OPTIMAL INVESTMENT IN THE PROTECTION OF A VULNERABLE BIOLOGICAL RESOURCE

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ABSTRACT

It is assumed that the probability of destruction of a biological asset by natural hazards can be reduced through investment in protection. Specifically a model, in which the hazard rate depends on both the age of the asset and the accumulated invested protection capital, is assumed. The protection capital is subject to depreciation through time and its effectiveness in reducing the hazard rate is subject to diminishing returns. It is shown how the investment schedule to maximize the expected net present value of the asset can be determined using the methods of deterministic optimal control, with the survival probability regarded as a state variable. The optimal investment pattern involves "bang–bang–singular" control. A numerical scheme for determining jointly the optimal investment policy and the optimal harvest (or replacement) age is outlined and a numerical example involving forest fire protection is given.

Keywords: Catastrophe; Protection Investment; Hazard Rate; Optimal Control; Forest Fire; Pests; Biological Resources.

1. INTRODUCTION

Most living resources are vulnerable to catastrophic destruction in one form or another. Fire can destroy rangeland and forest. Pest infestation can cause equally catastrophic damage to orchards, vineyards and plantations as well as to forests. All organisms are vulnerable to disease with commercial species of trees, fish and livestock being no exceptions. Windstorms can cause widespread destruction to forests and orchards, while unexpected frosts, hailstorms etc. frequently destroy crops. Unusual events in the aquatic environment can have equally catastrophic effects on fish populations, perhaps the best–known example of this being the effect of the El Niño incursion on the Peruvian anchoveta fishery in the early 1970's.

Most of these hazards are unpredictable. Some are unavoidable. For others however some degree of protection is possible provided that adequate resources are committed to the task. For example fruit trees can be protected against pests and fungal diseases through routine sprayings, and forests can be protected to some extent against fire, through expenditure on fire protection measures. Given that such protection is possible a question of some interest is how, optimally, financial resources should be allocated to the task of protection?

This question has been examined mathematically in two recent papers [Reed, 1987(a) & (b)]. The first paper deals with the optimal expenditure on fire protection, through time, for an even–aged forest stand, which produces revenue only when the stand is clear–cut harvested. The second paper extends the analysis to other resources capable of generating revenue continuously as well as possibly at some terminal time. Examples include orchards, vineyards, dairy herds etc., vulnerable to destruction through pest and disease, as well as uneven–aged forest stands and rangeland vulnerable to fire. In addition the model developed can be used to determine the optimal preventive maintenance schedule and replacement age for a machine subject to breakdown, a problem for which a considerable literature exists (see e.g. Kamien & Schwartz [1971], Alam, Lynn & Sarma [1977], Feichtinger [1983] and references therein). The optimal protection of exhaustible resources, such as topsoil, has recently been studied by Lamberson [1987].

In all of the above work it is assumed that the current probability of destruction or breakdown (the hazard rate) depends on the *current* expenditure on protection measures, as well as possible on the age of the asset. While this may be reasonable in many cases, it may be a less than adequate assumption in others, in so far as the probability of destruction may depend more on the accumulated investment in protection measures rather than on current expenditure. For example the probability of destruction of a forest by fire may be reduced permanently by the construction of fire–breaks, or by investment in observation towers, lightning–strike detectors etc. Similarly, inoculations of livestock may provide more or less permanent immunity against certain diseases. It is the purpose of this paper to consider an optimal protection model in which the hazard rate for destruction depends on the accumulated investment in protection, as well as, possibly, on the age of the asset. The possibility of natural depreciation in the invested protection capital is included.

In Section 2 the model is described and it is shown how the problem of determining the optimal investment pattern, and the optimal harvest age for the asset, can be formulated as one of deterministic optimal control, with two state variables, one representing the accumulated protection investment, and the other the survival probability. The optimal investment pattern is shown to be of a "bang–bang–singular" nature.

In Section 3, a special case of the model in which revenue and hazard rate functions are constant, is considered. Some simple numerical examples are given. In Section 4 a numerical scheme for determining the optimal protection investment, in the general case, is outlined. This method is applied in a numerical example to determine the optimal protection investment and cutting age for an even–aged forest stand vulnerable to destruction by fire. Section 5 briefly summarizes the results.

In an appendix it is shown how the model of Reed [1987(a), (b)], in which the probability of destruction depends on current protection expenditure, can be thought of as a limiting case of the investment model of this paper.

2. A MODEL FOR OPTIMAL PROTECTION INVESTMENT

We shall use notation and concepts similar to those employed in Reed [1987(b)] and assume that when the asset or resource is alive at age (or time) t it is capable of producing revenue continuously at the rate $r(t)$ dollars per unit time. Furthermore if the asset is harvested (to destruction) at age T, it will be assumed to produce a (once only) revenue (net of harvest costs) of size $R(T)$ dollars. If however the asset suffers catastrophic destruction (e.g. through fire, disease etc.) at some age t prior to the planned harvest age T it will be assumed to produce an instantaneous net revenue of size $q(t)$ $(< R(t))$, which can be thought of as representing a net salvage value. Note that $q(t)$ would be negative if the immediate costs associated with a catastrophe exceeded gross salvage value.

The probability of catastrophic destruction can be characterized by a *hazard rate function* $h(t)$:

$$h(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} P\,(\text{asset is destroyed between ages } t \text{ and } t + \Delta \,|\, \text{asset not destroyed by age } t) \tag{1}$$

Suppose that when the accumulated capital investment in protection is at the level K dollars, the hazard rate is modified to

$$h_p(t) = \psi(K)h(t), \tag{2}$$

where $\psi(K)$ is a function which we shall call the *response function*, and which will be assumed to be positive, convex and decreasing with $\psi(0) = 1$. Thus diminishing returns to increases in protection investment are assumed.

The dynamics of investment will be assumed to follow the differential equation

(3) $$\frac{dK}{dt} = -\gamma K + I(t)$$

where I(t) is the investment rate ($ per unit time) at age t, and γ (assumed ≥ 0) is a parameter reflecting the rate of depreciation of invested protection capital.

The *survivor function*, $S_p(x)$, which gives the probability that the asset survives until age x, is related to the hazard rate by

(4) $$S_p(x) = \begin{cases} \exp\left\{-\int_0^x h_p(t)dt\right\} & x < T \\ 0 & x \geq T \end{cases},$$

Now let X be a random variable denoting the time at which the asset is destroyed, either through catastrophe or through a harvest. The total discounted net revenue (net of costs and protection investment expenditure) depends, of course, on X and is given by

(5) $$\begin{array}{ll} \int_0^X e^{-\delta t}[r(t) - I(t)]dt + e^{-\delta X} q(X), & \text{if } X < T \\ \int_0^T e^{-\delta t}[r(t) - I(t)]dt + e^{-\delta T} R(T), & \text{if } X = T \end{array}$$

where δ is the instantaneous discount rate. The *expected* net revenue, Π, earned from the asset, can be found by integrating (5), as a Stieljes integral, with respect to the complement of (4). After carrying out the integration and simplifying (see Reed [1987(a)] for details), this gives:

(6) $$\Pi = \int_0^T e^{-\delta t}[r(t) - I(t) - \delta q(t) + q'(t)]S_p(t)\, dt$$

$$+ e^{-\delta T}[R(T) - q(T)]S_p(T) + q(0).$$

If, as in Reed [1987(a)] we let

(7) $$y(t) = -\log S_p(t),$$

then

(8) $$\frac{dy}{dt} = \psi(k)h(t)$$

and

(9) $$\Pi = \int_0^T e^{-\delta t - y(t)}[r(t) - I(t) - \delta q(t) + q'(t)]dt$$

$$+ e^{-\delta T - y(T)}[R(T) - q(T)] + q(0).$$

To determine an optimal investment schedule and an optimal harvest age, we need to find T and $I(t) : 0 \le t \le T$, to maximize (9), subject to the dynamic constraints (8) and (3) and the control constraint:

(10) $$0 \le I(t) \le I_{max} ,$$

where I_{max} represents an upper limit on the rate at which investment can be made. If pulses of investment are possible then there would be no upper bound on $I(t)$, i.e.

$I_{max} = \infty$.

Letting

(11) $$\tilde{r}(t) = r(t) - \delta q(t) + q'(t),$$

and

(12) $$\tilde{R}(T) = R(T) - q(T).$$

The above problem can be expressed as the following deterministic optimal control problem, with free terminal time:

Maximize:

(13) $$\Pi = \int_0^T e^{-\delta t - y(t)}[\tilde{r}(t) - I(t)]dt + e^{-\delta T - y(T)}\,\tilde{R}(T) + q(0)$$

subject to:

(14) $$\frac{dK}{dt} = -\gamma K + I(t)$$

(15) $$\frac{dy}{dt} = \psi(K)h(t)$$

and,

(16) $$0 \leq I(t) \leq I_{max}\,.$$

Standard methods (e.g. the Pontryagin maximum principle) can be used to solve this problem. The problem differs from the optimal control problem that arises in Reed [1987(a),(b)] (and in Kamien & Schwartz [1971], Feichtinger [1983] etc.) in so far as there is a second state variable K (with associated dynamic equation (3)), with the evolution of the survival probability (y–process) dependent on this second state variable rather than directly on the control variable. However this complication is offset by the fact that the optimization problem is now linear in the control I; it follows directly that the optimal policy will involve only bang–bang and singular controls.

The *Hamiltonian function* for this problem (see e.g. Kamien & Schwartz [1981]) is

(17) $$H = [\tilde{r}(t) - I(t)]e^{-\delta t - y(t)} + \lambda_1 \psi(K)h(t) + \lambda_2\left[-\gamma K(t) + I(t)\right]$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are *adjoint variables* satisfying the *adjoint equations*:

(18) $$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial y} = [\tilde{r}(t) - I(t)]e^{-\delta t - y(t)}$$

and

(19) $$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial K} = -\lambda_1 \psi'(K)h(t) + \gamma\lambda_2 ,$$

with terminal (*transversality*) conditions:

(20) $$\lambda_1(T) = \frac{\partial}{\partial y(T)}[e^{-\delta T - y(t)}\tilde{R}(T) + q(0)] = -e^{-\delta T - y(T)}\tilde{R}(T)$$

and

(21) $$\lambda_2(T) = \frac{\partial}{\partial K(T)}[e^{-\delta T - y(T)}\tilde{R}(T) + q(0)] = 0.$$

The Hamiltonian is linear in the control I, and can be written:

(22) $$H = [\lambda_2 - e^{-\delta t - y(t)}]I(t) + \text{other terms not involving } I(t).$$

Thus the optimal control is bang–bang–singular with the switching function given by

(23) $$\sigma(t) = \lambda_2 - e^{-\delta t - y(t)} .$$

Whenever the switching function is positive investment will take place at the maximum rate until the adjustment to $\sigma(t) = 0$ is achieved. In contrast if the switching function is negative there will be no investment until such an adjustment is accomplished. Thus the optimal investment plan prescribes investment at age t if the expected discounted marginal benefit of a dollar invested in protection at that age (i.e. *the shadow value* of protection capital, $\lambda_2(t)$) exceeds the expected discounted marginal cost (i.e. $e^{-\delta t}S_p(t)$). Furthermore such investment will continue at the maximum rate until the shadow price drops (which from (19) it will do) to the level where it equals expected marginal cost. The converse result holds when the shadow value exceeds expected marginal cost. On an interval of singular control, $\sigma(t) \equiv 0$, and $d\sigma(t)/dt \equiv 0$, so that on such a singular path,

(24) $$\dot{\lambda}_2 + (\delta + \dot{y})e^{-\delta t - y(t)} \equiv 0 .$$

Using the adjoint equation (19) and the dynamic equation (15) and the fact that $\sigma(t) \equiv 0$ on the singular path, it follows that on the singular path:

(25) $$\psi'(K)h(t)\lambda_1 = [\gamma + \delta + \psi(K)h(t)]e^{-\delta t - y(t)} .$$

Differentiating this equation with respect to t, and using the adjoint equation (18), the equation (25) and the state equations (14) and (15) gives an explicit differential equation for the level of capital investment K(t) on the singular path. It is:

$$(26)\quad \left\{\frac{\psi''(K)}{\psi'(K)}(\gamma+\delta+\psi(K)h(t)) - 2\psi'(K)h(t)\right\}\frac{dK}{dt}$$

$$= \gamma K\, \psi'(K)h(t) - \tilde{r}(t)\psi'(K)h(t) - \frac{\dot{h}(t)}{h(t)}(\delta+\gamma)$$

$$-(\delta+\psi(K)h(t))(\gamma+\delta+\psi(K)h(t)).$$

The optimal investment policy will in general comprise periods of investment at the maximum rate I_{max}, periods of singular control when equation (26) is satisfied, and periods of zero investment. In general it is difficult to determine the precise nature of the optimal policy. The reason for this is that it may not be feasible to stay on the singular path, either because it is decaying at a proportional rate faster than γ (i.e. the singular path has $dK/dt < -\gamma K$), or because it is growing at a proportional rate faster than $I_{max}/K - \gamma$. In this case a problem with *blocked intervals* (see e.g. Clark, [1976, p. 56]) would arise. Rather than attempt to deal with the problem in its full generality we shall simplify the problem to avoid the possibility of such complications. Specifically we shall assume there is no upper bound on the rate of investment $(I_{max} = \infty)$ so that discontinuous impulses of investment are permitted, and further that the singular path equation (26) always has $dK/dt > -\gamma K$. This latter assumption is the same as the assumption that the optimal policy would not involve disinvestment even in the case of perfectly malleable capital. It should hold in most cases, except possibly when h(t) drops very rapidly $(\dot{h}(t)/h(t)$ large negative) or $\tilde{r}(t)$ is very large.

Under these assumptions the optimal investment policy will result in a trajectory

$K^*(t)$ for the protection capital, following a path like that depicted in either Fig. 1(a) or Fig. 1(b). In both cases, at some age t_2 prior to the optimal harvest age T^*, it is optimal to stop further investment (set $I(t) \equiv 0$). This follows from the continuity of λ_2 and the transversality condition (21), which together indicate that the switching function (23) will be negative for some interval before the harvest age T^*. In Fig. 1(a) the optimal policy involves an initial pulse of investment of size I_0, followed by the level of investment required to stay on the singular path until the switch–off time t_2. In Fig. 1(b) the optimal policy involves initially a period of no investment, followed by investment just sufficient to stay on the singular path between ages t_1 and t_2.

In order to determine the optimal harvest time T^*, a further transversality condition is required. This is the *free terminal time condition* (e.g. Kamien & Schwartz [1981, p. 148])

(27) $$H(T) + \frac{\partial}{\partial T}\,[\tilde{R}(T)\, e^{-\delta T - y(T)}] = 0$$

which taken in conjunction with (20) and (21) requires that at the optimum harvest age T^*

(28) $$\tilde{r}(T^*) + \tilde{R}'(T^*) - \delta\tilde{R}(T^*) = 0,$$

or, upon using (11) and (12) that

(29) $$r(T^*) + R'(T^*) - \delta R(T^*) = 0.$$

Under the assumption that the optimal investment trajectory is of the form depicted in Fig. 1(a) or (b), the three transversality conditions (20), (21) and (29) provide sufficient information to determine the three unknown parameters: t_2 (the switch–off age); T^* (the harvest age); and either I_0 (the initial investment) or t_1 (the time of first investment). However numerical methods are required to accomplish this. We defer a discussion of this until Section 4.

3. TIME–HOMOGENEOUS MODELS

In this section a simple special case is considered in which the revenue functions and hazard rate function are assumed constant (independent of age), i.e.

$$\tilde{R}(t) \equiv \bar{R} ; \qquad \tilde{r}(t) \equiv \bar{r} ; \qquad h(t) \equiv \bar{h} .$$

This includes, for example, the situation of a machine capable of producing a constant revenue ($r(t) \equiv$ constant), suffering no detioration in reliability ($h(t) \equiv$ constant) and with constant resale and salvage values ($R(t) \equiv$ constant, $q(t) \equiv$ constant). Another example might be that of a rangeland which from year to year is capable of producing a constant economic rent but which also suffers a constant vulnerability to destruction by fire.

Clearly in this case, at <u>any</u> time t given that the asset is alive, one faces an identical problem in determining the optimal level of capital investment. It follows that this optimal level must be constant over time. There is no optimal "harvest" (or selling) age, T^* (no solution to (28)). Furthermore the optimal level of investment must either follow the singular path (26) or be zero. Thus the optimal investment policy is either:

(a) do not invest in protection; $I^*(t) \equiv 0$ and $K^*(t) \equiv 0$, or

(b) initially invest to the level K^* which satisfies:

$$\gamma \bar{h} K^* \psi'(K^*) - \bar{r} \bar{h} \psi'(K^*) - (\delta + \bar{h}\psi(K^*))(\gamma + \delta + \bar{h}\psi(K^*)) = 0 \qquad (30)$$

and subsequently invest at the rate $I^*(t) \equiv \gamma \kappa^*$ to maintain capital at level K^*.

As an example consider (c.f. Reed [1987(b), p. 248] the case in which:

$$\delta = .03; \quad \bar{h} = .01 \text{ per year}; \quad \bar{r} = \$1000 \text{ per year};$$

and

$$\psi(K) = e^{-\beta K}$$

with $\beta = 0.01$. Assume also that $\gamma = .02$ and $q(t) \equiv 0$. In this case the solution to (30) is $K^* = 418.32$. Thus optimally one initially invests $418.32 in protection and subsequently invests at the rate of $8.37 per year to maintain the protection capital at the $418.32 level. In consequence the hazard rate is reduced from .01 to .00015 and the expected net present value of the asset is increased from $25,000 to $32,469.

Automatic Replacement

Continuing with the time–homogeneous model we now consider the situation in which, once the asset has been destroyed, or has broken down, it is immediately replaced by another asset with similar characteristics. It will be assumed that all accumulated protection capital will be destroyed with the asset. Let C denote the cost of replacement, and J the optimal expected net present value over infinitely many cycles. As in Reed [1987(b)] we can think of the salvage value, after a breakdown as $q(t) \equiv J - C$, since after replacement future earnings have an expected present value of J. Thus the reward function $\tilde{r}(t)$ is constant and of the form

$$\tilde{r}(t) \equiv \bar{r} = r_0 - \delta(J-C)\,, \tag{31}$$

where r_0 is the constant rate at which revenue is earned. The optimal level of protection capital is at K^*, given by the solution to (30) with $\bar{r}$ as in (31). It also follows from (13) that

$$J = -K^* + \frac{\bar{r} - \gamma K^*}{\delta + \bar{h}\ \psi(K^*)} + (J-C)\ . \tag{32}$$

Equations (32) and (30) jointly determine K^* and J. Substituting for $\bar{r}$ from (32) in (30) gives

$$(K^*+C)\bar{h}\ \psi'(K^*) + \gamma + \delta + \bar{h}\ \psi(K^*) = 0 \tag{32}$$

which can be solved for K^*. Note that the optimal level of protection capital K^* does not now depend on r_0, the rate at which revenue is earned. This is because replacement is assumed to be immediate, thereby assuring a constant flow of revenue regardless of the frequency of breakdowns and replacements.

As an example, suppose replacement costs are $C = \$2,000$, with other parameter values as before. The optimal initial investment is (from (32)) \$140.64, and subsequent investment to maintain the protection capital at this level is at the rate of \$2.81 per year. In consequence the hazard rate is reduced from .01 to .0025 and the expected present value J, over infinitely many cycles, is increased from \$32,667 (no protection) to \$32,924 (optimal protection).

4. DETERMINATION OF THE OPTIMAL HARVEST AGE AND PROTECTION INVESTMENT SCHEDULE IN THE NON–HOMOGENEOUS CASE

As noted in Section 2, solution of the optimization problem, when any of the revenue or hazard functions are age–dependent, will in general require numerical methods. In this section we outline a numerical scheme for obtaining a solution and apply it to the problem of determining the optimal protection investment schedule and cutting age for a forest stand vulnerable to destruction by fire.

It was noted in Section 2 that provided certain plausible assumptions were met to avoid problems with blocked intervals, the optimal level of protection capital should follow a path like that depicted either in Fig. 1(a) or Fig. 1(b). In either case there are three unknown parameters to be determined, *viz.* T^*, t_2 and either I_0 or t_1 .

If the optimal protection pattern is as depicted in Fig. 1(a), then the singular path is followed from time 0 to t_2. The state variables K and y follow an evolution governed by the differential equations (26) and (15) respectively, with initial conditions $K(0) = I_0$ and $y(0) = 0$. Furthermore the values of the adjoint variables λ_1 and λ_2 can be determined on $[0,t_2]$ from (25) and from the fact that the switching function $\sigma(t)$ (in (23)) is equal to zero.

On the interval $[t_2,T^*]$ when optimally there is no further investment, the state variables follow

$$\frac{dK}{dt} = -\gamma K, \tag{33}$$

and

$$\frac{dy}{dt} = \psi(K)h(t), \tag{34}$$

while the adjoint variables follow

(35) $$\frac{d\lambda_1}{dt} = \tilde{r}(t)\, e^{-\delta t - y(t)} ,$$

(36) $$\frac{d\lambda_2}{dt} = -\lambda_1\, \psi'(K)h(t) + \gamma\, \lambda_2 \,.$$

At $t = T^*$ the three transversality conditions (20), (21) and (28) hold.

To determine the optimal values of I_0, t_2 and T^* the following scheme can be used. Given candidate values for I_0 (≥ 0) and t_2 (≥ 0) (26) and (15) can be solved numerically in a forward sense to obtain $K(t_2)$ and $y(t_2)$. These values determine (via (25) and (23)) $\lambda_1(t_2)$ and $\lambda_2(t_2)$. This enables the system (33)–(36) to be solved forward from these starting values, until the condition (28) is met. This determines a value of T^* and of $\lambda_1(T^*)$, $\lambda_2(T^*)$ and $y(T^*)$.

We require, from the transversality conditions (20) and (21), that:

(37) $$\lambda_1(T^*) + e^{-\delta T^* - y(T^*)}\, \tilde{R}(T^*) = 0,$$

and

(38) $$\lambda_2(T^*) = 0.$$

For given values of I_0 and t_2, values of the left–hand sides of (37) and (38) can be found numerically as outlined above. The roots I_0 and t_2 which solve the pair of non–linear equations (37) and (38) can be found numerically using a method such as Newton–Raphson with numerically computed derivatives (e.g. Press *et al.*, [1986, p. 269]) or some hybrid method such as that of Powell [1970].

If no positive solution can be found using this method, it may be because the optimal investment schedule is of the form illustrated in Fig. 1(b). In this case a method similar to the above can be used, except that one integrates (26) and (15) from t_1 to t_2 using initial values $y(t_1) = \int_0^{t_1} h(s)ds$ and $K(t_1) = 0$. A pair of values t_1 and t_2 which solve (37) and (38) will provide the optimum.

As an example we consider the problem of determining the optimal schedule of investment in fire protection for an even–aged forest stand. The net harvest revenue is assumed to be of the form

$$R(T) = b_1(1 - e^{-b_2 T})^{b_3} - C_1 \tag{39}$$

where

$$b_1 = 8.833 \times 10^5; \quad b_2 = .0516; \quad b_3 = 26.69; \quad C_1 = 10^5 .$$

This corresponds roughly to the net value of a 1000 ha stand of pure spruce (*Picea glauca* Moench Voss) on a good site in the Fort Nelson Timber Supply Area of northwestern British Columbia assuming the value of timber to be \$20 per m^3 and costs of harvesting to be \$100 per hectare. There is assumed to be no ongoing revenue $(r(t) \equiv 0)$, and no salvage value, although clean–up costs after a fire of size $c_2 = 2 \times 10^5$ (or \$200 per ha) are assumed to occur. Thus $q(t) \equiv -c_2$. The response function is assumed to be of the form $\psi(K) = \exp(-\beta K)$ with $\beta = 0.1$, the depreciation parameter is set at $\gamma = 0.02$, and the hazard rate is assumed constant with $h(t) \equiv 0.01$. The discount rate is set at $\delta = 0.03$.

Following the method outlined above, using NAG (Anonymous, 1984) Fortran subroutines the optimal trajectory for protection capital was found. It is as shown in Fig. 2. It involves an initial investment of size \$14,090, with the singular path being followed up until age 58.2 years. Henceforth no further investment occurs, with the capital depreciating slowly until the optimal cutting age at 76.6 years. The probability of the stand surviving until this age is 0.870. With no protection this probability would be only 0.465.

The above example is for a "once–and–for–all" forest (Wicksell paradigm). The extension to an "ongoing" forest (Faustmann paradigm) can be accomplished numerically by methods similar to those described in Reed [1987(a), p. 30].

5. SUMMARY AND CONCLUSIONS

It has been shown how the optimal pattern of investment in the protection of a vulnerable biological resource can be determined using the methods of optimal control theory. It is assumed that the hazard rate for catastrophic destruction can be reduced by capital investment in protection. The resulting hazard rate thus depends on both the age of the asset and the current accumulated level of capital invested. In this respect the model differs from that of Reed [(1987(a), (b)] in which the hazard rate depends on the age of the asset and the *current expenditure* on protection. In the investment model of this paper it is assumed that there is depreciation of invested protection capital, over time, and that there are diminishing returns in the effectiveness of such capital.

The optimal investment pattern is shown to be of a "bang–bang–singular" nature with a first–order differential equation for the singular path being derived. Except in simple time–homogeneous cases, numerical methods are required to determine precisely the optimal investment pattern. A numerical scheme for accomplishing this is outlined and an example given.

The model analyzed in this paper is a highly stylized one. In reality the responses to protection investment are likely to be complex and specific to the resource in question and the hazards it faces. For example sprayings and inoculations may be effective at some life–stages and seasons, but ineffective at others; forest fire protection may depend jointly on capital invested and on ongoing expenditure in protection etc., moreover forest fire protection plans will usually be made at the level of a multi–stand forest, rather than at the level of an individual stand. The results of this paper should be viewed in this light. Nonetheless it is hoped that the results and methodologies established may prove useful in the process of the development of more realistic and specific models for determining how best to allocate capital resources for the protection of vulnerable assets.

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APPENDIX

Relationship with the Protection Model in which the Hazard Rate Depends on Current Expenditure

The model developed in the body of the paper differs from that in Reed [1987(a) and (b)] in that the hazard rate depends on accumulated (but possibly depreciated) investment in protection rather than on the current expenditure on protection. One would expect that the latter model should arise as a limiting case of the investment model as the depreciation rate parameter γ tended to infinity.

Naively if one simply lets $\gamma \to \infty$ in the investment dynamic equation (14), one gets a solution $K(t) \equiv 0 + o(1)$, for all finite investment trajectories $I(t)$. In order to get $K(t)$ to converge to the current expenditure, a change of units is required. Specifically let

(A1) $$p(t) = I(t)/\gamma$$

and in order that revenues are measured in the same units, let

(A2) $$\tilde{v}(t) = \tilde{r}(t)/\gamma; \quad \tilde{V}(t) = \tilde{R}(t)/\gamma; \quad \text{and} \quad c(t) = q(t)/\gamma$$

The investment equation (14) then becomes

(A3) $$\frac{dK}{dt} = -\gamma K + \gamma p(t),$$

which as $\gamma \to \infty$ has the solution

(A4) $$K(t) = p(t) + o(1).$$

It follows that the survival probability evolves as

$$\text{(A5)} \qquad \frac{dy}{dt} = \psi(p(t))h(t) + o(1)$$

which is the model [Reed 1987(a), (b)] in which hazard rate depends on *current* protection expenditure.

If one now considers the limiting behavior of the singular path equation (26) as $\gamma \to \infty$, one gets:

$$\text{(A6)} \qquad \frac{\psi''(p)}{\psi'(p)}\frac{dp}{dt} = (p-\tilde{v}(t))\psi'(p)h(t) - \dot{h}(t)/h(t) - (\delta + \psi(p)h(t)) + o(1).$$

This equation is exactly the equation which determines the optimal expenditure in the model in which the hazard rate is dependent on current expenditure [Reed 1987(b); Eq. (16)]. That model therefore can be thought of as a limiting form of the investment model developed in Section 2.

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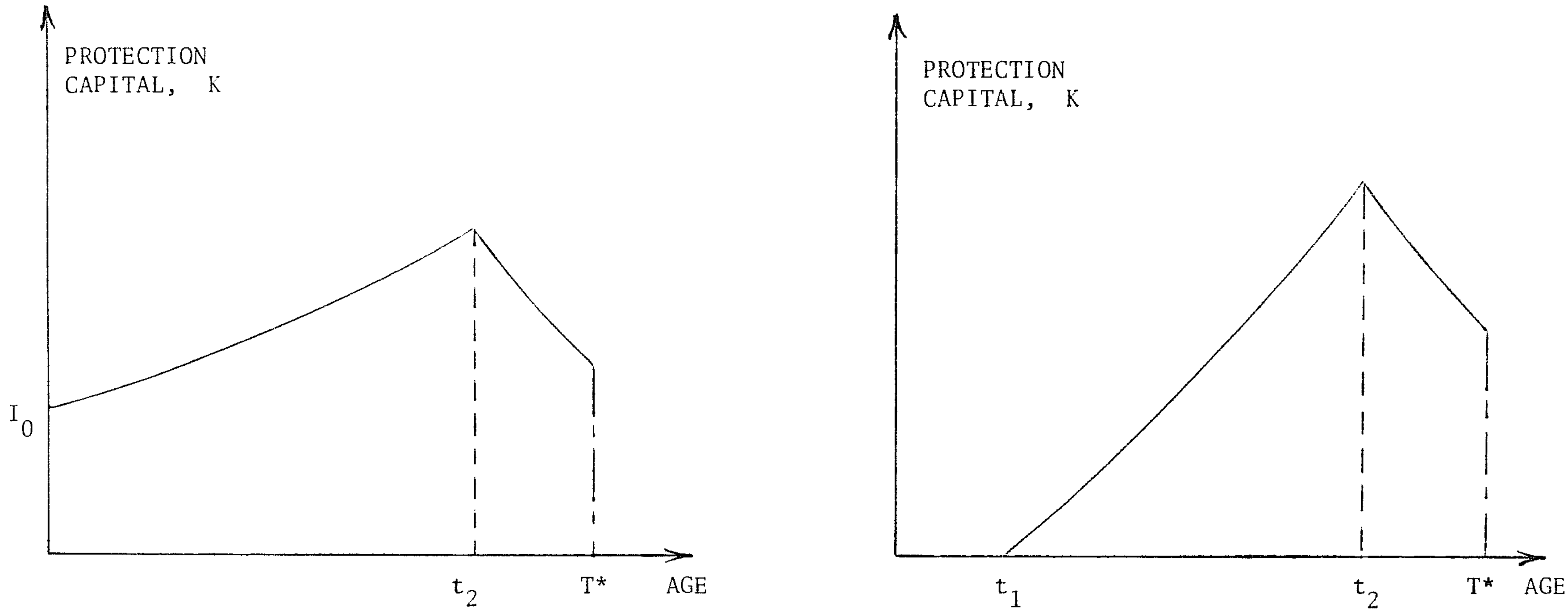
FIGURE CAPTIONS

Figure 1(a) and (b): Two possible trajectories for the optimal protection capital. In both cases there is no investment during a period immediately prior to the optimal harvest age, T*.

Figure 2: The trajectory of the optimal fire–protection capital (broken line) and protection investment flow (solid line) for the forestry example of Section 4. Units of age are years and units of capital and investment flow are thousands of dollars, and thousands of dollars per year respectively. There is no investment between the switch–off age of 58.2 years and the optimal cutting age of 76.6 years.

Figure 1.


PROTECTION
CAPITAL, K
I_0
t_2
T*
AGE
PROTECTION
CAPITAL, K
t_1
t_2
T*
AGE

(a)

(b)