

Dominating Broadcasts in Graphs

by

Sarada Rachelle Anne Herke

Bachelor of Science, University of Victoria, 2007

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Supervisory Committee

Dr. Kieka Mynhardt, Supervisor
(Department of Mathematics and Statistics)

Dr. Gary MacGillivray, Co-Supervisor or Departmental Member
(Department of Mathematics and Statistics)

Dr. Ernie Cockayne, Departmental Member
(Department of Mathematics and Statistics)

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Abstract

A *broadcast* is a function $f : V \rightarrow \{0, \dots, \text{diam } G\}$ that assigns an integer value to each vertex such that, for each $v \in V$, $f(v) \leq e(v)$, the eccentricity of v . The *broadcast number* of a graph is the minimum value of $\sum_{v \in V} f(v)$ among all broadcasts f for which each vertex of the graph is within distance $f(v)$ from some vertex v having $f(v) \geq 1$. This number is bounded above by the radius of the graph, as well as by its domination number. Graphs for which the broadcast number is equal to the radius are called *radial*. We prove a new upper bound on the broadcast number of a graph and motivate the study of radial trees by proving a relationship between the broadcast number of a graph and those of its spanning subtrees. We describe some classes of radial trees and then provide a characterization of radial trees, as well as a geometric interpretation of our characterization.

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Chapter 1

Introduction, Definitions and Notation

Suppose that a radio station wishes to broadcast at several locations so that its station may be heard by a certain region of the country. This situation can be modeled by a graph G whose vertices denote the sections of the region in which to broadcast, where an edge between two vertices indicates that these two areas are close to each other. If a broadcast tower is built at any of these locations, then the nearby neighbourhoods can hear the broadcast (vertices at distance 1). The goal for the company is to broadcast to the entire region using the fewest number of broadcasting towers. This goal is achieved by finding the minimum cardinality of a dominating set S , which is a set such that every vertex of the graph is either in S , or adjacent to a vertex in S . Finding such a set is a typical domination problem, a subject that has

been studied extensively in recent years. For an overview of domination, see [10]. Variations on domination include distance k -domination, in which vertices within distance k of a vertex in S are dominated by S . Distance domination is discussed in [12, 13, 14], for example. Broadcasting in graphs is another variation of domination where vertices in S dominate vertices within varying distances. Now we allow the radio station the option of building more powerful broadcast stations, but at an additional cost.

Any undefined terms and notations can be found in [3]. Let G be a graph. We assume throughout that G is nontrivial and connected. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of the vertices of G , respectively. The *eccentricity* of a vertex v , denoted $e(v)$, is the greatest distance between v and another vertex of G . We use $N(v)$ and $N[v]$ to denote the *open neighbourhood* and the *closed neighbourhood* of a vertex v , respectively. For $a, b \in \mathbb{Z}^+$ we use $[a, b]$ to denote the integer interval $\{a, a + 1, \dots, b\}$ if $a \leq b$, or the empty set if $a > b$.

A *broadcast* on a connected graph G is a function $f : V(G) \rightarrow [0, \text{diam}(G)]$ such that for every vertex $v \in V(G)$, $f(v) \leq e(v)$. Given a broadcast f , an *f -dominating vertex* or *broadcast vertex* is a vertex v for which $f(v) > 0$. The set of all f -dominating vertices is called the *f -dominating set* and is denoted $V_f^+(G)$, or V_f^+ when the graph under consideration is clear. An f -dominating vertex v *f -dominates* (or *broadcasts to*) every vertex u such that $d(u, v) \leq f(v)$. For a given $v \in V_f^+$, we define the *open f -neighborhood* of v as $N_f(v) = \{u \in V(G) - \{v\} : u \text{ is } f\text{-dominated by } v\}$. The *closed*

f -neighborhood of v is $N_f[v] = N_f(v) \cup \{v\}$. A vertex u is *overdominated* if $f(v) - d(u, v) > 0$ for some $v \in V_f^+$.

A broadcast f is a *dominating broadcast* if every vertex in $V(G) - V_f^+(G)$ is f -dominated by some vertex in $V_f^+(G)$. The cost of a broadcast f is defined as $\sum_{v \in V_f^+} f(v)$ and is denoted $\sigma(f)$. The *broadcast number* of a given graph G is thus defined as

$$\gamma_b(G) = \min\{\sigma(f) : f \text{ is a dominating broadcast of } G\}.$$

A broadcast f on G for which $\sigma(f) = \gamma_b(G)$ is called a *minimum dominating broadcast*, or a γ_b -broadcast.

The topic of broadcasting in graphs was first considered in a thesis by D.J. Erwin [8] in 2001, using the term cost domination. In his thesis, Erwin established some sharp upper and lower bounds on the broadcast number of a graph and characterized those graphs with broadcast number at most 3. He also discussed several other types of broadcasts, such as minimal broadcasts and independent broadcasts. Erwin's results can also be found in [9]. The following is a basic upper bound first noted by Erwin [8].

Proposition 1.1 [8] *For every nontrivial connected graph G ,*

$$\left\lceil \frac{\text{diam}(G) + 1}{3} \right\rceil \leq \gamma_b(G) \leq \min\{\text{rad}(G), \gamma(G)\}.$$

We call graphs for which $\gamma_b(G) = \text{rad}(G)$ Type 1 graphs or *radial* graphs. Graphs for which $\gamma_b(G) = \gamma(G)$ are called Type 2, and graphs for which

$\gamma_b(G) < \min\{\text{rad}(G), \gamma(G)\}$ are called Type 3. It was proved in [8] that there are infinitely many graphs of Type 3:

Proposition 1.2 [8] *For every $t \in \mathbb{Z}^+$, there exists a connected graph G for which*

$$\min\{\text{rad}(G), \gamma(G)\} - \gamma_b(G) \geq t.$$

In 2003 Dunbar, Hedetniemi and Hedetniemi [7] considered the problem of characterizing Type 1 and Type 2 trees and they achieved some partial results to this end. In 2005 Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [6] provided bounds on the minimum and maximum costs of broadcasts in graphs as well as for other types of broadcasts, and listed the characterization of Type 1 and Type 2 graphs as unsolved. In 2008 Seager [16] characterized caterpillars of Types 1, 2, and 3 respectively.

The focus of this thesis is to provide a characterization of radial trees and it is outlined as follows. In Chapter 2 we discuss relevant background material as well as the algorithmic complexity of the problem. We motivate the study of radial trees by providing a relationship between the broadcast number of a graph and those of its spanning subtrees in Chapter 3. Then in Chapter 4 we prove a new upper bound on the broadcast number of a graph, which leads to a characterization of radial caterpillars. We next provide some results about classes of radial trees with several long paths in Chapter 5. In Chapter 6 we motivate and prove a characterization of radial trees and discuss a geometrical interpretation of our characterization. We conclude Chapter

6 with an application of our characterization to general corona graphs. In Chapter 7 we list some open problems for further research.

Chapter 2

Background Results

In this chapter we begin with some basic background facts about broadcasts in Section 2.1. In Section 2.2 we discuss the work by Dunbar et al. [7] that begins to classify types of radial graphs. Then in Section 2.3 we provide a history of the study of the complexity of the broadcast problem.

2.1 Basic Facts

We begin with an important definition. An *efficient broadcast* f is a broadcast such that each vertex is f -dominated by exactly one vertex of V_f^+ .

Proposition 2.1 [6] *Every graph G has a γ_b -broadcast that is efficient.*

The above proposition is interesting because the same result is not true for domination; not every graph has an efficient dominating set. For example, the tree obtained by joining a new leaf to a central vertex of P_4 has no

efficient dominating set. The next result is frequently used to show that a given broadcast is not efficient.

Proposition 2.2 *Suppose f is a broadcast on a connected graph G . If, for some $v \in V_f^+(G)$, $G - N_f[v]$ contains an isolated vertex, then f is not efficient.*

Proof. Let f be a broadcast on G and let $v \in V_f^+$ such that $G - N_f[v]$ has an isolated vertex, w . In order to f -dominate w , either $f(w) \geq 1$ or there is a vertex q at distance ℓ from w such that $f(q) \geq \ell$. Let w' be any neighbour of w in T . Then in either case, w' is f -dominated by more than one vertex of V_f^+ ; therefore f is not an efficient broadcast.

□

The next two results concern the broadcast number of a tree and (certain types of) its subtrees, and mirror the corresponding results for the domination number of trees.

Proposition 2.3 [7] *If T is a tree with subtree T' , then $\gamma_b(T') \leq \gamma_b(T)$.*

The following two definitions are required for the next result. A *leaf* is a vertex of a tree with degree 1 and a *support vertex* is a non-leaf vertex that is adjacent to a leaf.

Proposition 2.4 *Let T be any tree. If a tree T' is obtained from T by joining a new leaf to a support vertex of T , then $\gamma_b(T') = \gamma_b(T)$.*

Proof. By Proposition 2.3, $\gamma_b(T) \leq \gamma_b(T')$. Amongst all γ_b -broadcasts on T , let f be one such that V_f^+ has the minimum number of leaves. Suppose that v is a leaf of T and that $v \in V_f^+$. Let u be the support vertex of v , and define the following broadcast on T :

$$g(x) = \begin{cases} 0 & \text{if } x = v \\ f(u) + f(v) & \text{if } x = u \\ f(x) & \text{otherwise.} \end{cases}$$

Then g is a γ_b -broadcast with fewer leaves as broadcast vertices than f , which is a contradiction. Thus V_f^+ contains no leaf of T , so V_f^+ contains all support vertices. Therefore f broadcasts to all vertices of T' as well, and $\gamma_b(T') \leq \gamma_b(T)$.

□

It is not surprising that the broadcast number of a path is equal to its domination number.

Proposition 2.5 [9] *For every integer $n \geq 2$,*

$$\gamma_b(P_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil. \quad (2.1)$$

It is easy to see that all trees with radius at most 2 are radial. From the results stated thus far, we obtain the following lemma.

Lemma 2.6 *If T is a central tree with radius 3, then T is radial.*

Proof. Since T is central with radius 3, P_7 is a subtree of T . It is clear that $\gamma_b(P_7) = 3$, so by Lemma 2.3, $3 \leq \gamma_b(T)$. However, by Proposition 1.1, $\gamma_b(T) \leq \min\{\text{rad}(T), \gamma(T)\}$. We are given that $\text{rad}(T) = 3$, and it is clear that $\gamma(T) \geq 3$ since at least three vertices are needed to dominate the subtree P_7 . Thus, $\min\{\text{rad}(T), \gamma(T)\} = 3$, and so $\gamma_b(T) \leq 3$. Therefore $\gamma_b(T) = 3 = \text{rad}(T)$ and T is radial.

□

We note that Lemma 2.6 does not hold for trees of radius 4. For example, P_9 has radius 4 but $\gamma_b(P_9) = \gamma(P_9) = 3$ (see Figure 2.1). Also, the path P_6 shows that bicentral trees of radius 3 are not necessarily radial (see Figure 2.2).

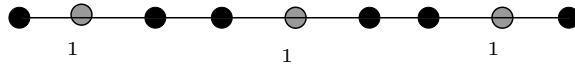


Figure 2.1: A γ_b -broadcast for P_9

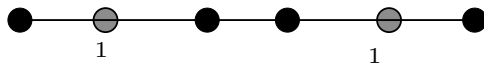


Figure 2.2: A γ_b -broadcast for P_6

2.2 Background on Radial Graphs

It is mentioned in [7] that of all nontrivial trees of order at most 9, only 11 are nonradial. This fact leads us to believe that there are more radial trees than nonradial ones of fixed order. In this section we state some previously discovered results about radial graphs.

The *corona* of two graphs G_1 and G_2 , denoted $G_1 \circ G_2$, is the graph obtained from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i th vertex of G_1 is adjacent to every vertex in the i th copy of G_2 . The *generalized corona* of a connected graph G with $V(G) = \{v_1, \dots, v_n\}$ and n arbitrary graphs G_1, \dots, G_n is the graph $H = G \circ (G_1, \dots, G_n)$; that is, v_i is adjacent to each vertex of G_i .

Proposition 2.7 [7] *For any connected graph G_1 and any graph G_2 , the graph $G = G_1 \circ G_2$ is radial.*

The proof of Proposition 2.7 also gives the following result.

Corollary 2.8 [7] *For any connected graph G of order n and any n graphs G_1, \dots, G_n , the generalized corona $G \circ (G_1, \dots, G_n)$ is radial.*

Proposition 2.9 [7] *If T_1 and T_2 are two radial trees, then the tree formed by adding an edge between a central vertex of T_1 and a central vertex of T_2 , is radial.*

Proposition 2.10 [7] *If T is a tree containing three vertices u, v, w satisfying $d(u, v) = \text{diam}(T)$ and $d(u, w), d(v, w) \geq \text{diam}(T) - 1$, then T is radial.*

However, Proposition 2.10 does not account for all radial trees. For example, Figure 2.3 shows three radial trees with radius 5 that do not satisfy the conditions of Proposition 2.10.

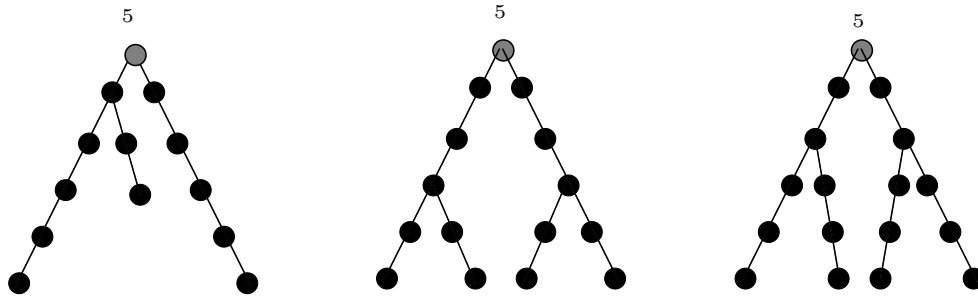


Figure 2.3: Radial trees not satisfying Proposition 2.10

For a graph G and a positive integer k , we define the k -subdivided graph of G , denoted $S_k(G)$, as the graph obtained from G by inserting k vertices into every edge of G . For positive integers k and t , we let $S_{k,t} = S_k(K_{1,t})$, where $K_{1,t}$ is the *star* consisting of a central vertex adjacent to t leaves. For example, see Figure 2.4.

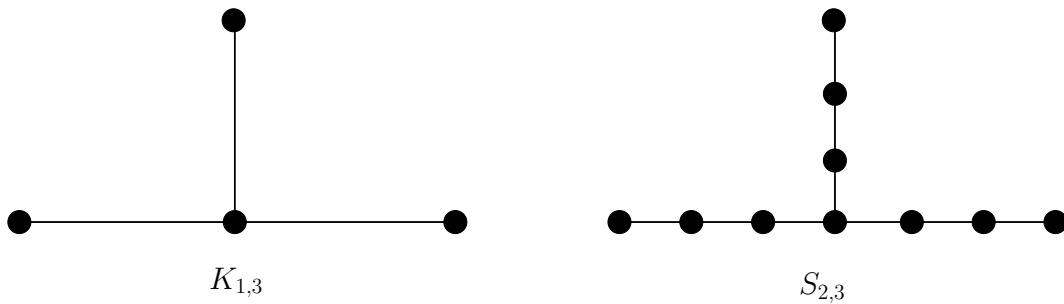


Figure 2.4: Star $K_{1,3}$ and the 2-subdivided graph $S_{2,3}$ of $K_{1,3}$

In 2001, Erwin [8] proved that $S_{k,t}$ is radial for $k \geq 0$ and $t \geq 5$, and he conjectured this property for $k \geq 0$ and $t \in \{3, 4\}$. In 2009, Bouchemakh and Sahbi [2] proved the following proposition, thus proving Erwin's conjecture.

Proposition 2.11 *For every integer $k \geq 0$ and $t \geq 3$,*

$$\gamma_b(S_{k,t}) = \text{rad}(S_{k,t}) = k + 1.$$

However, we note that Proposition 2.11 follows immediately from Proposition 2.10 by taking u, v and w to be any three of the t leaves of $S_{k,t}$.

2.3 Algorithms and Complexity

There are many varieties of domination and many of these problems are NP-hard. Thus, when the topic of broadcast domination on graphs was introduced, it was generally believed that the computational complexity of finding $\gamma_b(G)$ for a general graph G would also be in the class NP. However, this is not the case. The complexity of computing γ_b was studied by Horton, Meneses, Mukhegjee and Ulucakli [15] and by Blair, Heggernes, Horton and Maine [1]. These two groups of authors found some polynomial time algorithms for specific types of graphs. Then, in 2006, Heggernes and Lokshantov [11] showed that minimum broadcast domination is solvable in polynomial time for any graph. Their algorithm runs in $O(n^6)$ time for a graph with n vertices. The algorithm depends largely on two properties of minimum dominating broadcasts. The first of these is Proposition 2.1 by Dunbar et

al. [6] that every graph has an efficient minimum dominating broadcast. In order to state the second of these results, a definition is required.

In [11], the closed f -neighbourhood $N_f[v]$ is also called the *ball* with centre v , where $v \in V_f^+$. For an efficient dominating broadcast f on a graph G , the *ball graph* $B(f)$ of G is the graph obtained by contracting the vertices in every ball $N_f[v]$ for $f(v) > 0$ down to a single vertex.

Lemma 2.12 [11] *For any graph G , there is an efficient minimum dominating broadcast f on G such that the ball graph $B(f)$ has maximum degree 2.*

Corollary 2.13 [4] *Every tree T has an efficient minimum dominating broadcast f on T such that the ball graph $B(f)$ is a path.*

Note that Corollary 2.13 does not imply that the broadcast vertices all lie on the same path. However, we will prove this stronger result in Chapter 6 as a crucial result for our eventual characterization of radial trees.

For example, Figure 2.5 shows an efficient broadcast f on a tree T , and the corresponding ball graph $B(f)$.

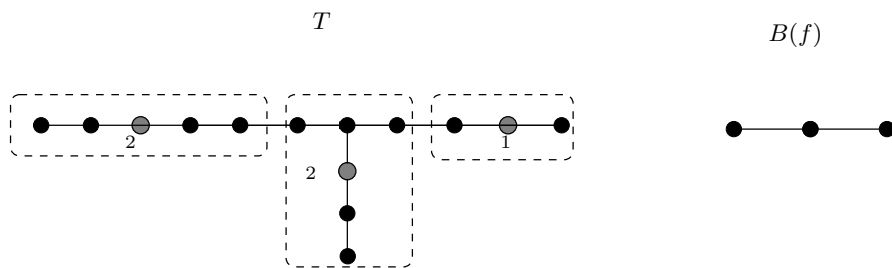


Figure 2.5: The ball graph of a broadcast f on a tree T

In 2007, J.R. Dabney [4] showed that for trees, γ_b can be found by an algorithm that runs in $O(n)$ time. To do this, Dabney required non-standard methods to make decisions based on non-local information. He also made use of the structure described by Corollary 2.13. This result is presented by Dabney, Dean and Hedetniemi in [5]. However, even with this algorithm, he was unable to determine a characterization of radial graphs.

Chapter 3

Broadcast Number of Graphs vs. Trees

In this chapter we motivate the study of broadcasts in trees by exploring the relationship between the broadcast number of a graph and those of its spanning subtrees. Let G be a connected graph and T a spanning tree of G with $\text{rad}(T) = \text{rad}(G)$. It is clear that if G is radial then T is radial. However, if T is radial then G is not necessarily radial. For example, T in Figure 3.1 has $\text{rad}(T) = 4$ and is a spanning subtree of G with $\text{rad}(T) = \text{rad}(G)$. But G is not radial, as illustrated by the dominating broadcast given in Figure 3.1 with cost 3.

We note that G does have a spanning subtree with the same broadcast number. Consider the spanning subtree T' of the graph G from Figure 3.1 obtained by deleting edge e of G . This spanning tree has the same radius as

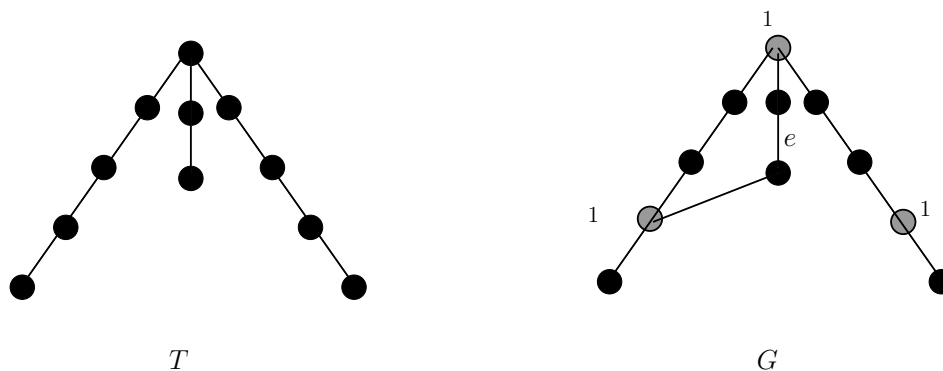


Figure 3.1: A radial spanning subtree T with $\text{rad}(T) = \text{rad}(G) = 4$

G but, like G , is not radial because the same broadcast used on G with cost 3 dominates T' .

If G is any connected graph and T a spanning tree of G with $\text{diam}(T) = \text{diam}(G)$, and if T is radial, one might wonder if it follows that G is radial. This is not the case, as illustrated by the same graphs in Figure 3.1. Here $\text{diam}(G) = \text{diam}(T) = 8$ and while T is radial, G is not.

The following theorem describes the relationship between the broadcast number of a graph and the broadcast numbers of its spanning subtrees. It is because of this relationship that the study of radial trees is vital to the characterization of radial graphs. For a connected graph G , we use $\mathcal{S}(G)$ to denote the set of spanning subtrees of G .

Theorem 3.1 *Suppose G is a connected graph. Then*

$$\gamma_b(G) = \min_{T \in \mathcal{S}(G)} \{\gamma_b(T)\}.$$

Proof. Let $\gamma_b(G) = k$ and $\min_{T \in \mathcal{S}(G)} \{\gamma_b(T)\} = t$. A minimum broadcast on a spanning subtree T also dominates the graph G , since G is obtained from T by adding more edges. Thus it is clear that $k \leq t$. We wish to show that $k \geq t$. For the purpose of deriving a contradiction, suppose $k < t$. By Proposition 2.1, G has an efficient γ_b -broadcast; call it f . Now consider the vertices of G partitioned into $V_f^+(G)$ and $V(G) - V_f^+(G)$. Since f is efficient, there are no edges between vertices of V_f^+ . For a given vertex $u \in V_f^+$, with $f(u) = r$, we define

$$L_i(u) = \{v \in N_f(u) : d(u, v) = i\},$$

for $i = 1, \dots, r$. We obtain a spanning subgraph H of G in the following way:

- For every $u \in V_f^+(G)$ with $f(u) = r$:
 - If there is an edge $vv' \in E(G)$ where $v, v' \in L_s(u)$ for some $s \leq r$, then delete this edge.
 - If for some $v \in L_s(u)$ there are ℓ edges $vv_1, vv_2, \dots, vv_\ell \in E(G)$ where $v_i \in L_{s'}(u)$ for some $s' \leq s$, then delete any $\ell - 1$ of these edges, so that only one edge remains.
- For every $u, u' \in V_f^+(G)$ with $f(u) = r, f(u') = r'$:

- If there are ℓ edges of the form $vv' \in E(G)$ where $v \in L_r(u), v' \in L_{r'}(u')$, then delete any $\ell - 1$ of these edges, so that only one edge remains.

Note that there are no edges between vertices $v \in L_s(u)$ and $v' \in L_{s'}(u')$ where $s < r, s' < r'$ since the existence of such an edge would contradict the fact that f is efficient.

It is clear that the graph H is a tree. Hence $H \in \mathcal{S}(G)$, so $\gamma_b(H) \geq t > k$. But by the way in which H was constructed, f dominates H , so $\gamma_b(H) \leq k$. This is a contradiction. So we have shown that $k \geq t$. Therefore $k = t$.

□

Example 3.2 *To illustrate the technique used in the proof of Theorem 3.1, consider the example of a graph G in Figure 3.2 and one possible resulting spanning subtree H in Figure 3.3.*

Let $\mathcal{R}(G)$ be the set of all spanning trees of a connected graph G such that $\text{rad}(T) = \text{rad}(G)$. It is of interest to notice that there is always such a spanning subtree.

Proposition 3.3 *Every connected graph G has a spanning tree T with $\text{rad}(T) = \text{rad}(G)$.*

Proof. Let x be any central vertex of G . For each $v \in V$, let P_v be a shortest $x - v$ path and define $G' = \langle \bigcup_{v \in V} E(P_v) \rangle$. Then G' is a connected spanning subgraph of G with $\text{rad}(G') = \text{rad}(G)$. If G' is acyclic, we are done,

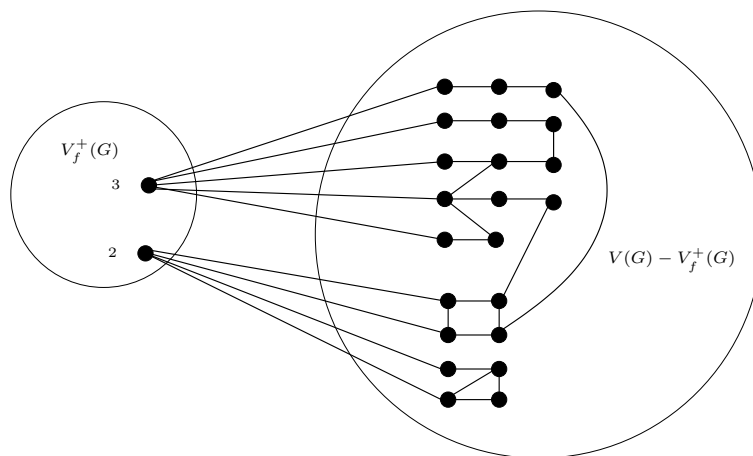


Figure 3.2: Graph G of Example 3.2

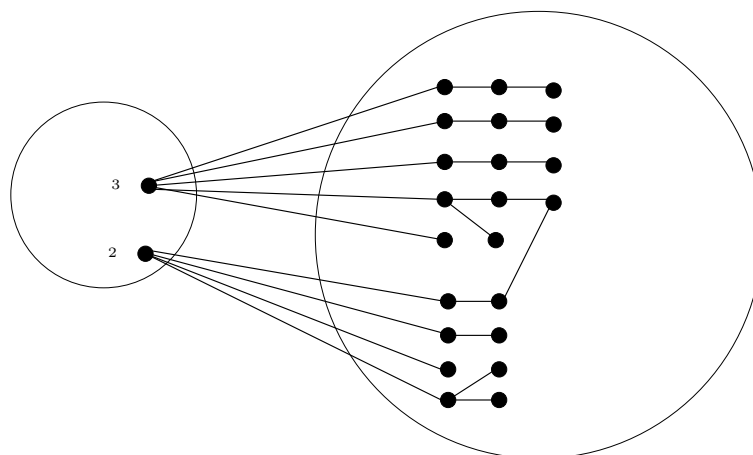


Figure 3.3: A spanning subgraph H obtained from G in Example 3.2

so assume C is a cycle of G' . Then C is a subgraph of $P_u \cup P_v$ for some $u, v \in V$. Let w_1, w_2 be the two vertices of C common to both P_u and P_v ; say $d(x, w_1) < d(x, w_2)$. Since P_u and P_v are shortest $x - u$ and $x - v$ paths, the two $w_1 - w_2$ paths on C have the same length. Let e be any of the two

edges of C incident with w_2 and define $G'' = G' - e$. Then G'' is connected and $e_{G''}(x) = e_{G'}(x)$, so that $\text{rad}(G'') = \text{rad}(G')$. Repeating this process until no cycles remain yields the desired tree T .

□

We might then wonder if Theorem 3.1 can be strengthened by answering the following question in the affirmative.

Question 3.4 *Is it true that for any connected graph G*

$$\gamma_b(G) = \min_{T \in \mathcal{R}(G)} \{\gamma_b(T)\}?$$

The answer to Question 3.4 is no. Consider the graph G given in Figure 3.4 with $\gamma_b(G) = 9$ and $\text{rad}(G) = 10$. By using the method of the proof of Theorem 3.1, we see that the only possible tree $T \in \mathcal{S}(G)$ for which the same (or any other) minimum broadcast of G will work is $T = G - e_1 - e_2$. However, $\text{rad}(T) = 11 \neq \text{rad}(G)$.

The situation for radial graphs is different, though.

Lemma 3.5 *If G is radial, then there exists $T \in \mathcal{R}(G)$ with $\gamma_b(T) = \gamma_b(G)$.*

Proof. Let $T \in \mathcal{R}(G)$. Then $\gamma_b(T) \leq \text{rad}(G) = \gamma_b(G)$. But $\gamma_b(G) = \min_{T \in \mathcal{S}(G)} \{\gamma_b(T)\}$, so $\gamma_b(T) \geq \text{rad}(G)$. Therefore $\gamma_b(T) = \gamma_b(G)$.

□

Another natural question concerns the monotonicity of the broadcast number of trees with the same order and different radii.

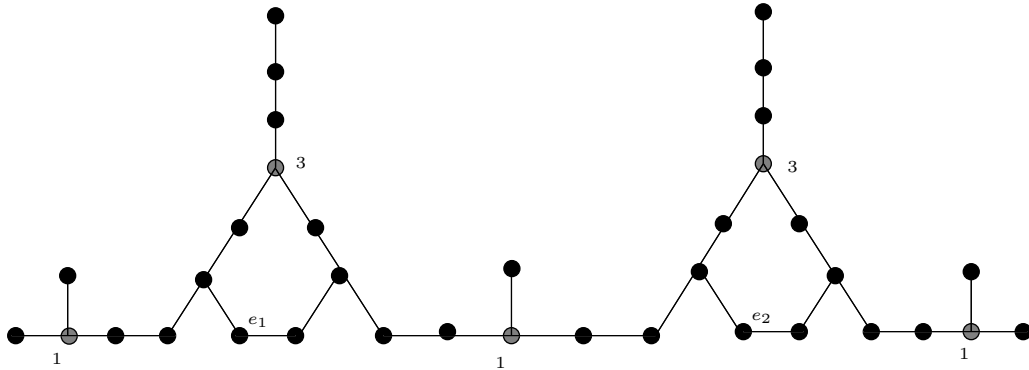


Figure 3.4: Counterexample to Question 3.4

Question 3.6 *Is it true that if T_1 and T_2 are trees of order n such that $\text{rad}(T_1) \leq \text{rad}(T_2)$, then $\gamma_b(T_1) \leq \gamma_b(T_2)$?*

The answer to Question 3.6 is also no. Consider the example given in Figure 3.5 in which two trees T_1 and T_2 are each of order 15, and $\text{rad}(T_1) = 5 < 6 = \text{rad}(T_2)$, but $\gamma_b(T_1) = 5 > 4 = \gamma_b(T_2)$.

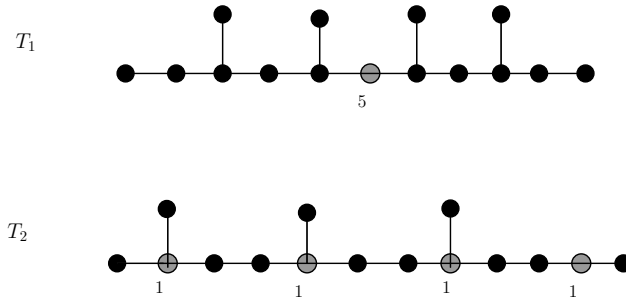


Figure 3.5: Counterexample to Question 3.6

Chapter 4

A New Upper Bound on the Broadcast Number

In Section 4.1 we show that the broadcast number of a tree of order n is bounded above by $\lceil \frac{n}{3} \rceil$, the exact value of $\gamma_b(P_n)$ (see Proposition 2.5). The same bound for general graphs then follows from Theorem 3.1. In Section 4.2 we consider other classes of trees for which the bound is exact. Our results here lead us to a characterization of radial caterpillars, which we prove in Section 4.3.

4.1 The Upper Bound

Theorem 4.1 *For any tree T of order n , $\gamma_b(T) \leq \lceil \frac{n}{3} \rceil$.*

Proof. The result is obviously true for $n \leq 3$. Suppose the result is not true in general and let T be a counterexample of minimum order $n \geq 4$. We first show that T has no adjacent vertices of degree two.

Suppose u_1 and u_2 are adjacent vertices of degree two. Let v_i be the other neighbour of u_i , $i = 1, 2$, so that v_1, u_1, u_2, v_2 is a path in T . Let T_i be the component of $T - u_1u_2$ containing u_i . If $|V(T_1)| \equiv 0 \pmod{3}$, let $T' = T_1$ and $T'' = T_2$. If $|V(T_1)| \equiv 1 \pmod{3}$, let $T' = T_1 - u_1$ and $T'' = T - T'$. If $|V(T_1)| \equiv 2 \pmod{3}$, let $T'' = T_2 - u_2$ and $T' = T - T''$. In each case T' and T'' are trees where $|V(T')| \equiv 0 \pmod{3}$; say $|V(T')| = 3t$. By the minimality of T , $\gamma_b(T') \leq t$ and $\gamma_b(T'') \leq \lceil \frac{n-3t}{3} \rceil$, so that

$$\gamma_b(T) \leq \left\lceil \frac{n-3t}{3} \right\rceil + t \leq \left\lceil \frac{n}{3} \right\rceil,$$

a contradiction.

Now assume T has radius k and let P be a diametrical path. Then $\gamma_b(T) \leq k$ and, since P is a subtree of T with the same radius, $\gamma_b(P) \leq k$.

If T is central, let $P = v_1, \dots, v_{2k+1}$. Since T does not have adjacent vertices of degree two, an application of the pigeonhole principle shows that at least $\lfloor \frac{2k-1}{2} \rfloor = k-1$ of the vertices v_i , $i = 2, \dots, 2k$, are adjacent to vertices not on P . Hence $n \geq 2k+1 + k-1 = 3k$. Therefore $\gamma_b(T) \leq k \leq \lceil \frac{n}{3} \rceil$, a

contradiction.

If T is bicentral, let $P = v_1, \dots, v_{2k}$. As above, at least $\frac{2k-2}{2} = k-1$ of the vertices v_i , $i = 2, \dots, 2k-1$, are adjacent to vertices not on P . So $n \geq 2k + k - 1 = 3k - 1$, i.e. $\gamma_b(T) \leq k \leq \lceil \frac{n}{3} \rceil$. This final contradiction proves the theorem.

□

By Theorem 3.1, we obtain the following corollary.

Corollary 4.2 *For any connected graph G of order n , $\gamma_b(G) \leq \lceil \frac{n}{3} \rceil$.*

Corollary 4.3 *If T is a radial tree of radius k , then T has at least $3k - 2$ vertices.*

Proof. If T has at most $3k - 3$ vertices, then by Theorem 4.1, $\gamma_b(T) \leq k - 1$ and so T is not radial.

□

4.2 Equality in the Upper Bound

We consider a class of radial trees that satisfy equality in the bound given in Theorem 4.1. We define a *caterpillar* to be a tree T consisting of a path with singleton vertices adjacent to any subset of the non-leaf vertices of the path. The path associated with a given caterpillar T is called its *spine*. Note that our definition differs from the conventional definition of a caterpillar in that

we require the spine to have the same diameter as the caterpillar. We define two specific classes of caterpillars. Seager [16] characterized caterpillars of Types 1, 2, and 3 respectively. Here we use a different approach which eventually leads to a characterization of radial trees.

Consider P_{2k+1} with a labeling of the vertices v_1, \dots, v_{2k+1} . Add a single leaf vertex to each of $v_3, v_5, v_7, \dots, v_{2k-1}$ of this path. Thus $\lceil \frac{2k+1-4}{2} \rceil$ vertices are added to P_{2k+1} , resulting in a tree with $2k+1 + \lceil \frac{2k-3}{2} \rceil = 2k+1+k-1 = 3k$ vertices. We call the resulting caterpillar F_{3k} , with spine P_{2k+1} . We define F_{3k-1} similarly, with spine P_{2k} and singletons adjacent to $v_3, v_5, v_7, \dots, v_{2k-1}$, so that the total number of vertices is $2k + \lceil \frac{2k-2}{2} \rceil = 3k - 1$. For example, F_8 and F_9 are given in Figure 4.1.

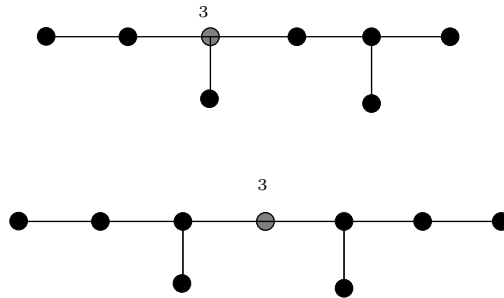


Figure 4.1: The caterpillars F_8 and F_9

Theorem 4.4 For any $k \in \mathbb{Z}^+$, $\gamma_b(F_{3k}) = \gamma_b(F_{3k-1}) = k$.

Proof. Consider $T = F_{3k}$ and let f be an efficient γ_b -broadcast of T . Let v_1, \dots, v_{2k+1} be a labeling of the spine of T and let u_i be the leaf adjacent to v_i ,

$i = 3, 5, \dots, 2k - 1$. Note that $\text{Cen}(T) = \{v_{k+1}\}$. We prove that $V_f^+ = \{v_{k+1}\}$; the result will follow immediately.

We first prove that

$$\begin{aligned} \text{if } v_i \in V_f^+ \text{ and } f(v_i) = m, \text{ where } i - m > 1 \text{ or } i + m < 2k + 1, \\ \text{then } i \equiv m \pmod{2}. \end{aligned} \quad (4.1)$$

Suppose $i \not\equiv m \pmod{2}$. Then $i + m$ and $i - m$ are odd. Assume $i + m < 2k + 1$; the proof in the case $1 < i - m$ is the same. Then v_i does not broadcast to v_{2k+1} . Since $i + m$ is odd, v_{i+m} is adjacent to u_{i+m} . Moreover, v_i broadcasts to v_{i+m} but not to u_{i+m} . But now u_{i+m} is an isolated vertex of $T - N_f[v_i]$, so that Proposition 2.2 provides a contradiction of the efficiency of f . Thus (4.1) holds.

We prove next that

$$\text{if } u_i \in V_f^+, \text{ then } f(u_i) = 1. \quad (4.2)$$

If $f(u_i) \geq 2$, define the broadcast g on T by

$$g(x) = \begin{cases} 0 & \text{if } x = u_i \\ f(u_i) - 1 & \text{if } x = v_i \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, g is a dominating broadcast of T , which is impossible because $\sigma(g) < \sigma(f) = \gamma_b(T)$. Therefore (4.2) holds.

Let v be the vertex that broadcasts to v_1 . Then (4.2) implies that $v = v_i$ for some i ; say $f(v_i) = m$. If $i = k + 1$, then $m = k$ and we are done, so assume $i \leq k$. Now $m \leq \sigma(f) \leq \text{rad}T = k$, so $i + m < 2k + 1$; hence $v_{i+m+1} \in V(T)$ and v_i does not broadcast to v_{i+m+1} . Moreover, Proposition 2.2 and the efficiency of f imply that $i + m + 1 \neq 2k + 1$. By (4.1), $i + m$ is even, so $i + m + 1$ is odd and hence v_{i+m+1} is adjacent to u_{i+m+1} . Let u be the vertex that broadcasts to v_{i+m+1} ; say $f(u) = l$. If $u = v_j$ for some j , then, by the efficiency of f , v_j does not broadcast to v_{i+m} and $j - l = i + m + 1$. But by (4.1), $j - l$ is even, a contradiction. Hence $u = u_{i+m+1}$ and by (4.2) $l = 1$.

Also note that since $i + m$ is even, $d(v_1, v_{i+m}) = i + m - 1$ is odd, so v_i overdominates v_1 .

Let w be the vertex that broadcasts to v_{i+m+2} . Again (4.2) implies that $w = v_r$ for some r ; say $f(v_r) = p$. If v_r also broadcasts to v_{2k+1} , then as in the case of v_1 , v_{2k+1} is overdominated. If v_r does not broadcast to v_{2k+1} , then $r + p$ is even and less than $2k$ (by Proposition 2.2 and the efficiency of f), and as before, v_{r+p+1} is f -dominated by u_{r+p+1} where $f(u_{r+p+1}) = 1$.

By repeating the above arguments we eventually obtain a sequence

$$v_i = v_{i_1}, u_{j_1}, v_{i_2}, u_{j_2}, \dots, u_{j_{s-1}}, v_{i_s} \in V_f^+, \quad s \geq 2,$$

where

- $j_\ell = i_\ell + f(v_{i_\ell}) + 1$ and $v_{i_{\ell+1}}$ broadcasts to $v_{j_\ell+1}$,

- for $\ell \notin \{1, s\}$, v_{i_ℓ} broadcasts to $2f(v_{i_\ell}) + 1$ vertices (including itself) on the spine of T , and
- v_{i_1} overdominates v_1 ; v_{i_s} overdominates v_{2k+1} .

Now

$$\begin{aligned}\sigma(f) &= \sum_{\ell=1}^s f(v_{i_\ell}) + \sum_{\ell=1}^{s-1} f(u_{j_\ell}) \\ &= \sum_{\ell=1}^s f(v_{i_\ell}) + s - 1.\end{aligned}$$

Also

$$\begin{aligned}2k + 1 &\leq 2(f(v_{i_1}) + f(v_{i_s})) + \sum_{\ell=2}^{s-1} (2f(v_{i_\ell}) + 1) + s - 1 \\ &= 2 \sum_{\ell=1}^s f(v_{i_\ell}) + 2s - 3.\end{aligned}$$

Therefore

$$\begin{aligned}\text{rad}(T) = k &\leq \left\lfloor \frac{2 \sum_{\ell=1}^s f(v_{i_\ell}) + 2(s-1) - 1}{2} \right\rfloor \\ &= \sum_{\ell=1}^s f(v_{i_\ell}) + s - 2 \\ &< \sigma(f),\end{aligned}$$

which is a contradiction. Thus $f(v_i) = 0$ for $i \leq k$. The same arguments show that $f(v_i) = 0$ for $i \geq k + 2$. Suppose $f(v_{k+1}) < k$. Then, even if $\{u_3, u_5, \dots, u_{2k-1}\} \subset V_f^+$, f is not a dominating broadcast by (4.2), which is a contradiction. So $V_f^+ = \{v_{k+1}\}$ where $f(v_{k+1}) = k$ and hence T is radial.

A similar argument holds for $T = F_{3k-1}$.

□

Proposition 2.3 now provides the following corollary.

Corollary 4.5 *If T is a tree of radius k that contains F_{3k} or F_{3k-1} as subgraph, then T is radial.*

Corollary 4.6 *If a new vertex is joined to every non-leaf vertex of P_n then the resulting tree T is radial.*

Proof. The result follows from Corollary 4.5.

□

4.3 A Characterization of Radial Caterpillars

The classes of radial caterpillars described by F_{3k} and F_{3k-1} are constructed such that no two vertices of degree two are adjacent. This property is significant for characterizing radial caterpillars. By considering adjacent vertices of degree two on the spine of a caterpillar, we are now able to determine exactly when a given caterpillar is radial.

Theorem 4.7 *Let T be any caterpillar with spine P of order n .*

1. *If n is even and there exists a vertex $v_i \in V(P)$ where $i \geq 3$ is odd such that $\deg(v_i) = \deg(v_{i+1}) = 2$; or*
2. *if n is odd and there exist vertices $v_i, v_j \in V(P)$ where $i \geq 3$ is odd and $j \geq i + 3$ is even such that*

$$\deg(v_i) = \deg(v_{i+1}) = \deg(v_j) = \deg(v_{j+1}) = 2,$$

then T is nonradial. Otherwise T is radial.

Proof. Suppose (1) holds. Let v_1, \dots, v_{2k} be a labeling of the vertices of the spine P . Suppose that $i = 2\ell + 1$, so that $\deg(v_{2\ell+1}) = \deg(v_{2\ell+2}) = 2$. Let $e = v_{2\ell+1}v_{2\ell+2}$ and let T_1 and T_2 be the components of $T - e$ containing $v_{2\ell+1}$ and $v_{2\ell+2}$, respectively. Then T_1 has a spine of order $2\ell + 1$ and central vertex $v_{\ell+1}$; T_2 has a spine of order $2k - 2\ell - 1$ and central vertex $v_{\ell+k+1}$. Define the broadcast f on T by $f(v_{\ell+1}) = \ell$, $f(v_{\ell+k+1}) = k - \ell - 1$, and $f(v) = 0$ otherwise. Then f is a dominating broadcast on T and $\sigma(f) = k - 1$. Therefore T is nonradial.

Suppose (2) holds. Let v_1, \dots, v_{2k+1} be a labeling of the vertices of the spine P . Suppose that $i = 2\ell + 1$ and $j = 2m$, so that $\deg(v_{2\ell+1}) = \deg(v_{2\ell+2}) = \deg(v_{2m}) = \deg(v_{2m+1}) = 2$. Let $e_1 = v_{2\ell+1}v_{2\ell+2}$, $e_2 = v_{2m}v_{2m+1}$ and let T_1, T_2 and T_3 be the components of $T - e_1 - e_2$ containing $v_{2\ell+1}$, $v_{2\ell+2}$, and v_{2m+2} respectively. Then T_1 has a spine of order $2\ell + 1$ and central vertex $v_{\ell+1}$; T_2 has a spine of order $2m - 2\ell - 1$ and central vertex $v_{\ell+m+1}$; T_3 has a spine of order $2k - 2m + 1$ and central vertex v_{m+k+1} . Define the broadcast

f on T by $f(v_{\ell+1}) = \ell$, $f(v_{\ell+m+1}) = m - \ell - 1$, $f(v_{m+k+1}) = k - m$ and $f(v) = 0$ otherwise. Then f is a dominating broadcast on T and $\sigma(f) = k - 1$. Therefore T is nonradial.

Now suppose that caterpillar T with spine P of order n is nonradial. Let f be an efficient γ_b -broadcast with $|V_f^+|$ minimized. Since T is nonradial, $|V_f^+| \geq 2$.

We wish to show that every $v \in V_f^+$ is on the spine P . Suppose not. Then for some $i \geq 2$, $v_i \in V(P)$ is adjacent to $w_i \in V(T) - V(P)$ and $w_i \in V_f^+$. If $f(w_i) > 1$, then the broadcast defined by $g(v_i) = f(w_i) - 1$, $g(w_i) = 0$, and $g(z) = f(z)$ otherwise is a dominating broadcast with lower cost than f , which is a contradiction. Hence $f(w_i) = 1$. Also, since f is efficient, there are vertices $u, v \in V_f$ such that $f(u) = \ell = d(u, v_{i-1})$ and $f(v) = m = d(v, v_{i+2})$. Then u, v, w_i broadcast to a subtree of T with spine P_1 of order at most $2(\ell + m) + 3$. Assume firstly that P_1 has order equal to $2(\ell + m) + 3$. Let x be the central vertex of P_1 and define a broadcast g by

$$g(z) = \begin{cases} \ell + m + 1 & \text{if } z = x \\ 0 & \text{if } z \in \{w_i, u, v\} \\ f(z) & \text{otherwise.} \end{cases}$$

Then g is an efficient γ_b -broadcast with fewer broadcast vertices than f , which is a contradiction. Now assume P_1 has order less than $2(\ell + m) + 3$. Then one (or both) leaves of P are overdominated by u or v . If one leaf of P is overdominated then we can choose a vertex x on P such that the

broadcast g as defined above is efficient, which is again a contradiction. If both leaves are overdominated, then the broadcast g' defined by $g'(x) = \ell + m$ and $g'(z) = 0$ otherwise, where x is the central vertex of P_1 , is a dominating broadcast with cost less than f , which is a contradiction.

So we may assume that f is an efficient γ_b -broadcast with $|V_f^+|$ minimized such that every $v \in V_f^+$ is a vertex of the spine P .

Suppose that a leaf of P , say v_1 , is overdominated by f . Let v_i be the vertex that f -dominates v_1 , and suppose $f(v_i) = \ell$. Then v_i broadcasts to a subtree T_1 of T with spine P_1 of order at most 2ℓ . Consider the nearest broadcasting vertex v_j to v_i and suppose $f(v_j) = m$. Then v_i and v_j together broadcast to a subtree T_2 of T with spine P_2 of order at most $2\ell + 2m + 1$. Let $y \in P_2$ be the vertex at maximum distance from v_1 and let $x \in P_2$ be the unique vertex with $d(x, y) = m + \ell$. Define the broadcast g by $g(x) = m + \ell$, $g(v_i) = g(v_j) = 0$ and $g(v) = f(v)$ otherwise. Then g is an efficient broadcast with $\sigma(g) = \sigma(f)$ but g has fewer broadcasting vertices, which is a contradiction.

So we may assume that f is an efficient γ_b -broadcast with $|V_f^+|$ minimized such that every $v \in V_f^+$ is a vertex of the spine P and no leaf of P is overdominated.

Suppose n is even. For every $v \in V_f^+$, $N_f[v]$ is a subtree of T with a spine of odd order. Since n is even, there is an even number of such balls, i.e. an even number of broadcasting vertices. Let $u, v \in V_f^+$ be such that $v_1 \in N_f[u]$ and $N_f[u]$ and $N_f[v]$ are adjacent in the ball graph of T . Since

v_1 is not overdominated, $v_{2f(u)+1}$ is the vertex at distance $f(u)$ from u , and thus $\deg(v_{2f(u)+1}) = \deg(v_{2f(u)+2}) = 2$. So (1) holds with $i = 2f(u) + 1$.

Suppose n is odd. For every $v \in V_f^+$, $N_f[v]$ is a subtree of T with a spine of odd order. Since n is odd, there is an odd number of such balls, i.e. an odd number of broadcasting vertices. Since T is nonradial, there are at least 3 broadcasting vertices. Let $u, v, w \in V_f^+$ be such that $v_1 \in N_f[u]$ and in the ball graph of T , $N_f[v]$ is adjacent to $N_f[u]$ and $N_f[w]$. Since v_1 is not overdominated, $v_{2f(u)+1}$ is the vertex at distance $f(u)$ from u . Let $i = 2f(u) + 1$ and $j = 2(f(u) + f(v)) + 2$. Thus $\deg(v_i) = \deg(v_{i+1}) = \deg(v_j) = \deg(v_{j+1}) = 2$, and so (2) holds. Therefore, T is nonradial if and only if (1) or (2) is satisfied.

□

Corollary 4.8 *If T is a caterpillar with no adjacent vertices of degree 2, then T is radial.*

In the proof of Theorem 4.7 we also proved that if f is an efficient γ_b -broadcast with the minimum number of broadcast vertices, then every broadcast vertex lies on the spine (thus on a diametrical path) and no leaf of the spine is overdominated. In Section 6.2 we generalize this crucial result to arbitrary trees. This generalization provides the key to characterizing radial trees (see Theorem 6.1).

Chapter 5

Long Paths Added at Vertices of P_n

In this chapter we prove stronger results than in [7]. We consider classes of trees obtained by joining a long path to vertices of P_n . In Sections 5.1 and 5.2 we deal with the case when n is odd and even, respectively. In Section 5.3 we provide some corollaries of these results.

We use the following notation for the vertices of P_n (see Figure 5.1). If $n = 2k + 1$, let x be the central vertex of P_n and let u_j and v_j be the vertices in P_n at distance j from x on the left and right, respectively. If $n = 2k$, let x and y be the two central vertices of P_n , on the left and right, respectively. Then let u_j be the vertex of P_n at distance j from x and distance $j + 1$ from y . Similarly, let v_j be the vertex of P_n at distance j from y and distance $j + 1$ from x . We say that we *add* a path of length k at vertex u_j of P_n if we

identify a leaf of P_{k+1} with u_j , and that a path added to vertex u_j of P_n is of *allowable length* if P_n is a diametrical path of the resulting tree.

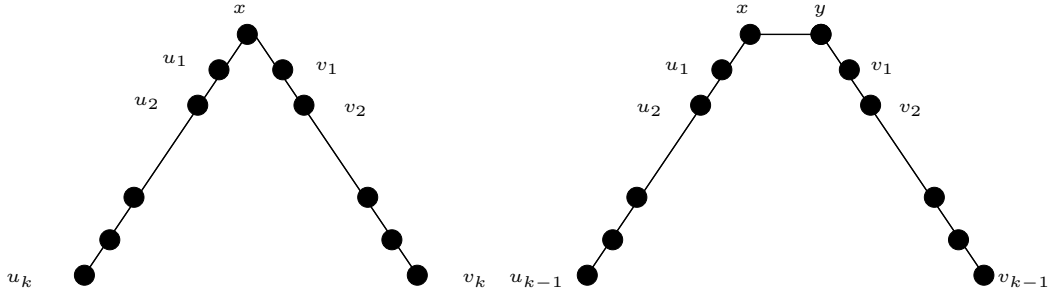


Figure 5.1: Labeling of vertices for P_{2k+1} and P_{2k}

5.1 The Central Case

Theorem 5.1 *Let $n = 2k + 1$ and consider P_n , which has radius k .*

- (i) *Adding a single path of length $\ell \leq k - 3$ at x does not make the resulting tree radial; adding a single path of any longer allowable length does make the resulting tree radial.*
- (ii) *Adding a single path of length $\ell \leq k - 4$ at u_1 or to u_2 does not make the resulting tree radial; adding a single path of any longer allowable length does make the resulting tree radial.*
- (iii) *Adding a single path of any allowable length at u_j , where $j \geq 3$, does not make the resulting tree radial.*

Proof.

(i) Add a path of length $\ell \leq k-3$ to vertex x of P_n and let T be the resulting tree. Now define the following broadcast on T :

$$f(v) = \begin{cases} k-3 & \text{if } v = x \\ 1 & \text{if } v \in \{u_{k-1}, v_{k-1}\} \\ 0 & \text{otherwise.} \end{cases}$$

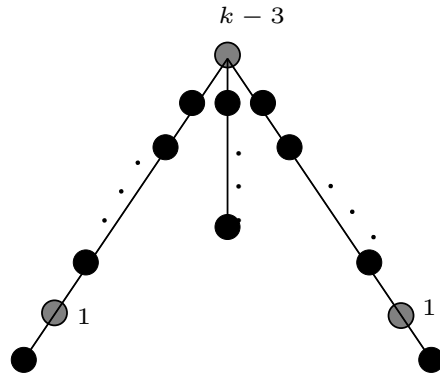


Figure 5.2: Broadcast f for Theorem 5.1 (i)

Then it is clear that f is a dominating broadcast with cost $\sigma(f) = k-1$, so T is not radial.

Now add a path of length $\ell = k-2$ to vertex x of P_n and let T be the resulting tree. Let w be the leaf of the added path. By Proposition 2.1, T has an efficient γ_b -broadcast f . Let $T_u = [u_k, x]$, the path from the leaf u_k to vertex x , of length k . Similarly, let $T_v = [v_k, x]$ and $T_w = [w, x]$, with lengths k and $k-2$, respectively. Now, let z be the vertex that f -dominates x and

let $d(z, x) = t$. Suppose that $z \in T_u$ (the proof is the same if $z \in T_v$). Then $T - N_f[z]$ is a collection $\mathbb{P} = \{P_u, P_v, P_w\}$ of three disjoint paths. Since f is efficient, it follows that

$$\sigma(f) = f(z) + \sum_{P \in \mathbb{P}} \gamma_b(P). \quad (5.1)$$

We note that P_u, P_v , and P_w have orders

$$k - f(z) - t, \quad k - f(z) + t, \quad k - f(z) + t - 2.$$

By Proposition 2.5,

$$\begin{aligned} \sum_{P \in \mathbb{P}} \gamma_b(P) &= \left\lceil \frac{k - f(z) - t}{3} \right\rceil + \left\lceil \frac{k - f(z) + t}{3} \right\rceil + \left\lceil \frac{k - f(z) + t - 2}{3} \right\rceil \\ &\geq \frac{3k - 3f(z) + t - 2}{3} \\ &\geq k - f(z) - 2/3. \end{aligned}$$

If $z \in T_w$, then the only difference in the above proof is that paths P_u, P_v , and P_w have orders

$$k - f(z) + t, \quad k - f(z) + t, \quad k - f(z) - t - 2.$$

In this case we also have $\sum_{P \in \mathbb{P}} \gamma_b(P) \geq k - f(z) - 2/3$. Now, by (5.1), $\sigma(f) \geq k - 2/3$. But f is a γ_b -broadcast, so $\sigma(f) = \gamma_b(T)$. Therefore, $\gamma_b(T) \geq k - 2/3$, and so $\gamma_b(T) = k = \text{rad}(T)$. Thus T is a radial tree. If a path of length $k - 1$ or k is added to x , then the result follows from Propo-

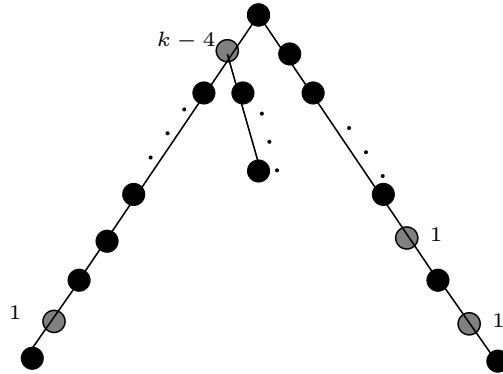


Figure 5.3: Broadcast f for Theorem 5.1 (ii)a

sition 2.3.

(ii) a) Add a path of length $\ell \leq k - 4$ to vertex u_1 of P_n and let T be the resulting tree. Now define the following broadcast on T :

$$f(v) = \begin{cases} k - 4 & \text{if } v = u_1 \\ 1 & \text{if } v \in \{u_{k-1}, v_{k-1}, v_{k-3}\} \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that f is a dominating broadcast with cost $\sigma(f) = k - 1$, so T is not radial.

Now add a path of length $\ell = k - 3$ to vertex u_1 of P_n and let T be the resulting tree. Let w be the leaf of the added path. By Proposition 2.1, T has an efficient γ_b -broadcast f . Let $T_u = [u_k, u_1]$, the path from the leaf u_k to vertex x , of length $k - 1$. Similarly, let $T_v = [v_k, u_1]$ and $T_w = [w, u_1]$,

with lengths $k + 1$ and $k - 3$, respectively. Now, let z be the vertex that f -dominates x and let $d(z, x) = t$. Suppose firstly that $z \in T_u$. Then $T - N_f[z]$ is a collection $\mathbb{P} = \{P_u, P_v, P_w\}$ of three disjoint paths. Since f is efficient, (5.1) holds. We note that P_u, P_v , and P_w have orders

$$k - f(z) - t - 1, \quad k - f(z) + t + 1, \quad k - f(z) + t - 3.$$

By Proposition 2.5,

$$\begin{aligned} \sum_{P \in \mathbb{P}} \gamma_b(P) &= \left\lceil \frac{k - f(z) - t - 1}{3} \right\rceil + \left\lceil \frac{k - f(z) + t + 1}{3} \right\rceil + \left\lceil \frac{k - f(z) + t - 3}{3} \right\rceil \\ &\geq \frac{3k - 3f(z) + t - 3}{3} \\ &= k - f(z) + \frac{t - 3}{3}. \end{aligned}$$

If $z \in T_v$, then the paths P_u, P_v , and P_w have orders

$$k - f(z) + t - 1, \quad k - f(z) - t + 1, \quad k - f(z) + t - 3.$$

If $z \in T_w$, then the paths P_u, P_v , and P_w have orders

$$k - f(z) + t - 1, \quad k - f(z) + t + 1, \quad k - f(z) - t - 3.$$

Therefore, in any case, $\sum_{P \in \mathbb{P}} \gamma_b(P) \geq k - f(z) + \frac{t-3}{3}$. Thus $\sigma(f) \geq k + \frac{t-3}{3}$, where $t \geq 0$. If $1 \leq t \leq 3$, then $\sigma(f) \geq k$ so $\sigma(f) = \gamma_b(T) = k = \text{rad}(T)$, and we're done. If $t \geq 4$, then $\sigma(f) > k$, which is impossible. So suppose $t = 0$ and let $s = k - f(z)$.

Then $\sigma(f) = f(z) + y$, where $y = \left\lceil \frac{s-1}{3} \right\rceil + \left\lceil \frac{s+1}{3} \right\rceil + \left\lceil \frac{s-3}{3} \right\rceil$. We consider the congruence classes of s modulo 3.

- If $s \equiv 0 \pmod{3}$ then $s = 3r$ for some $r \in \mathbb{Z}^+$. So $y = \lceil \frac{3r-1}{3} \rceil + \lceil \frac{3r+1}{3} \rceil + \lceil \frac{3r-3}{3} \rceil = r + r + 1 + r - 1 = 3r = s$.
- If $s \equiv 1 \pmod{3}$ then $s = 3r + 1$ for some $r \in \mathbb{Z}^+$. So $y = \lceil \frac{3r}{3} \rceil + \lceil \frac{3r+2}{3} \rceil + \lceil \frac{3r-2}{3} \rceil = r + r + 1 + r = 3r + 1 = s$.
- If $s \equiv 2 \pmod{3}$ then $s = 3r + 2$ for some $r \in \mathbb{Z}^+$. So $y = \lceil \frac{3r+1}{3} \rceil + \lceil \frac{3r+3}{3} \rceil + \lceil \frac{3r-1}{3} \rceil = r + 1 + r + 1 + r = 3r + 2 = s$.

Thus, $\sigma(f) = f(z) + s = f(z) + k - f(z) = k$. So $\gamma_b(T) = k = \text{rad}(T)$ and therefore T is a radial tree. Again, if an allowable path of length greater than $k - 3$ is added at u_1 , the result follows from Proposition 2.3.

(ii) b) Add a path of length $\ell \leq k - 4$ to vertex u_2 of P_n and let T be the resulting tree. Now define the following broadcast on T :

$$f(v) = \begin{cases} k - 4 & \text{if } v = u_2 \\ 1 & \text{if } v \in \{u_{k-1}, v_{k-1}, v_{k-4}\} \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that f is a dominating broadcast with cost $\sigma(f) = k - 1$, so T is not radial.

As in the proof of part (ii) a), the tree resulting from adding a path of allowable length at least $k - 3$ at vertex u_2 of P_n is radial.

(iii) Add a path of any allowable length at any vertex u_j , where $j \geq 3$, of P_n

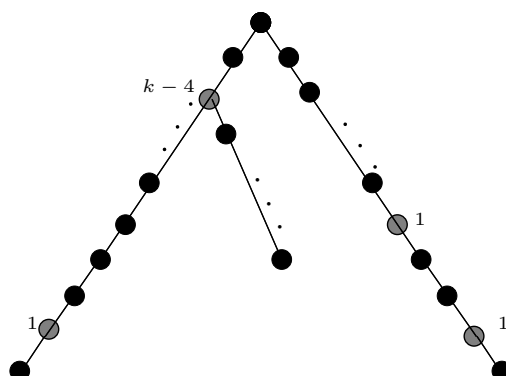


Figure 5.4: Broadcast f for Theorem 5.1 (ii)b

and let T be the resulting tree. Define the following broadcast on T :

$$f(v) = \begin{cases} k-3 & \text{if } v = u_3 \\ 1 & \text{if } v \in \{v_{k-1}, v_{k-4}\} \\ 0 & \text{otherwise.} \end{cases}$$

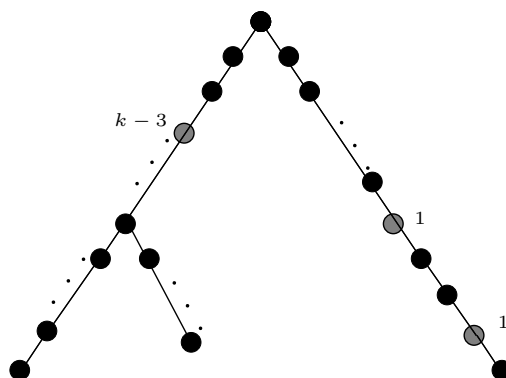


Figure 5.5: Broadcast f for Theorem 5.1 (iii)

Then it is clear that f is a dominating broadcast with cost $\sigma(f) = k - 1$,

so T is not radial.

□

5.2 The Bicentral Case

Theorem 5.2 *Let $n = 2k$ and consider P_n , which has radius k .*

- (i) *Adding a single path of length $\ell \leq k - 3$ at x does not make the tree radial; adding a single path of any longer allowable length does make the tree radial.*
- (ii) *Adding a single path of any allowable length at u_j , where $j \geq 1$, does not make the tree radial.*

Proof.

(i) Add a path of length $\ell \leq k - 3$ at vertex x of P_n and let T be the resulting tree. Define the following broadcast on T :

$$f(v) = \begin{cases} k - 3 & \text{if } v = x \\ 1 & \text{if } v \in \{u_{k-2}, v_{k-2}\} \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that f is a dominating broadcast with cost $\sigma(f) = k - 1$, so T is not radial.

Now add a path of length $\ell = k - 2$ to vertex x of P_n and let T be the resulting tree. Let w be the leaf of the added path. By Proposition 2.1, T

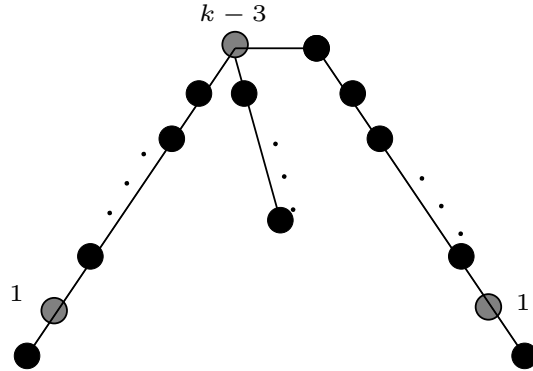


Figure 5.6: Broadcast f for Theorem 5.2 Part (i)

has an efficient γ_b -broadcast f . Let $T_u = [u_{k-1}, x]$, the path from the leaf u_{k-1} to vertex x , of length k . Similarly, let $T_v = [v_{k-1}, x]$ and $T_w = [w, x]$, with lengths $k + 1$ and $k - 1$, respectively. Now, let z be the vertex that f -dominates x and let $d(z, x) = t$. Suppose that $z \in T_u$ (the proof is the same in the other cases). Then $T - N_f[z]$ is a collection $\mathbb{P} = \{P_u, P_v, P_w\}$ of three disjoint paths. Since f is efficient, (5.1) from the proof of Theorem 5.1 holds. We note that P_u, P_v , and P_w have orders

$$k - f(z) - t - 1, \quad k - f(z) + t, \quad k - f(z) + t - 2.$$

By Proposition 2.5,

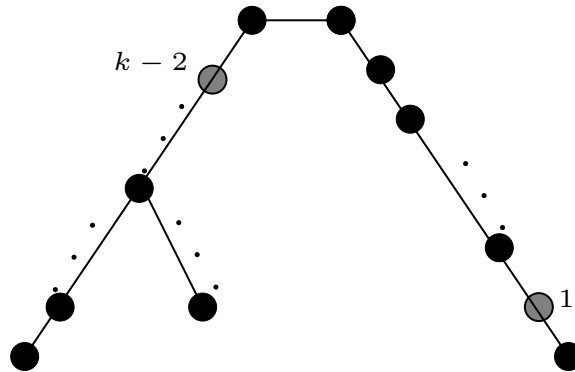
$$\begin{aligned}
\sum_{P \in \mathbb{P}} \gamma_b(P) &= \left\lceil \frac{k - f(z) - t - 1}{3} \right\rceil + \left\lceil \frac{k - f(z) + t}{3} \right\rceil + \left\lceil \frac{k - f(z) + t - 2}{3} \right\rceil \\
&> \frac{3k - 3f(z) + t - 3}{3} \\
&\geq k - f(z) - 1.
\end{aligned} \tag{5.2}$$

(Note that Inequality (5.2) is strict because at least one of the ceiling functions is rounded up.)

So, by (5.1), $\sigma(f) > k - 1$. But f is a γ_b -broadcast, so $\sigma(f) = \gamma_b(T)$. Therefore, $k - 1 < \gamma_b(T) \leq k$, and so $\gamma_b(T) = k = \text{rad}(T)$. Thus T is a radial tree. If a path of length $k - 2$ or $k - 1$ is added to x , then the result follows from Proposition 2.3.

(ii) Add a path of any allowable length to any vertex u_j , where $j \geq 1$, of P_n and let T be the resulting tree. Define the following broadcast on T :

$$f(v) = \begin{cases} k - 2 & \text{if } v = u_1 \\ 1 & \text{if } v = v_{k-2} \\ 0 & \text{otherwise.} \end{cases}$$

Figure 5.7: Broadcast f for Theorem 5.2 Part (ii)

Then it is clear that f is a dominating broadcast with cost $\sigma(f) = k - 1$, so T is not radial.

□

We saw in Section 4.3 that adjacent vertices of degree two on the spine of a caterpillar play an important role in determining whether the caterpillar is radial or not. When a single path is joined to a vertex of P_n , there are many adjacent vertices of degree two, and their role is not immediately clear. Note, however, that in the case where the resulting tree is bicentral and nonradial, we can delete the edge $v_{k-4}v_{k-3}$ (or $u_{k-4}u_{k-3}$), where $\deg(v_{k-4}) = \deg(v_{k-3}) = 2$, so that $u_{k-1}, \dots, u_1, x, y, v_1, \dots, v_{k-4}$ is a diametrical path of the new tree. In the central nonradial case we can similarly delete either three vertices from each end of P_n , or two consecutive paths P_3 from the same end. In all instances the endvertices of the edges where the “cut” is performed have degree two. This significant observation is explored in Section 6.1.

5.3 Corollaries of Theorems 5.1 and 5.2

Let P be a path in a tree T and $w \in V(T) - V(P)$. Define the *distance* $d(P, w)$ of w from P by $d(P, w) = \min\{d(v, w) : v \in V(P)\}$. We now state a corollary that is stronger than Proposition 2.10, which was proved in [7].

Corollary 5.3 *Let T be a tree and let P be a diametrical path of T . If $d(P, w) \geq k - 2$ for some $w \in V(T) - V(P)$, then T is radial.*

Corollary 5.4 *Let T be a central tree with diametrical path P and let v be a non-central vertex of P . If v is the initial vertex of a path of length at least $k - 3$ that is internally disjoint from P , then T is radial.*

The next result follows from the broadcast given in Figure 5.7.

Corollary 5.5 *Let $n = 2k$ and consider P_n with radius k . If T is formed by adding any number of paths of any allowable lengths to any subset of the vertices u_1, \dots, u_k , then T is nonradial.*

Similarly, the broadcasts given in Figures 5.2 to 5.7 imply the following results.

Corollary 5.6 *Let T be any central tree with $\text{rad}(T) = k \geq 4$, $\text{Cen}(T) = \{x\}$, and peripheral vertices u_k and v_k such that $d(u_k, v_k) = \text{diam}(T) = 2\text{rad}(T)$. Let $P : u_k, u_{k-1}, \dots, x, \dots, v_{k-1}, v_k$ be the $u_k - v_k$ path in T . If one of the following situations occurs, then T is nonradial:*

- *each vertex $w \in V(T) - V(P)$ is either adjacent to u_{k-1} or v_{k-1} , or satisfies $d(x, w) \leq k - 3$;*
- *each vertex $w \in V(T) - V(P)$ is either adjacent to u_{k-1}, v_{k-3} or v_{k-1} , or satisfies $d(u_1, w) \leq k - 4$;*
- *each vertex $w \in V(T) - V(P)$ is either adjacent to u_{k-1}, v_{k-4} or v_{k-1} , or satisfies $d(u_2, w) \leq k - 4$; or*
- *each vertex $w \in V(T) - V(P)$ is either adjacent to v_{k-4} or v_{k-1} , or satisfies $d(u_3, w) \leq k - 3$.*

Corollary 5.7 *Let T be any bicentral tree with $\text{rad}(T) = k \geq 4$, $\text{Cen}(T) = \{x, y\}$, and peripheral vertices u_{k-1} and v_{k-1} such that $d(u_{k-1}, v_{k-1}) = \text{diam}(T) = 2\text{rad}(T) - 1$. Let $P : u_{k-1}, u_{k-2}, \dots, x, \dots, v_{k-2}, v_{k-1}$ be the $u_{k-1} - v_{k-1}$ path in T . If one of the following situations occurs, then T is nonradial:*

- *each vertex $w \in V(T) - V(P)$ is either adjacent to u_{k-2} or v_{k-2} , or satisfies $d(x, w) \leq k - 3$;*
- *each vertex $w \in V(T) - V(P)$ is either adjacent to v_{k-2} , or satisfies $d(u_1, w) \leq k - 2$.*

We use Theorems 5.1 and 5.2 to prove a sufficient condition for a tree to be radial.

Theorem 5.8 *Let T be a tree. If $\text{rad}(T) = \text{rad}(T + uv)$ for every pair of distinct vertices $u, v \in V(T)$ such that $uv \notin E(T)$, then T is radial.*

Proof. Suppose T is nonradial. We show that there is a pair of vertices $u, v \in V(T)$ such that $uv \notin E(T)$ and $\text{rad}(T) > \text{rad}(T + uv)$. Let P be a diametrical path in T . Let $\text{rad}(T) = k$.

Case 1: T is central. Say $P = u_k, \dots, u_1, x, v_1, \dots, v_k$. Then $u_2v_2 \notin E(T)$, otherwise u_2, u_1, x, v_1, v_2 is a cycle. Root T at x and consider $T' = T + u_2v_2$. We calculate $e(u_2)$ in T' . If w is a descendant of u_2 , then $d(u_2, w) \leq k - 2$. If w is a descendant of u_1 but not u_2 , then $d(u_1, w) \leq k - 4$, by Theorem 5.1, so $d(u_2, w) \leq k - 3$. If w is a descendant of x but not of u_1 or v_1 , then $d(x, w) \leq k - 3$, by Theorem 5.1, so $d(u_2, w) \leq k - 1$. If w is a descendant of v_1 but not v_2 , then $d(v_1, w) \leq k - 4$, by Theorem 5.1, so $d(u_2, w) \leq k - 3$. Thus $e(u_2) \leq k - 1$ and so $\text{rad}(T') \leq k - 1$.

Case 2: T is bicentral. Say $P = u_{k-1}, \dots, u_1, x, y, v_1, \dots, v_{k-1}$. Then $u_1v_1 \notin E(T)$, otherwise u_1, x, y, v_1 is a cycle. Root T at x and consider $T' = T + u_1v_1$. We calculate $e(u_1)$ in T' . If w is a descendant of u_1 , then $d(u_1, w) \leq k - 2$. If w is a descendant of x but not u_1 or y , then $d(x, w) \leq k - 3$, by Theorem 5.2, so $d(u_1, w) \leq k - 2$. If w is a descendant of y but not v_1 , then $d(y, w) \leq k - 3$, by Theorem 5.2, so $d(u_1, w) \leq k - 1$. If w is a descendant of v_1 ,

then $d(v_1, w) \leq k - 2$, so $d(u_1, w) \leq k - 1$. Thus $e(u_1) \leq k - 1$ and so $\text{rad}(T') \leq k - 1$.

□

Note that the converse of Theorem 5.8 is not true. For example, consider the tree in Figure 5.8, where T is radial since $\gamma_b(T) = \text{rad}(T) = 2$, but $\text{rad}(T + uv) = 1 \neq \text{rad}(T)$.



Figure 5.8: T and $T + uv$

Chapter 6

A Characterization of Radial Trees

In this chapter we prove the main result of this thesis. In Section 6.1 we look at some examples that compare results from the previous chapters and motivate the characterization. We require a crucial result given in Section 6.2. Then in Section 6.3 we prove the characterization and obtain two related formulas for the broadcast number of a tree as corollaries. In Section 6.4 we provide a geometric interpretation of the characterization. In Section 6.5 we apply this characterization of radial trees to show that general coronas of graphs are radial, a result first proved in [7]. We also describe a method for calculating the broadcast number of a tree, and close with a proof that every tree T has an efficient broadcast with cost k , for any k between $\gamma_b(T)$ and $\text{rad}(T)$.

6.1 Motivation for the Characterization

We begin this section by comparing our results about caterpillars and long paths added to P_n . The central trees with radius 5 in Figure 6.1 are nonradial by Corollary 5.6 (for the trees on the left) and by Theorem 4.7 (for the caterpillars on the right). We notice that for each of these nonradial trees, the vertices can be covered by three isosceles right triangles whose hypotenuses have even integer lengths, as shown. Similarly, by Corollary 5.7 and Theorem 4.7, the bicentral trees with radius 5 in Figure 6.2 are nonradial, and we notice that the vertices of these trees can be covered by two isosceles right triangles whose hypotenuses have even integer lengths. In either the central or the bicentral case, when more vertices are added to the tree such that the tree becomes radial (by Theorem 5.1 or Theorem 4.7), it then becomes impossible to cover the vertices with triangles as described above. This rough description motivates our next definition, and is refined in Section 6.4.

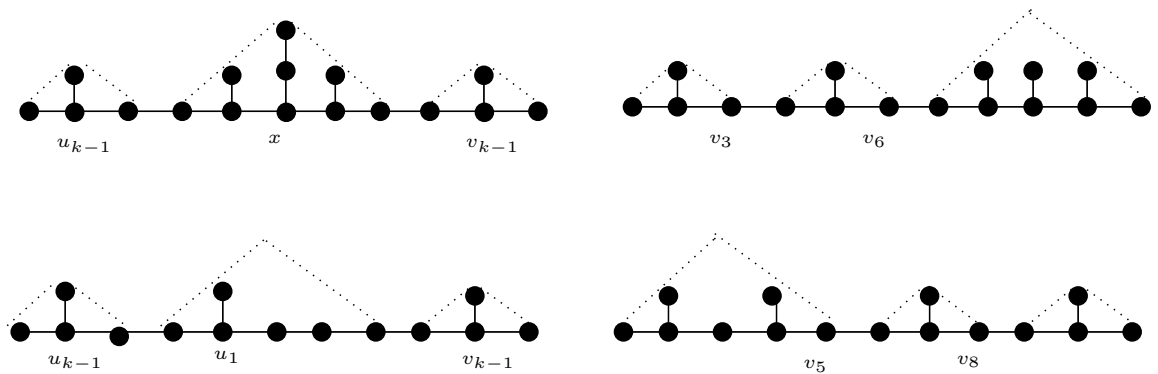


Figure 6.1: Nonradial central trees

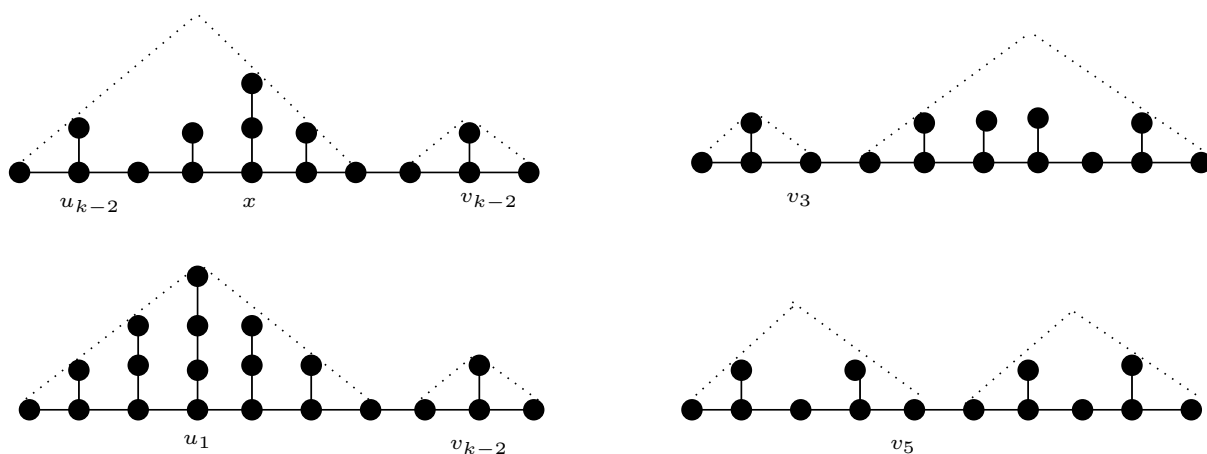


Figure 6.2: Nonradial bicentral trees

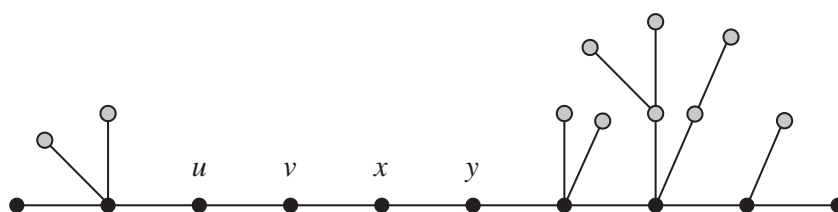


Figure 6.3: A tree with split-sets $M = \{uv\}$ and $M' = \{xy\}$

Let P be a diametrical path of the tree T . A set M of edges of P is a *split- P set* if the endvertices of each edge in M have degree two in T , and each component T' of $T - M$ has even positive diameter, the path $P' = P \cap T'$ being a diametrical path of T' . For example, the sets $M = \{uv\}$ and $M' = \{xy\}$ are split- P sets of the tree in Figure 6.3, where P is the path of black vertices. A *split-set* of T is a split- P set for some diametrical path P of T , and a *maximum split-set* of T is a split-set of maximum cardinality.

In Section 6.3 we prove that a tree T is radial if and only if it has no nonempty split-set.

6.2 Very Efficient Broadcasts

We now prove that a stronger result than Theorem 2.12 holds for trees (Theorem 6.1). Our characterization of radial trees, our formulas for the broadcast number of a tree, and our recursive method for calculating γ_b depend on this result. For these reasons, Theorem 6.1 is the most important result of this thesis.

Theorem 6.1 *Let P be a diametrical path of a tree T and amongst all γ_b -broadcasts of T , let f be one with the minimum number of broadcast vertices. Then*

- (i) *f is efficient,*
- (ii) *every broadcast vertex lies on P , and*
- (iii) *unless P is a bicentral radial tree, neither endvertex of P is overdominated.*

Conversely, every γ_b -broadcast that satisfies (i), (ii) and (iii) is a γ_b -broadcast with the minimum number of broadcast vertices.

Note that because of the conditions imposed on f , (i) does not follow from Proposition 2.1, and Lemma 2.12 does not imply that $\Delta(B(f)) \leq 2$.

Nevertheless, similar proofs establish these properties of f . We include them here for completeness.

Proof. Let f and $P = v_1, v_2, \dots, v_n$ satisfy the hypothesis of the theorem.

(i) Suppose, to the contrary, that $N_f[u] \cap N_f[w] \neq \emptyset$. Then this intersection contains a vertex v of the $u - w$ path (possibly an endvertex of the path) chosen as follows:

(a) if $f(w) \geq d(u, w)$, let $v = w$,

(b) otherwise choose v so that $d(u, v) = f(w)$.

In each case $d(u, v) \leq f(w)$. With choice (a), $d(v, w) = 0 < f(u)$. With choice (b), if $d(v, w) > f(u)$, then $d(u, w) = d(u, v) + d(v, w) > f(w) + f(u)$, which implies that $N_f[u] \cap N_f[w] = \emptyset$, a contradiction. Hence

$$d(u, v) \leq f(w) \text{ and } d(v, w) \leq f(u). \quad (6.1)$$

Define a broadcast g on T by

$$g(x) = \begin{cases} 0 & \text{if } x \in \{u, w\} - \{v\} \\ f(u) + f(w) & \text{if } x = v \\ f(x) & \text{otherwise.} \end{cases}$$

We show that v broadcasts to all of $N_f[u]$; the proof that v broadcasts to all of $N_f[w]$ is identical. If $x \in N_f[u]$, then $d(x, u) \leq f(u)$, hence

$$\begin{aligned} d(x, v) &\leq d(x, u) + d(u, v) \leq f(u) + f(w) \quad (\text{by (6.1)}) \\ &= g(v). \end{aligned}$$

Therefore g is a γ_b -broadcast of T with fewer broadcast vertices than f , a contradiction. This proves (i).

(ii) Suppose, to the contrary, that there is a broadcast vertex $u \in V(T) - V(P)$. Assume without loss of generality that there is no other broadcast vertex not on P that lies between u and P , and let v be the vertex on P at minimum distance from u .

Case 1: $N_f[u] \cap V(P) = \emptyset$. Let w be the vertex of T that broadcasts to v .

Subcase 1.1: $w = v$. We define a number of vertices as follows (see Figure 6.4):

- u' is the vertex with $d(u', u) = f(u)$ and $d(u', w) = f(w) + 1$,
- w'_1 and w'_2 are the vertices on P at distance $f(v) + 1$ from v on the $v - v_1$ and $v - v_n$ subpaths of P , respectively; w'_1 and w'_2 exist because P is a diametrical path,
- for $i = 1, 2$, w_i is the vertex that broadcasts to w'_i .

Note that w_1 and w_2 do not necessarily lie on P . Let Q, Q', Q'' be the paths in T between u and w_1 , u and w_2 , w_1 and w_2 , respectively. Denote the lengths

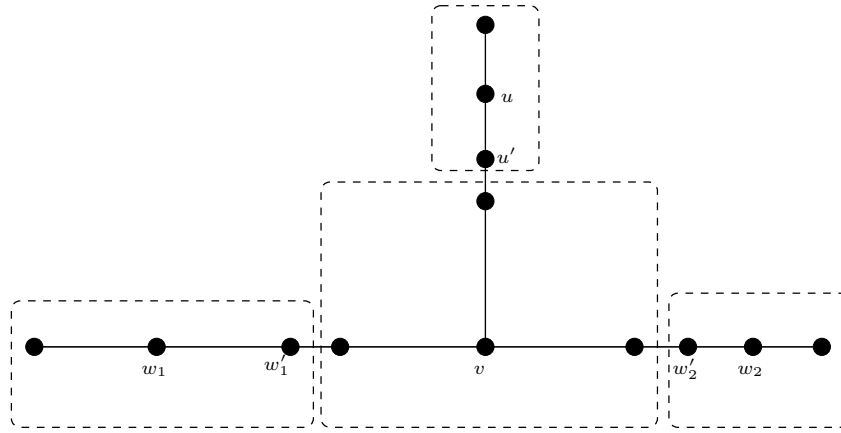


Figure 6.4: Subcase 1.1 of Theorem 6.1

of the paths by $\ell(Q), \ell(Q'), \ell(Q'')$, respectively.

Without loss of generality, assume that Q is the longest of these paths (the proof works the same in the other two cases). Then

$$f(w_2) \leq f(u) \tag{6.2}$$

and

$$\ell(Q) = f(u) + 2f(v) + f(w_1) + 2. \tag{6.3}$$

Let $v' \in V(Q)$ be such that $d(v', w_1) = f(u) + f(v) + 1$ and define a broadcast g on T as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \in \{u, v, w_1, w_2\} - \{v'\} \\ f(v) + f(u) + f(w_1) + f(w_2) & \text{if } x = v' \\ f(x) & \text{otherwise.} \end{cases}$$

For each $x \in N_f[w_1]$,

$$d(v', x) \leq d(v', w_1) + f(w_1) = f(u) + f(v) + 1 + f(w_1) \leq g(v') \quad (\text{since } f(w_2) \geq 1).$$

Thus v' broadcasts to all of $N_f[w_1]$. Also, for each $x \in N_f[u]$,

$$\begin{aligned} d(v', x) &\leq \ell(Q) + f(u) - d(w_1, v') \\ &= f(u) + 2f(v) + f(w_1) + 2 + f(u) - [f(u) + f(v) + 1] \quad (\text{by (6.3)}) \\ &= f(u) + f(v) + f(w_1) + 1 \leq g(v'), \end{aligned}$$

so v' broadcasts to all of $N_f[u]$. Since $N_f[v]$ lies between $N_f[w_1]$ and $N_f[u]$, it follows that v' broadcasts to all of $N_f[v]$. We show that v' broadcasts to all of $N_f[w_2]$. If $f(w_1) \geq f(u)$, then v' lies on the $w_1 - v$ path in T and $d(v', v) = f(w_1) - f(u)$. Hence for each $x \in N_f[w_2]$,

$$\begin{aligned} d(v', x) &= d(v', v) + d(v, x) \\ &\leq f(w_1) - f(u) + f(v) + 2f(w_2) + 1 \\ &\leq f(w_1) + f(w_2) + f(v) + 1 \quad (\text{by (6.2)}) \\ &\leq g(v') \quad (f(u) \geq 1). \end{aligned}$$

Similarly, when $f(w_1) < f(u)$, $d(v', x) \leq g(v')$ for each $x \in N_f[w_2]$. Therefore v' broadcasts to all of $N_f[w_2]$. Hence g is a γ_b -broadcast on T with fewer broadcast vertices than f , which is a contradiction.

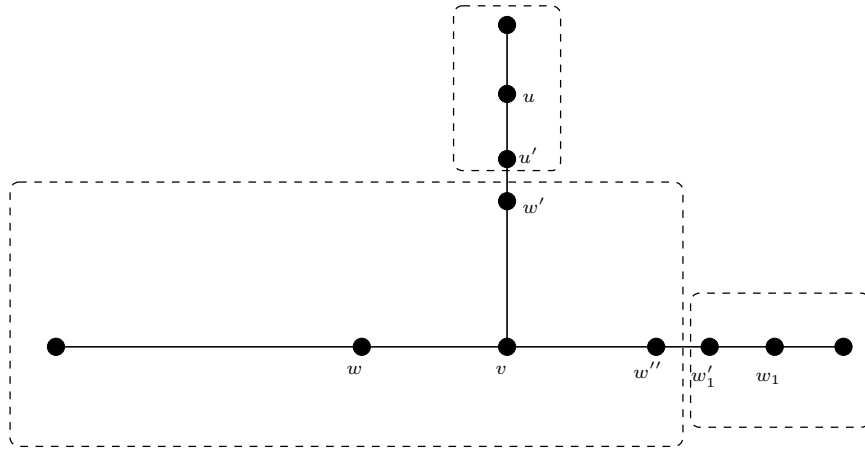


Figure 6.5: Subcase 1.2 of Theorem 6.1

Subcase 1.2: $w \neq v$. Define u' as in Subcase 1.1. We define a number of other vertices as follows (see Figure 6.5):

- $w' \in N_f[w]$ is the vertex adjacent to u' (i.e. $d(w, w') = f(w)$),
- $w'' \in N_f[w]$ is the vertex on P at maximum distance from w such that the $w - w''$ path contains v ,
- $w'_1 \in V(P)$ is the vertex adjacent to w'' such that $d(w, w'_1) = f(w) + 1$ (w'_1 exists because P is a diametrical path),
- w_1 is the vertex that broadcasts to w'_1 .

Then $d(w, w'') = f(w)$. Note that w and w_1 do not necessarily lie on P and that $w' = w'' = v$ is possible. Let Q, Q', Q'' be the paths in T between u and w , u and w_1 , w and w_1 , respectively.

- Suppose firstly that Q is the longest of these paths. (The proof is similar if Q'' is the longest path.)

Then

$$\ell(Q) = f(w) + f(u) + 1. \quad (6.4)$$

Let $v' \in V(Q)$ be such that $d(w, v') = f(u) + 1$ (and $d(u, v') = f(w)$) and define a broadcast g on T as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \in \{w, u, w_1\} - \{v'\} \\ f(w) + f(u) + f(w_1) & \text{if } x = v' \\ f(x) & \text{otherwise.} \end{cases} \quad (6.5)$$

For each $x \in N_f[w]$, $d(v', x) \leq f(w) + f(u) + 1 \leq g(v')$ since $f(w_1) \geq 1$, so v' broadcasts to all of $N_f[w]$. It also follows from (6.4) that for each $x \in N_f[u]$,

$$d(v', x) = \ell(Q) - d(w, v') + d(u, x) \leq f(w) + f(u) < g(v'),$$

so v' broadcasts to all of $N_f[u]$. We show that v' broadcasts to all of $N_f[w_1]$.

By the choice of Q ,

$$\left. \begin{array}{l} d(w, v) \\ d(u, v) \end{array} \right\} \geq d(v, w_1) = d(v, w'') + f(w_1) + 1. \quad (6.6)$$

If v' lies on the $v - u$ path in T , then

$$\begin{aligned} d(v', w'') &= d(v', w) - d(w, v) + d(v, w'') \\ &\leq f(u) + 1 - [d(v, w'') + f(w_1) + 1] + d(v, w'') \quad (\text{by (6.6)}) \\ &= f(u) - f(w_1), \end{aligned}$$

from which it follows that for each $x \in N_f[w_1]$,

$$\begin{aligned} d(v', x) &= d(v', w'') + d(w'', x) \leq [f(u) - f(w_1)] + [2f(w_1) + 1] \\ &= f(u) + f(w_1) + 1 \leq g(v'). \end{aligned}$$

If v' lies on the $w - v$ path in T , then

$$\begin{aligned} d(v', w'') &= d(v', u) - d(u, v) + d(v, w'') \\ &\leq f(w) - [d(v, w'') + f(w_1) + 1] + d(v, w'') \quad (\text{by (6.6)}) \\ &= f(w) - f(w_1) - 1, \end{aligned}$$

so for each $x \in N_f[w_1]$,

$$\begin{aligned} d(v', x) &= d(v', w'') + d(w'', x) \leq [f(w) - f(w_1) - 1] + [2f(w_1) + 1] \\ &= f(w) + f(w_1) < g(v'). \end{aligned}$$

Hence v' broadcasts to all of $N_f[w_1]$ and so g is a γ_b -broadcast on T with fewer broadcast vertices than f , which is a contradiction.

- Suppose Q' is the longest path.

Let $d = d(v, w') = d(v, w'')$. Then $d = f(w) - d(w, v) < f(w)$, i.e.

$$d + 1 \leq f(w), \quad (6.7)$$

and

$$\ell(Q') = f(w_1) + 2d + f(u) + 2. \quad (6.8)$$

We assume that $f(u) \geq f(w_1)$; the proof is similar if $f(u) < f(w_1)$. We choose a vertex v' as described below, and in each case define the broadcast g on T as in (6.5).

- ★ If $d(v, w) \leq 2f(w_1)$, choose $v' \in V(Q')$ such that $d(w_1, v') = d + f(u) + 1$ (and $d(u, v') = d + f(w_1) + 1$).

Then v' lies on the $v - u$ path in T . For each $x \in N_f[w_1]$,

$$\begin{aligned} d(v', x) &\leq d(v', w_1) + d(w_1, x) \\ &\leq d + f(u) + 1 + f(w_1) \\ &\leq f(w) + f(u) + f(w_1) \quad (\text{by (6.7)}) \\ &= g(v'). \end{aligned}$$

So v' broadcasts to all of $N_f[w_1]$. Moreover, for any $x \in N_f[u]$,

$$\begin{aligned} d(v', x) &\leq d(v', u) + d(u, x) \\ &\leq d + f(w_1) + 1 + f(u) \leq g(v') \quad (\text{by (6.7)}). \end{aligned}$$

Hence v' broadcasts to all of $N_f[u]$. We show that v' broadcasts to all of $N_f[w]$. Since

$$d(v, v') = d(v', w_1) - d(v, w_1) = d + f(u) + 1 - [d + f(w_1) + 1] = f(u) - f(w_1),$$

it follows that for any $x \in N_f[w]$,

$$\begin{aligned} d(v', x) &\leq d(v', v) + d(v, w) + d(w, x) \\ &\leq f(u) - f(w_1) + 2f(w_1) + f(w) \quad (\text{by the choice } \star) \\ &= f(u) + f(w_1) + f(w) = g(v') \end{aligned}$$

and so $N_f[w] \subseteq N_g[v']$.

★★ If $d(v, w) > 2f(w_1)$, choose v' on the $w - u$ path such that $d(w, v') = f(u) + f(w_1)$.

Obviously, $N_f[w] \subseteq N_g[v']$. We show that $N_f[u] \cup N_f[w_1] \subseteq N_g[v']$. Firstly,

$$\begin{aligned} d(v, v') &= |d(w, v') - d(w, v)| \\ &= \begin{cases} f(u) + f(w_1) - f(w) + d & \text{if } d(w, v') \geq d(w, v) \\ f(w) - f(u) - f(w_1) - d & \text{if } d(w, v') < d(w, v). \end{cases} \end{aligned}$$

Hence, if $d(w, v') \geq d(w, v)$, then v' lies on the $v - u$ path, and

$$\begin{aligned} d(v', w_1) &= d(v', v) + d(v, w_1) = [f(u) + f(w_1) - f(w) + d] + [d + 1 + f(w_1)] \\ &= f(u) + 2f(w_1) + 1 + d - [f(w) - d] \\ &\leq f(u) + d(v, w) + d - [f(w) - d] && \text{(by the choice } \star \star) \\ &= f(u) + f(w) - [f(w) - d] && (d(v, w) + d = f(w)) \\ &< f(u) + f(w) && \text{(by (6.7)),} \end{aligned}$$

so that for any $x \in N_f[w_1]$,

$$d(v', x) = d(v', w_1) + d(w_1, x) < f(u) + f(w) + f(w_1) = g(v').$$

Similarly,

$$\begin{aligned} d(v', u) &= d(v, u) - d(v', v) = [d + 1 + f(u)] - [f(u) + f(w_1) - f(w) + d] \\ &= f(w) - f(w_1) + 1, \end{aligned} \tag{6.9}$$

so that for any $x \in N_f[u]$,

$$d(v', x) = d(v', u) + d(u, x) \leq f(w) - f(w_1) + 1 + f(u) < g(v').$$

On the other hand, if $d(w, v') < d(w, v)$, then v' lies on the $w - v$ path, and

$$\begin{aligned} d(v', w_1) &= d(v', v) + d(v, w_1) = [f(w) - f(u) - f(w_1) - d] + [d + 1 + f(w_1)] \\ &= f(w) - f(u) + 1, \end{aligned}$$

so that for any $x \in N_f[w_1]$,

$$d(v', x) = d(v', w_1) + d(w_1, x) \leq f(w) - f(u) + 1 + f(w_1) < g(v').$$

Similarly,

$$\begin{aligned} d(v', u) &= d(v', v) + d(v, u) = [f(w) - f(u) - f(w_1) - d] + [d + 1 + f(u)] \\ &= f(w) - f(w_1) + 1, \end{aligned}$$

which is the same as (6.9) and so $d(v', x) < g(v')$ for all $x \in N_f[u]$. Therefore in either case $N_f[u] \cup N_f[w_1] \subseteq N_g[v']$.

But then, for both choices of v' , g is a γ_b -broadcast on T with fewer broadcast vertices than f , a contradiction. Thus the proof of Case 1 is complete.

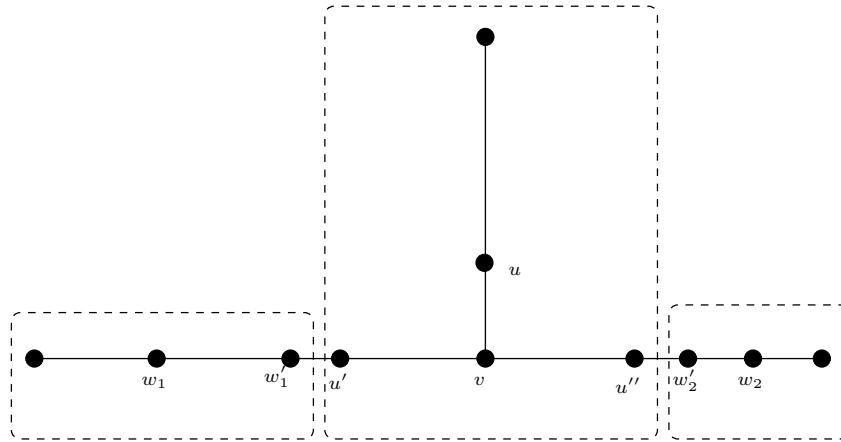


Figure 6.6: Case 2 of Theorem 6.1

Case 2: $N_f[u] \cap V(P) \neq \emptyset$. Now $v \in N_f[u]$. Once again we define a number of vertices (see Figure 6.6):

- $u', u'' \in N_f[u]$ are the vertices on P at maximum distance from v on the $v - v_1$ and $v - v_n$ subpaths of P , respectively; possibly $u' = u'' = v$,
- w'_1, w'_2 are the vertices on P adjacent to u', u'' , respectively; they exist because P is a diametrical path,
- for $i = 1, 2$, w_i is the vertex that broadcasts to w'_i .

Similar to the other cases, w_1 and w_2 do not necessarily lie on P . But Figure 6.6 can be redrawn as in Figure 6.7, which is the same as in Subcase 1.2 with some labels interchanged. Thus we obtain a contradiction as in Subcase 1.2. Therefore (ii) holds.

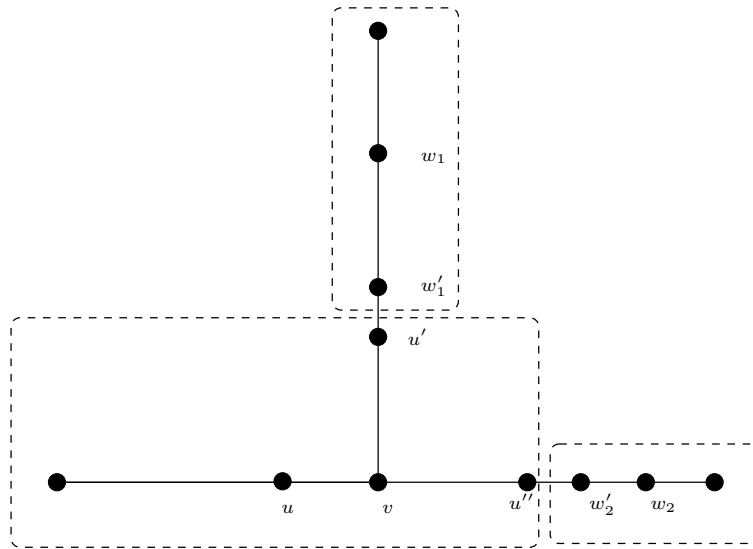


Figure 6.7: Case 2 of Theorem 6.1 redrawn

(iii) Clearly, when T is a bicentral radial tree, a leaf of P is overdominated. Assume that T is not a bicentral radial tree and suppose without loss of generality that v_1 is overdominated. Then T is also not a central radial tree, otherwise all peripheral vertices of T are overdominated and there exists a dominating broadcast g of T with $\sigma(g) < \sigma(f)$, a contradiction.

Assume therefore that T is not radial and let

- u be the vertex on P that broadcasts to v_1 ,
- u' be the vertex at maximum distance from v_1 such that $u' \in N_f[u] \cap V(P)$,
- w' be the vertex on P adjacent to u' such that $d(u, w') = f(u) + 1$; w' exists because T is nonradial,

- $w \in V(P)$ be the vertex that broadcasts to w' ,
- Q be the $u - w$ subpath of P , and
- v be the vertex on Q at distance $f(w) + 1$ from u .

Since f is efficient, v is an internal vertex of Q . Define the broadcast g on T

by

$$g(x) = \begin{cases} 0 & \text{if } x \in \{u, w\} \\ f(u) + f(w) & \text{if } x = v \\ f(x) & \text{otherwise.} \end{cases}$$

For each $x \in N_f[u]$ such that x is joined to u by a path internally disjoint from Q ,

$$\begin{aligned} d(x, u) &\leq d(v_1, u) && (v_1 \text{ is a peripheral vertex}) \\ &\leq f(u) - 1 && (v_1 \text{ is overdominated}). \end{aligned}$$

Hence for each such x , $d(x, v) = d(x, u) + d(u, v) \leq f(u) + f(w) \leq g(v)$. If $x \in N_f[u]$ is joined to u by a path that contains the internal vertex y of Q , then

$$d(x, v) < d(x, y) + d(y, u) + d(u, v) \leq f(u) + f(w) + 1$$

and so $d(x, v) \leq f(u) + f(w) = g(v)$ in this case as well. Thus v broadcasts to every vertex in $N_f[u]$. Moreover, for each $x \in N_f[w]$,

$$\begin{aligned} d(v, x) &\leq \ell(Q) - d(u, v) + d(w, x) \\ &\leq f(u) + f(w) + 1 - [f(w) + 1] + f(w) \\ &= f(u) + f(w) = g(v), \end{aligned}$$

so $N_f[w] \subseteq N_g[v]$. Therefore g is a γ_b -broadcast of T with fewer broadcast vertices than f . This contradiction concludes the proof of the necessity of conditions (i) – (iii).

For the converse, let P be a diametrical path of T and let f be a γ_b -broadcast that satisfies (i) – (iii); say $V_f^+ = \{u_1, \dots, u_r\}$. Let g be a γ_b -broadcast with the minimum number t of broadcast vertices w_1, \dots, w_t . As shown above, g also satisfies (i) – (iii). Then

$$\gamma_b(T) = \sum_{i=1}^r f(u_i) = \sum_{i=1}^t g(w_i)$$

and

$$\text{diam } T = \left(\sum_{i=1}^r 2f(u_i) \right) + r - 1 = \left(\sum_{i=1}^t 2g(w_i) \right) + t - 1,$$

from which it follows that $r = t$. The proof of the theorem is now complete.

□

Remark 6.2 We note that Theorem 6.1 does not hold for graphs in general. Consider C_n with $n \geq 9$ if n is odd or $n \geq 12$ if n is even. For example, consider C_9 . Each vertex v has $e(v) = 4$, so $\text{rad}(C_9) = \text{diam}(C_9) = 4$ and it is clear that $\gamma_b(C_9) = 3$. In Figure 6.8, a diametrical path $P : v_1, \dots, v_5$ is shown and it is straightforward to see that there is no γ_b -broadcast on C_9 such that every broadcast vertex lies on P .

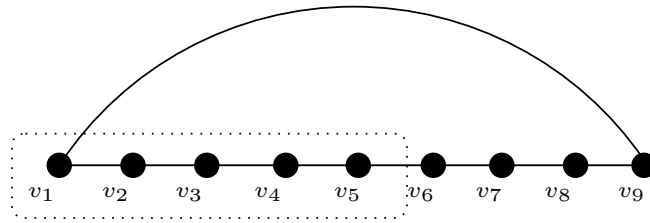
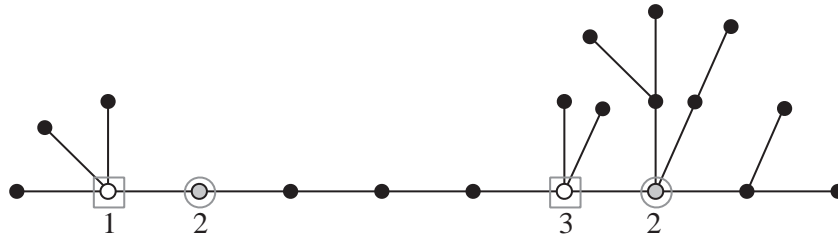


Figure 6.8: The cycle on 9 vertices

Remark 6.3 In the proof of Theorem 6.1, the fact that f is a γ_b -broadcast is not used. In fact, what is proved is the following:

Suppose the tree T has a dominating broadcast of cost k , where $\gamma_b(T) \leq k \leq \text{rad}(T)$. Amongst all such broadcasts, let f be one with the minimum number r of broadcast vertices. Let P be any diametrical path of T . Then

- (i) f is efficient,
- (ii) every broadcast vertex lies on P , and
- (iii) unless T is a bicentral tree and $r = 1$, neither endvertex of P is over-dominated.

Figure 6.9: A tree with two very efficient γ_b -broadcasts

Conversely, every dominating broadcast with cost k that satisfies (i), (ii) and (iii) is a dominating broadcast of cost k with the minimum number of broadcast vertices.

In view of Theorem 6.1 we henceforth call a broadcast of a tree T that satisfies (i), (ii) and (iii) a *very efficient* broadcast. Hence a γ_b -broadcast of T with the minimum number of broadcast vertices is a very efficient γ_b -broadcast. As can be seen by considering the tree in Figure 6.9, a tree may have more than one very efficient γ_b -broadcast. However, the following corollary of Theorem 6.1 is obvious.

Corollary 6.4 *If f is any very efficient γ_b -broadcast of a tree T , then every diametrical path of T contains all broadcast vertices of f .*

6.3 Proof of Characterization

We now prove the main result of this thesis.

Theorem 6.5 *A tree T is radial if and only if it has no nonempty split-set.*

Proof. Suppose M is a nonempty split- P set for a diametrical path P of T and that $T - M$ has r components T_1, \dots, T_r . Then $r \geq 2$ because $M \neq \emptyset$, and each T_i has positive even diameter $2\ell_i$ with $P_i = P \cap T_i$ being a diametrical path. Let v_i be the central vertex of P_i and define a broadcast f_M on T by $f_M(v_i) = \ell_i$ and $f_M(v) = 0$ otherwise. Then f_M is a very efficient dominating broadcast of T . The order of P is

$$n = \sum_{i=1}^r (2\ell_i + 1) = 2 \sum_{i=1}^r \ell_i + r. \quad (6.10)$$

So

$$\text{rad}(T) = \left\lfloor \frac{2 \sum_{i=1}^r \ell_i + r}{2} \right\rfloor = \sum_{i=1}^r \ell_i + \left\lfloor \frac{r}{2} \right\rfloor = \sigma(f_M) + \left\lfloor \frac{r}{2} \right\rfloor. \quad (6.11)$$

Hence $\sigma(f_M) < \text{rad}(T)$ since $r \geq 2$. Therefore T is nonradial.

Conversely, suppose that T is nonradial. Let P be a diametrical path of T and let f be a very efficient γ_b -broadcast of T . Since T is nonradial, Theorem 6.1 implies that $|V_f^+| \geq 2$, every broadcast vertex of f lies on P , and no leaf of P is overdominated. Let $V_f^+ = \{w_1, \dots, w_r\}$, $r \geq 2$, where these vertices have been labelled in order of their appearance on P . Since f is efficient, there exist consecutive vertices $u_i, u_{i+1} \in V(P)$ such that $u_i \in N_f[w_i] - N_f[w_{i+1}]$ and $u_{i+1} \in N_f[w_{i+1}] - N_f[w_i]$, $i = 1, \dots, r - 1$. Clearly, $\deg_T u_i = \deg_T u_{i+1} = 2$. Moreover, since no leaf of P is overdominated, $\langle N_f[w_i] \rangle$ has a diametrical

path $P_i = \langle V(P) \cap N_f[w_i] \rangle$ of length $2f(w_i)$ for each i . Thus, if $e_i = u_i u_{i+1}$, $i = 1, \dots, r-1$, then $M_f = \{e_1, \dots, e_{r-1}\}$ is a non-empty split- P set.

□

We now obtain two formulas for $\gamma_b(T)$ as corollaries to Theorem 6.1. We need some further definitions and a lemma.

If M is a split-set of a tree T and f_M is the broadcast as defined in the proof of Theorem 6.5, we call f_M the *broadcast associated with M* . Similarly, if f is a very efficient broadcast and M_f is the split-set as defined in the proof of Theorem 6.5, we call M_f the *split-set associated with f* .

By Theorem 6.1, a γ_b -broadcast is a very efficient γ_b -broadcast if and only if it is a γ_b -broadcast with the minimum number of broadcast vertices. We now show that a very efficient broadcast is a very efficient γ_b -broadcast if and only if it is a very efficient broadcast with the maximum number of broadcast vertices, that is, if and only if its associated split-set is a maximum split-set.

Lemma 6.6 *A split-set of a tree T is a maximum split-set if and only if its associated broadcast is a very efficient γ_b -broadcast.*

Proof. If M is a split-set, then obviously f_M is a very efficient dominating broadcast. Suppose M is a maximum split-set, and let g be a very efficient γ_b -broadcast of T with associated split-set M_g . Let $r = |V_{f_M}^+|$ and $s = |V_g^+|$. Then

$$r = |M| + 1 \geq |M_g| + 1 = s$$

and, similar to (6.10),

$$\text{diam}(T) = 2\sigma(f_M) + r - 1 = 2\sigma(g) + s - 1.$$

Therefore

$$2\sigma(f_M) = \text{diam}(T) - |M| \leq \text{diam}(T) - |M_g| = 2\sigma(g). \quad (6.12)$$

Since g is a γ_b -broadcast and f is a dominating broadcast, it follows that $\sigma(f) = \sigma(g)$ and so f_M is a very efficient γ_b -broadcast.

Conversely, assume that f is a very efficient γ_b -broadcast with associated split-set M_f . Let M be a maximum split-set of T with associated broadcast g_M . As proved above, g_M is a very efficient γ_b -broadcast, hence $\sigma(f) = \sigma(g_M)$. As in (6.12),

$$2\sigma(f) = \text{diam}(T) - |M_f| = \text{diam}(T) - |M| = 2\sigma(g_M)$$

and the result follows. □

The following corollaries of Theorem 6.1 and Lemma 6.6 are essentially restatements of (6.11) for very efficient γ_b -broadcasts.

Corollary 6.7 (i) Let f be a very efficient γ_b -broadcast of the tree T with r broadcast vertices. Then

$$\gamma_b(T) = \sigma(f) = \text{rad}(T) - \left\lfloor \frac{r}{2} \right\rfloor.$$

(ii) For any tree T , let M be a split-set of maximum cardinality m . Then

$$\gamma_b(T) = \text{rad}(T) - \left\lceil \frac{m}{2} \right\rceil.$$

Let $T_{m,n}$ denote the complete n -ary tree on m levels, defined as follows:

- Let $T_{1,n} = K_{1,n}$.
- For each $m \geq 2$, $T_{m,n}$ is obtained from n copies of $T_{m-1,n}$ whose roots are each joined to a new vertex r .

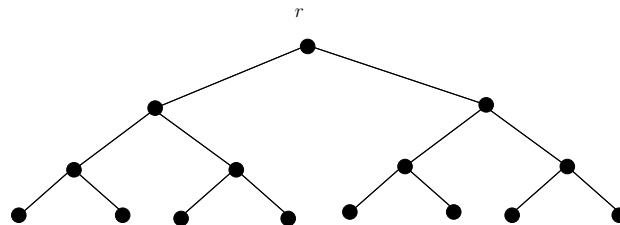


Figure 6.10: The complete binary tree on three levels

In [2], Bouchemakh and Sahbi proved that for $m \geq 1$, $T_{m,2}$ is radial. By Theorem 6.5, the following corollary is clear.

Corollary 6.8 For any $m \geq 1$ and $n \geq 2$, $T_{m,n}$ is radial.

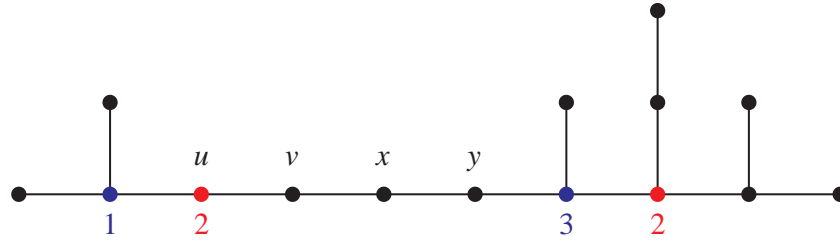


Figure 6.11: A shadow tree of the tree in Figure 6.9

6.4 A Geometrical Interpretation of the Characterization

Let $P = v_1, \dots, v_n$ be a diametrical path of the tree T . We define the *shadow tree* S_T of T with respect to P as follows. For each $v_i \in V(P)$, let V_i be the set of all vertices of T that are connected to v_i by a (possibly trivial) path internally disjoint from P . Let w_i be a vertex in V_i at maximum distance from v_i and let Q_i be the $v_i - w_i$ path in T . Then S_T is the subtree of T induced by $\bigcup_{i=1}^n Q_i$. Note that $\Delta(S_T) = 3$, and $\deg_{S_T} u = 3$ if and only if $u \in V(P)$ and $\deg_T u \geq 3$. The tree in Figure 6.11 is a shadow tree of the tree in Figure 6.9. Note that the shadow tree is not necessarily unique, and depends on the choice of P . For example, the tree in Figure 6.12 has many diametrical paths, and thus many shadow trees; two possible shadow trees are shown.

Since a set M of edges is a split- P set of T if and only if it is a split- P set of S_T , the next result follows immediately from Corollary 6.7(ii).

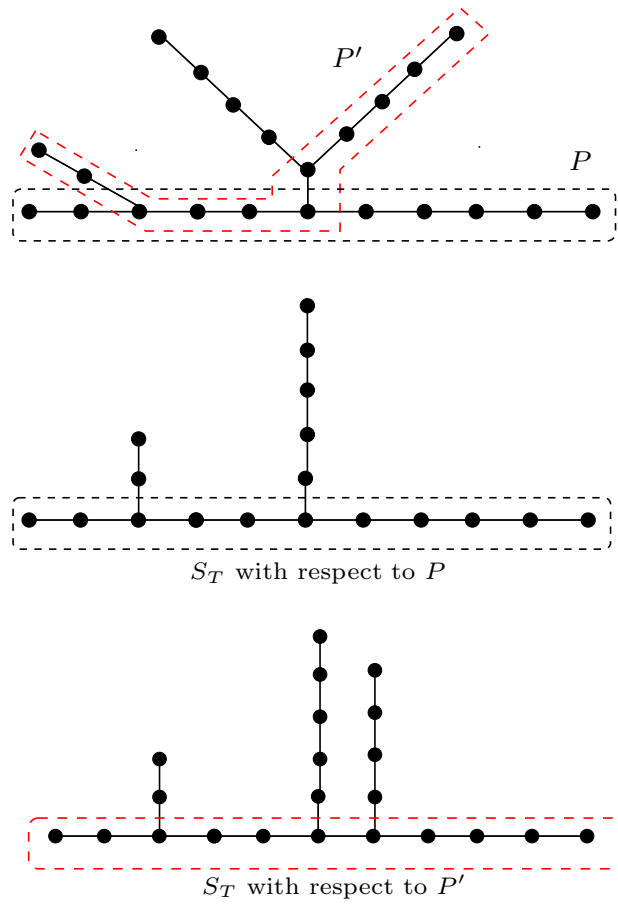


Figure 6.12: Tree with many diametrical paths

Corollary 6.9 For any tree T and any shadow tree S_T of T , $\gamma_b(T) = \gamma_b(S_T)$.

Let S_T be a shadow tree of T with respect to the diametrical path $P = v_1, \dots, v_n$ of T . Draw S_T in the positive $X - Y$ plane with P on the X -axis, v_1 at the origin, each edge having unit length, and each edge not on P being parallel to the Y -axis. Consider the vertices of S_T to have zero dimension – they are only points in the plane with integer coordinates. We call this

representation the *standard representation* of S_T . The shadow tree in Figure 6.11 is in standard representation. A region R in the $X - Y$ plane *covers* the vertices of S_T if each vertex of S_T lies on the boundary or in the interior of R .

Corollary 6.10 (i) *A central (bicentral, respectively) tree T with shadow tree S_T is nonradial if and only if the vertices of the standard representation of S_T can be covered by three (two, respectively) isosceles right triangles whose hypotenuses have even integer lengths and lie on the X -axis, one unit length apart, their vertices corresponding to vertices of S_T .*

(ii) *A tree T is radial if and only if the vertices of the standard representation of S_T cannot be covered by isosceles right triangles whose hypotenuses have even integer lengths that sum to less than $\text{diam}(T)$.*

(iii) *Let ρ be the maximum number of isosceles right triangles that can be used to cover the vertices of S_T as described in (i). Then $\gamma_b(T) = \text{rad}(T) - \lfloor \frac{\rho}{2} \rfloor$.*

Proof. (ii) Suppose S_T is nonradial. Then S_T has a non-empty split-set M . Let f_M be the broadcast associated with M , where $V_{f_M}^+ = \{v_1, \dots, v_r\}$ are the central vertices of the components T_i of $S_T - M$. Then a right triangle with base $2f(v_i)$ and height $f(v_i)$ will cover $N_f[v_i]$ for all $1 \leq i \leq r$. Also, since

the i^{th} triangle has diameter equal to the $\text{diam}(T_i)$ and

$$\text{diam}(S_T) = \sum_{i=1}^r \text{diam}(T_i) + r - 1,$$

it follows that the sum of the diameters of the triangles is less than the diameter of S_T .

Next, suppose that the vertices of the standard representation of S_T can be covered by isosceles right triangles whose hypotenuses have even integer lengths that sum to less than $\text{diam}(T)$. Let T_1, \dots, T_r be the components of S_T contained within the r triangles. Then $r \geq 2$ and $\text{diam}(T_i)$ is even, so let u_i be the central vertex of T_i , $i = 1, \dots, r$. Define f on S_T by $f(u_i) = \text{rad}(T_i) = \frac{\text{diam}(T_i)}{2}$. Then $N_f[u_i] = T_i$, so f is a dominating broadcast with cost $\sigma(f) = \sum_{i=1}^r \frac{\text{diam}(T_i)}{2}$. Thus

$$\begin{aligned} \text{rad}(S_T) &= \left\lceil \frac{\text{diam}(S_T)}{2} \right\rceil & (6.13) \\ &= \left\lceil \frac{\sum_{i=1}^r \text{diam}(T_i) + r - 1}{2} \right\rceil \\ &> \frac{\sum_{i=1}^r \text{diam}(T_i)}{2} \\ &= \sigma(f). \end{aligned}$$

Therefore S_T is nonradial. By Corollary 6.9, the proof of (ii) is complete.

(i) The the vertices of S_T can be covered by isosceles right triangles whose hypotenuses have even integer lengths that sum to less than $\text{diam}(T)$ if and only if there are at least two triangles. But if T (and thus S_T) is central,

then there are at least three triangles, since otherwise $r = 2$ and $\text{diam}(S_T)$ is odd, which is a contradiction. Thus, the proof of (i) is complete.

(iii) If T is radial, then $\rho = 1$ and the result holds. So suppose that T is nonradial. Then $\rho > 1$ and T has a maximum split-set M of size $\rho - 1$. Then by Lemma 6.6 the broadcast f_M associated with M is a very efficient γ_b -broadcast and f_M has ρ broadcast vertices. Hence, by Corollary 6.7, $\gamma_b(T) = \text{rad}(T) - \lfloor \frac{\rho}{2} \rfloor$.

□

Thus the trees in Figure 6.13 are nonradial, while the trees in Figure 6.14 are radial.

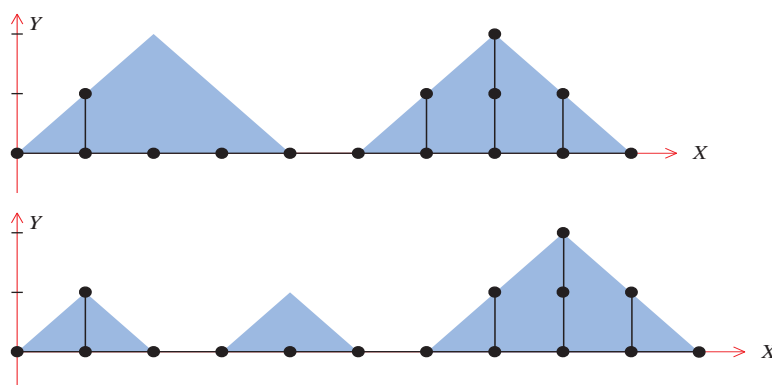


Figure 6.13: Vertices of nonradial trees covered by isosceles right triangles

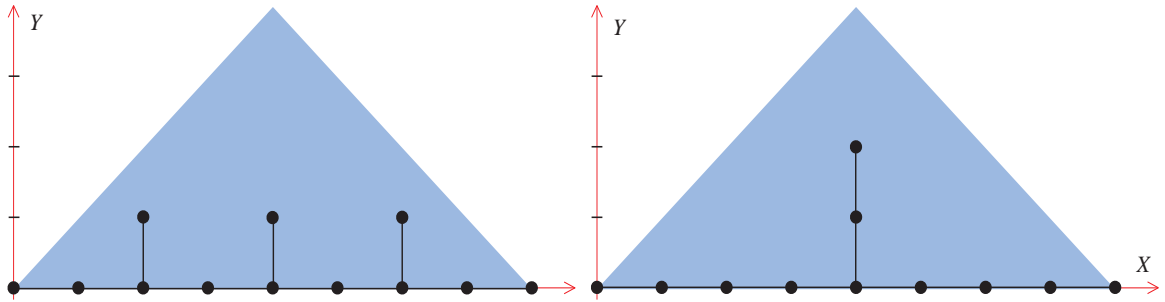


Figure 6.14: Vertices of radial trees covered by isosceles right triangles

6.5 Applications of Theorem 6.5

6.5.1 Generalized Coronas

Let G be any graph with $V(G) = \{v_1, \dots, v_n\}$, and let H_1, \dots, H_n be any graphs. Recall that the generalized corona $G \circ H_1, \dots, H_n$ is the graph obtained by joining v_i to all vertices of H_i , $i = 1, \dots, n$. As shown in [7], $G \circ H_1, \dots, H_n$ is radial for all graphs G and H_i (see Corollary 2.8). We obtain this result as a corollary to Theorem 6.5.

Corollary 6.11 *For any graph G of order n and any graphs H_1, \dots, H_n , the graph $G \circ H_1, \dots, H_n$ is radial.*

Proof. Let \mathcal{T} (\mathcal{S} , respectively) be the set of all spanning trees of G ($G \circ H_1, \dots, H_n$, respectively), let \mathcal{S}^* be the set of spanning subtrees of $G \circ H_1, \dots, H_n$ that consist of a tree in \mathcal{T} together with $|V(H_i)|$ leaves joined to v_i for each i , and let \mathcal{R} be the set of all trees that consist of a tree in \mathcal{T}

together with a leaf joined to v_i for each i . Then

$$\begin{aligned}
\gamma_b(G \circ H_1, \dots, H_n) &= \min_{T \in \mathcal{S}} \{\gamma_b(T)\} && \text{(Theorem 3.1)} \\
&\geq \min_{T \in \mathcal{R}} \{\gamma_b(T)\} && \text{(Proposition 2.3)} \\
&= \min_{T \in \mathcal{S}^*} \{\gamma_b(T)\} && \text{(Proposition 2.4)} \\
&\geq \min_{T \in \mathcal{S}} \{\gamma_b(T)\} && (\mathcal{S}^* \subseteq \mathcal{S}).
\end{aligned}$$

Therefore equality holds throughout and $\gamma_b(G \circ H_1, \dots, H_n) = \min_{T \in \mathcal{S}^*} \{\gamma_b(T)\}$. By Proposition 3.3 every graph has a spanning tree of the same radius, and for $G \circ H_1, \dots, H_n$ such a subtree is in \mathcal{S}^* . However, for any $T \in \mathcal{S}^*$ and any diametrical path $P = v_1, \dots, v_k$ of T , the only vertices that can possibly have degree two are v_2 and v_{k-2} . Hence T has no nonempty split-set and so is radial, by Theorem 6.5. Thus

$$\min_{T \in \mathcal{S}^*} \{\gamma_b(T)\} = \min_{T \in \mathcal{S}^*} \{\text{rad}(T)\} = \text{rad}(G \circ H_1, \dots, H_n)$$

and the result follows. □

6.5.2 Graphs with Radial Subtrees

Corollary 6.12 *Let G be a connected graph. If every spanning tree $T \in \mathcal{S}(G)$ is radial, then G is radial.*

Proof. Let $\text{rad}(G) = k$. For any $T \in \mathcal{S}(G)$, $\gamma_b(T) = \text{rad}(T) \geq k$. By Proposition 3.3, there exists $T' \in \mathcal{S}(G)$ with $\text{rad}(T') = k$, and T' has no nonempty split-set, so, by Theorem 6.5, $\gamma_b(T') = k$. Now, by Theorem 3.1, $\gamma_b(G) = \min_{T \in \mathcal{S}(G)} \{\gamma_b(T)\} = k$.

□

Note that the converse of Corollary 6.12 is false. In Figure 6.15, G is radial with radius 2, but T is a spanning subtree of G with a nonempty split-set $M = \{e\}$.

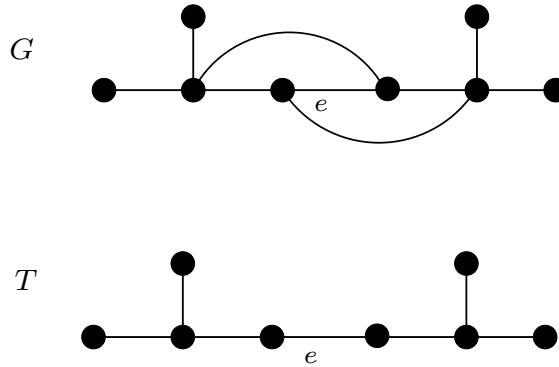


Figure 6.15: Counterexample to the converse of Corollary 6.12

6.5.3 Determining $\gamma_b(T)$

For a diametrical path $P = v_1, \dots, v_n$ of a tree T , let L_i denote the component of $T - v_i v_{i+1}$ that contains v_i , and $R_{i+1} = T - L_i$. We now describe an easy procedure to recursively determine a maximum split- P set $M = M_k$ of T . By Corollary 6.7 this also determines $\gamma_b(T)$.

Initially, let $T_0 = T$, $P_0 = v_{0,1}, \dots, v_{0,n_0}$ be a diametrical path of T_0 , and $M_0 = A_0 = \emptyset$.

Once the tree T_i , a diametrical path $P_i = v_{i,1}, \dots, v_{i,n_i}$ of T_i and the sets M_i and A_i have been constructed, construct T_{i+1} , P_{i+1} , M_{i+1} and A_{i+1} as follows:

1. Find the smallest odd integer t such that $\deg_{T_i} v_{i,t} = \deg_{T_i} v_{i,t+1} = 2$, and the paths $P_{i,t} = P_i \cap L_{i,t}$ and $P_{i,t+1} = P_i \cap R_{i,t+1}$ are diametrical paths of $L_{i,t}$ and $R_{i,t+1}$, respectively.
2. If
 - (a) $|V(P_{i,t+1})|$ is odd, then let $M_{i+1} = M_i \cup \{v_{i,t}v_{i,t+1}\} \cup A_i$, $T_{i+1} = R_{i,t+1}$, $P_{i+1} = P_{i,t+1} = v_{i+1,1}, \dots, v_{i+1,n_{i+1}}$, and $A_{i+1} = \emptyset$;
 - (b) $|V(P_{i,t+1})|$ is even, then let $M_{i+1} = M_i$, $T_{i+1} = R_{i,t+1}$, $P_{i+1} = P_{i,t+1} = v_{i+1,1}, \dots, v_{i+1,n_{i+1}}$, and $A_{i+1} = \{v_{i,t}v_{i,t+1}\}$.

Repeat 1 and 2 until no such integer t exists. The set $M = M_k$ is the last set M_{i+1} thus obtained.

6.5.4 An Interpolation Result

Consider $K_{1,n}$. It has a minimal dominating set of cardinality 1, and a minimal dominating set of cardinality n , and no minimal dominating set of any other size. The same is not true for broadcasts:

Proposition 6.13 *If k is any integer such that $\gamma_b(T) \leq k \leq \text{rad } G$, then T has a very efficient dominating broadcast of cost k .*

Proof. Consider a very efficient γ_b -broadcast $f = f_0$, and a diametrical path P of T . By Theorem 6.1, all broadcast vertices lie on P . If there are only two broadcast vertices, u_1 and u_2 , let $v \in \text{Cen}(T)$ and define the broadcast f_1 by $f_1(v) = f_0(u_1) + f_0(u_2) + 1$ and $f_1(x) = 0$ otherwise. If $|V_f^+| \geq 3$, let u_1, u_2 and u_3 be three consecutive broadcast vertices on P . Let P_1 be the subpath of P that is f -dominated by $\{u_1, u_2, u_3\}$ and note that P_1 has even length. Let v be the central vertex of P_1 and define f_1 by $f_1(v) = f_0(u_1) + f_0(u_2) + f_0(u_3) + 1$, $f_1(u_1) = f_1(u_2) = f_1(u_3) = 0$, and $f_1(x) = f_0(x)$ otherwise. Note that f_1 is a very efficient dominating broadcast of T with $\sigma(f_1) = \sigma(f_0) + 1$. Repeat the above steps until just one broadcast vertex remains. After the i^{th} step, the broadcast f_i is a very efficient dominating broadcast of T with $\sigma(f_i) = \sigma(f_{i-1}) + 1$. The process ends when the broadcast has only one broadcast vertex and cost $\text{rad}(T)$.

□

In Chapters 4 and 5 we considered special classes of trees; we compared and contrasted the properties of radial and nonradial trees. These properties

lead to further ideas and the thesis culminates in Chapter 6 with the characterization of radial trees. Many unsolved problems remain and we list some of them in Chapter 7.

Chapter 7

Future Research

After the characterization of radial trees there are some natural questions that arise. We conclude this thesis by describing a few related open problems for future research.

Problem 7.1 *Use the characterization of radial trees to find classes of radial graphs.*

A *uniquely radial tree* is a radial tree for which the only γ_b -broadcasts possible are such that $V_f^+ \subseteq \text{Cen}(T)$. For example, P_7 is not uniquely radial, but P_5 with a leaf added to the central vertex is uniquely radial.

Problem 7.2 *Characterize uniquely radial trees.*

Problem 7.3 *More generally, determine which trees have a unique γ_b -broadcast, and which trees have more than one γ_b -broadcast. Does the number of maximum split-sets play a role?*

By Corollary 4.3, a radial tree of radius k has at least $3k - 2$ vertices.

Problem 7.4 *Characterize radial trees with radius k and order $3k - 2$.*

Note that there does not exist a function $f(k)$ such that if T has radius k and at least $f(k)$ vertices, then T is radial, because for $k \geq 4$, P_{2k+1} together with any number of leaves added to its support vertices is nonradial.

Problem 7.5 *Characterize the class of trees T with $\gamma_b(T) = \gamma(T)$.*

Problem 7.6 *Characterize the class of trees T of order n with $\gamma_b(T) = \lceil \frac{n}{3} \rceil$.*

Problem 7.7 *Characterize the class of graphs G of order n with $\gamma_b(G) = \lceil \frac{n}{3} \rceil$, or find classes of such graphs.*

Problem 7.8 *Determine whether Proposition 6.13 holds for graphs in general. Does it hold for special classes of graphs?*

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