left(\frac{2^n}{2k^2}\right)^k \left(1-\frac{1}{k}\right)^{\frac{k-1}{2}} \left(1+\frac{1}{k}\right)^{\frac{k+1}{2}}.$$

Since $(1-\frac{1}{k})(1+\frac{1}{k}) = (1-\frac{1}{k^2}) < 1$,

$$\pi(n,k) \leq \left(1+\frac{1}{k}\right)\left(\frac{2^n}{2k}\right)^k$$

when k is odd. □

3.4 Some Exact Values of $\pi(n,k)$

For $n = 3, 4$ and $k = 3, 4$, we can determine the exact value of $\pi(n,k)$ using case analysis. We begin with some observations and prove a lemma to reduce the number of cases to consider.

First, certain sets cannot be in a cross-Sperner system. Since $[n]$ and $\emptyset$ are both comparable to every other set in $\mathcal{P}([n])$, having them in a family means that no other set can be in any other family. Since all families in a cross-Sperner system are non-empty, these sets cannot be in a family in a cross-Sperner system.

Observation 3.4.1. *If $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \subseteq \mathcal{P}([n])^k$ is cross-Sperner, then $[n] \notin \mathcal{F}_i$ and $\emptyset \notin \mathcal{F}_i$ for all $1 \leq i \leq k$.*

We also have that incomparability is preserved under taking complements. Suppose $X, Y \subseteq [n]$ are incomparable. Then $[n] \setminus X$ and $[n] \setminus Y$ are incomparable. To see this, suppose $[n] \setminus X \subseteq [n] \setminus Y$. Then $Y \subseteq X$, contradicting that X and Y are incomparable. This means that from every cross-Sperner system $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \subseteq \mathcal{P}([n])^k$, we can create another cross-Sperner system by considering the complement.

Observation 3.4.2. *Let* $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \subseteq \mathcal{P}([n])^k$ *be cross-Sperner. Define* $\mathcal{F}_i^c := \{[n] \setminus F : F \in \mathcal{F}_i\}$ *for* $1 \leq i \leq k$*. Then* $(\mathcal{F}_1^c, \mathcal{F}_2^c, \cdots, \mathcal{F}_k^c)$ *is cross-Sperner.*

The next lemma states that if a family contains a set of size 1, this restricts the size of the sets in the remaining families.

Lemma 3.4.3. *If* $(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ *is an* (n, k)*-best system where* $k \geq 3$*,* $n \geq k$ *such that* $\mathcal{F}_1$ *contains a set of size* 1*, then no other family contains a set of size* $(n-1)$*.*

Proof. Suppose $(\mathcal{F}_1, \ldots, \mathcal{F}_k) \subseteq \mathcal{P}([n])^k$ is an (n, k)-best family and $\mathcal{F}_1$ contains a set of size 1 and suppose, for a contradiction, $\mathcal{F}_2$ contains a set of size $(n-1)$. Notice that as $(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ is a cross-Sperner family, the set of size 1 and the set of size $(n-1)$ must be disjoint. It follows that all sets in $\mathcal{P}([n])$ must be comparable to at least one of these two sets. This implies all other families $\mathcal{F}_3, \ldots, \mathcal{F}_k$ must be empty, contradicting that $(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ is (n, k)-best. □

Lemma 3.4.3 allows us to assume that if a family in a cross-Sperner system has a set of size 1, no other family has a set of size $(n-1)$. This is especially useful when n is small. Now we can find the exact values of $\pi(3, 3), \pi(4, 4)$, and $\pi(4, 3)$.

Proposition 3.4.4. $\pi(3, 3) = 1$.

Proof. First we give a construction to show a lower bound.

Construction 3.4.5. *Let*

$$\begin{aligned} \mathcal{F}_1 &= \{\{1, 2\}\} \\ \mathcal{F}_2 &= \{\{1, 3\}\} \\ \mathcal{F}_3 &= \{\{2, 3\}\}. \end{aligned}$$

Then $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ *is cross-Sperner and*

$$\prod_{i=1}^{3} |\mathcal{F}_i| = 1.$$

Now we will show the upper bound. Let $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$. Notice that by Observation 3.4.1, there is no set of size 3 and no empty set in $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$.

Without loss of generality, up to complements, let $\{1\} \in \mathcal{F}_1$. Then by Lemma 3.4.3, no other family has a set of size 2. So $\mathcal{F}_2$ and $\mathcal{F}_3$ contain only elements from $\{2, 3\}$. Since no family is empty, we can suppose $\mathcal{F}_2 = \{\{2\}\}$ and $\mathcal{F}_3 = \{\{3\}\}$. Now, $\mathcal{F}_1$ cannot contain any other sets, so $\prod_{i=1}^{3} |\mathcal{F}_i| = 1$. Thus $\pi(3, 3) = 1$.

□

We now prove that $\pi(4, 4) = 4$. This is a counterexample to Conjecture 3.1.2 which gives $\pi(4, 4) \leq 1$.

Proposition 3.4.6. $\pi(4,4) = 4$

Proof. The following construction proves the lower bound on Proposition 3.4.6.

Construction 3.4.7. *Let*

$$\begin{aligned}\mathcal{F}_1 &= \{\{1,2\}\}\\ \mathcal{F}_2 &= \{\{1,3\},\{1,4\}\}\\ \mathcal{F}_3 &= \{\{2,3\},\{2,4\}\}\\ \mathcal{F}_4 &= \{\{3,4\}\}.\end{aligned}$$

Then $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$ *is a cross-Sperner system and*

$$\prod_{i-1}^{4} |\mathcal{F}_i| = 4.$$

Now we will show the upper bound. Let $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \mathcal{F}_4) \subseteq (\mathcal{P}([4]))^4$ be cross-Sperner. By Observation 3.4.1, no set has size 4 or is the empty set.

Case 1: Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ contain a set of size 1. Let $\{1\} \in \mathcal{F}_1$ and $\{2\} \in \mathcal{F}_2$ without loss of generality. Then the only sets that are incomparable to $\{1\}$ and $\{2\}$ are $\{3\}, \{4\}$ and $\{3,4\}$. The only way to maintain the cross-Sperner property without any family being empty is for $\mathcal{F}_3 = \{\{3\}\}$ and $\mathcal{F}_4 = \{\{4\}\}$. This means that we can add no other sets to $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ or $\mathcal{F}_4$, so $\prod_{i=1}^{4} |\mathcal{F}_i| = 1$.

Case 2: Suppose only $\mathcal{F}_1$ has a set of size 1, say $\{1\}$. By Lemma 3.4.3, no other family has a set of size 3, so the remaining families only have sets of size 2. There are three sets of size 2 that are incomparable to $\{1\}$, namely $\{2,3\}, \{2,4\}$, and $\{3,4\}$. To ensure that no family is empty, one of these sets belongs to each of $\mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$. We can add $\{1,2\}, \{1,3\}$ and $\{1,4\}$ to $\mathcal{F}_1$ without breaking the cross-Sperner property. Thus $\prod_{i=1}^{4} \mathcal{F}_i \leq 4$.

By Observation 3.4.2, and Lemma 2.3.8, Case 1 and 2 imply that if there are sets of size 3 in the family then $\prod_{i=1}^{4} |\mathcal{F}_i| \leq 4$. It remains to check the maximum product when all the sets have size 2.

Case 3: Suppose the cross-Sperner system only has sets of size 2. There are six sets of size 2 which make an antichain. To maximize the product we partition the sets into the four families as evenly as possible, giving us the construction of our lower bound.

Thus, $\pi(4,4) = 4$. □

Finally, we prove that $\pi(4,3) = 9$. This is again a counterexample to Conjecture 3.1.2 which gives $\pi(4,3) \leq 8$.

Proposition 3.4.8. $\pi(4,3) = 9$.

Proof. First we give a construction that prove the lower bound of Proposition 3.4.8.

Construction 3.4.9. *Let*

$$\begin{aligned} \mathcal{F}_1 &= \{\{1,3\},\{1,4\},\{1\}\} \\ \mathcal{F}_2 &= \{\{2,3\},\{2,4\},\{2\}\} \\ \mathcal{F}_3 &= \{\{3,4\}\}. \end{aligned}$$

Then $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \subseteq \mathcal{P}([4])^3$ *is a cross-Sperner system, and*

$$\prod_{i=1}^{3} |\mathcal{F}_i| = 9.$$

Now we will show the upper bound. Let $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \subseteq \mathcal{P}([4])^3$ be cross-Sperner. By Observation 3.4.1, no set has size 4 or is the empty set.

Case 1: Suppose $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ contains at least two singletons. Without loss of generality, suppose $\{1\} \in \mathcal{F}_1$. By Lemma 3.4.3, there will be no sets of size 3 in $\mathcal{F}_2, \mathcal{F}_3$. If one family contains two sets of size 1, say $\{1\}, \{2\} \in \mathcal{F}_1$, then the only remaining incomparable sets to $\mathcal{F}_1$ are $\{3\}, \{4\}, \{3,4\}$. Since all families must be non-empty, $\mathcal{F}_2 = \{\{3\}\}$ and $\mathcal{F}_3 = \{\{4\}\}$. The only set that can be added to $\mathcal{F}_1$ is $\{1,2\}$ so we get $\prod_{i=1}^{3} |\mathcal{F}_i| \leq 3$.

Now suppose the two singletons are in different families. Without loss of generality, let $\{1\} \in \mathcal{F}_1$ and $\{2\} \in \mathcal{F}_2$. By Lemma 3.4.3, $\mathcal{F}_3$ does not contain a set of size 3. If $\mathcal{F}_3$ contains a set of size 1, say $\{3\}$, then $\prod_{i=1}^{3} |\mathcal{F}_i| \leq 8$ since each family can have at most other set, namely $\{1,4\}, \{2,4\}$, and $\{3,4\}$ respectively. If $\mathcal{F}_3$ does not contain a set of size 1, it must contain a set of size 2, as it cannot be empty. The only set of size 2 that is incomparable to $\{1\}, \{2\}$ is $\{3,4\}$. So $\mathcal{F}_3 = \{3,4\}$. Now, $\{1,3\}, \{1,4\}$ can be added to $\mathcal{F}_1$ and $\{2,3\}, \{2,4\}$ can be added to $\mathcal{F}_2$. Thus $\prod_{i=1}^{3} |\mathcal{F}_i| \leq 9$.

Case 2: Suppose there is only one set of size 1 in $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$. Without loss of generality, let $\{1\} \in \mathcal{F}_1$. By Lemma 3.4.3, $\mathcal{F}_2$ and $\mathcal{F}_3$ do not contain a set of size 3. Then $\{2,3\}, \{2,4\}, \{3,4\}$ can be put in $\mathcal{F}_2$ and $\mathcal{F}_3$. Since no family can be empty, let $\mathcal{F}_2 = \{\{2,3\}, \{2,4\}\}$ and $\mathcal{F}_3 = \{\{3,4\}\}$. Then $\{1,2\}, \{1,3\}, \{1,4\}$ can be added to $\mathcal{F}_1$, so $\prod_{i=1}^{3} |\mathcal{F}_i| \leq 8$.

Note that by Observation 3.4.2, having at least one set of size 1 is equivalent to having at least one set of size 3 in the cross-Sperner system, and by Lemma 2.3.8, the product measure of the complement families will be the same. Thus, it suffices to consider families with only sets of size 2.

Case 3: Suppose $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ only contain set of size 2. All sets of size 2 are incomparable so to maximize the product, each family will have two sets and $\prod_{i=1}^{3} |\mathcal{F}_i| = 8$.

$\square$

When comparing the results of Propositions 3.4.4, 3.4.6, and 3.4.8 to the product upper bound in Theorem 1.3.7, we see that the upper bound could be improved. Theorem 1.3.7 gives $\pi(3,3) \leq 2$, $\pi(4,4) \leq 16$ and $\pi(4,3) \leq 22$.

Chapter 4
Bounding the Sum Measure of Cross-Sperner Systems

4.1 Introduction

In the previous chapter we considered the product measure of cross-Sperner systems. Another natural measure to consider is the sum. Recall from Section 1.3 that the definition of (n, k) is

$$(n, k) := \max \left\{ \sum_{i=1}^{k} |\mathcal{F}_i| : (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \text{ is cross-Sperner in } \mathcal{P}([n]) \right\}.$$

The simplest case to consider is the sum of cross-Sperner pairs. In 2021 Gerbner, Lemons, Palmer, Patkós, and Szécsi [19] considered bounding the sum of cross-Sperner pairs.

Theorem 4.1.1 (Gerbner, Lemons, Palmer, Patkós, Szécsi, Theorem 1.1 [19])**.** *There exists an n_0 such that if $n \geq n_0$ and $(\mathcal{F}, \mathcal{G}) \subseteq \mathcal{P}([n])^2$ are cross-Sperner families, then $|\mathcal{F}| + |\mathcal{G}| \leq 2^n - 2^{\lceil n/2 \rceil} - 2^{\lfloor n/2 \rfloor} + 2$.*

This bound is tight. To see this, let $\mathcal{F} = \{X\}$ for some $X \subseteq [n]$ where $|X| = \lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$. Then let $\mathcal{G}$ be all subsets of $[n]$ that are incomparable to X. There are $2^{\lceil n/2 \rceil} + 2^{\lfloor n/2 \rfloor} - 1$ sets in $\mathcal{P}([n])$ that are comparable to X. To see this, suppose $|X| = \lfloor n/2 \rfloor$. Notice $|u(X)| = 2^{\lfloor n/2 \rfloor}$ and $|d(X)| = 2^{\lceil n/2 \rceil}$. As X was counted in both the upset and downset, the number of sets incomparable to X is $2^n - 2^{\lceil n/2 \rceil} - 2^{\lfloor n/2 \rfloor} + 1$. Thus,

$$|\mathcal{F}| + |\mathcal{G}| = 2^n - 2^{\lceil n/2 \rceil} - 2^{\lfloor n/2 \rfloor} + 2.$$

Theorem 4.1.1 only applies to large enough n. In this section, we prove this result for all values of n using Theorem 1.3.9 (see Theorem 4.4.1). The main result of this chapter is bounding $\sigma(n, k)$ for $k \geq 3$. For this we prove the following theorem, restated here for convenience.

Theorem 1.3.8. *Let n, k be integers with $n \geq 2k$. Then*

$$2^n - \frac{3}{\sqrt{2}}\sqrt{2^n k} + 2(k-1) \;\leq\; \sigma(n,k) \;\leq\; 2^n - 2\sqrt{2^n(k-1)} + 2(k-1).$$

We will employ Theorem 1.3.9 on minimizing comparability to prove these bounds. When k is a power of 2 and $n - \log_2 k$ is even, we can further improve the lower bound to $2^n - 2\sqrt{2^n k} + 2(k-1)$, which is extremely close to the upper bound.

4.2 Lower Bound on $\sigma(n,k)$

The lower bound construction for $\sigma(n,k)$ is inspired by the star construction that minimizes comparability. Recall that a star of the form

$$\mathcal{F} = \{X, X \cup \{a_1\}, X \cup \{a_2\}, \cdots, X \cup \{a_{k-1}\}\}$$
$$\text{for some set } \{a_1, a_2, \cdots, a_{k-1}\} \subseteq [n] \setminus X, \text{ where } |X| = \frac{n-d}{2},$$

minimizes comparability. Ideally, there would exist families $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_{k-1} \subseteq \mathcal{P}([n])$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{k-1}$ is a star that minimizes comparability *and* is an antichain. Then, by Key Idea 3.1.3, the cross-Sperner system $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k) \subseteq \mathcal{P}([n])^k$, where $\mathcal{F}_k$ is all of the sets that are incomparable to $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{k-1}$, would maximize the sum. However, $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{k-1}$ is not an antichain. To get an antichain while remaining similar to a star construction, we take

$$\mathcal{F}_1 = X \cup \{a_1\}, \mathcal{F}_2 = X \cup \{a_2\}, \cdots, \mathcal{F}_{k-1} = X \cup \{a_{k-1}\}$$

for some set $\{a_1, a_2, \cdots, a_{k-1}\} \subseteq [n] \setminus X$. Notice that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{k-1}$ is an antichain. Let $\mathcal{F}_k$ be all the sets incomparable to $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{k-1}$. Then $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is cross-Sperner. We need the following counting lemma to determine the comparability number of an antichain of this form.

Lemma 4.2.1. *Let $\mathcal{A} := \{F_1, F_2, \cdots F_{k-1}\}$ be an antichain in $\mathcal{P}([n])$ where $F_i := \{i\} \cup \{n - \ell + 1, \ldots, n\}$. Then $c(\mathcal{A}) = k2^\ell + 2^{n-\ell}\left(1 - \frac{1}{2^{k-1}}\right) - (k-1)$.*

Proof. For each i, let $\mathcal{S}_i$ be the collection of sets comparable to F_i. For ease of notation, let $G := \{n - \ell + 1, \ldots, n\}$. Observe that, since $|F_i| = \ell + 1$ and $u(F_i) \cap d(F_i) = \{F_i\}$,

$$|\mathcal{S}_i| = |u(F_i) \cup d(F_i)| = 2^{\ell+1} + 2^{n-\ell-1} - 1. \tag{4.1}$$

Note that for each $i > 1$, we have

$$d(F_i) \setminus \bigcup_{i<j} d(F_j) = d(F_i) \setminus d(F_1) = \{\{i\} \cup Y : Y \subsetneq G\} \tag{4.2}$$

since $F_i = G \cup \{i\}$. Similarly, observe that for each $i > 1$, we have

$$u(F_i) \setminus \bigcup_{j<i} u(F_i) = \{Z \subseteq [n] : Z \supseteq F_i, Z \cap \{1, \dots, i-1\} = \emptyset\}. \tag{4.3}$$

So now putting together (4.1) (to bound $|\mathcal{S}_1|$), (4.2), and (4.3), we obtain

$$\begin{aligned} \left| \bigcup_{i=1}^{k-1} \mathcal{S}_i \right| &= |\mathcal{S}_1| + \sum_{i=2}^{k-1} \left| d(F_i) \setminus \bigcup_{i<j} d(F_j) \right| + \sum_{i=2}^{k-1} \left| u(F_i) \setminus \bigcup_{i<j} u(F_j) \right| - (k-2) \\ &= 2^{\ell+1} + 2^{n-\ell-1} - 1 + (k-2)2^{\ell} + \left(\sum_{i=2}^{k-1} 2^{n-\ell-i} \right) - (k-2). \end{aligned}$$

The last term comes from that fact that each set F_i is counted both in its downset and its upset, so we must subtract $(k-2)$ from $c(\mathcal{A})$. Simplifying we get

$$c(\mathcal{A}) = k2^{\ell} + 2^{n-\ell} \left(1 - \frac{1}{2^{k-1}} \right) - (k-1).$$

□

The lower bound given in Theorem 1.3.8 follows from the slightly stronger statement given below.

Lemma 4.2.2. *Let $n, k \in \mathbb{N}$ where $n \geq 2k - 1 - \log_2 k \geq 1$. Then*

$$(n, k) \geq 2^n - \frac{3}{\sqrt{2}} \left(1 - \frac{1}{2^{k-1}} \right)^{\frac{1}{2}} \sqrt{2^n k} + 2(k-1).$$

Proof. Let a be an integer with the same parity as n to be specified later. Let $G := \{n - \frac{n-a}{2} + 1, \dots, n\}$. Let $\mathcal{A} = \{F_1, F_2, \cdots F_{k-1}\}$ be an antichain in $\mathcal{P}([n])$, where $F_i = G \cup \{i\}$. This is possible provided $n - \frac{n-a}{2} \geq k - 1$, that is, $n \geq 2(k-1) - a$.

By Lemma 4.2.1 and setting $\ell = \frac{n-a}{2}$, we obtain

$$c(\mathcal{A}) = k2^{\frac{n-a}{2}} + 2^{\frac{n+a}{2}} \left(1 - \frac{1}{2^{k-1}} \right) - (k-1).$$

Define

$$\mathcal{F}_i := \{F_i\} \text{ for } 1 \leq i \leq k-1$$

and

$$\mathcal{F}_k := \{Z \subseteq [n] : Z \text{ incomparable to } F_i \text{ for all } 1 \leq i \leq k-1\}.$$

By construction, $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ is cross-Sperner in $\mathcal{P}([n])$. We have

$$\begin{aligned}\sum_{i=1}^{k} |\mathcal{F}_i| &= (k-1) + 2^n - c(\mathcal{A}) \\ &= 2^n - \sqrt{2^n}\left(\frac{k}{\sqrt{2^a}} + \left(1 - \frac{1}{2^{k-1}}\right)\sqrt{2^a}\right) + 2(k-1). \qquad (4.4)\end{aligned}$$

Differentiating this expression with respect to a gives

$$-\frac{\ln 2}{2}\sqrt{2^n}\left(-\frac{k}{\sqrt{2^a}} + \left(1 - \frac{1}{2^{k-1}}\right)\sqrt{2^a}\right).$$

Thus we can see that if there were no restrictions on a the maximum value of (4.4) would be achieved when $2^a = k\frac{2^{k-1}}{2^{k-1}-1}$; that is, $a = \log_2(k) + \log_2\left(\frac{2^{k-1}}{2^{k-1}-1}\right)$. However, we require a to be an integer with the same parity as n. Set a to be the unique integer such that

$$-1 < a - \log_2(k) - \log_2\left(\frac{2^{k-1}}{2^{k-1}-1}\right) \leq 1$$

and let $c = a - \log_2(k) - \log_2\left(\frac{2^{k-1}}{2^{k-1}-1}\right)$. Note that $n \geq 2k - 1 - \log_2 k \geq 1$ by hypothesis. This ensures that $n \geq 2(k-1) - a$ for any such value of a. We have

$$\begin{aligned}\sum_{i=1}^{k} |\mathcal{F}_i| &= 2^n - \sqrt{2^n}\left(\frac{k}{\sqrt{2^a}} + \left(1 - \frac{1}{2^{k-1}}\right)\sqrt{2^a}\right) + 2(k-1) \\ &= 2^n - \sqrt{2^n k}\left(1 - \frac{1}{2^{k-1}}\right)^{1/2}\left(\frac{1}{\sqrt{2^c}} + \sqrt{2^c}\right) + 2(k-1) \qquad (4.5) \\ &\geq 2^n - \sqrt{2^n k}\left(1 - \frac{1}{2^{k-1}}\right)^{1/2}\left(\frac{3}{\sqrt{2}}\right) + 2(k-1)\end{aligned}$$

where the last inequality follows from the fact that the bracketed expression in (4.5) is maximized when $c = 1$ for c in the range $-1 < c \leq 1$. □

For certain values of k we can prove a stronger lower bound which essentially matches the upper bound given in Lemma 4.3.1.

Corollary 4.2.3. *Let $n, k \in \mathbb{N}$ and suppose that $k = 2^a$ where a has the same parity as n and $n \geq 2(k-1) - a$. Then*

$$(n,k) \geq 2^n - 2\sqrt{2^n k}\left(1 - \frac{1}{2^k}\right) + 2(k-1).$$

Proof. Apply the proof of Lemma 4.2.2 with $a = \log_2 k$. □

4.3 Upper Bound on $\sigma(n, k)$

The following lemma gives the upper bound on (n, k).

Lemma 4.3.1. *For $k \geq 2$ and n such that $2^n \geq (k-1)(1+\sqrt{k-1})^2$,*

$$(n, k) \leq 2^n - 2\sqrt{2^n(k-1)} + 2(k-1).$$

Proof. Suppose $(\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k)$ is cross-Sperner in $\mathcal{P}([n])$. We may and will assume that $|\mathcal{F}_1| \leq |\mathcal{F}_2| \leq \cdots \leq |\mathcal{F}_k|$. Define $\mathcal{G} := \bigcup_{i=1}^{k-1} \mathcal{F}_i$. By Lemma 3.3.1, $(\mathcal{G}, \mathcal{F}_k)$ is cross-Sperner. Let $m = |\mathcal{G}|$ and observe that, as each family is non-empty, we have $m \geq k-1$.

By Key Idea 3.1.3, $|\mathcal{F}_k| \leq 2^n - c(n, m) \leq 2^n - 2^{n/2+1}\sqrt{m} + m = \left(\sqrt{2^n} - \sqrt{m}\right)^2$. Since the families are ordered by increasing size, $|\mathcal{F}_k| \geq \frac{m}{k-1}$. Putting this together gives

$$\frac{m}{k-1} \leq |\mathcal{F}_k| \leq \left(\sqrt{2^n} - \sqrt{m}\right)^2.$$

Rearranging, we obtain

$$\sqrt{m} \leq \sqrt{2^n}\left(\frac{\sqrt{k-1}}{1+\sqrt{k-1}}\right). \tag{4.6}$$

Now consider the sum

$$\sum_{i=1}^{k} |\mathcal{F}_i| = |\mathcal{G}| + |\mathcal{F}_k| \leq m + \left(\sqrt{2^n} - \sqrt{m}\right)^2. \tag{4.7}$$

Let $x = \frac{1}{2}\sqrt{2^n} - \sqrt{m}$. Substituting this into expression (4.7) gives

$$\left(\tfrac{1}{2}\sqrt{2^n} - x\right)^2 + \left(\tfrac{1}{2}\sqrt{2^n} + x\right)^2 = 2^{n-1} + 2x^2.$$

It is clear that (4.7) is maximized when $|x| = |\frac{1}{2}\sqrt{2^n} - \sqrt{m}|$ is as large as possible. Combining $m \geq k-1$ with (4.6) gives $\sqrt{k-1} \leq \sqrt{m} \leq \sqrt{2^n}\left(\frac{\sqrt{k-1}}{1+\sqrt{k-1}}\right)$, so we need only find which of these end values is further from $\frac{1}{2}\sqrt{2^n}$.

If we have $2^n \geq (k-1)(1+\sqrt{k-1})^2$ then

$$\tfrac{1}{2}\sqrt{2^n} - \sqrt{k-1} > \tfrac{1}{2}\sqrt{2^n} - \frac{\sqrt{2^n}}{1+\sqrt{k-1}} = \sqrt{2^n}\left(\frac{\sqrt{k-1}}{1+\sqrt{k-1}}\right) - \tfrac{1}{2}\sqrt{2^n}$$

and thus expression (4.7) is maximized when $m = k-1$. Substituting $m = k-1$ into

(4.7) gives

$$\begin{aligned}\sum_{i=1}^{k} |\mathcal{F}_i| &\leq (k-1) + \left(\sqrt{2^n} - \sqrt{k-1}\right)^2 \\ &= 2^n - 2\sqrt{2^n(k-1)} + 2(k-1),\end{aligned}$$

as desired. □

The proof of Lemma 4.3.1 uses a strategy that we will repeat in Section 4.4. We first ordered our families in increasing size and let $\mathcal{G}$ be the union of the first $k-1$. This reduced our system to a cross-Sperner pair. Then we can use the comparability result of Theorem 1.3.9 to bound the size of $|\mathcal{G}| + |\mathcal{F}_k|$. Recall that a similar technique was used to prove the upper bound of $\pi(n,k)$ in Theorem 1.3.7.

Now we can prove Theorem 1.3.8.

Proof of Theorem 1.3.8. Lemmas 4.2.2 and 4.3.1 together give the result. Observe that $2^{2k} \geq (k-1)(1+\sqrt{k-1})^2$ for $k \geq 2$ so the conditions of Lemma 4.3.1 hold. □

4.4 $\sigma(n,k)$ for Small Values of k

We begin this section by proving that Theorem 1.1 of Gerbner, Lemons, Palmer, Patkós, and Szécsi [19] on bounding the sum of cross-Sperner families pairs holds for all n. It is a simple consequence of Theorem 1.3.9.

Theorem 4.4.1. *Let $(\mathcal{F},\mathcal{G})$ be cross-Sperner in $\mathcal{P}([n])$. Then*

$$|\mathcal{F}| + |\mathcal{G}| \leq 2^n - 2^{\lfloor n/2 \rfloor} - 2^{\lceil n/2 \rceil} + 2.$$

Proof. Let $(\mathcal{F},\mathcal{G}) \in \mathcal{P}([n])^2$ be a cross-Sperner pair. Suppose $|\mathcal{F}| = m$. Since $\mathcal{F},\mathcal{G} \neq \emptyset$, $1 \leq m \leq 2^n - 1$. Moreover, we may assume without loss of generality that $|\mathcal{F}| \leq |\mathcal{G}|$. We know $|\mathcal{F}||\mathcal{G}| \leq 2^{2n-4}$ by Theorem 3.1.1. Using this, we have that

$$m^2 \leq |\mathcal{F}||\mathcal{G}| \leq 2^{2n-4},$$

which implies that $m \leq 2^{n-2}$.

Then, by Observation 2.1.8, $c(\mathcal{F}) = |u(\mathcal{F})| + |d(\mathcal{F})| - |u(\mathcal{F}) \cap d(\mathcal{F})|$. Since $\mathcal{F} \subseteq u(\mathcal{F}) \cap d(\mathcal{F})$ we have $m \leq |u(\mathcal{F}) \cap d(\mathcal{F})|$. Thus

$$c(\mathcal{F}) \geq |u(\mathcal{F})| + |d(\mathcal{F})| - m.$$

Since $\mathcal{G}$ must be incomparable to $\mathcal{F}$, we have that

$$|\mathcal{G}| \leq 2^n - c(\mathcal{F}) \leq 2^n - |u(\mathcal{F})| - |d(\mathcal{F})| + m.$$

Thus

$$|\mathcal{F}| + |\mathcal{G}| \leq 2^n - |u(\mathcal{F})| - |d(\mathcal{F})| + 2m. \tag{4.8}$$

We have the following two cases.

Case 1: Suppose $m = 1$. Since $\mathcal{F}$ only consists of one set, say F, we have $u(\mathcal{F}) = 2^{n-|F|}$ and $d(\mathcal{F}) = 2^{|F|}$. So $|u(\mathcal{F})||d(\mathcal{F})| = 2^n$, which, by the AM-GM inequality (Lemma 2.2.2) gives $|u(\mathcal{F})| + |d(\mathcal{F})| \geq 2^{n/2+1} \geq 2^{\lfloor n/2 \rfloor} + 2^{\lceil n/2 \rceil}$. So (4.8) yields

$$|\mathcal{F}| + |\mathcal{G}| \leq 2^n - 2^{\lfloor n/2 \rfloor} - 2^{\lceil n/2 \rceil} + 2,$$

as required.

Case 2: Now suppose $m \geq 2$. By Theorem 1.3.9, $|u(\mathcal{F})| + |d(\mathcal{F})| \geq 2^{\frac{n}{2}+1}\sqrt{m}$, so (4.8) gives

$$|\mathcal{F}| + |\mathcal{G}| \leq 2^n - 2^{\frac{n}{2}+1}\sqrt{m} + 2m.$$

By differentiating with respect to m we see that the expression on the right-hand side is decreasing in the range $2 \leq m \leq 2^{n-2}$. It is therefore maximized at $m = 2$, where we have

$$2^n - 2^{\frac{n}{2}+1}\sqrt{m} + 2m = 2^n - 2^{\frac{n+3}{2}} + 4.$$

Note that for all $n \geq 2$,

$$2^{\frac{n+3}{2}} - 4 \geq 2^{\lfloor n/2 \rfloor} + 2^{\lceil n/2 \rceil} - 2.$$

This implies that $2^n - 2^{\lfloor n/2 \rfloor} - 2^{\lceil n/2 \rceil} + 2 \geq 2^n - 2^{\frac{n}{2}+1}\sqrt{m} + 2m$ for all $2 \leq m \leq 2^{n-2}$ and $n \geq 2$.

We conclude that $|\mathcal{F}| + |\mathcal{G}| \leq 2^n - 2^{\lfloor n/2 \rfloor} - 2^{\lceil n/2 \rceil} + 2$, as desired. $\square$

Employing the technique used to prove Lemma 4.3.1 and the result of Lemma 2.4.2 we can improve the upper bound given in Theorem 1.3.8 for $(n, 3)$.

Proposition 4.4.2. *If $n \geq 4$, then*

$$(n, 3) \leq \begin{cases} 2^n - 3(2^{n/2}) + 4, & \text{if } n \text{ is even.} \\ 2^n - \frac{4}{\sqrt{2}}2^{n/2} + 4, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \subseteq \mathcal{P}([n])^3$ is a cross-Sperner system and $|\mathcal{F}_1| \leq |\mathcal{F}_2| \leq |\mathcal{F}_3|$. Let $\mathcal{G} = \mathcal{F}_1 \cup \mathcal{F}_2$ and let $|\mathcal{G}| = m$. Since all families are nonempty, $m \geq 2$.

Since the families are ordered in increasing size, $|\mathcal{F}_3| \geq m/2$. Notice that $(\mathcal{G}, \mathcal{F}_3)$ is cross-Sperner by Lemma 3.3.1. By Theorem 3.1.1,

$$\frac{m^2}{2} \leq |\mathcal{G}||\mathcal{F}_3| \leq 2^{2n-4}.$$

This gives $2 \leq m \leq 2^{n-3/2}$. Since $|\mathcal{F}_3|$ can be at most $2^n - c(n,m)$ by Key Idea 3.1.3,

$$\begin{aligned} |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| &\leq m + 2^n - c(n,m) \\ &\leq 2^n - 2^{n/2+1}\sqrt{m} + 2m := g(m). \end{aligned}$$

Since no family is empty, our purpose is to maximize $g(m)$ for $2 \leq m \leq 2^{n-3/2}$. This equation is decreasing on the interval $2 \leq m \leq 2^{n-2}$ and increasing on the interval $2^{n-2} \leq m \leq 2^{n-3/2}$. In order to find the maximum of $g(m)$, it suffices to compare the value at $m = 2$ and $m = 2^{n-3/2}$. So $(n,m) \leq \max\{g(2), g(2^{n-3/2})\}$. We have that

$$g(2) = 2^n + 4 - 2^{\frac{n+3}{2}}$$

and

$$g(2^{n-3/2}) = 2^n + 2^{n-1/2} - 2^{n-1/4}.$$

Notice that $g(2) \geq g(2^{n-3/2})$ when $n \geq 4$. This implies that the sum is maximized when $|\mathcal{F}_1| = |\mathcal{F}_2| = 1$, so $(n,3) \leq 2^n - c(n,2) + m$. Lemma 2.4.2 gives us the exact values of $c(n,2)$. When n is even, $c(n,2) = 3(2^{n/2}) - 2$, so

$$(n,3) \leq 2^n - 3(2^{n/2}) + 4.$$

When n is odd, $c(n,2) = \frac{4}{\sqrt{2}}2^{n/2} - 2$, so

$$(n,3) \leq 2^n - \frac{4}{\sqrt{2}}2^{n/2} + 4$$

as required. $\square$

Based on the construction of the extremal family for $c(n,2)$, when n is even, we can give an exact value of $(n,3)$.

Corollary 4.4.3. *When $n \geq 4$ and n is even*

$$(n,3) = 2^n - 3(2^{n/2}) + 4.$$

Proof. By Proposition 4.4.2, when $n \geq 4$ and even, $(n,3) \leq 2^n - 3(2^{n/2}) + 4$. To prove equality, we need a construction that obtains this bound. Let $F_1, F_2 \subseteq [n]$ be such that $|F_1| = |F_2| = n/2$ and $|F_1 \cap F_2| = n/2 - 1$. By Lemma 2.4.2, $c(\{F_1, F_2\}) = c(n,2)$. Let

$\mathcal{F}_1 = \{F_1\}$ and $\mathcal{F}_2 = \{F_2\}$. Define

$$\mathcal{F}_3 := \{X \subseteq [n] : X \text{ is incomparable to } F_1 \text{ and } F_2\}.$$

Then $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is cross-Sperner and

$$|\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| = 2 + 2^n - c(n, 2) = 2^n - 3(2^{n/2}) + 4$$

as required. $\square$

We can also improve the upper bound given in Theorem 1.3.8 for $(n, 4)$.

Proposition 4.4.4. *If $n \geq 5$ then*

$$(n, 4) \leq \begin{cases} 6 + 2^n - \frac{21}{2}(2^{n/2}), & \textit{if } n \textit{ is even.} \\ 6 + 2^n - \frac{5}{\sqrt{2}} 2^{n/2}, & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Suppose $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4) \subseteq \mathcal{P}([n])^4$ is a cross-Sperner system and let $|\mathcal{F}_1| \leq |\mathcal{F}_2| \leq |\mathcal{F}_3| \leq |\mathcal{F}_4|$. Let $\mathcal{G} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ and $|\mathcal{G}| = m$. All the families are nonempty, so $m \geq 3$. Since we ordered the families by increasing size, $|\mathcal{F}_4| \geq m/3$. Moreover, $(\mathcal{G}, \mathcal{F}_4)$ is cross-Sperner by Lemma 3.3.1, so $|\mathcal{G}||\mathcal{F}_4| \leq 2^{2n-4}$ by Theorem 3.1.1. This implies that

$$\frac{m^2}{3} \leq |\mathcal{G}||\mathcal{F}_4| \leq 2^{2n-4}$$

which gives $m \leq 2^{n-2}\sqrt{3}$. By Key Idea 3.1.3,

$$\begin{aligned} \sum_{i=1}^{4} |\mathcal{F}_i| &= |\mathcal{G}| + |\mathcal{F}_4| \\ &\leq 2^n - c(n, m) + m \\ &= 2^n - 2^{n/2+1}\sqrt{m} + 2m := g(m). \end{aligned}$$

Our purpose is to maximize $g(m)$ for $3 \leq m \leq 2^{n-2}\sqrt{3}$. Notice that $g(m)$ is decreasing from 3 to 2^{n-2} and increasing from 2^{n-2} to $2^{n-2}\sqrt{3}$. To maximize $g(m)$, it suffices to compare the values at $m = 3$ and $m = 2^{n-2}\sqrt{3}$. At $m = 3$ we have

$$g(3) = 6 + 2^n - 2^{n/2+1}\sqrt{3}$$

and, at $m = 2^{n-2}\sqrt{3}$,

$$g(2^{n-2}\sqrt{3}) = 2^{n-1}\sqrt{3} + 2^n - 2^n 3^{1/4},$$

so $g(3) \geq g(2^{n-2}\sqrt{3})$ when $n \geq 5$. This proves that to maximize $(n, 4)$, we want

$|\mathcal{F}_1| = |\mathcal{F}_2| = |\mathcal{F}_3| = 1$. So, $(n, 4) \leq 2^n - c(n, 3) + 3$. Lemma 2.4.3 gives us the exact value of $c(n, 3)$. We know that

$$c(n,3) = \begin{cases} \frac{21}{2}(2^{n/2}) - 3, & \text{if } n \text{ is even} \\ \frac{5}{\sqrt{2}}(2^{\frac{n}{2}}) - 3, & \text{if } n \text{ is odd.} \end{cases}$$

This implies

$$(n,4) \leq \begin{cases} 6 + 2^n - \frac{21}{2}(2^{n/2}), & \text{if } n \text{ is even} \\ 6 + 2^n - \frac{5}{\sqrt{2}}2^{n/2}, & \text{if } n \text{ is odd.} \end{cases}$$

□

The bound in Proposition 4.4.4 is not tight. The extremal construction for $c(n, 3)$ in both the even and odd case is not an antichain as it was for $c(n, 2)$. In the proof of Proposition 4.4.4, we show that $(n, 4)$ is maximized when $|\mathcal{F}_1| = |\mathcal{F}_2| = |\mathcal{F}_3| = 1$. Since the extremal constructions for $c(n, 3)$ cannot be split up into three families, each with size one, that are cross-Sperner, $(n, 4) < 2^n - c(n, 3) + 3$.

Chapter 5
Conclusion

Here we gather together some concluding thoughts and open questions. In Chapter 3 we provide upper and lower bounds on $\pi(n,k)$ in Theorems 1.3.6 and 1.3.7, restated here for convenience.

Theorem 1.3.6. *Let n and $k \geq 2$ be integers. For n sufficiently large,*

$$\left(\frac{2^n}{ek}\right)^k \leq \pi(n,k).$$

Theorem 1.3.7. *Let n and $k \geq 2$ be integers. Then*

$$\pi(n,k) \leq \left(\frac{2^n}{k^2}\right)^k \left\lfloor \frac{k}{2} \right\rfloor^{\lfloor k/2 \rfloor} \left\lceil \frac{k}{2} \right\rceil^{\lceil k/2 \rceil}.$$

Comparing these bounds shows that they differ by a factor of $\left(\frac{e}{2}\right)^k$ for k even and less than $\left(1+\frac{1}{k}\right)\left(\frac{e}{2}\right)^k$ for k odd. It would be interesting to tighten this gap. We believe that (for large n) the bound given in Lemma 3.2.1 ought to be essentially best possible.

Conjecture 5.0.1. *Let $k \geq 2$ be fixed and n be sufficiently large with respect to k. Then*

$$\pi(n,k) = (1+o(1))\left(\frac{(k-1)^{k-1}}{k^k}2^n\right)^k.$$

The technique used to find the upper bound given in Theorem 1.3.7 was quite wasteful. We combined the first $k-1$ families, and assumed that the comparability number of the union of the $k-1$ families was minimal. This is an unrealistic assumption, as the extremal families that minimize comparability tend to maximize the amount of containment between sets, where as in cross-Sperner systems, we want families that are incomparable with each other. It is most likely the case that the comparability number of the union of the first $k-1$ families of a cross-Sperner system is much higher than the minimal comparability number.

Recall that Seymour's Theorem (Lemma 1.3.3) and the AM-GM inequality (Lemma 2.2.2) induced a tight upper bound on $|\mathcal{F}||\mathcal{G}|$ where $\mathcal{F}, \mathcal{G}$ is a cross-Sperner pair. A generalization of Seymour's Theorem would perhaps give a proof of an improved upper bound on $\pi(n, k)$. If we can show that

$$|\mathcal{F}_1|^{1/k} + |\mathcal{F}_2|^{1/k} + \cdots + |\mathcal{F}_k|^{1/k} \leq c2^{n/k},$$

where $c = (1 + o(1))(k-1)^{1-\frac{1}{k}}$, then applying the AM-GM inequality will give the bound in Conjecture 5.0.1.

In Chapter 3 we also found the values of $\pi(3,3), \pi(4,4)$ and $\pi(4,3)$. We have found constructions that show $\pi(5,3) \geq 81, \pi(5,4) \geq 108$, and $\pi(6,3) \geq 810$. Determining if these bounds are tight would be interesting.

In Chapter 4 we proved Theorem 1.3.8, restated here.

Theorem 1.3.8. *Let n, k be integers with $n \geq 2k$. Then*

$$2^n - \frac{3}{\sqrt{2}}\sqrt{2^n k} + 2(k-1) \ \leq \ \sigma(n,k) \ \leq \ 2^n - 2\sqrt{2^n(k-1)} + 2(k-1).$$

Recall that when k is a power of 2, the lower bound is $2^n - 2\sqrt{2^n k} + 2(k-1)$, which is extremely close to the upper bound. It would be of particular interest to understand the extremal constructions of the cross-Sperner systems that maximize (n,k). We conjecture that one extremal construction is to take an antichain of size $k-1$ of the form $\{X \cup \{a_1\}, X \cup \{a_2\}, \cdots, X \cup \{a_{k-1}\}\}$ and let $\mathcal{F}_i = \{X_{a_i}\}$ for $1 \leq i \leq k$. Then, let $\mathcal{F}_k$ be all the sets incomparable to the antichain. One may ask what other extremal constructions maximize (n,k), if any.

Bibliography

[1] N. Alon, S. Das, R. Glebov, and B. Sudakov. Comparable pairs in families of sets. *J. Combin. Theory Ser. B*, 115:164–185, 2015.

[2] J. Balogh, R. Morris, and W. Samotij. The method of hypergraph containers. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures*, pages 3059–3092. World Sci. Publ., Hackensack, NJ, 2018.

[3] J. Balogh, R. Mycroft, and A. Treglown. A random version of Sperner's theorem. *J. Combin. Theory Ser. A*, 128:104–110, 2014.

[4] J. Balogh and A. Z. Wagner. Kleitman's conjecture about families of given size minimizing the number of k-chains. *Adv. Math.*, 330:229–252, 2018.

[5] M. A. Bashov. Minimization of two-sided shadow in the unit cube. *Diskret. Mat.*, 23(4):115–132, 2011.

[6] M. A. Bashov. Nonexistence of a Kruskal-Katona type theorem for double-sided shadow minimization in the boolean cube layer. *Acta Universitatis Sapientiae. Informatica*, 5, 07 2013.

[7] N. Behague, A. Kuperus, N. Morrison, and A. Wright. Improved bounds for cross-Sperner systems. submitted. arXiv:2302.02516, 2023.

[8] B. Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hungar.*, 16:447–452, 1965.

[9] M. Collares and R. Morris. Maximum-size antichains in random set-systems. *Random Structures Algorithms*, 49(2):308–321, 2016.

[10] S. Das, W. Gan, and B. Sudakov. Sperner's theorem and a problem of Erdős, Katona, and Kleitman. *Combin. Probab. Comput.*, 24(4):585–608, 2015.

[11] D. E. Daykin and P. Frankl. On Kruskal's cascades and counting containments in a set of subsets. *Mathematika*, 30(1):133–141, 1983.

[12] A. P. Dove, J. R. Griggs, R. J. Kang, and J.-S. Sereni. Supersaturation in the Boolean lattice. *Integers*, 14A:Paper No. A4, 7, 2014.

[13] P. Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, 51:898–902, 1945.

[14] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.

[15] P. Erdős and J. Spencer. *Probabilistic methods in combinatorics.* Probability and Mathematical Statistics, Vol. 17. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1974.

[16] P. Frankl. The Erdős-Ko-Rado theorem is true for $n = ckt$. In *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I*, volume 18 of *Colloq. Math. Soc. János Bolyai*, pages 365–375. North-Holland, Amsterdam-New York, 1978.

[17] D. Gerbner, B. Keszegh, N. Lemons, C. Palmer, D. Pálvölgyi, and B. Patkós. Saturating Sperner families. *Graphs Combin.*, 29(5):1355–1364, 2013.

[18] D. Gerbner, N. Lemons, C. Palmer, B. Patkós, and V. Szécsi. Almost intersecting families of sets. *SIAM J. Discrete Math.*, 26(4):1657–1669, 2012.

[19] D. Gerbner, N. Lemons, C. Palmer, B. Patkós, and V. Szécsi. Cross-Sperner families. *Studia Sci. Math. Hungar.*, 49(1):44–51, 2012.

[20] A. Gowty, D. Horsley, and A. Mammoliti. Minimising the total number of subsets and supersets. submitted. arXiv:2212.13112, 2022.

[21] G. Hansel. Sur le nombre des fonctions booléennes monotones de n variables. *C. R. Acad. Sci. Paris Sér. A-B*, 262:A1088–A1090, 1966.

[22] L. H. Harper. Optimal numberings and isoperimetric problems on graphs. *J. Combinatorial Theory*, 1:385–393, 1966.

[23] A. J. W. Hilton. A theorem on finite sets. *Quart. J. Math. Oxford Ser. (2)*, 27(105):33–36, 1976.

[24] G. Katona. On a conjecture of Erdős and a stronger form of Sperner's theorem. *Studia Sci. Math. Hungar.*, 1:59–63, 1966.

[25] D. Kleitman. On a lemma of Littlewood and Offord on the distribution of certain sums. *Math. Z.*, 90:251–259, 1965.

[26] D. Kleitman. Families of non-disjoint subsets. *J. Combinatorial Theory*, 1:153–155, 1966.

[27] D. Kleitman. A conjecture of Erdős-Katona on commensurable pairs among subsets of an n-set. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 215–218. Academic Press, New York, 1968.

[28] D. Kleitman. On a conjecture of Milner on k-graphs with non-disjoint edges. *J. Combinatorial Theory*, 5:153–156, 1968.

[29] D. Kleitman. On Dedekind's problem: The number of monotone Boolean functions. *Proc. Amer. Math. Soc.*, 21:677–682, 1969.

[30] J. B. Kruskal. The number of simplices in a complex. In *Mathematical optimization techniques*, pages 251–278. Univ. California Press, Berkeley, Calif., 1963.

[31] L. Lovász. Flats in matroids and geometric graphs. In *Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977)*, pages 45–86. Academic Press, London, 1977.

[32] M. Matsumoto and N. Tokushige. The exact bound in the Erdős-Ko-Rado theorem for cross-intersecting families. *J. Combin. Theory Ser. A*, 52(1):90–97, 1989.

[33] N. Morrison, J. A. Noel, and A. Scott. On saturated k-Sperner systems. *Electron. J. Combin.*, 21(3):Paper 3.22, 17, 2014.

[34] D. Osthus. Maximum antichains in random subsets of a finite set. *J. Combin. Theory Ser. A*, 90(2):336–346, 2000.

[35] L. Pyber. A new generalization of the Erdős-Ko-Rado theorem. *J. Combin. Theory Ser. A*, 43(1):85–90, 1986.

[36] A. Rényi. On random subsets of a finite set. *Mathematica (Cluj)*, 3:355–362, 1961.

[37] W. Samotij. Subsets of posets minimising the number of chains. *Trans. Amer. Math. Soc.*, 371(10):7259–7274, 2019.

[38] P. D. Seymour. On incomparable collections of sets. *Mathematika*, 20:208–209, 1973.

[39] E. Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.*, 27(1):544–548, 1928.

[40] R. M. Wilson. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica*, 4(2-3):247–257, 1984.