NORMAL SUBGROUPS OF INFINITE DIMENSIONAL LINEAR GROUPS

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DM-365-IR MAY 1985

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§1. Introduction. Let H be a complex separable infinite-dimensional Hilbert space, let $GL(H)$ denote the group of invertible operators on H and let $U(H)$ denote the group of unitary operators on H. In this paper we investigate the normal subgroups of these groups as well as their counterparts in the Calkin algebra.

We show that every proper normal subgroup of $GL(H)$ is contained in the normal subgroup $GK(H)$ consisting of all invertible operators which are scalars modulo the compact operators. Consequently, this latter subgroups is the only maximal normal subgroup. An identical result holds for the group of $U(H)$ and the subgroup $UK(H)$ of unitary operators which are scalars modulo the compacts. These results may be regarded as analogues of the classical result [3] that every proper ideal in the algebra $B(H)$ of all bounded operators on H is contained in the ideal $K(H)$ of all compact operators as well as the fact [11] (see also [6]) that every proper Lie ideal in $B(H)$ is contained in the Lie ideal $C1 + K(H)$.

We also characterize all normal subgroups of each of the group of invertible elements and the group of unitary elements in the Calkin algebra $B(H)/K(H)$. If G is either group, then a normal subgroup N of G is either a subgroup of the centre (i.e., the scalars) or is the inverse image of a subgroup of the integers under the index homomorphism, i.e. $N = \{a \in G\colon n \text{ divides } \operatorname{ind}(a)\}$ for some nonnegative integer n.

§2. The Groups $GL(H)$ and $U(H)$. We define $GK(H)$ and $UK(H)$ by

$$GK(H) = GL(H) \cap \big(\mathbb{C}1 + K(H)\big),$$

$$UK(H) = U(H) \cap \big(\mathbb{C}1 + K(H)\big).$$

We start by stating the main result about the unitary group.

THEOREM 1. *Every proper normal subgroup in the unitary group* $U(H)$ *is contained in the normal subgroup* $UK(H)$.

Another way to state this result is that if U is a unitary operator which is not a scalar modulo the compact operators, then the normal subgroup generated by U is all of $U(H)$. We prove a little more, namely that the "normal semigroup" generated by U is $U(H)$. More precisely, we prove the following proposition of which Theorem 1 is an immediate corollary.

PROPOSITION 1. *If* U *is a unitary operator and if* U *is not the sum of a scalar and a compact operator, then every unitary operator is a product of a finite number of operators each of which is unitarily equivalent to* U.

For instance we may take U to be a symmetry (i.e., a self-adjoint unitary operator) such that both ker(U-I) and ker(U+I) are infinite dimensional. Since U is not a scalar plus compact and since an operator unitarily equivalent to a symmetry is also a symmetry, it follows from Theorem 1 that every unitary operator is a product of symmetries. Thus, Theorem 2 may be regarded as a generalization of the following theorem of Halmos and Kakutani [8]: every unitary operator is a product of at most four symmetries.

Halmos and Kakutani's theorem has a "skew" version due to Radjavi [9]: every invertible operator is a product of at most seven involutions. (An involution is an operator whose square is the identity.) We also have the following "skew" version of Proposition 1, which may be regarded as a generalization of Radjavi's result.

PROPOSITION 2. *If* T *is an invertible operator which is not the sum of a scalar and a compact operator, then every invertible operator is a product of a finite number of operators similar to* T.

An immediate corollary of this is the following result.

THEOREM 2. *Every proper normal subgroup of* $GL(H)$ *is contained in* $GK(H)$.

In a different vein, Proposition 2 gives us a result about products of unipotent operators. An operator is said to be unipotent of order 2 if it is of the form $1 + N$ with $N^2 = 0$. If A is unipotent of order 2, then so is every operator similar to A. Thus we have the following consequence of Proposition 2.

COROLLARY. *Every invertible operator on* H *is a product of a finite number of unipotent operators of order* 2.

In [7], we have obtained a sharper result: every invertible operator is a product of at most six unipotent operators of order 2.

§3. Proof of Proposition 1. For an operator A on $\mathcal{H}$, we write $W_e(A)$ for its essential numerical range. For basic properties of the essential numerical range, the reader is referred to [5].

LEMMA 1. *If U is a unitary operator and if zero is in the interior of $W_e(U)$, then every unitary operator is a product of at most eight operators unitarily equivalent to U.*

Proof: Construct inductively an orthonormal sequence $\{e_n\}$ such that $(Ue_n, e_m) = 0$ for all n, m as follows. Since $0 \in W_e(U)^0 \subseteq W(U)$, there is a unit vector e_1 such that $(Ue_1, e_1) = 0$. Suppose now that we already have $e_1, \ldots, e_k$ such that $(Ue_n, e_m) = 0$ for all $n, m \leq k$. Let

$$\mathcal{M} = \{e_1,\ldots,e_k, Ue_1,\ldots,Ue_k, U^*e_1,\ldots,U^*e_k\}^{\perp}$$

and let V be the compression of U to $\mathcal{M}$. Since $\mathcal{M}^{\perp}$ is finite dimensional, we have $W_e(V) = W_e(U)$ and hence $0 \in W_e(V)^0$. Let e_{k+1} be a unit vector in $\mathcal{M}$ such that $(Ve_{k+1}, e_{k+1}) = 0$. Then $e_1, \ldots, e_{k+1}$ is a finite orthonormal sequence such that $(Ue_n, e_m) = 0$ for all $n, m \leq k + 1$.

Let $\mathcal{H}_1$ be the closed linear span of $\{e_n\colon n \text{ odd}\}$, let $\mathcal{H}_3 = U\mathcal{H}_1$ and let $\mathcal{H}_2 = (\mathcal{H}_1 \oplus \mathcal{H}_3)^{\perp}$. The unitary operator U maps $\mathcal{H}_1$ onto $\mathcal{H}_3$ and hence it maps $\mathcal{H}_2 \oplus \mathcal{H}_3 = \mathcal{H}_1^{\perp}$ onto $\mathcal{H}_3^{\perp} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and so the matrix of U relative to the decomposition: $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ takes the form

$$U = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ R & 0 & 0 \end{pmatrix}$$

where R is a unitary operator from $\mathcal{H}_1$ onto $\mathcal{H}_3$. We note that each of

H_1, H_2 and H_3 is isomorphic to H.

Now let V be any unitary operator on H_3 and let

$$V_0 = \begin{pmatrix} 0 & 0 & R^*V \\ 0 & 1 & 0 \\ R & 0 & 0 \end{pmatrix}.$$

So V_0 is a unitary operator on H and

$$UV_0UV_0^* = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & V \end{pmatrix} = \begin{pmatrix} V' & 0 \\ 0 & V \end{pmatrix}.$$

Identifying each of $H_1 + H_2$ and H_3 with H, the above computation shows that if V is a unitary operator on H, then there exists another unitary operator V' such that $V \oplus V'$ is a product of two operators unitarily equivalent to U. We now take V to be a bilateral shift of infinite multiplicity. The unitary operators V' can be written as a product V_1V_2 of two bilateral shifts of infinite multiplicity [8]. Let J be a unitary operator such that $V^* = JV_1J^*$ and let $S = JV_2J^*$. It follows that $V \oplus V'$ is unitarily equivalent to $V \oplus V^*S$ and so each of $V \oplus V^*S$ and $V^*S \oplus V$ is a product of two operators unitarily equivalent to U. So, there exists four operators unitarily equivalent to U whose product is the operator $(V \oplus V^*S)(V^*S \oplus V) = S \oplus V^*SV$ which is a bilateral shift of infinite multiplicity. Now the conclusion of the lemma follows by using, once again, the fact that every unitary operator is a product of two bilateral shifts of infinite multiplicity. □

Proof of Proposition 1. Suppose that U is a unitary operator which is not a scalar plus compact. The essential spectrum $\sigma_e(U)$ of U contains two distinct complex numbers λ_1 and λ_2. We may write U in the form

$$U = \begin{pmatrix} \lambda_1 1 & 0 & 0 \\ 0 & \lambda_2 1 & 0 \\ 0 & 0 & A \end{pmatrix} + K_1$$

where K_1 is a compact operator and where every direct summand is infinite dimensional, (see, e.g., [5, Theorem 4.2]). In view of Lemma 1, it suffices to show that there is a product V of a finite number of operators unitarily equivalent to U such that $0 \in W_e(V)^0$.

We consider two cases according as $\lambda_2 = -\lambda_1$ or not. In the first case,

$$U = \lambda_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & B \end{pmatrix} + K_1$$

which is unitarily equivalent to

$$\lambda_1 \begin{pmatrix} 0 & J & 0 \\ J^* & 0 & 0 \\ 0 & 0 & B \end{pmatrix} + K_1$$

for every unitary operator J. Now let R be a unitary operator such that $0 \in W_e(R)^0$, so we have that U is unitarily equivalent to each of the operators

$$U_1 = \lambda_1 \begin{pmatrix} 0 & R & 0 \\ R^* & 0 & 0 \\ 0 & 0 & B \end{pmatrix} + K_1 \quad \text{and} \quad U_2 = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & B \end{pmatrix} + K_1 .$$

It follows that

$$U_1 U_2 = \lambda_1^2 \begin{pmatrix} R & 0 & 0 \\ 0 & R^* & 0 \\ 0 & 0 & B^2 \end{pmatrix} + K_2$$

where K_2 is compact. Therefore $0 \in W_e(U_1U_2)^0$. This ends the proof in this case.

Finally, we consider the case $\lambda_2 \neq -\lambda_1$. Let $\mu = \lambda_2/\lambda_1$, so $\mu \neq \pm 1$. It is easy to see that there exists a positive integer n such that 0 belongs to the inerior of the convex hull of $\{1, \mu, \mu^2, \ldots, \mu^n\}$. For every positive integer m, we have

$$U^m = \lambda_1^m \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu^m & 0 \\ 0 & 0 & B^m \end{pmatrix} + K_m$$

where K_m is compact. So U^m is unitarily equivalent to the operator

$$V_m = \lambda_1^m \operatorname{diag}(1, \ldots, 1, \mu^m, 1, \ldots, 1, B^m) + K_m,$$

with $n + 2$ direct summands and with μ^m in the $(m+1)$st position. Now let $V = V_1 V_2 \ldots V_n$, so

$$V = \lambda \operatorname{diag}(1, \mu, \mu^2, \ldots, \mu^n, C) + K$$

for a unimodular complex number λ, a bounded operator C and a compact operator K. Therefore $0 \in W_e(V)^0$ and V is a product of $n(n+1)/2$ operators unitarily equivalent to U. $\square$

§4. Proof of Proposition 2. Before starting the proof, we state a well-known result (see [10, Corollary 0.15]).

LEMMA 2. *If* $\sigma(A) \cap \sigma(B) = \emptyset$, *then the operator* $\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$ *is similar to* $A \oplus B$.

To prove Proposition 2, assume that T is an invertible operator which is not a scalar modulo the compacts. By a result of Brown and Pearcy [1; Theorem 2], T is similar to an operator of the form

$$T_0 = \begin{pmatrix} 0 & A & B \\ 0 & C & D \\ 1 & E & F \end{pmatrix}$$

acting on $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$. Let S be an arbitrary invertible operator, let

$$L_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} S & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and let $T_j = L_j^{-1} T_0 L_j$ for $j = 1, 2$. So each of T_1 and T_2 is similar to T and

$$T_2T_1 = \begin{pmatrix} F(S) & 0 \\ * & S \end{pmatrix}$$

where

$$F(S) = \begin{pmatrix} S^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} VA & VB \\ C & D \end{pmatrix} \begin{pmatrix} D & C \\ VB & VA \end{pmatrix}.$$

For every invertible operator X, we will show that $\sigma(\alpha X) \cap \sigma(F(\alpha X)) = \emptyset$ if $|\alpha|$ is either large enough or small enough. To prove this, notice that $F(\alpha X = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} F(X)$, so $\|F(\alpha X)\| \le \|F(X)\|$ for $|\alpha| \ge 1$ and hence we can choose $|\alpha|$ large enough so that $\sigma(\alpha X)$ lies outside the disc $\{z: |z| \le \|F(X)\|\}$ which contains $\sigma(F(\alpha X))$. Similarly, for $|\alpha|$ small enough, $\sigma(\alpha X)$ is included in the disc $\{z\colon |z| < \|F(X)^{-1}\|^{-1}$ while $\sigma(F(\alpha X))$ lies outside the same disc since $\|F(\alpha X)^{-1}\| \le \|F(X)^{-1}\|$ for $|\alpha| \le 1$. Applying the above to $X = S$ and $X = 1$ and using Lemma 2, we conclude that there exists a scalar α such that each of the operators $\begin{pmatrix} F(\alpha S) & 0 \\ 0 & \alpha S \end{pmatrix}$ and $\begin{pmatrix} F(\alpha^{-1}1) & 0 \\ 0 & \alpha^{-1}1 \end{pmatrix}$ is a product of two operators similar to T, and so $S \oplus F(\alpha S)F(\alpha^{-1}1)$ is a product of four operators similar to T.

Now take S to be $U \oplus 1$ where U is a bilateral shift with infinite multiplicity and 1 is the identity operator on an infinite dimensional space. From the above, there exists an invertible operator Q on $\mathcal{H}$ such that $S \oplus Q$ is a product of four operators similar to T. The operator $S \oplus Q$ can be written as $U \oplus Q'$ where both U and Q' are operators on $\sum_{n\in Z} \oplus \mathcal{H}_n$ with $\mathcal{H}_n = \mathcal{H}_0$ for all n and

$$U(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, \ldots) = (\ldots, x_{-2}, \boxed{x_{-1}}, x_0, x_1, \ldots),$$

$$Q' = \mathrm{diag}\,(\ldots, 1, \boxed{Q}, 1, 1, \ldots),$$

that is,

$$Q'(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2 \ldots) = (\ldots, x_{-2}, x_{-1}, \boxed{Qx_0}, x_1, x_2, \ldots).$$

(The box "$\boxed{}$" is used to indicte the zero$^{\text{th}}$ position.) Now

$$(U \oplus Q')(Q' \oplus U) = UQ' \oplus Q'U,$$

$$UQ'(\ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2 \ldots) = (\ldots, x_{-2}, \boxed{x_{-1}}, Qx_0, x_1, x_2, \ldots).$$

Let $J = \mathrm{diag}\,(\ldots 1, 1, \boxed{1}, Q, Q, \ldots)$. By direct computation, we have

$J(UQ')J^{-1} = U$. In the same way, we can show that $Q'U$ is similar to U. Therefore $(U \oplus Q')(Q' \oplus U)$ is similar to a bilateral shift of infinite multiplicity. We have shown that a bilateral shift is a product of eight operators similar to T. Since each symmetry is a product of two bilateral shifts of infinite multiplicity, the theorem follows from Radjavi's result [9] which asserts that every invertible operator is a product at most seven involutions. $\square$

§5. Groups in the Calkin Algebra. Let $\mathcal{A}$ be the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and consider the two groups

$$GL_e = \{a \in \mathcal{A}: \ a \text{ is invertible in } \mathcal{A}\}$$

and

$$\mathcal{U}_e = \{u \in \mathcal{A}: \ u \text{ is unitary in } \mathcal{A}\}.$$

We describe all normal subgroups in each of these groups.

Before proceeding, we recall some facts about the Calkin algebra and index theory (see [4; Chapter 5]). The index of a Fredholm operator T is defined by $\mathrm{ind}(T) = \dim\ker(T) - \dim\ker(T^*)$. The index satisfies the equation $\mathrm{ind}(TS) = \mathrm{ind}(T) + \mathrm{ind}(S)$. Furthermore, it is invariant under compact perturbations. Let $\pi: \mathcal{B}(\mathcal{H}) \to \mathcal{A}$ be the canonical quotient map. Atkinson's theorem [4; Theorem 5.17] implies that the set of Fredholm operators is the inverse image under π of the set GL_e of invertible operators in $\mathcal{A}$. In view of this and the invariance of the index under compact perturbations, we define the index of an invertible element in $\mathcal{A}$ by $\mathrm{ind}(a) = \mathrm{ind}(A)$ for any $A \in \pi^{-1}(a)$. This gives a homomorphism from the group GL_e onto the

the group of integers Z.

Two facts about operators of index 0 are needed in the sequel.

(1) For a Fredholm operator T, $\mathrm{ind}(T) = 0$ if and only if T is a compact perturbation of an invertible operator.

(2) If $\pi(T)$ is unitary and if $\mathrm{ind}(T) = 0$, then T is a compact perturbation of a unitary operator [2; Theorem 3.1].

One more fact about the Calkin algebra A is that the centre of A is the scalars [3]. It follows immediately that the centre of the group GL_e is also the (nonzero) scalars. We can also easily establish the fact that the centre of the group U_e is $\{\lambda 1: |\lambda| = 1\}$ since every element of A is a linear combination of four unitary elements. (This follows from the fact that if a is self-adjoint with $\|a\| \leq 1$, then $a \pm (1-a^2)^{1/2}$ are unitaries.)

LEMMA 3. <u>If</u> a <u>is an invertible element in the Calkin algebra</u> A <u>such that</u> $a^{-1}u^{-1}au$ <u>is a scalar for every unitary element</u> u <u>in</u> A, <u>then</u> a <u>is a scalar</u>.

<u>Proof</u>: Let b be a self-adjoint element in A. Since e^{itb} is unitary for every real number t, there exist scalars λ_t such that $a^{-1}\, e^{-itb}\, a\, e^{itb} = \lambda_t 1$ for every scalar t. Taking the derivative at $t = 0$, we get that $b - a^{-1}\, b\, a = \lambda 1$ for a scalar λ. Consequently, b is similar to $b - \lambda 1$ and so $\sigma(b) = \sigma(b) - \lambda$. This implies that $\lambda = 0$, hence $b - a^{-1}\, b\, a = 0$ and $ab = ba$. Thus a commutes with every self-adjoint element in A, hence it commutes with every element in A and so a is scalar. $\square$

THEOREM 3. *Let* G *be either the group* GL_e *of invertible elements or the group* U_e *of unitary elements in the Calkin algebra. If* N *is a normal subgroups of* G, *then* G *is either contained in the centre (i.e. the scalars) or there exists a nonnegative integer* n *such that* $N = \{a \in G\colon n$ *divides* $\operatorname{ind}(a)\}$.

Proof: Let N be a normal subgroup in G which is not contained in the centre of G. Then N contains an element a which is not a scalar. By Lemma 3, there exists a unitary element u in A such that $b = a^{-1} u^{-1} a u$ is not a scalar. Since $u \in G$, we have that $b \in N$. Obviously, $\operatorname{ind}(b) = 0$, therefore there exists an invertible operator B in $\pi^{-1}(b)$; furthermore, in the case $G = U_e$, the operator B can be chosen to be unitary [2; Theorem 3.1]. Since B is not a scalar modulo the compacts, we get by Proposition 1 (or Proposition 2) that N contains $G_0 = \{c \in G\colon \operatorname{ind}(c) = 0\}$. Since G_0 is the kernel of the homomorphism $\operatorname{ind}\colon G \to Z$, the subgroup N is the inverse image under ind of a subgroup in Z. Therefore, these exists a nonnegative integer n such that $N = \{a \in G\colon n$ divides $\operatorname{ind}(a)\}$. $\square$

COROLLARY. *If* G *is either* GL_e *or* U_e, *then the maximal normal subgroups in* G *are the groups* $G_p = \{a \in G\colon p$ *divides* $\operatorname{ind}(a)\}$ *for a prime* p.

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