SUM OF A CERTAIN TRIPLE q–HYPERGEOMETRIC SERIES

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1. *Introduction*

Put

$$(1.1)\qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \forall\ n \in \{1,2,3,\cdots\}, \end{cases}$$

and let ${}_rF_s$ denote a generalized hypergeometric series with r numerator and s denominator parameters, defined by (*cf.* [7, Chapter 2] and [9, p. 19])

$$(1.2)\qquad {}_rF_s\left[\begin{matrix}(a_r);\\ \\(b_s);\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}\,\frac{z^n}{n!},$$

where, and in what follows, (a_r) abbreviates the array of r parameters $a_1,\cdots,a_r$, with similar interpretations for $(b)_s$, *et cetera*. Also let $F^{k:r;u}_{\ell:s;v}$ denote a generalized Kampé de Fériet double hypergeometric series defined by (*cf.* [1, p. 150] and [9, p. 27])

$$(1.3)\qquad F^{k:r;u}_{\ell:s;v}\left[\begin{matrix}(\alpha_k):\ (a_r);\ (c_u);\\ \\(\beta_\ell):\ (b_s);\ (d_v);\end{matrix}\ x,y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{k}(\alpha_j)_{m+n}\ \prod_{j=1}^{r}(a_j)_m\ \prod_{j=1}^{u}(c_j)_n}{\prod_{j=1}^{\ell}(\beta_j)_{m+n}\ \prod_{j=1}^{s}(b_j)_m\ \prod_{j=1}^{v}(d_j)_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!},$$

provided that the series converges absolutely (or terminates).

Making use of a known linear transformation for one of Lauricella's fourteen hypergeometric functions of three variables, *viz* (see [4, p. 114], [6], and [9, p. 43])

$$F_{13} = F_T(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\ \gamma_1,\gamma_1,\gamma_1;\ x,\ y,\ z) \tag{1.4}$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

Pradhan [5] derived the sum of a certain triple hypergeometric series. We recall Pradhan's result in the following *equivalent* form (*cf.* [5, p. 33, Equation (1)]):

$$\sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{p=0}^{P} \frac{(a)_{n+p}(c-b)_{m+p}(c-a)_m(b)_n(-M)_m(-N)_n(-P)_p}{m!\ n!\ p!\ (c)_{m+n+p}(1-d-M)_m(1+d-N)_n(1+a-b-P)_p} \tag{1.5}$$

$$= \frac{(c-a)_{N+P}(b)_{M+P}(a)_M(c-b)_N}{(c)_{M+N+P}(d)_M(-d)_N(b-a)_P},$$

where

$$\begin{cases} d = a + b - c \neq 0,\pm1,\pm2,\cdots; \\ \\ M,N,P \in \{0,1,2,\cdots\}. \end{cases} \tag{1.6}$$

Motivated by some recent applications of the summation formula (1.5) by Bohra *et al.* [2], we first present a simple and direct proof of (1.5). We then proceed to give its *basic* (or q-) extension which is believed to be new.

2. *Derivation of the Summation Formula* (1.5)

Our direct proof of the summation formula (1.5) is based upon the Pfaff–Saalschütz theorem [7, p. 49, Equation (2.3.1.3)]:

$$(2.1)\qquad {}_3F_2\left[\begin{matrix} a,\ b,\ -N; \\ c,\ a+b-c-N+1; \end{matrix}\ 1\right] = \frac{(c-a)_N(c-b)_N}{(c)_N(c-a-b)_N}$$

$$(N \in \{0,1,2,\cdots\})$$

together with the double hypergeometric sum [3]:

$$(2.2)\qquad F^{0:3;3}_{1:1;1}\left[\begin{matrix} -\colon\ c-a,\ c-b,\ -M;\ a,\ b,\ -N; \\ c\colon\ \qquad 1-d-M;\quad 1+d-N; \end{matrix}\ 1,\ 1\right]$$

$$= \frac{(a)_M(c-a)_N(b)_M(c-b)_N}{(c)_{M+N}(d)_M(-d)_N}\,,$$

where [*cf.* Equation (1.6)]

$$(2.3)\qquad \begin{cases} d = a+b-c \neq 0,\pm1,\pm2,\cdots; \\ M,N \in \{0,1,2,\cdots\}. \end{cases}$$

As already observed by Srivastava [8, p. 4, Section 2], the double hypergeometric sum (2.2) is essentially a consequence of the Pfaff–Saalschütz theorem (2.1) *and* another known result [7, p. 57, Equation (2.3.4.8)] summing a very well–poised terminating ${}_4F_3(-1)$ series.

Now let $\mathscr{S}$ denote the first member of the summation formula (1.5). Then, rewriting $\mathscr{S}$ as a single sum involving Kampé de Fériet series, we have

$$(2.4)\qquad \mathscr{S}=\sum_{p=0}^{P}\frac{(a)_p(c-b)_p(-P)_p}{p!\;(c)_p(1+a-b-P)_p}$$

$$\cdot\, F^{0:3;3}_{1:1;1}\left[\begin{matrix} \text{——}: c-a,\ c-b+p,\ -M;\ a+p,\ b,\ -N; \\ c+p: \qquad 1-d-M; \qquad 1+d-N; \end{matrix}\ 1,\ 1\right],$$

where d is given by (1.6).

The Kampé de Fériet series, occurring in the summand of (2.4), can be summed by means of (2.2) with c and a replaced by $c+p$ and $a+p$, respectively, and we thus find that

$$(2.5)\qquad \mathscr{S}=\frac{(a)_M(c-a)_N(b)_M(c-b)_N}{(c)_{M+N}(d)_M(-d)_N}$$

$$\cdot\, {}_3F_2\left[\begin{matrix} a+M,\ c-b+N,\ -P; \\ c+M+N,\ 1+a-b-P; \end{matrix}\ 1\right],$$

under the constraints involved in (1.6).

Finally, we apply the Pfaff–Saalschütz theorem (2.1) to sum the terminating ${}_3F_2(1)$ series in (2.5), and we obtain

$$(2.6)\qquad \mathscr{S}=\frac{(a)_M(c-a)_{N+P}(b)_{M+P}(c-b)_N}{(c)_{M+N+P}(d)_M(-d)_N(b-a)_P}\qquad (d=a+b-c),$$

which evidently completes our proof of the summation formula (1.5).

3. *A q–Extension of the Summation Formula* (1.5)

For a real or complex number q $(|q|<1)$, let

$$(3.1)\qquad (\lambda;q)_{\mu}=\prod_{j=0}^{\infty}\left[\frac{1-\lambda q^{j}}{1-\lambda q^{\mu+j}}\right]$$

for arbitrary λ and μ, so that

$$(3.2)\qquad (\lambda;q)_{n}=\begin{cases}1, \text{ if } n=0,\\ (1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{n-1}), \ \forall\ n\in\{1,2,3,\cdots\},\end{cases}$$

and [*cf*. Equation (1.1)]

$$(3.3)\qquad \lim_{q\to 1}\left\{\frac{(q^{\lambda};q)_{n}}{(q^{\mu};q)_{n}}\right\}=\frac{(\lambda)_{n}}{(\mu)_{n}}$$

$$(\mu\neq 0,-1,-2,\cdots;\ n\in\{0,1,2,\cdots\}).$$

A q–extension of the double hypergeometric sum (2.2) was given recently by Srivastava [8]. Making use of the familiar notations for q–hypergeometric functions of one and two variables, analogous to those for

$${}_{r}F_{s}\quad\text{and}\quad F_{\ell:s;v}^{k:r;u}$$

used in the preceding sections (*cf*., *e.g.*, [9, p. 347 *et seq*.]), we recall here Srivastava's result in the form [8, p. 7, Equation (3.8)]:

$$(3.4)\quad \Phi^{0:3;3}_{1:1;1}\left[\begin{matrix} -:\ c/a,\ c/b,\ q^{-M};\ a,\ b,\ q^{-N}; \\ c:\qquad cq^{1-M}/ab;\ \ abq^{1-N}/c; \end{matrix}\ q;q,q\right]$$

$$= \frac{(a;q)_M(c/a;q)_N(b;q)_M(c/b;q)_N}{(c;q)_{M+N}(ab/c;q)_M(c/ab;q)_N}$$

$$(M,N \in \{0,1,2,\cdots\}).$$

We also recall a well-known q-extension of the Pfaff-Saalschütz theorem (2.1) in the form [7, p. 97, Equation (3.3.2.2)]:

$$(3.5)\qquad {}_3\Phi_2\left[\begin{matrix} a,\ b,\ q^{-N}; \\ c,\ abq^{1-N}/c; \end{matrix}\ q,\ q\right] = \frac{(c/a;q)_N(c/b;q)_N}{(c;q)_N(c/ab;q)_N}$$

$$(N \in \{0,1,2,\cdots\}).$$

By employing the technique (already detailed in the preceding section) *mutatis mutandis*, we can deduce from (3.4) and (3.5) the following q-extension of the summation formula (1.5):

$$(3.6)\quad \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{p=0}^{P} \frac{(a;q)_{n+p}(c/b;q)_{m+p}(c/a;q)_m(b;q)_n}{(c;q)_{m+n+p}(cq^{1-M}/ab;q)_m(abq^{1-N}/c;q)_n(aq^{1-P}/b;q)_p}$$

$$\cdot\, q^{m+n+p}\ \frac{(q^{-M};q)_m}{(q;q)_m}\ \frac{(q^{-N};q)_n}{(q;q)_n}\ \frac{(q^{-P};q)_p}{(q;q)_p}$$

$$= \frac{(c/a;q)_{N+P}(b;q)_{M+P}(a;q)_M(c/b;q)_N}{(c;q)_{M+N+P}(ab/c;q)_M(c/ab;q)_N(b/a;q)_P}$$

$$(M,N,P \in \{0,1,2,\cdots\}),$$

provided that no zeros appear in the denominator.

The q-summation formula (3.6) is believed to be new. Indeed, in view of the limit relationship (3.3), it would reduce to the summation formula (1.5) upon replacing a, b, and c by q^a, q^b, and q^c, respectively, and letting $q \to 1$.

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