OPTIMAL PREVENTIVE MAINTENANCE, PROTECTION AND REPLACEMENT OF A REVENUE-EARNING ASSET

By

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ABSTRACT

The problem of determining the optimal pattern of expenditure on preventive maintenance or protection for a revenue-earning asset subject to catastrophic breakdown or destruction is discussed. It is assumed that the probability of breakdown at any time depends on the age of the asset and on the current rate of prevention expenditure. The objective considered is the maximization of the expected present value of revenues earned net of prevention and replacement costs. Three distinct cases are discussed: (a) when revenue is earned only until breakdown; (b) when there is automatic replacement following a breakdown; and (c) when there is the option of periodic replacement. Use of the Pontryagin Maximum Principle enables the determination of the optimal prevention schedules in all cases. In addition when periodic replacement is possible the optimal replacement interval can be determined. Numerical methods are required to obtain solutions.

1. INTRODUCTION

Consider a capital asset capable of producing revenue but subject to catastrophic breakdown or destruction. Suppose that the probability of breakdown depends on the age of the asset but that this probability can be reduced through expenditure on prevention or protection measures. The problem considered in this paper is the determination of the optimal pattern of prevention expenditure.

There are a number of possible situations where this problem might arise. For example the capital asset could be a machine or piece of equipment capable of producing revenue in an ongoing fashion, but subject to the possibility of a major breakdown which renders it useless. Money spent on preventive maintenance could reduce the probability of such breakdown. Alternatively the asset could be a living resource such as an orchard or vineyard, a dairy herd or flock of layer hens, capable of producing an ongoing revenue but subject to catastrophic destruction through pest or disease. The probability of such destruction could be reduced through protection measures such as sprayings, innoculations etc. The resource could also be a rangeland area, capable of earning a revenue during a certain part of the year through the rental of grazing rights, but susceptible to destruction by brush fire. The probability of such destruction could be reduced through fire protection measures.

In the above examples the revenue from the asset would be produced in an ongoing fashion with perhaps the rate of production depending on the age of the asset. This need not necessarily be the case. It could be that the revenue earned is an "instantaneous" terminal revenue rather than an ongoing one, which would be realized only when the asset was sold, harvested or slaughtered. A herd or flock of livestock raised for meat production but subject to destruction through disease would fall into this category. So also would an even-aged forest stand subject to the risk of catastrophic destruction through fire (see [11]). Revenue would be earned only when the stand was cut; expenditure on fire protection measures could reduce the probability of destruction through fire.

Finally it is possible that the asset be capable of producing both an ongoing revenue and a terminal revenue. An example of this would be an uneven-aged forest stand (see e.g. [1]) capable of providing an ongoing yield of timber, but which could also be clear-cut harvested to produce a terminal revenue.

A critical assumption of the paper is that the probability of destruction of the asset at any time depends only on the age of the asset and the current level of expenditure on prevention or protection, but not on past levels of expenditure. Such an assumption will likely be reasonable for the examples involving living assets described above; however it will impose some restrictions on the applicability of the model in the case where the asset is a machine. Another model in which the probability of destruction depends on the cumulative past maintenance expenditure subject to depreciation, will be considered in another paper ([12]) and may be more suitable in many instances for the case of preventive maintenance of a machine asset.

In this paper we first consider the problem of determining the pattern or schedule of protection or prevention expenditure which maximizes the expected net present value of the stream of revenues earned by the asset until such time as it suffers catastrophic destruction (or breakdown). Next we consider the situation of automatic replacement, in which the asset is replaced or reparied and overhauled whenever it suffers destruction or breakdown. The expenditure pattern which maximizes the expected net present value of revenues over many cycles of the process, is determined. Finally we consider the situation of periodic replacement in which the asset is replaced whenever it reaches a fixed age. In this case the optimal replacement age along with the optimal pattern of prevention expenditure is determined.

The models are formulated in continuous time and the optimization problem leads to an interesting application of the Pontryagin Maximum Principle (see e.g. [6]) in which the state of the system is a probability. The determination of the optimal solution involves the solution of a differential equation for which numerical methods must be employed. Numerical methods must be employed further in the problems with automatic and periodic replacement where the maximization takes place over infinitely many cycles of the process.

2. THE BASIC MODEL AND OPTIMIZATION PROBLEM

Suppose that when the asset is of age t, it can earn revenue at the rate $r(t)$ dollars per unit time, and that in the absence of prevention it is subject to the possibility of breakdown or destruction with the probability of this happening being characterized by a hazard function $h(t)$, where

$$(1)\qquad h(t) = \lim_{\Delta \to 0} \{P(\text{breakdown between ages } t \text{ and } (t+\Delta) \mid \text{no breakdown before age } t)/\Delta\}$$

Suppose that money is spent on protection or preventive maintenance at time t at the rate of $p(t)$ dollars per unit time, and that in consequence the effective hazard is reduced to $h_p(t)$, with

$$(2)\qquad h_p(t) = \psi(p(t))h(t)$$

where ψ is a function assumed to be decreasing and convex with $\psi(0) = 1$, which we shall call the response function. The resulting survivor function (or reliability function) $S_p(x)$, which gives the probability that the asset survives until age x is given by

$$(3)\qquad S_p(x) = \exp\left\{- \int_0^x h_p(t)dt\right\}$$

(see e.g. [9]) and the probability density function (p.d.f.), $f_p(x)$, of the time until breakdown, X, is

$$(4) \qquad f_p(x) = h_p(x)\ S_p(x) = h_p(x)\ \exp\left\{-\int_0^x h_p(t)dt\right\}$$

We shall refer to the function $p(t)$ as the <u>prevention schedule</u>.

The expected present value of revenues earned net of prevention expenditures, until the time the asset is destroyed is

$$(5) \qquad J = E_X\left\{\int_0^X e^{-\delta t}[r(t)-p(t)]dt\right\} = \int_0^\infty \int_0^x e^{-\delta t}[r(t)-p(t)]f_p(x)dt\ dx$$

where δ is the per annum instantaneous rate of discounting (related to the per annum yearly discount rate i by the formula $e^\delta = 1 + i$). After reversing the order of integration in the above integral and observing that $-S_p(x)$ is an anti-derivative of $f_p(x)$, J can be expressed as[1]

$$(6) \qquad J = \int_0^\infty [r(t)-p(t)]e^{-\delta t}\ S_p(t)dt$$

In order to find the optimal prevention schedule we seek a control function $p^*(t)$ to maximize (6) subject to the constraint (2) and the constraint $p(t) \geq 0$.

By making a simple variable transformation (see [11]) the above optimization problem can be re-expressed as a standard optimal control problem. Specifically let

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[1] The objective here is for revenue earned until breakdown, over an infinite time horizon. If, as in the case of a rangeland grazing resource, there is a finite time horizon, (e.g. the end of the grazing season), then the upper limit of the integrals in (5) and (6) will be finite.

(7) $$y(t) = -\log S_p(t),$$

so that (3) can be expressed

(8) $$y(t) = \int_0^t \Psi(p(x))h(x)dx,$$

which implies

(9) $$\frac{dy}{dt} = \Psi(p(t))h(t).$$

(Note that this is simply another way of writing (2), since $dy/dt = -S_p'(t)/S_p(t) = f_p(t)/S_p(t) = h_p(t)$, from (4)).

Upon replacing $S_p(t)$ by $e^{-y(t)}$ in the objective (6) we arrive at the following optimal control problem, with state variable y and control variable p:

maximize

(10) $$J(p) = \int_0^\infty [r(t)-p(t)]e^{-\delta t-y(t)}\, dt$$

subject to the dynamic equation

(11) $$\frac{dy}{dt} = \Psi(p)h(t)$$

and the constraint

(12) $$p(t) \geq 0 \quad \text{for} \quad t \geq 0.$$

To solve this control problem the <u>Pontryagin Maximum Principle</u> (see e.g. [6]) can be employed. To use the maximum principle we introduce an <u>adjoint variable</u>, $\lambda(t)$ and a <u>Hamiltonian function</u>

$$(13)\qquad H_t = [r(t)-p(t)]e^{-\delta t-y(t)} + \lambda(t)\Psi(p)h(t)$$

Necessary conditions for an optimum are that the variables p, y and λ satisfy the so-called adjoint-equation

$$(14)\qquad \frac{d\lambda}{dt} = -\frac{\partial H_t}{dy} = [r(t)-p(t)]e^{-\delta t-y(t)},$$

the dynamic equation (11), and that at all times $t \geq 0$, the control $p(t)$ maximizes the Hamiltonian H_t over the control set $\{p\colon p \geq 0\}$. From the concavity of H_t (in p) it follows that this last condition will be met at the solution to $\partial H_t/\partial p = 0$ if that solution is ≥ 0, otherwise it will be met at $p(t) = 0$. Now from (13) $\partial H_t/\partial p = 0$ is equivalent to

$$(15)\qquad \lambda(t) = \frac{e^{-\delta t-y(t)}}{\Psi'(p)h(t)}$$

provided, of course that $h(t) > 0$. Differentiating this equation with respect to t, and using the adjoint equation (14), to eliminate $d\lambda/dt$, provides an equation for dp/dt in terms of p, y and dy/dt. The dynamic equation (11) can be used to eliminate dy/dt, giving the following equation which the optimal control $p^*(t)$ must satisfy on regions where it is positive:

$$(16)\qquad \frac{dp}{dt} = -\frac{\Psi'(p)}{\Psi''(p)}[\delta + \Psi(p)h(t)] - \frac{\Psi'(p)}{\Psi''(p)}\cdot\frac{h'(t)}{h(t)} - \frac{[\Psi'(p)]^2}{\Psi''(p)}h(t)[r(t)-p(t)]$$

We shall call this equation the key equation.

Unfortunately we cannot simply solve the key equation to obtain the optimal prevention schedule. There are two reasons for this. Firstly we have no boundary condition specified, and secondly we have no guarantee that the optimal control $p^*(t)$ is positive for all $t \geq 0$. A boundary condition can be obtained by considering the optimization problem over a finite time horizon T, rather than over an infinite time horizon. Since there would be no constraint on the state of the system at time T (i.e. on $y(T)$) there would be present a free terminal value transversailty condition (see e.g. [6, p. 240])

$$(17) \qquad \lambda(T) = 0$$

Since there is no value of p that would make the right-hand side (r.h.s.) of (15) equal to zero, it follows that either $p^*(t) \equiv 0$ on $[0, T]$, or that there would exist a time t_0 such that the optimal control $p^*(t)$ would be identically zero on $[t_0, T]$, but would have $p^*(t_0^-) > 0$. The value of t_0 could be found numerically from the fact that on $[t_0, T]$, the optimal $y(t)$ and $\lambda(t)$ satisfy

$$(18) \qquad \frac{dy}{dt} = h(t),$$

and

$$(19) \qquad \frac{d\lambda}{dt} = r(t)e^{-\delta t - y(t)}$$

Solving this pair with the boundary conditions (17) and $y(t_0)$, and using the fact from (15) that

$$(20) \qquad \lambda(t_o^-) = \frac{e^{-\delta t_0 - y(t_0)}}{\Psi'(0)h(t_0)},$$

we obtain the following equation for t_0

$$(21) \qquad -\frac{e^{-\delta t_0 - H(t_0)}}{\Psi'(0)h(t_0)} = \int_{t_0}^{T} r(t)e^{-\delta t - H(t)}\,dt$$

where $H(t)$ is an anti-derivative of $h(t)$. In principle we can let $T \to \infty$ in the integral in the r.h.s. of (21) and then solve it for t_0. For example if $h(t) \equiv \nu$ and $r(t) = \rho e^{-\gamma t}$ then t_0 would satisfy

$$(22) \qquad e^{-\gamma t_0} = -\frac{\delta + \nu + \gamma}{\nu\rho\Psi'(0)}$$

Once t_0 has been determined the key equation (16) can be solved with the boundary condition $p(t_0) = 0$. If the resulting solution is positive on $[0, t_0)$, then it should provide the optimal prevention schedule (with $p^*(t) \equiv 0$ on $[t_0, \infty)$). However if the solution crosses the axis $p = 0$ again, then it will not provide the optimal solution on the whole of the interval $[0, t_0)$. Rather an argument of the above type will have to be used again to find the next interval where $p^*(t) \equiv 0$. Rather than pursue this further at this stage we shall consider some simple special cases for which the optimal prevention schedule can be fairly easily found.

(a) Constant hazard and constant revenue.

Consider the case when the revenue function and the hazard function are both constant in time:

$$(23) \qquad \begin{aligned} r(t) &\equiv \rho \\ h(t) &\equiv \nu, \end{aligned}$$

say. In this case the asset would produce revenue at a constant rate until a breakdown (occurring in a Poisson process) happened. The problem is time homogeneous. At any time $t \geq 0$, given that a breakdown has not occurred, one faces an identical problem in choosing the optimal current prevention expenditure. Thus $p^*(t)$ will be constant, (and equal to $\bar{p}$, say). One can determine the value of $\bar{p}$ from the key equation, since $dp^*/dt \equiv 0$. It follows from (16) that $\bar{p}$ satisfies

$$(24) \qquad \delta + \nu\Psi(\bar{p}) + \nu\Psi'(\bar{p})[\rho - \bar{p}] = 0$$

This equation can be solved, numerically if necessary, to obtain the optimal constant level of prevention expenditure.

For example if $\delta = 0.03$, and

$$(25) \qquad \Psi(p) = e^{-\beta p}$$

with $\beta = 0.1$ (which we shall assume in the numerical examples throughout the rest of this paper unless it is explicitly stated otherwise), and if $\rho = 1000$ and $\nu = 0.01$, we obtain from (24) $\bar{p} \equiv 34.609$. i.e. it is optimal to spend $34.61 per year of the $1000 per year revenue, on ongoing preventive maintenance or protection measures. The result of this is that the expected present value of the asset is increased from $25,000 to $31,846. If the asset were not susceptible to breakdown its present value would be $33,333. If the response to prevention expenditure is less strong (say, with $\beta = 0.01$) the optimal expenditure on prevention will be larger ($98.30 per year). In this case the resulting expected present value is $26,723.

(b) Constant hazard and constant revenue but with delay.

A slightly more complicated situation is when the revenue function is constant except for an initial period when it is zero. i.e.

$$(26) \qquad \begin{aligned} r(t) &= 0 && t \le t_1 \\ &= \rho && t > t_1 \end{aligned}$$

with again a constant hazard, $h(t) \equiv \nu$. Such a situation might prevail for a living resource such as an uneven-aged forest stand, for which there is an initial maturing period during which the asset is unproductive.

Clearly for $t > t_1$, the optimal prevention expenditure is again constant at the level $\bar{p}$ given by the solution to (24). For $t \le t_1$ we must solve the key equation

$$(27) \qquad \frac{dp}{dt} = -\frac{\varphi'(p)}{\varphi''(p)}[\delta + \nu\varphi(p)] + \nu\frac{[\varphi'(p)]^2}{\varphi''(p)}p$$

with the appropriate boundary (terminal) condition. Note that (27) is autonomous with a positive r.h.s. (from assumptions on φ). Thus $p^*(t)$ must either be increasing on the whole of $[0, t_1]$, or at first zero on some interval and then subsequently increasing on the remainder of $[0, t_1]$.

The boundary condition for p^* (at t_1) can be found by solving the adjoint (14) and the dynamic equation (11) on (t_1, ∞) with $p(t) \equiv \bar{p}$. This gives, from the continuity of λ and y,

$$(28) \qquad \lambda(t_1) = \lambda(t_1^+) = e^{-\delta t_1 - y(t_1)} \frac{\rho - \bar{p}}{\delta + \nu\varphi(\bar{p})}$$

Now the optimal $p^*(t_1)$ is given by solution to $\partial H_{t_0}/\partial p = 0$ if this is positive i.e. by the solution to:

$$(29) \qquad \lambda(t_1) = \frac{e^{-\delta t_1 - y(t_1)}}{\nu \Psi'(p)}$$

Equating (28) and (29) gives an equation for $p^*(t_1)$. Specifically

$$(30) \qquad -\Psi'(p^*(t_1)) = \frac{\delta + \nu \Psi(\bar{p})}{(\rho - \bar{p})\nu}$$

From (24) and (30) we see that $p^*(t_1) = \bar{p}$. Thus the optimal $p^*(t)$ is continuous at $t = t_1$. Solution of the key equation (27) with the boundary condition (30) can be carried out numerically. This has been done for the parameter values given above, and with $t_1 = 5$ years. Solution was carried out by means of a Runge-Kutta method (see e.g. [3, p. 366]) using NAG Library [2] software. Figure 1 shows the optimal prevention expenditure corresponding to a stronger response ($\beta = 0.1$) and a less strong response ($\beta = .01$). It can be seen that over the initial nonproductive interval (0 to 5 years) the optimal prevention expenditure is only slightly less than the steady-state rate of expenditure. With $\beta = 0.1$ the expected present value with optimal prevention is \$27,207 whereas with no prevention it is \$20,207.03. If there were no risk of breakdown the present value of the asset would be \$28,690.

3. AUTOMATIC REPLACEMENT

In the previous section we considered the maximization of the present value of revenues net of prevention costs, until the asset suffered catastrophic destruction or breakdown. In this section we shall assume that once the asset breaks down or is destroyed, it is immediately replaced or repaired to a "good-as-new" state. This replacement will of course cost something. We shall consider the problem of maximizing the net present value over infinitely many cycles of breakdown and replacement.

Let $J_{\max}$ denote the optimal expected net present value over infinitely many cycles. Then clearly $J_{\max}$ satisfies the Bellman-type equation

$$(31) \qquad J_{\max} = \max_{p(t)} \left\{ E_X \left[\int_0^X e^{-\delta t} [r(t)-p(t)] + E_X \left[e^{-\delta X} \right] \cdot [J_{\max}-C] \right\} \right.$$

where C is the cost of replacement or repair, and X (which depends on $p(t)$) is a random variable denoting the time at which breakdown occurs. Using the technique used in deriving (6) in the last section, and using the fact that, (from integration by parts),

$$(32) \qquad E\left[e^{-\delta X}\right] = \int_0^\infty e^{-\delta x} f_p(x)dx = 1 - \delta \int_0^\infty e^{-\delta x} S_p(x)dx,$$

equation (31) can be written

$$(33) \qquad C = \max_{p(t)} \left\{ \int_0^\infty \left[r(t)-p(t)-\delta \left[J_{\max}-C \right] \right] S_p(t) e^{-\delta t} \; dt \right\}$$

This equation can be solved numerically for $J_{\max}$ provided that the maximization problem,

maximize:

$$Q_L(p) = \int_0^\infty [r(t)-p(t)-\delta(L-C)]S_p(t)e^{-\delta t}\,dt \tag{34}$$

over functions $p(t)$ and subject to (2) and (12)

can be solved for any value of L. This maximization problem is essentially like the one solved in the last section, the only change being that $r(t)$ is replaced by $r(t) - \delta(L-C)$. Similar solution methods can thus be employed.

We consider the same special cases discussed in the previous section.

(a) <u>Constant hazard and constant revenue</u>.

For a given value of L and constant hazard and revenue functions as given in (23), $Q_L(p)$ is maximized in (34) by $p^*(t) \equiv \bar{p}_L$ where $\bar{p}_L$ satisfies

$$\delta + \nu\Psi(\bar{p}_L) + \nu\Psi'(\bar{p}_L)\left[\rho - \delta(L-C) - \bar{p}_L\right] = 0 \tag{35}$$

By carrying out the integration in (34) we get that the optimal value of $Q_L(p)$ is

$$Q_L(\bar{p}_L) = \frac{\rho - \delta(L-C) - \bar{p}_L}{\delta + \nu\Psi(\bar{p}_L)} \tag{36}$$

It follows from (33) and (36) that $J_{\max}$ and $\bar{p}_L$ are given by the values of L and $\bar{p}_L$ which solve (35) and

$$(37) \qquad C = \frac{\rho - \delta(L-C) - \bar{p}_L}{\delta + \nu\varphi(\bar{p}_L)}$$

From (35) and (37) we get that $\bar{p}_L$ is given by the solution to

$$(38) \qquad \varphi'(\bar{p}_L) = -1/\nu C$$

Using the numerical values given in the example in the last section with in addition the cost of replacement set at $C = \$2000$, we get $p^*(t) \equiv \$6.93$ per year. This should be compared to the constant prevention expenditure of $34.61 per year when there is no automatic replacement. Such a reduction in prevention expenditure when there is automatic replacement will always take place since the effective revenue function $r(t)$ (in (10)) is reduced to $r(t) - \delta(L-C)$ (in (34)). The expected present value of the asset under optimal prevention with automatic replacement is $32,768. This compares with the optimal expected present value of $31,846 when there is no replacement.

It should be noted that the optimal prevention schedule given by (38) does not depend on the constant revenue earning rate ρ. This makes sense because it is assumed that replacement/repair takes place instantaneously and thus one earns constantly at the rate ρ regardless of breakdowns. In effect one seeks a prevention schedule to minimize the total costs of repair or replacement following breakdowns plus the costs of prevention.

(b) Constant hazard and constant revenue but with delay.

For the revenue function given in (26) and for a given value of L, the prevention schedule to maximize (34) is, for $t \geq t_1$, given by $p^*(t) \equiv \bar{p}_L$ where $\bar{p}_L$ satisfies (35). For $t \leq t_1$ the prevention schedule to maximize

(34) is obtained by solving the key equation

$$(39) \qquad \frac{dp}{dt} = -\frac{\Psi'(p)}{\Psi''(p)}\,[\delta + \nu\Psi(p)] + \nu\,\frac{[\Psi'(p)]^2}{\Psi''(p)}\,[p(t)+\delta(L-C)]$$

on $[0, t_1]$ with boundary condition $p(t_1) = \bar{p}_L$. In order to determine the optimal value of the objective $Q_L(p)$ in (34) it is necessary to determine $S_p(t)$ (or $y(t)$). One way to do this is to solve (39) and (11) (with $h(t) = \nu$) as a two-point boundary-value problem with boundary conditions $y(0) = 0$ and $p(t_1) = \bar{p}_L$. An alternative method is to obtain the solution to (39) at a fine grid of values in the interval $[0, t_1]$, and then use these values to obtain a numerical solution to (11). The value of $Q_L(p)$ can then be obtained using for example Simpson's Rule (see e.g. [6, p. 219]) or some refinement of it.

Since the above procedure allows us to determine numerically the r.h.s. of (33) for any given value of J_{max}, the equation (33) can be solved iteratively for J_{max}, using a method such as the Secant Method (see e.g. [6, p. 48]). The optimal prevention schedule $p^*(t)$ is given by the solution to (39) with L set at J_{max}.

This procedure has been carried out using the parameter values described in (b) of Section 2, with $\beta = 0.1$. The boundary value problem was solved using a shooting method (see e.g. [8]) and the numerical integration was performed using a method of Gill and Miller [7]. NAG Library [2] software was again employed. The maximum expected net present value of the resource is $J_{max} = \$27,750.70$. This compares with an expected present value of \$27,207.03 when there is no automatic replacement. The optimal prevention schedule is as shown in Fig. 2 (trajectory (a)). Also shown is the optimal prevention schedule with no replacement (trajectory (b)). Again note how, when automatic replacement is operative, the optimal prevention schedule is lower. With automatic replacement, when the asset is productive $(t > t_1)$, the hazard, with optimal prevention in place is $h_p(t) \equiv .00151$ whereas with no replacement the greater level of prevention under optimal management leads to a reduced hazard of $h_p(t) = .000314$.

An interesting feature of the above solution is that the optimal <u>initial</u> prevention expenditure $p^*(0) = \$6.93$ is the same as in the case of constant revenue (case (a) above). In fact, when there is automatic or periodic replacement it can be shown by integrating the adjoint equation, that $\lambda(0) = C$ for any revenue function and any hazard function. In consequence the optimal initial prevention expenditure $p^*(0)$ is given by the solution to

$$(40) \qquad \Psi'(p) = -1/ch(0)$$

if this is positive, otherwise it is zero. This fact provides an alternative method of determining the optimal prevention schedule. One can solve (39) and (11) as an <u>initial value problem</u> with the above initial value for p, and $y(0) = 0$; then one can use numerical integration to evaluate $Q_L(p)$ in (34). Since this procedure can be performed for any value of L, the solution to $Q_L(p) = C$ (which provides J_{max}) can be determined by, for example, the Secant Method. The solution to (39) with $L = J_{max}$ will provide the optimal protection schedule.

The condition (40) for the optimal initial expenditure can be explained in the following way: if there is a breakdown and replacement at time $t = 0$, the only effect is the incurrence of a cost C for replacement; there is no change in the current and future hazard, nor in the rate at which current and future revenues can be earned. Since the probability of breakdown at time $t = 0$, is

$\Psi(p(0))h(0)$, the *expected* loss due to a breakdown at $t = 0$ is $-C\Psi(p(0))h(0)$. The condition (40) says that the optimal initial expenditure $p^*(0)$ is at the level where the marginal decrease in the expected loss due to a breakdown at time $t = 0$, corresponding to a \$1 increase in prevention expenditure, is exactly equal to one dollar. If such a trade-off cannot be achieved, the initial expenditure is zero.

4. PERIODIC REPLACEMENT

In this section we assume that the owner of the asset controls not only the prevention policy, but in addition the replacement policy for the asset. Specifically we shall assume that the owner can choose a replacement age T, and that the asset is replaced whenever it reaches age T, or immediately following a breakdown, if this occurs before age T. We shall suppose that there is some "scrap value" $R(T)$ for the asset, when it is replaced at age T. Again it will be assumed that the cost of replacement is C. Note that if the revenue function is set at $r(t) \equiv 0$, then the situation in which all revenue earned is as a terminal revenue, $R(T)$ prevails. This would correspond to the situations, discussed in the introduction, of livestock raised for meat production, and of an even-aged forest stand (see [11]) grown for timber production. The revenue function $r(t)$ could also be set negative to cover ongoing costs (other than prevention costs), such as those for feed, fertilizer, land rental etc.

The present value of the net revenue earned over one cycle (from immediately following one replacement to immediately following the next, regardless of whether that occurs after a breakdown or not) is

$$(41) \qquad \pi(T, p) = \int_0^X [r(t)-p(t)]e^{-\delta t} - ce^{-\delta X}, \quad \text{if} \quad X < T$$

$$= [R(T)-C]e^{-\delta T} + \int_0^T [r(t)-p(t)]e^{-\delta t}\, dt, \quad \text{if} \quad X = T$$

where X is a random variable representing the length of the cycle and with distribution function

$$(42) \qquad F_p(x) = P(X \leq x) = 1 - \exp\left\{-\int_0^x h_p(t)dt\right\} \qquad x < T$$
$$= 1 \qquad x \geq T.$$

Clearly the revenue $\pi(T, p)$ in (41) is a random variable. Its expected value $\bar{\pi}(T, p)$ can be obtained by integrating with respect to the distribution function (42). Performing this, and after doing some simplification one obtains

$$(43) \qquad \bar{\pi}(T, p) = [R(T)-C]e^{-\delta T} S_p(T) - C\left[1 - e^{-\delta T} S_p(T)\right] + \int_0^T [r(t)-p(t)+\delta C]e^{-\delta t} S_p(t)dt$$

If we let $J_{\max}(T)$ denote the optimal expected net present value over infinitely many cycles, but always using the replacement age T, then we have as before,

$$(44) \qquad J_{\max}(T) = \max_{p(t)} \left\{\bar{\pi}(T, p) + E_X\left[e^{-\delta X}\right]J_{\max}(T)\right\}$$

Now

$$(45) \qquad E\left[e^{-\delta X}\right] = 1 - \int_0^T \delta e^{-\delta x} S_p(x)$$

and thus (44) can be expressed as

$$(46) \qquad C = \max_{p(t)} \left\{ \int_0^T \left[r(t)-p(t)-\delta\left[J_{\max}(t)-C\right]\right] e^{-\delta t} \ S_p(t)dt + R(T)e^{-\delta T} \ S_p(t) \right\}$$

As before this can be solved numerically for $J_{\max}(T)$ provided that the maximization problem,

maximize

$$(47) \qquad Q_{L,T}(p) = \int_0^T [r(t)-p(t)-\delta(L-C)]S_p(t)e^{-\delta t} + R(T)e^{-\delta T} \ S_p(T)$$

subject to (2) and (12),

can be solved for any value of L. If this can be done, any maximization routine, that does not require the use of derivatives, can be used to find the value of T that maximizes $J_{\max}(T)$ over $T \geq 0$. This will provide the optimal replacement age T^*. The solution to the maximization problem in (47) with $T = T^*$, $L = J_{\max}(T^*)$ will provide the optimal prevention schedule.

Solution to the problem thus revolves around being able to solve the optimization problem of maximizing (47) subject to (2) and (12).

By carrying out the same transformation as in Section (equation (7)), this problem can be expressed as:

maximize

$$(48) \qquad Q_{L,T}(p) = \int_0^T [r(t)-p(t)-\delta(L-C)]e^{-\delta t-y(t)}dt + R(T)e^{-\delta T-y(T)}$$

subject to the dynamic equation (11)

and the constraint (12).

This optimal control problem is similar to that addressed in Section 3 except for the fact that a finite time horizon T, with a terminal payoff

$R(T)e^{-\delta T-y(T)}$, (which depends on the terminal state, $y(T)$) is in effect. The Pontryagin Maximum Principle can be used to obtain a solution as before. The transversality condition, corresponding to the terminal payoff is (see e.g. [6])

$$(49) \qquad \lambda(T) = \frac{\partial}{\partial y(T)}\left[R(T)e^{-\delta T-y(T)}\right] = -R(T)e^{-\delta T-y(T)}$$

Provided that the optimal prevention schedule is always positive, it can be determined by solving the key equation given by (16) only with $r(t)$ replaced by $r(t) - \delta(L-C)$, and with boundary condition

$$(50) \qquad \Psi'(p(T)) = -[h(T)R(T)]^{-1}$$

The optimal value of the objective $Q_{L,T}$ in (48) can then be obtained by the method described in the previous section.

If the optimal prevention schedule is identically zero on some segments of the interval $[0, T]$ one cannot obtain the solution directly from the key equation. Methods along the lines discussed in Section 2 must be employed. For example if there is no non-negative value of p to solve (50), then $p^*(t)$ must be identically zero on some interval $[t_0, T]$. Thus the adjoint equation on $[t_0, T]$ is

$$(51) \qquad \frac{d\lambda}{dt} = [r(t)-\delta(L-C)]e^{-\delta t-y(t)}$$

Solving this with boundary condition (49) and using the fact that $\lambda(t_0)$ is given by (20), the following equation for t_0 can be deduced (c.f. (21))

$$(52) \qquad -\frac{e^{-\delta t_0 - H(t_0)}}{\Psi'(0)h(t_0)} = R(T)e^{-\delta T - H(T)} + \int_{t_0}^{T} [r(t) - \delta(L-C)]e^{-\delta t - H(t)}dt$$

This can be solved numerically for t_0, and then the key equation with boundary condition $p(t_0) = 0$ can be solved on $[0, t_0]$. If the solution is always positive it will provide the optimal solution. If it crosses the axis $p = 0$ in the interval $[0, t_0]$, then a similar method to the above must be employed to determine the next segment on which $p(t) \equiv 0$.

The examples (a) and (b) of the last two sections, in which the asset can earn revenue at a constant rate, after possibly an initial non-productive maturing period, are not really suitable to illustrate the model with periodic replacement. The reason for this is that, except in exceptional cases where the scrap-value $R(T)$ varies greatly with age T, there would be no point in replacing the asset all the time that it is functioning and producing revenue. It would be optimal to replace the asset only when it breaks down. Thus the optimal replacement age T would be infinite.

To illustrate the methods described in this section we have chosen a different example.

(c) <u>Constant hazard but revenue function with maturing and senescence</u>.

The hazard function is assumed constant, $h(x) \equiv \nu$, with in the example $\nu = .01$. The revenue function assumed is piecewise defined with

$$(53) \qquad \begin{aligned} r(t) &= 200t, && \text{if } t \leq 5 \\ &= 1000, && \text{if } 5 < t \leq 10 \\ &= 1000\exp(-\tfrac{1}{2}(t-10)), && \text{if } 10 < t. \end{aligned}$$

The scrap value used is given by the function

$$(54) \qquad R(T) = 1500 \exp(-.09242T)$$

This corresponds to an immediate drop of 25% in the value of the asset once it is installed. Subsequently its value declines exponentially with a "half-life" of 7.5 years. Figure 3 shows the optimal expected net present value $J_{\max}(T)$ as a function of replacement age T, calculated using the methods described in this section. The optimal replacement age is $T^* = 11.13$ years. The corresponding optimal prevention schedule is shown in Fig. 4 (curve (a)) along with the corresponding survivor function (curve (b)). It can be seen that it is optimal to protect the asset during the earlier part of its life, but that the optimal prevention expenditure drops to zero, shortly before the end of its period of maximum productivity. The time at which prevention expenditure is greatest is shortly before the asset reaches its period of greatest productivity. The probability that there is no breakdown before the replacement age of 11.13 years is 0.941. With no prevention in place this probability would be 0.895.

Further examples of periodic replacement with $r(t) \equiv 0$, both with constant and non-constant hazard functions $h(t)$, are given in [11]. In that paper $R(T)$ is an increasing function representing the per hectare value of an even-aged forest stand of age T.

5. DISCUSSION AND CONCLUSIONS

In this paper the problem of determining the optimum schedule of expenditure on preventive maintenance or protection for a revenue earning asset subject to catastrophic breakdown has been discussed. Techniques have been presented for determining the solution in three cases: (a) when revenue is earned only until breakdown; (b) when there is automatic replacement, at a cost, following breakdown; and (c) when there is the option of periodic replacement.

In all cases it is shown how the solution revolves around a first order differential equation, termed the key equation, derived from the Pontryagin Maximum Principle. In the case when the optimal prevention policy involves a positive expenditure at all times, solution to the key equation can provide the optimum policy. However in other cases when the optimal prevention expenditure is zero for some intervals, more complex procedures must be employed. In all cases solution to the key equation requires in general, the use of numerical techniques. Furthermore, in the case when there is replacement, either automatic or periodic, iterative numerical methods must be further employed.

The model and methods developed in the paper lend themselves to extension in a number of ways. For example it could be assumed that the probability of a breakdown depends on the cumulative amount of expenditure to date, rather than on the current rate of expenditure. This leads to a more complicated optimal control problem with second-order dynamics (see [12]). Also it could be assumed that a complete breakdown would occur only after the asset has suffered a fixed number of shocks, occurring according to some hazard function. In this case the case the optimal prevention expenditure would depend on the number of shocks to date, in addition to the age of the asset.

While the literature in Reliability Theory contains a number of papers on optimal preventive maintenance (see e.g. [5], [10] and [4]), a basic assumption of that work appears to be that maintenance is carried out at discrete times. A novelty of this paper is that the expenditure on prevention or protection is assumed to occur continuously. This allows the problem to be expressed as one of dynamic optimal control with the state of the system being the probability of survival. Further applications of this idea in the area of reliability, preventive maintenance etc. may well be possible.

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FIGURE CAPTIONS

Figure 1. Optimal prevention schedules for the case (b) discussed in the text, of a constant revenue function but with delay. Curve (A) corresponds to a stronger response to protection expenditure (β = .01) while curve (B) corresponds to a weaker response (β = 0.01). Units of the prevention schedule are \$/year, and units of age are years.

Figure 2. Optimal prevention schedules for the case (b) discussed in the text, of a constant revenue function but with delay. Curve (A) corresponds to the case of automatic replacement, while curve (B) corresponds to the case of no replacement. In both cases the response parameter β is 0.1. Units of the prevention schedule are \$/year, and units of age are years.

Figure 3. The optimal expected net present value $J_{max}(T)$ using periodic replacement, as a function of replacement age, T. The revenue function and hazard function are as described under (c) in the text. Units of present value are dollars and units of replacement age are years.

Figure 4. The optimal prevention schedule and corresponding survivor function for the optimal replacement age of 11.1 years, and the revenue and hazard functions described under (c) in the text. Units of the prevention schedule are \$/year; units of age are years; survival probability is dimensionless.

Fig 1

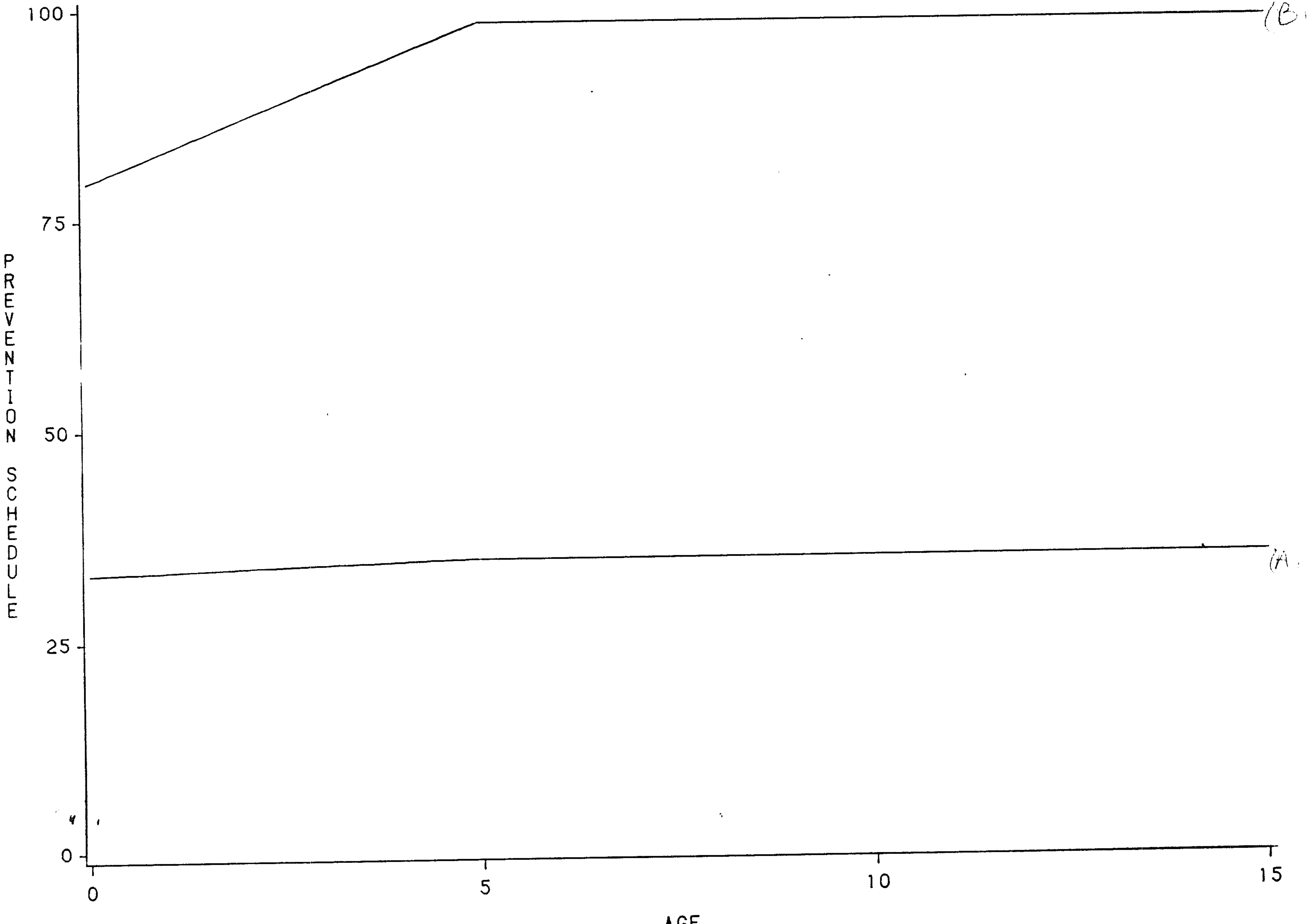


Fig 2

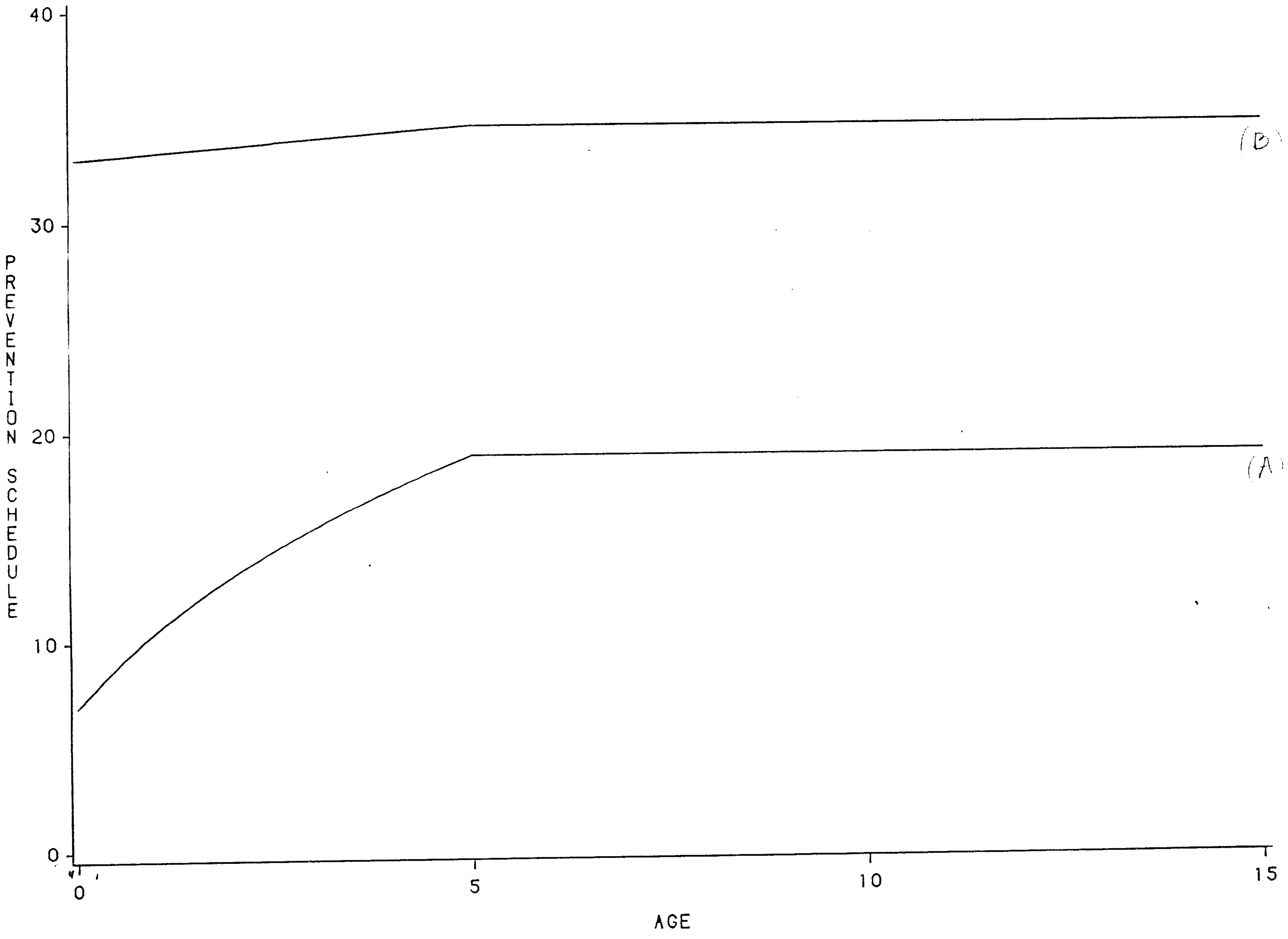


Fig 3

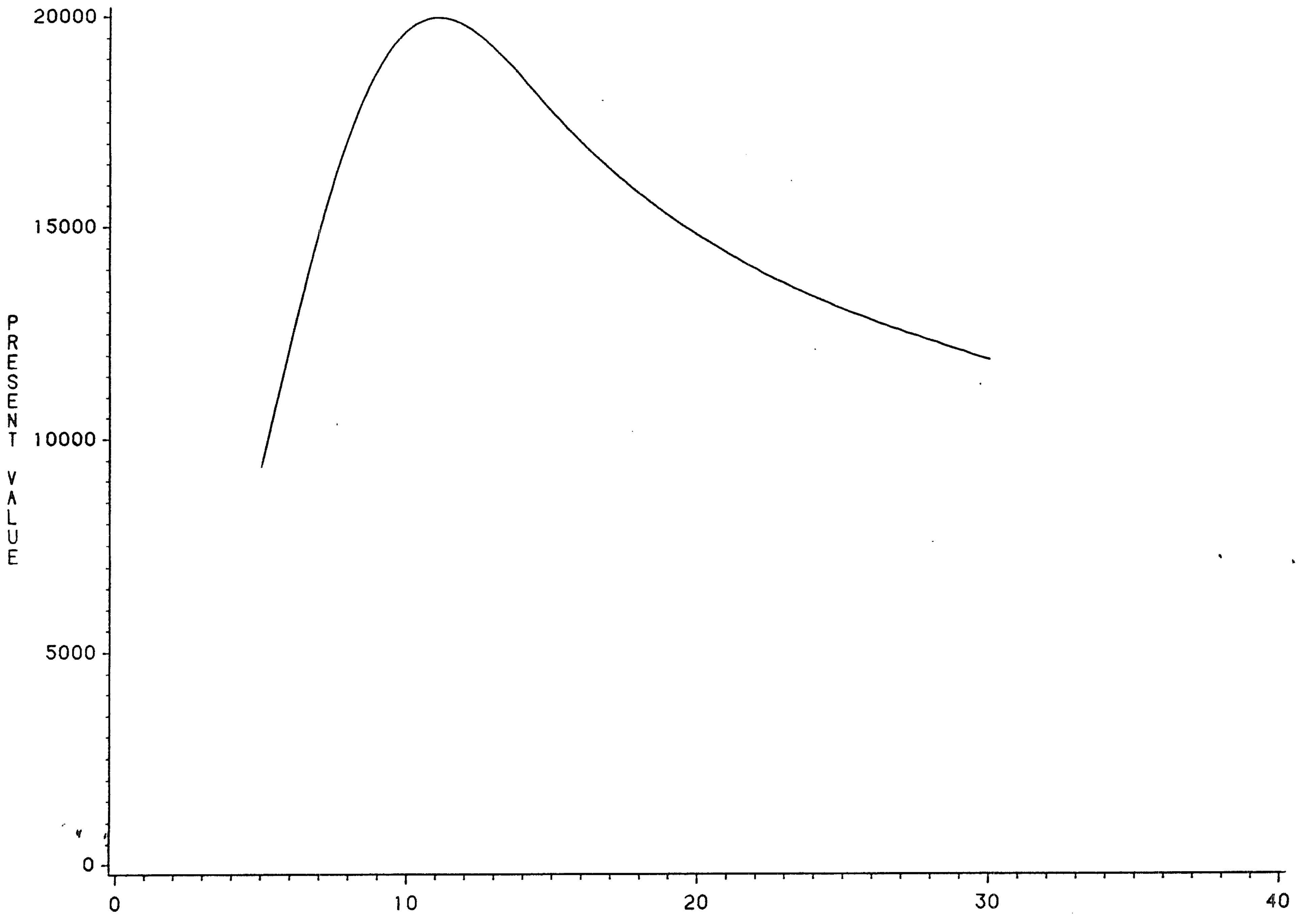
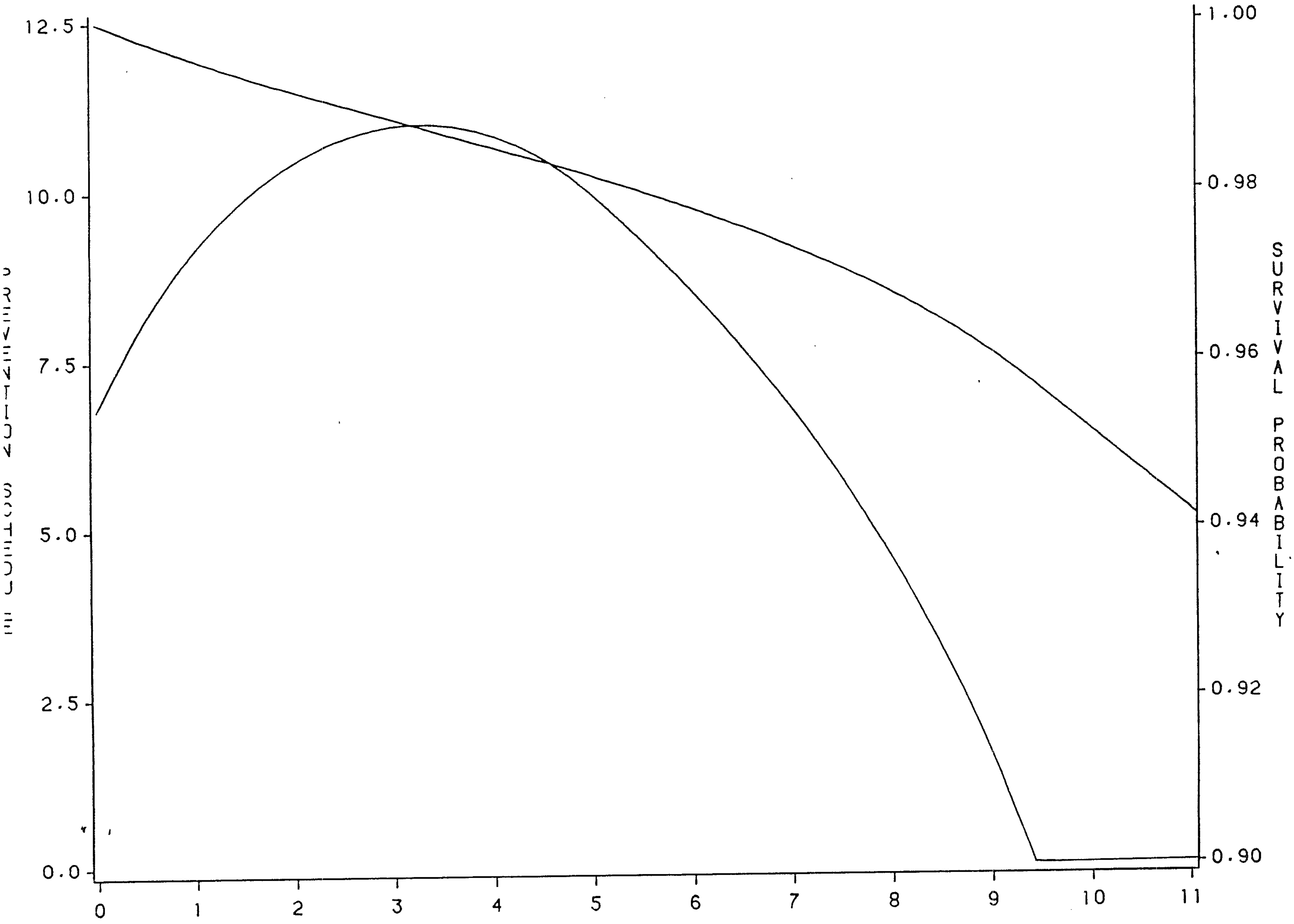


Fig 4


PREVENTION COSTS E
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10.0
7.5
5.0
2.5
0.0
0
1
2
3
4
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8
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11
1.00
0.98
0.96
0.94
0.92
0.90
SURVIVAL PROBABILITY