PERFORMANCE OF GRADE EIGHT MATHEMATICS STUDENTS ON SELECTED DISCOVERY TASKS

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ABSTRACT

A set of programmed booklets was prepared by the writer and presented to his mathematics classes to determine how well Grade Eight students perform on selected discovery tasks of a mathematical nature. The subjects were ninety-four students in three Mathematics 8 classes taught by the writer at Belmont-Fisher Secondary School, School District 62 (Sooke) in the 1973-74 school year.

Of nine programs tested in a pilot study, five were selected for the main experiment. The programmed booklets were presented to the pupils in their regular mathematics classes after prerequisite concepts had been studied. These booklets were used to assess skill in discovery and ability to verbalize discoveries.

For each of the selected tasks, the Chi-square statistic was used to test at the .05 level of significance whether rate of discovery was associated with mathematics achievement, mathematical ability, verbal ability, and sex. For two tasks in which the material was presented in alternate forms, the association between rate of discovery and method of presentation was also tested.

Scores on the Cooperative School and College Ability Test: Form 4-A were used as measures of mathematical ability and verbal ability. Mathematics achievement scores were available from the Stanford Achievement Test: Advanced Arithmetic Tests, Form W, which was written by the students in their Grade Seven classes.

For all of the discovery tasks investigated, rate of discovery was found to be independent of sex. For all tasks except one, rate of discovery was associated with mathematical ability and achievement. Rate of discovery was associated with verbal ability on two of the five tasks. For both topics in which method of presentation was varied, rate of discovery was found to be independent of method. It was also found that verbalization was not a reliable indication for the discovery of a concept.

The experimental findings were discussed with regard to amount of guidance, discovery readiness, the role of verbalization in discovery learning, the use of programmed materials, and the application of discovery techniques in the mathematics classroom.

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CHAPTER I

INTRODUCTION

Many attempts have been made to study various aspects of discovery methods of learning and teaching mathematics. Such studies usually compare a discovery treatment with an expository approach and generally find no significant difference in immediate achievement, retention, or transfer (Herman, 1969). Several studies have attempted to compare discovery and expository methods in terms of ability or achievement levels (Corman, 1957; Meconi, 1967; Michael, 1959), but no conclusions could be reached as to which was better suited to students with high, average, or low ability or achievement.

Although the results of research do not indicate a general superiority for either discovery or expository methods, there are a number of reasons which suggest that the discovery method should have its place in the mathematics classroom. Many great teachers from Socrates to Polya have relied heavily on heuristics. And classroom teachers testify to the benefits of discovery teaching in terms of the increased interest, enthusiasm, and participation of their pupils. That the active participation of the learner promotes learning is perhaps the most widely accepted principle of educational psychology. When the pupil is required to find his own answers, he can hardly avoid taking an active part. Finally, people tend to believe that they have better mastery and retention of what they discovered for themselves.

There is a further reason for advocating discovery learning in mathematics. Learning by discovery is particularly appropriate in mathematics because of its contribution in building intellectual skills applicable to all types of problem solving. In the modern world acquiring knowledge is not as important as learning how to learn. Plentiful experiences of the discovery type should be useful in teaching the student how to learn mathematics.

It has already been indicated that there is very little research evidence to support the seemingly convincing arguments in favor of discovery learning. Unless some explanation for the discrepancy between the anticipated benefits of learning by discovery and the empirical results can be suggested, further investigation would seem unwarranted. Much of the confusion results from lack of agreement on the meaning and implications of 'discovery'. It is not the kind of concept which can be precisely defined. As the review of the literature will clearly show, it is folly to try to establish meaningful generalizations about *the* discovery method for the simple reason that it does not exist. However, in fairness to researchers, it should be noted that they did describe the specific facets of the general problem on which they focussed. The real problem has been a human tendency to make unwarranted generalizations in order to arrive at simple answers to complex questions.

Although it can be argued that students should, whenever feasible, discover mathematical relationships for themselves, it has proved very difficult to simulate the ideal conditions for discovery in a controlled group experiment. The research does not refute the claims made on behalf of learning by discovery because discovery in the experimental situation is often artificial, contrived, and only remotely resembles the strategies of master teachers.

The discovery teaching versus expository teaching controversy has not been resolved, and it is pointless to pursue it. Both teaching strategies have a place. It is not a question of deciding which strategy is superior but of choosing the better strategy for a particular situation. However, there might be a serious flaw in the case for discovery learning; namely, the implicit assumption that whatever is to be discovered can be presented in a manner so that most pupils will discover.

Discovery usually grows out of extensive experience. Apart from rather elementary relationships, genuine discoveries are rare for most people. Because of their familiarity with mathematical concepts, teachers may be poor judges of the difficulty which pupils encounter in discovery tasks. The historical view of the growth of mathematical ideas suggests that today's commonplace in mathematics was originally strange and difficult. Therefore, it may be erroneous to assume that most ideas in school mathematics are readily discovered if properly presented.

Of course it would be a hopeless task to attempt to list all discoverable concepts, principles, problems, etc., but it is feasible, as was done in this study, to prepare discovery exercises on a number of topics of various types and difficulties, and to find out how well a typical group of pupils coped with these materials. The disadvantage of this procedure is the specificity of the results, which are meaningful only in conjunction with the materials used. This is not to say that the results cannot be generalized. Expected performance on other tasks could be inferred from a careful examination of the materials and the accompanying analyses.

The discussion in this chapter leads to several general conclusions. First, there is sufficient evidence to justify a place for discovery in the mathematics classroom. Secondly, it is not profitable to prolong the debate and research on the relative merits of expository and discovery strategies. Thirdly, the concept of discovery has not been precisely defined and no useful purpose would be served by attempting to do so. Finally, further studies concerning discovery learning should be directed to specific rather than general questions. These considerations led to the choice for the general purpose of this study, that is, to determine how well Grade Eight pupils perform on selected discovery tasks of a mathematical nature. The review of the literature raises further questions about discovery learning which will be formally stated in Chapter III.

CHAPTER II

REVIEW OF RELEVANT LITERATURE

Advantages and Disadvantages of Discovery Methods

From early experiments pertaining to discovery learning, the single conclusion which is still relevant to present research is that students taught by a discovery method seem to be more able to transfer the ideas learned to new material (Herman, 1969). The recent trend is to emphasize the importance of discovery in terms of attitude change, motivation, and creativity. It was hypothesized by Berger and Howitz (1967) that a general mathematics program using the discovery approach would produce a positive change in attitude. In general, they found no significant difference, but observed that the attitude scores of brighter students tended to increase on the Mathematics Inventory, a test developed by Cyril J. Hoyt and Donald G. MacEachern at the University of Minnesota in 1958. Price (1967), using his own questionnaire, also found a positive change in attitude in non-academic mathematics students.

Mathematics laboratories using discovery methods are currently popular because of the increases in motivation, creativity, and active learning which are supposedly achieved by this approach. Studies have been done both in Canada (Dawson, 1970; Sigurdson, 1970; Trivett, 1970; and Vance, 1970) and in Britain (Biggs, 1968) to show that the laboratory approach is an effective means of producing an active learning experience. Two of the most noted proponents of discovery, Polya (1965) and Bruner (1961) stress the importance of active learning as opposed to "learning about". The motivating effect of discovery learning was reported to be significant by Kersh (1962) and Davis (1966).

One of the most important advantages of the discovery approach is the direct involvement of the learner in the learning process. Sigurdson and Johnston (1970) quote Piaget as saying, "The goal in education is not to increase the amount of knowledge but to create the possibilities for a child to invent and discover [for] himself." Such a process, states Bruner (1960), should induce a feeling of self-reward, an important goal of education. Along with self-reward, adds Bruner, discovery should give the student a feeling of self-confidence and independence in the learning situation.

Despite the advantages mentioned, discovery learning does have certain drawbacks. The chief problem which Ausubel (1963, 1967) and Hess (1968) point out are time costs, inefficiency, and impracticality. With thirty to forty pupils in six or seven classes a day, there is little possibility of using a pure heuristic method which would require much time and attention to the individual. Ausubel feels that the discovery process is so time-consuming that it is not practical beyond the elementary school level. He goes on to state that propositions discovered through problem-solving methods are rarely significant and worth incorporating into the learner's subject-matter knowledge. Finally, Ausubel points out that many students do not have the ability to discover all they need to know. Hess (1968) goes even further to state that some teachers are incapable of using discovery methods effectively.

Problems Involved in Reviewing Literature on Discovery Learning

In reviewing the relevant literature on discovery methods versus expository methods, consideration must be given to the definitions of the terms involved. It is obvious that there is not just one discovery method or one expository method. And when several authors refer to *the* discovery method or *the* expository method, there is little wonder that seemingly similar experiments lead to contradictory findings. As others have found, the interpretation of research is made difficult by differences in terminology (Hendrix, 1961; Herman, 1969; Roughead, 1968; Scandura, 1964; Worthen, 1967). When precise definitions are given for each of the methods involved, findings are easier to interpret. For example, studies which provide explanatory phrases such as 'rule-given versus rule-not-given' or 'concept initially given versus concept delayed' (Kersh, 1958; Meconi, 1967; Roughead, 1968; Wittrock, 1963; Worthen, 1967) are more informative than those which merely refer to *the* expository and *the* discovery methods.

As Herman (1969) has shown in his excellent annotated review of literature on discovery techniques, the terms 'discovery/expository' are sometimes replaced by 'inductive/deductive'. However, Ausubel (1963) has pointed out that both discovery and expository methods can be inductive or deductive in their approaches.

Another example of conflicting or confusing terminology is the use of the phrase 'expository teaching methods' to imply 'rote learning'. The results of studies comparing discovery and expository methods under this assumption would obviously differ from those of studies which use 'rote' to describe the memorizing process which follows exposition or discovery (Ausubel, 1967; Kersh and Wittrock, 1962). 'Rote learning' suggests a repetitive process in which little attention is paid to the meaning of the material to be learned. Since 'expository' implies explanation and 'rote' implies disregard for meaning, it is clear that the terms are not interchangeable.

In addition to the problems involved in interpretation of terms, one must also bear in mind the characteristics of the subjects in a particular experiment. The results of experiments on college students enrolled in educational psychology courses (Haslerud and Meyers, 1958; Kersh, 1958; Sassenrath, 1959) are not necessarily applicable to the junior high school classroom. Nor would a study involving female students in a compulsory mathematics education course (Roughead, 1968) be directly comparable to a study involving grade nine boys (Ray, 1961).

In order to meet criteria for experimental control, many studies of discovery learning are based on novel, artificial tasks rather than typical classroom activities. One must be very careful in making uncritical generalizations from the results of such varied studies as match stick patterns (Corman, 1957; Katona, 1940), coding tests (Haslerud and Meyers, 1958; Wittrock, 1963), measuring with calipers (Ray, 1961), and word-number relationships (Sassenrath, 1959). Similarly, studies which do have direct application to the researcher in the field of mathematics (Gagné and Brown, 1961; Hendrix, 1961; Kersh, 1958, 1962; Michael, 1949; Roughead, 1968; Worthen, 1967) may not apply to other subject areas. Attention to the method of presenting the tasks is also important in comparing studies. In some studies the material is presented to individuals by programmed texts (Gagné and Brown, 1961; Kersh, 1962; Wittrock, 1963) while in others the material is presented orally to groups of students (Michael, 1949; Ray, 1961). Worthen (1967) suggests that the results of a carefully controlled classroom experiment which represents typical school behavior could be generalized to classroom practice with more confidence than those obtained from a short-term laboratory experiment.

Questions Raised by Research on Discovery Learning

Verbalization

One important question which stems from research on discovery learning is the role of verbalization. Hendrix (1961) argues that heuristic methods have been wrongly condemned because of the confusion between the verbalization of a discovery and the discovery itself. Correct verbalization of a concept, she claims, does not necessarily emerge immediately after the concept is discovered. Therefore it is important that teachers recognize responses other than "linguistic formulations" as being indicators of discovery. Becker and McLeod (1967) stress that ability to solve a task can be acquired *without* verbal formulation of what has been learned and performed successfully.

Intelligence and Achievement

The question of the relationships among intelligence, achievement, and ability to discover is often raised in the research with no general conclusions being formulated. Corman (1957) found that the less able student could do more match stick problems when given a rule, but that information about rules was detrimental. Meconi (1967) found no significant differences among ability levels on tests of transfer and retention when subjects were taught number sequence problems by programmed materials. In his experiment on methods of teaching the use of micrometer calipers, Ray (1961) did not find a significant interaction between method and mental ability although there was some indication that the discovery treatment was better for slower students. Michael (1949) found no significant differences among ability levels in an experiment involving signed numbers, although students of high and average ability did slightly better when taught by a deductive method while the lower-ability students did slightly better with an inductive method. Differences in terminology, tasks, samples, and procedures could account for the apparent contradiction with Ray's (1961) study.

Independent and Guided Discovery

Much of the research on discovery concerns the amount of direction given. Several studies involving a comparison of guided discovery, independent discovery, and expository techniques have found no significant differences between the three approaches (Roughead, 1968; Scandura, 1964; Schulman, 1968). Scandura (1964) and Corman (1957) claim that in guided discovery the timing of the presentation of instruction as well as the amount of guidance contributes to the learning. In investigating the process of learning in tasks involving arithmetical and geometrical relationships, Kersh (1958) found that although his self-directed group showed no superiority in terms of "meaningful learning", they were more highly motivated than the group that received guidance. Most studies, however, seem to conclude that a moderate amount of guidance is better than little guidance (Gagné and Brown, 1961; Kittle, 1957; Wittrock, 1963).

Sex

Educators have often asked whether or not boys are superior to girls in mathematics. Fennema (1974) concludes from her review of research into sex differences in mathematics that boys tend to rate higher than girls on higher-level cognitive tasks while girls are generally superior on lower-level cognitive tasks. In the specific area of discovery learning, Fennema cites one study (Sowder, 1971) in which there was no significant difference between boys and girls. Fennema also points out that there is some evidence to suggest that sex-related factors which influence mathematics achievement are environmental, and some evidence to support an argument that inherent factors are more important.

CHAPTER III

STATEMENT OF PURPOSE

Problems Investigated

In Chapter I, it was argued that discovery learning should have a place in the mathematics classroom. However, a review of the literature has shown that the voluminous research related to discovery learning has produced few unequivocal results. It would seem that further investigations should focus on specific practical questions and that detailed information be provided concerning the tasks, the subjects, and the experimental conditions.

It has been established that 'discovery learning' is an expression which cannot be precisely and unambiguously defined, especially if no mention is made of the nature of the learning task. A full appreciation of what is involved in a particular discovery task can be obtained only by examining the materials themselves. No synopsis or brief description will suffice. Therefore, a complete set of the task materials is reproduced in the appendices so that the reader can make his own inferences as to how applicable the results of this study would be in other circumstances.

As previously stated in very general terms, the main purpose of this study was to determine how well Grade Eight pupils perform on selected discovery tasks of a mathematical nature. Grade Eight is a particularly appropriate level to investigate discovery learning since the pupils not only have sufficient background to make significant discoveries, but, according to Piaget (1928), they also have reached the stage at which they can deal with formal operations. On the other hand, they have not yet reached the time when there is pressure on teacher and student to "cover" substantial amounts of content. Furthermore, Ausubel (1963) argues that, despite any benefits which might be claimed for it, learning by discovery is not a practical strategy in the senior secondary school.

The question as to how well the children performed can be put in at least two ways. What portion of the students could make specific discoveries? How much guidance was needed by students of varying characteristics to make specific discoveries? The experiment was designed to provide answers to both questions.

Since the study is concerned with the discovery of mathematical principles, the subjects' bent for mathematics is probably a relevant characteristic. But it is not clear whether mathematical achievement or aptitude would be the more important factor. Perhaps these factors are so highly correlated at the Grade Eight level that the distinction is unimportant. Nevertheless, in this study both relationships were investigated, i.e., between discovery and mathematical achievement and between discovery and mathematical aptitude.

Some degree of verbal skill is required in every academic school subject. Since the discovery tasks in this study were presented in programmed booklets, success on the tasks could well be affected by verbal ability. Therefore, it was decided to look into the relationship between discovery and verbal ability.

It was pointed out in Chapter II that there is a continuing controversy regarding the link between the discovery of a principle and the verbalization of it. An extreme position taken by some teachers is that inability to verbalize a concept, principle, or generalization implies lack of real understanding. In other words, the quality of the verbalization indicates the degree of understanding. But before pursuing sophisticated questions about the relationships between discovery and verbalization it would be appropriate to produce some evidence in absolute terms of children's skill in verbalization. Consequently an attempt was made to answer this question: How well do Grade Eight pupils verbalize their discoveries?

Finally, the relationship between discovery and sex was also investigated. Again research has failed to provide definite answers to the question of sex differences in mathematical achievement or aptitude. But if as Fennema (1974) suggests, boys at the junior high school level are superior on "more advanced problems of a cognitive nature", then they might also be better discoverers.

The Discovery Tasks

It was recognized at the outset that no small set of discovery tasks could be representative of all possible tasks suitable for Grade Eight pupils. Nevertheless, care was taken to choose exercises varying in difficulty and type. The discovery tasks selected for the study were:

1. To discover a rule for squaring a two-digit number ending in five;
2. To discover a method of finding the number of subsets of a given set;
3. To discover how to add two numbers with unlike signs;
4. To discover how to find, without dividing, the remainder when a number is divided by nine;
5. To discover how to classify a two-digit base five number as even or odd.

Statement of Hypotheses

In an exploratory study such as this, elaborate statistical analysis is not warranted. However, some clarification of intent is achieved by formally restating certain questions as hypotheses. It seemed more convenient to state the hypotheses in terms of rate of discovery rather than amount of guidance. Pupils who fail to discover a principle may be considered as very slow discoverers.

The following hypotheses are stated for each of the five discovery tasks:

1. Rate of discovery is independent of mathematical ability.
2. Rate of discovery is independent of mathematics achievement.
3. Rate of discovery is independent of verbal ability.
4. Rate of discovery is independent of sex.

CHAPTER IV

EXPERIMENTAL PROCEDURE AND DESIGN

Presentation of Tasks

Two basic questions had to be answered before considering other procedural details:

1. What would be the best method for presenting the tasks?
2. How would pupil performance on the tasks be evaluated?

Three methods of presentation were seriously considered:

1. Individual interview;
2. Oral presentations to a class;
3. Programmed booklets.

The major advantage of the interview is the opportunity for direct observation of each individual's behavior. The examiner can estimate the degree of participation of the subject; for example, his interest in the problem and the effort put forth. He can also probe if necessary to find out if genuine discovery has occurred. A second important advantage is that individual differences can be accommodated by suitable variations in pace and amount of guidance. However, since interviews are very time-consuming, the size of sample would be severely limited. Furthermore, it would be very difficult to control the experimental conditions adequately over an extended period of testing.

The main advantage of the oral presentation method is its similarity to what is commonly regarded as teaching by discovery. A "live" presentation by an enthusiastic and skilful teacher can be most effective, and virtually impossible to duplicate by other means. But pupil discovery measured under such conditions is inevitably confounded with teacher skill. And even the best teachers find it impossible to set a pace and difficulty level which is optimal for all their students. Class discussions are often dominated by the brighter students thus depriving others of their share of discovery experiences.

The third method, presentation by programmed materials, provides the best control of experimental conditions. The teacher variable is eliminated and each pupil has an opportunity to make the same discovery independently of his classmates. The presentation is identical for all, yet each student is able to complete the program at his own rate. Furthermore, with a branching program, the amount of guidance each student receives can be varied. If a task is clearly and unambiguously presented, and the time limits are sufficiently generous, then students who do not discover a specific principle are, from a practical point of view, unable to do so.

However, there are disadvantages too in the use of programmed booklets. The presentation is artificial and the vital spark of a forceful teacher is missing. Also sacrificed is the interaction with other pupils, probably a most important element in stimulating ideas. Nevertheless, although not ideal, the programmed method of presentation was selected as having more positive features than the other methods.

The booklets were designed to determine not only whether a student was able to make a specific discovery, but also how much help he needed. The criterion for discovery was success on an item, or on a specified number of a set of items, which required the application of the principle to be discovered. It is possible that the use of this criterion would occasionally credit pupils with having discovered principles which they did not fully understand. But on the other hand, it seemed that an acceptable verbalization of the principle would be too stringent a criterion.

Varying amounts of guidance were provided by means of branching programs. In the first presentation, little or no assistance was given. Additional help was provided in the second stage, and in some cases, in a third stage. Thus, for each principle, it was possible to determine how readily it was discovered.

Pilot Study

In the 1972-73 school year, preliminary versions of programmed material on nine discovery tasks were tested with three classes of Grade Eight students. The results were analyzed and three programs were judged unsuitable either because some students had prior knowledge of the concepts to be discovered or because the tasks proved to be too difficult. Six programs were thoroughly revised for use in the main experiment.

Careful consideration was given to choosing time limits which would be long enough for a substantial number of students to make non-trivial discoveries, yet not so long as to cause fatigue or boredom. The preliminary trials indicated that thirty minutes was an appropriate time limit for each program.

The Main Experiment

Sample

The revised versions of the selected programs were given to three Grade Eight classes at Belmont-Fisher Secondary School, School District 62 (Sooke). There were ninety-four students in these classes, all of which were taught by the writer. The classes were not homogeneously grouped although student choice of program may indirectly have operated as a selective factor. However, the sample obtained by pooling three classes can be considered reasonably representative of the total Mathematics 8 population (ten classes).

Instruments and Materials

Mathematics achievement was measured by the Stanford Achievement Test: Advanced Arithmetic Tests, Form W. This test was written by all Grade Seven students in the Sooke District in June 1973. Newcomers to the district wrote the test in September 1973. Students who scored in the top third of the sample were classified as high achievers, those in the middle third as average achievers, and those in the bottom third as low achievers.

Mathematical ability and verbal ability were measured by the Cooperative School and College Ability Test: Form 4-A, which was given in September 1973. The scores on the relevant sections of this test were used to classify students as high, average, or low in terms of mathematical ability and verbal ability.

The author's programmed learning booklets were used to assess skill in discovery of the selected mathematical principles and ability to verbalize these discoveries. To facilitate marking, an answer sheet was prepared for each booklet. The topics of the booklets were:

1. Squaring Numbers Ending in Five;
2. Sets and Subsets;

3. Adding Signed Numbers;
4. A Short-cut for Remainders (Casting out Nines);
5. Even and Odd Numbers, Base Ten;
6. Even and Odd Numbers, Base Five.

The booklets "Sets and Subsets" and "Adding Signed Numbers" were given in alternate forms in an attempt to determine whether the method of presentation affected the discovery scores. The task in "Sets and Subsets" was to determine the rule for finding the total number of subsets which can be selected from a set of n elements. The basic strategy in Form A was a systematic listing of all types of subsets, i.e., singletons, pairs, etc. In Form B, tree diagrams were used.

Form A of "Adding Signed Numbers" used the conventional symbols with the problems presented in a systematic order, i.e., one type at a time beginning with the simplest. Form B used the conventional symbols with the types presented randomly. Vector symbolism was used in Forms C and D, the types being presented systematically in Form C and randomly in Form D.

<u>Procedure</u>

Each booklet was given in a regular mathematics period. Since the half hour time limit was very generous, suitable activities were provided for pupils who finished early. For topics having multiple forms, the individuals in each mathematical ability level were randomly assigned to forms.

The pilot study established that the principles to be discovered were, in general, not already known to Grade Eight students. However, it was highly likely that a few students would have encountered some of the concepts in the booklets. To identify pupils with prior knowledge of the tested material, key questions were included with regular written assignments. If a student successfully answered one of the key questions, his answer sheet for the related booklet was discarded. The number of answer sheets eliminated for this reason ranged from none on "Even and Odd Numbers, Base Five" to twelve on "Adding Signed Numbers".

Statistical Analysis

For each of the five discovery tasks, the Chi-square statistic was used to test at the .05 level of significance the hypotheses listed in Chapter III.

For the discovery tasks in "Sets and Subsets" and "Adding Signed Numbers", one additional hypothesis was formulated:

5. Rate of discovery is independent of the method of presentation.

For both tasks, the Chi-square statistic was used to test this hypothesis at the .05 level of significance.

CHAPTER V

ANALYSIS OF DATA

Results of the Tests of the Hypotheses

A summary of the results of the tests of the hypotheses is shown in Table 1. The statistical tests lead to the following generalizations:

1. For all discovery tasks investigated, rate of discovery is independent of sex;
2. For all discovery tasks investigated, except finding the number of subsets for a given set, rate of discovery is associated with mathematical ability and achievement;
3. For two of the five topics ("Squaring Numbers Ending in Five" and "Short-cut for Remainders"), the results of the Chi-square tests suggest that rate of discovery is related to verbal ability;
4. For three of the five topics ("sets and Subsets", "Adding Signed Numbers", and "Even and Odd Numbers in Base Five"), the results of the Chi-square tests suggest that rate of discovery is independent of verbal ability;

TABLE 1

THE RELATIONSHIP BETWEEN RATE OF DISCOVERY AND SELECTED VARIABLES AS INDICATED BY CHI-SQUARE TESTS

Topic		Mathematics Achievement	Mathematical Ability	Verbal Ability	Sex	Method of Presentation
Squaring Numbers Ending in Five	Chi-square	31.2	28.8	16.0	1.7	
	Critical value (df)	12.6 (6)	12.6 (6)	12.6 (6)	7.8 (3)	
	Conclusion	dependent	dependent	dependent	independent	
Sets and Subsets	Chi-square	7.1	9.4	0.3	2.7	3.9
	Critical value (df)	9.5 (4)	9.5 (4)	9.5 (4)	6.0 (2)	6.0 (2)
	Conclusion	independent	independent	independent	independent	independent
Adding Signed Numbers	Chi-square	26.3	15.2	9.0	2.1	4.7
	Critical value (df)	9.5 (4)	9.5 (4)	9.5 (4)	6.0 (2)	12.6 (6)
	Conclusion	dependent	dependent	independent	independent	independent
Short-cut for Remainders	Chi-square	27.2	12.6	2.0	1.8	
	Critical value (df)	12.6 (6)	9.5 (4)	9.5 (4)	7.8 (3)	
	Conclusion	dependent	dependent	independent	independent	
Even and Odd Numbers in Base Five	Chi-square	12.2	16.7	16.6	0.07	
	Critical value (df)	9.5 (4)	9.5 (4)	9.5 (4)	6.0 (2)	
	Conclusion	dependent	dependent	dependent	independent	

5. For both topics in which method of presentation was varied ("Sets and Subsets" and "Adding Signed Numbers"), rate of discovery is independent of method.

Interpretation of Results

Since it is impossible to generalize over all discovery tasks, results on each of the five tasks are discussed separately. For each task, contingency tables are presented to show in detail the relationship between rate of discovery and mathematical ability and the relationship between rate of discovery and verbal ability. Contingency tables are not given for discovery and mathematics achievement because they do not contain significant additional information. Tables are also provided to show the results of responses to the other items in each booklet.

Squaring Numbers Ending in Five

Task 1: To discover a rule for squaring
a two-digit number ending in five.

As shown in Table 2, the rate of discovery patterns for the various ability groups are distinctly different. For example, 41% (12/29) of the high mathematical ability group was successful on the first presentation of the task, whereas none of the low mathematical ability group succeeded. And 90% (26/29) of the high ability students eventually discovered compared to 43% (12/28) of the low ability students. With minor variations similar patterns occur for all tasks. Furthermore, the number of low mathematical ability pupils making a discovery on a first presentation is consistently less than the number of low verbal ability pupils who are successful.

TABLE 2

THE RELATIONSHIP OF RATE OF DISCOVERY ON TASK 1 TO MATHEMATICAL ABILITY AND VERBAL ABILITY

	Rate of Discovery						
	No	Late	Ave.	Early	Total	x^2	Critical value p = .05
Math. Ability							
Low	16	9	3	0	28	28.8*	12.6 (df=6)
Average	6	13	7	6	32		
High	3	6	8	12	29		
	25	28	18	18	89		
Verbal Ability							
Low	14	9	4	2	29	16.0*	12.6 (df=6)
Average	8	11	4	8	31		
High	3	8	10	8	29		
	25	28	18	18	89		

*independence hypothesis rejected.

A summary of the success on task 1 items is given in Table 3. A high percentage (87%) of the discoverers also received credit for acceptable verbalizations. Many students used numerical examples to explain their method as follows:

> For 75 x 75, you put down 25 and then multiply 7 x 8 and put 56 in front of 25.

Although the pupil has not expressed himself in an elegant fashion, he has clearly demonstrated his knowledge of a short-cut method for squaring odd multiples of five. Another example of an acceptable verbalization is:

> For 75^2, put down 25 and multiply 7 x 7 and add 7. Put the answer (56) in front of 25.

TABLE 3

SUCCESS ON TASK 1 ITEMS

Task	Successful Students: Percentage of total group	Successful Students: Percentage of those successful on discovery task
To discover a rule for squaring a two-digit number ending in five	71	
To verbalize the discovery	62	87
To complete correctly:* 65 x 65 = (6 x 7 x 100) +25	62	87
To complete correctly: N5 x N5 = (N x (N+1) x 100) +25	19	27
To complete correctly: When a certain number ending in 5 is squared, the answer is 7225. The number must be 85.	34	48

*Henceforth, whenever completion items are quoted, the expected response is hand-lettered.

The percentage of students who were able to state how they squared a number ending in five was identical to the percentage of students who completed the following correctly:

$$65 \times 65 = (\underline{6} \times \underline{7} \times 100) + 25$$

Thus, this figure (62%) can be regarded as a reliable estimate of the percentage of the group which was successful on the discovery task. The higher figure (71%) probably included some pupils who multiplied in the ordinary way to find the answer.

However, only one-third of the discoverers were able to complete the following item correctly:

$$N5 \times N5 = (\underline{N} \times \underline{(N+1)} \times 100) + 25$$

Here is clear evidence that students must specifically be taught how to think in terms of variables, and, in this case, how to represent the integer following N. The poor performance on this item was unexpected because the pilot study was done late in the school year after the students had had experience with algebraic problems involving consecutive numbers.

It is rather disturbing to note that almost half of the students who demonstrated their ability to use the short-cut procedure could not answer this item:

> When a certain number ending in 5
> is squared, the answer is 7225.
> The number must be 85.

Apparently these students did not see how this item was related to the problem which they had just solved, i.e., that essentially they were required to find the smaller of two consecutive numbers whose product is 72. Thus there is some question about the value of adroitly guiding students to the discovery of a specific bit of knowledge if they are unable to apply that knowledge in a simple problem.

Discovery tasks can be more difficult than they seem. For example, 28% of the pupils were unable to complete 55 x 55 = 3025 correctly even after this hint:

> The underlined digits have been placed in a table on the right. Try to see how you can get the number in column B by using the number in column A.

	A	B
(a) 15 x 15 = 2 25	1	2
(b) 25 x 25 = 6 25	2	6
(c) 35 x 35 = 12 25	3	12
(d) 45 x 45 = 20 25	4	20

Sets and Subsets

Task 2: To discover a method of finding the number of subsets of a given set.

The patterns in Table 4 showing the relationship between discovery on task 2 and mathematical ability are much like the comparable patterns for task 1. Nevertheless, the hypothesis that rate of discovery is independent of mathematical ability is *not* rejected because the probability associated with the computed Chi-square is .052, which slightly exceeds the stipulated value of .05. In contrast, the evidence concerning the relationship between discovery and verbal ability is unequivocal. In this case, the patterns are strikingly similar for all ability groups thus strongly suggesting that for task 2, rate of discovery is not related to verbal ability.

TABLE 4

THE RELATIONSHIP OF RATE OF DISCOVERY ON TASK 2 TO MATHEMATICAL ABILITY, VERBAL ABILITY, AND METHOD OF PRESENTATION

	Rate of Discovery					
	No	Late	Early	Total	X^2	Critical value p = .05
Mathematical Ability						
Low	15	9	4	28		
Average	20	7	4	31		
High	9	9	11	29		
	44	25	19	88	9.4*	9.5 (df=4)
Verbal Ability						
Low	14	8	7	29		
Average	14	9	6	29		
High	16	8	6	30		
	44	25	19	88	.3*	9.5 (df=4)
Method of Presentation						
Form A	20	10	13	43		
Form B	24	15	6	45		
	44	25	19	88	3.9*	6.0 (df=2)

*independence hypothesis *not* rejected.

Although it was reported earlier that the Chi-square test suggests that rate of discovery on task 2 and method of presentation are not related, the data in Table 4 indicate that perhaps the tree diagram approach (Form B) is somewhat more difficult than the roster approach (Form A). On Form B, only 13% (6/45) of the subjects succeeded on the first presentation of the task compared to 30% (13/43) on Form A. A possible explanation of the greater difficulty of the tree diagram approach might be that the discovery task was made more complex by the introduction of an unfamiliar schema along with the counting problem itself.

The results of the students' responses on the questions related to the discovery task are presented in Table 5. Although about half of the students were able to discover a method of finding the number of subsets of a given set, none explained how to do so directly in terms of the number of elements in the set. Many students did not attempt any explanation of how they arrived at their answer. Of those who did, a typical answer was:

> Take 2 times the number of subsets for the number which came before.

When the papers were originally scored, this kind of answer received no credit since, as the following program excerpt shows, the instructions specified that the number of subsets be found from the number of elements:

Here are some more examples:

	A	B
1.	A set with 2 members has	4 subsets.
2.	A set with 3 members has	8 subsets.
3.	A set with 4 members has	16 subsets.
4.	A set with 5 members has	32 subsets.
5.	A set with 6 members has	64 subsets.

Study these examples very carefully. Try to see how the numbers in column B are related to the numbers in column A, *not* to other numbers in column B.

It will be noted that the student is specifically instructed to relate the numbers in column B to the numbers in column A. Two conjectures are offered to explain why these instructions were ignored: (1) the basic strategy of the program was to consider, in turn, sets with two, three, and four members, and (2) the format in which the numerals were displayed draws attention to the relationship within column B rather than between columns. Under the circumstances, it seems reasonable to allow credit for explanations based on the relationship in column B.

The percentages of successful students on the related items were extremely low (0% - 4%). The failure of this program indicates the importance of ensuring a thorough understanding of underlying concepts before expecting students to apply them to a novel situation. In the tryouts, the subjects had the advantage of additional familiarity with exponents derived from their Grade Eight work.

"Sets and Subsets" is an example of a carefully structured program which not only did not lead students to the intended discovery but actually produced a set which distracted them from it.

TABLE 5

SUCCESS ON TASK 2 ITEMS

Task	Percentage Successful	
	Form A	Form B
To discover a method of finding the number of subsets of a given set	57	50
To verbalize the discovery	36	28
To name or describe two subsets of any set "S"	2	4
To complete correctly: If a set has n members, there are $\underline{2^n}$ subsets	0	0
To complete correctly: A set with 15 members has $\underline{2^{15}}$ subsets	0	0

Adding Signed Numbers

Task 3: To discover how to add two numbers with unlike signs.

To receive credit for discovery, students had to correctly answer five out of six questions of the following type:

$$-30 + {}^{+}25 = \underline{-5}$$

$$\text{or} \quad \overrightarrow{40} + \overleftarrow{30} = \underline{\overrightarrow{10}}$$

The methods of presentation (symbols and order) do not have a significant effect on the rate of discovery of the method for adding numbers with unlike signs. However, students who were presented material with vector notation in a random order (Form D) had the most difficulty discovering the principle. This is not unexpected since students are more familiar with the + and - signs.

TABLE 6

THE RELATIONSHIP OF RATE OF DISCOVERY ON TASK 3 TO MATHEMATICAL ABILITY, VERBAL ABILITY, AND METHOD OF PRESENTATION

	Rate of Discovery					
	No	Late	Early	Total	X^2	Critical value p = .05
Mathematical Ability						
Low	8	6	8	22	15.2*	9.5 (df=4)
Average	2	8	17	27		
High	1	4	21	26		
	11	18	46	75		
Verbal Ability						
Low	5	7	13	25	9.0	9.5 (df=4)
Average	3	9	11	23		
High	3	2	22	27		
	11	18	46	75		
Method of Presentation						
Form A	2	8	10	20	4.7	12.6 (df=6)
Form B	4	3	13	20		
Form C	2	3	12	17		
Form D	3	4	11	18		
	11	18	46	75		

*independence hypothesis rejected.

The relationship of the rate of discovery on task 3 to mathematical ability, verbal ability, and method of presentation is shown in Table 6. The patterns observed here between discovery and mathematical ability exhibit two features unique to this discovery task. The percentage of discoverers is abnormally high (96% of the high ability group and 85% of the total group) and the percentage of the low ability group successful on the "first" presentation is also unusually high (36%). The principle to be discovered may be inherently easier than the others, but a combination of other factors is perhaps more significant in accounting for the discrepancy in the results.

The signed numbers program was the longest and most carefully structured. In the first part, which dealt with like signs only, the basic ideas and the model for representing addition examples on a number line were developed. In the second part, which dealt with unlike signs, each of the four cases was presented as a question and followed by the answer together with a representation on the number line. After this, there was a trial set of six items for which the answers were supplied on the following page. Only then was the pupil presented with the "discovery" task. In short, the booklet contained a large element of programmed *instruction*.

In the second place, it is reasonable to assume that pupils had prior experience with related activities, even if they had not previously been formally taught how to add signed numbers. This conjecture also explains the relatively high success of the low ability group.

In this task, discovery does not seem to be related to verbal ability. Two suggestions for this can be offered. Because of prior experience, the students were less dependent on the program for an explanation of the problem. Furthermore, the number line provided an effective supplement to verbal explanations.

An example of an acceptable verbalization of the method for adding numbers with unlike signs follows:

> Ignore the signs, subtract the smaller number from the larger, and put in the sign of the larger.

Although the above is not expressed in elegant terms, it is clear that the pupil can find the answers. Many students did not even attempt to state their method in words, but most who did used examples as follows:

> For $-6 + +8$, take 6 from 8 and put in a plus sign.

Since instructions specifically excluded the use of the number line, explanations involving it were unacceptable.

A table was prepared in an attempt to uncover relationships between method of presentation and discovery, verbalization, and success in transferring to a different notation. No clear trends were apparent from the examination of these data. Perhaps there was a slight indication that the success rates for conventional symbols and systematic order were somewhat higher than others.

Almost invariably, students who discovered how to add directed numbers using one notation had no difficulty using the other notation. A few students failed to make the transfer from conventional to vector symbolism.

A Short-cut for Remainders

Task 4: To discover how to find the remainder, without dividing, when a number is divided by nine.

To receive credit for early discovery, students had to answer correctly two of the following questions:

Find the remainder from the digits
when the given number is divided by 9.

Number	Remainder
1723	4
5404	4
1334	2

Although verbal ability and sex are not related to discovery in task 4, mathematical ability is associated with ability to discover. These results are presented in Table 7. The observed frequencies for the high mathematical ability group deviate markedly from the expected frequencies.

TABLE 7

THE RELATIONSHIP OF RATE OF DISCOVERY ON TASK 4 TO MATHEMATICAL ABILITY AND VERBAL ABILITY

	Rate of Discovery					
	No	Late	Early	Total	X^2	Critical value p = .05
Mathematical Ability						
Low	17	6	5	28	12.6*	9.5 (df=4)
Average	16	13	3	32		
High	8	9	12	29		
	41	28	20	89		
Verbal Ability						
Low	16	7	6	29	2.0	9.5 (df=4)
Average	12	10	8	30		
High	13	11	6	30		
	41	28	20	89		

*independence hypothesis rejected.

The results of the students' responses on the questions related to the discovery task are presented in Table 8. An example of an acceptable verbalization for task 4 follows:

> Add all the digits in the number. If the sum is less than 9, that is the remainder. If it is equal to 9, the remainder is 0. If it is more than 9, divide by 9 to get a remainder.

None of the students discovered the more elegant procedure of adding the digits in the remainder if it was a two-digit number. Students were not required to mention the case of zero remainders since only one test set contained such an item.

TABLE 8

SUCCESS ON ITEMS RELATED TO TASK 4

Task	Successful Students	
	As a percentage of total group	As a percentage of those successful on discovery task
To discover how to find the remainder when a number is divided by 9 (without dividing)	54	
To verbalize the discovery	56	104
To write a 4-digit number whose remainder is 6 when the number is divided by 9	55	102
To explain *why* the method for finding remainders works	0	0

Those students who discovered the method of casting out nines to determine the remainder when a number is divided by 9, had no difficulty in verbalizing the concept or in using it to write a number with a given remainder. However, no one was able to explain why the method works.

Even and Odd Numbers, Base Five

Task 5: To discover how to classify two-digit base five numbers as even or odd.

The booklet "Even and Odd Numbers, Base Ten" was used as an introduction to task 5. It provided a review of place value in the decimal numeration system. Since students already knew how to classify base ten numbers as even or odd, their only task was to discover why a number must be even if the last digit is even. No student was able to give a satisfactory explanation. An analysis of results on the introductory booklet is therefore unnecessary.

TABLE 9

THE RELATIONSHIP OF RATE OF DISCOVERY ON TASK 5 TO MATHEMATICAL ABILITY AND VERBAL ABILITY

	Rate of Discovery					
	No	Late	Early	Total	X^2	Critical value p = .05
Mathematical Ability						
Low	22	5	1	28	16.7*	9.5 (df=4)
Average	12	10	8	30		
High	10	6	12	28		
	44	21	21	86		
Verbal Ability						
Low	21	4	4	29	16.6*	9.5 (df=4)
Average	16	4	7	27		
High	7	13	10	30		
	44	21	21	86		

*independence hypothesis rejected.

To receive credit for discovery on task 5 students had to correctly label five out of six base five numbers as even or odd. The data in Table 9 suggest that rate of discovery on task 5 is related to mathematical ability and to verbal ability. Only one person in the low mathematical ability group was classified as an early discoverer, but twelve in the high mathematical ability group were.

The following is an example of an acceptable verbalization of the method of identifying base five numerals as even or odd:

> Add the digits. If the sum is even, the number is even. If the sum is odd, the number is odd.

Any explanation involving the expanded form of the base five numeral was not acceptable, since this method was specifically excluded in the instructions.

The results of the students' responses on the questions related to the discovery task are presented in Table 10. Although over half of the students were able to discover a method of classifying two-digit base five numbers as even or odd, only about one quarter were able to verbalize the concept. Perhaps some students did not find a short-cut but used the expanded form to convert to base ten, thereby classifying the numbers correctly. This conjecture is supported by the fact that only 14% of the students were able to classify four-digit numbers correctly. Again only half of this group provided satisfactory

TABLE 10

SUCCESS ON ITEMS RELATED TO TASK 5

Task	Successful Students: As a percentage of total group	Successful Students: As a percentage of those successful on discovery task
1. (a) To discover how to classify 2-digit base five numbers as even or odd	50	
(b) To verbalize this discovery	23	47
2. (a) To discover how to classify 4-digit base five numbers as even or odd	14	
(b) To verbalize this discovery	7	50
3. (a) To discover that the base ten method of classifying numbers as even or odd will work for any even base	21	
(b) To discover that the base five method of classifying numbers as even or odd will work for any odd base	21	
(c) To verbalize these discoveries	13	61

explanations. However, 21% of all subjects did respond correctly to the following item:

> Question 11
>
> B. The short-cut method for deciding even or odd in base 5 will work for which other bases?
>
> 3 6 7 9 12

But only 13% of all subjects gave adequate explanations. Taking all factors into consideration, it seems clear that less than 20% of the total group thoroughly understood the principle that they were expected to discover.

General Comments

Although the risk involved in attempting to generalize from the specific tasks has been stressed, the data were examined to see if any overall trends were apparent. An attempt was also made to find plausible explanations for the varying degrees of success among tasks.

The data of Table 11 reveal little consistency of performance on the various discovery and verbalization tasks. There is, however, a marked uniformity in the percentage of students succeeding on the first presentation of four of the discovery tasks. The values vary between 20% and 24% and perhaps suggest that somewhat less than one fourth of an average group will be able to make *independent* discoveries in *novel* situations. However, if suitable guidance is given, the percentage will usually rise to 50% or more.

TABLE 11

COMPARATIVE SUCCESS ON DISCOVERY AND VERBALIZATION TASKS

	Percentage discovering		Percentage verbalizing	
Topic	At first presentation	At any stage	Percentage of group	Percentage of those discovering
Squaring numbers ending in five	20	72	62	87
Sets and subsets	20	50	32	64
Adding signed numbers	61	85	33	40
Short-cut for remainders	22	54	56	100
Even and odd numbers in base five	24	49	21	39

It is not difficult to offer an explanation for the superior performance on addition of signed numbers, the only topic in the mainstream of the curriculum. If twelve students had somehow learned how to add signed numbers before they were exposed to the programmed booklet, then others may have had similar opportunities. Thus it is likely that students performed well on this task partly because the ideas were not entirely new to them.

There was a wide variation in the percentage making adequate verbalizations of discovery tasks of comparable difficulty. In three booklets about half of the group eventually satisfied the criterion for discovery. But of the successful discoverers, the percentage which also gave acceptable verbalizations of what they discovered ranged from 39% to 100%. And although 85% of the students discovered a procedure for adding signed numbers, only 40% of the discoverers were able to state clearly how to add signed numbers. These results suggest that the ability to verbalize is not a reliable index of understanding. The quality of a verbalization is influenced to some degree by the pupil's general verbal facility and his familiarity with concepts, language, and symbolism associated with a specific principle.

CHAPTER VI

CONCLUSIONS AND IMPLICATIONS

Conclusions

The following conclusions refer specifically to the tasks, materials, students, and experimental conditions of this study:

1. For four of the five tasks, discovery was related to mathematical ability.
2. For some tasks, discovery was related to verbal ability. The inherent nature of the task and the mode of presentation probably contributed to the equivocal results.
3. Success on the discovery tasks was not related to sex differences. This result supports the contention that when sex differences in mathematics achievement are found, they have been culturally induced.
4. For the two topics in which method of presentation was varied, success on the discovery tasks was not significantly different. However, the data indicated that students found the systematic presentation in familiar language and symbolism somewhat easier. It should be noted in passing that ease of discovery does

not necessarily imply better learning and retention.

5. On novel tasks, a minority of students succeeded in making independent discoveries on the first presentation. And even when given very specific guidance, a substantial number were still unsuccessful.
6. Students were more successful when thoroughly familiar with the language, symbolism and concepts related to the principle to be discovered. In other words, success is related to *discovery readiness*.
7. Satisfying the criterion for discovery of a principle did not imply its mastery. Some pupils were unable to apply or generalize principles for which they received credit for discovering.
8. Ability to verbalize was not a reliable criterion of discovery.

Implications for the Classroom Teacher

A thoughtful consideration of the results of this study, especially in conjunction with a critical appraisal of the booklets themselves, could lead to better appreciation of the problems associated with heuristic techniques, to more realistic expectations, and to an appropriate balance between discovery and other teaching strategies. The case for discovery is somewhat weakened by the relatively limited success which pupils had on the tasks. Unless an individual is reasonably successful, he is hardly likely to enjoy the benefits claimed for discovery learning. It may well be that for such a person, frequent failure would depress interest and motivation by destroying his confidence in his ability to learn mathematics.

This study has produced further evidence that pupils may not discover much on their own. Most pupils need some guidance, but the amount is related to mathematical aptitude. It is a real challenge to develop strategies for providing each pupil with the optimum level of assistance. It is by no means certain that giving very specific guidance is the best strategy for developing full understanding. Furthermore, as in the case of "Sets and Subsets", the particular approach used may inhibit a broader appreciation of the principle.

On certain items the subjects did not have a reasonable opportunity for success because they had inadequate mastery of relevant ideas. Thus it is clear that the teacher must provide for discovery readiness by ensuring, insofar as possible, that his pupils have the knowledge, language, and symbolism needed for the task at hand.

The use of programmed booklets, except in an experimental situation, is far from ideal. Although each student does have the opportunity of making the discovery on his own, he is deprived of the stimulation of other minds

The argument that programmed booklets ensure that each student is actively involved is not entirely convincing. Many students learn to go through the motions with minimal mental effort. In general, the printed page is less effective than a good teacher in creating interest, in providing motivation, and in focussing on the critical features of the problem. Furthermore, it is extremely difficult to provide for all possible contingencies without producing a very dull program. Finally, the physical, mental, and emotional state of the individual at the time the booklet was presented could easily make the difference between success and failure. Since the booklet was to be completed at one sitting there was no opportunity to postpone dealing with the problem until a more favorable time.

The written responses of the students indicated a general weakness in expressing themselves clearly. However, the necessity to verbalize a concept can be questioned since it was found that students can discover and use a principle without being able to put it into words.

Despite the arguments in favor of learning by discovery in the classroom, there are limitations. Even when using carefully prepared materials some students do not discover. Teachers should be aware of the strengths and limitations of discovery techniques so that they may use them effectively in the classroom.

The effort involved in preparing the programmed materials makes such a process prohibitive for the classroom teacher. The preparation of each booklet involved several trial runs and revisions, and even then the final versions of the booklets were by no means flawless. It proved to be extremely difficult to prepare material which avoided all possible misunderstandings. In contrast, it is relatively easy in a class discussion to deal with difficulties as they arise.

Suggestions for Further Research

The present study has shown that certain principles presented by programmed booklets can be discovered by Grade Eight mathematics students. The study could be replicated with different topics and different grade levels.

As a follow-up to this study, one might wish to test rate of discovery against other variables such as attitude towards mathematics or personality. Furthermore, attitude change resulting from learning by discovery could be studied.

Although this study concerns discovery as an individualized process within the classroom setting, other methods of presenting the material could be investigated. One might present the material to small groups and observe each student's ability to discover in a group situation. By using their own materials and strategies, teachers can determine which methods are appropriate and meaningful for their pupils.

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APPENDIX I

SQUARING NUMBERS ENDING IN FIVE

Task 1: To discover a rule for squaring a two-digit number ending in five.

1

To square a number, multiply it by itself. For example, to square 4, multiply 4 by 4.

In this booklet, you are going to look for a fast way of squaring numbers ending in '5'. As you do each question, be looking for a short-cut method.

Remember, you are to follow the booklet as instructed, write only on the answer sheet, and try as hard as you can to find a short-cut. Now, turn to page 2 and begin the search for a quick way of squaring numbers ending in '5'.

2

Here are some examples. Study them.

(a) 15 x 15 = 225

(b) 25 x 25 = 625

(c) 35 x 35 = 1225

Do you see a short-cut already? If so, try your method on 45 x 45= 2025 . If it works, turn to page 8.

If you would like more help, turn to page 3.

3

Here are the examples again:

(a) 15 x 15 = 225

(b) 25 x 25 = 625

(c) 35 x 35 = 1225

Question 1

Complete on your answer sheet:

The thing I notice about all of the answers is ______________.

(Turn to page 4.)

4

Answer

You probably noticed that the answers all end in 25. For example 15 x 15 = 225

Now, study these examples, noting the underlined digits.

(a) 1 5 x 1 5 = 2 25

(b) 2 5 x 2 5 = 6 25

(c) 3 5 x 3 5 = 12 25

Do you see a short-cut now? If so, try your method on 4 5 x 4 5 = 20 25 . If it works, turn to page 7.

If you would like more help, turn to page 5.

5

Here are some more examples to help you find a short-cut. The underlined digits have been placed in a table on the right. Try to see how you can get the number in column 'B' by using the number in column 'A'.

(a) 1 5 x 1 5 = 2 25

(b) 2 5 x 2 5 = 6 25

(c) 3 5 x 3 5 = 12 25

(d) 4 5 x 4 5 = 20 25

A	B
1	2
2	6
3	12
4	20

Turn to page 6.

Question 2

On your answer sheet, write the answer to the following question <u>without</u> multiplying out in full.

55 x 55 = ________

Turn to page 9.

Question 3

On your answer sheet, write the answer to the following question <u>without</u> multiplying out in full.

65 x 65 = ________

Turn to page 9.

Question 4

On your answer sheet, write the answer to the following question <u>without</u> multiplying out in full.

75 x 75 = ______

Turn to page 9.

Question 5

In a sentence or two, explain <u>how</u> to use a short-cut method for multiplying 75 x 75. (Use your answer sheet.)

Turn to page 10.

The number 3625 can be written as

3600 + 25
or (36 x 100) + 25
or (6 x 6 x 100) + 25
or (12 x 3 x 100) + 25 etc.

Question 6

Use this idea to express how the short-cut method of squaring works for 65 x 65 = 4225 . Complete the following by filling the blanks on your answer sheet.

65 x 65 = 4225 or 4200 + 25

or (________ x ________ x 100) + 25

Turn to page 11.

Consider the number N5 where N can be any digit except zero.

For example: N5 = 45 if N is 4.
N5 = 95 if N is 9.

Question 7

Complete by filling in the blanks on your answer sheet.

N5 x N5 = (_____ x _____ x 100) + 25

Turn to page 12.

Question 8 Complete by filling in the blank on your answer sheet.

When a certain number ending in 5 is squared, the answer is 7225. The number must be ______.

Now, hand in your booklet and your answer sheet.

Squaring Numbers Ending in Five

Answer Sheet Name ____________

Question 1 ____________ ____________ ____________

Question 2 ________

Question 3 ________

Question 4 ________

Question 5

____

____

Question 6

(___ x ___ x 100) + 25

Question 7

N5 x N5 = (___ x ___ x 100) + 25

Question 8

____

Score Chart

		Page	Points
Rate of Discovery	No discovery		
	2. Late discovery	6	
	3. Average discovery	7	
	4. Early discovery	8	
	5. Verbalize	9	
	6. Place Value	10	
	7. Generalize	11	
	8. Apply	12	

APPENDIX 2

SETS AND SUBSETS

(FORM A)

Task 2: To discover a method of finding the number of subsets of a given set.

1

Al and Bob were playing basketball in the school yard.

"Do you think that we'll make the team?," said Al.

"Hard to say ", replied Bob. "Mr. Jones might choose both of us, but..."

Question 1

Complete the following statement on your answer sheet.

There are some other decisions Mr. Jones, the coach, could make concerning Al and Bob. He could choose both Al and Bob or ______________________________.

Turn to page 2 and check your answer.

2

Answer

Here are the other decisions the coach could make:

1. Just Al, 2. Just Bob, 3. Neither Al nor Bob.

Did you get them all?

The four choices could be written by using the following symbols, where A stands for Al and B stands for Bob.

1. {A}, the set which contains A only.
2. {B}, the set which contains B only.
3. {A,B} the set which contains A and B.
4. { }, the empty set, which contains neither A nor B.

Now, turn to page 3.

3

While Al and Bob were practising, Charlie arrived. He is also trying out for the team.

"I see you two are trying out for the team", he said. "I wonder what the chances are of us being picked."

Albert smiled and said, "I know how many choices Mr. Jones could make about the three of us."

Question 2

On your answer sheet, put down the number of choices you think the coach could make.

Turn to page 4 to check with Albert's answer.

4

Answer

"There are 8 choices the coach could make concerning the three of us," said Albert.

Question 3

Check Albert's answer by listing on your answer sheet 8 choices the coach could make regarding the three boys. Use A, B, C, for Al, Bob, and Charlie. For example, {A,C} means that Al and Charlie are chosen, but Bob is not.

Turn to page 5 to check your answer.

5

Answer

Here is Al's list of the 8 possible decisions:

{ }, {A}, {B}, {C}, {A,B}, {A,C} , {B,C} ,{A,B,C}

Did you get them all? The order is not important.

We have seen that for 2 boys there are 4 choices and for 3 boys there are 8 choices.

Now, turn to page 6.

6

Dan arrived to find Al, Bob, and Charlie playing basketball and talking about the chances of making the team.

"What about me?" Dan questioned. "I'm going to try out, too."

"This is getting complicated," said Charlie. "There are 12 possibilities."

"No," said Bob, "there are more than 12!"

Question 4

On your answer sheet, put down the number of choices _you_ think the coach could make concerning the four boys.

Check with Bob's answer on page 7.

7

Answer

"There are _16_ possibilities," said Bob. (And Bob is right!)

Question 5

On your answer sheet, list the 16 possible choices concerning the four boys, A, B, C, D.

Turn to page 8 to check your answer.

8

Answer

Here are the 16 possibilities:

{ }, {A} , {B} , {C} , {D} , {A,B} , {A,C} , {A,D} , {B,C}
{B,D} , {C,D} , {A,B,C} , {A,B,D} ,{A,C,D} ,{B,C,D} ,{A,B,C,D}

So far we have seen the following examples:

For 2 boys, there are 4 choices.

For 3 boys, there are 8 choices.

For 4 boys, there are 16 choices.

Now, turn to page 9.

9

It seems to be getting more difficult to determine the number of possible ways of being selected, as the number of boys in the group increases, because it takes a long time to list them. You're going to look for a method of figuring out how many possibilities there are for _any_ number of boys in the group.

Turn to page 10.

10

We have already seen that the group of boys Al, Bob, Charlie, and Dan can be expressed as the _set_ {A,B,C,D} . The set {B,C} is called a _subset_ of {A,B,C,D} because each of its members belongs to the set {A,B,C,D} .

Question 6

Any non-empty set, call it S, must have at least 2 subsets. Name or describe these two subsets on your answer sheet.

Turn to page 11.

11

The number of ways in which Mr. Jones can make a choice from four boys is the total number of subsets which can be formed from a set with four members.

We have determined from our examples the following information:

1. If a set has 2 members, there are 4 subsets.
2. If a set has 3 members, there are 8 subsets.
3. If a set has 4 members, there are 16 subsets.

Study these 3 statements <u>very carefully</u> and try to find a method of getting the number of subsets from the number of members. If you can do this for a set with <u>any</u> number of members, turn to page 14 now.

If you would like more help, turn to page 12.

12

Here are some more examples:

	A	B
1.	A set with 2 members has	4 subsets.
2.	A set with 3 members has	8 subsets.
3.	A set with 4 members has	16 subsets.
4.	A set with 5 members has	32 subsets.
5.	A set with 6 members has	64 subsets.

Study these examples very carefully. Try to see how the numbers in column B are related to the numbers in column A, <u>not</u> to other numbers in column B.

Turn to page 13.

13

Question 7

On your answer sheet, complete the following statement.

A set with 7 members has _____ subsets.

Turn to page 15.

14

Question 8

On your answer sheet, complete the following statement.

A set with 8 members has _____ subsets.

Turn to page 15.

15

Question 9

The answer to this question is a large number. Do not calculate the exact answer, but use mathematical symbols to show what the answer will be. Use your answer sheet.

A set with 15 members has ________ subsets.

Turn to page 16.

Question 10

Complete on your answer sheet, using mathematical symbols.

If a set has 'n' members, there are ______ subsets.

Turn to page 17.

Question 11

If you were able to find a method of calculating the number of subsets for a set, explain in words how you got your answer. Use your answer sheet.

Hand in your booklet and your answer sheet.

Score Chart

Form ___ Name ______________

	Page	Points	Value
7. Late discovery	13		1
8. Early discovery	14		2
9. Use of exponents, specific case	15		1
10. Use of exponents, general case	16		1
11. Verbalization	17		1
Total discovery score			5

Sets and Subsets Form A

Answer Sheet Name ______________

Question Number

1. ______________________
____

2. ______________

3. ______________________
____
____

4. ______________

5. ______________________
____
____
____

6. ______________
____
____
____

7. ________ 8. ________ 9. ________ 10. ________

11. ______________________________
____
____

APPENDIX 3

ADDING SIGNED NUMBERS

(FORM A)

Task 3: To discover how to add two numbers with unlike signs.

1

If you cycle one mile along a perfectly straight road, and then walk a mile on the same road, how far are you from your starting point?

When you have solved the problem in your head, turn to the next page.

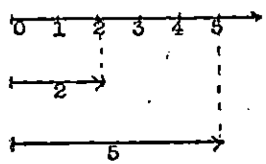
2

The distance depends on the direction you went! If you continued in the same direction, the answer is two miles. If you walked in the opposite direction, the answer is zero miles.

For this type of problem and many others, it is necessary to consider direction as well as distance. Turn the page to see how this works.

3

Let us replace the road by a number line, labelling the starting point as zero (0).

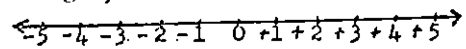


These examples show that from 0 to 2 is a trip of 2, and from 0 to 5 is a trip of 5.

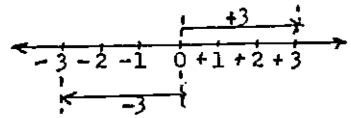
(Turn to page 4.)

4

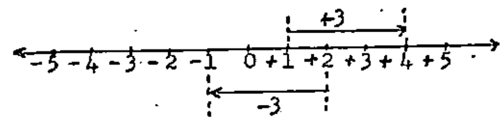
To be able to tell the direction and distance from the starting point, called the origin, we label our new number line as follows:



A trip from 0 to +3 is a trip of +3 and a trip from 0 to -3 is a trip of -3.



Furthermore, any trip of 3 to the right will be labelled +3 and any trip of 3 to the left will be labelled -3.



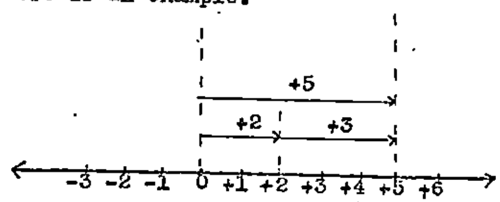
Our new set of numbers is called the set of integers,

(.....-3,-2,-1, 0,+1,+2,+3.....).

In this booklet we hope that you will discover how to add integers. Now, turn to page 5.

5

Let us think of integers as trips on a number line. Here is an example.



So, a trip of +5 is the same as a trip of +2 followed by a trip of +3. You will be able to try out this idea on the next page.

6

Question 1

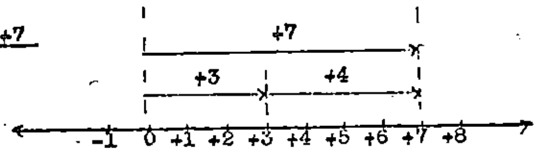
On your answer sheet, using the number line that is given, complete this statement:

A trip of +3 followed by a trip of +4 is the same as a trip of______.

(Check your answer on page 7.)

7

Answer: +7



So, a trip of +7 is the same as a trip of +3 followed by a trip of +4.

Now you are going on another trip. We have agreed that (+3) means 3 units to the right and (-3) means 3 units to the left.

Question 2

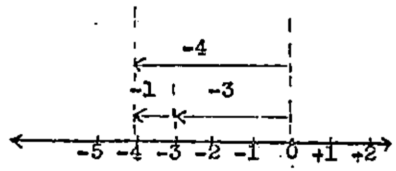
Complete this statement on your answer sheet.

A trip of -3 followed by a trip of -1 is the same as a trip of ______.

(Check your answer on page 8.)

8

Answer: -4 Study the following diagram to see how this answer is obtained.



This can be written as $(-3) + (-1) = (-4)$

Now, turn to page 9.

9

Use the number line to help you to add the following integers. Think of the answer as a single trip which will take you to the same place as two trips.

Question 3 Complete on your answer sheet.

(a) $(+5) + (+3) =$ ___

(b) $(+2) + (+4) =$ ___

(c) $(-7) + (-2) =$ ___

(d) $(-5) + (-4) =$ ___

Check your answers on page 10.

10

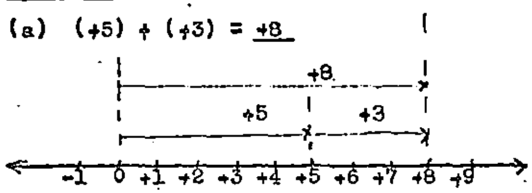
Answers: (a) +8 (b) +6 (c) -9 (d) -9

If all of your answers are correct, turn to page 11.

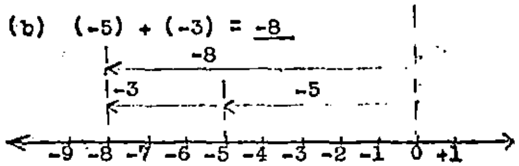
If you made any mistakes, study the following examples.

Examples:

(a) (+5) + (+3) = +8



(b) (-5) + (-3) = -8



Now, turn to page 11.

If you think that you are ready to answer more questions of this type <u>without using the number line</u>, turn to page 14.

If you would like more practice, turn to page 12.

Here are several examples. Try to see how to get the solution (with the correct sign) by doing something with the two integers you are adding. Study these examples very carefully.

(a) (+3) + (+6) = <u>+9</u> (d) (-2) + (-4) = <u>-6</u>

(b) (+4) + (+2) = <u>+6</u> (e) (+3) + (+5) = <u>+8</u>

(c) (-3) + (-6) = <u>-9</u> (f) (-5) + (-3) = <u>-8</u>

Think you've got it? Try the following <u>without</u> using the number line.

<u>Question 4</u> Complete on your answer sheet.

(a) (-2) + (-5) = ___

(b) (+6) + (+2) = ___

(c) (-4) + (-6) = ___

Check your answers from page 13.

<u>Answers</u>: (a) <u>-7</u> (b) <u>+8</u> (c) <u>-10</u>

Did you get these? Now, try the following questions.

<u>Question 5</u> Complete on your answer sheet <u>without</u> using the number line. No answers are given.

(a) (+12) + (+15) = ___

(b) (-13) + (- 5) = ___

(c) (-30) + (-40) = ___

(d) (+45) + (+15) = ___

(e) (-20) + (-60) = ___

Now, turn to page 15.

Complete the following question on your answer sheet <u>without using the number line</u> . Include the correct sign.

<u>Question 6</u> Complete on your answer sheet. No answers are given.

(a) (+20) + (+45) = ___

(b) (-25) + (-30) = ___

(c) (+35) + (+15) = ___

(d) (-40) + (-20) = ___

(e) (-80) + (-40) = ___

(Now turn to page 15.)

15

<u>Question 7</u>

On your answer sheet, explain briefly how to add two integers with the same sign without using the number line. Don't forget to tell how you decide which sign to use.

When you have completed the question as best you can, turn to page 16.

16

Suppose we use $\overrightarrow{3}$ to mean 3 units to the right and $\overleftarrow{3}$ to mean 3 units to the left. Answer the following questions, using the new symbols for your answers.

<u>Question 8</u> Complete on your answer sheet. No answers are given.

(a) $\overrightarrow{5} + \overrightarrow{16} =$ ____

(b) $\overleftarrow{34} + \overleftarrow{23} =$ ____

(c) $\overleftarrow{35} + \overleftarrow{46} =$ ____

(d) $\overrightarrow{50} + \overrightarrow{25} =$ ____

(e) $\overleftarrow{75} + \overleftarrow{75} =$ ____

(Now, turn to page 17.)

17

Let us now look at somewhat harder questions dealing with the addition of integers.

<u>Question 9</u> Complete on your answer sheet.

(+9) + (-6) = ____

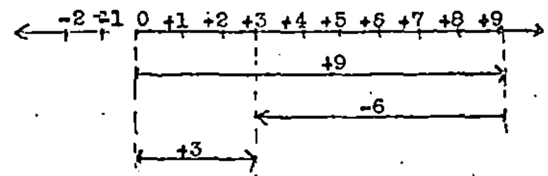
<u>HINT</u>: As a problem in trips along the number line, we ask ourselves "what single trip can replace a trip of +9 followed by a trip of -6 ?"

(Check your answer from page 18.)

18

<u>Answer</u> (+9) + (-6) = <u>+3</u>

Study the following diagram to see how this works on the number line.



The diagram shows that a trip of +9 followed by a trip of -6 can be replaced by a trip of +3.

Now, try the question on page 19.

19

<u>Question 10</u> Complete on your answer sheet.

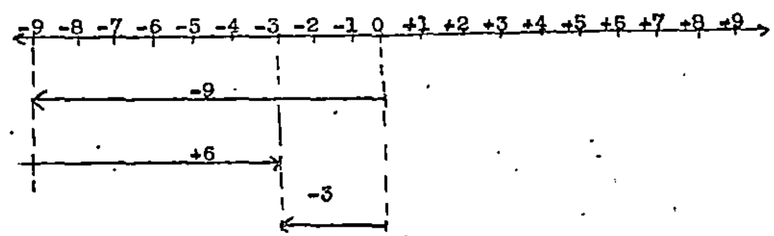
(-9) + (+6) = ____

(Or, a trip of -9 followed by a trip of +6 can be replaced by what single trip?)

Check your answer on page 20.

20

<u>Answer</u> <u>-3</u> Study the diagram to see how the answer was obtained.



The diagram shows that a trip of -9 followed by a trip of +6 can be replaced by a trip of -3. Study this example again if you did not get the answer.

Now, turn to page 21.

21

Question 11 Complete on your answer sheet.

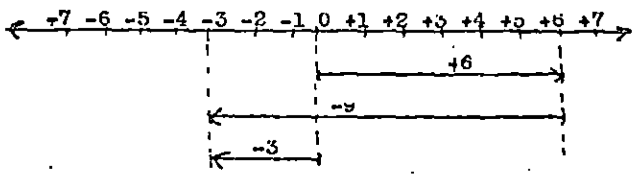
(+6) +(-9) = ____

(Or, a trip of +6 followed by a trip of -9 can be replaced by what single trip?)

Turn to page 22 to check the answer.

22

Answer -3 The diagram should help you to see how to get this answer.



Read the diagram as follows: a trip of +6 followed by a trip of -9 can be replaced by a trip of -3.

(Turn to page 23)

23

Question 12 Complete on your answer sheet.

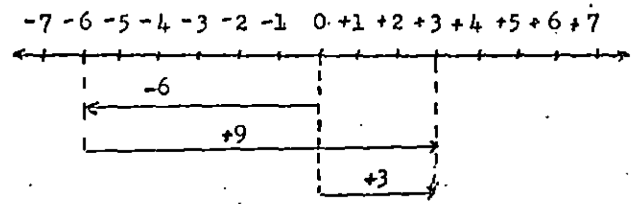
(-6) + (+9) = ____

(Or, a trip of -6 followed by a trip of +9 can be replaced by what single trip?)

The answer is on page 24.

24

Answer +3 The diagram should help you to see that a trip of -6 followed by a trip of +9 is the same as a trip of +3.



You have seen all possibilities of adding with different signs, using 9 and 3. Now, turn the page.

25

Question 13 Complete on your answer sheet. Use the number line.

(a) (+4) + (-5) = ____

(b) (-3) + (+2) = ____

(c) (+6) + (-2) = ____

(d) (-3) + (+4) = ____

(e) (+2) + (-7) = ____

(f) (-4) + (+6) = ____

Turn the page to check your answers.

26

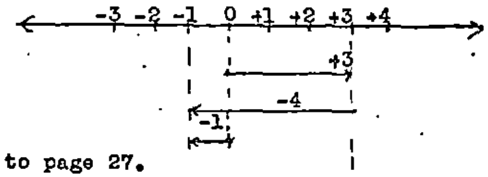
Answers: (a) -1 (b) -1 (c) +4 (d) +1 (e) -5 (f) +2

If all answers are correct, turn to page 27.

If you made any mistakes, study the example below.

<u>Example</u>: (+3) + (-4) = <u>-1</u>

Remember, +3 means a trip of 3 units to the right, starting at 0. Then you take the second trip of -4, or 4 units to the left, starting from where the first trip ends.



Turn to page 27.

27

Do you think you can now add any <u>two integers</u> without using the number line? If so, turn to page 30.

If you would like more help, turn to page 28.

28

Here are some examples. Try to see how to get the solution, <u>with the correct sign</u>, by doing something with the two numbers you are adding. Study the examples to find a pattern:

(a) (-9) + (+6) = <u>-3</u>

(b) (-6) + (+2) = <u>-4</u>

(c) (-2) + (+5) = <u>+3</u>

(d) (+4) + (-7) = <u>-3</u>

Turn to page 29.

29

<u>Question 14</u> Complete on your answer sheet. No answers are given.

(a) (-20) + (+25) = ___

(b) (+35) + (-15) = ___

(c) (-50) + (+25) = ___

(d) (+90) + (-50) = ___

(e) (+80) + (-90) = ___

(f) (-85) + (+35) = ___

Turn to page 31 when you have completed this question.

30

<u>Question 15</u> Complete on your answer sheet. No answers are given.

(a) (-30) + (+25) = ___

(b) (+40) + (-30) = ___

(c) (-80) + (+20) = ___

(d) (+90) + (-25) = ___

(e) (+60) + (-80) = ___

(f) (-65) + (+35) = ___

When you've completed the question, turn to page 31.

31

<u>Question 16</u> On your answer sheet, explain briefly how to add integers, each with a different sign, without using a number line. Don't forget to say how to decide which sign to use in your answer.

When you have completed this question, turn to page 32.

Answer the following questions using the new "arrow" symbols. No answers are given.

Question 17

(a) $\overrightarrow{20} + \overleftarrow{5}$ = ___

(b) $\overleftarrow{16} + \overrightarrow{8}$ = ___

(c) $\overrightarrow{45} + \overleftarrow{5}$ = ___

(d) $\overrightarrow{23}$ $+\overleftarrow{17}$ = ___

(e) $\overleftarrow{46} + \overrightarrow{34}$ = ___

Turn in your answer sheet and this booklet.

Score Chart   Name ____________________

	Page	Points	Value
5. Late discovery, like signs	13		1
6. Early discovery, like signs	14		2
7. Verbalization, like signs	15		1
8. Transfer, like signs	16		1
14. Late discovery, unlike signs	29		1
15. Early discovery, unlike signs	30		2
16. Verbalization, unlike signs	31		1
17. Transfer, unlike signs	32		1
Total discovery score			8

Adding Signed Numbers   A B C D

Answer sheet   Name ____________________

Question 1 ______   Question 2 ______   Question 3 (a) ___ (b) ___ (c) ___ (d) ___

Question 4 (a) ___ (b) ___ (c) ___

Question 5 (a) ___ (b) ___ (c) ___ (d) ___ (e) ___

Question 6 (a) ___ (b) ___ (c) ___ (d) ___ (e) ___

Question 7

____

____

Question 8 (a) ___ (b) ___ (c) ___ (d) ___ (e) ___

Question 9 ______

Question 10 ______   Question 11 ______   Question 12 ______

Question 13 (a) ___ (b) ___ (c) ___ (d) ___ (e) ___ (f) ___

Question 14 (a) ___ (b) ___ (c) ___ (d) ___ (e) ___ (f) ___

Question 15

(a) ____ (b) ____ (c) ____ (d) ____ (e) ____ (f) ____

Question 16

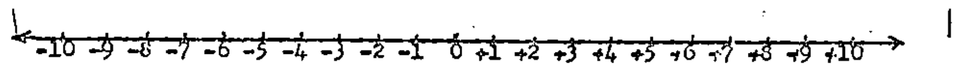
____

____

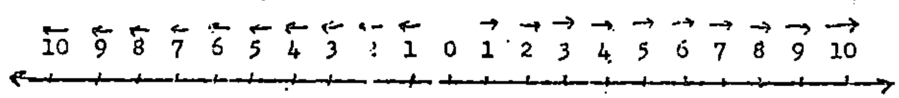
Question 17

(a) ____ (b) ____ (c) ____ (d) ____ (e) ____

Number Line Forms A and B



Number Line Forms C and D



APPENDIX 4

A SHORT-CUT FOR REMAINDERS

Task 4: To discover how to find the remainder when a number is divided by nine (without dividing).

1

In this booklet you are going to try to discover a quick way of finding remainders when dividing by 9. You will see later how this short-cut will help you to check difficult multiplication questions.

Follow the instructions in the booklet, complete only the questions which apply to you, and write all answers on your answer sheet.

Now, turn to page 2.

2

Study the following list of numbers and the remainder when divided by 9. Try to see how you can get the remainder without actually dividing the given number by 9.

Number	Remainder
54	0
144	0
70	7
35	8
13	4
85	4
21	3
678	3

If you have found a short-cut, try it out with the number, 1121. If the method works, turn to page 7.

If you would like some more help in finding the short-cut, turn to page 3.

3

Here are some more examples. Study them to see how you can get the remainder just by looking at the given number.

Number	Remainder
11	2
29	2
38	2
12	3
57	3
363	3
23	5
869	5
4568	5

If you can see a short-cut, try it on the number 2112. If it works, turn to page 6.

If you would like some more help, turn to page 4.

4

The number 2312 contains the digits 2,3,1,2. When 2312 is divided by 9, the remainder is 8. Try to see how the remainder can be found from the digits.

Example	Number	Remainder
	611	8
	11	2
	3413	2
	17	8
	4445	8
	21	3
	6780	3

(Turn to page 5)

Question 1

On your answer sheet, find the remainder from the digits when the given number is divided by 9.

Number	Remainder
2135	______
1472	______
3450	______

Now, turn to page 8.

Question 2

On your answer sheet, find the remainder from the digits when the given number is divided by 9.

Number	Remainder
2334	______
1143	______
3335	______

Now, turn to page 8.

Question 3

on your answer sheet, find the remainder from the digits when the given number is divided by 9.

Number	Remainder
1723	______
5404	______
1334	______

Now, turn to page 8.

Question 4

On your answer sheet, explain in a few words how you can find the remainder of a number divided by 9, without actually dividing the number by 9.

Turn to page 9.

Question 5

On your answer sheet, write a 4 digit number whose remainder is 6 when the number is divided by 9.

Turn to page 10.

Question 6

On your answer sheet, explain why the short-cut method works for 9 in our numeration system.

Now, hand in your booklet and answer sheet.

A Short-cut for Remainders

Answer Sheet — Name ______________

Question 1 ________

Question 2 ________

Question 3 ________

Question 4

____

____

____

Question 5 ________

Question 6

____

____

____

____

Score Chart

		Page	Points
Rate of Discovery	No discovery		
	1. Late discovery	5	
	2. Average discovery	6	
	3. Early discovery	7	
	4. Verbalize	8	
	5. Apply	9	
	6. Explain	10	

APPENDIX 5

EVEN AND ODD NUMBERS, BASE FIVE

Task 5: To discover how to classify two-digit base five numbers as even or odd.

1

We have studied odd and even numbers in base 10. Now we will look at base 5 numerals.

Question 1 (Circle the correct answer on your answer sheet.

Is 11_5 odd or even ?

(Now, turn to page 2.)

2

Answer 11_5 is even.

This can be shown by writing the numeral 11_5 in base 10.

eg. $11_5 = (1 \times 5) + 1$

$= 5 + 1$

$= 6$ which is even.

Notice that the last digit is odd, but the number is even. You are going to discover a method for deciding if a number in base 5 is even or odd and why the method works.

Now, turn to page 3.

3

Question 2.

Complete the following example on your answer sheet by filling each blank with the word even or odd as is done in the second line.

$31_5 = (3 \times 5) + 1$

= (odd x odd) + odd

= (________) + ______

The answer will be ______.

(Now, turn to page 4 .)

4

Answer

$31_5 = (3 \times 5) + 1$

= (odd x odd) + odd

= (odd) + odd

The answer will be even.

(Now, turn to page 5 .)

5

Question 3.

(Circle the letter of the correct answer on your answer sheet)

If a 2-digit base 5 numeral ends in 1 , then

(a) it must be odd.

(b) it must be even.

(c) it could be either odd or even.

(Now turn to page 6.)

6

Answer

(c) it could be either odd or even.

For example, 21_5 is odd but

31_5 is even.

(Turn to page 7 .)

7

Question 4.

Complete the following example on your answer sheet by filling each blank with the word even or odd .

$12_5 = (1 \times 5) + 2$

= (_____ x _____) + _____

= (_________) + ______

The answer will be ________.

(Now, turn to page 8 .)

8

Answer $12_5 = (1 \times 5) + 2$

$= (\underline{odd} \times \underline{odd}) + \underline{even}$

$= (\underline{odd}) + \underline{even}$

The answer will be odd .

If your answers are all correct, turn to page 11.

If you made any mistakes, turn to page 9 .

9

Review:

Even x Even = Even	Even + Even = Even
Odd x Even = Even	Odd + Even = Odd
Even x Odd = Even	Even + Odd = Odd
Odd x Odd = Odd	Odd + Odd = Even

Question 5. On your answer sheet, fill the blanks with even or odd .

$24_5 = (2 \times 5) + 4$

$= (\_\_\_\_ \times \_\_\_\_) + \_\_\_\_$

$= (\_\_\_\_\_\_\_\_) + \_\_\_\_$

The answer will be ______.

(Now, turn to page 10 .)

10

Answer $24_5 = (2 \times 5) + 4$

$= (\underline{even} \times \underline{odd}) + \underline{even}$

$= (\underline{even}) + \underline{even}$

The answer will be even .

(Now, turn to page 11.)

11

Question 6. In the numeral $A3_5$, the letter A represents a digit.

eg. If A = 4, then $A3_5$ represents 43_5 .

Since $A3_5 = (A \times 5) + 3$, for any value of A ,

Then (A x 5)

(a) must be odd .

(b) must be even .

(c) could be either odd or even .

(Circle the letter of one of the above choices ,on the answer sheet, then turn to page 12.)

12

Answer (c) could be either odd or even .

If your answer is correct, turn to page 14.

If your answer is incorrect, turn to page 13.

13

Study the following examples:

Example 1: $32_5 = (3 \times 5) + 2$

$= (\text{odd} \times \text{odd}) + 2$

$= (\text{odd}) + 2$

Example 2; $42_5 = (4 \times 5) + 2$

$= (\text{even} \times \text{odd}) + 2$

$= (\text{even}) + 2$

Note that the value of the expression within the parentheses ( ), was odd in example 1 and even in example 2 .

(Turn to page 14 .)

If you think you can determine whether any number like 42_5 is even or odd, without using the expanded form, 42_5 = (4x5)+2; turn to page 15.

If you would like more help, turn to page 16.

Question 7. On your answer sheet, label the following numerals odd or even. Do not use the expanded form ie. (2 x 5) + 3

(a) 23_5
(b) 44_5
(c) 14_5
(d) 21_5
(e) 13_5

(Now, turn to page 18.)

Study the following examples. The underlining may help you to see an important relationship.

(a) 23_5 = (2 x 5) + 3
= (even) + odd = odd

(b) 13_5 = (1 x 5) + 3
= (odd) + odd = even

(c) 44_5 = (4 x 5) + 4
= (even) + even = even

(d) 14_5 = (1 x 5) + 4
= (odd) + even = odd

(e) 21_5 = (2 x 5) + 1
= (even) + odd = odd

(Turn to page 17 .)

Question 8. On your answer sheet label the following numerals odd or even.

(a) 11_5
(b) 34_5
(c) 12_5
(d) 41_5
(e) 24_5

(Now, turn to page 18 .)

Question 9

To decide whether a base 5 numeral is even or odd, it could first be written in expanded form, eg. 34_5=(3x5) +4 = 19. If you can think of an easier method, describe it for question 9 on your answer sheet.

Turn to page 19.

Question 10 Circle the letter of the correct answer.

A. Would the method you've described work for a four digit number in base 5? (a) yes (b) no

B. Explain why you think it would or would not work.

C. Label the following numerals as even or odd.

(a) 2132_5
(b) 1322_5
(c) 1441_5
(d) 2143_5
(e) 1444_5

Turn to page 20.

Question 11

A. The short-cut method for deciding even or odd in base 10 will work for which other bases ? (Circle all correct answers on your answer sheet.)

3 6 7 9 12

B. The short-cut method for deciding even or odd in base 5 will work for which other bases ?

3 6 7 9 12

C. Explain how you decided which method to use for the bases in parts A and B.

Hand in your booklet and your answer sheet.

Score Chart		
	Page	Points
7 Early discovery	15	
8 Late discovery	17	
No discovery		
9 Verbalize	18	
10 Apply method to 4-digit number	19	
Explain method for 4-digit number	19	
11 Discover method for even and odd bases	20	
Explain method for even and odd bases	20	

Even and Odd Numbers in Base Five

Answer Sheet Name ______________

1. odd even

2. $31_5 = (3 \times 5) + 1$
$= (\text{odd} \times \text{odd}) + \text{odd}$
$= (\_\_\_\_) + \_\_\_\_$
The answer will be ______.

3. a b c

4. $12_5 = (1 \times 5) + 2$
$= (\_\_\_)\times(\_\_\_) + \_\_\_\_$
$= (\_\_\_\_) + \_\_\_\_$
The answer will be ______.

5. $24_5 = (2 \times 5) + 4$
$= (\_\_\_ \times \_\_\_) + \_\_\_\_$
$= (\_\_\_\_) + \_\_\_\_$
The answer will be ______.

6. a b c

7. (a) _____ (b) _____ (c) _____ (d) _____ (e) _____

8. (a) _____ (b) _____ (c) _____ (d) _____ (e) _____

9. ______________________________________

____

10. A. a b B. ______________________________

____

C. (a) _____ (b) _____ (c) _____ (d) _____ (e) _____

11. A. 3 6 7 9 12 B. 3 6 7 9 12

C. ______________________________________

____

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