

Smale Spaces with Totally Disconnected Local Stable Sets

by

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B.Sc., University of Winnipeg, 2005

M.Sc., University of Victoria, 2007

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ABSTRACT

A Smale space is a chaotic dynamical system with canonical coordinates of contracting and expanding directions. The basic sets for Smale's Axiom A systems are a key class of examples. R.F. Williams considered the special case where the basic set had a totally disconnected contracting set and a Euclidean expanding one. He provided a construction using inverse limits of such examples and also proved that (under appropriate hypotheses) all such basic sets arose from this construction. We will be working in the metric setting of Smale spaces, but the goal is to extend Williams' results by removing all hypotheses on the unstable sets. We give criteria on a stationary inverse limit which ensures the result is a Smale space. We also prove that any irreducible Smale space with totally disconnected local stable sets is obtained through this construction.

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Chapter 1

Introduction

All of our work deals with topological dynamical systems. A dynamical system is specified by a set X together with a map, f , from X to X . In our case, the topology on X will always come from a metric, d , which is simply a way to measure the distance between any two points in X . The dynamics of such a system refers to what happens to points $x \in X$ as f is applied to x repeatedly; we denote by $f^n(x)$ the image of x after n iterations of f have been applied to x . Dynamics are often used to model the time evolution of physical systems, such as the flow of water in a pipe. If a mapping f describes how a physical system changes from time $t = 0$ to $t = 1$, then then $f^n(x)$ gives us the state of a point x at $t = n$. This field of mathematics also has applications in many other disciplines, such as information technology and cognitive science.

However, the dynamical systems that we are going to be studying are chaotic, meaning that the long-term behaviour of a point is difficult to predict. As a result, looking at the orbits of individual points in the system is not practical. Instead we try to understand the global nature of the system. The introduction of chaos theory in the 1960s and 1970s caused a revolution in the physical and social sciences. While chaotic behaviour has long been observed in complex systems, such as weather, it was surprising to find chaos within almost trivial systems. Chaos involves a complicated mix of contraction and expansion. But this is not where the difficulty lies - it is the geometric rates at which the contraction and expansion happen that are hard to understand. Consider the example of taking a penny and every day doubling what you have; in a month, you'll have over ten million dollars. As humans, we're simply not familiar with that kind of evolution.

Manifolds are topological spaces that on a small enough scale look like \mathbb{R}^n , for some $n \geq 0$, where n is called the dimension of the manifold. For example, the surface

of the earth is a two-dimensional manifold since it can be portrayed by a collection of two-dimensional maps. Furthermore, manifolds typically come with a differentiable structure on which one can do calculus, as well as a metric with which to measure distances and angles. The study of dynamical systems on manifolds is a very classical subject with important applications and a history going back to Poincaré.

Hyperbolic dynamical systems are defined on manifolds, and model the properties seen in chaotic systems. In hyperbolic systems, the tangent space of the manifold can be divided into two parts: one on which the derivative of the map is contracting (stable direction) and one on which it is expanding (unstable direction). A dynamical system on a manifold possessing hyperbolic structure everywhere is called an Anosov system. One of S. Smale's great insights was that one should only expect hyperbolic structure where there is recurrence; that is, where after a sufficiently long time the system returns to a state very similar to its initial state. There are several types of recurrence and Smale identified the appropriate notion of recurrence in this case to be "non-wandering". The non-wandering set of a dynamical system is usually some kind of fractal. Smale [14] defined an Axiom A system as a dynamical system on a manifold where the non-wandering set has hyperbolic structure and is the closure of the periodic points. Smale's Spectral Decomposition Theorem states that Axiom A systems can be decomposed into finitely many "basic sets" with some desirable topological properties.

D. Ruelle [13] defined Smale spaces in an effort to axiomatize the topological dynamics of the basic sets of an Axiom A system. It is these spaces that we are concerned with. The idea of moving from an Axiom A system to a Smale space is motivated by the fact that the basic sets themselves are merely topological spaces and not submanifolds. This comes at the price of giving up the derivative, which some would regard as completely foolish if you want to discuss contractions and expansions. But for constructing examples as we will be doing, this is important as it can't be done with manifolds. It is an interesting question to ask whether or not our examples could actually be put into a manifold somehow. But that goes beyond the scope of this thesis.

It is well-known that all totally disconnected Smale spaces are shifts of finite type. And shifts of finite type are inverse limits of one-sided shifts of finite type, which were characterized by W. Parry [7] as positively expansive open mappings of compact, totally disconnected metrizable spaces. Since shifts of finite type are useful and well-understood systems, the natural next step is to consider Smale spaces which

are totally disconnected in only one direction and to work towards a characterization of these as inverse limits of spaces satisfying certain conditions.

R.F. Williams [18] looked at hyperbolic dynamical systems called expanding attractors. (In our research, we replace his expanding attractors with Smale spaces.) He defined an n -solenoid as a stationary inverse limit, where the space in the limit is a branched n -manifold. The stable sets of an n -solenoid are totally disconnected, and the unstable sets are Euclidean. His “construction” theorem states that under a certain technical condition, an expanding attractor is conjugate to an n -solenoid. And his “realization” theorem states that any n -solenoid is conjugate to an expanding attractor.

I. Yi [19] gave a topological development of the systems arising as Williams’ 1-solenoids by ignoring their differentiable structure. We take an altogether different approach.

Our results generalize those of Williams in that we do not put any restrictions on the local unstable sets. It should be noted, however, that he relied very heavily on the smooth structures of branched manifolds in his conditions and proofs, and to adapt to the metric setting of Smale spaces, we really needed a whole new set of ideas. We give criteria on a stationary inverse limit of a topological space which ensures that the result is a Smale space with totally disconnected local stable sets. Moreover, we prove that an irreducible Smale space with totally disconnected local stable sets is topologically conjugate to a stationary inverse limit. The space in the limit is a quotient of the original space. All of our results depend on a metric and do not involve differential topology.

The organization of this thesis is as follows.

In Chapter 2, we provide some background on Smale spaces. We also prove some technical results that we will need to prove our main results.

Chapter 3 is a review of Williams’ work on expanding attractors, including some examples and a statement of his results.

Chapter 4 contains a statement of our two main results, and some examples illustrating these results.

Chapters 5 and 6 contain the proofs of the first and second of our main results, respectively.

In Chapter 7, we outline some ideas of how our results could be applied in future research.

Throughout this work, $B(x, \epsilon)$ denotes the *closed* ϵ -ball centered at x .

Chapter 2

Introduction to Smale Spaces

In this chapter, we state the technical definition of a Smale space, an s -resolving factor map, a Markov partition for a Smale space, and a (stationary) inverse limit of topological spaces. We also prove some technical results that we will need in Chapters 5 and 6.

D. Ruelle's definition of a Smale space was motivated by what S. Smale called the basic sets of Axiom A systems. Let us review the definition of an Axiom A system.

Let (X, f) be a dynamical system. That is, X is a metric space, and $f : X \rightarrow X$ is continuous. The point $x \in X$ is said to be *non-wandering* if for any neighborhood U of x , there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. We denote by $\Omega(f)$ the set of all non-wandering points of X ; this set is closed and f -invariant. The mapping f is *topologically transitive* if for any open sets U and V in X , there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$.

Let M be a compact manifold with a Riemannian metric. For our purposes, it suffices to say that this means that we can measure the length, $|v|$, of any tangent vector v (for a thorough treatment of Riemannian geometry, see [4]). Let $f : M \rightarrow M$ be a diffeomorphism. A closed invariant subset $\Lambda \subseteq M$ is *hyperbolic* if the tangent bundle $T(M)$ restricted to Λ splits as a direct sum, $T(M)|_\Lambda = E^u \oplus E^s$, invariant under the derivative Df of f and such that $Df|_{E^u}$ is an expansion and $Df|_{E^s}$ is a contraction. That is, there exist constants $A, B > 0$ and $\mu > 1$ such that for all $n \in \mathbb{N}$, $v \in E^u$, and $w \in E^s$ we have $|Df^n(v)| \geq A\mu^n|v|$ and $|Df^n(w)| \leq B\mu^{-n}|w|$. Note that hyperbolicity is independent of the Riemannian metric used.

In [14], S. Smale defined an *Axiom A system* as a diffeomorphism $f : M \rightarrow M$ of a compact manifold which satisfies the following two properties:

1. the nonwandering set $\Omega(f)$ is hyperbolic, and
2. the periodic points of f are dense in $\Omega(f)$.

Theorem 2.1 (Smale’s Spectral Decomposition Theorem [14]). *Suppose (M, f) is an Axiom A system. Then there is a unique way of writing $\Omega(f)$ as the finite union of disjoint, closed, invariant indecomposable subsets (or “basic sets”) on each of which f is topologically transitive.*

Motivated by Smale’s observation that the basic sets need not be manifolds, Ruelle [12] introduced the notion of a Smale space as an attempt to axiomatize the topological dynamics on a basic set of an Axiom A system.

2.1 Definition of a Smale Space

Definition 2.2. *Let (X, d) be a compact metric space and $f : X \rightarrow X$ be a homeomorphism. The triple (X, d, f) is a Smale space if there exist constants $\epsilon_X > 0$ and $0 < \lambda < 1$, as well as a mapping*

$$[\cdot, \cdot] : \{(x, y) \in X \times X \mid d(x, y) \leq \epsilon_X\} \mapsto [x, y] \in X$$

satisfying properties (S1) through (S7) below. For $x \in X$ and $0 < \epsilon \leq \epsilon_X$, we denote

$$\begin{aligned} X^s(x, \epsilon) &= \{y \mid [x, y] = y, d(x, y) \leq \epsilon\}, \text{ and} \\ X^u(x, \epsilon) &= \{y \mid [y, x] = y, d(x, y) \leq \epsilon\}; \end{aligned}$$

these are called the local stable and unstable sets of x .

(S1) $[\cdot, \cdot]$ is continuous

(S2) $[x, x] = x$ for all $x \in X$

(S3) $[[x, y], z] = [x, z]$ whenever both sides are defined

(S4) $[x, [y, z]] = [x, z]$ whenever both sides are defined

(S5) $f([x, y]) = [f(x), f(y)]$ whenever both sides are defined

(S6) $d(f(y), f(z)) \leq \lambda d(y, z)$ if $y, z \in X^s(x, \epsilon_X)$

(S7) $d(f^{-1}(y), f^{-1}(z)) \leq \lambda d(y, z)$ if $y, z \in X^u(x, \epsilon_X)$

Note that the metric $d_2((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}$ gives the product topology on $X \times X \supset \text{Domain}([\cdot, \cdot])$.

Let us have a closer look at the “bracket map”, $[\cdot, \cdot]$, on a Smale space (X, d, f) . The continuity in (S1) is actually uniform continuity since the domain of $[\cdot, \cdot]$ is compact. So we can choose $0 < \epsilon \leq \epsilon_X$ such that if $d_2((x, y), (u, v)) \leq \epsilon$ then $d([x, y], [u, v]) \leq \epsilon_X$. Let us show that for $d(x, y) \leq \epsilon$,

$$X^s(x, \epsilon_X) \cap X^u(y, \epsilon_X) = \{[x, y]\}.$$

First, $[x, [x, y]] = [x, y]$ by (S4). Moreover, $d(x, [x, y]) = d([x, x], [x, y]) \leq \epsilon_X$ since $d(x, y) \leq \epsilon$, so $[x, y] \in X^s(x, \epsilon_X)$. Similarly $[x, y] \in X^u(y, \epsilon_X)$. On the other hand, suppose $u \in X^s(x, \epsilon_X) \cap X^u(y, \epsilon_X)$ as well. Then $[x, u] = u = [u, y]$, and hence $u = [u, u] = [[x, u], [u, y]] = [x, [u, y]] = [x, y]$. Specifically, for any $x \in X$, we have

$$X^s(x, \epsilon_X) \cap X^u(x, \epsilon_X) = \{x\}.$$

The given definition of the local stable and unstable sets is generally inconvenient to work with. By Ruelle [12], we can choose $\epsilon_X > 0$ small enough such that for $0 < \epsilon \leq \epsilon_X$,

$$X^s(x, \epsilon) = \{y \in X \mid d(f^n(x), f^n(y)) \leq \epsilon \forall n \geq 0\}, \text{ and} \quad (2.1)$$

$$X^u(x, \epsilon) = \{y \in X \mid d(f^{-n}(x), f^{-n}(y)) \leq \epsilon \forall n \geq 0\}. \quad (2.2)$$

We will always assume that ϵ_X was chosen like this. Notice then that $[\cdot, \cdot]$ is uniquely determined by (X, d, f) .

Moreover, Theorem 1.3.4 of [11] describes the local product structure given by $[\cdot, \cdot]$. This theorem involves a constant, and as seen in the proof of the theorem, this constant satisfies an additional technical property. We will need this property later, so we state it here along with the theorem.

Proposition 2.3. *There is $0 < \epsilon'_X \leq \epsilon_X/3$ such that, for every $0 < \epsilon \leq \epsilon'_X$, the map $[\cdot, \cdot] : X^u(x, \epsilon) \times X^s(x, \epsilon) \rightarrow X$ is a homeomorphism to its image, which is a neighbourhood of x . Moreover, for any $x, y \in X$ with $d(x, y) \leq \epsilon'_X$, we have $[x, y] \in X^s(x, \frac{1}{3}\epsilon_X) \cap X^u(y, \frac{1}{3}\epsilon_X)$.*

A homeomorphism $f : X \rightarrow X$ is said to be *expansive* if there exists a constant $c > 0$ such that for any two distinct points $x, y \in X$, there is some $n \in \mathbb{Z}$ such that

$d(f^n(x), f^n(y)) \geq c$. Any number $c > 0$ with this property is called an *expansiveness constant* for f .

Proposition 2.4. *Let (X, d, f) be Smale space. Then f is expansive, and the Smale space constant $\epsilon_X > 0$ is an expansiveness constant for f .*

Proof. Suppose $d(f^n(x), f^n(y)) < \epsilon_X$ for all $n \in \mathbb{Z}$. By (2.1) and (2.2), this means that $y \in X^s(x, \epsilon_X)$ and $y \in X^u(x, \epsilon_X)$. Hence $[y, x] = y = [x, y]$, so that

$$y = [y, y] = [[x, y], [y, x]] = [x, [y, x]] = [x, x] = x.$$

□

A dynamical system, (X, d, f) , is *non-wandering* if every point $x \in X$ is non-wandering; that is, if $\Omega(f) = X$. The *forward orbit* of a point $x \in X$ is defined as the set $\{f^n(x) \mid n \geq 0\}$. We say that (X, d, f) is *irreducible* if it is non-wandering and contains a dense forward orbit; this is equivalent to topological transitivity in the case where f is a homeomorphism. Recall that the basic sets of Axiom A systems are topologically transitive. Since Smale spaces are modelled on these, it seems natural to always require irreducibility; but this would rule out a lot of important examples, including many shifts of finite type. Furthermore, Theorem 3.3.1 of [11] states that any non-wandering Smale space (X, d, f) may be decomposed into finitely many clopen, pair-wise disjoint, f -invariant subsets on which the restriction of f is irreducible.

Corollary 3.2.5 of [11] gives the following important property of non-wandering Smale spaces.

Proposition 2.5. *If (X, d, f) is a non-wandering Smale space, then the set of periodic points is dense in X .*

2.1.1 Example 1: Hyperbolic Toral Automorphism

Let X denote the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. Together with the usual metric, X is compact. And the mapping induced by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is a homeomorphism on X since $\det A = -1$. The eigenvalues of A are γ and $-\frac{1}{\gamma}$, where $\gamma = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden mean. Let \mathbf{v}_1 be an eigenvector of length 1 corresponding to γ and \mathbf{v}_2 be an eigenvector of length 1 corresponding to $-\frac{1}{\gamma}$.

Let $q : \mathbb{R}^2 \rightarrow X$ denote the usual quotient map. Then for $\mathbf{v} \in X$ and $\epsilon > 0$, we have

$$X^s(\mathbf{v}, \epsilon) = \{q(\mathbf{v} + t\mathbf{v}_2) \mid |t| \leq \epsilon\}, \text{ and}$$

$$X^u(\mathbf{v}, \epsilon) = \{q(\mathbf{v} + t\mathbf{v}_1) \mid |t| \leq \epsilon\}.$$

For the Smale space constants, we can take $\epsilon_X = \frac{1}{2}$ and $\lambda = \frac{1}{\gamma}$. And $[\mathbf{x}, \mathbf{y}]$ is simply defined as the unique point in the intersection $X^s(\mathbf{x}, \frac{1}{2}) \cap X^u(\mathbf{y}, \frac{1}{2})$ for $d(\mathbf{x}, \mathbf{y}) \leq \frac{1}{2}$. See Figure 2.1.

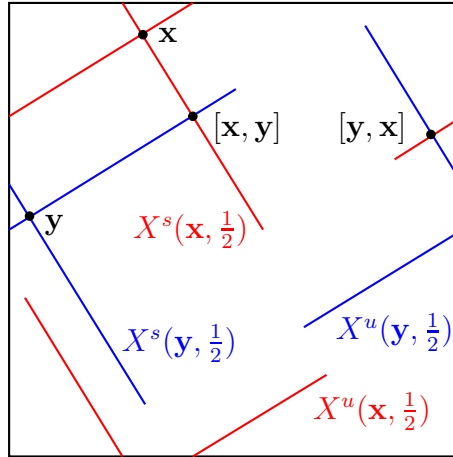


Figure 2.1: Canonical coordinates for (X, A)

2.1.2 Example 2: Shifts of Finite Type

There are several equivalent definitions of a shift of finite type; we will work with the following one.

A directed graph, G , consists of a set of vertices V and a set of directed edges E . Since the edges are directed, each edge $e \in E$ has an initial vertex $i(e)$ and a terminal vertex $t(e)$. We define Σ as the set of all bi-infinite paths in the graph $G = (V, E)$;

each element of Σ is an edge list. That is,

$$\Sigma = \{\mathbf{x} = (\cdots x_{-2}x_{-1} \cdot x_0x_1x_2 \cdots) \in E^{\mathbb{Z}} \mid t(x_n) = i(x_{n+1}) \forall n \in \mathbb{Z}\}.$$

For example, if G is the graph in Figure 2.2, then $\cdots e_1e_2 \cdot e_1e_2e_1 \cdots$ is an element of Σ while $\cdots e_3e_3 \cdot e_3e_3e_3 \cdots$ is not.

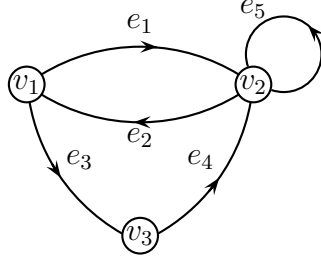


Figure 2.2: A directed graph

We give Σ the metric $d(\mathbf{x}, \mathbf{y}) = 2^{-\min\{|n| \mid x_n \neq y_n\}}$ for $\mathbf{x} \neq \mathbf{y}$. The left shift map $S : \Sigma \rightarrow \Sigma$ is given by

$$S : (\cdots x_{-2}x_{-1} \cdot x_0x_1x_2 \cdots) \mapsto (\cdots x_{-2}x_{-1}x_0 \cdot x_1x_2 \cdots),$$

and is clearly a homeomorphism on Σ . The pair (Σ, S) is a *shift of finite type*.

For $\mathbf{x} \in \Sigma$ and $m \in \mathbb{N}$,

$$\begin{aligned} \Sigma^s(\mathbf{x}, 2^{-m}) &= \{\mathbf{y} \in \Sigma \mid y_n = x_n \text{ for all } n > -m\} \\ &= \{(\cdots * * * x_{-m+1}x_{-m+2}x_{-m+3} \cdots) \in \Sigma\} \end{aligned}$$

and

$$\begin{aligned} \Sigma^u(\mathbf{x}, 2^{-m}) &= \{\mathbf{y} \in \Sigma \mid y_n = x_n \text{ for all } n < m\} \\ &= \{(\cdots x_{m-3}x_{m-2}x_{m-1} * * * \cdots) \in \Sigma\}. \end{aligned}$$

For $d(\mathbf{x}, \mathbf{y}) \leq \frac{1}{2}$ we have $x_0 = y_0$, and so

$$[\mathbf{x}, \mathbf{y}] = (\cdots y_{-2}y_{-1} \cdot x_0x_1x_2 \cdots).$$

Theorem 2.2.8 of [10] states that a Smale space is a shift of finite type if and only if it is totally disconnected. [6] is an excellent reference for shifts of finite type.

2.2 s -Resolving Factor Maps

Let (X, f) and (Y, g) be Smale spaces. A *factor map* is a continuous surjective function $\pi : X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow \pi & & \downarrow \pi \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes; that is, $g \circ \pi = \pi \circ f$. We will write this as $\pi : (X, f) \rightarrow (Y, g)$.

We say that the factor map π is *s-resolving* if $\pi|_{X^s(x, \epsilon)}$ is injective for every $x \in X$ and some $\epsilon > 0$. And we say that π is *finite-to-one* if there is a constant $m \in \mathbb{N}$ such that $\#\pi^{-1}\{y\} \leq m$ for every $y \in Y$. (The notation $\#A$ denotes the cardinality of the set A .) For a finite-to-one map π , we define the *degree* of π by $\deg(\pi) = \min\{\#\pi^{-1}\{y\} \mid y \in Y\}$.

Putnam [11] proves the following useful properties of s -resolving factor maps. We see from these properties that s -resolving factor maps are much nicer than general factor maps.

Proposition 2.6 (Putnam [11]). *Let $\pi : (X, f) \rightarrow (Y, g)$ be an s -resolving factor map between irreducible Smale spaces. Then*

1. π is a homeomorphism on the local stable sets $X^s(x, \epsilon)$,
2. π is finite-to-one, and
3. for every point y in Y with a dense forward orbit we have $\#\pi^{-1}\{y\} = \deg(\pi)$.

Furthermore, there exists $\epsilon_\pi > 0$ such that

4. for all $x_1, x_2 \in X$ with $d_X(x_1, x_2) \leq \epsilon_\pi$, we have $[x_1, x_2]$ and $[\pi(x_1), \pi(x_2)]$ both defined and

$$[\pi(x_1), \pi(x_2)] = \pi([x_1, x_2]),$$

5. if $\pi(x_1) \in Y^u(\pi(x_2), \epsilon_Y)$ and $d(x_1, x_2) \leq \epsilon_\pi$, then $x_1 \in X^u(x_2, \epsilon_\pi)$, and
6. if $x, x' \in X$ with $\pi(x) = \pi(x')$ and $\liminf_{n \rightarrow \infty} d(f^n(x), f^n(x')) < \epsilon_\pi$, then $x = x'$.

2.3 Markov Partitions

Let (X, d, f) be a Smale space, and let $[\cdot, \cdot]$ and $\epsilon_X > 0$ be as in Definition 2.2.

We will call a non-empty set $R \subseteq X$ is called a *rectangle* if $R = \overline{\text{Int}(R)}$ and $[x, y] \in R$ whenever $x, y \in R$. The second condition tells us that we must have $\text{diam}(R) \leq \epsilon_X$.

For a rectangle R and $x \in R$, we will denote $X^s(x, R) = X^s(x, \epsilon_X) \cap R$ and $X^u(x, R) = X^u(x, \epsilon_X) \cap R$.

A finite cover $\mathcal{P} = \{R_1, R_2, \dots, R_n\}$ of X by rectangles is a *Markov partition* provided that

1. $\text{Int}(R_i) \cap \text{Int}(R_j) = \emptyset$ for $i \neq j$, and
2. $f(X^s(x, R_i)) \subseteq X^s(f(x), R_j)$ and $f^{-1}(X^u(f(x), R_j)) \subseteq X^u(x, R_i)$ whenever $x \in \text{Int}(R_i) \cap f^{-1}(\text{Int}(R_j))$ (see Figure 2.3). This is called the ‘‘Markov property’’.

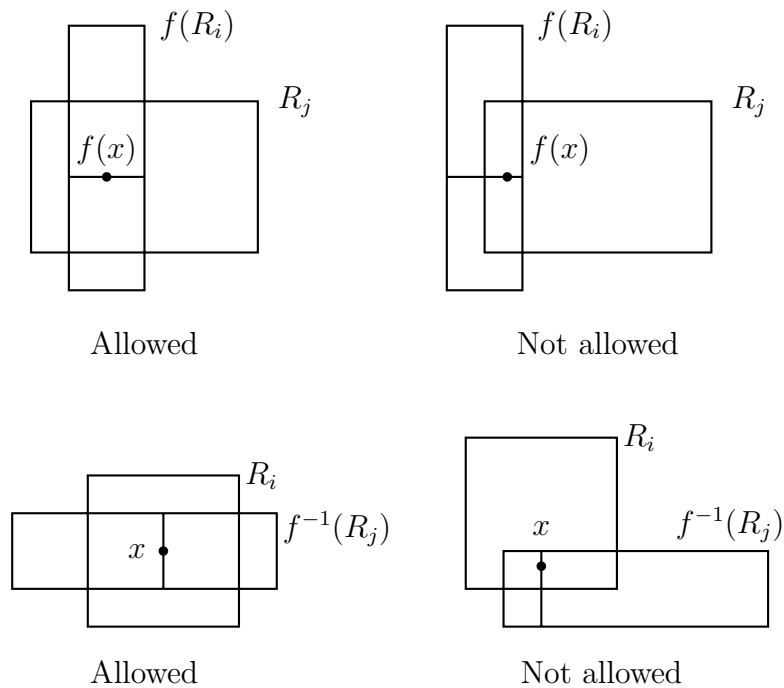


Figure 2.3: Markov partition rectangles

Bowen [3] proved that all irreducible Smale spaces have Markov partitions. But a generic Markov partition is not sufficient in our case. We need a Markov partition

where each rectangle has a clopen stable direction¹. We will use Corollary 1.3 of [9] to prove the existence of such a Markov partition.

Theorem 2.7 (Putnam [9]). *Let (X, f) be an irreducible Smale space such that $X^s(x, \epsilon)$ is totally disconnected for every $x \in X$ and $0 < \epsilon \leq \epsilon_X$. Then there is an irreducible shift of finite type (Σ, S) and an s -resolving factor map $\pi : (\Sigma, S) \rightarrow (X, f)$.*

The metric on Σ is given by $d_\Sigma(\mathbf{s}, \mathbf{t}) = \sum_{n \in \mathbb{Z}} 2^{-|n|} \chi(s_n, t_n)$, where

$$\chi(s_n, t_n) = \begin{cases} 0 & \text{if } s_n = t_n \\ 1 & \text{if } s_n \neq t_n \end{cases}.$$

This metric is equivalent to the one given in Section 2.1.2.

Proposition 2.8. *Let (X, f) be an irreducible Smale space such that $X^s(x, \epsilon)$ is totally disconnected for every $x \in X$ and $0 < \epsilon \leq \epsilon_X$. Then there exists a Markov partition, \mathcal{P} , for (X, f) such that if $x \in R \in \mathcal{P}$, then $X^s(x, R)$ is clopen in $X^s(x, \epsilon_X)$.*

Proof. Let $\pi : (\Sigma, S) \rightarrow (X, f)$ be the s -resolving factor map given by Theorem 2.7. By Proposition 2.6, π is finite-to-one. Let $d = \deg(\pi) \equiv \min\{\#\pi^{-1}\{x\} \mid x \in X\}$.

Let $\epsilon_\pi > 0$ be as in Proposition 2.6. Choose $N \in \mathbb{N}$ such that

$$\sum_{|n| > N} 2^{-n} < \epsilon_\pi.$$

Let P_{2N+1} be the set of all paths of length $2N + 1$ which appear in elements of Σ .

For $w \in P_{2N+1}$, let $R_w = \{\mathbf{a} \in \Sigma \mid a_{-N} \cdots a_N = w\}$. Then R_w is a clopen rectangle in Σ with diameter less than ϵ_π , and $\mathcal{P} = \{R_w \mid w \in P_{2N+1}\}$ is a Markov partition for (Σ, S) .

Since each $R_w \in \mathcal{P}$ is compact in Σ , it follows that $\pi(R_w)$ is compact in X , and hence closed. Moreover, since π is s -resolving and each R_w is clopen, it follows that each $\pi(R_w)$ is clopen in the stable direction. Let's show that $[x, y] \in \pi(R_w)$ whenever $x, y \in \pi(R_w)$. Suppose $x = \pi(\mathbf{a})$ and $y = \pi(\mathbf{b})$ for some $\mathbf{a}, \mathbf{b} \in R_w$. Since $\text{diam}(R_w) \leq \epsilon_\pi$, it follows from Proposition 2.6 (4) that we must have $[x, y] = [\pi(\mathbf{a}), \pi(\mathbf{b})] = \pi([\mathbf{a}, \mathbf{b}]) \in \pi(R_w)$.

We will show that a subset of

$$\{\pi(R_{w_1}) \cap \pi(R_{w_2}) \cap \cdots \cap \pi(R_{w_d}) \mid R_{w_1}, R_{w_2}, \cdots, R_{w_d} \in \mathcal{P} \text{ distinct}\}$$

¹For a constructive proof that such a Markov partition exists, see [15]

is a Markov partition for (X, f) .

Let us define a map $n : X \rightarrow \mathbb{N}$ by

$$n(x) = \#\{R_w \in \mathcal{P} \mid x \in \pi(R_w)\}.$$

Since the R_w are disjoint, it follows that

$$n(x) \leq \#\pi^{-1}\{x\} \tag{2.3}$$

for all $x \in X$.

We have the following estimate of continuity of n . Suppose we have a convergent sequence x_k with limit point x . Since each x_k lies in $n(x_k)$ elements of the finite set $\{\pi(R_w) \mid R_w \in \mathcal{P}\}$, we may pass to a subsequence where every term is contained in the same $\pi(R_w)$'s. Since they are closed, x also lies in these $\pi(R_w)$'s. Hence

$$n(x) \geq \limsup n(x_k). \tag{2.4}$$

Let us show that $n(x) \geq d$ for all $x \in X$, and that equality holds if x has a dense forward orbit. Let x be any point in X and let $x_0 \in X$ have a dense forward orbit (such a point exists since (X, f) is irreducible). By Theorem 2.6, $\#\pi^{-1}\{x_0\} = d$; let $\pi^{-1}\{x_0\} = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_d\}$. Choose a sequence of positive integers so that $f^{m_k}(x_0)$ converges to x . Pass to a subsequence where $S^{m_k}(\mathbf{a}_j)$ converges, for each $1 \leq j \leq d$. If two of the limit points (for different values of j) are in the same rectangle, then they are within ϵ_π of each other. So by Theorem 2.6 (6), these two \mathbf{a}_j 's are equal. Since this isn't the case, we see that no two limit points of the sequences can be in the same rectangle, but they all clearly lie in $\pi^{-1}\{x\}$. As a result, $n(x) \geq d$. It follows from (2.3) that $n(x_0) = d$.

That is, $n^{-1}\{d\}$ is non-empty and $n^{-1}\{k\}$ is empty for $k < d$. From (2.4) we also see that $n^{-1}\{d+1, d+2, \dots\}$ is closed and so $n^{-1}\{d\}$ is open. We claim it is also dense. But that follows from the fact that it contains all points with a dense forward orbit. One of them is enough, since each point in its forward orbit also has a dense forward orbit.

Let

$$\begin{aligned} \mathcal{S} = \{ \{R_{w_1}, R_{w_2}, \dots, R_{w_d}\} \subseteq \mathcal{P} \mid \exists x \in n^{-1}\{d\} \text{ with} \\ x \in \pi(R_w) \Leftrightarrow w \in \{w_1, w_2, \dots, w_d\} \}. \end{aligned}$$

We will show that

$$\mathcal{R} = \{\pi(R_{w_1}) \cap \pi(R_{w_2}) \cap \cdots \cap \pi(R_{w_d}) \mid \{R_{w_1}, R_{w_2}, \dots, R_{w_d}\} \in \mathcal{S}\}$$

is a Markov partition for (X, f) . We already observed above that each $\pi(R_w)$ is clopen in the stable direction; it is clear that a finite intersection of these sets would have the same property.

First we need to know that the elements of \mathcal{R} are rectangles. That they have dense interiors follows from the fact that $n^{-1}\{d\}$ is open and dense in X . Moreover, we observed above that for any $R_w \in \mathcal{P}$, we have $[x, y] \in \pi(R_w)$ whenever $x, y \in \pi(R_w)$.

That \mathcal{R} covers X and that the elements of \mathcal{R} have disjoint interiors also follows from the fact that $n^{-1}\{d\}$ is open and dense in X .

So it remains to prove that \mathcal{R} satisfies the Markov property. It suffices to prove this for the set of points in X with dense forward orbits, since these points (and their orbits) are clearly contained in the interiors of elements of \mathcal{R} .

Let $x \in \text{Int}(\bigcap_{i=1}^d \pi(R_{w_i})) \cap f^{-1}(\text{Int}(\bigcap_{i=1}^d \pi(R_{v_i})))$, where $\bigcap_{i=1}^d \pi(R_{w_i})$ and $\bigcap_{i=1}^d \pi(R_{v_i})$ are elements of \mathcal{R} . Since $\pi^{-1}\{x\} = \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$ and $n(x) = d$, it follows that for each \mathbf{a}_k there are $1 \leq i, j \leq d$ such that $\mathbf{a}_k \in R_{w_i} \cap S^{-1}(R_{v_j})$. Therefore $S(\Sigma^s(\mathbf{a}_k, R_{w_i})) \subseteq \Sigma^s(S(\mathbf{a}_k), R_{v_j})$ and $S^{-1}(\Sigma^u(S(\mathbf{a}_k), R_{v_j})) \subseteq \Sigma^u(\mathbf{a}_k, R_{w_i})$. Since π is a homeomorphism on local stable sets,

$$f(X^s(x, \pi(R_{w_i}))) = f(\pi(\Sigma^s(\mathbf{a}_k, R_{w_i}))) = \pi(S(\Sigma^s(\mathbf{a}_k, R_{w_i}))) \subseteq \pi(R_{v_j}).$$

And by Proposition 2.6 (5), we also have

$$\begin{aligned} f^{-1}(X^u(f(x), \pi(R_{w_j}))) &\subseteq f^{-1}(\pi(\Sigma^u(S(\mathbf{a}_k), R_{w_j}))) \\ &= \pi(S^{-1}(\Sigma^u(S(\mathbf{a}_k), R_{w_j}))) \\ &\subseteq \pi(R_{v_i}). \end{aligned}$$

Since $f(X^s(x, \epsilon_X)) \subseteq X^s(f(x), \epsilon_X)$ and $f^{-1}(X^u(f(x), \epsilon_X)) \subseteq X^u(x, \epsilon_X)$ hold trivially, we are done. \square

2.4 Inverse Limits

The general definition of an inverse limit involves a sequence of dynamical systems, but we only consider the case where this sequence is constant. Such an inverse limit is called “stationary”.

The *inverse limit* for a mapping $g : (Y, d) \rightarrow (Y, d)$ on a metric space is defined as

$$\begin{aligned} \varprojlim Y \xleftarrow{g} Y \xleftarrow{g} Y \xleftarrow{g} \cdots &= \{(y_0, y_1, y_2, \cdots) \mid y_n \in Y, y_n = g(y_{n+1}) \forall n \geq 0\} \\ &\subseteq \prod_{n \geq 0} Y. \end{aligned}$$

For ease of notation, we will denote $\varprojlim Y \xleftarrow{g} Y \xleftarrow{g} Y \xleftarrow{g} \cdots$ by \hat{Y} .

The mapping g induces a natural mapping on $\prod_{n \geq 0} Y$, namely

$$\hat{g}(y_0, y_1, y_2, \cdots) = (g(y_0), g(y_1), g(y_2), \cdots).$$

We give $\prod_{n \geq 0} Y$ the product topology. It is easy to see that \hat{Y} is a closed subspace. Moreover \hat{g} restricted to \hat{Y} is a homeomorphism, with inverse

$$\hat{g}^{-1}(y_0, y_1, y_2, \cdots) = (y_1, y_2, \cdots).$$

Chapter 3

Williams' Expanding Attractors

Since our results generalize R.F. Williams' work towards characterizing attractors with hyperbolic structure, we begin with an overview of Williams' results. To state the results of [18], we will need to look at his definitions of solenoids and expanding attractors first.

3.1 Solenoids

Before we get into Williams' abstract definition of an n -solenoid, let's consider some examples.

3.1.1 Example of a 1-solenoid

In [16], Williams constructs inverse limits from one-dimensional non-wandering sets. The following example belongs to this class.

Let Y be a wedge of two circles, a and b , joined at a single point v . Let both circles have circumference 1.

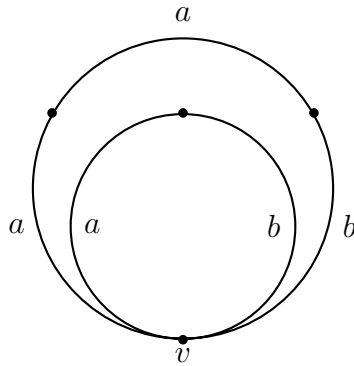
Divide a into thirds and b into halves. Let $g : Y \rightarrow Y$ be the map described by

$$a \mapsto aab$$

$$b \mapsto ab.$$

That is, g scales a by a factor of 3 and b by a factor of 2. See Figure 3.1.

It is easy to see that the image of any small ball is an arc.

Figure 3.1: $a \mapsto aab$, $b \mapsto ab$

The inverse limit,

$$\hat{Y} = \varprojlim Y \xleftarrow{g} Y \xleftarrow{g} Y \xleftarrow{g} \dots$$

together with the usual map, \hat{g} , is a 1-solenoid.

3.1.2 Example of a 2-solenoid

Anderson and Putnam [2] construct a 2-solenoid from the well-known Penrose tiling. The variation of this tiling that they use is the one with forty triangular prototiles. However, there are only two prototiles up to rotation. The substitution rule inflates the prototiles by a factor of $\phi = \frac{1+\sqrt{5}}{2}$ and subdivides as in Figure 3.2.

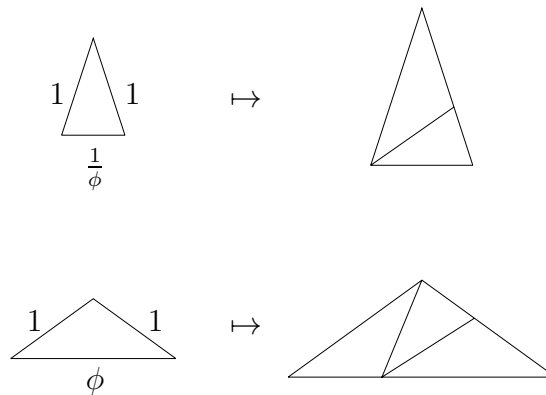


Figure 3.2: Penrose substitution

A cell complex, Γ_0 , is constructed by gluing together the forty prototiles in all the ways in which the substitution rule allows them to be adjacent. The substitution map induces a continuous surjection γ_0 on Γ_0 .

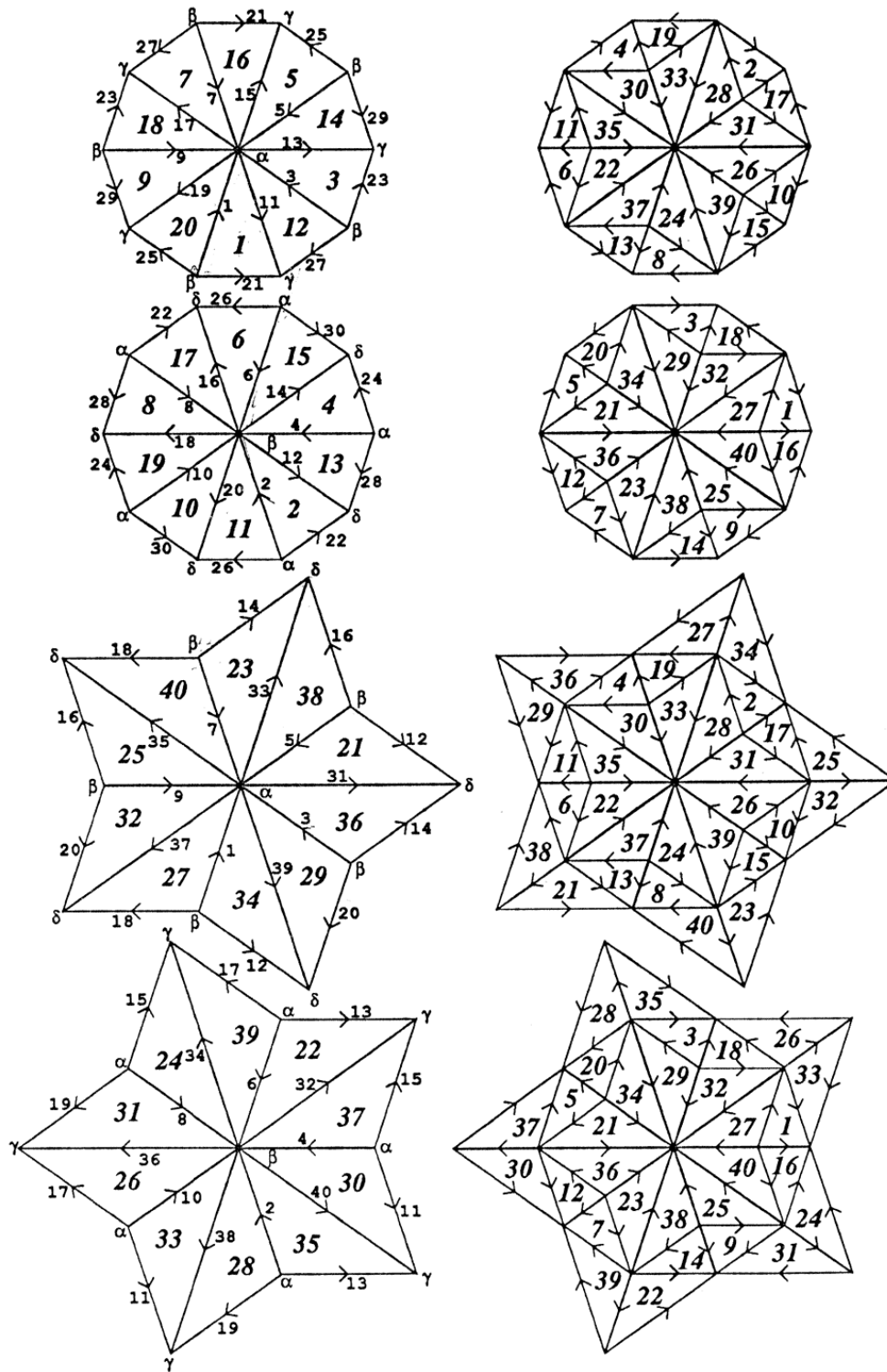


Figure 3.3: Cell complex (L) and substitution map (R) for the 40-prototile Penrose tiling [2]

The cell complex Γ_0 has 40 faces, 40 edges, and 4 vertices. The left side of Figure 3.3 shows how the faces are glued together, and the right side illustrates γ_0 . By identifying only the edges at the boundary of each group of ten tiles, one obtains $\mathbb{T}^2 \# \mathbb{T}^2$, a genus-two oriented surface.

The inverse limit

$$\widehat{\Gamma}_0 = \varprojlim \Gamma_0 \xleftarrow{\gamma_0} \Gamma_0 \xleftarrow{\gamma_0} \Gamma_0 \xleftarrow{\gamma_0} \dots,$$

together with the usual map, is a 2-solenoid.

3.1.3 Definition of an n -solenoid

Intuitively, smooth branched n -manifolds are “complexes” embedded in some higher dimensional Euclidean space, in such a manner that there is a unique tangent n -plane at each point. An abstract definition is given in [18]. The wedge Y in the first example is a branched 1-manifold with one branch point, and the cell complex Γ_0 in the second example is a branched 2-manifold.

An n -solenoid is an inverse limit

$$\widehat{K} = \varprojlim K \xleftarrow{g} K \xleftarrow{g} K \xleftarrow{g} \dots,$$

where K is a compact Riemannian branched C^r n -manifold and $g : K \rightarrow K$ is a C^r immersion¹ satisfying the following axioms:

1. $\Omega(g) = K$,
2. g is an expansion: there exist constants $A > 0$ and $\mu > 1$ such that for all $n \in \mathbb{N}$ and $k \in T(K)$, we have $|Dg^n(k)| \geq A\mu^n|k|$, where $T(K)$ is the tangent space of K and Dg is the derivative of g ,² and
3. for each $x \in K$ there is a neighborhood N of x and $j \in \mathbb{Z}$ such that $g^j(N)$ is contained in a subset diffeomorphic to an open ball in \mathbb{R}^n

¹An *immersion* is a differentiable map whose derivative is injective.

²The Riemannian structure on K means that we are measuring the length of the tangent vectors.

The map $\hat{g} : \hat{K} \rightarrow \hat{K}$ is the usual map

$$\begin{aligned}\hat{g}(x_0, x_1, x_2, \dots) &= (g(x_0), g(x_1), g(x_2), \dots) \\ &= (g(x_0), x_0, x_1, \dots).\end{aligned}$$

Consider the branched 1-manifold in the first example. We observe that something very subtle is happening at the branch point: a neighborhood is being flattened (Axiom 3) but also stretched (Axiom 2). Intuitively it seems that we shouldn't be able to have a flattening condition on an expanding map, since one's natural idea of expanding would imply that the map must be one-to-one. The way this seeming contradiction is resolved is that the derivative Dg cleverly fails to notice that the map g isn't one-to-one.

3.2 Expanding Attractors

Let M be a compact Riemannian manifold. For $r \geq 1$, the set of bijective maps $f : M \rightarrow M$ such that both f and f^{-1} are r times continuously differentiable, is denoted by $\text{Diff}^r(M)$.

A subset $\Lambda \subseteq M$ is an *expanding attractor* for $f \in \text{Diff}^r(M)$, $r \geq 1$, if there is a closed neighborhood N of Λ such that:

1. $f(N) \subseteq \text{Int}N$,
2. $\Lambda = \bigcap_{i \geq 0} f^i(N)$,
3. $\Lambda = \Omega(f|N)$,
4. Λ has a hyperbolic structure $E^u \oplus E^s$, and
5. the topological, or covering, dimension of Λ is the same as the linear dimension of a fibre of E^u .

In our own work, we replace expanding attractors with Smale spaces. One advantage of this angle is that we don't need to consider manifolds or derivatives.

3.3 The Main Results

Williams' Theorem A. *Each point of an n -solenoid has a neighborhood of the form $(\text{Cantor set}) \times (n\text{-disk})$.*

We observe that the Cantor set is the local stable set and n -disk is the local unstable set. In our own work, the local unstable sets may be anything. The following two results are the “construction” theorem and the “realization” theorem that we aim to generalize for Smale spaces.

Williams' Theorem B. *Let (\hat{Y}, \hat{g}) be an n -solenoid. Then there is a manifold M and $f \in \text{Diff}^r(M)$, $r \geq 1$, having an expanding attractor Λ such that $(\Lambda, f|_{\Lambda})$ is conjugate to (\hat{Y}, \hat{g}) .*

Williams' Theorem C. *Let Λ be an expanding attractor for $f \in \text{Diff}^r(M)$, $r \geq 1$, such that the foliation $\{W^s(x, f) \mid x \in \Lambda\} = \{y \in M \mid \lim_{m \rightarrow \infty} d(f^m(x), f^m(y)) = 0\}$ is C^1 on some neighborhood of Λ . Then $(\Lambda, f|_{\Lambda})$ is conjugate to an n -solenoid.*

Observe that in Theorems B and C, n is equal to the linear dimension of a fibre of E^u .

Chapter 4

Motivation and Statement of Results

Our aim is to generalize Williams' results on expanding attractors for topological systems with metrics. That is, we are looking for conditions on a pair (Y, g) such that the inverse limit is a Smale space with totally disconnected local stable sets; this is our analogy to Williams' Theorem B. Moreover, we would like to prove that every Smale space with totally disconnected local stable sets results from this construction; this is our analogy to Williams' Theorem C. By "generalize" we refer to the point that Williams' local unstable sets were Euclidean, and we drop this condition. However, we do not put the inverse limit into a manifold, and in this sense our results do not reflect Williams'.

4.1 Statement of Results

Recall that we are using the notation $B(x, r)$ to denote a closed ball.

Let (Y, d) be a compact metric space, and let $g : Y \rightarrow Y$ be continuous and surjective. We will say that (Y, d, g) satisfies Axioms 1 and 2 if there exist constants $\beta > 0$, $K \geq 1$, and $0 < \gamma < 1$ such that

Axiom 1 *if $d(x, y) \leq \beta$ then*

$$d(g^K(x), g^K(y)) \leq \gamma^K d(g^{2K}(x), g^{2K}(y)),$$

and

Axiom 2 for all $x \in Y$ and $0 < \epsilon \leq \beta$,

$$g^K(B(g^K(x), \epsilon)) \subseteq g^{2K}(B(x, \gamma\epsilon)).$$

Intuitively, Axioms 1 and 2 could be viewed as weakened versions of the conditions that g be locally expanding and open, respectively. Locally expanding would be Axiom 1 with the two g^K 's removed and the two g^{2K} 's replaced by g^K 's. And Axiom 2 with the first g^K removed and the g^{2K} replaced by g^K , would imply that g is open.

In addition, we define

Axiom 3 (Y, d, g) is non-wandering, and

Axiom 4 (Y, d, g) has a dense forward orbit.

Recall the inverse limit space (\hat{Y}, \hat{g}) defined in Section 2.4. We define a metric \hat{d} on \hat{Y} by

$$\hat{d}(\mathbf{x}, \mathbf{y}) = d'(\mathbf{x}, \mathbf{y}) + \gamma^{-1}d'(\hat{g}^{-1}(\mathbf{x}), \hat{g}^{-1}(\mathbf{y})) + \dots + \gamma^{-(K-1)}d'(\hat{g}^{-(K-1)}(\mathbf{x}), \hat{g}^{-(K-1)}(\mathbf{y})),$$

where $d'(\mathbf{x}, \mathbf{y}) = \sup_{n \geq 0} \{\gamma^n d(x_n, y_n)\}$.

The following two theorems are the main results of this thesis.

Construction Theorem. *If (Y, d, g) satisfies Axioms 1 and 2 then $(\hat{Y}, \hat{d}, \hat{g})$ is a Smale space with totally disconnected local stable sets. Moreover, $(\hat{Y}, \hat{d}, \hat{g})$ is a non-wandering Smale space if and only if (Y, d, g) also satisfies Axiom 3; and it is an irreducible Smale space if and only if (Y, d, g) also satisfies Axioms 3 and 4.*

Realization Theorem. *Let (X, d, f) be an irreducible Smale space with totally disconnected local stable sets. Then (X, d, f) is topologically conjugate to an inverse limit space $(\hat{Y}, \hat{\delta}, \hat{\alpha})$ such that (Y, δ, α) satisfies Axioms 1 and 2 (and hence 3 and 4).*

We note that the Realization Theorem could also be proved for non-wandering Smale space. The reason we need irreducibility is to obtain a certain Markov partition, and we pointed out earlier that any non-wandering Smale space may be decomposed into finitely many irreducible pieces.

The proof of the Realization Theorem is constructive. Let us consider what the inverse limit space looks like when we start with a shift of finite type as our Smale space.

Let (Σ, S) be the SFT on the directed graph $G = (V, E)$. For each $e \in E$, the set $R_e = \{x \in \Sigma \mid x_0 = e\}$ is a clopen rectangle. And $\mathcal{P} = \{R_e \mid e \in E\}$ is a Markov partition for (Σ, S) .

We define an equivalence relation

$$x \sim y \iff x_n = y_n, n \geq 0.$$

Observe that the quotient space X/\sim is simply the one-sided shift on G . We denote equivalence classes by $[[\cdot]]$. That is,

$$[[\cdots x_{-2}x_{-1} \cdot x_0x_1x_2 \cdots]] = (x_0x_1x_2 \cdots).$$

And the metric is the natural one, $\delta([[x]], [[y]]) = 2^{-\min\{n \geq 0 \mid x_n \neq y_n\}}$. We define $\alpha : X/\sim \rightarrow X/\sim$ by $\alpha([[x]]) = [[S(x)]]$, which is simply the usual left shift map.

We observe that the inverse limit

$$\begin{aligned} \widehat{X/\sim} &= \varprojlim X/\sim \xleftarrow{\alpha} X/\sim \xleftarrow{\alpha} X/\sim \xleftarrow{\alpha} \cdots \\ &= \{([x]_0, [x]_1, [x]_2, \cdots) \mid \alpha([x]_{n+1}) = [x]_n \forall n \geq 0\} \\ &= \left\{ \left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}, \begin{pmatrix} x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{pmatrix}, \begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \\ \vdots \end{pmatrix}, \cdots \right) \mid x \in \Sigma \right\} \end{aligned}$$

simply recovers Σ .

In the category of Smale spaces, the morphisms can be taken to be s -resolving factor maps. As we saw in Section 2.2, these maps have some very nice properties (for example, they are finite-to-one). Our results indicate that in such a category, the closest object to a shift of finite type is an inverse limit space $(\hat{Y}, \hat{d}, \hat{g})$, where (Y, d, g) satisfies Axioms 1-4.

Before we consider an example of a system satisfying Axioms 1 and 2, let us consider a system that fails Axiom 2.

4.2 An Example Failing Axiom 2

Let $\Sigma_{\{0,1\}}^+$ be the full one-sided shift on the symbol set $\{0,1\}$. In the terminology of Section 2.1.2, this is the set of all infinite paths in the graph consisting of one vertex and two edges, denoted 0 and 1, from that vertex to itself. Similarly, let $\Sigma_{\{0,2\}}^+$ be the full one-sided shift on the symbol set $\{0,2\}$. The metric on these one-sided shifts is analogous to the one on shifts of finite type: $d(\mathbf{x}, \mathbf{y}) = 2^{-\min\{n \mid x_n \neq y_n\}}$.

Let $Y = \Sigma_{\{0,1\}}^+ \cup \Sigma_{\{0,2\}}^+$, and g be the usual left shift map. Then g is clearly a continuous and surjective map on Y .

Let us show that Axiom 2 fails for (Y, d, g) . Choose $K \geq 1$, $N \geq 2K$ and $0 < \gamma < 1$. Consider the points $\mathbf{x}, \mathbf{y} \in Y$ given by

$$x_n = \begin{cases} 1 & \text{if } n = N + K \\ 0 & \text{if } n \neq N + K \end{cases}$$

and

$$y_n = \begin{cases} 2 & \text{if } n = N \\ 0 & \text{if } n \neq N \end{cases}.$$

Then $d(g^K(\mathbf{x}), \mathbf{y}) = 2^{-N}$. However, $g^K(\mathbf{y}) \notin g^{2K}(B(\mathbf{x}, \gamma 2^{-N}))$ since for any point $\mathbf{z} \in B(\mathbf{x}, \gamma 2^{-N})$ we have $g^{2K}(\mathbf{z})_{N-2K} = z_N = x_N = 0$.

It is easy to see that the inverse limit (\hat{Y}, \hat{g}) is conjugate to $(\Sigma_{\{0,1\}} \cup \Sigma_{\{0,2\}}, S)$, where $\Sigma_{\{0,1\}}$ and $\Sigma_{\{0,2\}}$ are the full two-sided shifts on their respective symbol sets. However this system is not a Smale space: we can find points $\mathbf{x} \in \Sigma_{\{0,1\}}$ and $\mathbf{y} \in \Sigma_{\{0,2\}}$ that are arbitrarily close, yet $[\mathbf{x}, \mathbf{y}] = (\cdots y_{-2}y_{-1} \cdot x_0x_1x_2 \cdots)$ is not a point in the space.

4.3 An Example Satisfying Axioms 1 and 2

We will construct a quotient space using six copies of the Sierpinski gasket. This example was suggested by Ian Putnam.

To construct the Sierpinski gasket, take an equilateral triangle in the plane, with side length 1. Remove the interior of the “middle triangle”, that is, the triangle whose vertices are the midpoints of the original triangle. This results in three equilateral triangles with side length $\frac{1}{2}$. For each of these three triangles, remove the interior of its “middle triangle”, resulting in nine equilateral triangles with side length $\frac{1}{4}$; and so on. The Sierpinski gasket is the intersection of this sequence of spaces. See Figure 4.1.

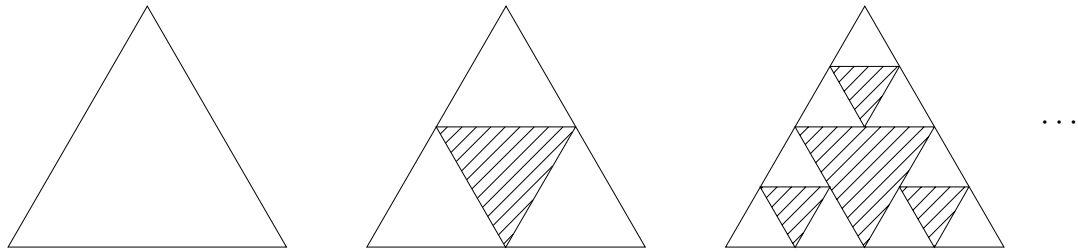


Figure 4.1: Construction of the Sierpinski gasket

Now take six copies, Y_1, Y_2, \dots, Y_6 , of the Sierpinski gasket with distinguished vertices, as in Figure 4.2. Let \sim be the equivalence relation on $Y_1 \cup Y_2 \cup \dots \cup Y_6$ identifying the six vertices labeled A , the six vertices labeled B , and the six vertices labeled C . We will not use $[\cdot]$ notation since the only equivalence classes containing more than one point are A , B , and C . We define

$$Y = (Y_1 \cup Y_2 \cup \dots \cup Y_6) / \sim.$$

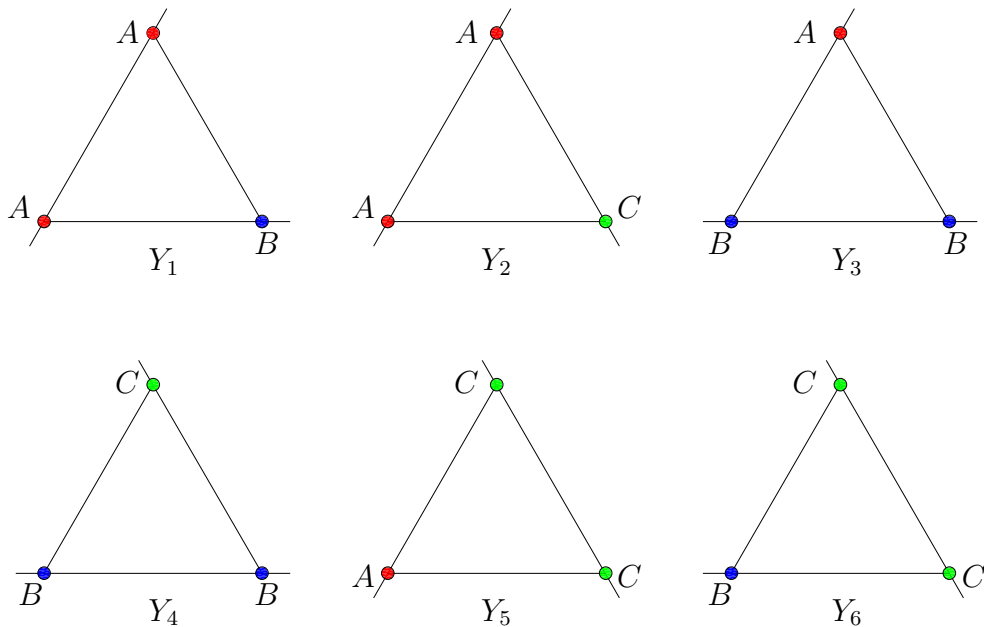


Figure 4.2: Three distinguished vertices

We will use the standard “shortest path” metric on Y . That is, for $x, y \in Y$, define

$$P(x, y) = \{(p_1, \dots, p_n) \mid n \in \mathbb{N}, p_1 = x, p_n = y \text{ and } \forall 1 \leq i < n, p_i, p_{i+1} \in Y_j \text{ some } j\}.$$

Let $|x - y|$ denote the standard Euclidean distance between x and y on the individual Sierpinski gaskets, Y_1, \dots, Y_6 . For $x, y \in Y$, we define the length of a path $p = (p_1, \dots, p_n) \in P(x, y)$ by

$$l(p) = \sum_{i=1}^{n-1} |p_{i+1} - p_i|,$$

and we define a metric on Y by

$$d(x, y) = \inf\{l(p) \mid p \in P(x, y)\}.$$

It is not hard to see that d is a metric on Y . Indeed, for any $x, y \in Y$, the shortest path in $P(x, y)$ is either (x, y) or (x, v, y) where $v \in \{A, B, C\}$. See Figure 4.3 for an example of a neighborhood of A .

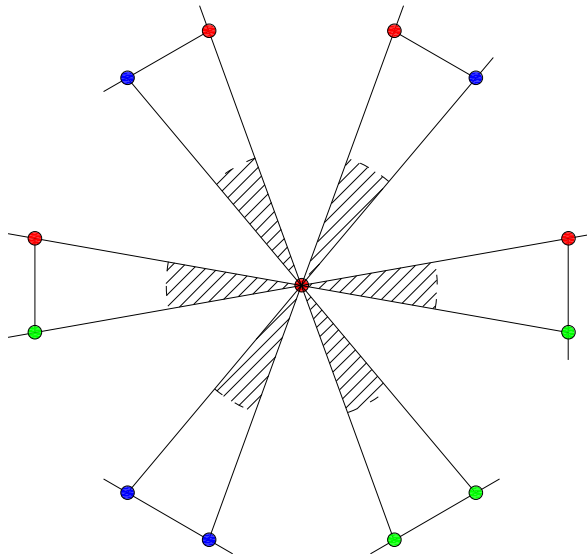
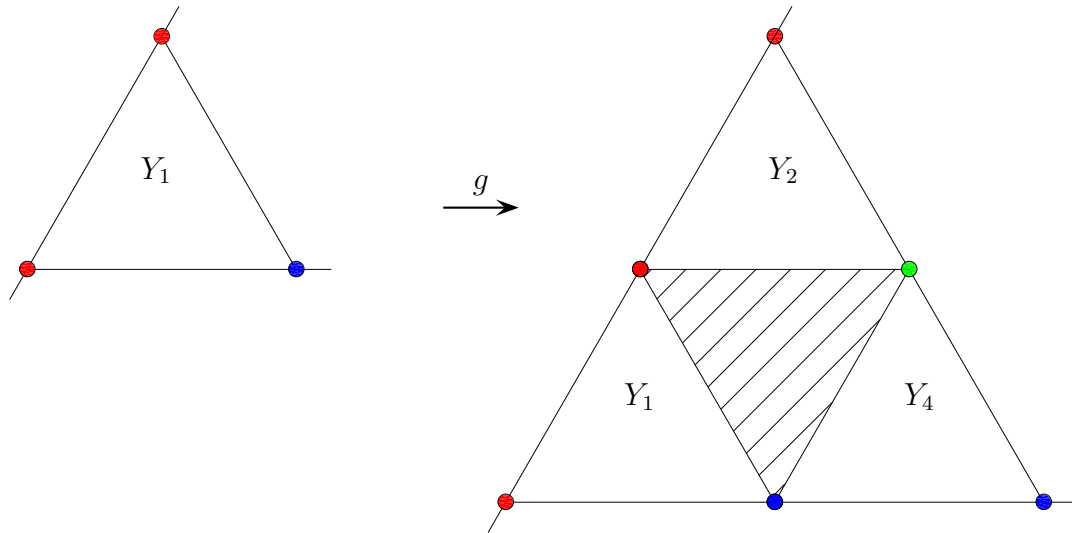


Figure 4.3: A neighborhood of A

It is clear that (Y, d) is compact.

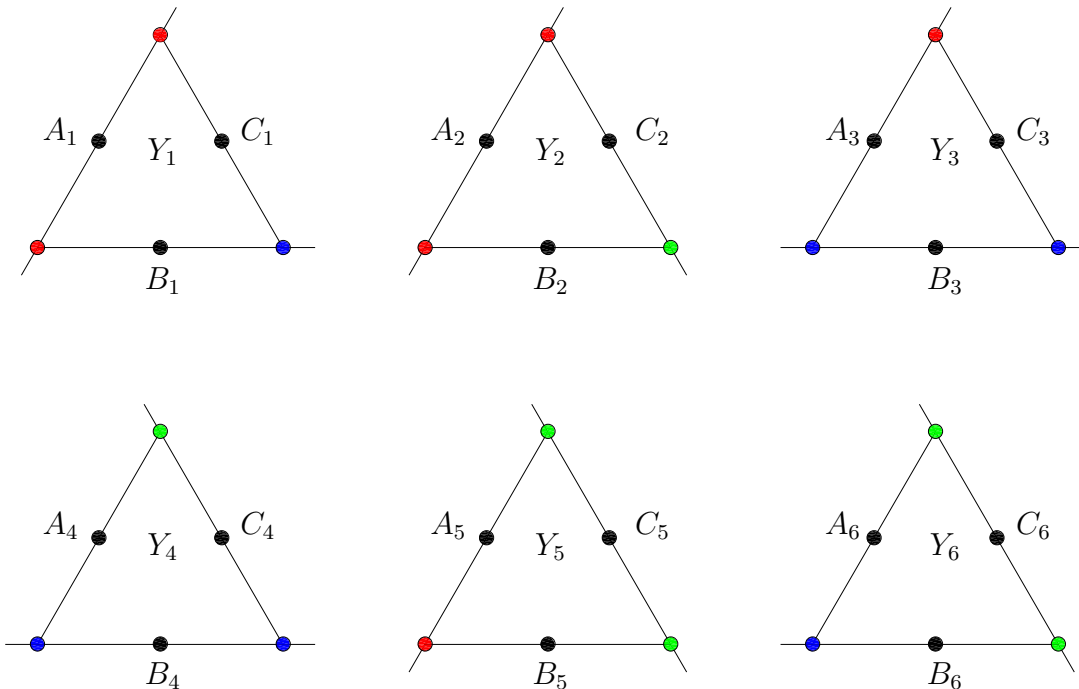
We define a mapping $g : Y \rightarrow Y$ as follows: g fixes A , B , and C , and on each Sierpinski gasket, Y_1, \dots, Y_6 , simply scale by a factor of 2 and map the left, bottom, and right midpoints to A , B , and C , respectively. This subdivides the image of each gasket into three gaskets (see Figure 4.4). The relation \sim ensures that g is well-defined on Y . This mapping is clearly continuous on Y .

Figure 4.4: $g(Y_1)$

Observe that the map g is not locally injective, and hence not locally expanding either. Moreover, for every $k \geq 1$, the map g^k is not an open map.

We will show that (Y, d, g) , along with the constants $\beta = \frac{1}{9}$, $K = 1$, and $\lambda = \frac{1}{2}$, satisfies Axioms 1 and 2.

We denote $V = \{A, B, C\}$. For each $i = 1, \dots, 6$ and each $v \in V$, denote by v_i the unique point in $Y_i \cap g^{-1}\{v\} \setminus \{v\}$ (see Figure 4.5).

Figure 4.5: V' labeled

We will use the notation $V' = \{A_i, B_i, C_i \mid 1 \leq i \leq 6\}$.

Consider Figure 4.6. We see that $g(B(A_4, \frac{1}{4})) \subseteq Y_3 \cup Y_5$, and that g maps $g(B(A_4, \frac{1}{4})) \cap Y_3$ bijectively onto Y_2 , and $g(B(A_4, \frac{1}{4})) \cap Y_5$ bijectively onto Y_1 .

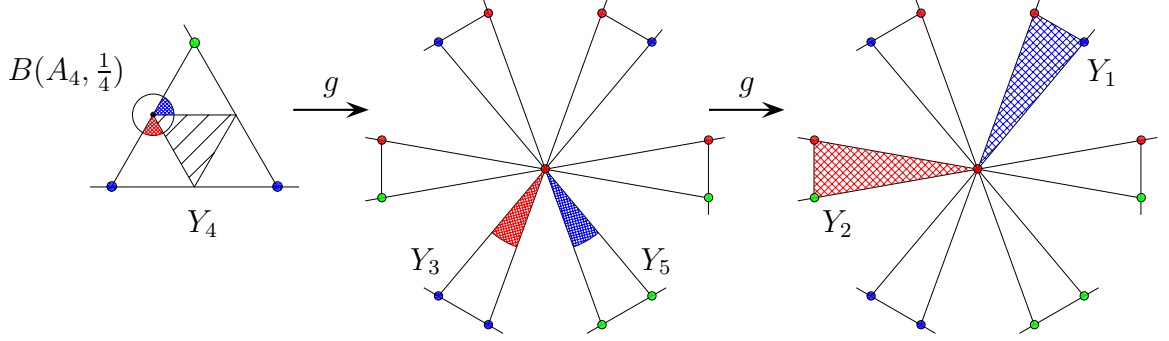


Figure 4.6: $g^2(B(A_4, \frac{1}{4}))$

Analogous observations can be made about the other 20 points of $V \cup V'$.

Observation 4.1. *Let $w \in V \cup V'$, and $g(w) = v$. Then there exist $1 \leq i \neq j \leq 6$ such that*

$$g(B(w, \frac{1}{4})) \subseteq Y_i \cup Y_j.$$

Moreover, g maps $g(B(w, \frac{1}{4})) \cap Y_i$ and $g(B(w, \frac{1}{4})) \cap Y_j$ bijectively onto distinct Y_i and Y_j .

Claim 4.2. *If $d(x, y) \leq \frac{1}{9}$ then $d(g(x), g(y)) \leq \frac{1}{2}d(g^2(x), g^2(y))$. That is, Axiom 1 holds with $\beta = \frac{1}{9}$, $K = 1$, and $\lambda = \frac{1}{2}$.*

Proof. We will consider two cases.

Case 1: $B(x, \frac{1}{9}) \cap (V \cup V') \neq \emptyset$

Since $\text{diam}(B(x, \frac{1}{9})) \leq \frac{2}{9} < \frac{1}{4}$, and since any two distinct points of $V \cup V'$ are at least distance $\frac{1}{2}$ apart, it follows that there is a unique point $w \in B(x, \frac{1}{9}) \cap (V \cup V')$.

Let $v = g(w)$.

Since $B(x, \frac{1}{9}) \subseteq B(w, \frac{1}{4})$, it follows from Observation 4.1 that there exist $1 \leq i \neq j \leq 6$ such that

$$g(B(x, \frac{1}{9})) \subseteq Y_i \cup Y_j.$$

Assume without loss of generality that $g(x) \in Y_i$. Let $\epsilon = |g(x) - v|$. Then we have

$$g(B(x, \frac{1}{9})) = \{z \in Y_i \mid |g(x) - z| \leq \frac{2}{9}\} \cup \{z \in Y_j \mid |v - z| \leq \frac{2}{9} - \epsilon\}.$$

Moreover, there exist $1 \leq i' \neq j' \leq 6$ such that the restrictions

$$g : \{z \in Y_i \mid |g(x) - z| \leq \frac{2}{9}\} \mapsto \{z \in Y_{i'} \mid |g^2(x) - z| \leq \frac{4}{9}\}$$

and

$$g : \{z \in Y_j \mid |v - z| \leq \frac{2}{9} - \epsilon\} \mapsto \{z \in Y_{j'} \mid |v - z| \leq \frac{4}{9} - 2\epsilon\}$$

are bijective.

So if $g(y) \in Y_i$ then

$$d(g^2(x), g^2(y)) = |g^2(x) - g^2(y)| = 2|g(x) - g(y)| = 2d(g(x), g(y)).$$

Otherwise $g(y) \in Y_j$, in which case

$$\begin{aligned} d(g^2(x), g^2(y)) &= |g^2(x) - v| + |v - g^2(y)| \\ &= 2|g(x) - v| + 2|v - g(y)| \\ &= 2d(g(x), g(y)). \end{aligned}$$

Case 2: $B(x, \frac{1}{9}) \cap (V \cup V') = \emptyset$

Then $g(B(x, \frac{1}{9})) \cap V = \emptyset$, so that $g(B(x, \frac{1}{9})) \subseteq Y_i$ for some $1 \leq i \leq 6$. Since the Euclidean metric $|\cdot|$ makes sense on all of $g(Y_i)$, we have

$$\begin{aligned} d(g(x), g(y)) &= |g(x) - g(y)| \\ &= \frac{1}{2}|g^2(x) - g^2(y)| \\ &\leq \frac{1}{2}d(g^2(x), g^2(y)). \end{aligned}$$

□

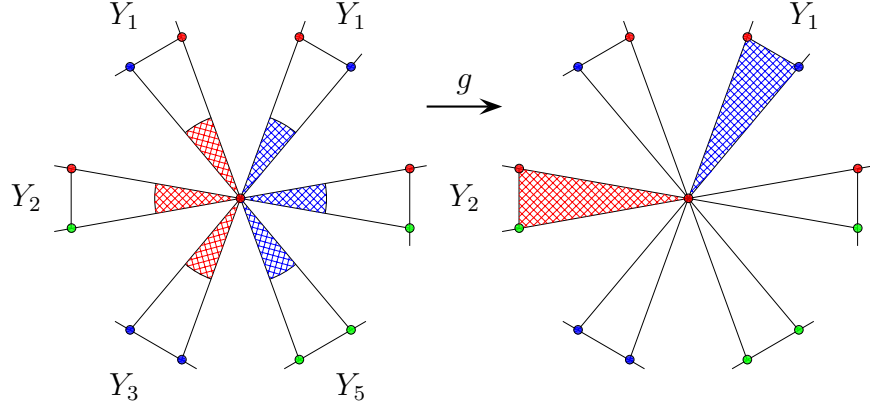
To prove that Axiom 2 holds, let us begin by making another observation. We see in Figure 4.6 above and Figure 4.7 that

$$g^2(B(A_4, \frac{1}{4})) = Y_1 \cup Y_2 = g(B(A, \frac{1}{2})).$$

Analogous observations can be made about each $v \in V$ and $w \in g^{-1}\{v\}$.

Observation 4.3. *Let $v \in V$ and $0 < \epsilon \leq \frac{1}{2}$. For any $w \in g^{-1}\{v\}$, we have*

$$g^2(B(w, \frac{1}{2}\epsilon)) = g(B(v, \epsilon)).$$

Figure 4.7: $g(B(A, \frac{1}{2}))$

Claim 4.4. *Let $x \in Y$ and $0 < \epsilon \leq \frac{1}{9}$. Then*

$$g(B(g(x), \epsilon)) = g^2(B(x, \frac{1}{2}\epsilon)).$$

In particular, Axiom 2 holds for $\beta = \frac{1}{9}$, $K = 1$, and $\lambda = \frac{1}{2}$.

Proof. Let $0 < \epsilon \leq \frac{1}{9}$. We consider two cases.

Case 1: $B(x, \frac{1}{2}\epsilon) \cap (V \cup V') = \emptyset$

Then clearly $B(x, \frac{1}{2}\epsilon) \subseteq Y_i \setminus V$ and $g(B(x, \frac{1}{2}\epsilon)) \subseteq Y_j \setminus V$ for some $1 \leq i, j \leq 6$. Since g simply scales Y_i by a factor of 2, we have

$$\begin{aligned} g(B(x, \frac{1}{2}\epsilon)) &= g(\{z \in Y_i \mid |x - z| \leq \frac{1}{2}\epsilon\}) \\ &= \{z \in Y_j \mid |g(x) - z| \leq \epsilon\} \\ &= B(g(x), \epsilon). \end{aligned}$$

Case 2: $B(x, \frac{1}{2}\epsilon) \cap (V \cup V') \neq \emptyset$

Since any two distinct points of $V \cup V'$ are at least distance $\frac{1}{2}$ apart, it follows that there is a unique point $w \in B(x, \frac{1}{2}\epsilon) \cap (V \cup V')$. It is clear that x and w are in the same Y_{i_1} for some $1 \leq i_1 \leq 6$, and that $d(x, w) = |x - w|$. Let $\epsilon' = \epsilon - |x - w|$ and let $v = g(w)$. Further suppose that $g(x) \in Y_{i_2}$ and $g^2(x) \in Y_{i_3}$. We have

$$B(x, \frac{1}{2}\epsilon) = \{y \in Y_{i_1} \mid |y - x| \leq \frac{1}{2}\epsilon\} \cup B(w, \epsilon').$$

Since $d(g^2(x), v) = |g^2(x) - v| \leq 2\epsilon \leq \frac{2}{9}$, it follows that

$$g^2(\{y \in Y_{i_1} \mid |y - x| \leq \frac{1}{2}\epsilon\}) \subseteq Y_{i_3}.$$

Therefore

$$g^2(B(x, \frac{1}{2}\epsilon)) = \{y \in Y_{i_3} \mid |y - g^2(x)| \leq 2\epsilon\} \cup g^2(B(w, \epsilon')). \quad (4.1)$$

Similarly, we have

$$B(g(x), \epsilon) = \{y \in Y_{i_2} \mid |y - g(x)| \leq \epsilon\} \cup B(v, 2\epsilon'),$$

and hence

$$g(B(g(x), \epsilon)) = \{y \in Y_{i_3} \mid |y - g^2(x)| \leq 2\epsilon\} \cup g(B(v, 2\epsilon')). \quad (4.2)$$

The result follows from (4.1), (4.2), and Observation 4.3. \square

Chapter 5

Proof of the Construction Theorem

Suppose that (Y, d, g) , together with the constants $\beta > 0$, $K \geq 1$, and $0 < \gamma < 1$, satisfies Axioms 1 and 2. Recall that $\hat{g} : \prod_{n \geq 0} Y \rightarrow \prod_{n \geq 0} Y$ denotes the map $\hat{g}(y_0, y_1, y_2, \dots) = (g(y_0), g(y_1), g(y_2), \dots)$.

We define two metrics on $\prod_{n \geq 0} Y$ as follows:

$$d'(\mathbf{x}, \mathbf{y}) = \sup_{n \geq 0} \{\gamma^n d(x_n, y_n)\},$$

and

$$\hat{d}(\mathbf{x}, \mathbf{y}) = d'(\mathbf{x}, \mathbf{y}) + \gamma^{-1} d'(\hat{g}^{-1}(\mathbf{x}), \hat{g}^{-1}(\mathbf{y})) + \dots + \gamma^{-(K-1)} d'(\hat{g}^{-(K-1)}(\mathbf{x}), \hat{g}^{-(K-1)}(\mathbf{y})).$$

While d' is the natural metric to use, it doesn't give us the Smale space property (S7) for \hat{Y} . Using \hat{d} solves this problem.

It is clear that d' is in fact a metric.

Lemma 5.1. *The metric d' gives the product topology on $\prod_{n \geq 0} Y$.*

Proof. Let $\mathcal{T}_{d'}$ denote the metric topology, and \mathcal{T}_p denote the product topology on $\prod_{n \geq 0} Y$.

Let $\mathbf{x} \in \prod_{n \geq 0} Y$, and choose $\epsilon > 0$. Then choose $N \in \mathbb{N}$ such that $\lambda^N \text{diam} Y < \epsilon$, and let

$$U_n = \begin{cases} B(x_n, \epsilon) & 0 \leq n < N \\ Y & n \geq N \end{cases}.$$

Then $\mathbf{x} \in \prod_{n \geq 0} U_n \subseteq B_{d'}(\mathbf{x}, \epsilon)$, so that $\mathcal{T}_{d'} \subseteq \mathcal{T}_p$.

For the other direction, we start with a basis element $\prod_{n \geq 0} U_n \in \mathcal{T}_p$ containing \mathbf{x} ; that is, U_n is an open neighborhood of x_n for $n = i_1, i_2, \dots, i_m$ and for all other n , U_n is

Y . Then for each $n = i_1, i_2, \dots, i_m$, there exists an $\epsilon_n > 0$ such that $B(x_n, \epsilon_n) \subseteq U_n$. Let $\epsilon = \min\{\lambda^n \epsilon_n \mid n = i_1, i_2, \dots, i_m\}$. Then $\mathbf{x} \in B_{d'}(\mathbf{x}, \epsilon) \subseteq \prod_{n \geq 0} U_n$, so that $\mathcal{T}_p \subseteq \mathcal{T}_{d'}$. \square

By Tychonoff's Theorem, the compactness of (Y, d) implies the compactness of $\prod_{n \geq 0} Y$ with the product topology. And by Lemma 5.1, this implies that $(\prod_{n \geq 0} Y, d')$ is compact. We observed in 2.4 that the inverse limit \hat{Y} is a closed subset of $\prod_{n \geq 0} Y$. It follows that (\hat{Y}, d') is compact.

The following three lemmas are easy observations about the metrics d' and \hat{d} on \hat{Y} .

Lemma 5.2. *If $x_0 = y_0$ then $d'(\hat{g}(\mathbf{x}), \hat{g}(\mathbf{y})) = \gamma d'(\mathbf{x}, \mathbf{y})$.*

Proof. We have

$$\begin{aligned} d'(\hat{g}(\mathbf{x}), \hat{g}(\mathbf{y})) &= \sup_{n \geq 0} \{ \gamma^n d(\hat{g}(\mathbf{x})_n, \hat{g}(\mathbf{y})_n) \} \\ &= \sup_{n \geq 1} \{ d(g(x_0), g(y_0)), \gamma^n d(x_{n-1}, y_{n-1}) \} \\ &= \gamma \sup_{n \geq 0} \{ \gamma^n d(x_n, y_n) \} \\ &= \gamma d'(\mathbf{x}, \mathbf{y}). \end{aligned}$$

\square

Lemma 5.3. *There exists a constant $c > 0$ such that for any $\mathbf{x}, \mathbf{y} \in \hat{Y}$, $d'(\mathbf{x}, \mathbf{y}) \leq \hat{d}(\mathbf{x}, \mathbf{y}) \leq c d'(\mathbf{x}, \mathbf{y})$. That is, d' and \hat{d} are strongly equivalent metrics.*

Proof. The first inequality follows immediately from the definition of \hat{d} .

The second inequality follows from Lemma 5.2:

$$\begin{aligned} \hat{d}(\mathbf{x}, \mathbf{y}) &= \sum_{n=0}^{K-1} \gamma^{-n} d'(\hat{g}^{-n}(\mathbf{x}), \hat{g}^{-n}(\mathbf{y})) \\ &= \sum_{n=0}^{K-1} \gamma^{-2n} d'(\mathbf{x}, \mathbf{y}) \\ &= \left(\sum_{n=0}^{K-1} \gamma^{-2n} \right) d'(\mathbf{x}, \mathbf{y}). \end{aligned}$$

\square

Strong equivalence of metric implies topological equivalence, and preserves uniform continuity of mappings. We observed following Lemma 5.1 that (\hat{Y}, d') is compact. It follows that (\hat{Y}, \hat{d}) is compact.

Lemma 5.4. *For any $\mathbf{x}, \mathbf{y} \in \hat{Y}$, we have $d'(\hat{g}^{-1}(\mathbf{x}), \hat{g}^{-1}(\mathbf{y})) \leq \frac{1}{\gamma} d'(\mathbf{x}, \mathbf{y})$.*

Proof. Since $\hat{g}^{-1}(\mathbf{x}) = (x_1, x_2, \dots)$, we have

$$\begin{aligned} d'(\hat{g}^{-1}(\mathbf{x}), \hat{g}^{-1}(\mathbf{y})) &= \sup_{n \geq 1} \{\gamma^{n-1} d(x_n, y_n)\} \\ &\leq \frac{1}{\gamma} \sup_{n \geq 0} \{\gamma^n d(x_n, y_n)\} \\ &= \frac{1}{\gamma} d'(\mathbf{x}, \mathbf{y}). \end{aligned}$$

□

Let us denote

$$\hat{Y}^s(\mathbf{x}, \epsilon)' = \{\mathbf{y} \in \hat{Y} \mid \hat{d}(\hat{g}^n(\mathbf{x}), \hat{g}^n(\mathbf{y})) \leq \epsilon \forall n \geq 0\}$$

and

$$\hat{Y}^u(\mathbf{x}, \epsilon)' = \{\mathbf{y} \in \hat{Y} \mid \hat{d}(\hat{g}^{-n}(\mathbf{x}), \hat{g}^{-n}(\mathbf{y})) \leq \epsilon \forall n \geq 0\}$$

for $\epsilon > 0$ and $\mathbf{x} \in \hat{Y}$. Recall from Definition 2.2 that the definition of $\hat{Y}^s(\mathbf{x}, \epsilon)$ and $\hat{Y}^u(\mathbf{x}, \epsilon)$ depends on the bracket map $[\cdot, \cdot]$ of a Smale space. Since we do not know at this point that $(\hat{Y}, \hat{d}, \hat{g})$ is in fact a Smale space, we use the extra ' decoration. We will prove later that these sets are in fact the usual local stable and unstable sets.

We will show that there exists $\epsilon_{\hat{Y}} > 0$ such that for $\hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon_{\hat{Y}}$,

$$\hat{Y}^s(\mathbf{x}, \epsilon'_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{y}, \epsilon'_{\hat{Y}})'$$

is a singleton. We will then use this property to define a mapping

$$[\cdot, \cdot] : \{(\mathbf{x}, \mathbf{y}) \in \hat{Y} \times \hat{Y} \mid \hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon_{\hat{Y}}\} \rightarrow \hat{Y}.$$

Our first tasks will be to obtain more useful descriptions of the sets $\hat{Y}^s(\mathbf{x}, \epsilon)'$ and $\hat{Y}^u(\mathbf{x}, \epsilon)'$.

Choose $0 < \epsilon'_{\hat{Y}} \leq \frac{\beta}{2}$ such that $\hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon'_{\hat{Y}}$ implies $\hat{d}(\hat{g}^{-n}(\mathbf{x}), \hat{g}^{-n}(\mathbf{y})) \leq \beta$ for $n = 0, \dots, 2K - 1$.

Lemma 5.5. *For any $0 < \epsilon \leq \epsilon'_{\hat{\gamma}}$, $\mathbf{y} \in \hat{Y}^s(\mathbf{z}, \epsilon)'$ if and only if $y_m = z_m$ for $m = 0, \dots, K-1$ and $\hat{d}(\mathbf{y}, \mathbf{z}) \leq \epsilon$.*

Proof. Let $0 < \epsilon \leq \epsilon'_{\hat{\gamma}}$.

First, suppose that $\mathbf{y} \in \hat{Y}^s(\mathbf{z}, \epsilon)'$. By our choice of $\epsilon'_{\hat{\gamma}}$, we have

$$\hat{g}^{-(2K-1)}(\mathbf{y}) \in \hat{Y}^s(\hat{g}^{-(2K-1)}\mathbf{z}, \beta)'.$$

So for each $m = 0, \dots, K-1$ and any $n \geq 0$, we have

$$\begin{aligned} d(g^n(y_{K+m}), g^n(z_{K+m})) &= d(g^n(\hat{g}^{-(K+m)}(\mathbf{y})_0), g^n(\hat{g}^{-(K+m)}(\mathbf{z})_0)) \\ &= d(\hat{g}^{n-(K+m)}(\mathbf{y})_0, \hat{g}^{n-(K+m)}(\mathbf{z})_0) \\ &\leq d'(\hat{g}^{n-(K+m)}(\mathbf{y}), \hat{g}^{n-(K+m)}(\mathbf{z})) \\ &\leq \hat{d}(\hat{g}^{n-(K+m)}(\mathbf{y}), \hat{g}^{n-(K+m)}(\mathbf{z})) \\ &\leq \beta. \end{aligned}$$

Applying Axiom 1, we get

$$d(g^{K+n}(y_{K+m}), g^{K+n}(z_{K+m})) \leq \gamma^K d(g^{2K+n}(y_{K+m}), g^{2K+n}(z_{K+m}))$$

for all $n \geq 0$. That is,

$$d(y_m, z_m) = d(g^K(y_{K+m}), g^K(z_{K+m})) \leq \gamma^{sK} d(g^{(s+1)K}(y_{K+m}), g^{(s+1)K}(z_{K+m})) \leq \gamma^{sK} \beta$$

for all $s \geq 1$, so that $y_m = z_m$.

For the converse, suppose $y_m = z_m$ for $m = 0, \dots, K-1$ and $\hat{d}(\mathbf{y}, \mathbf{z}) \leq \epsilon$. Since for each $m = 0, \dots, K-1$ we have $\hat{g}^{-m}(\mathbf{y})_0 = y_m = z_m = \hat{g}^{-m}(\mathbf{z})_0$, it follows from Lemma 5.2 that

$$d'(\hat{g}^{-m+1}(\mathbf{y}), \hat{g}^{-m+1}(\mathbf{z})) = \gamma d'(\hat{g}^{-m}(\mathbf{y}), \hat{g}^{-m}(\mathbf{z})),$$

and hence

$$\hat{d}(\hat{g}(\mathbf{y}), \hat{g}(\mathbf{z})) = \gamma \hat{d}(\mathbf{y}, \mathbf{z}). \quad (5.1)$$

That is, $y_m = z_m$ for $m = 0, \dots, K-1$ implies $\hat{d}(\hat{g}(\mathbf{y}), \hat{g}(\mathbf{z})) = \gamma \hat{d}(\mathbf{y}, \mathbf{z})$. Let us

apply this result to $\hat{g}^n(\mathbf{y})$ and $\hat{g}^n(\mathbf{z})$, where $n \geq 0$. We have

$$\hat{g}^n(\mathbf{y})_m = g^n(y_m) = g^n(z_m) = \hat{g}^n(\mathbf{z})_m$$

for $m = 0, \dots, K - 1$, hence

$$\hat{d}(\hat{g}^{n+1}(\mathbf{y}), \hat{g}^{n+1}(\mathbf{z})) = \gamma \hat{d}(\hat{g}^n(\mathbf{y}), \hat{g}^n(\mathbf{z})).$$

It follows that

$$\hat{d}(\hat{g}^n(\mathbf{y}), \hat{g}^n(\mathbf{z})) = \gamma^n \hat{d}(\mathbf{y}, \mathbf{z}) \leq \epsilon$$

for all $n \geq 0$. □

The following property follows easily from the proof of Lemma 5.5.

Corollary 5.6. *If $\mathbf{y}, \mathbf{z} \in \hat{Y}^s(\mathbf{x}, \epsilon'_Y)$, then $\hat{d}(\hat{g}(\mathbf{y}), \hat{g}(\mathbf{z})) \leq \gamma \hat{d}(\mathbf{y}, \mathbf{z})$.*

Now let us consider $\hat{Y}^u(\mathbf{x}, \epsilon)$.

Lemma 5.7. *For any $0 < \epsilon \leq \epsilon'_Y$, $\mathbf{y} \in \hat{Y}^u(\mathbf{z}, \epsilon)$ if and only if $d(y_n, z_n) \leq \epsilon$ for every $n \geq 0$ and $\hat{d}(\mathbf{y}, \mathbf{z}) \leq \epsilon$.*

Proof. Let $0 < \epsilon \leq \epsilon'_Y$.

If $\mathbf{y} \in \hat{Y}^u(\mathbf{z}, \epsilon)$ then

$$\begin{aligned} d(y_n, z_n) &= d(\hat{g}^{-n}(\mathbf{y})_0, \hat{g}^{-n}(\mathbf{z})_0) \\ &\leq d'(\hat{g}^{-n}(\mathbf{y}), \hat{g}^{-n}(\mathbf{z})) \\ &\leq \hat{d}(\hat{g}^{-n}(\mathbf{y}), \hat{g}^{-n}(\mathbf{z})) \\ &\leq \epsilon \end{aligned}$$

for all $n \geq 0$.

Conversely, suppose $d(y_n, z_n) \leq \epsilon$ for all $n \geq 0$ and $\hat{d}(\mathbf{y}, \mathbf{z}) \leq \epsilon$. Since $\epsilon \leq \epsilon'_Y < \beta$, we can apply Axiom 1 to get $d(g^K(y_n), g^K(z_n)) \leq \gamma^K d(g^{2K}(y_n), g^{2K}(z_n))$ for all $n \geq 0$.

Hence

$$\begin{aligned}
d'(\hat{g}^{-K}(\mathbf{y}), \hat{g}^{-K}(\mathbf{z})) &= \sup_{n \geq 0} \{\gamma^n d(y_{K+n}, z_{K+n})\} \\
&= \sup_{n \geq 0} \{\gamma^n d(g^K(y_{2K+n}), g^K(z_{2K+n}))\} \\
&\leq \gamma^K \sup_{n \geq 0} \{\gamma^n d(g^{2K}(y_{2K+n}), g^{2K}(z_{2K+n}))\} \\
&= \gamma^K \sup_{n \geq 0} \{\gamma^n d(y_n, z_n)\} \\
&= \gamma^K d'(\mathbf{y}, \mathbf{z}),
\end{aligned}$$

which gives

$$\begin{aligned}
\hat{d}(\hat{g}^{-1}(\mathbf{y}), \hat{g}^{-1}(\mathbf{z})) &= \sum_{m=0}^{K-1} \gamma^{-m} d'(\hat{g}^{-m-1}(\mathbf{y}), \hat{g}^{-m-1}(\mathbf{z})) \\
&\leq \gamma^{-(K-1)} \gamma^K d'(\mathbf{y}, \mathbf{z}) + \sum_{m=0}^{K-2} \gamma^{-m} d'(\hat{g}^{-m-1}(\mathbf{y}), \hat{g}^{-m-1}(\mathbf{z})) \\
&= \gamma \left(d'(\mathbf{y}, \mathbf{z}) + \sum_{m=1}^{K-1} \gamma^{-m} d'(\hat{g}^{-m}(\mathbf{y}), \hat{g}^{-m}(\mathbf{z})) \right) \\
&= \gamma \hat{d}(\mathbf{y}, \mathbf{z}).
\end{aligned}$$

We have just shown that $d(y_n, z_n) \leq \beta$ for all $n \geq 0$ implies that

$$\hat{d}(\hat{g}^{-1}(\mathbf{y}), \hat{g}^{-1}(\mathbf{z})) \leq \gamma \hat{d}(\mathbf{y}, \mathbf{z}). \quad (5.2)$$

Let us apply this result to $\hat{g}^{-s}(\mathbf{y})$ and $\hat{g}^{-s}(\mathbf{z})$, where $s \geq 0$. We have

$$d(\hat{g}^{-s}(\mathbf{y})_n, \hat{g}^{-s}(\mathbf{z})_n) = d(y_{n+s}, z_{n+s}) \leq \epsilon'_Y$$

for all $n \geq 0$. It follows that

$$\hat{d}(\hat{g}^{-s-1}(\mathbf{y}), \hat{g}^{-s-1}(\mathbf{z})) \leq \gamma \hat{d}(\hat{g}^{-s}(\mathbf{y}), \hat{g}^{-s}(\mathbf{z})),$$

and this is for any $s \geq 0$. Therefore

$$\hat{d}(\hat{g}^{-n}(\mathbf{y}), \hat{g}^{-n}(\mathbf{z})) \leq \gamma^n \hat{d}(\mathbf{y}, \mathbf{z}) \leq \epsilon$$

for every $n \geq 0$. □

The following property follows easily from the proof of Lemma 5.7.

Corollary 5.8. *If $\mathbf{y}, \mathbf{z} \in \hat{Y}^u(\mathbf{x}, \epsilon'_Y)'$, then $\hat{d}(\hat{g}^{-1}(\mathbf{y}), \hat{g}^{-1}(\mathbf{z})) \leq \gamma \hat{d}(\mathbf{y}, \mathbf{z})$.*

We now describe how to choose our parameter ϵ_Y . Choose

$$0 < \epsilon''_Y \leq \frac{1}{2} \epsilon'_Y$$

such that $\hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon''_Y$ implies $\hat{d}(\hat{g}(\mathbf{x}), \hat{g}(\mathbf{y})) \leq \epsilon'_Y$. Then choose

$$0 < \epsilon_Y \leq \frac{1}{2K} \gamma^K \epsilon''_Y$$

such that $d(x, y) \leq \epsilon_Y$ implies $d(g^n(x), g^n(y)) \leq \frac{1}{2K} \gamma^{K-1} \epsilon''_Y$ for $n = K, \dots, 2K - 1$.

Lemma 5.9. *If $\hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon_Y$ then $\hat{Y}^s(\mathbf{x}, \epsilon'_Y)' \cap \hat{Y}^u(\mathbf{y}, \epsilon'_Y)'$ is a singleton.*

Proof. Let $\hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon_Y$. Notice that we have

$$\begin{aligned} \gamma^{-(K-1)}(\gamma^K d(x_{2K-1}, y_{2K-1})) &\leq \gamma^{-(K-1)} d'(\hat{g}^{-(K-1)}(\mathbf{x}), \hat{g}^{-(K-1)}(\mathbf{y})) \\ &\leq \hat{d}(\mathbf{x}, \mathbf{y}) \\ &\leq \epsilon_Y. \end{aligned}$$

That is,

$$d(x_{2K-1}, y_{2K-1}) \leq \gamma^{-1} \epsilon_Y < \beta. \quad (5.3)$$

Let us define a point \mathbf{z} by defining $z_{sK}, \dots, z_{(s+1)K-1}$ inductively on s . Let $z_m = x_m$ for $m = 0, \dots, K - 1$. By (5.3) and Axiom 2, we have

$$\begin{aligned} z_{K-1} &= x_{K-1} \\ &= g^K(x_{2K-1}) \\ &\in g^K(B(y_{2K-1}, \gamma^{-1} \epsilon_Y)) \\ &= g^K(B(g^K(y_{3K-1}), \gamma^{-1} \epsilon_Y)) \\ &\subseteq g^{2K}(B(y_{3K-1}, \epsilon_Y)), \end{aligned}$$

so $z_{K-1} = g^{2K}(u_{3K-1})$ for some

$$u_{3K-1} \in B(y_{3K-1}, \epsilon_Y). \quad (5.4)$$

Define

$$\begin{aligned}
z_{2K-1} &= g^K(u_{3K-1}) \\
z_{2K-2} &= g(z_{2K-1}) = g^{K+1}(u_{3K-1}) \\
&\vdots \\
z_K &= g(z_{K+1}) = g^{2K-1}(u_{3K-1}).
\end{aligned}$$

Observe that we have $g(z_K) = g^{2K}(u_{3K-1}) = z_{K-1}$.

We can now apply Axiom 2 to (5.4) to get

$$g^K(u_{3K-1}) \in g^K(B(y_{3K-1}, \epsilon_{\hat{Y}})) = g^K(B(g^K(y_{4K-1}), \epsilon_{\hat{Y}})) \subseteq g^{2K}(B(y_{4K-1}, \epsilon_{\hat{Y}})).$$

That is, $g^K(u_{3K-1}) = g^{2K}(u_{4K-1})$ for some

$$u_{4K-1} \in B(y_{4K-1}, \epsilon_{\hat{Y}}). \quad (5.5)$$

As above, define

$$\begin{aligned}
z_{3K-1} &= g^K(u_{4K-1}) \\
z_{3K-2} &= g(z_{3K-1}) = g^{K+1}(u_{4K-1}) \\
&\vdots \\
z_{2K} &= g(z_{2K+1}) = g^{2K-1}(u_{4K-1}).
\end{aligned}$$

Observe that we have $g(z_{2K}) = g^{2K}(u_{4K-1}) = g^K(u_{3K-1}) = z_{2K-1}$.

We then use (5.5) and Axiom 2 to get $u_{5K-1} \in B(y_{5K-1}, \epsilon_{\hat{Y}})$, which we use to define z_{3K}, \dots, z_{4K-1} ; and so on. Our construction ensures that $\mathbf{z} \equiv (z_0, z_1, \dots) \in \hat{Y}$.

Let us show that $\mathbf{z} \in \hat{Y}^s(\mathbf{x}, \epsilon'_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{y}, \epsilon'_{\hat{Y}})'$. We'll start with showing $\mathbf{z} \in \hat{Y}^u(\mathbf{y}, \epsilon'_{\hat{Y}})'$. Notice that for $m = 0, \dots, K-1$, we have

$$\begin{aligned}
\gamma^{-m}d(x_m, y_m) &\leq \gamma^{-m}d'(\hat{g}^{-m}(\mathbf{x}), \hat{g}^{-m}(\mathbf{y})) \\
&\leq \hat{d}(\mathbf{x}, \mathbf{y}) \\
&\leq \epsilon_{\hat{Y}}.
\end{aligned}$$

That is,

$$d(z_m, y_m) = d(x_m, y_m) \leq \gamma^m \epsilon_{\hat{Y}} \leq \epsilon_{\hat{Y}} < \frac{1}{2K} \gamma^{K-1} \epsilon''_{\hat{Y}}$$

for $m = 0, \dots, K-1$. Furthermore, we have $d(u_{sK-1}, y_{sK-1}) \leq \epsilon_{\hat{Y}}$ for all $s \geq 3$. By our choice of $\epsilon_{\hat{Y}}$, it follows that

$$d(z_{sK-1-N}, y_{sK-1-N}) = d(g^N(u_{sK-1}), g^N(y_{sK-1})) \leq \frac{1}{2K} \gamma^{K-1} \epsilon_{\hat{Y}}''$$

for $N = K, \dots, 2K-1$ and $s \geq 3$. Since for all $n \geq K$, we defined $z_n = g^N(u_{sK-1})$ for some $N = K, \dots, 2K-1$ and $s \geq 3$, we have

$$d(z_n, y_n) \leq \frac{1}{2K} \gamma^{K-1} \epsilon_{\hat{Y}}''$$

for all $n \geq K$, and hence for all $n \geq 0$. This gives

$$d'(\hat{g}^{-n}(\mathbf{y}), \hat{g}^{-n}(\mathbf{z})) = \sup_{k \geq 0} \{\gamma^k d(y_{n+k}, z_{n+k})\} \leq \sup_{k \geq 0} \{d(y_{n+k}, z_{n+k})\} \leq \frac{1}{2K} \gamma^{K-1} \epsilon_{\hat{Y}}''$$

for all $n \geq 0$. Therefore,

$$\begin{aligned} \hat{d}(\mathbf{y}, \mathbf{z}) &= \sum_{m=0}^{K-1} \gamma^{-m} d'(\hat{g}^{-m}(\mathbf{y}), \hat{g}^{-m}(\mathbf{z})) \\ &\leq \sum_{m=0}^{K-1} \frac{1}{2K} \gamma^{K-1-m} \epsilon_{\hat{Y}}'' \\ &\leq \sum_{m=0}^{K-1} \frac{1}{2K} \epsilon_{\hat{Y}}'' \\ &= \frac{1}{2} \epsilon_{\hat{Y}}''. \end{aligned}$$

By Lemma 5.7, it follows that

$$\mathbf{z} \in \hat{Y}^u(\mathbf{y}, \frac{1}{2} \epsilon_{\hat{Y}}'')' \subseteq \hat{Y}^u(\mathbf{y}, \epsilon_{\hat{Y}}')'.$$

For inclusion in $\hat{Y}^s(\mathbf{x}, \epsilon_{\hat{Y}}')'$, recall that we defined $z_m = x_m$ for $m = 0, \dots, K-1$. And we also have

$$\hat{d}(\mathbf{z}, \mathbf{x}) \leq \hat{d}(\mathbf{z}, \mathbf{y}) + \hat{d}(\mathbf{y}, \mathbf{x}) \leq \frac{1}{2} \epsilon_{\hat{Y}}'' + \epsilon_{\hat{Y}} \leq \epsilon_{\hat{Y}}''.$$

So by Lemma 5.5,

$$\mathbf{z} \in \hat{Y}^s(\mathbf{x}, \epsilon_{\hat{Y}}'')' \subseteq \hat{Y}^s(\mathbf{x}, \epsilon_{\hat{Y}}')'.$$

Finally, let us show that \mathbf{z} is the only point in $\hat{Y}^s(\mathbf{x}, \epsilon'_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{y}, \epsilon'_{\hat{Y}})'$. Suppose

$$\mathbf{v} \in \hat{Y}^s(\mathbf{x}, \epsilon'_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{y}, \epsilon'_{\hat{Y}})'.$$

Since $\mathbf{v}, \mathbf{z} \in \hat{Y}^s(\mathbf{x}, \epsilon'_{\hat{Y}})'$, we have by Lemma 5.5 that $v_m = x_m = z_m$ for $m = 0, \dots, K-1$. And by Lemma 5.7, $\mathbf{v}, \mathbf{z} \in \hat{Y}^u(\mathbf{y}, \epsilon'_{\hat{Y}})'$ implies

$$d(v_n, z_n) \leq d(v_n, y_n) + d(y_n, z_n) \leq 2\epsilon'_{\hat{Y}} \leq \beta \quad (5.6)$$

for all $n \geq 0$. We will complete the proof by induction, showing that $v_m = z_m$ implies $v_{m+K} = z_{m+K}$. So suppose that $v_m = z_m$. From (5.6) we have $d(v_{m+2K}, z_{m+2K}) \leq \beta$, and we have assumed that $d(g^{2K}(v_{m+2K}), g^{2K}(z_{m+2K})) = d(v_m, z_m) = 0$. It follows by Axiom 1 that $v_{m+K} = g^K(v_{m+2K}) = g^K(z_{m+2K}) = z_{m+K}$. \square

Corollary 5.10. $\epsilon_{\hat{Y}}$ is an expansive constant for \hat{g} ; that is, if $\hat{d}(\hat{g}^n(\mathbf{x}), \hat{g}^n(\mathbf{y})) \leq \epsilon_{\hat{Y}}$ for every $n \in \mathbb{Z}$ then $\mathbf{x} = \mathbf{y}$.

Proof. If $\hat{d}(\hat{g}^n(\mathbf{x}), \hat{g}^n(\mathbf{y})) \leq \epsilon_{\hat{Y}}$ for every $n \in \mathbb{Z}$ then

$$\mathbf{y} \in \hat{Y}^s(\mathbf{x}, \epsilon'_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{x}, \epsilon'_{\hat{Y}})' = \{\mathbf{x}\}.$$

\square

As a result of Lemma 5.9, we can define a mapping

$$[\cdot, \cdot] : \{(\mathbf{x}, \mathbf{y}) \in \hat{Y} \times \hat{Y} \mid \hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon_{\hat{Y}}\} \rightarrow \hat{Y}$$

by

$$[\mathbf{x}, \mathbf{y}] = \hat{Y}^s(\mathbf{x}, \epsilon'_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{y}, \epsilon'_{\hat{Y}})'.$$

We see in the proof of Lemma 5.9 that in fact

$$[\mathbf{x}, \mathbf{y}] \in \hat{Y}^s(\mathbf{x}, \epsilon''_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{y}, \epsilon''_{\hat{Y}})'. \quad (5.7)$$

Notice that $\{(\mathbf{x}, \mathbf{y}) \in \hat{Y} \times \hat{Y} \mid \hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon_{\hat{Y}}\}$ is clearly closed.

We have already chosen $\epsilon_{\hat{Y}} > 0$. For the constant $0 < \lambda_{\hat{Y}} < 1$, simply let $\lambda_{\hat{Y}} = \gamma$. Let us show that $\epsilon_{\hat{Y}}$, $\lambda_{\hat{Y}}$, and $[\cdot, \cdot]$ satisfy properties (S1) through (S7) of Definition 2.2.

(S1) $[\cdot, \cdot]$ is continuous

Let $C \subseteq \hat{Y}$ be closed. We will prove that $[\cdot, \cdot]^{-1}(C)$ is closed as well.

Suppose $(\mathbf{x}^n, \mathbf{y}^n) \in [\cdot, \cdot]^{-1}(C)$ and $(\mathbf{x}^n, \mathbf{y}^n) \rightarrow (\mathbf{x}, \mathbf{y})$. Since C is compact, there exists a convergent subsequence $[\mathbf{x}^{n_k}, \mathbf{y}^{n_k}] \rightarrow \mathbf{z}$ of $([\mathbf{x}^n, \mathbf{y}^n]) \subseteq C$. Since C is closed, $\mathbf{z} \in C$.

Let $m \geq 0$. Then for every n_k we have

$$\begin{aligned} \hat{d}(\hat{g}^m(\mathbf{x}), \hat{g}^m(\mathbf{z})) &\leq \hat{d}(\hat{g}^m(\mathbf{x}), \hat{g}^m(\mathbf{x}^{n_k})) + \hat{d}(\hat{g}^m(\mathbf{x}^{n_k}), \hat{g}^m([\mathbf{x}^{n_k}, \mathbf{y}^{n_k}])) \\ &\quad + \hat{d}(\hat{g}^m([\mathbf{x}^{n_k}, \mathbf{y}^{n_k}]), \hat{g}^m(\mathbf{z})). \end{aligned}$$

Since $\hat{g}^m(\mathbf{x}^{n_k}) \rightarrow \hat{g}^m(\mathbf{x})$, $[\mathbf{x}^{n_k}, \mathbf{y}^{n_k}] \in \hat{Y}^s(\mathbf{x}^{n_k}, \epsilon'_{\hat{Y}})'$, and $\hat{g}^m([\mathbf{x}^{n_k}, \mathbf{y}^{n_k}]) \rightarrow \hat{g}^m(\mathbf{z})$, it follows that

$$\hat{d}(\hat{g}^m(\mathbf{x}), \hat{g}^m(\mathbf{z})) \leq \epsilon'_{\hat{Y}}.$$

Therefore $\mathbf{z} \in \hat{Y}^s(\mathbf{x}, \epsilon'_{\hat{Y}})'$. Similarly $\mathbf{z} \in \hat{Y}^u(\mathbf{y}, \epsilon'_{\hat{Y}})'$. That is,

$$\mathbf{z} \in \hat{Y}^s(\mathbf{x}, \epsilon'_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{y}, \epsilon'_{\hat{Y}})' = \{[\mathbf{x}, \mathbf{y}]\},$$

so that $[\mathbf{x}, \mathbf{y}] = \mathbf{z} \in C$.

(S2) $[\mathbf{x}, \mathbf{x}] = \mathbf{x}$ for all $\mathbf{x} \in \hat{Y}$

This follows immediately from the definition of $[\cdot, \cdot]$.

(S3) $[[\mathbf{x}, \mathbf{y}], \mathbf{z}] = [\mathbf{x}, \mathbf{z}]$ whenever both sides are defined

Suppose $\hat{d}(\mathbf{x}, \mathbf{y}), \hat{d}([\mathbf{x}, \mathbf{y}], \mathbf{z}), \hat{d}(\mathbf{x}, \mathbf{z}) \leq \epsilon_{\hat{Y}}$. Then by (5.7), we have

$$[[\mathbf{x}, \mathbf{y}], \mathbf{z}] \in \hat{Y}^s([\mathbf{x}, \mathbf{y}], \frac{1}{2}\epsilon'_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{z}, \frac{1}{2}\epsilon'_{\hat{Y}})' \subseteq \hat{Y}^s(\mathbf{x}, \epsilon'_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{z}, \epsilon'_{\hat{Y}})' = \{[\mathbf{x}, \mathbf{z}]\};$$

that is, $[[\mathbf{x}, \mathbf{y}], \mathbf{z}] = [\mathbf{x}, \mathbf{z}]$.

(S4) $[\mathbf{x}, [\mathbf{y}, \mathbf{z}]] = [\mathbf{x}, \mathbf{z}]$ whenever both sides are defined

This is analogous to (S3).

(S5) $\hat{g}([\mathbf{x}, \mathbf{y}]) = [\hat{g}(\mathbf{x}), \hat{g}(\mathbf{y})]$ whenever both sides are defined

Suppose $\hat{d}(\mathbf{x}, \mathbf{y}), \hat{d}(\hat{g}(\mathbf{x}), \hat{g}(\mathbf{y})) \leq \epsilon_{\hat{Y}}$. We have from (5.7) that

$$[\mathbf{x}, \mathbf{y}] \in \hat{Y}^s(\mathbf{x}, \epsilon''_{\hat{Y}})' \cap \hat{Y}^u(\mathbf{y}, \epsilon''_{\hat{Y}})';$$

that is, $\hat{d}([\mathbf{x}, \mathbf{y}], \mathbf{y}) \leq \epsilon_Y''$. By our choice of $\epsilon_Y'' > 0$, we have $\hat{d}(\hat{g}([\mathbf{x}, \mathbf{y}]), \hat{g}(\mathbf{y})) \leq \epsilon_Y'$. As a result,

$$\hat{g}([\mathbf{x}, \mathbf{y}]) \in \hat{Y}^s(\hat{g}(\mathbf{x}), \epsilon_Y')' \cap \hat{Y}^u(\hat{g}(\mathbf{y}), \epsilon_Y')' = \{\hat{g}(\mathbf{x}), \hat{g}(\mathbf{y})\};$$

that is, $\hat{g}([\mathbf{x}, \mathbf{y}]) = [\hat{g}(\mathbf{x}), \hat{g}(\mathbf{y})]$.

As in Definition 2.2, we denote

$$\hat{Y}^s(\mathbf{x}, \epsilon) = \{\mathbf{y} \mid [\mathbf{x}, \mathbf{y}] = \mathbf{y}, \hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon\}$$

and

$$\hat{Y}^u(\mathbf{x}, \epsilon) = \{\mathbf{y} \mid [\mathbf{y}, \mathbf{x}] = \mathbf{y}, \hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon\}$$

for $\mathbf{x} \in \hat{Y}$ and $0 < \epsilon \leq \epsilon_Y$.

Let us show that

$$\hat{Y}^s(\mathbf{x}, \epsilon_Y) = \hat{Y}^s(\mathbf{x}, \epsilon_Y)' \quad (5.8)$$

for any $\mathbf{x} \in \hat{Y}$. First, suppose that $\mathbf{y} \in \hat{Y}^s(\mathbf{x}, \epsilon_Y)$. Then $\mathbf{y} = [\mathbf{x}, \mathbf{y}] \in \hat{Y}^s(\mathbf{x}, \epsilon_Y)'$. So by Lemma 5.5, it follows that $y_m = x_m$ for $m = 0, \dots, K-1$. Since $\hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon_Y$, we can apply the other direction of Lemma 5.5 to get $\mathbf{y} \in \hat{Y}^s(\mathbf{x}, \epsilon_Y)'$.

Now suppose $\mathbf{y} \in \hat{Y}^s(\mathbf{x}, \epsilon_Y)'$. Then $\hat{d}(\mathbf{x}, \mathbf{y}) \leq \epsilon_Y$, and

$$\mathbf{y} \in \hat{Y}^s(\mathbf{x}, \epsilon_Y)' \cap \hat{Y}^u(\mathbf{y}, \epsilon_Y)' = \{[\mathbf{x}, \mathbf{y}]\};$$

that is, $\mathbf{y} = [\mathbf{x}, \mathbf{y}]$. Hence $\mathbf{y} \in \hat{Y}^s(\mathbf{x}, \epsilon_Y)$.

Similarly,

$$\hat{Y}^u(\mathbf{x}, \epsilon_Y) = \hat{Y}^u(\mathbf{x}, \epsilon_Y)' \quad (5.9)$$

for any $\mathbf{x} \in \hat{Y}$.

$$(S6) \quad \hat{d}(\hat{g}(\mathbf{y}), \hat{g}(\mathbf{z})) \leq \lambda_Y \hat{d}(\mathbf{y}, \mathbf{z}) \text{ if } \mathbf{y}, \mathbf{z} \in \hat{Y}^s(\mathbf{x}, \epsilon_Y)$$

This follows immediately from (5.8) and Corollary 5.6.

$$(S7) \quad \hat{d}(\hat{g}^{-1}(\mathbf{y}), \hat{g}^{-1}(\mathbf{z})) \leq \lambda_Y \hat{d}(\mathbf{y}, \mathbf{z}) \text{ if } \mathbf{y}, \mathbf{z} \in \hat{Y}^u(\mathbf{x}, \epsilon_Y)$$

This follows immediately from (5.9) and Corollary 5.8.

To prove that the Smale space $(\hat{Y}, \hat{d}, \hat{g})$ has totally disconnected local stable sets, we first need the following lemma.

Lemma 5.11. *Axiom 1 implies that g is finite-to-one.*

Proof. Suppose that Y contains an infinite sequence (y_n) of distinct points all having the same image under g . As g is onto, so is g^K . For each n , pick z_n with $g^K(z_n) = y_n$. Then (z_n) must have an accumulation point, so we may find z_m and z_n with $m \neq n$ and $d(z_m, z_n) \leq \beta$. So we have $g^{2K}(z_m) = g^{2K}(z_n)$, but $g^K(z_m) = y_m$ and $g^K(z_n) = y_n$ are distinct; this contradicts Axiom 1. \square

Proposition 5.12. *If (Y, d, g) satisfies Axioms 1 and 2, then the Smale space $(\hat{Y}, \hat{d}, \hat{g})$ has totally disconnected local stable sets.*

Proof. For $n \geq 0$, denote by $\pi_n : \hat{Y} \rightarrow Y$ the projection map $\pi_n(y_0, y_1, y_2, \dots) = y_n$. Choose $\mathbf{y} \in \hat{Y}$. By Lemma 5.5 and (5.8), every point in $\hat{Y}^s(\mathbf{y}, \epsilon_{\hat{Y}})$ has the same first coordinate, y_0 . Therefore, for any $n \geq 0$, the set $\pi_n(\hat{Y}^s(\mathbf{y}, \epsilon_{\hat{Y}})) \subseteq g^{-n}\{y_0\}$ is finite by Lemma 5.11. So the π_n preimage of any point in this finite set is clopen in $\hat{Y}^s(\mathbf{y}, \epsilon_{\hat{Y}})$. As a result, for any two distinct points in $\hat{Y}^s(\mathbf{y}, \epsilon_{\hat{Y}})$, we can find a clopen set containing one but not the other. \square

Now let us consider the matter of irreducibility.

Proposition 5.13. *$(\hat{Y}, \hat{d}, \hat{g})$ is non-wandering if and only if (Y, d, g) is non-wandering.*

Proof. “ \Rightarrow ” Let $y \in Y$ and $\epsilon > 0$. Since g is surjective, there exists $\mathbf{y} \in \hat{Y}$ such that $y_0 = y$. Since $(\hat{Y}, \hat{d}, \hat{g})$ is non-wandering, there exists $k \geq 1$ and $\mathbf{z} \in \hat{Y}$ such that

$$\mathbf{z} \in B(\mathbf{y}, \epsilon) \cap \hat{g}^k(B(\mathbf{y}, \epsilon)).$$

Then $\mathbf{z} = \hat{g}^k(\mathbf{x})$ for some $\mathbf{x} \in B(\mathbf{y}, \epsilon)$. So we have

$$d(z_0, y) = d(z_0, y_0) \leq d'(\mathbf{z}, \mathbf{y}) \leq \hat{d}(\mathbf{z}, \mathbf{y}) \leq \epsilon$$

and similarly $d(x_0, y) \leq \epsilon$. That is,

$$z_0 = g^k(x_0) \in B(y, \epsilon) \cap g^k(B(y, \epsilon)).$$

“ \Leftarrow ” Let $\mathbf{y} \in \hat{Y}$ and $\epsilon > 0$. Choose $N \geq 1$ such that $\gamma^N < \frac{\epsilon}{c}$, where $c > 0$ is the constant from Lemma 5.3. Then choose $\epsilon' > 0$ such that $d(x, y) < \epsilon'$ implies that $d(g^n(x), g^n(y)) < \frac{\epsilon}{c}$ for each $n = 0, 1, \dots, N$.

Since (Y, d, g) is non-wandering, there exists $m \geq 1$ and $z \in Y$ such that

$$z \in B(y_N, \epsilon') \cap g^m(B(y_N, \epsilon')).$$

So $z = g^m(u)$ for some $u \in B(y_N, \epsilon')$. Since g is surjective, there exists $\mathbf{z} \in \hat{Y}$ such that

$$\begin{aligned} z_N &= z = g^m(u) \\ z_{N+1} &= g^{m-1}(u) \\ &\vdots \\ z_{N+m} &= u \end{aligned}$$

Since $d(z_N, y_N) < \epsilon'$, it follows that $d'(\mathbf{z}, \mathbf{y}) < \frac{\epsilon}{c}$, and hence $\hat{d}(\mathbf{z}, \mathbf{y}) < \epsilon$. Similarly, $d(\hat{g}^{-m}(\mathbf{z})_N, y_N) = d(u, y_N) < \epsilon'$ gives us that $\hat{d}(\hat{g}^{-m}(\mathbf{z}), \mathbf{y}) < \epsilon$. That is,

$$\mathbf{z} \in B(\mathbf{y}, \epsilon) \cap \hat{g}^m(B(\mathbf{y}, \epsilon)).$$

□

Proposition 5.14. $(\hat{Y}, \hat{d}, \hat{g})$ has a dense forward orbit if and only if (Y, d, g) does.

Proof. “ \Rightarrow ” Suppose $\mathbf{y} \in \hat{Y}$ has a dense forward orbit. We will show that y_0 has a dense orbit in Y .

Let $z \in Y$ and $\epsilon > 0$. Since g is surjective, there is a point $\mathbf{z} \in \hat{Y}$ such that $z_0 = z$. Since \mathbf{y} has a dense forward orbit, there exists $k \geq 0$ such that $\hat{d}(\hat{g}^k(\mathbf{y}), \mathbf{z}) < \epsilon$. It follows that $d(g^k(y_0), z) < \epsilon$.

“ \Leftarrow ” Suppose $y \in Y$ has a dense (forward) orbit. Since g is surjective, there exists a point $\mathbf{y} \in \hat{Y}$ such that $y_0 = y$. We will show that \mathbf{y} has a dense forward orbit in \hat{Y} .

Let $\mathbf{z} \in \hat{Y}$ and $\epsilon > 0$. Choose $N \geq 1$ such that $\gamma^N < \frac{\epsilon}{c}$, where $c > 0$ is the constant from Lemma 5.3. Then choose $\epsilon' > 0$ such that $d(x, y) < \epsilon'$ implies that $d(g^n(x), g^n(y)) < \frac{\epsilon}{c}$ for each $n = 0, 1, \dots, N$. Since y has a dense orbit, there exists $m \geq 0$ such that $d(z_N, g^m(y)) < \epsilon'$. It follows that $d'(\mathbf{z}, \hat{g}^{m+N}(\mathbf{y})) < \frac{\epsilon}{c}$, and hence $\hat{d}(\mathbf{z}, \hat{g}^{m+N}(\mathbf{y})) < \epsilon$. □

Chapter 6

Proof of the Realization Theorem

We will construct the inverse limit space from a quotient space of our Smale space, (X, f) . We will use a certain Markov partition to define an equivalence relation \sim on X , and consider the quotient X/\sim . The relation \sim has the effect of collapsing each Markov partition rectangle to a single unstable set (see Figure 6). These unstable sets may intersect on the boundaries, making the definition of an appropriate metric on X/\sim rather difficult. The other aspects of our construction of the inverse limit space are quite intuitive.

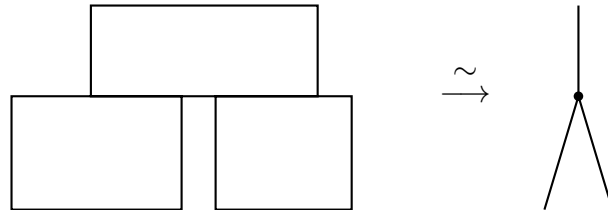


Figure 6.1: The equivalence relation \sim

6.1 Construction of a Quotient Space

Let (X, d, f) be a Smale space with totally disconnected stable sets. In this section, we will use a Markov partition for (X, d, f) to define an equivalence relation, \sim , on X . We then define a metric, δ , and a mapping, α , on the quotient X/\sim . In Section 6.2, we will show that

$$\varprojlim X/\sim \xleftarrow{\alpha} X/\sim \xleftarrow{\alpha} X/\sim \xleftarrow{\alpha} \cdots,$$

together with the usual shift map, is conjugate to (X, d, f) . Moreover, we prove that $(X/\sim, \delta, \alpha)$ satisfies Axioms 1 and 2, so that the inverse limit space is also a Smale space by the Construction Theorem.

6.1.1 An Equivalence Relation on X

Let $\mathcal{P} = \{R_1, \dots, R_M\}$ be a Markov partition for (X, f) such that for every $x \in R_i \in \mathcal{P}$, $X^s(x, R_i)$ is clopen in $X^s(x, \epsilon_X)$. Such a Markov partition exists by Proposition 2.8. By Propositions 5.8 and 6.2 of [1], the diameters of the rectangles in \mathcal{P} can be chosen arbitrarily small. So we will assume that for each $R \in \mathcal{P}$, $\text{diam}(R) \leq \frac{1}{2}\epsilon'_X$, where $0 < \epsilon'_X \leq \frac{\epsilon_X}{3}$ is the constant from Proposition 2.3.

It is an easy consequence of Proposition 2.3 that for any $x \in \text{Int}(R_i)$ we have $X^s(x, R_i) \subseteq \text{Int}(R_i)$.

We define a relation \approx on X as follows:

$$x \approx y \text{ if and only if } x, y \in R_i \text{ for some } R_i \in \mathcal{P} \text{ and } x \in X^s(y, \epsilon_X).$$

We observe that this relation is reflexive and symmetric, but not transitive. Let \sim be the equivalence relation generated by \approx , with equivalence classes denoted $[[\cdot]]$. That is, $x \sim y$ if and only if there are $x_1, x_2, \dots, x_n \in X$ such that $x \approx x_1 \approx \dots \approx x_n \approx y$. It is clear that \sim is simply the transitive closure of \approx .

We observe that $[[x]] \subseteq X^s(x) \equiv \{y \in X \mid \lim_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0\}$. Moreover, if $x \in \text{Int}R_i$ then $x \approx X^s(x, R_i) \subseteq \text{Int}R_i$, and $\text{Int}R_i \cap R_j = \emptyset$ if $i \neq j$, so that $[[x]] = X^s(x, R_i)$.

We will spend the rest of this section proving various properties of \mathcal{P} , \approx , and \sim . We will need these properties to define a metric on X/\sim .

It is a well-known fact that if a metric space A is compact and $C \subseteq A$ is a clopen subset, then there exists $\epsilon > 0$ such that $B(C, \epsilon) \subseteq C$. Since we know that the rectangles are clopen in the stable direction, we would like a uniform constant satisfying this property.

Lemma 6.1. *There exists $0 < \epsilon_0 \leq \frac{1}{3}\epsilon_X$ such that if $x \in R_i \in \mathcal{P}$ then*

$$X^s(x, \epsilon_0) \subseteq R_i.$$

Proof. Choose $x_i \in R_i \in \mathcal{P}$. Since $X^s(x_i, R_i)$ is clopen in the compact set $X^s(x_i, \epsilon_X)$,

there exists $0 < \epsilon_i \leq \frac{1}{3}\epsilon_X$ such that

$$B(X^s(x_i, R_i), \epsilon_i) \cap X^s(x_i, \epsilon_X) \subseteq X^s(x_i, R_i) \quad (6.1)$$

The collection $\{X^s(y, \frac{1}{2}\epsilon_i) \mid y \in X^s(x_i, R_i)\}$ covers $X^s(x_i, R_i)$, so there is a finite subcover, with centers y_1, \dots, y_n .

By the uniform continuity of $[\cdot, \cdot]$, there exists $0 < \eta_i \leq \frac{1}{2}\epsilon_i$ such that if $d(a, b) \leq \eta_i$ then $d([c, a], [c, b]) \leq \frac{1}{2}\epsilon_i$ for any $c \in B(a, \epsilon_X) \cap B(b, \epsilon_X)$. We will show that

$$X^s(x, \eta_i) \subseteq R_i$$

for any $x \in R_i$.

Choose $x \in R_i$ and $y \in X^s(x, \eta_i)$. Since $R_i = [X^u(x_i, R_i), X^s(x_i, R_i)]$, it follows that $x = [u, s]$ for some $u \in X^u(x_i, R_i)$ and $s \in X^s(x_i, R_i)$. And $s \in X^s(y_j, \frac{1}{2}\epsilon_i)$ for some $1 \leq j \leq n$. Moreover, since $d(x, y) \leq \eta_i$, it follows that we have $d(s, [y_j, y]) = d([y_j, x], [y_j, y]) \leq \frac{1}{2}\epsilon_i$. Hence

$$\begin{aligned} d([y_j, y], y_j) &\leq d([y_j, y], s) + d(s, y_j) \\ &\leq \frac{1}{2}\epsilon_i + \frac{1}{2}\epsilon_i \\ &= \epsilon_i, \end{aligned}$$

so that $[y_j, y] \in X^s(y_j, \epsilon_i)$. Moreover, since $y_j \in X^s(x_i, R_i) \subseteq X^s(x_i, \frac{1}{2}\epsilon_X)$, it follows that $X^s(y_j, \epsilon_i) \subseteq X^s(y_j, \frac{1}{3}\epsilon_X) \subseteq X^s(x_i, \epsilon_X)$. That is,

$$[y_j, y] \in B(y_j, \epsilon_i) \cap X^s(x_i, \epsilon_X) \subseteq B(X^s(x_i, R_i), \epsilon_i) \cap X^s(x_i, \epsilon_X).$$

It follows from (6.1) that $[y_j, y] \in R_i$. By the definition of a rectangle, $x, [y_j, y] \in R_i$ implies

$$y = [x, y] = [x, [y_j, y]] \in R_i.$$

Let $\epsilon_0 = \min\{\eta_i \mid i = 1, \dots, M\}$. □

Next, we find a bound on the transitive closure, \sim , of the relation \approx .

Lemma 6.2. *There exists $N \in \mathbb{N}$ such that if $y \sim x$ then there are y_1, \dots, y_N with*

$$y \approx y_1 \approx \dots \approx y_N \approx x.$$

Proof. Let $\epsilon_0 > 0$ be as in Lemma 6.1 and $0 < \lambda < 1$ as in Definition 2.2, and choose $m \in \mathbb{N}$ such that

$$\lambda^m \epsilon_X \leq \epsilon_0.$$

Then choose $\eta > 0$ such that $d(x, y) < \eta$ implies $d(f^{-k}(x), f^{-k}(y)) < \epsilon_X$ for all $k = 0, \dots, m$. Cover X by $\frac{1}{2}\eta$ -balls and extract a finite subcover $\{B_1, \dots, B_n\}$.

We claim that $N = 2nM - 2$ satisfies the conclusion.

Let $x, y \in X$ with $y \sim x$. By definition of \sim , we know that $y \approx y_1 \approx \dots \approx y_L \approx x$ for some $y_1, \dots, y_L \in X$. Denote $y_0 = y$ and $y_{L+1} = x$. Suppose $L > 2nM - 2$. We will show that $y_j \approx y_{j'}$ for some non-consecutive j and j' .

Since P covers X , $f^m(y_0) \in R_i$ for some (not necessarily unique) $1 \leq i \leq M$. By definition of \approx , we have $y_{j+1} \in X^s(y_j, \epsilon_X)$ for all $j = 0, \dots, L$, so that

$$f^m(y_{j+1}) \in X^s(f^m(y_j), \lambda^m \epsilon_X) \subseteq X^s(f^m(y_j), \epsilon_0).$$

Arguing inductively we see that

$$f^m(y_j) \in X^s(f^m(y_0), \epsilon_X) \cap R_i \tag{6.2}$$

for all $j = 1, \dots, L + 1$.

Since $L + 2 \geq 2nM + 1$, at least $2n + 1$ of the y_j are in the same $R_{i'}$ for some $1 \leq i' \leq M$. Look at those and apply f^m to all of them. Since there are at least $2n + 1$ of these, at least 3 of them are in the same B_r for some $1 \leq r \leq n$. Of these 3, choose 2 that have non-consecutive indices. That is, there are y_j and $y_{j'}$ with j and j' not consecutive, such that $y_j, y_{j'} \in R_{i'}$ and $f^m(y_j), f^m(y_{j'}) \in B_r$. So

$$d(f^m(y_j), f^m(y_{j'})) < \eta,$$

and we have from (6.2) that $f^m(y_j), f^m(y_{j'}) \in X^s(f^m(y_0), \epsilon_X)$, hence

$$f^m(y_j) \in X^s(f^m(y_{j'}), \eta).$$

It follows from our choice of $\eta > 0$ that

$$y_j \in X^s(y_{j'}, \epsilon_X).$$

Therefore $y_j \approx y_{j'}$. □

Choose $0 < \epsilon_0'' \leq \epsilon_0$ such that $d(x, y) \leq \epsilon_0''$ implies that $[x, y] \in X^s(x, \epsilon_0) \cap X^u(y, \epsilon_0)$ (the existence of such a constant follows from the continuity of $[\cdot, \cdot]$).

Let $\epsilon_0' = \min\{\frac{1}{3}\epsilon_X', \frac{1}{3}\epsilon_0''\}$.

We noted that $[[x]] \subseteq X^s(x)$. It follows that for each $y \in [[x]]$ there exists $n \geq 0$ such that $f^n(y) \in X^s(f^n(x), \epsilon_0')$. We now show that we can do this in uniform time.

Corollary 6.3. *There exists $K \in \mathbb{N}$ such that if $y \in [[x]]$ then $f^K(y) \in X^s(f^K(x), \epsilon_0')$.*

Proof. Suppose $y \sim x$. Then by Lemma 6.2, there exist y_1, \dots, y_N with

$$y \approx y_1 \approx \dots \approx y_N \approx x.$$

Denote $y_0 = y$ and $y_{N+1} = x$. We have

$$y_i \in X^s(y_{i+1}, \epsilon_X)$$

for all $i = 0, \dots, N$. Choose $K \in \mathbb{N}$ such that

$$\lambda^K(N+1)\epsilon_X \leq \epsilon_0'.$$

We have

$$f^K(y_i) \in X^s(f^K(y_{i+1}), \lambda^K \epsilon_X)$$

for all $i = 0, \dots, N$, so that

$$\begin{aligned} f^K(y_0) &\in X^s(f^K(y_1), \lambda^K \epsilon_X) \\ &\subseteq X^s(f^K(y_2), 2\lambda^K \epsilon_X) \\ &\quad \vdots \\ &\subseteq X^s(f^K(y_{N+1}), (N+1)\lambda^K \epsilon_X) \\ &\subseteq X^s(f^K(y_{N+1}), \epsilon_0'). \end{aligned}$$

That is, $f^K(y) \in X^s(f^K(x), \epsilon_0')$. □

Corollary 6.4. *The equivalence classes $[[\cdot]]$ are closed.*

Proof. Let $(z_n) \subseteq [[x]]$, with $z_n \rightarrow z$. By Corollary 6.3,

$$f^K(z_n) \in X^s(f^K(x), \epsilon_0') \tag{6.3}$$

for all n .

Choose $\eta > 0$ such that $d(x, y) \leq \eta$ implies $d(f^{-k}(x), f^{-k}(y)) \leq \epsilon_X$ for all $k = 0, \dots, K$. Let $\{B_1, \dots, B_s\}$ be a finite cover of X by $\frac{1}{2}\eta$ -balls.

Infinitely many of the z_n are in the same R_i for some $1 \leq i \leq M$. Consider those and apply f^K to all of them. Infinitely many of these are in the same B_r for some $1 \leq r \leq s$. That is, there is a subsequence (z_{n_k}) of (z_n) such that $(z_{n_k}) \subseteq R_i$ and $(f^K(z_{n_k})) \subseteq B_r$. So

$$d(f^K(z_{n_k}), f^K(z_{n_1})) < \eta$$

for all n_k . It follows from (6.3) that

$$f^K(z_{n_k}) \in X^s(f^K(z_{n_1}), \eta),$$

so by our choice of $\eta > 0$ we have

$$z_{n_k} \in X^s(z_{n_1}, \epsilon_X)$$

for all n_k . That is $z_{n_k} \in X^s(z_{n_1}, R_i)$ for all n_k . Since $z_{n_k} \rightarrow z$ and $X^s(z_{n_1}, R_i)$ is closed, it follows that $z \in X^s(z_{n_1}, R_i) \subseteq [[z_{n_1}]] = [[x]]$. \square

Lemma 6.5. *Let (x_n) and (a_n) be sequences in X . If $x_n \rightarrow x$, $a_n \rightarrow a$, and $x_n \approx a_n$ for all n , then $x \approx a$.*

Proof. Without loss of generality, we may assume that $d(a_n, a) \leq \frac{1}{2}\epsilon'_X$ for all n . Then $d(x_n, a) \leq d(x_n, a_n) + d(a_n, a) \leq \epsilon'_X \leq \epsilon_X$, so that $[x_n, a]$ is defined for all n . In addition, $d(x, a) \leq d(x, x_n) + d(x_n, a)$ for all n , so it follows that $d(x, a) \leq \epsilon_X$, and hence $[x, a]$ is also defined.

Observe that $[x_n, a] \rightarrow [x, a]$ by the continuity of $[\cdot, \cdot]$. But since $d(x_n, a) \leq \epsilon'_X$ and $x_n \in X^s(a_n, \frac{1}{2}\epsilon'_X) \subseteq X^s(a_n, \frac{1}{2}\epsilon_X)$, we have

$$[x_n, a] \in X^s(x_n, \frac{1}{2}\epsilon_X) \cap X^u(a, \epsilon_X) \subseteq X^s(a_n, \epsilon_X) \cap X^u(a, \epsilon_X) = \{[a_n, a]\}.$$

Therefore $[x_n, a] = [a_n, a] \rightarrow [a, a] = a$; i.e. $[x, a] = a$. It follows that $x \in X^s(a, \epsilon_X)$.

By the pigeonhole principle, a subsequence $(x_{n_k}) \subseteq R_i$ for some $1 \leq i \leq M$. Since $x_{n_k} \approx a_{n_k}$, it follows that $(a_{n_k}) \subseteq R_i$. Since R_i is closed, $x, a \in R_i$ as well. So $x \approx a$. \square

Lemma 6.6. *If $x_n \rightarrow x$, $y_n \rightarrow y$, and $x_n \sim y_n$ for all n , then $x \sim y$.*

Proof. By Lemma 6.2, for each n there exist $a_{n,1}, \dots, a_{n,N}$ such that

$$x_n \approx a_{n,1} \approx \dots \approx a_{n,N} \approx y_n.$$

Since X is compact, it follows by Tychonoff's Theorem that X^{N+2} with the product topology is also compact. As a result, $(x_n, a_{n,1}, \dots, a_{n,N}, y_n)$ has a convergent subsequence $(x_{n_k}, a_{n_k,1}, \dots, a_{n_k,N}, y_{n_k}) \rightarrow (x, a_1, \dots, a_N, y)$. By Lemma 6.5,

$$x \approx a_1 \approx \dots \approx a_N \approx y.$$

Hence $x \sim y$. □

6.1.2 A Metric on X/\sim

We observed earlier that the equivalence classes, $[[\cdot]]$, are larger on the boundaries of the Markov partition rectangles than they are on the interiors; by “larger” we mean that they intersect more rectangles. As a result we have the following intuitive sense of lower semi-continuity on the local unstable sets: for a sequence (x_n) converging to x , $[[x]]$ can be larger than $[[x_n]]$ but not smaller.

To define our metric on X/\sim , we will enlarge the equivalence classes $[[\cdot]]$ near the boundaries of the Markov partition rectangles, and then define paths using these enlarged classes. The distance between $[[x]]$ and $[[y]]$ will be defined to be the length of the shortest path between $[[x]]$ and $[[y]]$. A distinctive feature of our paths is that they are concatenations of very short moves within local stable or unstable sets, where the moves in the stable sets do not contribute to the length of the path. A variation of our metric appears in [9].

Recall that we chose the following constants:

1. $0 < \epsilon'_X \leq \frac{\epsilon_X}{3}$ such that $d(x, y) \leq \epsilon'_X$ implies

$$[x, y] \in X^s(x, \frac{1}{3}\epsilon_X) \cap X^u(y, \frac{1}{3}\epsilon_X),$$

2. $0 < \epsilon_0 \leq \frac{\epsilon_X}{3}$ such that if $x \in R_i \in \mathcal{P}$ then

$$X^s(x, \epsilon_0) \subseteq R_i,$$

3. $0 < \epsilon''_0 \leq \epsilon_0$ such that $d(x, y) \leq \epsilon''_0$ implies that $[x, y] \in X^s(x, \epsilon_0) \cap X^u(y, \epsilon_0)$

$$4. \epsilon'_0 = \min\left\{\frac{\epsilon_X}{3}, \frac{\epsilon''_0}{3}\right\},$$

5. $K \in \mathbb{N}$ such that if $y \in [[x]]$ then

$$f^K(y) \in X^s(f^K(x), \epsilon'_0).$$

Now choose

$$0 < \epsilon''_X \leq \epsilon_0 \tag{6.4}$$

such that $d(x, y) \leq \epsilon''_X$ implies that $d(f^k(x), f^k(y)) \leq \epsilon_X$ for each $k = 0, \dots, K$. Then choose

$$0 < \epsilon_1 \leq \epsilon_0 \tag{6.5}$$

such that $d(x, y) \leq \epsilon_1$ implies that $d(f^k(x), f^k(y)) \leq \epsilon'_0$ for each $k = -K, \dots, K$, and that $[x, y] \in X^s(x, \epsilon''_X) \cap X^u(y, \epsilon''_X)$ (a constant satisfying the second property exists by the continuity of $[\cdot, \cdot]$).

Suppose $d([[y]], [[x]]) = \inf\{d(y', x') \mid y' \in [[y]], x' \in [[x]]\} \leq \epsilon_1$. Then there exist $u \in [[y]]$ and $v \in [[x]]$ with $d(u, v) \leq \epsilon_1$. Let $z \in [[y]]$ and $x' \in [[x]]$. We observe that

$$d(f^K(z), f^K(x')) \leq d(f^K(z), f^K(u)) + d(f^K(u), f^K(v)) + d(f^K(v), f^K(x')) \leq 3\epsilon'_0 < \epsilon_X, \tag{6.6}$$

so that $[f^K(z), f^K(x')]$ is defined. Moreover, since $d(f^K(z), f^K(x')) \leq \epsilon'_X$, it follows that

$$\begin{aligned} [f^K(z), f^K(x')] &\in X^s(f^K(z), \frac{1}{3}\epsilon_X) \cap X^u(f^K(x'), \epsilon_X) \\ &\subseteq X^s(f^K(y), \epsilon_X) \cap X^u(f^K(x'), \epsilon_X) \\ &= \{[f^K(y), f^K(x')]\}; \end{aligned}$$

that is, $[f^K(z), f^K(x')] = [f^K(y), f^K(x')]$. Define

$$\langle y, [[x]] \rangle = \{f^{-K}[f^K(y), f^K(x')] \mid x' \in [[x]]\}.$$

We just showed that $\langle y, [[x]] \rangle = \langle z, [[x]] \rangle$. And we observe that

$$\begin{aligned} \langle x, [[x]] \rangle &= \{f^{-K}[f^K(x), f^K(x')] \mid x' \in [[x]]\} \\ &= \{f^{-K}[f^K(x'), f^K(x')] \mid x' \in [[x]]\} \\ &= \{x' \mid x' \in [[x]]\} \\ &= [[x]]. \end{aligned}$$

By the uniform continuity of f , f^{-1} , and $[\cdot, \cdot]$, for each $0 < \epsilon \leq \epsilon_1$ there exists

$$0 < \beta(\epsilon) \leq \epsilon$$

such that $d_2((a, b), (c, d)) \leq \beta(\epsilon)$ and $(f^K(a), f^K(b)), (f^K(c), f^K(d)) \in \text{Domain}[\cdot, \cdot]$ implies

$$d(f^{-K}[f^K(a), f^K(b)], f^{-K}[f^K(c), f^K(d)]) \leq \epsilon.$$

Lemma 6.7. *For each $x \in X$, there exists $0 < \epsilon_{[[x]]} \leq \beta(\epsilon_1)$ such that $d(y, [[x]]) \leq \epsilon_{[[x]]}$ implies $[[y]] \subseteq \langle y, [[x]] \rangle$.*

Proof. Suppose that for each $n \in \mathbb{N}$ with $\frac{1}{n} \leq \beta(\epsilon_1)$ there exists $y_n \in X$ with $d(y_n, [[x]]) \leq \frac{1}{n}$ and $[[y_n]] \not\subseteq \langle y_n, [[x]] \rangle$; i.e. there exists $y'_n \in [[y_n]]$ with $y'_n \notin \langle y_n, [[x]] \rangle$. We claim that $d(y'_n, [[x]]) \geq \epsilon_1$.

Suppose $d(y'_n, [[x]]) < \epsilon_1$. Then there exists $x' \in [[x]]$ such that $d(y'_n, x') < \epsilon_1$. By our choice of $\epsilon_1 > 0$, it follows that $[x', y'_n] \in X^s(x', \epsilon'_X) \subseteq X^s(x', \epsilon_0)$. By our choice of ϵ_0 , this implies $[x', y'_n] \approx x'$; that is, $[x', y'_n] \in [[x]]$. Furthermore, we also have $[x', y'_n] \in X^u(y'_n, \epsilon''_X)$. Therefore $d(f^k(y'_n), f^k([x', y'_n])) \leq \epsilon_X$ for each $0 \leq k \leq K$, so that $[f^k(y'_n), f^k([x', y'_n])]$ is defined for each $0 \leq k \leq K$. It follows from (S5) of Definition 2.2 that

$$f^K[y'_n, [x', y'_n]] = [f^K(y'_n), f^K[x', y'_n]].$$

Therefore

$$\begin{aligned} y'_n &= [y'_n, [x', y'_n]] \\ &= f^{-K}[f^K(y'_n), f^K[x', y'_n]] \\ &\in \langle y'_n, [[x]] \rangle \\ &= \langle y_n, [[x]] \rangle, \end{aligned}$$

a contradiction.

Since X is compact, there exists a convergent subsequence $y_{n_k} \rightarrow y$ of (y_n) , and a convergent subsequence $y'_{n_{k_j}} \rightarrow y'$ of (y'_{n_k}) . So by Lemma 6.6, $y \sim y'$. Moreover, since each $d(y'_{n_{k_j}}, [[x]]) \geq \epsilon_1$, it follows that $d(y', [[x]]) \geq \epsilon_1$.

However, since $[[x]]$ is closed as well and $d(y, [[x]]) \leq d(y, y_{n_k}) + d(y_{n_k}, [[x]]) \rightarrow 0$, it follows that $y \in [[x]]$, and hence $y' \in [[x]]$. So we have $0 = d(y', [[x]]) \geq \epsilon_1$. \square

Since $d([[y]], [[x]]) \leq \epsilon_1$ implies that $\langle y, [[x]] \rangle = \langle y', [[x]] \rangle$ for every $y' \in [[y]]$, we get the following easy consequence of Lemma 6.7.

Corollary 6.8. *If $d([[y]], [[x]]) \leq \epsilon_{[[x]]}$ then $[[y]] \subseteq \langle y, [[x]] \rangle$.*

Lemma 6.9. *If $d([[y]], [[x]]) \leq \epsilon_{[[x]]}$ and $d([[z]], [[y]]), d([[z]], [[x]]) \leq \epsilon_1$ then*

$$\langle z, [[y]] \rangle \subseteq \langle z, [[x]] \rangle.$$

Proof. Recall from (6.6) that $d([[z]], [[x]]), d([[y]], [[x]]) \leq \epsilon_1$ implies that for every $x' \in [[x]]$, we have $d(f^K(z), f^K(x')), d(f^K(y), f^K(x')) \leq \epsilon'_X$. Hence

$$\begin{aligned} [f^K(z), f^K(x')] &\in X^s(f^K(z), \epsilon_X) \cap X^u(f^K(x'), \frac{1}{3}\epsilon_X) \\ &\subseteq X^s(f^K(z), \epsilon_X) \cap X^u([f^K(y), f^K(x')], \frac{2}{3}\epsilon_X) \\ &= \{[f^K(z), [f^K(y), f^K(x')]]\}; \end{aligned}$$

that is, $[f^K(z), [f^K(y), f^K(x')]] = [f^K(z), f^K(x')]$ for every $x' \in [[x]]$.

By Corollary 6.8, we have $[[y]] \subseteq \langle y, [[x]] \rangle$, hence

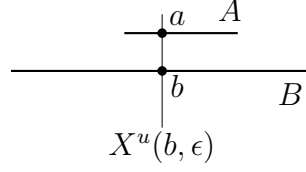
$$\begin{aligned} \langle z, [[y]] \rangle &= \{f^{-K}[f^K(z), f^K(y')] \mid y' \in [[y]]\} \\ &\subseteq \{f^{-K}[f^K(z), f^K(f^{-K}[f^K(y), f^K(x')])] \mid x' \in [[x]]\} \\ &= \{f^{-K}[f^K(z), [f^K(y), f^K(x')]] \mid x' \in [[x]]\} \\ &= \{f^{-K}[f^K(z), f^K(x')] \mid x' \in [[x]]\} \\ &= \langle z, [[x]] \rangle. \end{aligned}$$

\square

For sets $A, B \subseteq X$ and $0 < \epsilon \leq \epsilon_X$, let

$$X^u(A, B, \epsilon) = \{(a, b) \mid a \in A, b \in B, a \in X^u(b, \epsilon)\}$$

(see Figure 6.2).

Figure 6.2: $(a, b) \in X^u(A, B, \epsilon)$

We define

$$d^u(A, B) = \begin{cases} \sup\{d(a, b) \mid (a, b) \in X^u(A, B, \epsilon_1)\} & \text{if } X^u(A, B, \epsilon_1) \neq \emptyset \\ \epsilon_1 & \text{otherwise} \end{cases}.$$

As a distance function on X/\sim , d^u is clearly symmetric and we will show that it is reflexive, but the triangle inequality fails. To prove the reflexivity of d^u on X/\sim , let $(y, z) \in X^u([[x]], [[x]], \epsilon_1)$. Then by our choice of K ,

$$f^K(y) \in X^s(f^K(z), \epsilon'_0). \quad (6.7)$$

Since we also have $d(y, z) \leq \epsilon_1$, it follows from (6.7) and our choice of ϵ_1 that $y \in X^s(z, \epsilon'_0)$. So we have

$$y \in X^s(z, \epsilon'_0) \cap X^u(z, \epsilon_1) \subseteq X^s(z, \epsilon_X) \cap X^u(z, \epsilon_X) = \{z\};$$

that is, $y = z$. It follows that

$$d^u([[x]], [[x]]) = 0. \quad (6.8)$$

The following result is a version of continuity for the mapping $\langle \cdot, [[x]] \rangle$.

Lemma 6.10. *Let $0 < \epsilon \leq \epsilon_1$. If $d([[y]], [[x]]) \leq \epsilon_1$, $d([[z]], [[x]]) \leq \epsilon_1$, and $d([[y]], [[z]]) \leq \beta(\epsilon)$ then*

$$X^u(\langle y, [[x]] \rangle, \langle z, [[x]] \rangle, \epsilon_1) = \{(f^{-K}[f^K(y), f^K(x')], f^{-K}[f^K(z), f^K(x')]) \mid x' \in [[x]]\}$$

and $d^u(\langle y, [[x]] \rangle, \langle z, [[x]] \rangle) \leq \epsilon$.

Proof. Suppose $(f^{-K}[f^K(y), f^K(x')], f^{-K}[f^K(z), f^K(x'')]) \in X^u(\langle y, [[x]] \rangle, \langle z, [[x]] \rangle, \epsilon_1)$ for some $x', x'' \in [[x]]$; we want to show that $x' = x''$. By our choice of ϵ_1 , it follows

that

$$[f^K(y), f^K(x')] \in X^u([f^K(z), f^K(x'')], \epsilon'_0) \subseteq X^u([f^K(z), f^K(x'')], \frac{1}{3}\epsilon_X). \quad (6.9)$$

Furthermore, since $d([[y]], [[x]]) \leq \epsilon_1$ and $[[y]]$ and $[[x]]$ are closed, it follows that $d(y', x''') \leq \epsilon_1$ for some $y' \in [[y]]$ and $x''' \in [[x]]$. Therefore

$$\begin{aligned} d(f^K(y), f^K(x')) &\leq d(f^K(y), f^K(y')) + d(f^K(y'), f^K(x''')) + d(f^K(x'''), f^K(x')) \\ &\leq 3\epsilon'_0 \\ &\leq \epsilon'_X, \end{aligned}$$

so that

$$[f^K(y), f^K(x')] \in X^u(f^K(x'), \frac{1}{3}\epsilon_X). \quad (6.10)$$

Similarly, we have

$$[f^K(z), f^K(x'')] \in X^u(f^K(x''), \frac{1}{3}\epsilon_X). \quad (6.11)$$

Combining (6.10), (6.9), and (6.11) we get

$$f^K(x') \in X^u([f^K(y), f^K(x')], \frac{1}{3}\epsilon_X) \subseteq X^u([f^K(z), f^K(x'')], \frac{2}{3}\epsilon_X) \subseteq X^u(f^K(x''), \epsilon_X).$$

But then we have $f^K(x') \in X^s(f^K(x''), \epsilon'_0) \cap X^u(f^K(x''), \epsilon_X) = \{f^K(x'')\}$; that is, $x' = x''$.

Now, let $u \in [[x]]$. Clearly $f^{-K}[f^K(y), f^K(u)], f^{-K}[f^K(z), f^K(u)] \in X^u(u, \epsilon_X)$. Moreover, since $d([[y]], [[z]]) \leq \beta(\epsilon)$, we have $d(y'', z') \leq \beta(\epsilon)$ for some $y'' \in [[y]]$ and $z' \in [[z]]$. So by our choice of $\beta(\epsilon)$, we have

$$\begin{aligned} d(f^{-K}[f^K(y), f^K(u)], f^{-K}[f^K(z), f^K(u)]) \\ = d(f^{-K}[f^K(y''), f^K(u)], f^{-K}[f^K(z'), f^K(u)]) \\ \leq \epsilon, \end{aligned} \quad (6.12)$$

and hence $f^{-K}[f^K(y), f^K(u)] \in X^u(f^{-K}[f^K(z), f^K(u)], \epsilon)$. That is,

$$(f^{-K}[f^K(y), f^K(u)], f^{-K}[f^K(z), f^K(u)]) \in X^u(\langle y, [[x]] \rangle, \langle z, [[x]] \rangle, \epsilon_1).$$

It follows from (6.12) that $d^u(\langle y, [[x]] \rangle, \langle z, [[x]] \rangle) \leq \epsilon$. \square

Since $\langle x, [[x]] \rangle = [[x]]$ for all x , we get the following easy consequence of Lemma

6.10.

Corollary 6.11. *Let $0 < \epsilon \leq \epsilon_1$. If $d([[y]], [[x]]) \leq \beta(\epsilon)$ then*

$$X^u(\langle y, [[x]] \rangle, [[x]], \epsilon_1) = \{(f^{-K}[f^K(y), f^K(x')], x') \mid x' \in [[x]]\}$$

and $d^u(\langle y, [[x]] \rangle, [[x]]) \leq \epsilon$.

It is clear that the equivalence classes $[[\cdot]]$ are larger on $\partial\mathcal{P}$. For x near $\partial\mathcal{P}$, we want to enlarge $[[x]]$ using $\langle x, [[y]] \rangle$ for some $y \in \partial\mathcal{P}$ (see Figure 6.3).

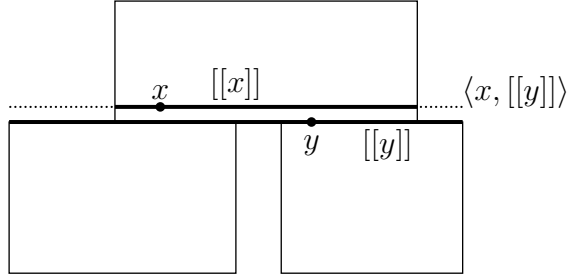


Figure 6.3: Enlarging the $[[\cdot]]$'s near $\partial\mathcal{P}$

Let $B^o(S, \epsilon)$ denote the open ball around the set S of radius ϵ .

The collection $\{B^o([[x]], \beta(\frac{1}{4}\epsilon_{[[x]]})) \mid x \in \partial\mathcal{P}\}$ clearly covers $\partial\mathcal{P}$, so there exists a finite subcover consisting of balls around $[[x_1]], \dots, [[x_L]]$. That is, for any $x \in \partial\mathcal{P}$ there exists $1 \leq l \leq L$ such that $d(x, [[x_l]]) < \beta(\frac{1}{4}\epsilon_{[[x_l]]})$. Hence, by Lemma 6.7 and Corollary 6.11,

$$d^u([[x]], [[x_l]]) \leq d^u(\langle x, [[x_l]] \rangle, [[x_l]]) \leq \frac{1}{4}\epsilon_{[[x_l]]}.$$

Denote $\mathcal{C} \equiv \bigcup_{l=1}^L B^o([[x_l]], \beta(\frac{1}{4}\epsilon_{[[x_l]]}))$.

Define

$$[x] = [[x]] \cup \left(\bigcup \{ \langle x, [[x_l]] \rangle \mid 1 \leq l \leq L, d^u([[x]], [[x_l]]) < \frac{1}{2}\epsilon_{[[x_l]]} \} \right).$$

Denote $\mathcal{O}(\partial\mathcal{P}) = \{x \in X \mid d^u([[x]], [[x_l]]) < \frac{1}{2}\epsilon_{[[x_l]]} \text{ for some } 1 \leq l \leq L\}$. We observed above that $\partial\mathcal{P} \subseteq \mathcal{C} \subseteq \mathcal{O}(\partial\mathcal{P})$, and so $\mathcal{O}(\partial\mathcal{P})^c \subseteq \mathcal{C}^c \subseteq \text{Int}(\mathcal{P})$. In addition, notice that

$$[x] = [[x]]$$

if $x \in \mathcal{O}(\partial\mathcal{P})^c$.

In fact, each $x \in \mathcal{O}(\partial\mathcal{P})$ is only enlarged by at most one $[[x_m]]$.

Lemma 6.12. *If $x \in \mathcal{O}(\partial\mathcal{P})$ then*

$$[x] = \langle x, [[x_m]] \rangle$$

for some $1 \leq m \leq L$ with $d^u([[x]], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]}$.

Proof. Suppose that $d^u([[x]], [[x_l]]) < \frac{1}{2}\epsilon_{[[x_l]]}$ and $d^u([[x]], [[x_k]]) < \frac{1}{2}\epsilon_{[[x_k]]}$, where $1 \leq l, k \leq L$. Since $d([[x]], [[x_l]]) \leq d^u([[x]], [[x_l]]) < \frac{1}{2}\epsilon_{[[x_l]]}$, we have from Corollary 6.8 that $[[x]] \subseteq \langle x, [[x_l]] \rangle$. That is,

$$x = f^{-K}[f^K(x), f^K(x'_l)] \quad (6.13)$$

for some $x'_l \in [[x_l]]$. Since $\epsilon_{[[x_l]]} \leq \beta(\epsilon_1)$, it follows from Lemma 6.10 that

$$d(x, x'_l) \leq d^u(\langle x, [[x_l]] \rangle, \langle x_l, [[x_l]] \rangle) \leq \epsilon_1.$$

Since (6.13) implies $x \in X^u(x'_l, \epsilon_X)$, we have $x \in X^u(x'_l, \epsilon_1)$. Similarly $x \in X^u(x'_k, \epsilon_1)$ for some $x'_k \in [[x_k]]$. Without loss of generality, $d^u([[x]], [[x_k]]) \leq d^u([[x]], [[x_l]])$. Hence

$$\begin{aligned} d(x'_k, [[x_l]]) &\leq d(x'_k, x'_l) \\ &\leq d(x'_k, x) + d(x, x'_l) \\ &\leq d^u([[x_k]], [[x]]) + d^u([[x]], [[x_l]]) \\ &\leq 2d^u([[x]], [[x_l]]) \\ &< \epsilon_{[[x_l]]}, \end{aligned}$$

that is, $d([[x_k]], [[x_l]]) < \epsilon_{[[x_l]]}$. It follows from Lemma 6.9 that

$$\langle x, [[x_k]] \rangle \subseteq \langle x, [[x_l]] \rangle.$$

So we choose $1 \leq m \leq L$ such that

$$d^u([[x]], [[x_m]]) = \max\{d^u([[x]], [[x_l]]) \mid 1 \leq l \leq L, d^u([[x]], [[x_l]]) < \epsilon_{[[x_l]]}/2\}.$$

□

For $x, y \in X$, let $P(x, y)$ consist of finite paths $p = (p_0, p_1, \dots, p_I)$ satisfying $p_0 = x$, $p_I = y$, and $X^u([p_i], [p_{i+1}], \epsilon_1) \neq \emptyset$ for each $0 \leq i < I$. We define the length of a path $p = (p_0, p_1, \dots, p_I)$ to be

$$l(p) = \sum_{i=0}^{I-1} d^u([p_i], [p_{i+1}]),$$

and we define a function δ on $X \times X$ by

$$\delta(x, y) = \inf\{1, l(p) \mid p \in P(x, y)\}.$$

We aim to prove that $\delta([x], [y]) = \delta(x, y)$ defines a metric on X/\sim .

Choose

$$0 < \epsilon'_1 \leq \beta(\epsilon_1)$$

such that $d_2((a, b), (c, d)) \leq \epsilon'_1$ implies that $d([a, b], [c, d]) \leq \epsilon_1$ for all $(a, b), (c, d) \in \text{Domain}[\cdot, \cdot]$, and such that $d(x, y) \leq \epsilon'_1$ implies that $d(f(x), f(y)) \leq \epsilon_1$.

Since $\partial\mathcal{P}$ and $\mathcal{C}^c = \bigcap_{l=1}^L B^o([x_l], \beta(\frac{1}{4}\epsilon_{[x_l]}))^c$ are closed and disjoint,

$$d(\partial\mathcal{P}, \mathcal{C}^c) > 0.$$

Let

$$\eta_0 = \min\{\frac{1}{2}d(\partial\mathcal{P}, \mathcal{C}^c), \epsilon'_1\}.$$

Then

$$B(\mathcal{C}^c, \eta_0) \cap \partial\mathcal{P} = \emptyset \tag{6.14}$$

Choose

$$0 < \eta'_0 \leq \beta(\eta_0)$$

such that $d(x, y) \leq \eta'_0$ implies that $[x, y] \in X^s(x, \eta_0) \cap X^u(y, \eta_0)$.

For $A \subseteq X$ and $0 < \epsilon \leq \epsilon_X$, we will denote $X^u(A, \epsilon) = \bigcup_{x \in A} X^u(x, \epsilon)$.

As expected, if $x \in \mathcal{C}^c \cap \text{Int}R_j$ then all short enough paths starting at x will be entirely contained in $\text{Int}R_j$.

Lemma 6.13. *Suppose $x \in \mathcal{C}^c \cap R_j$. If $p = (p_0, \dots, p_I)$ with $p_0 = x$ and $l(p) \leq \eta'_0$, then $p_i \in \text{Int}R_j$ for all i .*

Proof. Since $x \in \mathcal{C}^c \cap R_j$, it follows that $x \in \text{Int}R_j$. Let us show that the intersection $X^u([[x]], \eta'_0) \cap \partial R_j$ is empty. Let $x' \in [[x]] \subseteq \text{Int}R_j$ and suppose that there exists $y \in X^u(x', \eta'_0) \cap \partial R_j$. Then $[y, x] \in \partial R_j$ and $d(x, [y, x]) \leq \eta_0$, which contradicts (6.14). Therefore $X^u([[x]], \eta'_0) \subseteq \text{Int}R_j$.

Let $p \in P(x, \cdot)$ and suppose $l(p) \leq \eta'_0$. We have already observed that $p_i \in \text{Int}R_j$ for $i = 0$. Suppose it is true for $i = 0, \dots, k$, so that $[[p_i]] = X^s(p_i, R_j) \subseteq \text{Int}R_j$ for $i = 0, \dots, k$. Since $d^u([[p_i]], [[p_{i+1}]]) \leq d^u([p_i], [p_{i+1}]) \leq \eta'_0$ for all i , it follows that there exist

$$(q_i, q'_i) \in X^u([[p_i]], [[p_{i+1}]], \eta'_0)$$

for all i . Choose $(p'_k, p'_{k+1}) \in X^u([[p_k]], [[p_{k+1}]], \epsilon_1)$. Since $p'_k, q_i, q'_i \in R_j$ for $i = 0, \dots, k-1$, it follows that $[q_i, p'_k] \in [[p_i]]$ and $[q'_i, p'_k] \in [[p_{i+1}]]$ for $i = 0, \dots, k-1$. Moreover, since $d(q_i, q'_i) \leq \epsilon'_1$, it follows that $([q_i, p'_k], [q'_i, p'_k]) \in X^u([[p_i]], [[p_{i+1}]], \epsilon_1)$ for $i = 0, \dots, k-1$. Therefore

$$\begin{aligned} d(p'_{k+1}, [q_0, p'_k]) &\leq d(p'_{k+1}, p'_k) + \sum_{i=0}^{k-1} d([q_i, p'_k], [q'_i, p'_k]) \\ &\leq \sum_{i=0}^k d^u([[p_i]], [[p_{i+1}]]) \\ &\leq \sum_{i=0}^k d^u([p_i], [p_{i+1}]) \\ &\leq l(p) \\ &\leq \eta'_0. \end{aligned}$$

Combining this with $p'_{k+1} \in X^u(p'_k, \epsilon_1) \subseteq X^u([q_0, p'_k], \epsilon_X)$ gives

$$p'_{k+1} \in X^u([q_0, p'_k], \eta'_0) \subseteq X^u([[x]], \eta'_0) \subseteq \text{Int}R_j.$$

Hence $p_{k+1} \in [[p'_{k+1}]] \subseteq \text{Int}R_j$. □

Corollary 6.14. *If $x \in \mathcal{C}^c$, and $p \in P(x, y)$ with $l(p) \leq \eta'_0$ then*

$$(x, [y, x]) \in X^u([[x]], [[y]], l(p)).$$

Proof. Let $p = (p_0, \dots, p_I) \in P(x, y)$ with $l(p) \leq \eta'_0$. Say $x \in R_j$. Then by Lemma 6.13, $p_i \in \text{Int}R_j$ for all $i = 0, \dots, I$. Therefore $[p_i, x] \approx p_i$ for $i = 0, \dots, I$.

Since $l(p) \leq \eta'_0 \leq \epsilon'_1$, it follows that for each $i = 0, \dots, I - 1$ there exist

$$(q_i, q'_i) \in X^u([p_i], [[p_{i+1}]], \epsilon'_1).$$

By our choice of ϵ'_1 , it follows that

$$([p_i, x], [p_{i+1}, x]) = ([q_i, x], [q'_i, x]) \in X^u([p_i], [[p_{i+1}]], \epsilon_1) \subseteq X^u([p_i], [p_{i+1}], \epsilon_1)$$

for each $i = 0, \dots, I - 1$. Therefore

$$d(x, [y, x]) \leq \sum_{i=0}^{I-1} d([p_i, x], [p_{i+1}, x]) \leq \sum_{i=0}^{I-1} d^u([p_i], [p_{i+1}]) \leq l(p);$$

that is, $(x, [y, x]) \in X^u([x], [[y]], l(p))$. □

Lemma 6.15. *Suppose $d^u([x], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]}$ for some $1 \leq m \leq L$. If $p \in P(x, \cdot)$ and $l(p) < \frac{1}{2}\epsilon_{[[x_m]]} - d^u([x], [[x_m]])$, then $d^u([p_i], [[x_m]]) \leq d^u([x], [[x_m]]) + l(p)$ for all i .*

Proof. Let $p \in P(x, \cdot)$ with $l(p) < \frac{1}{2}\epsilon_{[[x_m]]} - d^u([x], [[x_m]])$.

It is trivial that

$$d^u([p_i], [[x_m]]) \leq d^u([x], [[x_m]]) + l(p)$$

holds for $i = 0$. Suppose that it holds for $i = 0, \dots, n$. Then

$$d([p_i], [[x_m]]) \leq d^u([p_i], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]},$$

so that $[p_i] \subseteq \langle p_i, [[x_m]] \rangle \subseteq [p_i]$ for $i = 0, \dots, n$.

We begin by showing that $X^u([p_{n+1}], [[x_m]], \epsilon_1)$ is nonempty. Choose

$$(p'_n, p'_{n+1}) \in X^u([p_n], [[p_{n+1}]], \epsilon_1).$$

Since $[p_n] \subseteq \langle p_n, [[x_m]] \rangle$, we have

$$p'_n = f^{-K}[f^K(p_n), f^K(x'_m)]$$

for some $x'_m \in [[x_m]]$. Define

$$p'_i = f^{-K}[f^K(p_i), f^K(x'_m)]$$

for $i = 0, \dots, n-1$. Then $p'_i \in \langle p_i, [[x_m]] \rangle \subseteq [p_i]$ for each $i = 0, \dots, n$. Moreover, since

$$d([[p_i]], [[x_m]]) \leq d^u([[p_i]], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]} < \epsilon_1 \quad (6.15)$$

and

$$d([[p_i]], [[p_{i+1}]]) \leq d^u([[p_i]], [[p_{i+1}]]) \leq d^u([p_i], [p_{i+1}]) \leq l(p) < \beta(\epsilon_1), \quad (6.16)$$

it follows from Lemma 6.10 that

$$(p'_i, p'_{i+1}) \in X^u(\langle p_i, [[x_m]] \rangle, \langle p_{i+1}, [[x_m]] \rangle, \epsilon_1) \subseteq X^u([p_i], [p_{i+1}], \epsilon_1),$$

$i = 0, \dots, n-1$. By Corollary 6.11, $d([[x]], [[x_m]]) \leq d^u([[x]], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]} < \beta(\epsilon_1)$ implies $(x'_m, p'_0) \in X^u([[x_m]], [[x]], \epsilon_1)$, and hence

$$\begin{aligned} d(x'_m, p'_{n+1}) &\leq d(x'_m, p'_0) + \sum_{i=0}^n d(p'_i, p'_{i+1}) \\ &\leq d^u([[x_m]], [[x]]) + \sum_{i=0}^n d^u([p_i], [p_{i+1}]) \\ &\leq d^u([[x_m]], [[x]]) + l(p) \\ &< d^u([[x_m]], [[x]]) + \frac{1}{2}\epsilon_{[[x_m]]} - d^u([[x]], [[x_m]]) \\ &= \frac{1}{2}\epsilon_{[[x_m]]}. \end{aligned}$$

Since we also have $p'_{n+1} \in X^u(p'_n, \epsilon_1) \subseteq X^u(x'_m, \epsilon_X)$, this gives us that $p'_{n+1} \in X^u(x'_m, \frac{1}{2}\epsilon_{[[x_m]]}) \subseteq X^u(x'_m, \epsilon_1)$; that is, $(x'_m, p'_{n+1}) \in X^u([[p_{n+1}]], [[x_m]], \epsilon_1)$.

Now choose arbitrary $(p''_{n+1}, x''_m) \in X^u([[p_{n+1}]], [[x_m]], \epsilon_1)$. By our choice of ϵ_1 , it follows that

$$\begin{aligned} [f^K(p''_{n+1}), f^K(x''_m)] &\in X^s(f^K(p''_{n+1}), \frac{1}{3}\epsilon_X) \cap X^u(f^K(x''_m), \frac{1}{3}\epsilon_X) \\ &\subseteq X^s(f^K(p''_{n+1}), \epsilon_X) \cap X^u(f^K(p''_{n+1}), \frac{2}{3}\epsilon_X) \\ &= \{f^K(p''_{n+1})\}; \end{aligned}$$

that is, $p''_{n+1} = f^{-K}[f^K(p''_{n+1}), f^K(x''_m)] = f^{-K}[f^K(p_{n+1}), f^K(x''_m)]$. Define

$$p''_i = f^{-K}[f^K(p_i), f^K(x''_m)]$$

for $i = 0, \dots, n$. Then $p''_i \in \langle p_i, [[x_m]] \rangle \subseteq [p_i]$ for each $i = 0, \dots, n$, and $p''_{n+1} \in [[p_{n+1}]] \subseteq [p_{n+1}]$.

Since (6.16) holds for all i and (6.15) holds for $i = 0, \dots, n$, it follows from Lemma 6.10 that

$$(p''_i, p''_{i+1}) \in X^u(\langle p_i, [[x_m]] \rangle, \langle p_{i+1}, [[x_m]] \rangle, \epsilon_1) \subseteq X^u([p_i], [p_{i+1}], \epsilon_1)$$

for $i = 0, \dots, n-1$. Similarly, since $d([[p_{n+1}]], [[x_m]]) \leq d(x'_m, p'_{n+1}) < \frac{1}{2}\epsilon_{[[x_m]]} \leq \epsilon_1$, it follows that $(p''_n, p''_{n+1}) \in X^u(\langle p_n, [[x_m]] \rangle, \langle p_{n+1}, [[x_m]] \rangle, \epsilon_1)$; that is,

$$(p''_n, p''_{n+1}) \in X^u(\langle p_n, [[x_m]] \rangle, [[p_{n+1}]], \epsilon_1) \subseteq X^u([p_n], [p_{n+1}], \epsilon_1).$$

And finally, since $d([[x]], [[x_m]]) \leq d^u([[x]], [[x_m]]) < \epsilon_{[[x_m]]} \leq \beta(\epsilon_1)$, it follows from Corollary 6.11 that

$$(x''_m, p''_0) \in X^u([[x_m]], [[x]], \epsilon_1).$$

Therefore

$$\begin{aligned} d(p''_{n+1}, x''_m) &\leq d(x''_m, p''_0) + \sum_{i=0}^n d(p''_i, p''_{i+1}) \\ &\leq d^u([[x_m]], [[x]]) + \sum_{i=0}^n d^u([p_i], [p_{i+1}]) \\ &\leq d^u([[x_m]], [[x]]) + l(p). \end{aligned}$$

Since (p''_{n+1}, x''_m) were arbitrary, it follows that

$$d^u([[p_{n+1}]], [[x_m]]) \leq d^u([[x_m]], [[x]]) + l(p).$$

□

Corollary 6.16. *Suppose $d^u([[x]], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]}$ for some $1 \leq m \leq L$. If $p \in P(x, \cdot)$ and $l(p) < \min\{\epsilon_{[[x]]}, \frac{1}{2}\epsilon_{[[x_m]]} - d^u([[y]], [[x_m]])\}$, then for all i ,*

$$[[p_i]] \subseteq \langle p_i, [[x]] \rangle \subseteq \langle p_i, [[x_m]] \rangle \subseteq [p_i].$$

Proof. Let $p \in P(x, \cdot)$ with $l(p) < \min\{\epsilon_{[[x]]}, \frac{1}{2}\epsilon_{[[x_m]]} - d^u([y], [[x_m]])\}$.

It is clear that

$$d([[p_i]], [[x]]) \leq \epsilon_{[[x]]} \quad (6.17)$$

holds for $i = 0$. Suppose that it holds for $i = 0, \dots, n$.

Choose $(p'_n, p'_{n+1}) \in X^u([p_n], [p_{n+1}], \epsilon_1)$. Since $[p_n] \subseteq \langle p_n, [[x]] \rangle$, we have

$$p'_n = f^{-K}[f^K(p_n), f^K(x')]$$

for some $x' \in [[x]]$. And $[[x]] \subseteq \langle x, [[x_m]] \rangle$, so $x' = f^{-K}[f^K(x), f^K(x'_m)]$ for some $x'_m \in [[x_m]]$. Define

$$p'_i = f^{-K}[f^K(p_i), f^K(x'_m)]$$

for $i = 0, \dots, n-1$. Then $p'_i \in \langle p_i, [[x_m]] \rangle \subseteq [p_i]$ for each $i = 0, \dots, n$. Moreover, since $d^u([p_i], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]} < \epsilon_1$ and $d^u([p_i], [p_{i+1}]) \leq l(p) \leq \epsilon_{[[x]]} \leq \beta(\epsilon_1)$, it follows from Lemma 6.10 that

$$(p'_i, p'_{i+1}) \in X^u(\langle p_i, [[x_m]] \rangle, \langle p_{i+1}, [[x_m]] \rangle, \epsilon_1) \subseteq X^u([p_i], [p_{i+1}], \epsilon_1)$$

for $i = 0, \dots, n-1$. Therefore

$$\begin{aligned} d([p_{n+1}], [[x]]) &\leq d(x', p'_{n+1}) \\ &\leq \sum_{i=0}^n d(p'_i, p'_{i+1}) \\ &\leq \sum_{i=0}^n d^u([p_i], [p_{i+1}]) \\ &\leq l(p) \\ &< \epsilon_{[[x]]}. \end{aligned}$$

Therefore (6.17) holds for all i .

Choose i . We have

$$[[p_i]] \subseteq \langle p_i, [[x]] \rangle$$

by (6.17) and Corollary 6.8. Moreover, $d^u([p_i], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]}$ by Lemma 6.15, so that

$$\langle p_i, [[x_m]] \rangle \subseteq [p_i]$$

by the definition of $\lceil \cdot \rceil$. To get the middle containment, we want to apply Lemma 6.9. Our hypothesis gives $d(\lceil [x] \rceil, \lceil [x_m] \rceil) \leq d^u(\lceil [x] \rceil, \lceil [x_m] \rceil) < \epsilon_{\lceil [x_m] \rceil}$, and we showed that $d(\lceil [p_i] \rceil, \lceil [x] \rceil) \leq \epsilon_{\lceil [x] \rceil} \leq \epsilon_1$ and $d(\lceil [p_i] \rceil, \lceil [x_m] \rceil) \leq d^u(\lceil [p_i] \rceil, \lceil [x_m] \rceil) < \epsilon_{\lceil [x_m] \rceil} \leq \epsilon_1$. Therefore $\langle p_i, \lceil [x] \rceil \rangle \subseteq \langle p_i, \lceil [x_m] \rceil \rangle$. \square

Observe that if $x' \in \lceil [x] \rceil$, $y' \in \lceil [y] \rceil$ and $p = (p_0, \dots, p_I) \in P(x, y)$, then

$$q = (x', p_0, \dots, p_I, y') \in P(x', y')$$

and $l(p) = l(q)$. It follows that $\delta(x', y') = \delta(x, y)$. As a result, we can define δ on X/\sim by $\delta(\lceil [x] \rceil, \lceil [y] \rceil) = \delta(x, y)$. We want to show that δ is a metric on X/\sim . We will need the following.

Lemma 6.17. *If $\delta(x, y) = 0$ then $\lceil [x] \rceil = \lceil [y] \rceil$.*

Proof. Case 1: $x \in \mathcal{O}(\partial\mathcal{P})^c$

Then $x \in \text{Int}(R_j)$ for some j . Since $\delta(x, y) = 0$, there exists a path $p \in P(x, y)$ with $l(p) \leq \eta_0$. Recall that $\mathcal{O}(\partial\mathcal{P})^c \subseteq \mathcal{C}^c$. By Lemma 6.13, $y \in R_j$, and by Corollary 6.14,

$$d(x, [y, x]) \leq l(p).$$

Since this is true for all $l(p) \leq \eta_0$, letting $l(p) \rightarrow 0$ we see that $x = [y, x]$, and hence $x \in X^s(y, \epsilon_X)$. So $x \approx y$.

Case 2: $x \in \mathcal{O}(\partial\mathcal{P})$

By Lemma 6.12, $\lceil x \rceil = \langle x, \lceil [x_m] \rceil \rangle$ for some $1 \leq m \leq L$ with $d^u(\lceil [x] \rceil, \lceil [x_m] \rceil) < \frac{1}{2}\epsilon_{\lceil [x_m] \rceil}$. Since $\delta(x, y) = 0$, there is a path $p = (p_0, \dots, p_I) \in P(x, y)$ with $l(p) < \frac{1}{2}\epsilon_{\lceil [x_m] \rceil} - d^u(\lceil [x] \rceil, \lceil [x_m] \rceil)$. By Lemma 6.15,

$$d(\lceil [p_i] \rceil, \lceil [x_m] \rceil) \leq d^u(\lceil [p_i] \rceil, \lceil [x_m] \rceil) \leq d^u(\lceil [x] \rceil, \lceil [x_m] \rceil) + l(p) < \frac{1}{2}\epsilon_{\lceil [x_m] \rceil}$$

for all $i = 0, \dots, I$, so that

$$\lceil [p_i] \rceil \subseteq \langle p_i, \lceil [x_m] \rceil \rangle \subseteq \lceil p_i \rceil$$

for all $i = 0, \dots, I$.

Therefore $y = f^{-K}[f^K(y), f^K(x'_m)]$ for some $x'_m \in \lceil [x_m] \rceil$. For $i = 0, \dots, I$, let

$$p'_i = f^{-K}[f^K(p_i), f^K(x'_m)].$$

Since $d(\lceil p_i \rceil, \lceil p_{i+1} \rceil) \leq d^u(\lceil p_i \rceil, \lceil p_{i+1} \rceil) \leq d^u(\lceil p_i \rceil, \lceil p_{i+1} \rceil) \leq l(p) \leq \beta(\epsilon_1)$ and $d(\lceil p_i \rceil, \lceil x_m \rceil) < \frac{1}{2}\epsilon_{\lceil x_m \rceil} < \epsilon_1$, it follows from Lemma 6.10 that

$$(p'_i, p'_{i+1}) \in X^u(\langle p_i, \lceil x_m \rceil \rangle, \langle p_{i+1}, \lceil x_m \rceil \rangle, \epsilon_1) \subseteq X^u(\lceil p_i \rceil, \lceil p_{i+1} \rceil, \epsilon_1)$$

for $i = 0, \dots, I$. So

$$\begin{aligned} d(y, \lceil x \rceil) &\leq d(p'_0, y) \\ &\leq \sum_{i=0}^{I-1} d(p'_i, p'_{i+1}) \\ &\leq \sum_{i=0}^{I-1} d^u(\lceil p_i \rceil, \lceil p_{i+1} \rceil) \\ &\leq l(p). \end{aligned}$$

Letting $l(p) \rightarrow 0$, we see that $d(y, \lceil x \rceil) = 0$; that is, $y \in \lceil x \rceil$. \square

Proposition 6.18. δ defines a metric on X/\sim .

Proof. (i) $\delta(\lceil x \rceil, \lceil y \rceil) = 0 \Leftrightarrow \lceil x \rceil = \lceil y \rceil$

If $\lceil x \rceil = \lceil y \rceil$ then $\lceil x \rceil = \lceil y \rceil$ and $(x, y) \in P(x, y)$. So $\delta(\lceil x \rceil, \lceil y \rceil) \leq d^u(\lceil x \rceil, \lceil y \rceil)$. Let's show that $d^u(\lceil x \rceil, \lceil x \rceil) = 0$ for all $x \in X$. From Lemma 6.12, we see that either $\lceil x \rceil = \lceil x \rceil$ or $\lceil x \rceil = \langle x, \lceil x_m \rceil \rangle$ for some $1 \leq m \leq L$ with $d^u(\lceil x \rceil, \lceil x_m \rceil) < \frac{1}{2}\epsilon_{\lceil x_m \rceil}$. We have from (6.8) that $d^u(\lceil x \rceil, \lceil x \rceil) = 0$ for all x , and from Lemma 6.10, we have

$$\begin{aligned} d^u(\langle x, \lceil x_m \rceil \rangle, \langle x, \lceil x_m \rceil \rangle) &= \sup\{d(f^{-K}[f^K(x), f^K(x'_m)], f^{-K}[f^K(x), f^K(x'_m)]) \mid \\ &\quad x'_m \in \lceil x_m \rceil\} \\ &= 0. \end{aligned}$$

Conversely, we proved in Lemma 6.17 that $\delta(\lceil x \rceil, \lceil y \rceil) = \delta(x, y) = 0$ implies $\lceil x \rceil = \lceil y \rceil$.

(ii) $\delta(\lceil x \rceil, \lceil y \rceil) = \delta(\lceil y \rceil, \lceil x \rceil)$ for all $\lceil x \rceil, \lceil y \rceil \in X/\sim$

This follows immediately from the observation that $(p_0, \dots, p_I) \in P(x, y)$ if and only if $(p_I, \dots, p_0) \in P(y, x)$, and the symmetry of d^u .

(iii) $\delta(\lceil x \rceil, \lceil y \rceil) \leq \delta(\lceil x \rceil, \lceil z \rceil) + \delta(\lceil z \rceil, \lceil y \rceil)$ for all $\lceil x \rceil, \lceil y \rceil, \lceil z \rceil \in X/\sim$

If $\delta([[x]], [[z]]) = 1$ or $\delta([[z]], [[y]]) = 1$, this holds trivially. So assume that $\delta([[x]], [[z]]) < 1$ and $\delta([[z]], [[y]]) < 1$. If $p = (p_0, \dots, p_I) \in P(x, z)$ and $q = (q_0, \dots, q_J) \in P(z, y)$, then $(p_0, \dots, p_I, q_0, q_1, \dots, q_J) \in P(x, y)$, that is, $P(x, y) \neq \emptyset$. Therefore

$$\begin{aligned}
\delta([[x]], [[y]]) &\leq \inf\{l(p) \mid p \in P(x, y)\} \\
&\leq \inf\{l(p) \mid p = (p_0, \dots, p_I) \in P(x, y), p_i = z \text{ for some } 0 \leq i \leq I\} \\
&= \inf\{l(p_0, \dots, p_i) + l(p_i, \dots, p_I) \mid p = (p_0, \dots, p_I) \in P(x, y), p_i = z\} \\
&= \inf\{l(p') + l(p'') \mid p' \in P(x, z), p'' \in P(z, y)\} \\
&= \inf\{l(p') \mid p' \in P(x, z)\} + \inf\{l(p'') \mid p'' \in P(z, y)\} \\
&= \delta([[x]], [[z]]) + \delta([[z]], [[y]]).
\end{aligned}$$

□

We finish this section by showing that δ gives the quotient topology on X/\sim , from which it follows that $(X/\sim, \delta)$ is compact. We will need two more lemmas first. Recall that we chose $0 < \epsilon'_1 \leq \beta(\epsilon_1)$ such that $d_2((a, b), (c, d)) \leq \epsilon'_1$ implies $d([a, b], [c, d]) \leq \epsilon_1$ for all $(a, b), (c, d) \in \text{Domain}([\cdot, \cdot])$.

Lemma 6.19. *Let $y \in \mathcal{C}^c \cap R_i$. Then for every $z \in \text{Int}(R_i)$ with $d([[z]], [[y]]) \leq \epsilon'_1$ we have*

$$X^u([z], [[y]], \epsilon_1) = \{([z, y'], y') \mid y' \in [[y]]\}.$$

Proof. Since $y \in \mathcal{C}^c \cap R_i \subseteq \text{Int}R_i$, it follows that $[[y]] = X^s(y, R_i)$. Similarly, $z \in \text{Int}(R_i)$ implies $[[z]] = X^s(z, R_i) \subseteq [z]$. And since $d([[z]], [[y]]) \leq \epsilon'_1$, it follows that there are $z_0 \in [[z]]$ and $y_0 \in [[y]]$ with $d(z_0, y_0) \leq \epsilon'_1$.

“ \supseteq ” Let $y' \in [[y]]$. Since $z, y' \in R_i$, it follows that $[z, y'] \in [[z]]$. Moreover, $z_0 \in X^s(z, R_i)$ and $y_0 \in X^s(y, R_i)$ gives

$$d([z, y'], y') = d([z_0, y'], [y_0, y']) \leq \epsilon_1.$$

That is, $[z, y'] \in X^u(y', \epsilon_1)$.

“ \subseteq ” Let $(z', y') \in X^u([z], [[y]], \epsilon_1)$. Then $y' \in [[y]]$, so it follows from “ \supseteq ” that we also have $([z, y'], y') \in X^u([z], [[y]], \epsilon_1)$.

If $[z] = [[z]]$, then we have

$$z' \in X^s(z, \frac{1}{2}\epsilon'_X) \cap X^u(y', \epsilon_1) \subseteq X^s(z, \epsilon_X) \cap X^u(y', \epsilon_X) = \{[z, y']\};$$

that is, $z' = [z, y']$.

Otherwise $[z] = \langle z, [[x_m]] \rangle$ for some $1 \leq m \leq L$ with $d^u([z], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]}$. So we have $z' = f^{-K}[f^K(z), f^K(x'_m)]$ and $[z, y'] = f^{-K}[f^K(z), f^K(x''_m)]$ for some $x'_m, x''_m \in [[x_m]]$.

Since $d([z], [[x_m]]) \leq d^u([z], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]} < \epsilon_1$, it follows that $d(z'', x'''_m) \leq \epsilon_1$ for some $z'' \in [z]$ and $x'''_m \in [[x_m]]$. Therefore

$$\begin{aligned} d(f^K(z), f^K(x''_m)) &\leq d(f^K(z), f^K(z'')) + d(f^K(z''), f^K(x'''_m)) + d(f^K(x'''_m), f^K(x''_m)) \\ &\leq 3\epsilon'_0 \\ &\leq \epsilon'_X, \end{aligned}$$

so that

$$[f^K(z), f^K(x''_m)] \in X^u(f^K(x''_m), \frac{1}{3}\epsilon_X).$$

Furthermore, by our choice of ϵ_1 ,

$$f^{-K}[f^K(z), f^K(x'_m)], f^{-K}[f^K(z), f^K(x''_m)] \in X^u(y', \epsilon_1)$$

implies

$$[f^K(z), f^K(x'_m)], [f^K(z), f^K(x''_m)] \in X^u(f^K(y'), \epsilon'_0) \subseteq X^u(f^K(y'), \frac{1}{3}\epsilon_X),$$

and hence

$$[f^K(z), f^K(x'_m)] \in X^u([f^K(z), f^K(x''_m)], \frac{2}{3}\epsilon_X).$$

So we have

$$\begin{aligned} [f^K(z), f^K(x'_m)] &\in X^s(f^K(z), \epsilon_X) \cap X^u([f^K(z), f^K(x''_m)], \frac{2}{3}\epsilon_X) \\ &\subseteq X^s(f^K(z), \epsilon_X) \cap X^u(f^K(x''_m), \epsilon_X) \\ &= \{[f^K(z), f^K(x''_m)]\}; \end{aligned}$$

that is, $f^K(z') = f^K([z, y'])$, and hence $z' = [z, y']$. □

Lemma 6.20. *Suppose $d^u([y], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]}$, where $1 \leq m \leq L$. If $d([z], [[x_m]]) \leq \epsilon_1$ and $d([z], [y]) \leq \beta(\epsilon_1)$, then*

$$X^u([z], \langle y, [[x_m]] \rangle, \epsilon_1) \subseteq \{(f^{-K}[f^K(z), f^K(x'_m)], f^{-K}[f^K(y), f^K(x'_m)]) \mid x'_m \in [[x_m]]\}.$$

Proof. Let $(z', y') \in X^u(\lceil z \rceil, \langle y, \llbracket x_m \rrbracket \rangle, \epsilon_1)$. Then $y' = f^{-K}[f^K(y), f^K(x'_m)]$ for some $x'_m \in \llbracket x_m \rrbracket$. We have from Lemma 6.10 that

$$f^{-K}[f^K(z), f^K(x'_m)] \in X^u(f^{-K}[f^K(y), f^K(x'_m)], \epsilon_1) = X^u(y', \epsilon_1).$$

That is, $z', f^{-K}[f^K(z), f^K(x'_m)] \in X^u(y', \epsilon_1)$. By our choice of ϵ_1 , it follows that

$$f^K(z') \in X^u([f^K(z), f^K(x'_m)], \frac{2}{3}\epsilon_X).$$

And we showed in the proof of Lemma 6.10 that $d(\llbracket z \rrbracket, \llbracket x_m \rrbracket) \leq \epsilon_1$ implies

$$[f^K(z), f^K(x'_m)] \in X^u(f^K(x'_m), \frac{1}{3}\epsilon_X).$$

Therefore

$$f^K(z') \in X^u(f^K(x'_m), \epsilon_X).$$

We claim that

$$f^K(z') \in X^s(f^K(z), \epsilon_X).$$

If $\lceil z \rceil = \llbracket z \rrbracket$ then this follows immediately by our choice of K . Otherwise $\lceil z \rceil = \langle z, \llbracket x_n \rrbracket \rangle$ for some $1 \leq n \leq L$. So we have $z' = f^{-K}[f^K(z), f^K(x'_n)]$ for some $x'_n \in \llbracket x_n \rrbracket$; that is, $f^K(z') = [f^K(z), f^K(x'_n)] \in X^s(f^K(z), \epsilon_X)$.

Therefore we have $f^K(z') \in X^s(f^K(z), \epsilon_X) \cap X^u(f^K(x'_m), \epsilon_X) = \{[f^K(z), f^K(x'_m)]\}$; that is, $z' = f^{-K}[f^K(z), f^K(x'_m)]$. \square

Proposition 6.21. *The metric δ gives the quotient topology on X/\sim .*

Proof. Let \mathcal{T}_q denote the quotient topology on X/\sim . We will show that the quotient map $X \rightarrow (X/\sim, \delta)$ is continuous. Since \mathcal{T}_q is defined to be the finest topology on X which makes the quotient map $X \rightarrow X/\sim$ continuous, it follows that the identity mapping $\text{id} : (X/\sim, \mathcal{T}_q) \rightarrow (X/\sim, \delta)$ is continuous. Since this identity map is a bijection from a compact space to a Hausdorff space, it follows that it is in fact a homeomorphism. That is, the two topologies are the same.

Let $(y_n) \subseteq X$ with $y_n \rightarrow y$. We want to show $\llbracket y_n \rrbracket \rightarrow \llbracket y \rrbracket$.

Case 1: $y \in \mathcal{O}(\partial\mathcal{P})^c$

Then $y \in \text{Int}R_i$ for some i . Without loss of generality, $(y_n) \subseteq \text{Int}R_i \cap B(y, \epsilon'_1)$. Therefore $[y_n, y] \approx y_n$ for all n , and $([y_n, y], y) \in X^u(\llbracket y_n \rrbracket, \llbracket y \rrbracket, \epsilon_1)$; and hence

$(y_n, y) \in P(y_n, y)$. Moreover, by Lemma 6.19, we have

$$X^u(\lceil y_n \rceil, \lceil y \rceil, \epsilon_1) = \{([y_n, y'], y') \mid y' \in \llbracket y \rrbracket\}.$$

Therefore, as $n \rightarrow \infty$,

$$\begin{aligned} \delta(\llbracket [y_n] \rrbracket, \llbracket [y] \rrbracket) &\leq d^u(\lceil y_n \rceil, \lceil y \rceil) \\ &= \sup\{d([y_n, y'], y') \mid y' \in \llbracket [y] \rrbracket\} \\ &= \sup\{d([y_n, y'], [y, y']) \mid y' \in \llbracket [y] \rrbracket\} \\ &\rightarrow 0. \end{aligned}$$

Case 2: $y \in \mathcal{O}(\partial\mathcal{P})$

Then $\lceil y \rceil = \langle y, \llbracket [x_m] \rrbracket \rangle$ for some $1 \leq m \leq L$ with $d^u(\llbracket [y] \rrbracket, \llbracket [x_m] \rrbracket) < \frac{1}{2}\epsilon_{\llbracket [x_m] \rrbracket} < \beta(\epsilon_1)$. So by Corollary 6.11,

$$X^u(\langle y, \llbracket [x_m] \rrbracket \rangle, \llbracket [x_m] \rrbracket, \epsilon_1) = \{(f^{-K}[f^K(y), f^K(x'_m)], x'_m) \mid x'_m \in \llbracket [x_m] \rrbracket\}.$$

Since $y \in \llbracket [y] \rrbracket \subseteq \langle y, \llbracket [x_m] \rrbracket \rangle$, it follows that $(y, x'_m) \in X^u(\llbracket [y] \rrbracket, \llbracket [x_m] \rrbracket, \epsilon_1)$ for some $x'_m \in \llbracket [x_m] \rrbracket$, and hence $d(y, x'_m) \leq d^u(\llbracket [y] \rrbracket, \llbracket [x_m] \rrbracket) < \frac{1}{2}\epsilon_{\llbracket [x_m] \rrbracket}$. Without loss of generality, $d(y_n, y) < \min\{\epsilon'_1, \frac{1}{2}\epsilon_{\llbracket [x_m] \rrbracket} - d(y, x'_m)\}$ for all n . Then

$$\begin{aligned} d(y_n, x'_m) &\leq d(y_n, y) + d(y, x'_m) \\ &< \frac{1}{2}\epsilon_{\llbracket [x_m] \rrbracket} - d(y, x'_m) + d(y, x'_m) \\ &= \frac{1}{2}\epsilon_{\llbracket [x_m] \rrbracket} \\ &< \epsilon_1; \end{aligned}$$

that is, $d(\llbracket [y_n] \rrbracket, \llbracket [x_m] \rrbracket) < \epsilon_1$. Moreover, $d(y_n, y) < \epsilon'_1$ gives us that $[y_n, y] \in X^s(y_n, \epsilon_1) \cap X^u(y, \epsilon_1)$. Since $\epsilon_1 \leq \epsilon_0$, it follows that $[y_n, y] \approx y_n$. Therefore $([y_n, y], y) \in X^u(\llbracket [y_n] \rrbracket, \llbracket [y] \rrbracket, \epsilon_1)$, so that $(y_n, y) \in P(y_n, y)$. Applying Lemma 6.20, we get

$$\begin{aligned} \delta(\llbracket [y_n] \rrbracket, \llbracket [y] \rrbracket) &\leq d^u(\lceil y_n \rceil, \lceil y \rceil) \\ &= \sup\{d(f^{-K}[f^K(y_n), f^K(x'_m)], f^{-K}[f^K(y), f^K(x'_m)]) \mid x'_m \in \llbracket [x_m] \rrbracket\} \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ □

Corollary 6.22. *The metric space $(X/\sim, \delta)$ is compact.*

6.1.3 A Mapping on X/\sim

Let us show that the mapping $\alpha : X/\sim \rightarrow X/\sim$ given by

$$\alpha([x]) = [[f(x)]]$$

is well-defined.

We begin by showing that $y \in [[x]]$ implies $f(y) \in [[f(x)]]$. First, consider the case where $x \in \text{Int}(R_i) \cap f^{-1}(\text{Int}(R_j))$, and suppose that $y \approx x$. Then $y \in X^s(x, R_i)$, so by the definition of a Markov partition, it follows that

$$f(y) \in f(X^s(x, R_i)) \subseteq X^s(f(x), R_j);$$

that is, $f(y) \approx f(x)$. Since \sim is generated by \approx , we also have $x \sim y$ implies that $f(x) \sim f(y)$.

Now choose any $x \in X$, and suppose $y \approx x$. Then $x, y \in R_i$ for some $R_i \in \mathcal{P}$. Bowen [3] proves that $f(x) \in R_j$ for some j with $\text{Int}(R_i) \cap f^{-1}(\text{Int}(R_j)) \neq \emptyset$, and moreover that

$$f(X^s(x, R_i)) \subseteq X^s(f(x), R_j).$$

Therefore $f(y) \approx f(x)$. Since \sim is generated by \approx , we also have $x \sim y$ implies $f(x) \sim f(y)$.

We will show that α is continuous and finite-to-one. That α is surjective follows immediately from the surjectivity of f .

Proposition 6.23. *$\alpha : X/\sim \rightarrow X/\sim$ is continuous.*

Proof. Let $[[y_n]] \rightarrow [[y]]$. We want to show that $[[f(y_n)]] \rightarrow [[f(y)]]$.

We begin by showing that there exist

$$(a_n, b_n) \in X^u([[f(y_n)]], [[f(y)]], \epsilon_1) \text{ such that } d(a_n, b_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (6.18)$$

Case 1: $y \in \mathcal{O}(\partial\mathcal{P})^c$

Then $\lceil y \rceil = \llbracket y \rrbracket$ and $y \in \text{Int}R_j$ for some j , so that $\llbracket y \rrbracket = X^s(y, R_j) \subseteq \text{Int}R_j$. Without loss of generality, $\delta(\llbracket y \rrbracket, \llbracket y_n \rrbracket) < \eta_0$ for all n .

Choose n , and let $p = (p_0, \dots, p_I) \in P(y, y_n)$ with $l(p) \leq \eta_0$. By Corollary 6.14,

$$(y, [y_n, y]) \in X^u(\llbracket y \rrbracket, \llbracket y_n \rrbracket, l(p)) \subseteq X^u(\llbracket y \rrbracket, \llbracket y_n \rrbracket, \epsilon_1),$$

and hence $(y, y_n) \in P(y, y_n)$. Moreover, $d(\llbracket y \rrbracket, \llbracket y_n \rrbracket) \leq d(y, [y_n, y]) \leq l(p) \leq \epsilon'_1$ implies by Lemma 6.19 that

$$X^u(\lceil y \rceil, \lceil y_n \rceil, \epsilon_1) = \{(y', [y_n, y']) \mid y' \in \llbracket y \rrbracket\}.$$

Now, since we have $l(p) \leq \epsilon'_1$, it follows that for each $i = 0, \dots, I-1$ and $y' \in \llbracket y \rrbracket$, $([p_i, y'], [p_{i+1}, y']) \in X^u(\llbracket p_i \rrbracket, \llbracket p_{i+1} \rrbracket, \epsilon_1) \subseteq X^u(\lceil p_i \rceil, \lceil p_{i+1} \rceil, \epsilon_1)$. Therefore

$$\begin{aligned} d^u(\lceil y \rceil, \lceil y_n \rceil) &= \sup\{d(y', [y_n, y']) \mid y' \in \llbracket y \rrbracket\} \\ &\leq \sup\left\{\sum_{i=0}^{I-1} d([p_i, y'], [p_{i+1}, y']) \mid y' \in \llbracket y \rrbracket\right\} \\ &\leq \sum_{i=0}^{I-1} d^u(\lceil p_i \rceil, \lceil p_{i+1} \rceil) \\ &= l(p). \end{aligned}$$

It follows that $d^u(\lceil y \rceil, \lceil y_n \rceil) \leq \inf\{l(p) \mid p \in P(y, y_n), l(p) \leq \eta_0\} = \delta(\llbracket y \rrbracket, \llbracket y_n \rrbracket)$. Since $(y, y_n) \in P(y, y_n)$, it follows that

$$d^u(\lceil y \rceil, \lceil y_n \rceil) = \delta(\llbracket y \rrbracket, \llbracket y_n \rrbracket).$$

Therefore

$$d(y, [y_n, y]) \leq d^u(\llbracket y \rrbracket, \llbracket y_n \rrbracket) \leq d^u(\lceil y \rceil, \lceil y_n \rceil) = \delta(\llbracket y \rrbracket, \llbracket y_n \rrbracket) \rightarrow 0.$$

By our choice of ϵ'_1 ,

$$(y, [y_n, y]) \in X^u(\llbracket y \rrbracket, \llbracket y_n \rrbracket, \epsilon'_1)$$

implies

$$(f(y), f([y_n, y])) \in X^u(\llbracket f(y) \rrbracket, \llbracket f(y_n) \rrbracket, \epsilon_1).$$

Moreover, we have $d(f(y), f([y_n, y])) \rightarrow 0$ by the continuity of the f .

Case 2: $y \in \mathcal{O}(\partial\mathcal{P})$

Then $[y] = \langle y, [[x_k]] \rangle$ for some $1 \leq k \leq L$ with $d^u([y], [[x_k]]) < \frac{1}{2}\epsilon_{[[x_k]]}$. Without loss of generality, we have $\delta([y_n], [[y]]) < \min\{\frac{1}{2}\epsilon_{[[x_k]]} - d^u([y], [[x_k]]), \epsilon_{[[y]]}, \epsilon'_1\} \equiv \mu$ for all n .

Choose n and let $p \in P(y, y_n)$ with $l(p) < \mu$. By Corollary 6.16, we have

$$[[p_i]] \subseteq \langle p_i, [[y]] \rangle \subseteq \langle p_i, [[x_k]] \rangle \subseteq [p_i]$$

for all i .

Since $[y_n] \subseteq \langle y_n, [[y]] \rangle$, we have $y_n = f^{-K}[f^K(y_n), f^K(z_n)]$ for some $z_n \in [[y]]$. Since $l(p) \leq \beta(\epsilon_1)$, it follows that

$$(f^{-K}[f^K(p_i), f^K(z_n)], f^{-K}[f^K(p_{i+1}), f^K(z_n)]) \in X^u([p_i], [p_{i+1}], \epsilon_1)$$

for all i . Therefore

$$\begin{aligned} d(z_n, y_n) &\leq \sum d(f^{-K}[f^K(p_i), f^K(z_n)], f^{-K}[f^K(p_{i+1}), f^K(z_n)]) \\ &\leq \sum d^u([p_i], [p_{i+1}]) \\ &\leq l(p), \end{aligned}$$

so that $(z_n, y_n) \in X^u([y], [y_n], l(p)) \subseteq X^u([y], [y_n], \epsilon_1)$. It follows that $(y, y_n) \in P(y, y_n)$, and hence $\delta([y], [y_n]) \leq d^u([y], [y_n])$.

Since $d([y_n], [y]) \leq d(y_n, z_n) \leq l(p) \leq \beta(\epsilon_1)$, we can apply Lemma 6.20 to get

$$\begin{aligned} d^u([y], [y_n]) &\leq \sup\{d(f^{-K}[f^K(y), f^K(x'_k)], f^{-K}[f^K(y_n), f^K(x'_k)]) \mid x'_k \in [[x_k]]\} \\ &\leq \sup\{\sum d(f^{-K}[f^K(p_i), f^K(x'_k)], f^{-K}[f^K(p_{i+1}), f^K(x'_k)]) \mid x'_k \in [[x_k]]\} \\ &\leq \sum d^u([p_i], [p_{i+1}]) \\ &= l(p). \end{aligned}$$

Therefore $d^u([y], [y_n]) \leq \inf\{l(p) \mid p \in P(y, y_n), l(p) < \mu\} = \delta([y], [y_n])$. And hence $d^u([y], [y_n]) = \delta([y], [y_n])$.

Further observe that

$$d(z_n, y_n) \leq d^u([y], [y_n]) \leq d^u([y], [y_n]) = \delta([y], [y_n]) \rightarrow 0.$$

By the continuity of f , $d(f(z_n), f(y_n)) \rightarrow 0$ as well. Moreover,

$$(z_n, y_n) \in X^u([y], [y_n], l(p)) \subseteq X^u([y], [y_n], \epsilon'_1)$$

implies that

$$(f(y_n), f(z_n)) \in X^u([f(y_n)], [f(y)], \epsilon_1).$$

Now we will show that (6.18) implies $[[f(y_n)]] \rightarrow [[f(y)]]$.

Case A: $f(y) \in \mathcal{O}(\partial\mathcal{P})^c$

Then $f(y) \in \text{Int}R_k$ for some k , so that $[[f(y)]] = X^s(f(y), R_k) \subseteq \text{Int}R_k$. Moreover, $[f(y)] = [[f(y)]]$.

Recall that $\eta_0 \leq \epsilon'_1$ and

$$X^u([f(y)], \eta_0) \subseteq \text{Int}R_k.$$

It follows from (6.18) that for large enough n , we have

$$(a_n, b_n) \in X^u([f(y_n)], [f(y)], \eta_0),$$

hence $a_n \in \text{Int}R_k$. So for large n we have $(f(y), f(y_n)) \in P(f(y), f(y_n))$, and by Lemma 6.19,

$$\begin{aligned} \delta([f(y)], [f(y_n)]) &\leq d^u([f(y)], [f(y_n)]) \\ &= \sup\{d([z, f(y_n), z]) \mid z \in [f(y)]\} \\ &= \sup\{d([b_n, z], [a_n, z]) \mid z \in [f(y)]\} \\ &\rightarrow 0. \end{aligned}$$

Case B: $[f(y)] = \langle f(y), [x_m] \rangle$ for some $1 \leq m \leq L$ with $d^u([f(y)], [x_m]) < \frac{1}{2}\epsilon_{[x_m]}$

It follows from (6.18) that

$$(a_n, b_n) \in X^u([f(y_n)], [f(y)], \frac{1}{2}\beta(\epsilon_1))$$

for large enough n .

Now, $d^u([f(y)], [x_m]) < \frac{1}{2}\epsilon_{[x_m]} < \beta(\epsilon_1)$ implies $b_n \in [f(y)] \subseteq \langle f(y), [x_m] \rangle$, so

that $b_n = f^{-K}[f^K(f(y)), f^K(z_n)]$ for some $z_n \in [[x_m]]$. By Corollary 6.11,

$$(b_n, z_n) \in X^u([[f(y)]], [[x_m]], \epsilon_1),$$

so

$$\begin{aligned} d([[f(y_n)]], [[x_m]]) &\leq d(a_n, z_n) \\ &\leq d(a_n, b_n) + d(b_n, z_n) \\ &\leq \frac{1}{2}\beta(\epsilon_1) + d^u([[f(y)]], [[x_m]]) \\ &< \epsilon_1. \end{aligned}$$

Since we also have $d([[f(y_n)]], [[f(y)]]) \leq d(a_n, b_n) \leq \frac{1}{2}\beta(\epsilon_1)$ for large enough n , we can apply Lemma 6.20 to get

$$\begin{aligned} \delta([[f(y_n)]], [[f(y)]]) &\leq d^u([f(y_n)], [f(y)]) \\ &\leq \sup\{d(f^{-K}[f^K(f(y_n)), f^K(x''_m)], f^{-K}[f^K(f(y)), f^K(x''_m)]) \mid \\ &\quad x''_m \in [[x_m]]\} \\ &= \sup\{d(f^{-K}[f^K(a_n), f^K(x''_m)], f^{-K}[f^K(b_n), f^K(x''_m)]) \mid \\ &\quad x''_m \in [[x_m]]\} \\ &\rightarrow 0. \end{aligned}$$

□

Proposition 6.24. $\alpha : X/\sim \rightarrow X/\sim$ is finite-to-one.

Proof. Let $\pi : X \rightarrow X/\sim$ denote the quotient map. Since X is compact, there exists a finite cover $\{B_1, \dots, B_n\}$ of X by $\frac{1}{2}\epsilon_1$ -balls. We claim that $\#\alpha^{-1}\{[[x]]\} \leq n$ for all $[[x]] \in X/\sim$.

Let $[[x]] \in X/\sim$. Then $\pi^{-1}\{\alpha^{-1}\{[[x]]\}\} = \bigcup_{i=1}^n (\pi^{-1}\{\alpha^{-1}\{[[x]]\}\} \cap B_i)$, and hence

$$\alpha^{-1}\{[[x]]\} = \bigcup_{i=1}^n \pi(\pi^{-1}\{\alpha^{-1}\{[[x]]\}\} \cap B_i).$$

For all $1 \leq i \leq n$ with $\pi^{-1}\{\alpha^{-1}\{[[x]]\}\} \cap B_i \neq \emptyset$, choose $y_i \in \pi^{-1}\{\alpha^{-1}\{[[x]]\}\} \cap B_i$. We will show that $\pi(\pi^{-1}\{\alpha^{-1}\{[[x]]\}\} \cap B_i) = \{[[y_i]]\}$, so that

$$\alpha^{-1}\{[[x]]\} = \{[[y_i]] \mid 1 \leq i \leq n, \pi^{-1}\{\alpha^{-1}\{[[x]]\}\} \cap B_i \neq \emptyset\}.$$

Choose $1 \leq i \leq n$ with $\pi^{-1}\{\alpha^{-1}\{[[x]]\}\} \cap B_i \neq \emptyset$, and let $z \in \pi^{-1}\{\alpha^{-1}\{[[x]]\}\} \cap B_i$. Then $f(z), f(y_i) \in [[x]]$, so that $f^{K+1}(z) \in X^s(f^{K+1}(y_i), 2\epsilon'_0) \subseteq X^s(f^{K+1}(y_i), \epsilon_X)$ by Lemma 6.3. And since $z, y_i \in B_i$, we have $d(z, y_i) \leq \epsilon_1$. It follows that

$$z \in X^s(y_i, \epsilon'_0) \subseteq X^s(y_i, \epsilon_0),$$

so that $z \approx y_i$ by Lemma 6.1. □

6.1.4 The Quotient Space Satisfies Axioms 1 and 2

We have already shown that $(X/\sim, \delta)$ is a compact metric space, and that the mapping $\alpha : X/\sim \rightarrow X/\sim$ is continuous and surjective.

Choose K as in Lemma 6.3 and let $\gamma = \lambda$, the expansive constant for the Smale space (X, d, f) . We will show that there exists $\beta > 0$ such that

Axiom 1 if $\delta([[x]], [[y]]) \leq \beta$ then

$$\delta([[f^K(x)]], [[f^K(y)]]) \leq \gamma^K \delta([[f^{2K}(x)]], [[f^{2K}(y)]]),$$

and

Axiom 2 for all $[[x]] \in X/\sim$ and $0 < \epsilon \leq \beta$,

$$\alpha^K(B([[f^K(y)]], \epsilon)) \subseteq \alpha^{2K}(B([[y]]], \gamma\epsilon)$$

Lemma 6.25. For any $[[x]] \in X/\sim$, $f^K[x] \subseteq [[f^K(x)]]$.

Proof. CASE 1. $x \in \mathcal{O}(\partial\mathcal{P})^c$

Then $[x] = [[x]]$, so that $f^K[x] = f^K[[x]] \subseteq [[f^K(x)]]$ by Lemmas 6.1 and 6.3.

CASE 2. $x \in \mathcal{O}(\partial\mathcal{P})$

Then $[x] = \langle x, [[x_m]] \rangle$ for some $1 \leq m \leq L$ with $d^u([[x]], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]}$. Choose $x'_m \in [[x_m]]$. Recall from (6.6) that

$$d(f^K(x), f^K(x'_m)) \leq 3\epsilon'_0 \leq \epsilon''_0.$$

By our choice of ϵ''_0 , it follows that

$$[f^K(x), f^K(x'_m)] \in X^s(f^K(x), \epsilon_0).$$

So by Lemma 6.1, we have $[f^K(x), f^K(x'_m)] \approx f^K(x)$. □

Let

$$\epsilon_2 = \min \left\{ \frac{1}{2} \epsilon_{[[x_i]]} \mid 1 \leq l \leq L \right\}.$$

Then choose

$$0 < \epsilon_3 \leq \frac{1}{2} \epsilon_2$$

such that $d(x, y) \leq \epsilon_3$ implies $d(f^k(x), f^k(y)) \leq \beta(\epsilon_1)$ for $k = -K, \dots, K$. And let

$$\eta_1 = \min\{\epsilon_3, \eta'_0\}.$$

Lemma 6.26. *If $\delta([[x]], [[y]]) < \eta_1$ then $X^u([[f^K(x)]], [[f^K(y)]], \beta(\epsilon_1)) \neq \emptyset$.*

Proof. Let $p = (p_0, \dots, p_l) \in P(x, y)$ with $l(p) < \eta_1$.

CASE 1: $x \in \mathcal{C}^c$

By Corollary 6.14, $(x, [y, x]) \in X^u([[x]], [[y]], l(p)) \subseteq X^u([[x]], [[y]], \epsilon_3)$, so that

$$(f^K(x), f^K([y, x])) \in X^u([[f^K(x)]], [[f^K(y)]], \beta(\epsilon_1)).$$

CASE 2: $x \in \mathcal{C}$

Then $d(x, [[x_m]]) < \beta(\frac{1}{4} \epsilon_{[[x_m]])}$ for some $1 \leq m \leq L$. So $d^u([[x]], [[x_m]]) \leq \frac{1}{4} \epsilon_{[[x_m]]}$ by Corollary 6.11.

Since $\mathcal{C} \subseteq \mathcal{O}(\partial\mathcal{P})$ and $l(p) + d^u([[x]], [[x_m]]) < \frac{1}{4} \epsilon_2 + \frac{1}{4} \epsilon_{[[x_m]]} = \frac{1}{2} \epsilon_{[[x_m]]}$, we can apply Lemma 6.15 to get $d^u([p_i], [[x_m]]) \leq d^u([[x]], [[x_m]]) + l(p) < \frac{1}{2} \epsilon_{[[x_m]]}$ for all i . Therefore $[p_i] \subseteq \langle p_i, [[x_m]] \rangle \subseteq [p_i]$ for all i . So we have $f^K \langle p_i, [[x_m]] \rangle \subseteq f^K [p_i] \subseteq [[f^K(p_i)]]$ for all i by Lemma 6.25.

Since $x \in [[x]] \subseteq \langle x, [[x_m]] \rangle$, we have

$$x = f^{-K}[f^K(x), f^K(x'_m)]$$

for some $x'_m \in [[x_m]]$. Moreover,

$$(f^{-K}[f^K(p_i), f^K(x'_m)], f^{-K}[f^K(p_{i+1}), f^K(x'_m)]) \in X^u([p_i], [p_{i+1}], \epsilon_1)$$

for all $i = 0, \dots, I - 1$ by Lemma 6.10, so that

$$\begin{aligned} d(x, f^{-K}[f^K(y), f^K(x'_m)]) &\leq \sum_{i=0}^{I-1} d(f^{-K}[f^K(p_i), f^K(x'_m)], f^{-K}[f^K(p_{i+1}), f^K(x'_m)]) \\ &\leq l(p) \\ &\leq \epsilon_3. \end{aligned}$$

Hence

$$d(f^K(x), [f^K(y), f^K(x'_m)]) \leq \beta(\epsilon_1).$$

Since $f^K(x), [f^K(y), f^K(x'_m)] \in X^u(f^K(x'_m), \epsilon_X)$, it follows that

$$f^K(x) \in X^u([f^K(y), f^K(x'_m)], \beta(\epsilon_1)).$$

By Lemma 6.25, $[f^K(y), f^K(x'_m)] \in f^K[y] \subseteq [[f^K(y)]]$, so we have

$$(f^K(x), [f^K(y), f^K(x'_m)]) \in X^u([[f^K(x)]], [[f^K(y)]]), \beta(\epsilon_1)).$$

□

By the uniform continuity of α and f^{-1} , there exists

$$0 < \eta_2 < \eta_1$$

such that $\delta([[x]], [[y]]) \leq \eta_2$ implies $\delta(\alpha^K[[x]], \alpha^K[[y]]) < \eta_1$, and such that $d(x, y) \leq \eta_2$ implies $d(f^{-K}(x), f^{-K}(y)) \leq \epsilon_1$.

Corollary 6.27. *If $\delta([[x]], [[y]]) \leq \eta_2$ then*

$$d^u([[f^K(x)]], [[f^K(y)]]) \leq \delta([[f^K(x)]], [[f^K(y)]]).$$

Proof. By Lemma 6.26, $X^u([[f^K(x)]], [[f^K(y)]], \epsilon_1) \neq \emptyset$. Moreover, $\delta([[x]], [[y]]) \leq \eta_2$ implies $\delta([[f^K(x)]], [[f^K(y)]]) < \eta_1$. So let $p \in P(f^K(x), f^K(y))$ with $l(p) < \eta_1$.

CASE 1: $f^K(x) \in \mathcal{C}^c$

Then $f^K(x) \in \text{Int}R_j$ for some j . By Lemma 6.13, $p_i \in \text{Int}R_j$ for each i . Since

$l(p) < \epsilon'_1$, it follows from Lemma 6.19 that

$$\begin{aligned}
d^u([[f^K(x)]], [[f^K(y)]]) &= \sup\{d(z', [f^K(y), z']) \mid z' \in [[f^K(x)]]\} \\
&\leq \sup\left\{\sum_i d([p_i, z'], [p_{i+1}, z']) \mid z' \in [[f^K(x)]]\right\} \\
&\leq \sum_i d^u([p_i], [p_{i+1}]) \\
&\leq l(p).
\end{aligned}$$

Therefore

$$\begin{aligned}
d^u([[f^K(x)]], [[f^K(y)]]) &\leq \inf\{l(p) \mid p \in P(f^K(x), f^K(y)), l(p) < \eta_1\} \\
&= \delta([[f^K(x)]], [[f^K(y)]]).
\end{aligned}$$

CASE 2: $f^K(x) \in \mathcal{C}$

Then $d(f^K(x), [[x_m]]) < \beta(\frac{1}{4}\epsilon_{[[x_m]])}$ for some $1 \leq m \leq L$. By Corollary 6.11, it follows that $d^u([[f^K(x)]], [[x_m]]) \leq \frac{1}{4}\epsilon_{[[x_m]]}$.

Since $\mathcal{C} \subseteq \mathcal{O}(\partial\mathcal{P})$ and $l(p) + d^u([[f^K(x)]], [[x_m]]) < \frac{1}{4}\epsilon_2 + \frac{1}{4}\epsilon_{[[x_m]]} = \frac{1}{2}\epsilon_{[[x_m]]}$, we can apply Lemma 6.15 to get $d^u([p_i], [[x_m]]) \leq d^u([[f^K(x)]], [[x_m]]) + l(p) < \frac{1}{2}\epsilon_{[[x_m]]}$ for all i . Therefore $[p_i] \subseteq \langle p_i, [[x_m]] \rangle \subseteq [p_i]$ for all i .

By Lemma 6.26, $d([[f^K(x)]], [[f^K(y)]]) \leq \beta(\epsilon_1)$, so we can apply Lemma 6.10 to get

$$\begin{aligned}
d^u([[f^K(x)]], [[f^K(y)]]) &\leq d^u(\langle f^K(x), [[x_m]] \rangle, \langle f^K(y), [[x_m]] \rangle) \\
&= \sup\{d(f^{-K}[f^{2K}(x), f^K(x'_m)], f^{-K}[f^{2K}(y), f^K(x'_m)]) \mid x'_m \in [[x_m]]\} \\
&= \sup\left\{\sum d(f^{-K}[f^K(p_i), f^K(x'_m)], f^{-K}[f^K(p_{i+1}), f^K(x'_m)]) \mid x'_m \in [[x_m]]\right\} \\
&\leq \sum d^u([p_i], [p_{i+1}]) \\
&= l(p).
\end{aligned}$$

Therefore

$$\begin{aligned}
d^u([[f^K(x)]], [[f^K(y)]]) &\leq \inf\{l(p) \mid p \in P(f^K(x), f^K(y)), l(p) < \eta_1\} \\
&= \delta([[f^K(x)]], [[f^K(y)]]).
\end{aligned}$$

□

Choose

$$0 < \eta_3 < \eta_2$$

such that $\delta([x], [y]) \leq \eta_3$ implies $\delta(\alpha^K[x], \alpha^K[y]) \leq \eta_2$, and such that $d(x, y) \leq \eta_3$ implies that $d([x, z], [y, z]) \leq \eta_2$ for all z such that $(x, z), (y, z) \in \text{domain}([\cdot, \cdot])$.

Suppose that $\delta([x], [y]) \leq \eta_3 < \eta_1$. By Lemma 6.26, we have that $(f^K(x), f^K(y)) \in P(f^K(x), f^K(y))$, so that

$$\delta([f^K(x)], [f^K(y)]) \leq d^u(\lceil f^K(x) \rceil, \lceil f^K(y) \rceil). \quad (6.19)$$

Since $\delta([x], [y]) \leq \eta_3$ implies $\delta([f^K(x)], [f^K(y)]) \leq \eta_2$, we have by Corollary 6.27 that

$$d^u([f^{2K}(x)], [f^{2K}(y)]) \leq \delta([f^{2K}(x)], [f^{2K}(y)]). \quad (6.20)$$

And by Lemma 6.25,

$$\lceil f^K(x) \rceil \subseteq f^{-K}([f^{2K}(x)]) \text{ and } \lceil f^K(y) \rceil \subseteq f^{-K}([f^{2K}(y)]). \quad (6.21)$$

Combining (6.19), (6.21), and (6.20), we get

$$\begin{aligned} \delta([f^K(x)], [f^K(y)]) &\leq d^u(\lceil f^K(x) \rceil, \lceil f^K(y) \rceil) \\ &\leq d^u(f^{-K}([f^{2K}(x)]), f^{-K}([f^{2K}(y)])) \\ &= \sup\{d(u, v) \mid (u, v) \in X^u(f^{-K}([f^{2K}(x)]), f^{-K}([f^{2K}(y)]), \epsilon_1)\} \\ &= \sup\{d(f^{-K}(u), f^{-K}(v)) \mid (u, v) \in X^u([f^{2K}(x)], [f^{2K}(y)], \epsilon_1)\} \\ &= \sup\{\lambda^K d(u, v) \mid (u, v) \in X^u([f^{2K}(x)], [f^{2K}(y)], \epsilon_1)\} \\ &= \lambda^K d^u([f^{2K}(x)], [f^{2K}(y)]) \\ &\leq \lambda^K \delta([f^{2K}(x)], [f^{2K}(y)]). \end{aligned}$$

This proves that Axiom 1, with $\beta = \eta_3$, is satisfied.

Now let's prove that Axiom 2 is satisfied; we'll prove that

$$\alpha^K(B([f^K(y)], \epsilon)) \subseteq \alpha^{2K}(B([y], \gamma\epsilon))$$

for all $0 < \epsilon \leq \eta_3$.

Let $\delta([[z]], [[f^K(y)]]) \leq \eta_3$.

CASE 1: $f^K(y) \in \mathcal{C}^c$

Since $\delta([[z]], [[f^K(y)]]) \leq \eta_3$, it follows from Lemma 6.13 that $f^K(y), z \in \text{Int}R_i$ for some i , and from Corollary 6.14 that

$$(f^K(y), [z, f^K(y)]) \in X^u([[f^K(y)]], [[z]], \eta_3) \subseteq X^u([[f^K(y)]], [[z]], \eta_2).$$

So by our choice of η_2 ,

$$(y, f^{-K}([z, f^K(y)])) \in X^u([[y]], [[f^{-K}([z, f^K(y)])]], \epsilon_1),$$

hence $\delta([[y]], [[f^{-K}([z, f^K(y)])]]) \leq d^u(\lceil y \rceil, \lceil f^{-K}([z, f^K(y)]) \rceil)$. Moreover, we have

$$d([[f^K(y)]], [[z]]) \leq d(f^K(y), [z, f^K(y)]) \leq \eta_3 \leq \epsilon'_1.$$

So by Lemma 6.19 and our choice of η_3 ,

$$\begin{aligned} d^u([[f^K(y)]], [[z]]) &= \sup\{d(u, [z, u]) \mid u \in [[f^K(y)]]\} \\ &= \sup\{d([f^K(y), u], [[z, f^K(y)], u]) \mid u \in [[f^K(y)]]\} \\ &\leq \eta_2. \end{aligned}$$

So by Lemma 6.25,

$$\begin{aligned} \delta([[y]], [[f^{-K}([z, f^K(y)])]]) &\leq d^u(\lceil y \rceil, \lceil f^{-K}([z, f^K(y)]) \rceil) \\ &\leq d^u(f^{-K}[[f^K(y)]], f^{-K}[[z, f^K(y)]]) \\ &= d^u(f^{-K}[[f^K(y)]], f^{-K}[[z]]) \\ &\leq \lambda^K d^u([[f^K(y)]], [[z]]) \\ &\leq \eta_2 \end{aligned}$$

It follows by Lemma 6.27 that $d^u([[f^K(y)]], [[z]]) \leq \delta([[f^K(y)]], [[z]])$. Therefore

$$\begin{aligned} \delta([[y]], [[f^{-K}([z, f^K(y)])]]) &\leq \lambda^K d^u([[f^K(y)]], [[z]]) \\ &\leq \lambda^K \delta([[f^K(y)]], [[z]]). \end{aligned}$$

Moreover, we have $\alpha^{2K}([[f^{-K}([z, f^K(y)])]]) = \alpha^K([[z, f^K(y)]]) = \alpha^K[[z]]$.

CASE 2: $f^K(y) \in \mathcal{C}$

Then $d(f^K(y), [[x_m]]) < \beta(\frac{1}{4}\epsilon_{[[x_m]])}$ for some $1 \leq m \leq L$. So by Corollary 6.11, we have $d^u([[f^K(y)]], [[x_m]]) \leq \frac{1}{4}\epsilon_{[[x_m]]}$.

Let $p = (p_0, \dots, p_I) \in P(f^K(y), z)$ such that $l(p) < \eta_1$.

Since $\mathcal{C} \subseteq \mathcal{O}(\partial\mathcal{P})$ and $l(p) + d^u([[f^K(y)]], [[x_m]]) < \frac{1}{4}\epsilon_2 + \frac{1}{4}\epsilon_{[[x_m]]} = \frac{1}{2}\epsilon_{[[x_m]]}$, we can apply Lemma 6.15 to get $d^u([[p_i]], [[x_m]]) \leq d^u([[f^K(y)]], [[x_m]]) + l(p) < \frac{1}{2}\epsilon_{[[x_m]]}$ for all $i = 0, \dots, I$. Therefore $[[p_i]] \subseteq \langle p_i, [[x_m]] \rangle \subseteq [p_i]$ for all $i = 0, \dots, I$.

So we have

$$f^K(y) = f^{-K}[f^{2K}(y), f^K(x'_m)]$$

for some $x'_m \in [[x_m]]$. Define

$$u_i = f^{-K}[f^K(p_i), f^K(x'_m)]$$

for all $i = 0, \dots, I$. Then by Lemma 6.25,

$$f^K(u_i) \in f^K\langle p_i, [[x_m]] \rangle \subseteq f^K[p_i] \subseteq [[f^K(p_i)]]$$

for all $i = 0, \dots, I$.

It is easy to check that $f^{-K}[f^K(u_I), f^K(x''_m)] = f^{-K}[f^K(p_I), f^K(x''_m)]$ for all $x''_m \in [[x_m]]$.

Let us check that the hypothesis of Lemma 6.20 is satisfied for $[[x_m]]$, $[[f^K(y)]]$ and $[[u_I]]$. We already have $d^u([[f^K(x)]], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]}$. Since

$$d([[z]], [[x_m]]) \leq d^u([[z]], [[x_m]]) < \frac{1}{2}\epsilon_{[[x_m]]} \leq \beta(\epsilon_1),$$

it follows that

$$d([[u_I]], [[x_m]]) \leq d(u_I, x'_m) \leq \epsilon_1.$$

And

$$\begin{aligned} d([[f^K(y)]], [[u_I]]) &\leq d(f^K(y), u_I) \\ &\leq \sum_i d(u_i, u_{i+1}) \\ &\leq l(p) \\ &< \beta(\epsilon_1). \end{aligned}$$

So we can apply Lemma 6.20, and we have

$$(f^K(y), u_I) \in X^u([[f^K(y)]], [[u_I]], \eta_1). \quad (6.22)$$

Therefore

$$\begin{aligned} d^u([[f^K(y)]], [[u_I]]) &\leq d^u(\langle f^K(y), [[x_m]] \rangle, [u_I]) \\ &= \sup\{d(f^{-K}[f^{2K}(y)], f^K(x''_m)), f^{-K}[f^K(u_I), f^K(x''_m)] \mid \\ &\quad x''_m \in [[x_m]]\} \\ &= \sup\{d(f^{-K}[f^{2K}(y)], f^K(x''_m)), f^{-K}[f^K(z), f^K(x''_m)] \mid \\ &\quad x''_m \in [[x_m]]\} \\ &\leq \sup\{\sum d(f^{-K}[f^{2K}(p_i)], f^K(x''_m)), f^{-K}[f^K(p_{i+1}), f^K(x''_m)] \mid \\ &\quad x''_m \in [[x_m]]\} \\ &\leq \sum_i d^u(\langle p_i, [[x_m]] \rangle, \langle p_{i+1}, [[x_m]] \rangle) \\ &\leq \sum_i d^u([p_i], [p_{i+1}]) \\ &= l(p), \end{aligned}$$

and so $d^u([[f^K(y)]], [[u_I]]) \leq \inf\{l(q) \mid q \in P(f^K(y), z), l(q) < \eta_1\} = \delta([[f^K(y)]], [[z]])$.

Moreover, (6.22) gives

$$(y, f^{-K}(u_I)) \in X^u([[y]], [[f^{-K}(u_I)]], \epsilon_1)$$

by our choice of η_1 .

So by Lemma 6.25,

$$\begin{aligned} \delta([[y]], [[f^{-K}(u_I)]]) &\leq d^u([y], [f^{-K}(u_I)]) \\ &\leq d^u(f^{-K}[[f^K(y)]], f^{-K}[[u_I]]) \\ &\leq \lambda^K d^u([[f^K(y)]], [[u_I]]) \\ &\leq \lambda^K \delta([[f^K(y)]], [[z]]). \end{aligned}$$

And we had $f^K(u_I) \in f^K([z]) \subseteq [[f^K(z)]]$, so that

$$\alpha^{2K}[[f^{-K}(u_I)]] = [[f^K(u_I)]] = [[f^K(z)]] = \alpha^K[[z]].$$

6.2 The Conjugacy

As usual, we denote

$$\widehat{X/\sim} = \varprojlim X/\sim \xleftarrow{\alpha} X/\sim \xleftarrow{\alpha} X/\sim \xleftarrow{\alpha} \dots$$

Recall that $\hat{\alpha}$ denoted the usual map on $\widehat{X/\sim}$ and that

$$\hat{\delta}(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{K-1} \lambda^{-k} \delta'(\hat{\alpha}^{-k}(\mathbf{x}), \hat{\alpha}^{-k}(\mathbf{y})),$$

where

$$\delta'(\mathbf{x}, \mathbf{y}) = \sup\{\lambda^n \delta([[x_n]], [[y_n]]) \mid n \geq 0\}$$

and $0 < \lambda < 1$ is the Smale space constant for (X, d, f) .

Define $\omega : X \rightarrow \widehat{X/\sim}$ by

$$\omega(x) = ([[x]], [[f^{-1}(x)]], [[f^{-2}(x)]], \dots).$$

Lemma 6.28. $\omega : (X, d) \rightarrow (\widehat{X/\sim}, \delta')$ is a homeomorphism.

Proof. Since (X, d) is compact and $(\widehat{X/\sim}, \delta')$ is a metric space, it suffices to prove that ω is continuous and bijective.

continuity: Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\lambda^N < \epsilon$. Since the quotient map $X \rightarrow X/\sim$ is uniformly continuous (see the proof of Proposition 6.21), there exists $\epsilon' > 0$ such that $\delta([[x]], [[y]]) < \epsilon$ if $d(x, y) < \epsilon'$. And there exists $\epsilon'' > 0$ such that $d(x, y) < \epsilon''$ implies that $d(f^{-n}(x), f^{-n}(y)) < \epsilon'$ for $n = 0, \dots, N-1$.

So let $d(x, y) < \epsilon''$. Then for $n = 0, \dots, N-1$ we have

$$\lambda^n \delta([[f^{-n}(x)]], [[f^{-n}(y)]]) < \lambda^n \epsilon < \epsilon.$$

And for $n \geq N$, we have

$$\lambda^n \delta([[f^{-n}(x)]], [[f^{-n}(y)]]) \leq \lambda^n < \epsilon.$$

Hence $\delta'(\omega(x), \omega(y)) = \sup\{\lambda^n \delta([[f^{-n}(x)]], [[f^{-n}(y)]]) \mid n \geq 0\} \leq \epsilon$.

surjectivity: Let $\mathbf{z} = ([[z_0]], [[z_1]], \dots) \in \widehat{X/\sim}$. Observe that for all $N \in \mathbb{N}$ and each

$$0 \leq m \leq N,$$

$$[[z_m]] = \alpha^{N-m}([[z_N]]) = [[f^{N-m}(z_N)]].$$

Hence

$$\begin{aligned} \delta'(\omega(f^N(z_N)), \mathbf{z}) &= \sup\{\lambda^n \delta([[f^{N-n}(z_N)]], [[z_n]]) \mid n \geq 0\} \\ &= \sup\{\lambda^n \delta([[f^{N-n}(z_N)]], [[z_n]]) \mid n > N\} \\ &< \lambda^N. \end{aligned}$$

That is, $\omega(f^N(z_N)) \rightarrow \mathbf{z}$ as $N \rightarrow \infty$.

However, since $(f^N(z_N))$ is a sequence in the compact space X , it has a convergent subsequence $f^{N_k}(z_{N_k}) \rightarrow y$. By the continuity of ω ,

$$\omega(f^{N_k}(z_{N_k})) \rightarrow \omega(y),$$

hence $\mathbf{z} = \omega(y)$.

injectivity: Suppose $\omega(x) = \omega(y)$. Then $f^{-n}(x) \sim f^{-n}(y)$ for all $n \geq 0$. In particular,

$$f^{-(K+n)}(x) \sim f^{-(K+n)}(y)$$

for all $n \geq 0$, so that $f^{-n}(x) \in X^s(f^{-n}(y), \epsilon'_0)$ by Lemma 6.3. This implies

$$x \in X^s(y, \lambda^n \epsilon'_0),$$

and hence $d(x, y) \leq \lambda^n \epsilon'_0$, for all $n \geq 0$. So $x = y$. □

It follows immediately that $\omega : (X, d) \rightarrow (\widehat{X/\sim}, \hat{\delta})$ is a homeomorphism.

Now let's show that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow \omega & & \downarrow \omega \\ \widehat{X/\sim} & \xrightarrow{\hat{\alpha}} & \widehat{X/\sim} \end{array}$$

Let $x \in X$. Then

$$\begin{aligned}
 \omega \circ f(x) &= \omega(f(x)) \\
 &= ([[f(x)]], [[x]], [[f^{-1}(x)]], \dots) \\
 &= \hat{\alpha}([[x]], [[f^{-1}(x)]], \dots) \\
 &= \hat{\alpha} \circ \omega(x).
 \end{aligned}$$

Therefore (X, d, f) and $(\widehat{X}/\sim, \hat{\delta}, \hat{\alpha})$ are topologically conjugate.

Chapter 7

Future Directions

In this chapter, we outline some ideas of how our results could be applied in future research.

7.1 Homology for Smale Spaces

This section closely follows Sections 2.6 and 5.1 of [10].

In [17], R.F. Williams defines the notion of “shift equivalence” to classify shifts of finite type. Motivated by Williams’ results, W. Krieger [5] defined two dimension groups for shifts of finite type, and proved that they are invariant under shift equivalence. The dimension group that we are interested in is denoted $D^s(\Sigma, S)$ for a shift of finite type (Σ, S) . Putnam [10] extends the definition of Krieger’s invariants to all non-wandering Smale spaces.

For a Smale space (X, d, f) and $x \in X$, we call the set

$$X^s(x) = \{y \in X \mid \lim_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0\}$$

the *stable equivalence class* of x . Similarly, the set

$$X^u(x) = \{y \in X \mid \lim_{n \rightarrow \infty} d(f^{-n}(x), f^{-n}(y)) = 0\}$$

is called the *unstable equivalence class* of x .

Let $\pi : (X, f) \rightarrow (Y, g)$ be a factor map between Smale spaces. We say that π is *s-bijective* if for each $x \in X$, π maps the stable equivalence class of x bijectively to the stable equivalence class of $\pi(x)$. Recall that an *s-resolving* map was only required

to be injective on the local stable sets. Similarly, π is *u-bijective* if for each $x \in X$, π maps the unstable equivalence class of x bijectively to the unstable equivalence class of $\pi(x)$.

Theorem 2.6.3 of [10] gives the following.

Theorem 7.1 (Putnam [10]). *Let (X, f) be a non-wandering Smale space. Then there exist a Smale space (Y, g) with $Y^u(y)$ totally disconnected for all $y \in Y$, a Smale space (Z, h) with $Z^s(z)$ totally disconnected for all $z \in Z$, an *s-bijective* factor map $\pi_s : (Y, g) \rightarrow (X, f)$, and a *u-bijective* factor map $\pi_u : (Z, h) \rightarrow (X, f)$.*

The 6-tuple $(Y, g, \pi_s, Z, h, \pi_u)$ in Theorem 7.1 is called an *s/u-bijective pair*, and is denoted simply by π .

Let $\pi = (Y, g, \pi_s, Z, h, \pi_u)$ be an *s/u-bijective* pair for the Smale space (X, f) . For each $L, M \geq 0$, Putnam defines

$$\Sigma_{L,M}(\pi) = \{(y_0, \dots, y_L, z_0, \dots, z_M) \mid y_l \in Y, z_m \in Z, \pi_s(y_l) = \pi_u(z_m), \\ 0 \leq l \leq L, 0 \leq m \leq M\}.$$

One naturally defines a mapping σ on $\Sigma_{L,M}(\pi)$ by

$$\sigma(y_0, \dots, y_L, z_0, \dots, z_M) = (g(y_0), \dots, g(y_L), h(z_0), \dots, h(z_M)).$$

By Theorem 2.6.6 of [10], $(\Sigma_{L,M}, \sigma)$ is a shift of finite type for all $L, M \geq 0$.

Furthermore, for $L \geq 1$ and $0 \leq l \leq L$, Putnam defines $\delta_{l, \cdot} : \Sigma_{L,M}(\pi) \rightarrow \Sigma_{L-1,M}(\pi)$ as the map which deletes entry y_l . Similarly, for $M \geq 1$ and $0 \leq m \leq M$, he defines $\delta_{\cdot, m} : \Sigma_{L,M}(\pi) \rightarrow \Sigma_{L,M-1}(\pi)$ as the map which deletes entry z_m . These factor maps induce maps on Krieger's dimension groups and by summing, with alternating signs, we obtain boundary maps in the following double complex:

$$\begin{array}{ccccc} & & \uparrow & & \uparrow & & \\ & & D^s(\Sigma_{0,1}) & \longleftarrow & D^s(\Sigma_{1,1}) & \longleftarrow & \\ & & \uparrow & & \uparrow & & \\ & & D^s(\Sigma_{0,0}) & \longleftarrow & D^s(\Sigma_{1,0}) & \longleftarrow & \end{array}$$

For a general Smale space (X, d, f) , this complex is very complicated. However, things improve drastically if we assume that the local stable sets of X are totally disconnected. In this case, we can let $Z = X$ and $\pi_u = \text{id}$ in the *s/u-bijective* pair for

(X, f) . Furthermore, recall the Markov partition we used to define \sim on X in our inverse limit construction. This Markov partition yields a shift of finite type (Σ, S) , and an s -resolving factor map $\pi_s : (\Sigma, S) \rightarrow (X, f)$ (see [3]). We let $(Y, g) = (\Sigma, S)$ in the s/u -bijective pair for (X, f) . The result is that all the rows in the double complex are identical. So the idea is that if we can understand this special case, it will help us to understand the general case.

Now, we can understand $\Sigma_{L,0}$ from the model $(X/\sim, \alpha)$ we constructed in the proof of the Realization Theorem. We conjecture that it is possible to compute $H_*(X)$ just by looking at $(X/\sim, \alpha)$. The idea is to decompose X/\sim into a finite number of “rectangles”, and using the data of which collections of rectangles have non-empty intersection to compute $H_*(X)$.

This was in fact the original goal of this research. However, the task of finding the correct model proved much more substantial than anticipated.

7.2 Ultrametrics

Let d be a metric on X . Then d is an *ultrametric* if the triangle inequality is strengthened as follows:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}, \quad x, y, z \in X.$$

It seems a worthwhile question to ask whether the metric \hat{d} on the inverse limit space \hat{Y} in Chapter 5 is an ultrametric on sufficiently small local stable sets. (It can't be overall, since the existence of an ultrametric means the space is totally disconnected.) If it is not, could it be adjusted to be one?

Ultrametric spaces have much more structure than general metric spaces, and there are a number of applications in physics. Our own curiosity was motivated by a paper of Pearson and Bellissard [8]. They define a spectral triple on ultrametric Cantor sets as part of a program for applying the techniques of noncommutative geometry to fractals.

7.3 Constructing Examples

Our inverse limit construction could also be used to produce additional interesting examples of Smale spaces.

Recall our example from Section 4.3, where we identified vertices of six Sierpinski

gaskets. Suppose we started with N gaskets instead of six. It is conceivable that there is a general procedure to decide which vertices to identify, and where to send the midpoints of the edges of the gaskets. This would result in a system (Y, g) , and we are interested in knowing under what conditions such a system would satisfy Axioms 1 and 2.

In a different direction, an alternative method of construction the Sierpinski gasket uses an iterated function system (IFS). Iterated function systems are a well-researched way of constructing self-similar fractals. It seems worthwhile to look for a systematic approach to use a general IFS to produce a system satisfying Axioms 1 and 2.

And finally, for a system satisfying Axioms 1 and 2, there may be conditions under which the restriction to certain subspaces also satisfies these axioms. For example, consider the following. Suppose (Y, g) satisfies Axioms 1 and 2, and let Z_0 be a closed subspace of Y such that $g(Z_0) \supseteq Z_0$. Let

$$Z = \{z \in Z_0 \mid g^n(z) \in Z_0 \forall n \geq 0\} = \bigcap_{n \geq 0} g^{-n}(Z_0).$$

It is easy to show that Z is non-empty and $g : Z \rightarrow Z$ is surjective. Moreover, $(Z, g|_Z)$ satisfies Axiom 1 quite trivially. We ask for a condition on Z_0 that would also make this system satisfy Axiom 2. For additional motivation, take the simple example $g(x, y) = (3x, 3y) \bmod 1$ on \mathbb{T}^2 with $Z_0 = \mathbb{T}^2 \setminus (\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$. Then $(Z, g|_Z)$ is the Sierpinski carpet, and satisfies Axioms 1 and 2.

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