FLUID DYNAMICS WITH NEARLY CONSTANT DENSITY

By

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Abstract. The purpose of this article is to describe fluid flow with nearly constant density from the dynamical systems viewpoint. To do so, pressure is regarded as a state variable along with the three velocity components. To the usual Navier-Stokes velocity equation for $\partial v_i/\partial t$ is added an equation for $\partial p/\partial t$. In turn, regarding a region of fluid flow as partitioned into a large number of small cells (cubes), with inlet cells, outlet cells, and boundary cells, leads in a natural way to a dynamical system of large (but finite) dimension.

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1. Introduction.

As described by Landau and Lifshitz, [LL, pp. 13-49], the velocity components $\{v_i\}$ of a fluid with nearly constant density ρ satisfy

$$\partial v_i/\partial t = -\rho^{-1}\,\partial p/\partial x_i - \sum_{j=1}^{3} v_j\,\partial v_i/\partial x_j + \nu\,\partial^2 v_i/\partial x_j^2 + \left(\frac{\zeta}{\rho}+\frac{\nu}{3}\right)\partial\mathcal{D}/\partial x_i \tag{1}$$

where $\{x_j\}$ are the usual spatial coordinates, p is pressure, ρ is density, ν is kinematic viscosity, ζ is the second coefficient of viscosity, and $\mathcal{D}$ is dilation, that is,

$$\mathcal{D} = \sum_{j=1}^{3} \partial v_j/\partial x_j .$$

The central assertion of this paper is that for such fluids

$$\partial p/\partial t = \underbrace{-\delta\mathcal{D}}_{\text{dilation effect}} + \underbrace{\kappa\left\{\sum_{i=1}^{3}\frac{\partial^2 p}{\partial x_i^2} - 2\rho\sum_{\substack{j,k=1\\ j<k}}^{3}\left(\frac{\partial v_j}{\partial x_j}\frac{\partial v_k}{\partial x_k} - \frac{\partial v_j}{\partial x_k}\frac{\partial v_k}{\partial x_j}\right)\right\}}_{\text{pressure diffusion}} \tag{2}$$

Here $-\delta$ is a negative dilation constant in units $ML^{-1}T^{-2}$ and κ is a positive diffusion constant in units L^2T^{-1}. Both $-\delta$ and κ could be measured for a real fluid in experiments not involving flow. In fact, $-\delta \cong -2.18 \times 10^9$ Pa for water at standard conditions. For water, κ appears to be on the order of $1\ m^2\ sec^{-1}$, as explained in section 6.

<u>We do not assume $\mathcal{D} \equiv 0$</u>. Rather, for steady flows, straightforward algebraic manipulations show that (1) and (2) imply

$$\frac{\delta}{\rho\kappa}\mathcal{D} + \mathcal{D}^2 + \underline{v} \cdot \mathrm{grad}(\mathcal{D}) = \left(\frac{\zeta}{\rho} + \frac{4\nu}{3}\right)\nabla^2\mathcal{D} \tag{3}$$

The simplest solution for (3) is, of course, $\mathcal{D} \equiv 0$, and indeed $\mathcal{D} \equiv 0$ for most - but not all - steady flows described below.

Of course, $\mathcal{D} \neq 0$ and $\rho \equiv$ constant imply violation of conservation of mass. Strictly speaking, the assumption that $\rho \equiv$ constant is the source of the trouble since it implies that with the passage of a pressure wave through any finite cell in any such fluid there will be, at times, violation of conservation of mass. Nonetheless the density of water, say, is given by the empirical formula

$$\rho = \frac{10^3}{1 - (4.58 \times 10^{-10}\ \mathrm{Pa}^{-1})p}\ \mathrm{kg\ m}^{-3} \tag{4}$$

and is obviously nearly constant in routine flows. To avoid excessive complexity we shall take $\rho \equiv$ constant and so endure violations (very small violations, it turns out, because $|-\delta|$ is so large for liquids) of conservation of mass for unsteady flows.

Roughly speaking, significant dilation (expansion or compression) would cause large pressure changes and gradients. Such gradients acting through (1) would cause large accelerations which would tend to correct $\mathcal{D} \neq 0$. Limiting excursions of $\mathcal{D}$ could be thought of as limiting the severity of turbulence.

A dynamical system can be associated with equations (1) and (2) by regarding the fluid region as partitioned into a large number of small cubes, each with edge length ℓ. Such associations are described in Walter [W, pp. 275-303]. Let each cell have index $\iota = (i,j,k)$. Let adjacent indices $+\alpha(\iota)$ and $-\alpha(\iota)$ for $\alpha = 1, 2, 3$ be specified by adding one or subtracting one to the α component of index combination ι. Thus $+1(2,2,2) = (3,2,2)$, $-1(2,2,2) = (1,2,2)$, and so on. We may then write the rate of change of the $\alpha = 1, 2, 3$ component of velocity of cell ι as

$$\dot{v}_\alpha^\iota = \frac{1}{\rho}\left(\frac{p^{-\alpha(\iota)} + p^{+\alpha(\iota)}}{2\ell}\right) + \sum_{\beta=1}^{3} v_\beta^\iota \left(\frac{v_\alpha^{-\beta(\iota)} - v_\alpha^{+\beta(\iota)}}{2\ell}\right) + \nu \sum_{\beta=1}^{3} \frac{v_\alpha^{-\beta(\iota)} + v_\alpha^{+\beta(\iota)} - 2v_\alpha^\iota}{\ell^2}$$

$$+ \left(\frac{\zeta}{\rho} + \frac{\nu}{3}\right)\left(\frac{\mathscr{D}^{+\alpha(\iota)} - \mathscr{D}^{-\alpha(\iota)}}{2\ell}\right) \tag{5}$$

where

$$\mathscr{D}^\iota = \sum_{\beta=1}^{3} \frac{v_\beta^{+\beta(\iota)} - v_\beta^{-\beta(\iota)}}{2\ell}$$

is dilation in cell ι. The pressure equation becomes

$$\dot{p}^\iota = -\delta\mathscr{D}^\iota$$

$$+ \kappa\left\{\sum_{\alpha=1}^{3} \frac{p^{-\alpha(\iota)} + p^{+\alpha(\iota)} - 2p^\iota}{\ell^2} - 2\rho \sum_{\substack{\alpha,\beta=1 \\ \alpha<\beta}}^{3} \frac{\left(v_\alpha^{+\alpha(\iota)} - v_\alpha^{-\alpha(\iota)}\right)\left(v_\beta^{+\beta(\iota)} - v_\beta^{-\beta(\iota)}\right) - \left(v_\alpha^{+\beta(\iota)} - v_\alpha^{-\beta(\iota)}\right)\left(v_\beta^{+\alpha(\iota)} - v_\beta^{-\alpha(\iota)}\right)}{4\ell^2}\right\} \tag{6}$$

If equations (5) and (6) could be shown to be correct for fluid dynamics with nearly constant density, their usefulness would not be lost on fluid dynamicists. Together with virtial inlet, outlet, and boundary cells, a considerable variety of flows could be studied using routine difference equation approximations and computing machines. Because (5) and (6) clearly satisfy Lipschitz conditions, we would avoid the notorious existence and uniqueness difficulties associated with the usual Navier-Stokes approach (namely (1) and $\mathcal{D} \equiv 0$, together with some constitutive law connecting ρ and p). Also, it should be possible to predict the onset of turbulence from computer simulations using the geometry of the fluid region and the constants ρ, ν, δ, and κ —all of which could in principle be determined in static or low speed (nonturbulent) experiments. In particular, some interest would be attached to determination of ζ, since in this paper we estimate $-(13/3)\rho\nu < \zeta < -(4/3)\rho\nu$ in contrast to the usual assertion that ζ is positive [LL, p. 186].

Our first task is to justify equation (2). We submit that there are two mechanisms by which pressure changes at a point with time. Dilation effect refers to pressure changes due to $\mathcal{D}$ not exactly zero. Pressure diffusion refers to the maintenance of pressure gradients due to acceleration relative to a moving reference frame (as in steady rotation). These concepts are developed in the following sections.

2. Dilation Effect.

An empirical formula for the density of water at 20°C is again

$$\rho = \frac{10^3}{1 - \left(4.58 \times 10^{-10} \frac{1}{\mathrm{Pa}}\right) p} \frac{\mathrm{kg}}{\mathrm{m}^3}$$

For a small test cube of water of volume V in a static region of water subject to changing pressure,

$$\mathcal{D} = \frac{\dot{V}}{V} = \frac{-4.58 \times 10^{-10} \dot{p}}{1 - 4.58 \times 10^{-10} p} \cong -4.58 \times 10^{-10} \mathrm{Pa}^{-1} \times \dot{p}$$

Thus

$$\dot{p} \cong -2.18 \times 10^9 \text{ Pa} \times \mathscr{D} \tag{7}$$

Thus negative dilation (decrease of V) leads to a marked increase in pressure.

Considering this, we include in $\partial p/\partial t$ the term (a linear approximation) $-\delta\mathscr{D}$.

3. Pressure Diffusion.

For illustrative purposes, consider a cubic metre of water at room temperature which fills an inflexible tank. Suppose that at time zero the pressure throughout the water is one bar except in a central cubic centimetre in which the pressure is two bar. As time progresses, the pressure in the tank would quickly approach a constant value slightly greater than one bar. All velocity components would remain virtually zero and the temperature of the water would remain virtually constant. It seems clear that a diffusion model

$$\frac{\partial p}{\partial t} = \kappa \nabla^2 p$$

where κ is a (positive) constant would be a simple model with which to predict the outcome of this experiment and, indeed, many related experiments with zero flow and a nonuniform initial pressure distribution.

Next let us consider water which fills a cylindrical tank, say, one metre long and one metre in diameter. Suppose the water and tank rotate about the axis of the tank with constant angular speed ω. Of course, we expect a pressure distribution like

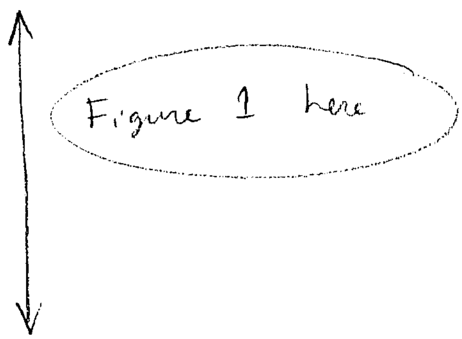
$$p(r) = p_0 + \frac{\rho r^2 \omega^2}{2}$$

Assuming again that pressure anomalies would disappear without creating significant velocity or temperature disturbances, the model

$$\frac{\partial p}{\partial t} = \kappa\left(\nabla^2 p - 2\rho\omega^2\right)$$

can be derived. One need only assume that pressure diffuses spatially through a small piece of an imaginary plane at a rate proportional to the product of surface area and the pressure gradient across the surface.

Let us now turn to three examples of planar flow.



Again, a small pressure anomaly in each of the three flows (of, say, water) would quickly diffuse without significant velocity or temperature disturbances. Clearly we need a function f of the deformation tensor $\{\partial v_i/\partial x_j\}$ so that

$$\frac{\partial p}{\partial t} = \kappa\left(\nabla^2 p + f(\partial v_i/\partial x_j)\right)$$

adequately describes such diffusions. The function f should be invariant with respect to a rotation of coordinate frame (as is ∇^2, of course) and as simple as possible. An obvious candidate is the second elementary symmetric function of $\{\partial v_i/\partial x_j\}$ multiplied by an appropriate constant. Hence pressure diffusion and maintenance of pressure gradients by acceleration within steady flows (with zero dilation) might be modelled by

$$\frac{\partial p}{\partial t} = \kappa\left\{\nabla^2 p - 2\rho \sum_{\substack{j,k=1\\ j<k}}^{3} \frac{\partial v_j}{\partial x_j}\frac{\partial v_k}{\partial x_k} - \frac{\partial v_j}{\partial x_k}\frac{\partial v_k}{\partial x_j}\right\} \tag{8}$$

Combining terms in (7) and (8) yields (2) and (6).

Of course, the three flows in Fig. 1 conform to equations (1) and (2). In three dimensions any rotational flow $\underline{v} = \underline{v}_0 + \underline{\omega} \times \underline{x}$ satisfies equations (1) and (2) with

$$\frac{\partial p}{\partial t} = \kappa\left\{\nabla^2 p - 2\rho\left(\omega_1^2 + \omega_2^2 + \omega_3^2\right)\right\} = \kappa\left\{\nabla^2 p - 2\rho\|\omega\|^2\right\}$$

and $p = p_0 + \rho \dfrac{\|\omega\|^2\|x\|^2}{4} - \rho \displaystyle\sum_{\substack{j,k=1\\ j<k}}^{3} \omega_j\omega_k x_j x_k.$

Second, consider the laminar shear flow $v_i = \sum_{j=1}^{3} a_{ij}x_j$ where $\{a_{ij}\}$ is a matrix of one of the forms shown below:

$$\begin{pmatrix}0 & * & *\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & *\\ 0 & 0 & *\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & * & 0\\ 0 & 0 & 0\\ 0 & * & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ * & 0 & 0\\ * & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ * & * & 0\end{pmatrix}, \text{ or } \begin{pmatrix}0 & 0 & 0\\ * & 0 & *\\ 0 & 0 & 0\end{pmatrix}$$

Here $*$ denotes a possibly nonzero entry in $\{a_{ij}\}$. Equations (1) and (2) are satisfied with $p = p_0$.

Third, consider a hyperbolic flow of the form $v_i = \sum_{j=1}^{3} a_{ij}x_j$ where $\{a_{ij}\}$ is a matrix with zero nondiagonal entries and zero trace. Suppose also that $\sum_{i=1}^{3} a_{ii}^2 = -2(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33})$. Then setting

$$p = p_0 - \frac{\rho}{2}\sum_{i=1}^{3} a_{ii}^2 x_i^2$$

gives us a flow satisfying (1) and (2). As an example, let the diagonal entries of $\{a_{ij}\}$ be 2, -1, and -1, or, extending the pattern in Fig. 1, 1, -1, and 0.

Fourth, consider laminar flow between two parallel planes. It is well known that v_1 depends quadratically on x_2, $v_2 \equiv 0$, $v_3 \equiv 0$, and p depends linearly on x_1, fulfilling (1). Again (2) is also satisfied for any such flow.

Finally we refer to a model due to Ladyzhenskaya [L, xi-xii] of planar flow outside $x_1^2 + x_2^2 = 1$. Here (1) and (2) are solved by

$$v_1 = cx_1 r^{-2}$$

$$v_2 = xc_2 r^{-2}$$

$$p = p_0 - \frac{1}{2}\rho c^2 r^{-2}$$

where $r^2 = x_1^2 + x_2^2$ and c is the outward radial velocity of the fluid at $r = 1$. Of the other flows mentioned by Ladyzhenskaya — an infinite number parameterized by an arbitrary constant $c_1 \neq 0$, all solution flows of the Navier-Stokes equations — not one satisfies (1) and (2).

4. Dilation in Steady Flow.

Assuming each v_i is suitably differentiable for a flow,

$$\frac{\partial \mathcal{D}}{\partial t} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial v_i}{\partial t}\right)$$

$$= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left\{-\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} + \nu \sum_{j=1}^{3} \frac{\partial^2 v_i}{\partial x_j^2} + \left(\frac{\zeta}{\rho} + \frac{\nu}{3}\right)\frac{\partial \mathcal{D}}{\partial x_i}\right\}$$

$$= \sum_{i=1}^{3} -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2} - \sum_{i,j=1}^{3}\left\{\frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j} - v_j\frac{\partial^2 v_i}{\partial x_j \partial x_i} + \nu\frac{\partial^2}{\partial x_j^2}\left(\frac{\partial v_i}{\partial x_i}\right) + \left(\frac{\zeta}{\rho}+\frac{\nu}{3}\right)\frac{\partial^2}{\partial x_j^2}\left(\frac{\partial v_i}{\partial x_i}\right)\right\} \quad (9)$$

Now

$$-\sum_{i,j=1}^{3}\frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j} = -\sum_{i=1}^{3}\left(\frac{\partial v_i}{\partial x_i}\right)^2 - 2\sum_{\substack{i,j=1\\ i<j}}^{3}\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} + (-2+2)\sum_{\substack{i,j=1\\ i<j}}^{3}\frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}$$

$$= -\left(\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i}\right)^2 + 2\sum_{\substack{i,j=1\\ i<j}}^{3}\frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} \quad (10)$$

Hence (2), (9), and (10) yield

$$\frac{\partial \mathscr{D}}{\partial t} = -\frac{1}{\rho}\left\{\sum_{i=1}^{3}\frac{\partial^2 p}{\partial x_i^2} - 2\rho\sum_{\substack{i,j=1\\ i<j}}^{3}\frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\right\}$$

$$-\mathscr{D}^2 - \underline{v}\cdot \mathrm{grad}(\mathscr{D}) + \left(\frac{\zeta}{\rho}+\frac{4\nu}{3}\right)\nabla^2\mathscr{D}$$

$$= -\frac{(\partial p/\partial t + \delta\mathscr{D})}{\rho\kappa} - \mathscr{D}^2 - \underline{v}\cdot \mathrm{grad}(\mathscr{D}) + \left(\frac{\zeta}{\rho}+\frac{4\nu}{3}\right)\nabla^2\mathscr{D} \quad (11)$$

For steady flow, (11) becomes

$$\frac{\delta}{\rho\kappa}\mathscr{D} + \mathscr{D}^2 + \underline{v}\cdot \mathrm{grad}(\mathscr{D}) = \left(\frac{\zeta}{\rho}+\frac{4\nu}{3}\right)\nabla^2\mathscr{D}$$

as asserted in (3).

5. Roller Assisted Flow.

In this section we consider a flow experiment as sketched in Fig. 2.

(Figure 2 here)

By roller assisted flow we mean one-dimensional flow between two reservoirs as shown in Fig. 2. It is assumed that the gap between the two parallel, flexible sheets is sufficiently narrow to insure the flow remains laminar and so one-dimensional. We anticipate that along x_1, p will fall, v_1 will asymptotically approach some limit value, and $\partial v_1/\partial x_1$ will asymptotically approach zero.

For the case of steady, roller assisted flow, (1) and (2) become

$$0 = -\rho^{-1}p' - vv' + \gamma v'' \tag{12}$$

and

$$0 = -\delta v' + \kappa\rho'' \tag{13}$$

Here v denotes v_1, $'$ denotes differentiation with respect to x_1, and $\gamma = [(\zeta/\rho)+(4\nu/3)]$. From (12) we see $p = p_0 - \rho v^2/2 + \rho\gamma v'$. Hence to understand roller assisted flow, we need only solve for v in

$$\frac{\delta}{\kappa\rho} v' = -\left(\frac{v^2}{2}\right)'' + \gamma v''' , \tag{14}$$

specify p, and so solve (12) and (13). Integrating (14) we have

$$v = -\frac{\kappa\rho}{\delta}\left(\frac{v^2}{2}\right)' + \frac{\kappa\rho\gamma}{\delta}v'' + \mathscr{C} \tag{15}$$

Here the integration constant $\mathscr{C}$ is the limit value of v (so $\mathscr{C} > 0$).

Consider the two-dimensional dynamical system derived from (15) if $\gamma \neq 0$ or from (14) if $\gamma = 0$ by regarding x_1 as an independent variable and v and v' as state variables. Thus

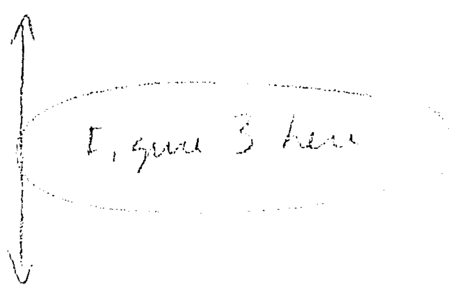
$$(v)' = v'$$

$$(v')' = \begin{cases} -\dfrac{v'}{v}\left(\dfrac{\delta}{\rho\kappa} + v'\right) & \text{if } \gamma = 0 \\[2ex] \dfrac{\delta}{\kappa\rho\gamma}(v - \mathscr{C}) + \dfrac{1}{\gamma}vv' & \text{if } \gamma \neq 0 \end{cases} \tag{16}$$

We shall determine qualitatively the solutions of (14) or (15) by studying (16).

Typical trajectories for (16) are shown in Fig. 3. One can readily appreciate that for real fluids, $\gamma < 0$. Thus the second coefficient of viscosity ζ must satisfy

$$\zeta < -4\rho\nu/3 \tag{17}$$



Since $\gamma < 0$ we can define a Lyapunov function

$$\Lambda = \left(-\frac{\delta}{\kappa\rho\gamma}\right)\frac{(v-\mathscr{C})^2}{2} + \frac{(v')^2}{2}$$

Along trajectories of (16) the rate of change of Λ is

$$\Lambda' = \gamma^{-1}\, v(v')^2$$

Thus any initial (v, v') value inside the ellipse $\Lambda = -\frac{\delta\mathscr{C}^2}{2\kappa\rho\gamma}$ must asymptotically approach $(\mathscr{C}, 0)$. Roughly speaking, if $v > 0$ at the inlet of the device in Fig. 2 and if v' is "reasonable" at the inlet, then along the channel v approaches $\mathscr{C}$ and v' approaches zero.

As shown in the previous section, for any steady flow

$$\frac{\delta}{\rho\kappa}\mathscr{D} + \mathscr{D}^2 + \underline{v}\cdot \operatorname{grad}(\mathscr{D}) = \gamma\,\nabla^2\mathscr{D} \tag{3}$$

We now consider the consequence of $\gamma = (\zeta/\rho) + (4\nu/3) < 0$.

Let R be the region in three-dimensional space occupied by a fluid. We recall that if the region is partitioned into small cubes indexed as $\iota = (i,j,k)$ with edge length ℓ, then at the centre of cube ι,

$$\nabla^2\mathscr{D} \cong \sum_{\alpha=1}^{3}\frac{\mathscr{D}^{+\alpha(\iota)} + \mathscr{D}^{-\alpha(\iota)} - 2\mathscr{D}^{\iota}}{\ell^2}$$

Thus if $\nabla^2\mathscr{D}$ is negative, $\mathscr{D}^{\iota}$ is greater than the average of the values of $\mathscr{D}$ in the six adjacent cubes; likewise $\nabla^2\mathscr{D}^{\iota} > 0$ implies $\mathscr{D}^{\iota}$ is less than the average of other local $\mathscr{D}$ values.

Let us define $Z^+ = \{\underline{x} \in R \mid \mathscr{D}(\underline{x}) > 0\}$. Suppose at a point $\underline{z}$ in the interior of Z^+, $\mathscr{D}$ has a local minimum value. Then $\operatorname{grad}(\mathscr{D}) = 0$ and $\nabla^2\mathscr{D} > 0$ at $\underline{z}$. But this contradicts (3).

Let us define $Z^- = \{\underline{x} \in R \mid -\delta/\rho\kappa < \mathscr{D} < 0\}$. Suppose at a point $\underline{z}$ in the interior of Z^-, $\mathscr{D}$ has a local maximum value. Then grad $(\mathscr{D}) = 0$ and $\nabla^2\mathscr{D} < 0$ at $\underline{z}$. But again (3) is contradicted. (We shall show that for water, $\delta/\rho\kappa$ is a large number, so a flow with $\mathscr{D} \leq -\delta/\rho\kappa$ is not physically possible with $\rho \cong$ constant.)

Thus for steady flows modelled by (1) and (2), $\mathscr{D}$ can have neither a local minimum value in the interior of the region Z^+ nor a local maximum value in the interior of the region Z^-.

6. Estimating κ for Water.

Suppose the pressure of water at room temperature in a large tank is given (in Pa) by

$$p(x,t) = 10^5 + (4\kappa t+10^{-4})^{-1/2} \exp[-x^2/(4\kappa t+10^{-4})]$$

satisfying

$$\frac{\partial p}{\partial t} = \kappa \frac{\partial^2 p}{\partial x^2}$$

where we abbreviate x_1 as x. Thus at $t = 0$ and $x = 0$, p is 1.001 bar; at $t = 0$ and $x = \pm 1$ cm, p is 1.0004 bar; and at $t = 0$ and $x = \pm 2$ cm, p is 1.00002 bar. As time progresses, p quickly approaches 1 bar for all x.

At $t = 0$, $\partial^2 p/\partial x^2$ is -2×10^6 at $x = 0$ and about $+10^6$ at $x = \pm\sqrt{3}/2$ cm. Thus at $t = 0$, p is decreasing at $x = 0$ about twice as fast as p is increasing at $\pm\sqrt{3}/2$ cm. Now the speed of sound in water is about 400 m/sec. Thus the time required for pressure at $x = 0$ to equal pressure at

$x = \pm\sqrt{3}/2$ cm should be about $(\sqrt{3}/2) \times 10^{-2}/400 \cong 2 \times 10^{-5}$ sec. Hence we estimate at $x = 0$ that

$$\frac{\Delta p}{\Delta t} \cong \frac{-50 \text{ Pa}}{2 \times 10^{-5} \text{ sec}} \cong \kappa\left(-2 \times 10^{6}\right)$$

Thus a rough estimate of κ for water at room temperature is 1.

7. The Stability of Low Speed Steady Flows.

We proceed to calculate the linear approximation matrix associated with the dynamical system (5), (6), assuming all v_α^ι are small. We can order cells $\iota = (i,j,k)$ just as the natural numbers ijk are ordered. We group the state variables as $(v_1^{111}, \ldots, v_1^{nnn}, v_2^{111}, \ldots, v_3^{nnn}, p^{111}, \ldots, p^{nnn})$. Thus the linear approximation matrix is naturally partitioned into 16 square blocks, A_{11}, $A_{12}, \ldots, A_{44}$, each $n^3 \times n^3$.

Let us ignore indices of cells adjacent to boundary cells for the moment.

Each block $A_{\alpha 4}$, $\alpha = 1, 2, 3$ is of the form

$$\frac{1}{\rho 2\ell}\begin{pmatrix} \ddots & & & & \\ \ldots -1 & \ldots & 0 & \ldots & 1 \ldots \\ & & & & \ddots \end{pmatrix}$$

and each block $A_{4\alpha}$, $\alpha = 1, 2, 3$, is of the form

$$\frac{\delta}{2\ell}\begin{pmatrix} \ddots & & & & \\ \ldots +1 & \ldots & 0 & \ldots & -1 \ldots \\ & & & & \ddots \end{pmatrix}$$

If velocity terms are neglected, the block A_{44} is of the form

$$\frac{\kappa}{\ell^2}\left[\ldots 1 \ldots 1 \ldots 1 \ldots -6 \ldots 1 \ldots 1 \ldots 1 \ldots\right]$$

It remains to consider viscosity terms in $A_{\alpha\alpha}$ and dilation terms in $A_{\alpha\beta}$ where $\alpha, \beta = 1, \ldots, 3$. Viscosity terms amount to

$$\frac{\nu}{\ell^2}\left[\ldots 1 \ldots 1 \ldots 1 \ldots -6 \ldots 1 \ldots 1 \ldots 1 \ldots\right]$$

Now the dilation term in $\dot{v}_1^{333}$ is

$$\frac{(\zeta/\rho) + (\nu/3)}{4\ell^2}\left\{-2v_1^{333} + v_1^{133} + v_1^{533} + v_2^{223} - v_2^{243} - v_2^{423} + v_2^{443} + v_3^{232} - v_3^{234} - v_3^{432} + v_3^{434}\right\}$$

Thus dilation terms contribute to each $A_{\alpha\alpha}$

$$\frac{(\zeta/\rho) + (\nu/3)}{4\ell^2}\left[\ldots 1 \ldots -2 \ldots 1 \ldots\right]$$

as well as +, - pairs of terms to $A_{\alpha\beta}$ and $A_{\beta\alpha}$, $\alpha \neq \beta$.

Consider the linear dynamical system $\dot{x} = Ax$ and the Lyapunov function

$$\Lambda = \sum_{i=1}^{3n^3} \frac{x_i^2}{2} + \sum_{i=3n^3+1}^{4n^3} \frac{\rho\delta x_i^2}{2}$$

Ignoring cells adjacent to boundary cells, the rate of change of Λ along an arbitrary trajectory of $\dot{x} = Ax$ is

$$\dot{\Lambda} \cong -\frac{\nu}{\ell^2}\sum_{i=1}^{3n^3}\sum_{\alpha=1}^{3}\left(x_i - x_{-\alpha(i)}\right)^2 + \left(x_i - x_{+\alpha(i)}\right)^2$$

$$-\left(\frac{(\zeta/\rho)+(\nu/3)}{4\ell^2}\right)\sum_{i=1}^{3n^3}\left(x_i - x_{+\beta(i)}\right)^2 + \left(x_i - x_{-\beta(i)}\right)$$

$$- \frac{\kappa\rho\delta}{\ell^2} \sum_{i=3n^3+1}^{4n^3} \left(x_i - x_{-\alpha(i)}\right)^2 + \left(x_i - x_{+\alpha(i)}\right)^2 \tag{18}$$

Here each number i corresponds to an index triplet ι and $\{-\alpha(i), +\alpha(i)\}$, $\alpha = 1, 2, 3$, are the numbers in $1, \ldots, 3n^3$ corresponding to the neighbouring cubes of ι. Heavy use is made of the symmetries and skew symmetries of A. If i is the number corresponding to the place of v_β^ι, then $+\beta(i)$ is the number corresponding to the place of $v_\beta^{+\beta(\iota)}$; similarly $-\beta(\iota)$ is defined.

Note that $\dot{\Lambda} \le 0$ (ignoring boundary cells) for all x if

$$-4\nu < \zeta/\rho + \nu/3$$

that is

$$-13\nu/3 < \zeta/\rho \tag{19}$$

Since low speed flows are known to be stable, (18) and (17) enable us to estimate a range for ζ sufficient for local stability of low speed steady flows:

$$(-13/3)\rho\nu < \zeta < -(4/3)\rho\nu$$

8. Conclusion

The model (1), (2) or (5), (6) conforms with a few analytically known flows. Numerical simulations of complex flows are needed to assess the value of our approach.

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