On the Mathematical Foundations of Quantum Theory

by

Gerald William Charlwood

B.A., University of Toronto, 1973
M.A., University of Toronto, 1974
Ph.D., University of Toronto, 1978

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[redacted]
Gary G. Miller

[redacted]
Charles E. Murley

[redacted]
Albert E. Hurd

[redacted]
Colin D. Scarfe

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Supervisor: Professor Gary G. Miller

ABSTRACT

Beginning with the work of Birkhoff and von Neumann, the lattice of closed subspaces of Hilbert space has been studied by mathematicians, physicists, and philosophers. Since the 1960's, work of Gudder and of Kochen has focussed on partial structures extracted from, or related to, the lattice mentioned above.

This thesis follows in the same tradition. Starting from a new point of view, a particular sort of partial algebra is constructed. The relations of these structures, quasi boolean algebras, to Gudder's work on various sorts of posets and lattices are studied. Their relation to the partial boolean algebras studied by Kochen is also described. In this connection, his definition of partial boolean algebras is refined.

A study of quasi boolean algebras themselves is started. Various properties of these partial algebras are isolated and studied. The theory of relations between these partial algebras is then investigated. This theory is modelled after both standard developments in quantum mechanics proper, and some recent work by

Kochen. In particular, a theory of "projections" and "interactions" in the context of these algebras is begun.

Examiners: [redacted]
Albert E. Hurd
[redacted]
Charles E. Murley
[redacted]
Colin D. Scarfe
[redacted]
Gary G. Miller

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INTRODUCTION:

We follow standard practice in giving von Neumann credit for originating the mathematical foundations of quantum theory. Heisenberg and Schrödinger [10,24] had already in 1925 provided unified formalisms for studying quantum mechanical phenomena. Von Neumann abstracted the common structure from these equivalent formulations [29] and set down axioms which link the Hilbert space formalism with physical reality.

These axioms bring the subspaces of Hilbert space to a key position. Von Neumann linked observable quantities with subspaces in such a way that simultaneous observability of quantities is equivalent to orthogonality of the disjoint parts of the corresponding subspaces. Observable quantities are said to be compatible exactly when they are simultaneously observable. The existence of incompatible observables is one of the revolutionary features of quantum theory. By studying structures which arise from the subspaces of Hilbert space, a general characterization of the features of incompatibility is possible. Such a study might be expected to permit an understanding of how an electron or a photon can appear to be both a discrete particle and a continuous wave.

Von Neumann and Birkhoff published "The Logic of Quantum Mechanics" in 1936 [30]. This article pioneered the mathematical study of structures arising from the subspaces of Hilbert space. Their lead was chiefly ignored for almost thirty years. Then in the 1960's the work of Mackey, Jauch, and Piron [20,14,15, respectively] renewed interest in "the lattice of closed subspaces of separable infinite dimensional Hilbert space over the field of complex numbers"-henceforth, LH(C). This interest has continued and mushroomed until today a reasonably comprehensive bibliography on the subject is of article length itself. (Extensive bibliography may be gleaned from [12, 13]). Kochen in particular has worked in this area since shortly after the work of Mackey cited above. In the sequel we will lean heavily on his work with Specker [16] and later [17].

The body of this thesis is divided into three parts. It begins with a review of some work in the area. Touching only the highlights, we present the mathematical setting of the work: describing LH(C) and related structures in some detail. A few of the connections established among these structures are original.

The second part begins with an informal discussion intended to motivate and render intelligible the subsequent formal development. This yields an interpretation of a class of structures, quasi-boolean algebras: QBA, arising from a generalization of LH(C). This section ends with an adaption of Kochen and Specker's "no hidden variable" proof [16] to the constructed QBA.

The third section begins to develop a dynamical theory on QBA and a theory of relations between QBA. These are intended to mirror parts of quantum mechanics proper: in particular, its dynamics and its consequences for measurement. This section and the preceding one are largely original, while owing a great debt to Kochen's [17].

PART I:

One place to begin a review of the mathematical foundations of quantum theory is with the work of Schrödinger [24] and of Heisenberg[10]. They provided unified formal ways of describing the evolution of (states of) quantum mechanical systems.

Schrödinger's formulation is often described as the wave mechanical model, and may be described as the model in which changes are attributed to the evolution of a system relative to a fixed descriptive framework. See for example [1].

Heisenberg's formulation is the matrix mechanical model which may be thought of as accounting for changes by evolving the descriptive framework or "frame of reference", keeping the object of study constant.

In Dirac's presentation [4], observed changes are attributed to the combined changes of the frame of reference and the object of study.

Following von Neumann's [29], note that it is useful to regard the formulations of Heisenberg and Schrödinger as not being about the spaces underlying the formulations, the discrete space of index values $Z = 1,2,3,\ldots$for the matrices, and the continuous state space Ω of the system in question, respectively, but as being about certain classes of functions, $F\Omega$ and FZ, on these spaces.

These two classes of functions are isomorphic, for a given system.

Von Neumann then provided a model of these two classes, freed from their respective non-isomorphic foundations. FZ and FΩ are isomorphic to Hilbert space H: a separable inner product space over the field C, which is complete with respect to the norm. Almost all subsequent mathematical study of the foundations of quantum theory has started from H.

After von Neumann, a state of a quantum mechanical system may be encoded as a collection of unit vectors in a suitable Hilbert space. The evolution of the system becomes a continuous differentiable path, up to phase shift, of the endpoints of the chosen vectors on the (surface of the) unit ball in the space. These paths are of central concern in quantum mechanics.

At this point our attention shifts from the laws of evolution of quantum mechanical systems, and their encoding, to the structure of H. Familiarity with the following is assumed: the fields R and C of real and complex numbers, respectively; finite dimensional vector spaces; and inner products defined on such spaces over R.

A vector space with a defined inner product is also called an inner product space. Although C as a vector space over R is isomorphic to R^2, they are not isomorphic as inner product spaces. The reason is that a map preserving the R-vector space structure does not preserve the inner products, and vice versa.

A set will be said to be countable if and only if its cardinality is not larger than that of N, the set of natural numbers. A set is denumerable if and only if it is countable and infinite in cardinality.

An inner product space over R or C is separable if and only if it has a countable dense subset. The space H is the complete, separable, inner product space over C, of infinite dimensionality.

Let V be a finite dimensional vector space over R or C and $\{v_1,\dots,v_n\}$ an orthonormal basis for V. Then $\{v_1,\dots,v_n\}$ is also an algebraic basis for V in that every element of V may be written as a finite linear combination of elements of $\{v_1,\dots,v_n\}$. This is not true however if V is infinite dimensional: a basis for H, a "Hilbert basis", is not an algebraic basis for H.

Finally, observe that vectors in an n-dimensional inner product space over C may be expressed as n-tuples $\langle \rho_1 e^{i\theta_1},\dots,\rho_n e^{i\theta_n}\rangle$, where elements of C are expressed in "polar" coordinates. Vectors and rays in H which differ only in phase are often identified in quantum mechanics.

At this point we develop a little of the theory of H. A linear transformation (operator) on H is a function A from H into H such that for all vectors v, w in H and all scalars a, b in C, A(av + bw) = aA(v) + bA(w). Then PROPOSITION 1.1: The set of all linear transformations on H is itself a vector space, [8].

Linear transformations of H into H may also be referred to as linear "functionals". If A and B are linear transformations on H, so is their product, functional composition. Things begin to take on a distinctly quantum air when we notice that the operation on functional composition is not commutative in general. PROPOSITION 1.2: Every subspace of H is the range of an idempotent linear operator A (AA = A) and vice versa.

Another way of approaching this is the PROPOSITION 1.3: Linear operator A on H is a projection on some subspace of H if and only if A is idempotent [8].

Then an idempotent linear transformation may be identified with the subspace which is its range, and may be referred to as a projection (operator) on H. Conversely, it is sometimes useful to identify a subspace of H with the projection which has it for its range.

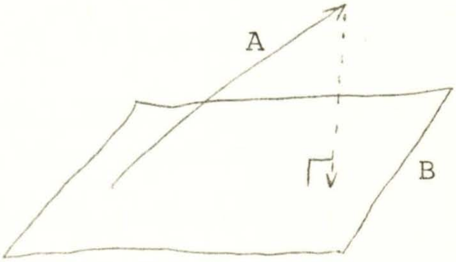

Fig. 1

Projection of A on B

Focus attention now on the subspaces of H; in particular, on the closed subspaces of H[1]. These subspaces form a lattice, henceforth referred to as LH(C). The properties of this lattice have been carefully studied [21,13,28,31]. LH(C) is an atomic, atomistic, orthomodular lattice of denumerable dimensionality, such that the elements of LH(C) below any element of LH(C) of finite dimension form a projective geometry. We now proceed to define these terms and review some of the highlights of present LH(C) theory.

Let a and b be in LH(C). Define $a^\perp = \{v \in H |$ for all w in a, v is orthogonal to w$\}$. Then $a^\perp$ is in LH(C). Define $a \wedge b = a \cap b$, the set theoretic intersection of a and b. Then $a \wedge b$ is in LH(C). Define $a \vee b$ as lub$\{a,b\}$, the smallest element of LH(C) that includes a and b. Then $a \vee b$ is obviously in LH(C). Define $a \leq b$ if and only if $a \vee b = b$. Then $\leq$ is a partial order on LH(C).

With $\vee$ and $\wedge$ so defined, LH(C) is a lattice. To establish this it suffices to note that $\leq$ as defined

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[1] While evolution of a quantum mechanical system may be expressed in terms of linear transformations on H, it is not necessary to suppose that every application of a linear transformation, projections in particular, in using quantum theory corresponds to an evolution of the system studied. The change may be only epistemic.

agrees with the subset relation, and that $\wedge$ and $\vee$ are glb and lub respectively in LH(C). Where 1 is H itself and 0 the zero subspace of H we have $1 = \sup(LH(C))$ and $0 = \inf(LH(C))$. Further, LH(C) is a complete lattice since every subset of LH(C) has a glb and lub in LH(C).

LH(C) is orthocomplemented. That is, for any a in LH(C), $a^{\perp}$ is in LH(C) and the following all hold: $a^{\perp\perp} = a$, $a \vee a^{\perp} = 1$, $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$, $(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$.

LH(C) is orthomodular: for all a, b in LH(C), if $a \leq b$ then $b = a \vee (b \wedge a^{\perp})$. Since in any orthocomplemented lattice $a \leq b$ implies $b \geq a \vee (b \wedge a^{\perp})$, orthomodularity amounts to the condition that if $a \leq b$, then $b \leq a \vee (b \wedge a^{\perp})$.

An element a of LH(C) <u>covers</u> an element b if and only if $b \leq a, a \neq b$, and for all c in LH(C), if $a \geq c \geq b$ then $c = a$ or $c = b$. An element a of LH(C) is an <u>atom</u> if and only if a covers 0. The atoms of LH(C) are the one dimensional subspaces of H.

LH(C) is <u>atomistic</u> since every element of LH(C) is the join of the atoms below it.

Element a of LH(C) is <u>dimensional</u> if and only if every maximal chain of elements of LH(C) below a has the same number of elements. LH(C) is dimensional, in that all its elements are dimensional.

Element a of LH(C) <u>has dimension n</u> if and only if it is dimensional and every (some) maximal chain below it has cardinality $n+1$, $n = -1, 0, 1, \ldots$

PROPOSITION 1.4: If b is an element of LH(C) such that some maximal chain below b is finite in length, then b has dimension n for some n in $N \cup \{0\}$. [2, p. 273 passim]

A lattice L is modular if and only if whenever a,b are in L and $a \leq b$, it follows that for all c in L, $a \vee (c \wedge b) = (a \vee c) \wedge b$. LH(C) as a whole is not modular although its finite dimensional parts and cofinite dimensional parts are. This limited modularity is a necessary and sufficient condition for LH(C) to be dimensional [28]. Since LH(C) is not modular, it is not distributive. It is however orthomodular, and for any a,b,c in LH(C), $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$.

It remains then to consider the fact that the set of elements of LH(C) below any finite dimensional element of LH(C) forms a projective geometry.

As noted, the lattice of elements of LH(C) below any element of LH(C) is modular. Atoms in LH(C) may also be referred to as points. Lines in LH(C) are elements of LH(C) that cover points. In general an n-flat is an element of LH(C) that has n orthogonal points below it, for $n \geq 0$, and is above all points on line $a \vee b$ whenever it is above points a and b. Observe that an n-flat has dimension n. Let a be in LH(C) with dimension $n > 1$, and let $PG(a) = \{b \in LH(C) \mid a \geq b\}$. Then the following hold [2, p.92]:

PG1: Two distinct points are in one and only one line.

PG2: If a line i intersects two sides of a triangle (not at their intersection), then it also intersects the third side.

PG3: Every line contains at least three points.

PG4: The set of all points in PG(a) is spanned by n points, but not by fewer than n points.

But these are the defining conditions of a projective geometry. Condition PG4 may equivalently be expressed as: any n-flat is an m-flat if and only if m = n. This shows more clearly the need for dimensionality and modularity.

Orthocomplemented lattices, "ortholattices", play a central role in lattice theory. Birkhoff refers to them as "nondistributive analogs of Boolean algebras" [2, p.52]. Extensive results concerning ortholattices and their neighbours are scattered throughout [2]. In particular, note that "Any complemented modular lattice of finite length is a product of simple (no proper congruence relations) lattices" [op. cit. p.71] and [op. cit. p.93]

> Each of the following conditions on a modular geometric lattice L is necessary and sufficient that it be a projective geometry:
> (i) L is simple;
> (ii) L is directly indecomposable,
> (iii) all points in L are perspective.

In the sequel we will be more concerned with orthomodular lattices (OML) which may or may not be atomic or atomistic; that is, with orthocomplemented, orthomodular lattices. We will also look at orthoposets and orthomodular posets. An orthoposet is a poset P with first and last elements 0,1, an orthocomplement, $^{\perp}$, with standard properties, and finally if $a \leq b^{\perp}$ then $a \vee b$ exists, for all a,b in P. Orthomodular posets are defined in the obvious way. Two other sorts of structures will be of concern. They are partial boolean algebras (PBA) and quasi boolean algebras (QBA).

A PBA is a sextuple $\langle B,0,1,\$,\sim,\vee \rangle$ where B is a set containing 0 and 1, $1 \neq 0$, $\$ \subseteq B^2$, $\sim$ is a unary operator on B, v is a mapping from \$ to H such that for all a,b, c in B:

i $\sim 0 = 1$, $\sim 1 = 0$

ii $\sim\sim a = a$

iii $\$(1,a)$, also written $1\$a$

iv $\$(b,a)$ only if $\$(a,b)$

v $\$(\sim a,b)$ only if $\$(a,b)$

vi $1 \vee a = 1 = a \vee 1$

vii $0 \vee a = a = a \vee 0$

viii $\sim a \vee b = 1 = \sim b \vee a$ implies $a = b$

ix $\$(a,b)$ implies $\$(\sim b,a)$, $\$(\sim a,\sim b \vee a)$,
$\sim a \vee (\sim b \vee a) = 1$

x $a\$b$, $a\$c$, and $b\$c$ jointly imply $\sim a\$\sim b \vee c$,
$\sim(\sim a \vee b)\$\sim a \vee c$, $\sim(\sim a \vee (\sim b \vee c))\$\sim(\sim a \vee b) \vee (\sim a \vee c))$,
and $\sim(\sim a \vee (\sim b \vee c)) \vee (\sim(\sim a \vee b) \vee (\sim a \vee c)) = 1$

A QBA is like a PBA except that condition x is replaced by x': whenever a,b,c are in QBA G and all their polynomials in v, & and $\sim$ exist in G, all boolean identities hold for these polynomials. ($a\&b = \sim(\sim a \vee \sim b)$.)

We now develop some theory of these various sorts of structures. A pasted boolean algebra is a collection G of boolean algebras, $G = \{B_i \mid i \in \lambda\}$, such that

(1) Whenever B,B' are in G, so is their intersection,

(2) Whenever $b_1, \ldots, b_n$ are in $\cup G$ such that for all $1 \leq i < j \leq n$ there exists $B \in G$ such that b_i, b_j are in B, then there exists $B' \in G$ such that $\{b_1, \ldots, b_n\} \subseteq B'$.

PROPOSITION 1.5: Every PBA is isomorphic to a pasted boolean algebra, and vice versa.

Proof may be found in [12]. A glued boolean algebra is like a pasted boolean algebra except that condition (2) need not hold.

PROPOSITION 1.6: Every QBA is isomorphic to a glued boolean algebra, and vice versa.

The straightforward proof is given in the appendix.

Let G be a QBA. For any b,c in G, define $b \leq c$ if and only if $c \vee b = b$. Then $\leq$ is reflexive and antisymmetric. When it is transitive as well, describe G as transitive. In an arbitrary QBA it can happen that $a \vee b = b$, $b \vee c = c$, but $a \vee c$ is not defined. No changes result if G is a PBA. In such cases it is easy however to form the transitive closure of $\leq$ and to assume, if desired that when $a \leq c$, $a \vee c$ is defined. Likewise, in orthoposets we may assume, without loss of generality, that if $a \leq b$, then $a \vee b$ and $a\&b$

are defined.

Let G be an orthoposet or QBA. Any b,c in G are *comparable* if and only if $b \leq c$ or $c \leq b$. Write $b \perp c$ if b and c are *disjoint* ($b \leq \sim c$). Any b,c in G are *compatible* if and only if there exist pairwise disjoint a, b_1, c_1 in G such that $b = avb_1$ and $c = avc_1$. Write bCc if b and c are compatible.

Condition (C): Whenever a,b,c in G, a QBA or orthoposet, are pairwise compatible, (avb)Cc.

PROPOSITION 1.7: Every QBA satisfying condition (C) is a PBA.

Proof: It suffices to establish that conditions (C) and x' together yield condition x. Details are in appendix.

PROPOSITION 1.8: Every transitive QBA is isomorphic to an orthomodular poset.

The proof is routine, and given in the appendix.

PROPOSITION 1.9: Every orthomodular poset satisfying condition (C) is isomorphic to a (transitive) PBA.

Proof: Establish first that in an orthomodular poset P satisfying (C), a subset S of P is contained in a boolean subalgebra of P if and only if the elements of S are pairwise compatible [6]. Then adapt the argument which showed that every pasted boolean algebra is isomorphic to a PBA. The details are filled in in the appendix.

PROPOSITION 1.10: An orthoposet satisfying (C) is orthomodular only if comparable elements are compatible. Proof: Such a poset is isomorphic to a transitive PBA, and in all PBA comparable elements are compatible.

PROPOSITION 1.11: Every ortholattice satisfies (C). [7]

Note that on the class of QBA transitivity does not imply (C), and vice versa.

PROPOSITION 1.12: Let G be an ortholattice and define compatibility on G by setting aCb if and only if $a = (a \& b) \vee (a \& {\sim} b)$, for all a,b in G. Then C is symmetric if and only if G is orthomodular. [9].

Two interesting studies related to this material require a review of some mathematical logic.

Let FL denote a (classical) propositional language. That is, let L be a denumerable set of (syntactic) variables and perhaps some constants. Let $\&$ and $\vee$ be binary connectives (operators) and $-$ a unary connective. Then FL is the smallest set G such that $L \subseteq G$ and whenever A and B are in G so are $(A \vee B)$, $(A \& B)$, and $-A$.

As usual define $(A \supset B) = (-A) \vee B$, $A \equiv B = (A \supset B) \& (B \supset A)$. Adopt the conventions of [3] for the omission and restoration of parentheses. Thus, for example, $A \vee B \& C$ is $A \vee (B \& C)$, $-A \supset (B \supset C) \supset (A \supset B \supset (A \supset C))$ is $((-A) \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$, and $-A \& B \vee C \supset -A \vee B$ is $(((-A \& B) \vee C) \supset ((-A) \vee B))$. Classical propositional logic may be axiomatized by taking a set of axioms for boolean algebra, letting the variables range over FL, letting &, v, and - be glb, lub, and complement, respectively, and replacing "=" by "≡". Marrying FL and boolean axioms as above yields classical propositional logic.

There are two common ways to tinker with this logic. One way is to change the axioms. Another is to change both the language and the axioms. At the same time, the language may be interpreted, given a (formal) semantics or model. Judicious tinkering with models usually yields characteristic models for any given language and axioms. For example, [22].

One common addition to the language is so-called "modal" operators; for example, [19]. The characteristic property of modal operators is that the semantic value of a modal formula is not determined in general by the values of its parts. There was a delay of almost thirty years from Lewis' invention of modal calculi to Kripke's invention of their models [18]. Kripke's innovation may be described as combining a number of classical models with a (binary) relation connecting them. The resulting structure is not unlike a subdirect product. Modal formulas are evaluated at classical models in the larger modal model. Their value is determined by the value of their parts at (other) classical parts of the model. By varying the properties of the parts, and the relation(s) among them, modal models have been devised for a large range of modal logics. Elements of these modal models are usually called 'possible worlds'. We will prefer the term 'reference frames', both because it is less metaphysical and because the frames are not much like worlds.

Both Dishkant and Hardegree draw on this tradition in their work; which we now review.

Following Dishkant's [5], let OL be an ortholattice and FL a classical propositional language. Let h be a *morphism* of FL into OL. That is, for all A in FL, h(A) is in OL and i $h(-A) = h(A)^{\perp}$

ii $h(A\&B) = h(A) \wedge h(B)$

iii $h(A \vee B) = h(A) \vee h(B)$

Define an algebraic model *A* as an ordered pair <OL,h> where OL and h are as above. *A makes* A *algebraically true* if and only if h(A) = 1 (if and only if A is in the kernel of h, which is a dual ideal in FL). A is *algebraically true* if and only if all algebraic models make A true. *A makes* A and B *algebraically equivalent* if and only if h(A) = h(B). A and B are *algebraically equivalent* if and only if all algebraic models make them equivalent.

Two models are *conformable* if and only if they make the same sentences (formulas) true, and the same pairs of sentences equivalent.

A *semantic model G* is a triple $\langle G, R, \vdash \rangle$ where G is a nonempty set, $R \subseteq G^2$ such that R is reflexive and symmetric, and $\vdash \subseteq G \times FL$ such that where g^* signifies that gRg^*, for any g,g^* in G, and writing $\vdash$ as an infix:

F1 $g \vdash -A$ if and only if for all g^*, $g^* \nvdash A$

F2 $g \vdash A \& B$ if and only if $g \vdash A$ and $g \vdash B$

F3 $g \vdash A \vee B$ if and only if for all g^* there is g^{**} such that $g^{**} \vdash A$ or $g^{**} \vdash B$

We have streamlined Dishkant's presentation by incorporating his 'note at publication'.

> G is intended to be a collection of possible states of knowledge. Thus a particular g in G may be considered as a collection of physical facts known at a particular time. The relation R represents the possible time succession. g* is intended to be any state of knowledge, which can come after g. The transition from g to g* is connected with fulfilling of an experiment. And we assume that it is a simple experiment (quantum observation). That is, gRg* means: if we now know g, it is possible that later, when an experiment will have fulfilled, we shall know g*...The quantum logic is a logic of slowly changing restorable facts. [5, p. 19,20]

PROPOSITION 1.13: $\vdash$ is uniquely determined by its action on L [5, p. 21]

Semantic model *G makes* A *true* if and only if for all g in G, $g \vdash A$. A is *semantically true* if and only if it is true on all semantic models. *G makes* A and B *semantically equivalent* if and only if for all g in G, $g \vdash A$ if and only if $g \vdash B$. A and B are *semantically equivalent* if and only if they are equivalent on all semantic models.

PROPOSITION 1.14: For every semantic model there is a conformable algebraic model.

PROPOSITION 1.15: For every algebraic model there is a conformable semantic model.

PROPOSITION 1.16: Formulas are semantically true (equivalent) if and only if they are algebraically true (equivalent).

Proofs may be found in Dishkant's paper. Without going into the details, remark that in Dishkant's proof of proposition 1.15 above the relation R in the constructed semantic model corresponds to non-orthogonality in the original ortholattice.

While Dishkant does not mention implication at all in his "logic", Hardegree makes it the center of his [9]. He begins by distinguishing between implication as a relation between formulas and implication as an operator on them. In general we may read the implication relation off a lattice by looking at its ordering relation. The implication operation is less obvious.

After lengthy, detailed, and careful discussion, Hardegree opts for defining the implication operator "$\to$" on orthomodular lattices by setting $a \to b = a^{\perp} \vee (a \wedge b)$. This operation had been previously discussed in the literature 23 and Hardegree notes that it is the only "conditional that is both residual and locally Boolean" on orthomodular lattices [9, p. 65], calling it "Sasaki hook".

Quite independently, various logicians have presented formal analyses of counterfactual conditionals [19,26]. In particular they analyzed the conditional occurring in the schema "if it were the case that A, then it would be the case that B".

Referring to this conditional as analyzed by Stalnaker [26], Hardegree remarks that

> A natural Stalnaker conditional is defined in terms of the standard Hilbert space metric, and it is moreover shown to coincide with the Sasaki hook. [9, p. 52]

Hardegree then lucidly discusses the EPR paradox in terms of counterfactuals.

Another approach to quantum logic begins with the assumption that every person has their own frame of reference with their own (propositional) language. Frames of reference, and languages, of different people may be more or less the same and to that extent, more or less compatible. Correlations between frames of reference may be expressed as partial translations between languages. These translations may be formalized as partial morphisms between languages. Very little more structure yields a class of languages and translations which when algebraized becomes a QBA.

Observe that in standard developments of formal semantics languages are interpreted or evaluated by means of morphisms into algebraic structures of some sort.

In the case of quantum logic, languages may be interpreted by collections of partial interpretations into other languages. Further, if different people look at the same thing from their different points of view, it may be expected to appear in different ways. The next part of this thesis begins to develop these ideas.

PART II

Any thing may be looked at in a variety of ways. Some intelligible assumptions about the way these points of view are related lead to QBA. Another approach to this begins with the idea that a thing can interact with its environment in a variety of ways. Simple assumptions about the way these possible interactions are related to one another lead to the consideration of QBA. At some point it might be possible to identify an object with the sum of its possible interactions with the environment. To be more concrete: any countable propositional language L of the usual classical sort is taken to be (codify, express, signify) a way of looking at things. Then a family $\underline{L}$ of such L can codify (all the) ways of looking at a given thing.

These languages are related by translations between them. Formally, these translations are partial morphisms. Given $\underline{L}$, fix a class T of morphisms compatible with one another. Then the pair $<\underline{L},T>$ may be considered as a category and studied as such.

More concretely, $<\underline{L},T>$ gives rise to a QBA. Thus many QBA may be looked at as systems of meshed languages. Consider a structure $<\underline{L},T>$ where $\underline{L}$ is a nonempty set of propositional languages and T is a collection of 1:1 functions such that:

(1) For all L, L' in $\underline{L}$, there is exactly one f in T with $\mathrm{dom}(f)$ in L and $\mathrm{range}(f)$ in L'. $\mathrm{Dom}(f) \neq \emptyset$.

(2) For all L in $\underline{L}$, the f in T mapping L into itself is the identity map defined everywhere on L.

(3) For all L, L' in $\underline{L}$, if f, f' are in T such that $f: L \to L'$ and $f': L' \to L$ then $f' = f^{-1}$.

(4) For all f in T, if A, B are in $\mathrm{dom}(f)$ then $f(-A)$ is $-f(A)$ and $f(A \vee B) = f(A) \vee f(B)$.

(5) If $f_1: L_1 \to L_2$, $f_2: L_2 \to L_3$, and $f_3: L_1 \to L_3$, for any f_1, f_2, f_3 in T and any L_1, L_2, L_3 in $\underline{L}$, then f_3 extends $f_2 \circ f_1$.

(6) Whenever $A \vdash B$ and A, B are in $\mathrm{dom}(f)$, f in T, $f(A) \vdash f(B)$ where $A \vdash B$ if and only if $\vdash A \supset B$ and $\vdash A$ says that A is a theorem, a boolean tautology.

PROPOSITION 2.1: Each f in T preserves theoremhood. That is, if $\vdash A$ and A is in $\mathrm{dom}(f)$, then $\vdash f(A)$.

Proof: Suppose $\vdash A$ and A is in $\mathrm{dom}(f)$. Since $A \vee -A \vdash A$, $f(A \vee -A) \vdash f(A)$. But $f(A \vee -A) = f(A) \vee -f(A)$ and so $\vdash f(A)$.

Let G be a subset of L in $\underline{L}$. Define $G \vdash A$ if and only if there exist $A_1, \ldots, A_n$ in G such that $\vdash (A_1 \& \ldots \& A_n) \supset A$, for some n in N. Say G is consistent if and only if there is B in L such that $G \nvdash B$. It is routine to show that G is consistent if and only if there is not B in G such that $G \vdash B \& -B$.

PROPOSITION 2.2: Each f in T preserves consistency.

Proof: Let G be a consistent subset of L in $\underline{L}$ and let

$f{:}L\to L'$, and suppose G is consistent. If f does not preserve consistency then there is some A in f(G) such that $f(G)\vdash A\&-A$. Hence A is in dom(f^{-1}) and so $G\vdash f^{-1}(A\&-A)$. Then $G\vdash f^{-1}(A)\&-f^{-1}(A)$ and so G must be inconsistent.

For G as above, say G is an *L-theory* if and only if G is properly contained in some L in $\underline{L}$ and if $G\vdash A$ then $A\in G$, for all A in L. Where $\not{L}$ denotes the set of theorems of L, whenever G is an L-theory it follows that $\not{L}\subseteq G$.

PROPOSITION 2.3: Let f be in T, $f{:}L\to L'$. Then f preserves theoryhood if and only if $\not{L}'\subseteq\not{L}$ and whenever $\nvdash -A\vee -B$ and $A, A\supset B$ are in dom(f^{-1}), then so is B in dom(f^{-1}).

Proof: *only if*: To show $\not{L}'$ is included in $f(\not{L})$, note that $\not{L}$ is a theory and so, so is $f(\not{L})$. Since $\not{L}'$ is in all L'-theories, the result follows. For the rest, suppose $\nvdash -A\vee -B$ and $A, A\supset B$ are in dom(f^{-1}). Then $f^{-1}(A\&(A\supset B)) = -f^{-1}(A)\vee -f^{-1}(A\supset B)$. But $\vdash(-A\vee -B)\equiv(-A)\vee-(A\supset B)$ and so $\nvdash(-f^{-1}(A))\vee -f^{-1}(A\supset B)$. So $\{f^{-1}(A), f^{-1}(A\supset B)\}$ is consistent. Hence there is an L-theory G such that $G\vdash f^{-1}(A)$ and $G\vdash f^{-1}(A\supset B)$. Then $f^{-1}(A)$ is in G and $f^{-1}(A\supset B)$ is in G. Since f(G) is a theory, and $A, A\supset B$ are in f(G), B is in f(G). So B is in dom(f^{-1}) as required.

if: Let G be an L-theory. It suffices to show f(G) is an L'-theory. First, $f(G)\neq L'$ by proposition 2.2. $\not{L}'$ is contained in f(G) since $\not{L}$ is in G and thus $\not{L}'\subseteq f(\not{L})\subseteq f(G)$ as desired. Finally, suppose $A, A\supset B$ are in f(G). Then

$f^{-1}(A)$ and $f^{-1}(A \supset B)$ are in G. Hence $\nvdash (-A) v - (-A v B)$ since f(G) is consistent. Then $f^{-1}(B)$ exists and so $f^{-1}(A \supset B) = f^{-1}(A) \supset f^{-1}(B)$. Then G contains $f^{-1}(B)$ since G is a theory. Then B is in f(G) as required.

Starting with a propositional language with classical propositional logic as described above, the identification of provably equivalent formulas produces a structure which is (isomorphic to) the free boolean algebra with as many generators as the language has atoms, the partial order is derived from $\supset$, the equivalence class of theorems is the unit element, and the equivalence class of their negations is the required zero.

Our present purpose is to turn a given $<\underline{L},T>$ into an algebra and examine its properties.

Fix $<\underline{L},T>$ and write ATB if for some f in T f(A) = B. then the infixed relation T is an equivalence relation by conditions (1), (2), and (5) on $<\underline{L},T>$. Define $[A] = \{ B$ in L in $\underline{L} \mid$ ATB$\}$, for all A in L in $\underline{L}$. Define QL = $\{[A] \mid$ A is in L in $\underline{L}\}$. QL is a "quasi propositional language" with connectives - and v defined in it by: $-[A] = [-A]$ and $[A]v[B] = [AvB]$ if and only if A and B are in some one L in $\underline{L}$.

PROPOSITION 2.4: As defined in QL, - and v are well-defined.

Proof: If ATB then -AT-B. Suppose A,B are in L in $\underline{L}$, A',B' are in L', and ATA', BTB'. Then (AvB)T(A'vB').

Definitions of &, ⊃, and ≡ in terms of - and v are as usual. Notice that the following can fail:

(i) [A] ≡ [B],[B] ≡ [C] are defined implies [A] ≡ [C] is defined.

(ii) [A]v[C],[B]v[C],[A]v[B] are defined implies that [A]v[B]v[C] is defined.

(iii) Like (i) but with ⊃ for ≡.

When (iii) holds, and therefore (i) holds, QL is said to be transitive. Define ≡" by setting A≡"B if and only if ⊢(A ≡ B). Let ≡' be the transitive closure of ≡". Then ≡' is an equivalence relation on QL, Define, for all A in QL, [A] = {B in QL| A ≡' B}. Define QA = {[A]| A is in QL}. Then define ~[A] = [-A] and [A]v[B] =[A'vB'] whenever A'vB' is defined for some A' in [A] and some B' in [B].

PROPOSITION 2.5: v and ~ are well defined on QA.

Proof: Whenever A ≡' B, -A ≡' -B and whenever AvB and A'vB' are defined and A ≡' A', B ≡' B', it follows that (AvB) ≡' (A'vB').

Thus, beginning with <$\underline{L}$,T> and modding out the two equivalence relations ≡' and T, QA results. It is equally possible to mod out T and ≡ in the reverse order; in this case T needs to be made transitive on $\underline{L}_{/\equiv}$. Definitions of the connectives at each stage are essentially as before.

Let $\underline{L} = \{L_1, L_2, L_3\}$ where L_1 is constructed from atoms $p_1, p_2, \ldots$; L_2 from atoms $q_1, q_2, \ldots$; and L_3 from atoms $r_1, r_2, \ldots$. Suppose T contains, in addition to the identity maps, $f_1 : L_1 \to L_2$, $f_2 : L_2 \to L_3$ and $f_3 : L_1 \to L_3$. Let f_1 be generated by $f_1(p_1) = q_1$; f_2 is generated by $f_2(q_2) = r_1$; f_3 is generated by $f_3(p_2) = r_2$. Then in the resulting QBA we have $[p_1]\$[r_1]$, $[p_1]\$[r_2]$, and $[r_1]\$[r_2]$, but not $[p_1]\$[r_1 \vee r_2]$.

Thus not all QBA are PBA. This example might lead to the idea that the addition of carefully chosen elements to $\underline{L}$ and T would remedy the situation. We will see, in connection with $QQ_{/\simeq}$, to be defined, that a mere abundance of elements of $\underline{L}$ and T is not enough.

Kochen and Specker reach the conclusion that

> A necessary condition for the existence of hidden variables for quantum mechanics is the existence of an imbedding of the partial algebra Q of quantum mechanical observables into a commutative algebra. [16, p. 66]

Restricting attention to the idempotent elements of Q, the commutative algebra in question becomes boolean. They then state

> Theorem 0. Let $\underline{A}$ be a partial Boolean algebra. A necessary and sufficient condition that $\underline{A}$ is imbeddable in a Boolean algebra B is that for every pair of distinct elements a,b in $\underline{A}$ there is a homomorphism $h : \underline{A} \to Z_2$ such that $h(a) \neq h(b)$. [16, p. 67][2]

Consider the following structure $<\underline{L},T>$ which yields a PBA not imbeddable in a boolean algebra. The strategy is adapted from Kochen and Specker's article just cited. Notice in particular that Kochen and Specker produced their non-imbeddability result from an examination of experimentally testable propositions which arise in studying orthohelium II. Their non-imbeddable PBA is not just a mathematical oddity, it expresses an empirically verifiable situation. The example produced below is a simplification of the structure they consider.

Let $\underline{L} = \{L_1,L_2,L_3,L_4\}$. L_1 (L_2,L_3,L_4) is constructed from p_i $(q_i,r_i,s_i$, respectively), $i = 1,2,\ldots$

T contains, in addition to the four identity maps,

$f_1:L_2 \rightarrow L_1$,

$f_2:L_3 \rightarrow L_2$,

$f_3:L_4 \rightarrow L_1$,

$f_4:L_3 \rightarrow L_4$,

$f_5:L_4 \rightarrow L_2$,

$f_6:L_1 \rightarrow L_3$.

Suppose that these functions are generated by the following identifications:

f_1: p_2 — q_2
p_1 — q_6

f_2: $-r_4\&r_6$ — $s_3\&s_4$
r_5 — $s_3\&s_1$
r_2 — $s_5\&-s_3$
$-r_1\&r_3$ — $s_2\&-s_3$

f_3: p_3 — s_3
p_1 — s_6

f_4: $q_2\&q_4$ — r_4
$q_2\&q_3$ — $-r_5\&-r_6$
$-q_2\&q_5$ — $-r_2\&-r_3$
$-q_2\&q_1$ — r_1

f_6: p_4 — r_7

Then it follows from condition 5 on $<\underline{L},T>$ that $f_5(q_6) = s_6$. Let QB be the resulting structure of the form $<\underline{L},T>$. The closure of the domains and ranges of the f_i under clauses 4 and 5 in defining QB suffices for QB to satisfy conditions 1 to 5. Then it only remains to show 6 is satisfied. It is immediate that all f in T satisfy 6 except for f_2 and f_4.

PROPOSITION 2.10: f_2 and f_4 satisfy 6.

Proof: Define $D^2 = \mathrm{dom}(f_2)$, $R^2 = \mathrm{range}(f_2)$, and D^4 and R^4 likewise for f_4. Define g^2_o: $-r_4 \& r_6$ — r_8

r_5 — r_9

r_2 — r_{10}

$-r_1 \& r_3$ — r_{11} and set

g^2_o: $s_3 \& s_4$ — s_8

$s_3 \& s_1$ — s_9

$-s_3 \& s_5$ — s_{10}

$-s_3 \& s_2$ — s_{11}

Close the domain and range of g^2_o under - and & to get g2, satisfying conditions 4 and 5, _mutatis mutandis_. Extend f_2 by setting $f_2(r_i) = s_i$, for i = 8,9,10,11, and closing off the domain and range under - and & as usual.

Then it is straightforward to prove that for all A and B in the domain of g2 or f_2, $A \dashv\vdash B$ if and only if $g2(A) \vdash g2(B)$; and that $g2^{-1}(f(g(A)) = f(A)$. Then it is possible to derive $A \dashv\vdash B$ if and only if $g2(A) \vdash g2(B)$

if and only if $f_2(g2(A))\vdash f_2(g2(B))$ if and only if $g2^{-1}(f_2(g2(A))\vdash g2^{-1}(f_2(g2(B))$ if and only if $f(A)\vdash f(B)$ as required. The argument for f_4 is similar.

Then, with reference to the following diagram, it is easy to prove, following Kochen and Specker, that where h is any homomorphism from QB to Z_2, $h(p_1\&p_2) = h(p_1\&-p_3)$. Hence, the QBA constructed from QB, which is a PBA since condition (C) holds trivially, is not imbeddable in a BA.

In the diagram, formulas in any one balloon are identified by T. A solid line between two balloons indicates that no homomorphism to Z_2 can map the two balloons to 1.

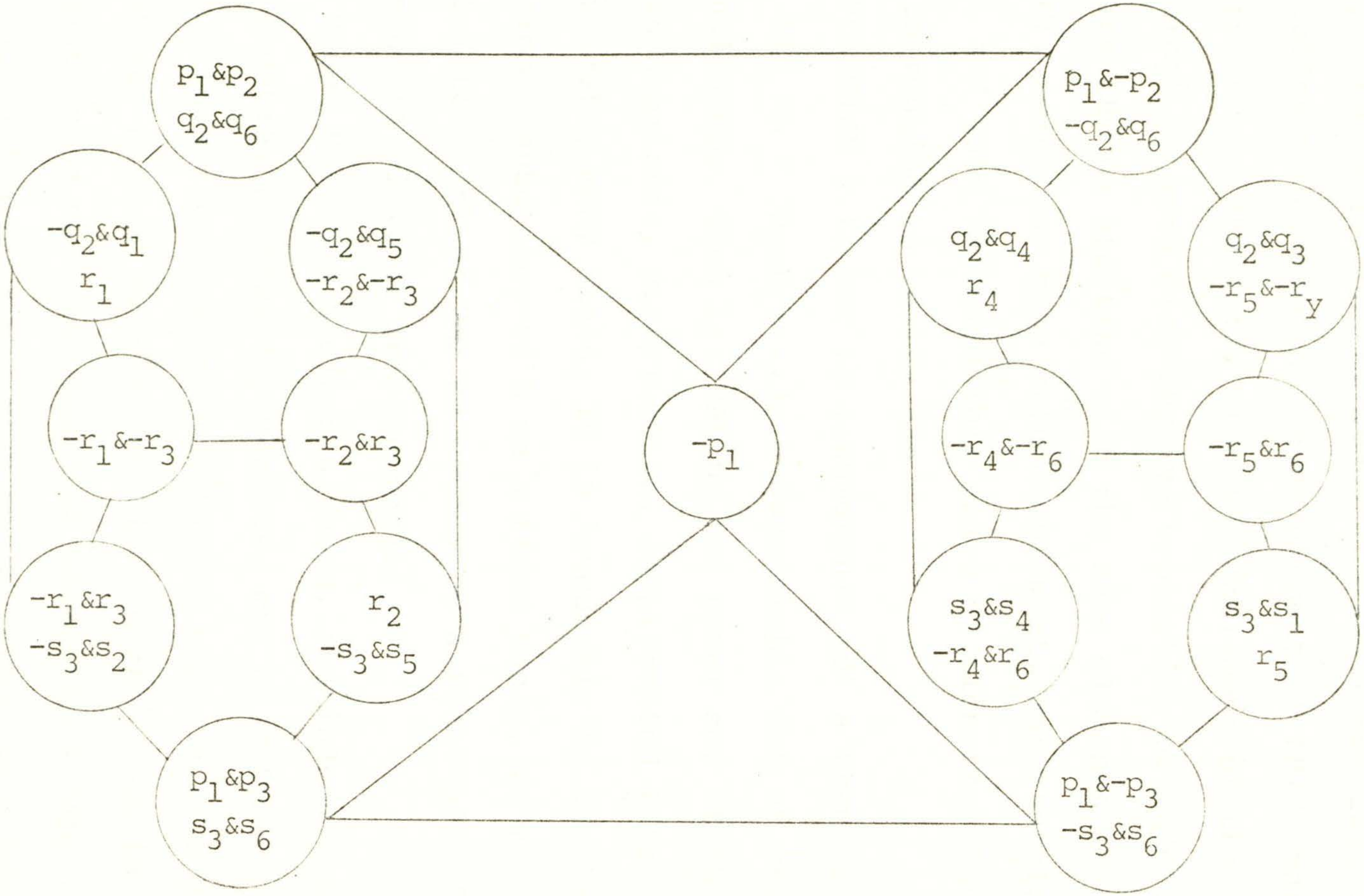
$p_1\&p_2$
$q_2\&q_6$
$-q_2\&q_1$
r_1
$-q_2\&q_5$
$-r_2\&-r_3$
$-r_1\&-r_3$
$-r_2\&r_3$
$-r_1\&r_3$
$-s_3\&s_2$
r_2
$-s_3\&s_5$
$p_1\&p_3$
$s_3\&s_6$
$-p_1$
$p_1\&-p_2$
$-q_2\&q_6$
$q_2\&q_4$
r_4
$q_2\&q_3$
$-r_5\&-r_y$
$-r_4\&-r_6$
$-r_5\&r_6$
$s_3\&s_4$
$-r_4\&r_6$
$s_3\&s_1$
r_5
$p_1\&-p_3$
$-s_3\&s_6$

Fig. 2

Exploration of the theory of <$\underline{L}$,T> and of QBA more generally is continued in this part of the thesis. Note first that LH(C) may be readily transformed into a PBA as follows. Observe that each element of LH(C) may be expressed as the join, or orthocomplement of the join, of atoms in some distributive sublattice of LH(C). Then define LHC by starting with the same elements as are in LH(C), using the same orthocomplement, but restricting meet and join to compatible pairs of elements. Then it is routine to verify the

PROPOSITION 3.1: LHC is (isomorphic to) a transitive PBA.

Let AL = $\{p_1, p_2, \ldots\}$, a denumerable set of propositional atoms. Then let PAL be the power set of AL. PAL is an atomistic, distributive, complemented lattice with denumerably many atoms. It is isomorphic to the lattice of closed subspaces of H on a fixed basis. It may be useful to think of PAL as codifying a frame of reference, a point of view, or an observable.

Since PAL is a boolean algebra, and indeed a power set algebra, probability measures may be defined on it. Following Kolmogorov, a probability measure is a non-negative, normed, weight function on PAL, that is σ-additive on pairwise disjoint elements of PAL.

Such a function may also be described as a homomorphism from the Borel subsets of the unit interval in R onto PAL. Such functions are often called <u>observables</u> in the context of quantum theory.

Let PR be the class of all such functions on PAL. It is useful to think of PR as a number of copies of PAL, indexed by the various probability measures. Let QQ be the class of all PALi such that i is in PR. Let hij be the natural isomorphism from PALi onto PALj, for all i,j in PR. That is hij is generated from $hij(\{p_k^i\}) = \{p_k^j\}$, for all k in N, where p_k^i denotes the instance of p_k in PALi. Note also that for all PALi and all a in PALi, ia, also written i(a), is in [0,1] and is the probability assigned to a by i in PR.

Then for all i,j in PR and all a in PAL, define tij(a) = hij(a) if and only if i(a) = j(hij(a)). Otherwise tij(a) is undefined. Then one readily verifes the PROPOSITION 3.2: The tij are partial morphisms from Pali to PALj, for all i,j in PR, which satisfy all the conditions on T in $\langle \underline{L}, T \rangle$ ($a \vdash b$ holds if and only if $a \subseteq b$).

Let TQ be the class of all tij as above, and $\simeq$ the resulting equivalence relation on QQ. Then follows the PROPOSITION 3.3: $QQ_{/\simeq}$ is a QBA.

However $QQ_{/\simeq}$ is not a PBA. For an example of the failure of (C), consider a,b,c in PAL such that $a \cup b \cup c$ is AL, $a \not\subseteq b \cup c$, $b \not\subseteq a \cup c$, $c \not\subseteq a \cup b$. Then there exist i,j,k in PR such that ia = ja, ib = kb, jc = kc, and ia+ib+jc<1. Note that the projective geometry conditions also fail.

So far two ways of generating QBA have been explored. Now they may be put together. Given AL as before and FAL the usual propositional language on AL, let PF be the class of all probabilities on FAL. Probabilities are defined as before, <u>mutatis</u> <u>mutandis</u>.

PROPOSITION 3.4: Let Pr be in PF, and let A and B be in FAL. Then $Pr(A\vee -A) = 1$, $Pr(-A) = 1-Pr(A)$, and $Pr(A\vee B) = Pr(A)+Pr(B)-Pr(A\&B)$.

Let QFL be the analogue of QQ that results from using FAL for PAL and PF for PR. Then let TF be the analogue of TQ and $QFL_{/\simeq}$ the resulting analogue of $QQ_{/\simeq}$. Then follows

PROPOSITION 3.5: The ordered pair <QFL,TF> satisfies all the constraints on <<u>L</u>,T>.

Then QFL is the resulting pasted language and let LQ be the resulting QBA.

A surjective morphism or epimorphism of LQ is a function f from LQ onto LQ such that for all a,b in LQ, $f(a\vee b) = f(a)\vee f(b)$ whenever either side is defined, and $f(\sim a) = \sim f(a)$.

PROPOSITION 3.6: Let f be an epimorphism of LQ. Then for all a,b in LQ, $f(a\wedge b) = f(a)\wedge f(b)$ and where $a\leq b$ if and only if $a\vee b = b$, $a\leq b$ only if $f(a)\leq f(b)$.

Proof: $f(a\wedge b) = f(\sim(\sim a\vee\sim b)) = \sim(\sim f(a)\vee\sim f(b)) = f(a)\wedge f(b)$. For the rest, suppose $a\leq b$. Then $a\vee b = b$ and so $f(a\vee b) = f(b) = f(a)\vee f(b)$. Hence $f(a)\leq f(b)$.

Observe that $\leq$ as defined on LQ is reflexive and antisymmtric but not transitive.

PROPOSITION 3.7: For all b in LQ such that $b \neq 1$, there exists a in LQ such that $b<a<1$.

Proof: Let b in LQ contain formula A. Then A contains some definite collection of syntax atoms and lacks, for example, atom p. Then $A \vee (p \& -p)$ is in b and suppose a contains $A \vee p$. Then $b<a<1$ so long as it is assumed that the probability assigned to a is at least that of b.

PROPOSITION 3.8: Epimorphisms of LQ preserve compatibility when this is defined as for $<\underline{L},T>$.

Proof: Let a and b be in LQ. Then a,b are compatible if and only if $a \vee b$ is defined, if and only if $f(a) \vee f(b)$ is defined, if and only if $f(a),f(b)$ are compatible.

PROPOSITION 3.9: Let f be an epimorphism of LQ. Then f is 1:1 if and only if $1 = f^{-1}(1)$.

Proof: Note first that $f(1) = 1$ since $f(a \vee {\sim} a) = f(a) \vee {\sim} f(a)$. Therefore if f is 1:1 then $1 = f^{-1}(1)$.

Conversely, suppose f is not 1:1. Then for some distinct a,b in LQ, $f(a) = f(b)$. Then $f(a) \vee f(b) = f(a)$ and so $f(a \vee b)$ is defined. Since $a \neq b$ and $a \vee b$ exists, ${\sim} a \vee b \neq 1$ or ${\sim} b \vee a \neq 1$. Suppose, without loss of generality, that ${\sim} a \vee b \neq 1$. But $f({\sim} a \vee b) = {\sim} f(a) \vee f(b) = {\sim} f(a) \vee f(a) = 1$. Hence $\{1, {\sim} a \vee b\} \subseteq f^{-1}(1)$ and so $1 \neq f^{-1}(1)$ as required.

Bijective (1:1) epimorphisms are called <u>automorphisms</u>.

PROPOSITION 3.10: The class of all epimorphisms of LQ is a semigroup with the operation of function composition and the identity element being the identity function on LQ. The class of all automorphisms of LQ is a group, as above, with the obvious inverse operation. Both the group and the semigroup are non-abelian.

Proof: Immediate from definitions.

It is interesting to explore a topology on LQ. Consider $a\downarrow = \{b \text{ in } LQ \mid a>b\}$ for all a in LQ, and $a\uparrow = \{b \text{ in } LQ \mid b>a\}$ for all a in LQ. Let $O\downarrow = \{a\downarrow \mid a\in LQ\}$, $O\uparrow$ is analogous. Then $O\downarrow \cup O\uparrow$ is a subbase for a topology on LQ. In particular, let T(LQ) be the closure of $O\downarrow \cup O\uparrow$ under finite intersections and arbitrary unions. With T(LQ) understood, LQ may now be referred to as a topological space.

Recall that a topological space is a Hausdorff space if and only if for all distinct points a and b in the space, there exist disjoint open sets A and B such that $a\in A$ and $b\in B$. In this case say that a and b are Hausdorff.

PROPOSITION 3.11: All comparable a,b in LQ are Hausdorff.

Proof: Suppose, without loss of generality, that $a<b$. Consider two cases depending on whether there exists c such that $a<c<b$. If such a c exists, then $c\downarrow$ and $c\uparrow$ show that a and b are Hausdorff. If there is no such c, then $a\uparrow$ and $b\downarrow$ show a and b are Hausdorff.

Note that every a in LQ may be considered as an ordered pair $<I,v>$ in which Ia is the equivalence class of formulas underlying a, and va is the probability assigned to this instance of Ia. In fact, LQ may be constructed from the class of all $<I,v>$ such that I is in $FAL_{/\equiv}$ and v is in PF.

PROPOSITION 3.12: Let a and b in LQ be compatible and not comparable. Then a and b are Hausdorff.

Proof: Since a and b are compatable there exists f in PF such that $f(I(a)) = va$ and $f(I(b)) = vb$. Let A and B be in Ia and Ib respectively. Let p be a syntactic atom foreign to A and B and consider elements c and d in LQ such that Avp is in Ic, B&-p is in Id, $vc = f(Avp)$, and $vd = f(B\&-p)$. Then $a<c$ and $d<b$. Further, $d \not\leq c$ for if $d \not\leq c$ then $\vdash (B\&-p) \supset (Avp)$. Then $\vdash --BvpvA$ and so $\vdash B \supset A$. Hence $b \leq a$, contrary to assumption. Hence b is in $d{\uparrow}$, a is in $c{\downarrow}$, and $d{\uparrow} \cap c{\downarrow} = \emptyset$ as required.

PROPOSITION 3.13: Let a and b be in LQ and not compatible. Then a and b are Hausdorff.

Proof: Since a and b are incompatible, they are not comparable. Consider cases depending on the relations between Ia and Ib. Suppose first that $Ia = Ib$. Then $va \neq vb$ and we may suppose $va<vb$, without loss of generality. Let A be in Ia and suppose p in AL is not in A. Then consider c and d in LQ such that Avp is in Ic, A&-p is in Id, $vc = va$, and $vd = vb$. Then c and d exist and are not

compatable since $Id \leq Ic$ and $vc < vd$. Further, it follows that $c \not\leq d$ and $d \not\leq c$ since c and d are incompatable. Hence $c{\downarrow} \cap d{\uparrow} = \emptyset$, a is in $c{\downarrow}$ and b is in $d{\uparrow}$ as required.

Notice that if a and b are incompatible then $Ia \leq Ib$ or $Ib \leq Ia$. For if not then consider c, a_1 and b_1 in LQ and f in PF such that $Ic = Ia \wedge Ib$, $Ia_1 = Ia \wedge {\sim} Ib$, $Ib_1 = Ib \wedge {\sim} Ia$, $f(Ic) = \min(va, vb)$, $f(Ia_1) = va - f(Ic)$ and $f(Ib_1) = vb - f(Ic)$. Such f exist in PF and show a and b not to be incompatible. Hence we may suppose $Ia \leq Ib$ or $Ib \leq Ia$. Then suppose, without loss of generality, that $Ia \leq Ib$ and hence $va > vb$ or else a and b would be compatible.

Let A be in Ia, B in Ib, p in AL and in neither A nor B. Since $Ia \leq Ib$, $\vdash A \supset B$. Consider a_1, a_2, b_1, b_2 in LQ such that $A \& p$ is in Ia_1, $A \vee -B$ is in Ia_2, $B \& -A$ is in Ib_1, and $B \vee p$ is in Ib_2. Then $Ia_1 < Ia < Ia_2$ and $Ib_1 < Ib < Ib_2$. Suppose further that $va_1 = va = va_2$ and $vb_1 = vb = vb_2$. Then a is in $a_1{\uparrow} \cap a_2{\downarrow}$ and b is in $b_1{\uparrow} \cap b_2{\downarrow}$. Then to complete the argument it suffices to show that $b_1 \not< a_2$, since then $a_1{\uparrow} \cap a_2{\downarrow} \cap b_1{\uparrow} \cap b_2{\downarrow} = \emptyset$.

But $Ib_1 = {\sim} Ia_2$ and ${\sim} Ia_2 \neq 0$, ${\sim} Ia_2 \neq 1$. Hence $b_1 \not< a_2$.

PROPOSITION 3.14: LQ is a Hausdorff space under T(LQ).

Proof: The preceding three propositions establish that all pairs of elements of LQ are Hausdorff.

Following standard, say that automorphism f on LQ is <u>continuous</u> if and only if for all open X in LQ $f^{-1}(X)$ is open. In other words, if X is in T(LQ), so is $f^{-1}(X)$.

To show all automorphisms on LQ are continuous it suffices to show that $f^{-1}(X)$ is open for all X in $O{\downarrow} \cup O{\uparrow}$ since all elements of aut(LQ) preserve intersections and unions.

PROPOSITION 3.15: If f is in aut(LQ) and a is in LQ, then $f^{-1}(a{\downarrow}) = f^{-1}(a){\downarrow}$.

Proof: Since f is in aut(LQ), so is f^{-1}. Since f^{-1} preserves , $f^{-1}(a{\downarrow}) \subseteq f^{-1}(a){\downarrow}$. Conversely, suppose b is in $f^{-1}(a){\downarrow}$. Then $b<f^{-1}(a)$. Then $f(b)<a$ and so b is in $f^{-1}(a{\downarrow})$ as required.

PROPOSITION 3.16: All members of aut(LQ) are continuous.

Proof: Given the preceding proposition it is routine to show that for all f in aut(LQ) and all a in LQ, $f^{-1}(a{\uparrow}) = f^{-1}(a){\uparrow}$. Combining this with the previous result, the general claim follows.

Define now some partial operations on aut(LQ). Let f and g be in aut(LQ) and set: $-f$ is defined by $(-f)(a) = f({\sim}a)$, for all a in LQ. Define $f \vee g$ by setting $(f \vee g)(a) = f(a) \vee g(a)$ if and only if the right hand side exists for all a in LQ. Otherwise $f \vee g$ is undefined. Define also function 1_f such that for all a in LQ, $1_f(a) = 1$. Let Qaut(LQ) be the closure of aut(LQ)$\cup\{1_f\}$ under $-$ and $\vee$, where -1_f and $1_f \vee g$ are defined in the obvious way.

PROPOSITION 3.17: Where f&g is defined in the obvious (boolean) way for all f and g in Qaut(LQ), f&g is in Qaut(LQ) if and only if $f \vee g$ is. Furthermore, $-(f \vee g)$ is

$(-f)\&(-g)$.

Proof: It suffices to observe that for all a in LQ,

$(-(f \vee g))(a) = \sim(f(a) \vee g(a)) = \sim f(a) \wedge \sim g(a) = ((-f)\&(-g))(a)$

For all f and g in Qaut(LQ), define $\$(f,g)$ to hold if and only if $f \vee g$ exists. Also let $0_f = -1_f$.

PROPOSITION 3.18: As defined, Qaut(LQ), that is, $<\mathrm{Qaut}(LQ), 0_f, 1_f, \$, -, \vee>$, is a QBA.

Proof: Deal with the conditions in turn. That (i) and (ii) hold is immediate from the definitions of $-$, 1_f and 0_f.

Ad iii: To show $1_f \$ g$ it suffices to show $1_f \vee g$ exists. But it is clear that for all a in LQ, $(g \vee 1_f)(a) = 1_f(a) = (1_f \vee g)(a)$, since 1 in LQ is compatable with all a. This also establishes (vi).

Ad iv: This follows from the symmetry of the definition of $f\$g$.

Ad v: This holds since it holds in LQ.

Ad vii: For all f in Qaut(LQ) and all a in LQ, $(0_f \vee f)(a) = 0_f(a) \vee f(a) = 0 \vee f(a) = f(a)$.

Ad viii: Let f and g be in Qaut(LQ) and suppose $-f \vee g = 1_f = -g \vee f$. That is, for all a in LQ, $(-f \vee g)(a) = 1 = (-g \vee f)(a)$. Then $\sim f(a) \vee g(a) = \sim g(a) \vee f(a)$ and since (viii) holds in LQ, $g(a) = f(a)$ for all a in LQ. Hence $f = g$.

Ad ix: Suppose f and g are in Qaut(LQ) and $f\$g$. By conditions (iv) and (v), $g\$-f$. Hence, for all a in LQ, (gv-f)(a) exists. Then ~g(a)v(-fvg)(a) exists. But (-gv(-fvg))(a) = ~g(a)v(~f(a)vg(a) = 1.

Ad x': Routine argument as above suffices.

Then it follows that where G is any QBA, Qaut(G) is a QBA as well. In particular, let V be a map from FAL into {0,1} such that for all A and B in FAL, V(-A) = 1-V(A) and V(A&B) = V(A)V(B). Then V is a probability assignment on FAL, but is better known as a (classical, bivalent) valuation. Let PV be the class of all valuations on FAL. Then replace PF by PV and generate QVL, TV, and $\mathrm{QVL}_{/\equiv}$ on direct analogy with QFL, TF, and $\mathrm{QFL}_{/\equiv}$. With these definitions, <QVL,TV> satisfies all the conditions on <$\underline{\mathrm{L}}$,T> and so gives rise to a QBA, called CLQ. Then Qaut(CLQ) is a QBA. The reason for introducing CLQ is to show that it is not necessary to be concerned with probabilities or with limited information in order to generate partial algebraic structures.

As a further generalization, let G and G_1 be two QBA and let GG_1 be the class of all morphisms from G into G_1. That is, if f is in GG_1 and a and b are in G, then f(~a) = ~f(a) and f(avb) = f(a)vf(b), if and only if either side is defined. Then for all f,g in GG_1 define -f and fvg as in Qaut(LQ). Let 1_{GG_1} be the function such that for all a in G, $1_{GG_1}(a) = 1$, in G_1. Then define $Q(GG_1)$ as the closure of $GG_1 \cup \{1_{GG_1}\}$ under - and v. Similarly define $\mathrm{epi}(GG_1)$, $\mathrm{iso}(GG_1)$, $\mathrm{Qepi}(GG_1)$, and $\mathrm{Qiso}(GG_1)$ from epimorphisms and isomorphisms in the obvious way.

PROPOSITION 3.19: For all QBA G and G_1, $Q(GG_1)$, $Qepi(GG_1)$, and $Qiso(GG_1)$ are QBA. Also, $Qiso(GG_1)$ is isomorphic to $Qiso(G_1G)$.

Proof: It suffices to establish the final isomorphism. Define h: $Qiso(GG_1) \to Qiso(G_1G)$ by setting

i for all f in $iso(GG_1)$, $h(f) = f^{-1}$, in $iso(G_1G)$;

ii $h(1_{GG_1}) = 1_{G_1G}$;

iii $h(fvg) = h(f)vh(g)$, for all f,g in $Qiso(GG_1)$;

iv $h(-f) = -h(f)$, for all f in $Qiso(GG_1)$.

Then it is routine to check that h is an isomorphism.

Let G and G_1 be QBA and let f be a morphism of G into G_1. That is, dom(f) is G, range$(f) \subseteq G_1$ and for all a,b in G, $f(\sim a) = \sim f(a)$ and $f(a \vee b) = f(a) \vee f(b)$. These morphisms differ from the previous surjective morphisms in that morphisms need not be surjective and $f(a) \vee f(b)$ may be defined though $a \vee b$ is not.

A morphism f from G into G_1, two QBA, is an <u>action</u> of G on G_1 if and only if range(f) is contained in some element of $Max(G_1)$ which is the class of maximal algebras in G_1. An idempotent action of a QBA on itself is a <u>projection</u>. Let $act(GG_1)$ be the class of actions of G on G_1 and Proj(G) the class of projections on G. A <u>partial isomorphism</u> between G and G_1 is an isomorphism between one subalgebra of G and one of G_1. Say that G and G_1 are <u>partially isomorphic</u> if and only if for every algebra A in G there is an isomorphic algebra in G_1, and vice versa.

Let f be in $\mathrm{act}(GG_1)$. Describe f as *decomposable* if and only if there exists a partial isomorphism h from G to G_1 and some g in $\mathrm{Proj}(G)$ such that $\mathrm{range}(g) = \mathrm{dom}(h)$, $\mathrm{range}(h) = \mathrm{range}(f)$, and $f = h \circ g$.

PROPOSITION 3.20: Action f of G on G_1 is decomposable if and only if there exists a boolean subalgebra B of G isomorphic to $\mathrm{range}(f)$.

Proof: The necessity is clear from the existence of the partial isomorphism in the decomposition. Conversely, let f be as described and suppose subalgebra B of G is isomorphic to $\mathrm{range}(f)$. Let h be an isomorphism from B to $\mathrm{range}(f)$. Let g be a map from G onto B such that for all a in G, $h \circ g(a) = f(a)$. That g exists is not problematic. It remains to show that g is in $\mathrm{Proj}(G)$. For this it suffices to show g is a morphism. Suppose a is in G and $f(a) = h \circ g(a)$. Then $h \circ g({\sim}a) = f({\sim}a) = {\sim}f(a) = {\sim}h(g(a))$ $= h({\sim}g(a))$. Since h is a partial isomorphism, $g({\sim}a) = {\sim}g(a)$.

Suppose a,b are in G and $a \vee b$ exists. Then $h \circ g(a \vee b)$ $= f(a \vee b) = f(a) \vee f(b) = h(g(a)) \vee h(g(b)) = h(g(a) \vee g(b))$. Since h is still a partial isomorphism, $g(a \vee b) = g(a) \vee g(b)$ as desired. This completes the proof.

Proceeding as in the definition of $\mathrm{Qaut}(LQ)$, *mutatis mutandis*, define $\mathrm{Qact}(GG_1)$, another QBA. Describe $\mathrm{act}(GG_1)$ and $\mathrm{Qact}(GG_1)$ as decomposable if and only if every element of act(GG_1) is decomposable.

It is now possible to describe sufficient conditions on G and G_1 for Qact(GG_1) and Qact(G_1G) to be isomorphic. The choice of these conditions is largely motivated by analogy with quantum theory. Assume from here on that only QBA G with Max(G) nonempty are considered. Describe QBA G as homogeneous if and only if all elements of Max(G) are isomorphic. Where G is homogeneous and B is an algebra isomorphic to the elements of Max(G), say that G is homogeneous in B. Describe G and G_1 as coexpressive if and only if $|\text{Max}(G)| = |\text{Max}(G_1)|$ and they are homogeneous in B, for some boolean algebra B.

PROPOSITION 3.21: If G and G_1 are QBA such that $|\text{Max}(G)| = |\text{Max}(G_1)|$ then G and G_1 are coexpressive if and only if they are partially isomorphic.

The proof is immediate.

PROPOSITION 3.22: If QBA G and G_1 are coexpressive then Qact(GG_1) and Qact(G_1G) are isomorphic.

Proof: It suffices to give a bijection from act(GG_1) onto act(G_1G), assuming G and G_1 are coexpressive QBA. Note first that Qact(GG_1) and Qact(G_1G) are decomposable, given proposition 3.20.

Since G and G_1 are coexpressive, there is a bijective map f from the class of subalgebras of G onto the class of subalgebras of G_1, such that for all subalgebras B of G, f(B) is isomorphic to B.

For each subalgebra B of G_1, let inv(B) be the class of elements f of act(GG_1) such that range(f) = B. Then since G and G_1 are coexpressive, for all algebras B in G, $|\text{inv}(B)| = |\text{inv}(f(B))|$. Let h be a 1:1 map from inv(B) onto inv(f(B)) and let BG be the class of all subalgebras of G. Let inv(BG) be the class of all elements of act(G_1G) that have their range in BG. Then inv induces a partition of act(G_1G) (= inv(BG)). Extend h defined on inv(B) in the obvious way to inv(BG) and the proof is complete.

The preceding argument also establishes that if G and G_1 are coexpressive QBA then if f is in act(GG_1) there exists g in act(G_1G) such that range(f) and range(g) are isomorphic. Further, where B is any subalgebra of G_1, there exists subalgebra B' of G, f in act(GG_1), and f' in act(G_1G) such that range(f) = B, range(f') = B', and f and f'are isomorphisms when restricted to B' and B respectively. Note finally that f and f' may be chosen as above so that in addition $f \circ f'$ and $f' \circ f$ are identity maps when restricted to B' and B respectively. In this case refer to the pair <f,f'> as an interaction.

Consider now the question of specifying conditions under which the decomposition of an action is unique. A QBA G is irreducible if and only if for all a in G, if a\$b for all b in G then either a = 0 or a = 1.

PROPOSITION 3.23: Let G and G_1 be coexpressive, irreducible QBA. Then f in act(GG_1) is uniquely decomposable if and only if range(f) = {0,1}.

Proof: Note first that if f is an action of G on G_1, with G and G_1 as described above, and if h∘g, h∘g' are decompositions of f, then g = g'. If G is irreducible and homogeneous then either G is homogeneous in {0,1} or $|Max(G)|>1$.

Then it suffices to observe that if B is an algebra in G_1, B has a unique isomorphic image in G if and only if B = {0,1}. Homogeneity ensures that there is at least one algebra in G isomorphic to B in each element of Max(G). Irreducibility ensures that these copies of B in G are not identified unless B = {0,1}. For suppose a is in B, $a \neq 0$, $a \neq 1$, and let $\{B_i | i \text{ is in } W\}$ be a set of copies of B, exactly one B_i in each element of Max(G).

Then let a_i be an image of a in B_i, for all i in W. If all B_i were identified in G, then all a_i would be compatible with all elements of G. Hence G would not be irreducible.

Partial isomorphisms are second order partial translations: where partial translations connect languages and hence, their algebras, partial isomorphisms connect the connected algebras.

PROPOSITION 3.24: Let G and G_1 be QBA and suppose f is in act(GG_1). If h∘g decomposes f then h and g uniquely determine each other.

Proof: Suppose first that h∘g and h∘g' decompose f. That is, dom(h) = range(g) = range(g'), range(h) = range(f) and, for all a in G, $h \circ g(a) = f(a) = h \circ g'(a)$. Then since h is a partial isomorphism, $g(a) = g'(a)$, as desired.

Conversely, suppose $h \circ g$ and $h' \circ g$ decompose f. Then for all a in G, $h \circ g(a) = h' \circ g(a)$ and so $h = h'$.

PROPOSITION 3.25: Let G and G_1 be coexpressive QBA and let h be an isomorphism from G onto G_1. Then h induces a 1:1 correspondence between act(GG_1) and Proj(G).

Proof: Suppose f is in act(GG_1). If g and g' are in Proj(G) such that $h \circ g = f = h \circ g'$, then $g = g'$. Conversely, if g is in Proj(G) and f, f' are in act(GG_1) such that $f = h \circ g = f'$, then $f = f'$.

Thus, in the presence of a specific isomorphism between coexpressive QBA G and G_1, each action of G on G_1 determines a unique projection in G, and vice versa.[1]

Let G be a QBA and let f and g be in Proj(G). Say that f and g <u>commute</u> if and only if $g \circ f = f \circ g$.

PROPOSITION 3.26: Let f and g be in Proj(G). Then f and g commute only if f(range(g)) = g(range(f)).

Proof: Since f is idempotent, $f(a) = a$ for all a in range(f). If not, then where b is in $f^{-1}(a)$, $f(f(b)) \neq f(b)$ contrary to idempotence. The same holds for g.

Suppose f(range(g)) $\neq$ g(range(f)). Suppose, without

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[1] More contentiously: relative to a specific isomorphism between G and G_1, each f in act(GG_1) determines the state of G relative to G_1, and vice versa. The same isomorphism makes an interaction between G and G_1 determine both states.

loss of generality, that a is in f(range(g)) and a is not in g(range(f)). Then f(g(a)) = a and g(f(a)) $\neq$ a.

PROPOSITION 3.27: Let f and f' be in act(GG_1), where G and G_1 are coexpressive QBA and such that range(f) is isomorphic to range(f'). Suppose h is in iso(GG_1) and f* is in Proj(G) such that h∘f* = f. Then there exist h" in iso(GG_1) and f" in Proj(G) such that h"∘f* = f' = h∘f".

The proof is straightforward. This proposition is intended to mirror the difference between Schrödinger's wave analysis and Heisenberg's transition analysis. The difference between the actions f and f' of G on G_1 may be expressed as a difference in projections on G, or as a difference between the correlations between the two quasi boolean algebras.

CONCLUSION:

The study of partial structures in mathematics is relatively new, category theory being probably the best known field using such structures. Often in mathematics investigation of more general structures yields both new insights concerning the original objects of study, and a theory of far reaching depth and scope.

This thesis is a contribution to the theory of partial structures which arise in connection with quantum mechanics. Parts II and III above are essentially original, while working from material due to Gudder and Kochen.

Future work may be hoped to develop theories of change and of interaction, and to investigate motivation for the satisfaction of condition (C). Partial structures are a significant and non-trivial generalization of previously studied structures. As such, their mathematical interest is clear. Light shed on the interpretation of quantum mechanical phenomena would also be welcome.

APPENDIX

Details of the proofs of propositions 1.6, 1.7, 1.8, and 1.9 are given.

Proof of proposition 1.6: First establish that every QBA is isomorphic to a glued boolean algebra. Let <B, 0, 1, \$, ~, v> be a QBA. Consider any set A_i which is a subset of B such that for all members of A_i, all their polynomials in ~ and v exist in B. By condition x', all A_i must be boolean algebras. Let F be the class of all such A_i. Then whenever A_i, A_j are in F, so is their intersection, as required for F to be a glued boolean algebra. It is not difficult to show that B and F, as defined, are naturally isomorphic.

Conversely, let F be a glued boolean algebra. Let B' = {b| b is a member of B for some boolean algebra B in F}. Let 0 be the common zero element of the boolean algebras in F. Let 1 be the common unit element of these algebras. Define \$ as a subset of B'^2 by setting, for all a, b in B', a\$b if and only if there exists boolean algebra B in F such that a and b are in B. For all a in B', define ~a as the complement of a in an algebra in F containing a. As defined, ~a is well-defined, for suppose B_i and B_j are distinct boolean algebras in F which both contain a. Then a is in their intersection which is also in F. Hence the complements of a in B_i and in B_j are identical, since they are in the intersection of B_i and B_j, since a is.

Define $\vee$ on B' by setting: $a\vee b$ exists if and only if $a\$b$. In this case, $a\vee b$ is in some B_i in F and we may set $a\vee b$ in B' to be $a\vee b$ in B_i. Argument as for $\sim a$ shows $a\vee b$ is well-defined in B'.

Proof of *proposition 1.7*: It suffices to show that condition (C) and condition x' imply condtion x, in the presence of conditions i to ix.

Suppose that in some QBA satisfying (C), $a\$b$, $a\$c$, and $b\$c$, for some a, b, c in the QBA. By ix, $c\$\sim b$, $a\$\sim b$, $\sim a\$c$. Then c, $\sim b$ and $\sim a$ are pairwise compatible. By (C), $\sim a\$\sim b\vee c$. Likewise, $\sim a\$\sim a\vee c$, $b\$\sim a\vee c$, $\sim a\vee b\$\sim a\vee c$, and $\sim(\sim a\vee b)\$\sim a\vee c$. Repetition of applications of conditions ix and (C) yields $\sim(\sim a\vee(\sim b\vee c))\$\sim(\sim a\vee b)\vee(\sim a\vee c))$. In general, we may conclude that all boolean polynomials in a, b, and c exist. Then by x', all boolean identities hold in these polynomials. In particular, as required, $\sim(\sim a\vee(\sim b\vee c))\vee(\sim(\sim a\vee b)\vee(\sim a\vee c)) = 1$.

Proof of *proposition 1.8*: Let G be a transitive QBA. Then there is a natural partial order, $\leq$, on G. Then $<G,\leq>$ is a partially ordered set. Clearly infG and supG exist. They are 0 and 1, respectively, in QBA G. An orthocomplement, $^\perp$, on $<G,\leq>$ is provided by the complement,

~, on QBA G. Since G is a transitive QBA, comparable elements of G are compatible. Hence, whenever $a \leq b^{\perp}$, a\$~b. Hence a\$b and so avb exists in QBA G if $a \leq b^{\perp}$. Define avb in $<G, \leq>$ if and only if a\$b in QBA G, and in this case avb in $<G, \leq>$ is avb in QBA G. Then whenever $a \leq b^{\perp}$ in $<G, \leq>$, avb exists, as required.

Finally we must show that orthomodularity holds in $<G, \leq>$. Suppose $a \leq b$ in $<G, \leq>$. Then $a \leq b$ in QBA G and so b = ~av(b&~a). But then orthomodularity in $<G, \leq>$ follows easily from the definitions of $^{\perp}$ and v in the poset.

Proof of proposition 1.9: Let G be an orthomodular poset satisfying condition (C). Gudder proved in [6] that in such a poset, G, a subset S of G is contained in a boolean subalgebra of G if and only if the elements of S are pairwise compatible. Let A_i be any subset of G such that for all elements of A_i, all their polynomials in v and + exist. Then all such A_i are boolean algebras. Let F be the class of all such A_i. Then whenever A_i and A_j are in F, so is their intersection. Thus F is a glued boolean algebra, and is isomorphic to G. By proposition 1.6 we may regard F as a QBA. Since (C) holds in G, it holds in F. Hence F is in fact a PBA. Finally, since F is isomorphic to G, it is transitive. This completes the proof.

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VITA

Surname: CHARLWOOD Given Names: GERALD WILLIAM

Place of Birth: Toronto, Ontario

Date of Birth: October 18, 1950

Degrees:

Ph.D. (Philosophy)	1978	University of Toronto
M.A. (Philosophy)	1974	University of Toronto
B.A. (Arts & Science)	1973	University of Toronto

Scholarships & Awards:

1969	Ontario Scholar (Grade 13)
1969-71	Clara Flavelle Memorial Scholarship (Tuition)
1970-71	Alumni Scholarship (General Proficiency)
1972-73	Clara Flavelle Memorial Scholarship (Tuition)
1972-73	William Neil Hanna Scholarship
1972-73	Susan D. Massey Scholarship in Philosophy
1973-77	Ontario Graduate Scholarships

Publications:

"Positive Semilattice Relevance Logic", The Journal for Symbolic Logic, Vol. 46, #2, 1981.

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Author [redacted]

Gerald William Charlwood

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