DECOMPOSITION OF THE FREE (ϵ, G)-LIE COLORALGEBRA

by

Paul Sadro

B.Sc., Simon Fraser University, 1998

A Thesis Submitted in Partial Fullfillment of the
Requirements for the Degree of
MASTER OF SCIENCE
in the Departement of Mathematics and Statistics

We accept this thesis as conforming
to the required standard

[redacted]

Dr. C. R. Miers, Supervisor (Department of Mathematics and Statistics)

[redacted]

Dr. J. Phillips, Department Member (Department of Mathematics and Statistics)

[redacted]

Dr. A. Sourour, Department Member (Department of Mathematics and Statistics)

[redacted]

Dr. W. S. Martindale 3rd, External Examiner (University of Massachusetts)

University of Victoria

Supervisor: Dr. C. R. Miers

Abstract

The Hall basis of the free Lie algebra has inspired extensive research into the module properties of the free Lie algebra. The (ϵ, G)-Lie coloralgebra is a generalization of the Lie algebra. Naturally, many of the results for Lie algebras can be generalized for (ϵ, G)-Lie coloralgebras. In particular, the free (ϵ, G)-Lie coloralgebra and an analogue of the Hall basis can be constructed.

Examiners:

[redacted]

Dr. C. R. Miers, Supervisor (Department of Mathematics and Statistics)

[redacted]

Dr. J. Phillips, Department member (Department of Mathematics and Statisitics)

[redacted]

Dr. A. Sourour, Department member (Department of Mathematics and Statisitics)

[redacted]

Dr. W. S. Martindale 3rd, External Examiner (University of Massachusetts)

Table of Contents

Acknowledgements

I would like to thank the Department of Mathematics and Statistics at the University of Victoria for providing me with the opportunity to continue studying mathematics and supporting me in my studies. In particular, I would like to express my gratitude to my supervisor, Dr. C. R. Miers for his instruction, constructive criticism and patience during the past two years. I would also like to thank my former instructors Dr. J. Phillips and Dr. A. Sourour and Dr. W. S. Martindale for agreeing to take part in the examination of this thesis. Finally, my parents deserve more thanks than I could possibly give them for everything.

To whom it may concern

Chapter 1

Preliminaries

In this chapter we define the algebraic objects required to define and construct a free (ϵ, G)-Lie coloralgebra. We will also state some basic results which are needed in the chapters to follow. Much of the material in this chapter can be found in [2] and [7].

1.1 Magmas

A *magma* is a set M with a binary operation or product $(,) : M \times M \longrightarrow M$. M is called *associative* when $((x,y),z) = (x,(y,z))$ for all elements $x, y, z \in M$ and *commutative* or *abelian* when $(x,y) = (y,x)$ for all $x, y \in M$. If M and M' are magmas, then $h : M \longrightarrow M'$ is a *magma homomorphism* if and only if $h((x,y)) = (h(x), h(y))$ for all $x, y \in M$. A magma M is *freely generated* by a set A if and only if $A \subseteq M$ and every map from A into a magma M' extends uniquely to a magma homomorphism from M into M'.

Let A be a nonempty set. We construct a magma $M(A)$ which is freely generated by A. The sets $M^k(A)$ for $k > 0$ are defined recursively by letting $M^1(A) = A$ and $M^n(A) = \bigcup_{0<j<n} M^j(A) \times M^{n-j}(A)$ when $n > 1$. For any $x \in M^p(A)$ and $y \in M^q(A)$, let (x,y) denote the ordered pair of x and y in $M^{p+q}(A)$. $M(A)$ is defined as the set $\bigcup_{n>0} M^n(A)$ with the binary operation $(\cdot, \cdot)$. That is, the product of x and y is the ordered pair (x, y) for all $x, y \in M(A)$. Note that this binary operation is injective. For every $x \in M(A)$, let $|x|$ be the *length of* x which we define as the unique integer $n > 0$ such that $x \in M^n(A)$. Thus $|\cdot| : M(A) \longrightarrow N^+$ is a magma homomorphism where N^+ denotes the positive integers under addition. Let M be a magma and f be a mapping of A into M. The functions $f_k : M^k(A) \longrightarrow M$ for $k > 0$ are recursively defined as $f_1 = f$ and $f_n((x, y)) = (f_p(x), f_q(y))$ for all $(x, y) \in M^p(A) \times M^q(A) \subseteq M^n(A)$ when $n > 1$. The function $\overline{f} : M(A) \longrightarrow M$ defined as $\overline{f}(x) = f_{|x|}(x)$ for all $x \in M(A)$ is a magma homomorphism. Suppose that $h : M(A) \longrightarrow M$ is magma homomorphism extending f which does not agree with $\overline{f}$ on all of $M(A)$. Thus we have $x \in M(A) \setminus A$ such that $h(x) \neq \overline{f}(x)$ and $h(w) = \overline{f}(w)$ for all $w \in M(A)$ with $|w| < |x|$. Thus there exists $y, z \in M(A)$ such that $|y|, |z| < n$ and $x = (y, z)$ which implies that $h(x) = (h(y), h(z)) = (\overline{f}(y), \overline{f}(z)) = \overline{f}(x)$. This is a contradiction. Hence $\overline{f}$ is unique.

1.2 Monoids

An element 1 contained in a magma M is called an *identity element* if and only if $(1, x) = (x, 1) = x$ for every element $x \in M$. Magmas that contain an identity element are called *unital magmas*. A *monoid* is an associative unital magma. When the symbol 1 is used, it will always denote the identity element in a magma. There are cases where the identity element will be denoted by a different symbol. For example when we consider a monoid with addition as the binary operation, we call the identity element zero and denote it by the symbol 0. A map $h : M \longrightarrow M'$ between two monoids is called a *monoid homomorphism* if and only if it is a magma homomorphism mapping the identity element of M to the identity element of M'. A monoid M is *freely generated* by a set A if and only if $A \subseteq M$ and every map from A into a monoid M' extends uniquely to a monoid homomorphism from M into M'.

Let A be a nonempty set. We will construct a monoid A^* which is freely generated by A. We call A an *alphabet* and the elements of A *letters*. The set of all *words on A* is the set of all finite sequences of letters including the empty sequence. The *length* of a sequence $w \in A^*$ is denoted by $|w|$. Note that every letter is a word of length one. For any words w and v on A we denote the product of w and v by wv and define it as the word of length $|w| + |v|$ which agrees with w for the first $|w|$ letters and agrees with v for the last $|v|$ letters. Let A^* denote the magma consisting of the set of all words on A with this product. Clearly, A^* is a monoid with the empty sequence as the identity element. A^* is freely generated by A. Indeed, for every map f of A into a monoid M we can define a monoid homomorphism $\overline{f} : A^* \longrightarrow M$ such that $\overline{f}(1) = 1$ and $\overline{f}(a_1 \cdots a_n) = f(a_1) \cdots f(a_n)$ for all $\emptyset \neq \{a_1, \cdots, a_n\} \subseteq A$. Clearly, $\overline{f}$ extends f and is unique because any monoid homomorphism $h : A^* \longrightarrow M$ satisfies $h(1) = 1$ and $h(a_1 \cdots a_n) = h(a_1) \cdots h(a_n)$ for all $\emptyset \neq \{a_1, \cdots, a_n\} \subseteq A$.

Many common algebraic objects are monoids. For example a *group* is a monoid with the additional property that every element has an inverse. That is, for every element x there exists an element y such that $xy = yx = 1$. A *commutative ring with unity* is a set R with two binary operations addition, $+$, and multiplication, $*$, such that R is an abelian group under $+$, a commutative monoid under $*$ and $x*(y+z) = x*y+x*z$ for all $x, y, z \in R$. We will often denote $x * y$ by xy.

1.3 K-modules

Let K be a commutative ring with unity We define a $K-module$ W as an abelian group under addition such that the elements of K are unary operations on W satisfying the following four properties

$$a(x+y) = a(x) + a(y) \tag{1.3.1}$$

$$(a+b)(x) = a(x) + b(x) \tag{1.3.2}$$

$$(ab)(x) = a(b(x)) \tag{1.3.3}$$

$$1(x) = x \tag{1.3.4}$$

for all $x, y \in W$ and $a, b \in K$ When no confusion can arise we write $a(x) = ax$ for $a \in K$ and $x \in W$ The simplest example of a K-module is K itself where every operation $x \in K$ is defined as $x(y) = xy$ for all $y \in K$ A *$K-$module homomorphism* or *$K-$linear mapping* is a monoid homomorphism h from a K-module W into a K-module such that $h(ax) = ah(x)$ for all $x \in W$ Note that K-module homomorphisms always map zero to zero A *submodule* V of a K-module W is a K-module consisting of a subset V of W with the operations of the K-module W on V This is equivalent to saying that $V \subseteq W$ and $kx - y \in V$ for all $x, y \in V$ and $k \in K$ If $h \quad V \longrightarrow W$ is a K-linear map, then the inverse image of a submodule of W is a submodule of V and the image of a submodule of V is a submodule of W The inverse image of the submodule $\{0\}$ is called the *kernel of h* Let S be a submodule of the K-module T and T/S be the set of all equivalence classes obtained from the equivalence relation $\sim$ on T defined as $x \sim y$ if and only if $x - y \in S$ For each $x \in T$, $\widetilde{x}$ denotes the equivalence class containing x Note that $\sim$ is a congruence on T Indeed, if $x \sim x'$ and $y \sim y'$, then $kx + y \sim kx' + y'$ for all $k \in K$ Thus T/S is a K-module under the following operations:

$$k\widetilde{x} = \widetilde{kx}$$

$$\widetilde{x} + \widetilde{y} = \widetilde{x+y}$$

for all $x, y \in T$ and $k \in K$. T/S is called the *quotient module of T by the submodule S*. The canonical map $f : T \longrightarrow T/S$ defined as $f(x) = \widetilde{x}$ for all $x \in T$ is called the *quotient map of T/S*. Note that f is K-linear and $kernel(f) = S$.

The following proposition is known as the Homomorphism Theorem for K-modules.

Proposition 1.3.1 *Let $f : T \longrightarrow T/S$ be the quotient map of T/S. For every K-module R and K-linear map $h : T \longrightarrow R$ such that $S \subseteq kernel(h)$, there exists a unique K-linear map $\overline{h}$ such that $\overline{h}(f(x)) = h(x)$ for all $x \in T$.*

$$\begin{array}{ccc} T & \xrightarrow{f} & T/S \\ \downarrow h & & \downarrow \overline{h} \\ R & \xrightarrow{id} & R \end{array}$$

Proof. This result is an application of Corollary 3.8 in Chapter 2 of [4] to K-modules. Alternatively, this proposition is explicitly stated as Theorem 5.9 in Chapter 5 of [9]. □

Whenever we write $\sum_{x \in X} k_x x$ for some subset X of a K-module it will be understood that this is a finite sum, that is $k_x x = 0$ for all but finitely many $x \in X$ and $\sum_{x \in X} k_x x$ equals to zero plus the sum of all the nonzero summands. Note that when X is empty $\sum_{x \in X} k_x x = 0$. A K-module V is a *direct sum* of a set $\{V_j : j \in J\}$ of submodules of V if and only if $V = \bigoplus_{j \in J} V_j$ which means that every $x \in V$ can be expressed uniquely as a sum $\sum_{j \in J} x_j$ such that $x_j \in V_j$ for all $j \in J$, the map $proj_j : V \longrightarrow V_j$ defined as $proj_j(x) = x_j$ is K-linear. The set of *homogeneous* elements of V are the elements contained in $\bigcup_{j \in J} V_j$. Note that for any $x \in V$, $x = \sum_{j \in J} proj_j(x)$. A binary operation or function $f : V \times V \longrightarrow V$ on a K-module

V is called $K-$ *bilinear* if and only if for all $x \in V$ the maps $y \mapsto f(x,y)$ and $y \mapsto f(y,x)$ are K-linear. If $R = \bigoplus_{j \in J} R_j$ and $f : \bigcup_{j \in J} R_j \times \bigcup_{j \in J} R_j \longrightarrow R$ is such that $f(kx+y,z) = kf(x,y) + f(y,z)$ and $f(z,kx+y) = kf(z,x) + f(z,y)$ for all $j,i \in J$, $x,y \in R_j$, $z \in R_i$ and $k \in K$, then f extends uniquely to a K-bilinear operation $\overline{f} : R \times R \longrightarrow R$. Henceforth it will be sufficient to define a K-bilinear function on a direct sum by defining it on the pairs of homogeneous elements.

Proposition 1.3.2. *Let h be a surjective K-linear map from $V = \bigoplus_{j \in J} V_j$ into W. $W = \bigoplus_{j \in J} h(V_j)$ iff $kernel(h) = \bigoplus_{j \in J}(V_j \cap kernel(h))$.*

Proof. Let $W = \bigoplus_{j \in J} h(V_j)$. Since $V_j \cap kernel(h)$ is a submodule of V_j for all $j \in J$, it is enough to show that for every $x \in kernel(h)$ and $j \in J$, $proj_j(x) \in kernel(h)$. For all $x \in kernel(h)$, $0 = h(x) = \sum_{j \in J} h(proj_j(x))$ which implies that $proj_j(x) = 0$ for all $j \in J$. To prove the converse let $kernel(h) = \bigoplus_{j \in J}(V_j \cap kernel(h))$. Since h is surjective and $h(V_j)$ is a submodule of W for every $j \in J$, it is enough to show that the representation is unique or equivalently to show that $0 = \sum_{j \in J} x_j$ where $x_j \in h(V_j)$ for all $j \in J$ implies that $x_j = 0$ for all $j \in J$. We have $0 = \sum_{j \in J} x_j = \sum_{j \in J} h(v_j) = h(\sum_{j \in J} v_j)$ where $v_j \in V_j$ for all $j \in J$. Thus $\sum_{j \in J} v_j \in kernel(h)$ which implies that each $v_j \in V_j \cap kernel(h)$ by the uniqueness of the representation in V. Therefore, $x_j = h(v_j) = 0$ for all $j \in J$. □

If W is a K-module and $X \subseteq W$, the submodule $\{\sum_{x \in X} k_x x : k_x \in K\}$ is denoted by $K \cdot X$. A K-module V has a *basis* B if and only if every element $x \in V$ can be expressed uniquely as $\sum_{b \in B} k_b b$ where $k_b \in K$ for all $b \in B$. V is *freely generated by* B if and only if $B \subseteq V$ and every map from B into a K-module W extends uniquely to a K-module homomorphism from V into W. Note that a K-module V is the submodule generated by B when it is freely generated by B.

Proposition 1.3.3. *Let B be a nonempty set. V is freely generated by B if and only if B is a basis of V.*

Proof. Let V be freely generated by B. $V = K \cdot B$ because $K \cdot B$ is a submodule of V containing B and V is generated by B. So it enough to show that if $0 = \sum_{b \in B} k_b b$, then all of the elements k_b are zero. Let $0 = \sum_{b \in B} k_b b$. Consider K as a K-module and for all $b \in B$ let $f_b : B \longrightarrow K$ extend the zero map on $B \setminus \{b\}$ with $f_b(b) = 1$ and $\overline{f_b}$ be the unique K-linear extension of f_b to V. For all $b \in B$, $0 = \overline{f_b}(\sum_{c \in B} k_c c) = \sum_{c \in B} k_c \overline{f_b}(c) = \sum_{c \in B} k_c f_b(c) = k_b 1 = k_b$. To show the converse let B be a basis of V and $h : B \longrightarrow W$ where W is a K-module. Since B is a basis of V, $\overline{h} : V \longrightarrow W$ is well defined by $\overline{h}(\sum_{b \in B} k_b b) = k_b h(b)$ for all $\sum_{b \in B} k_b b \in K \cdot B$. It is easy to verify that $\overline{h}$ is K-linear and that any K-linear map extending h will satisfy our definition for $\overline{h}$. So $\overline{h}$ is the unique K-linear map extending h. □

We say that V is a $free\ K-module$ if and only if V has a basis.

Let B be a nonempty set. We construct a K-module $V(B)$ that is freely generated by B. For each $b \in B$ define $K_b = K$ and let W be the subset of the direct product $\prod_{b \in B} K_b$ consisting of all elements $(x_b)_{b \in B}$ such that $x_b = 0$ for all but finitely many $b \in B$. The addition and K operations are defined as $(x_b)_{b \in B} + (y_b)_{b \in B} = (x_b + y_b)_{b \in B}$ and $k(x_b)_{b \in B} = (kx_b)_{b \in B}$ for all $k \in K$. W is a K-module with these operations. For each $a \in B$ let $(e^a_b)_{b \in B} \in W$ be such that $e^a_b = 0$ for all $b \in B \setminus \{a\}$ and $e^a_a = 1$. Let $V(B)$ be the set obtained from W by replacing $(e^a_b)_{b \in B}$ with a for each $a \in B$ and let $r : V(B) \longrightarrow W$ be the bijection extending the identity on $V(B) \cap W$ such that $r(a) = (e^a_b)_{b \in B}$ for all $a \in B$. Thus $V(B)$ is a K-module with addition and K operations defined as $kx = r^{-1}(kr(x))$ and $x + y = r^{-1}(r(x) + r(y))$ for all $k \in K$ and $x, y \in V(B)$. Thus r is K-linear and we note that $(x_b)_{b \in B} = \sum_{b \in B} x_b(b)$ for all $(x_b)_{b \in B} \in V(B) \cap W$. So it is enough to show that the representation is unique which is equivalent to showing that $0 = \sum_{b \in B} k_b b$ implies that $k_b = 0$ for all $b \in B$. Towards a contradiction, suppose that $0 = k_c c + \sum_{b \in B \setminus \{c\}} k_b b$ and $k_c \neq 0$. Thus $r(\sum_{b \in B \setminus \{c\}} k_b b)$ has $-k_c$ as its c^{th} component but $r(b)$ is zero in its c^{th} component for all $b \in B \setminus \{c\}$ this is contradicts the K-linearity of r.

1.4 K-Algebras

A $K - algebra$ is a K-module with a K-bilinear operation called *multiplication* which is not necessarily associative. We will often denote multiplication in K-algebras by placing elements in juxtaposition. The K-linear maps between K-algebras which are magma homomorphisms with respect to multiplication are called $K - algebra\ homomorphisms$. A submodule R of a K-algebra S is a called a *subalgebra* of S if and only if R is closed under the multiplication in S. That is, for all $x, y \in S$, $xy \in R$. The *subalgebra of S generated by A* is the least subalgebra of S containing A or equivalently the intersection of all subalgebras of S containing A. A subalgebra R of a K-algebra S such that $xy, yx \in R$ for every $x \in R$ and $y \in S$ is called an *ideal* of S. The *ideal of S generated by A* is the least ideal of S containing A or equivalently the intersection of all ideals of S containing A. Let R be an ideal of S. The K-module S/R is a K-algebra with multiplication defined as $\widetilde{x}\widetilde{y} = \widetilde{xy}$ for all $x, y \in S$ and the canonical map $h : S \longrightarrow S/R$ defined as $h(x) = \widetilde{x}$ for all $x \in S$ is a K-algebra homomorphism. S is a *freely generated by A* if and only if $A \subseteq S$ and every map from A into a K-algebra T extends uniquely to a K-algebra homomorphism from S into T. Note that S is the subalgebra of S generated by A when S is freely generated by A.

Let A be a nonempty set. We will construct a K-algebra $S(A)$ which is freely generated by A. Let $S(A)$ be the K-module $V(M(A))$ which is freely generated by the magma $M(A)$ which is freely generated by A. For each $x \in M(A)$ let f_x be the unique K-linear map on $V(M(A))$ extending $y \mapsto (x, y)$ for all $y \in M(A)$. We define the multiplication $*$ on $V(M(A))$ as $(\sum_{x \in M(A)} k_x(x)) * (z) = \sum_{x \in M(A)} k_x(f_x(z))$ for all $\sum_{x \in M(A)} k_x(x)$ & $z \in V(M(A)) \equiv K$ $M(A)$ It easily verified that $*$ is a K-bilinear extension of the product on $M(A)$ To show that $S(A)$ is freely generated by A as a K-algebra, consider a map h from A into a K-algebra R We have a unique magma homomophism extending h to $M(A)$ which extends uniquely to a K-linear map $\overline{h}$ from $V(M(A))$ into R We can verify that $\overline{h}$ is a K-algebra homomorphism Let g $S(A) \longrightarrow R$ be a K-algebra homomorphism extending h Thus g is the unique K-linear map extending g restricted to $M(A)$ Since g restricted to $M(A)$ is a magma homomorphism from $M(A)$ into R extending h, we have that g agrees with $\overline{h}$ on $M(A)$ Thus g is the unique K-linear map extending $\overline{h}$ restricted to $M(A)$, which is $\overline{h}$

Let Δ be a commutative monoid with addition as its binary operation A K-algebra R is called $\Delta-$ *graded* if and only if there exist submodules R_δ of R for all $\delta \in \Delta$ such that $R = \bigoplus_{\delta \in \Delta} R_\delta$ and $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for all $\alpha, \beta \in \Delta$ Let R and S be Δ-graded K-algebras, a $K -$ *algebra homomorphism* h $R \longrightarrow S$ is called $\Delta -$ *graded* when $h(R_\delta) \subseteq S_\delta$ for all $\delta \in \Delta$ Note that multiplication is often defined on a Δ-graded K-algebras by defining it on the homogeneous elements We say that a subalgebra S of a Δ-graded K-algebra R is *graded* when $S = \bigoplus_{\delta \in \Delta}(R_\delta \cap S)$ If S is a graded ideal of R and h $R \longrightarrow R/S$ is the canonical K-algebra homomorphism, then $R/S = \bigoplus_{\delta \in \Delta} h(R_\delta)$ is a Δ-graded K-algebra by Proposition 1 3 2

We have the Homomorphism Theorem for Δ-graded K-algebras

Proposition 1.4.1 *Let R be a graded ideal of the Δ-graded K-algebra S and h $S \longrightarrow S/R$ be the canonical Δ-graded K-algebra homomorphism For any Δ-graded K-algebra homomorphism f from S into a Δ-graded K-module T with $R \subseteq kernel(f)$, there exists a unique Δ-graded K-module homomorphism $\overline{f}$ $S/R \longrightarrow T$ such that $\overline{f} \circ h = f$*

$$\begin{array}{ccc} S & \xrightarrow{h} & S/R \\ \Big\downarrow f & & \Big\downarrow \overline{f} \\ T & \xrightarrow{id} & T \end{array}$$

Proof. Applying Corollary 3.8 in Chapter 2 of [4] to K-algebras, we have a unique K-algebra homomorphism $\overline{f} : S/R \longrightarrow T$ such that $\overline{f} \circ h = f$. For all $\delta \in \Delta$, $\overline{f}((S/R)_\delta) = \overline{f}(h(S_\delta)) = f(S_\delta) \subseteq T_\delta$. Thus $\overline{f}$ is a Δ-graded K-algebra homomorphism. Since every Δ-graded K-algebra homomorphism is a K-algebra homomorphism, $\overline{f}$ is unique. □

Proposition 1.4.2. *Let I be an ideal of a Δ-graded K-algebra R generated by a set of homogeneous elements X. Then I is a graded ideal.*

Proof. Let H denote the set of homogeneous elements in R. We define the subsets J_n of R for $n \geq 0$ by recursion. Let $J_0 = X$ and $J_n = J_{n-1}H \cup HJ_{n-1}$ for all $n > 0$ where $J_{n-1}H = \{x \in R : x = yz \;\&\; y \in J_{n-1} \;\&\; z \in H\}$ and $HJ_{n-1} = \{x \in R : x = zy \;\&\; y \in J_{n-1} \;\&\; z \in H\}$. It is easy to verify that $I = K \cdot J$ where $J = \bigcup_{n\geq 0} J_n$. Note that J is a set of homogeneous elements because H is closed under multiplication. For each $\delta \in \Delta$ let $F_\delta = (R_\delta \cap J) \setminus \{0\}$. Since $I \cap R_\delta = K \cdot F_\delta$ for all $\delta \in \Delta$ and $I = \bigoplus_{\delta\in\Delta} K \cdot F_\delta$, I is a graded ideal. □

A set A which is a disjoint union of the subsets A_δ where $\delta \in \Delta$ is called a Δ – *graded set*, a map f from A into a Δ-graded K-algebra R is called a Δ – *map* when $f(A_\delta) \subseteq R_\delta$ for all $\delta \in \Delta$. Let R be a Δ-graded K-algebra and $A = \bigcup_{\delta\in\Delta} A_\delta$ be a Δ-graded set, R is *freely generated* by A if and only if $\forall\delta \in \Delta$, $A_\delta \subseteq R_\delta$ and every Δ-map from A into a Δ-graded K-algebra S extends uniquely to a Δ-graded K-algebra homomorphism from R into S.

Let $A = \bigcup_{\delta\in\Delta} A_\delta$ be a nonempty Δ-graded set. We will construct a Δ-graded K-algebra $D(A)$ which is freely generated by A by defining a Δ-grading on $S(A)$. Let $g : M(A) \longrightarrow \Delta$ be the unique magma homomorphism extending the map which assigns each element in A_δ the element δ for all $\delta \in \Delta$. For each $\delta \in \Delta$ we let $M_\delta(A) = \{x \in M(A) : g(x) = \delta\}$. Thus $S(A) = \bigoplus_{\delta\in\Delta}(K \cdot M_\delta(A))$ and $xy \in K \cdot M_{\alpha+\beta}(A)$ for all $\alpha, \beta \in \Delta$, $x \in K \cdot M_\alpha(A)$ and $y \in K \cdot M_\beta(A)$. We define $D(A)$ to be this

Δ-graded K-algebra. Any K-algebra homomorphism f extending a Δ-map from A into a Δ-graded K-algebra T is a Δ-graded K-algebra homomorphism. Indeed, by induction on length in $M(A)$ we have that $f(M_\delta(A)) \subseteq T_\delta$ for all $\delta \in \Delta$ which implies that $f(D_\delta(A)) \in T_\delta$ for all $\delta \in \Delta$. So it is clear that $D(A)$ is freely generated by A as a Δ-graded K-algebra because it is freely generated by A as a K-algebra.

Note that $D(A)$ has $M(A)$ as a basis and thus $D(A) = \bigoplus_{n\geq 1} D(A)_n$ as a K-module where $D(A)_n = K \cdot M^n(A)$ for all $n \geq 1$.

1.5 Associative K-algebras

An *associative $K-$ algebra* is a K-algebra such that the nonzero elements form a monoid under the multiplication operation. Note that this definition implies that every associative K-algebra has a multiplicative identity. An *associative $K-$algebra homomorphism* is K-algebra homomorphism from an associative K-algebra T into an associative K-algebra U which maps the identity element of T to the identity element of U. Let A be a nonempty set and $K\langle A\rangle = V(A^*)$ with multiplication defined by the multiplication in A^* extended to $V(A^*)$ by K-bilinearity. It is clear that $K\langle A\rangle$ is an associative K-algebra. If f maps A into an associative K-algebra R, then there is a unique monoid homomorphism from A^* into R extending f which extends uniquely to a K-module homomorphism $\overline{f}$. It is clear that this $\overline{f}$ is the only associative K-algebra homomorphism extending f. Thus $K\langle A\rangle$ is an associative K-algebra which is *freely generated* by A. That is, every map from A into an associative K-algebra R extends uniquely to an associative K-algebra homomorphism from $K\langle A\rangle$ into R.

An *associative $\Delta-$graded $K-$algebra* is a Δ-graded K-algebra which is a monoid with respect to multiplication.

Example 1.5.1 *Let G be an abelian group under addition and $R = \bigoplus_{g \in G} R_g$ be a G-graded K-module. $End(R)$ is an associative K-algebra consisting of the set of all K-linear maps on R with*

$$(f + h)(x) = f(x) + h(x) \tag{1.5.1}$$
$$(kf)(x) = kf(x) \tag{1.5.2}$$
$$f \circ h(x) = f(h(x)) \tag{1.5.3}$$

for all $f, h \in End(R)$ and $k \in K$ and $x \in R$. Note that $0 \in End(R)$ is the zero map on R and $1 \in End(R)$ is the identity map on R. For each $g \in G$ we define $End_g(R) = \{f \in End(R) : \forall h \in G,\ f(R_h) \subseteq R_{g+h}\}$. We note that $End_g(R)$ is a submodule of $End(R)$ for each $g \in G$ and that $End(R) = \bigoplus_{g \in G} End_g(R)$ as a K-module. For all $f \in End_{g_f}(R)$, $h \in End_{g_h}(R)$ and $x \in R_{g_x}$, $f \circ h(x) \in R_{g_f + g_h + g_x}$. Thus $f \circ h \in End_{g_f + g_h}$ which implies that $End(R)$ is an associative G-graded K-algebra.

Chapter 2

Introduction

A K-algebra L with multiplication [ ] is called a *Lie algebra* if and only $[x, x] = 0$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. Note that $[x, y] + [y, x] = [x + y, x + y] = 0$ for all $x, y \in L$. A *Lie algebra homomorphism* is a K-algebra homomorphism between two Lie algebras. We say a Lie algebra L is *free* on a set A if and only if there exists a map $\imath : A \longrightarrow L$ such that for every map f from A into a Lie algebra L' there exists a unique Lie algebra homomorphism $\overline{f} : L \longrightarrow L'$ satisfying $\overline{f} \circ \imath = f$. The least K-module in $K\langle A\rangle$ containing A which is closed under the bracket operation defined as $[x, y] = xy - yx$ for all $x, y \in K\langle A\rangle$ is a free Lie algebra which we will denote L_A, the map $\imath$ associated with L_A is the inclusion map. Note that every element in $M(A)$ can be represented in L_A under the unique magma homomorphism $\psi : M(A) \longrightarrow L_A$ such that $\psi(a) = a$ for all $a \in A$ where L_A is considered a magma under the bracket operation. For example if $a, b \in A$, then $\psi((a, b)) = [a, b]$. In [8], Hall constructed a basis for L_A over a field of characteristic zero. Hall's result is easily generalized to L_A over any commutative ring with unity and the basis can be written as $\psi(B)$ where $B \subseteq M(A)$. In [12], Lazard defined a set of subsets of $M(A)$ called the Lazard sets (see Definition 5.0.2) and used Lazard elimination to prove that $\psi(Z)$

is a basis of L_A for each Lazard sets Z In [20], Viennot generalized Hall's basis by defining a set of subsets of $M(A)$ called the Hall sets which contain B and showed that it was in fact equivalent to the much less explicit Lazard sets That is, every Hall set is a Lazard set and every Lazard set is a Hall set

Definition 2.0.1. A *Hall set* is defined to be a subset H of $M(A)$ which is ordered by a binary relation $\leq$ such that

- H is totally ordered by $\leq$
- A is contained in H
- For all $h = (h', h'') \in H \setminus A$, $h' \in H$ and $h' < h$
- For all $h = (h', h'') \in M(A) \setminus A$, $h \in H$ if and only if $h', h'' \in H$ and $h' < h''$ and either $h'' \in A$ or $h'' = (x, y)$ and $x \leq h'$

The following example of a Hall set is due to Viennot

Example 2.0.2 *Consider the total ordering $\leq$ of A^* which extends a total ordering of A defined as $x < y$ if and only if $x = uy$ for some $u \in A^* \setminus \{1\}$ or $x = vau$ & $y = zbu$ where $a < b$, $a, b \in A$ & $u, v, z \in A^*$ The Lyndon words under $\leq$ are defined to be the subset of A^* consisting of all $w \in A^* \setminus \{1\}$ such that if $w = uv$ and $u, v \in A^* \setminus \{1\}$, then $u < w$ Let Y denote the Lyndon words in A^* with respect to $\leq$ We define the map $t : A^* \setminus \{1\} \longrightarrow M(A)$ as $t(a) = a$ for all $a \in A$ and $t(w) = (t(u), t(v))$ for all $w \in A^* \setminus A$ where u is the greatest word such that $w = uv$ and u, v are not empty In Appendix B it is proven that $t(Y)$ is a Hall set under the total ordering defined as $t(x) < t(y)$ if and only if $x < y$ in A^* $\psi(t(Y))$ is called the Lyndon basis of L_A*

In [17], Ree constructed a generalization of the free Lie algebra over a field of characteristic zero. Melancon derived a basis for Ree's construction using Hall sets in [14]. In [15], Mikhalev gives a more general construction of the free Lie algebra over any commutative ring with unity K such that 2 and 3 are units. In this work, Mikhalev also defines a basis for his construction which is analogous to the Lyndon basis of the free Lie algebra. Melancon's basis is more general than Mikhalev's in the case that K is a field of characteristic zero.

The goal of the following work is to provide a generalization of Mikhalev's construction without assuming that 2 and 3 are units in K, to generalize Lazard elimination for this construction and to derive a basis for this construction in terms of Hall sets via Viennot's results. This basis agrees with Melancon's basis for the case that K is a field of characteristic zero and is more general than Mikhalev's basis in the case that 2 and 3 are units in K.

Chapter 3

(ϵ, G)-Lie Coloralgebras

In this chapter we give a definition for an (ϵ, G)-Lie coloralgebra over a commutative ring with unity which agrees with that of Mikhalev in [15] when 2 and 3 are units. The results in the last two sections of this chapter are generalizations of results for Lie algebras that can be found in [18]. The results for Lie algebras are illustrated in examples.

3.1 Definitions and Examples

Let K be a commutative ring with group $U(K)$ of units, G be an abelian group under addition and $\epsilon : G \times G \longrightarrow U(K)$ be an *antisymmetric bicharacter*. That is, $\epsilon(a+b,c) = \epsilon(a,c)\epsilon(b,c)$, $\epsilon(c,a+b) = \epsilon(c,a)\epsilon(c,b)$ and $\epsilon(a,b)\epsilon(b,a) = 1$ for all $a, b, c \in G$. Moreover, let $\epsilon(g,g) \in \{1,-1\}$ for all $g \in G$.

Definition 3.1.1. An $(\epsilon, G)-$*Lie coloralgebra* is a G-graded K-algebra $L = \bigoplus_{g \in G} L_g$ with a K-bilinear product $[\,]$ such that for all homogeneous elements $x \in L_{g_x}$, $y \in L_{g_y}$

and $z \in L_{g_z}$,

$$[x, y] + \epsilon(g_x, g_y)[y, x] = 0 \tag{3.1.1}$$

$$\epsilon(g_z, g_x)[x, [y, z]] + \epsilon(g_x, g_y)[y, [z, x]] + \epsilon(g_y, g_z)[z, [x, y]] = 0 \tag{3.1.2}$$

$$\epsilon(g_x, g_x) = 1 \Rightarrow [x, x] = 0 \tag{3.1.3}$$

$$\epsilon(g_x, g_x) = -1 \Rightarrow [x, [x, x]] = 0 \tag{3.1.4}$$

Note that if $2 \in U(K)$, then 3.1.1 implies 3.1.3 and if $3 \in U(K)$, then 3.1.2 implies 3.1.4. Also note that this definition is a generalization of the definition of a Lie algebra over K. Indeed, every Lie algebra can be considered an (ϵ, G)-Lie coloralgebra with $G = \{0\}$ and $\epsilon(0, 0) = 1$.

Example 3.1.1. *Let F be a field of characteristic zero, Z_2 be the integers modulo 2 under addition and $\epsilon' : Z_2 \times Z_2 \longrightarrow K$ be defined as $\epsilon'(n, m) = (-1)^{nm}$ for all $n, m \in Z_2$. (ϵ', Z_2)-Lie coloralgebras (over F) are called Lie superalgebras.*

Definition 3.1.2. If L and L' are (ϵ, G)-Lie coloralgebras then $h : L \longrightarrow L'$ is a *$(\epsilon, G) -$ Lie coloralgebra homomorphism* if and only if h is K-linear, $h([x, y]) = [h(x), h(y)]$ for all $x, y \in L$ and $h(L_g) \subseteq L'_g$ for all $g \in G$.

An (ϵ, G)-Lie coloralgebra homomorphism which is a bijection is called an *$(\epsilon, G) -$ Lie coloralgebra isomorphism*. When an (ϵ, G)-Lie coloralgebra isomorphism exists between two (ϵ, G)-Lie coloralgebras we say that they are *isomorphic*.

Example 3.1.2. *Consider $End(R)$ from Example 1.5.1. We define a new multiplication, $[\,]$, on $End(R)$ as the unique K-bilinear map such that $[x, y] = x \circ y - \epsilon(g_x, g_y) y \circ x$ for all $x \in End_{g_x}(R)$ & $y \in End_{g_y}(R)$. Under this multiplication $End(R)$ is an (ϵ, G)-Lie coloralgebra which we denote $pl(R) = \bigoplus_{g \in G} pl_g(R)$.*

Example 3.1.3. *Let Q be the rational numbers and Q_2 be the set of 2×1 matrices with entries in Q. Q_2 is a vector space over Q in the usual sense and has the basis $\{e_0, e_1\}$ where e_0 is the transpose of $(1,0)$ and e_1 is the transpose of $(0,1)$. Thus $Q_2 = Q \cdot \{e_0\} \oplus Q \cdot \{e_1\}$. End$(Q_2)$ is the set of Q-linear maps on Q_2 which can be realized as the set of all 2×2 matrices over Q under matrix addition and matrix multiplication. Thus End(Q_2) = End$_0(Q_2) \oplus$ End$_1(Q_2)$ where End$_0(Q_2)$ is the set of all matrices of the form.*

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

with $x, y \in Q$, and End$_1(Q_2)$ is the set of all matrices of the form.

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$$

with $x, y \in Q$. Consider the Lie superalgebra $pl(Q_2)$ with respect to Z_2 and ϵ' as they are defined in Example 3.1.1. We have [ ] defined on $pl(Q_2)$ as:

$$[\begin{pmatrix} r & s \\ t & u \end{pmatrix}, \begin{pmatrix} v & w \\ x & y \end{pmatrix}]$$

$$= [\begin{pmatrix} r & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} 0 & w \\ x & 0 \end{pmatrix}] + [\begin{pmatrix} 0 & s \\ t & 0 \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & y \end{pmatrix}] + [\begin{pmatrix} 0 & s \\ t & 0 \end{pmatrix}, \begin{pmatrix} 0 & w \\ x & 0 \end{pmatrix}]$$

$$= \begin{pmatrix} 0 & rw - wu \\ ux - xr & 0 \end{pmatrix} + \begin{pmatrix} 0 & sy - vs \\ tv - yt & 0 \end{pmatrix} + \begin{pmatrix} sx + wt & 0 \\ 0 & tw + xs \end{pmatrix}$$

$$= \begin{pmatrix} sx + wt & rw - wu + sy - vs \\ ux - xr + tv - yt & tw + xs \end{pmatrix}$$

for all $r, s, t, u, v, w, x, y \in Q$

3.2 L-derivations

Let L be an (ϵ, G)-Lie coloralgebra and $pl(L)$ be defined for L as in Example 3 1 2 We define the $L-derivations$ $Der(L)$, as $Der(L) = \bigoplus_{g \in G} Der_g(L)$ where $Der_g(L) = \{D \in pl_g(L) \; D(xy) = D(x)y + \epsilon(g, g_x)xD(y)$ for all $x \in L_{g_x}\}$ This definition is justified by the fact that $\forall g \in G$, $kx - y \in Der_g(L)$ for all $x, y \in Der_g(L)$ and $k \in K$

Proposition 3 2 1 *Let* $L = \bigoplus_{g \in G} L_g$ *be a* (ϵ, G)*-Lie coloralgebra Then* $Der(L)$ *is a subalgebra of* $pl(L)$

Proof For all $D_a \in Der_a(L)$ & $D_b \in Der_b(L)$ we have that

$$[D_a, D_b] = (D_a \circ D_b - \epsilon(a, b)D_b \circ D_a) \in Der_{a+b}(L)$$

This follows from the fact that $[D_a, D_b] \in pl_{a+b}(L)$ and the following calculation,

$$\begin{aligned}(D_a \circ D_b - \epsilon(a,b)D_b \circ D_a)[x,y] &= ([D_a \circ D_b(x), y] + \epsilon(a, g_x + b)[D_b(x), D_a(y)] \\ &+ \epsilon(b, g_x)\epsilon(a, g_x)[x, D_a \circ D_b(y)] + \epsilon(b, g_x)[D_a(x), D_b(y)]) \\ &- \epsilon(a,b)([D_b \circ D_a(x), y] + \epsilon(b, g_x + a)[D_a(x), D_b(y)] \\ &+ \epsilon(a, g_x)\epsilon(b, g_x)[x, D_b \circ D_a(y)] + \epsilon(a, g_x)[D_b(x), D_a(y)]) \\ &= [D_a \circ D_b(x), y] + \epsilon(b, g_x)\epsilon(a, g_x)[x, D_a \circ D_b(y)] \\ &- \epsilon(a,b)([D_b \circ D_a(x), y] + \epsilon(a, g_x)\epsilon(b, g_x)[x, D_b \circ D_a(y)]) \\ &= [(D_a \circ D_b - \epsilon(a,b)D_b \circ D_a)(x), y] \\ &+ \epsilon(a+b, g_x)[x, (D_a \circ D_b - \epsilon(a,b)D_b \circ D_a)(y)]\end{aligned}$$

for all $x \in L_{g_x}$ & $y \in L_{g_y}$ Thus $Der(L)$ is a subalgebra of $pl(L)$ □

3.3 Semidirect Products

We define the adjoint $ad : L \longrightarrow Der(L)$ as $ad(x)(y) = [x,y]$ for all $x, y \in L$ Note that $ad(x)$ is an L-derivation by 3 1 2

Theorem 3 3 1 *Let* $[\ ]$ & $[\ ]'$ *denote the brackets on the* (ϵ, G)*-Lie coloralgebras* L & L' *respectively and* $h : L \longrightarrow Der(L')$ *be an* (ϵ, G)*-Lie coloralgebra homomorphism There is a unique* (ϵ, G)*-Lie coloralgebra* $L'' = L \oplus L'$ *with bracket* $[\ ]''$ *extending* $[\ ]$ & $[\ ]'$ *such that* $[x, x']'' = h(x)x'$

Proof Let $L'' = L \oplus L'$ Note that $L'' = \bigoplus_{g \in G} L''_g$ where $L''_g = L_g \oplus L'_g$ for all $g \in G$ Let $[\ ]''$ be defined on L'' as

$$[\sum_{g \in G} x_g, \sum_{h \in G} y_g]'' = \sum_{g,h \in G} [x_g, y_h]''$$

for all direct sum decompositions $\sum_{g\in G} x_g$ & $\sum_{h\in G} y_g \in L''$ and

$$[x+x', y+y']'' = [x,y] + h(x)y' - \epsilon(g_x, g_y)h(y)x' + [x', y']'$$

for all direct sum decompositions $x+x'$ & $y+y'$ in L''_{g_x} & L''_{g_y} respectively. It is clear that $[\ ]''$ extends $[\ ]$ & $[\ ]'$ and is K-bilinear. Since $[L_g, L_h] \in L_{g+h}$ & $[L_g, L_h]' \in L'_{g+h}$ for all $g, h \in G$ and h is an (ϵ, G)-Lie coloralgebra homomorphism, $[L''_g, L''_h]'' \in L''_{g+h}$ for all $g, h \in G$. It is now necessary to show that $[\ ]''$ satisfies properties 3.1.1, 3.1.2, 3.1.3 and 3.1.4. Let $x+x'$, $y+y'$ & $z+z'$ be direct sum decompositions in L''_{g_x}, L''_{g_y} & L''_{g_z} respectively. Thus

$$\begin{aligned}
[x+x', y+y']'' &= [x,y] + h(x)y' - \epsilon(g_x, g_y)h(y)x' + [x', y']' \\
&= -\epsilon(g_x, g_y)[y,x] + h(x)y' - \epsilon(g_x, g_y)h(y)x' - \epsilon(g_x, g_y)[y', x']' \\
&= -\epsilon(g_x, g_y)([y,x] - \epsilon(g_y, g_x)h(x)y' + h(y)x' + [y', x']') \\
&= -\epsilon(g_x, g_y)[y+y', x+x']''
\end{aligned}$$

So $[\ ]''$ satisfies property 3.1.1. To prove property 3.1.2 we first prove the following two cases,

Case 1. $\epsilon(g_z, g_x)[x, [y', z']'']'' + \epsilon(g_x, g_y)[y', [z', x]'']'' + \epsilon(g_y, g_z)[z', [x, y']'']'' = 0$

Case 2. $\epsilon(g_z, g_x)[x', [y, z]'']'' + \epsilon(g_x, g_y)[y, [z, x']'']'' + \epsilon(g_y, g_z)[z, [x', y]'']'' = 0$

The following calculation proves Case 1,

$$\begin{aligned}
\epsilon(g_z, g_x)[x, [y', z']'']'' &= \epsilon(g_z, g_x)h(x)[y', z']'' \\
&= \epsilon(g_z, g_x)([h(x)y', z']'' + \epsilon(g_x, g_y)[y', h(x)z']'') \\
&= \epsilon(g_z, g_x)([[x, y']'', z']'' + \epsilon(g_x, g_y)[y', [x, z']'']'') \\
&= \epsilon(g_z, g_x)(-\epsilon(g_x + g_y, g_z)[z', [x, y']'']'' - \epsilon(g_x, g_y)\epsilon(g_x, g_z)[y', [z', x]'']'') \\
&= -\epsilon(g_y, g_z)[z', [x, y']'']'' - \epsilon(g_x, g_y)[y', [z', x]'']''
\end{aligned}$$

Similarly we prove Case 2,

$$\begin{aligned}\epsilon(g_z, g_x)[x', [y, z]'']'' &= \epsilon(g_z, g_x)(-\epsilon(g_x, g_y + g_z)[[y, z]'', x']'') \\ &= -\epsilon(g_x, g_y)h([y, z])x' \\ &= -\epsilon(g_x, g_y)[h(y), h(z)]x' \\ &= -\epsilon(g_x, g_y)(h(y) \circ h(z) - \epsilon(g_y, g_z)h(z) \circ h(y))x' \\ &= -\epsilon(g_x, g_y)[y, [z, x']'']'' + \epsilon(g_x, g_y)\epsilon(g_y, g_z)[z, [y, x']'']'' \\ &= -\epsilon(g_x, g_y)[y, [z, x']'']'' - \epsilon(g_y, g_z)[z, [x', y]'']''\end{aligned}$$

By K-bilinearity we have that,

$$\epsilon(g_z, g_x)[x + x', [y + y', z + z']'']''$$

$= \epsilon(g_z, g_x)([x, [y, z]] + [x, [y, z']'']'' + [x, [y', z]'']'' + [x, [y', z']'']'' + [x', [y, z]'']'' + [x', [y, z']'']'' +$

$[x', [y', z]'']'' + [x', [y', z']']')$

When we distribute $\epsilon(g_z, g_x)$ on the right side of the above equation, apply Case 1 and Case 2 to the six summands on the right and apply the K-bilinearity of $[\ ]''$ we have that,

$$= -\epsilon(g_x, g_y)[y + y', [z + z', x + x']'']'' - \epsilon(g_y, g_z)[z + z', [x + x', y + y']'']''$$

Thus we have proven 3 1 2 If $\epsilon(g_x, g_x) = 1$, then $[x+x', x+x']'' = 0$ because $[x, x] = 0$, $[x', x']' = 0$ and $h(x)x' - \epsilon(g_x, g_x)h(x)x' = 0$ Therefore property 3 1 3 holds Also, if $\epsilon(g_x, g_x) = -1$, then

$$[x + x', [x + x', x + x']'']''$$

$= [x, [x, x]'']'' + [x, [x, x']'']'' + [x, [x', x]'']'' + [x, [x', x']'']'' + [x', [x, x]'']'' + [x', [x, x']'']'' +$

$[x', [x', x]'']'' + [x', [x', x']'']''$

$$= ([x, [x, x']'']'' + [x, [x', x]'']'' + [x', [x, x]'']'') + ([x, [x', x']'']'' + [x', [x', x]'']'' + [x', [x, x']'']'')$$

The last equation equals zero by property 3 1 2 which proves that property 3 1 4 holds The uniqueness of $[\ ]''$ is implicit in its definition □

The (ϵ, G)-Lie coloralgebra L'' in Theorem 3 3 1 is called the *semidirect product* of L and L' The following example shows how this theorem is a generalization of the analogous result for Lie algebras (see [18] Lemma 0 7(ii))

Example 3.3.1. *Consider a Lie algebra homomorphism h $L \longrightarrow Der(L')$ where L and L' are Lie algebras When we let G be the trivial group $\{0\}$ under addition and $\epsilon(0,0) = 1$, we have that L and L' are (ϵ, G)-Lie coloralgebras and h is an (ϵ, G)-Lie coloralgebra homomorphism Any Lie algebra $L \oplus L'$ extending the multiplication on L and L' with $[x, y] = h(x)(y)$ for all $x \in L$ and $y \in L'$, is an (ϵ, G)-Lie coloralgebra Thus we have a unique Lie algebra $L \oplus L'$ extending the multiplication on L and L' with $[x, y] = h(x)(y)$ for all $x \in L$ and $y \in L'$ by Theorem 3 3 1*

Chapter 4

Free (ϵ, G)-Lie Coloralgebras

This chapter is divided into two sections. In the first section we give the definition of a free (ϵ, G)-Lie coloralgebra on a G-graded set, prove its existence and uniqueness (up to isomorphism) and prove some technical results about it. The second section consists of a theorem and a corollary on derivations of the free (ϵ, G)-Lie coloralgebra. These results generalize a result for free Lie algebras found in [18] which we state as an example at the end of the section. Note that the corollary in the second section is not required later in the thesis.

4.1 Definition and Construction

Let $A = \bigcup_{g \in G} A_g$ be a G-graded set, $\mathcal{L}$ be an (ϵ, G)-Lie coloralgebra and $i : A \longrightarrow \mathcal{L}$ be a G-map, that is $i(a) \in \mathcal{L}_{g_a}$ when $a \in A_{g_a}$. The (ϵ, G)-Lie coloralgebra $\mathcal{L}$ is called *free on A under* i if and only if for any (ϵ, G)-Lie coloralgebra L and G-map $f : A \longrightarrow L$ there exists a unique (ϵ, G)-Lie coloralgebra homomorphism $\overline{f} : \mathcal{L} \longrightarrow L$ such that $\overline{f}(i(a)) = f(a)$ for all $a \in A$. This can be represented by the following commutative diagram where id_L is the identity map on L.

$$\begin{array}{ccc} A & \xrightarrow{f} & L \\ \downarrow{\scriptstyle \imath} & & \downarrow{\scriptstyle \imath d_L} \\ \mathcal{L} & \xrightarrow{\bar{f}} & L \end{array}$$

For simplicity, we say that $\mathcal{L}$ is a *free* $(\epsilon, G) -$ *Lie coloralgebra on* A if and only if there exists a G-map $\imath$ such that $\mathcal{L}$ is free on A under $\imath$. Note that every free Lie algebra on A is a free (ϵ, G)-Lie coloralgebra on A where $G = \{0\}$, $A = A_0$ and $\epsilon(0,0) = 1$.

Theorem 4.1.1. *For each nonempty G-graded set $A = \bigcup_{g \in G} A_g$, there exists an (ϵ, G)-Lie coloralgebra $\mathcal{L}(A)$ which is free on A under an injective G-map. Moreover, $\mathcal{L}(A)$ is unique up to isomorphism.*

Proof. We will first construct the free (ϵ, G)-Lie coloralgebra. Let $D(A)$ be the G-graded K-algebra which we constructed in Chapter 1. Recall that $D(A)$ is freely generated by A as a G-graded K-algebra and that $D(A) = \bigoplus_{g \in G} K \cdot M_g(A)$. Consider the following four sets of homogeneous elements in $D(A)$:

$$H_1 = \{(x,y) + \epsilon(g_x, g_y)(y,x) : x \in M_{g_x}(A) \ \& \ y \in M_{g_y}(A)\}$$

$H_2 = \{\epsilon(g_z, g_x)(x,(y,z)) + \epsilon(g_x, g_y)(y,(z,x)) + \epsilon(g_z, g_x)(z,(x,y)) : x \in M_{g_x}(A),\ y \in M_{g_y}(A) \ \& \ z \in M_{g_z}(A)\}$

$$H_3 = \{(x,x) : x \in M_{g_x}(A) \ \& \ \epsilon(g_x, g_x) = 1\}$$

$$H_4 = \{(x(x,x)) : x \in M_{g_x}(A) \ \& \ \epsilon(g_x, g_x) = -1\}$$

Let I be the ideal generated by the union of H_1, H_2, H_3, H_4 and $h : D(A) \longrightarrow D(A)/I$ be the canonical homomorphism. I is G-graded by Proposition 1.4.2. It is clear that $D(A)/I$ satisfies properties 3.1.1, 3.1.2, 3.1.3, 3.1.4 and $D(A)/I$ is G-graded. Thus $D(A)/I$ is an (ϵ, G)-Lie coloralgebra. Let $id : A \longrightarrow D(A)$ be the identity map. We will show that the quotient algebra $D(A)/I$ is free on A under $h \circ id$. Let L be an (ϵ, G)-Lie coloralgebra and $f : A \longrightarrow L$ be a G-map. Since $D(A)$ is a free G-graded K-algebra, we have a unique G-graded K-algebra homomorphism $\theta : D(A) \longrightarrow L$ such that $\theta(id(a)) = f(a)$ for all $a \in A$. Because L is a (ϵ, G)-Lie coloralgebra, we have that the kernel of θ contains the sets $H_1, \ldots, H_4$ by properties 3.1.1, 3.1.2, 3.1.3, 3.1.4. Thus $I \subseteq \text{kernel}(\theta)$. By Proposition 1.4.1 we have a unique G-graded K-algebra homomorphism $\overline{f} : D(A)/I \longrightarrow L$ such that $\overline{f}(h(x)) = \theta(x)$ for all $x \in D(A)$. Note that $\overline{f}$ is an (ϵ, G)-Lie coloralgebra homomorphism. Thus $\overline{f}(h \circ id(a)) = \theta(id(a)) = f(a)$ for all $a \in A$. This argument is summarized in the following commutative diagram:

$$\begin{array}{ccccc}
A & \xrightarrow{id} & D(A) & \xrightarrow{h} & D(A)/I \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \overline{f}} \\
L & \xrightarrow{id_L} & L & \xrightarrow{id_L} & L
\end{array}$$

To show the uniqueness of $\overline{f}$ suppose that $\psi : D(A)/I \longrightarrow L$ is an (ϵ, G)-Lie coloralgebra homomorphism such that $\psi(h(id(a)) = f(a)$ for all $a \in A$. Since $\psi \circ h : D(A) \longrightarrow L$ is a G-graded K-algebra homomorphism, we have that $\psi \circ h = \theta$. Since $\psi(h(x)) = \theta(x)$ for all $x \in D(A)$, $\psi = \overline{f}$ by Proposition 1.4.1. Thus $D(A)/I$ is a free (ϵ, G)-Lie coloralgebra.

To see that $h \circ id$ is injective, it is enough to prove that $h(a) = h(b)$ implies that $a = b$ for all $a, b \in A$. $I = K \cdot J$ where $J = \bigcup_{n \geq 0} J_n$, $J_0 = \bigcup_{i=1,\ldots,4} H_i$ and $J_{n+1} = M(A)J_n \cup J_n M(A)$ for $n > 0$. Suppose that $h(a) - h(b) = 0$ where $a \neq b$ and

$a, b \in A$ Thus $a - b \in I$ This is not possible because J only contains elements of length greater than one and a, b are basis elements of $D(A)$

Now we show that $D(A)/I$ is unique up to isomorphism Let $\imath = h \circ \imath d$ Let L be another (ϵ, G)-Lie coloralgebra which is free on A under l By definition we have a unique (ϵ, G)-Lie coloralgebra homomorphism $\bar{\imath}$ satisfying the property that $\bar{\imath} \circ l = \imath$ Similarly, there is a unique (ϵ, G)-Lie coloralgebra homomorphism $\bar{l}$ $D(A)/I \longrightarrow L$ such that $\bar{l} \circ \imath = l$ Thus $(\bar{l} \circ \bar{\imath}) \circ l = \bar{l} \circ (\bar{\imath} \circ l) = l$ which implies that $\bar{l} \circ \bar{\imath}$ is the identity (ϵ, G)-Lie coloralgebra homomorphism on L Therefore $\bar{l}$ is surjective Likewise, $\bar{\imath} \circ \bar{l}$ is the identity (ϵ, G)-Lie coloralgebra homomorphism on $D(A)/I$ which implies that $\bar{l}$ is is injective Thus $\bar{l}$ is a (ϵ, G)-Lie coloralgebra isomorphism Thus any two (ϵ, G)-Lie coloralgebras which are free on A are isomorphic as (ϵ, G)-Lie coloralgebras □

Example 4 1 1 *Let $G = 0$, $A = A_0$ and $\epsilon(0, 0) = 1$ We have constructed an (ϵ, G)-Lie coloralgebra $\mathcal{L}(A)$ which is free on A under a G-map $\imath$ We claim that $\mathcal{L}(A)$ is a Lie algebra which is free on A Indeed, every map f from A into a Lie algebra L is a G-map when we consider L G-graded as $L = L_0$ Hence there is a unique (ϵ, G)-Lie coloralgebra homomorphism $\overline{f}$ $\mathcal{L}(A) \longrightarrow L$ such that $\overline{f} \circ \imath = f$ It is clear that $\overline{f}$ is a Lie algebra homomorphism and that any Lie algebra homomorphism f' $\mathcal{L}(A) \longrightarrow L$ is an (ϵ, G)-Lie coloralgebra homomorphism So $\overline{f}$ is unique as a Lie algebra homomorphism*

$\mathcal{L}(A)$ will always denote an (ϵ, G)-Lie coloralgebra that is free on A under $\imath$ Let ψ $M(\imath(A)) \longrightarrow \mathcal{L}(A)$ be defined as $\psi(a) = a$ for all $a \in \imath(A)$ and $\psi((xy)) = [\psi(x), \psi(y)]$ for all $x, y \in M(\imath(A))$ Intuitively one can think about $\psi(M(\imath(A))$ as being the set of all elements in $\mathcal{L}(A)$ which can be represented as a bracketed word in $(\imath(A))^*$ For example consider $a, b, c \in A$ the element $\psi((\imath(a), (\imath(b), \imath(c)))) = [\imath(a), [\imath(b), \imath(c)]]$

is represented by the word $\imath(a)\imath(b)\imath(c)$ bracketed in a certain way. For each positive natural number n let $\mathcal{L}_n(A) = K \cdot \psi(M^n(\imath(A)))$. We define the right bracketing function $r : \imath(A)^*\backslash\{1\} \longrightarrow \mathcal{L}(A)$ as $r(a) = a$ and $r(ax) = [ar(x)]$ for all $x \in \imath(A)^*\backslash\{1\}$ and $a \in \imath(A)$. For example r(abc)=[a[bc]] when $a, b, c \in \imath(A)$.

Proposition 4.1.2. *Let A be a nonempty G-graded set.*

(i)$\mathcal{L}(A) = \bigoplus_{n>0} \mathcal{L}_n(A)$ as a K-module.

(ii)$\mathcal{L}_n(A)$ is spanned by $\{r(x) : x \in \imath(A)^*_n\}$ where $\imath(A)^*_n$ are the words of $\imath(A)^*$ having length n.

(iii)Let $Y \subseteq M(\imath(A))$ be nonempty. The subalgebra of $\mathcal{L}(A)$ generated by $\psi(Y)$ is $K \cdot \psi(M(Y))$.

(iv)Let $m \geq 0$, $a, b \in A$ and $a \neq b$. Then $ad(\imath(a))^m(\imath(b)) \neq 0$.

Proof. (i)Because any two isomorphic (ϵ, G)-Lie coloralgebras are isomorphic as K-modules it is enough to prove the result for the case that $\mathcal{L}(A) = D(A)/I$ where $D(A)/I$ is as in the proof of Theorem 4.1.1. $h(M^n(A))$ consists of all elements of $D(A)/I$ which can be represented as a bracketed word in $(h(A))^*$ of length n. So $h(M^n(A)) = \psi(M^n(h(A)))$ for all $n > 0$. From the direct sum $D(A) = \bigoplus_{n>0} D(A)_n$ we have that $h(D(A)_n) = \mathcal{L}_n(A)$ for all $n > 0$. Since I is generated by homogeneous elements with respect to the direct sum $D(A) = \bigoplus_{n>0} D(A)_n$, $\mathcal{L}= \bigoplus_{n>0} h(D(A)_n)$.

(ii)It is clear that $D(A)/I$ is spanned by $h(M(A))$. Again we let $\imath = h \circ \imath d$. Towards a contradiction suppose that we have a least $n > 0$ such that $\mathcal{L}_n(A)$ is not spanned by $\{r(x) : x \in \imath(A)^*_n\}$. Clearly $n \geq 3$ so for all $[a, [x]] \in \psi(M^n(\imath(A)))$ such that $a \in \imath(A)$, $[a, [x]]$ is in the span of $\{r(x) : x \in \imath(A)^*_n\}$. Thus we have a least $m \geq 2$

such that $m < n$ and there exists $y \in \psi(M^m(\imath(A)))$ and $z \in \psi(M^{n-m}(\imath(A))$ such that $[y, z]$ is not in the span of $\{r(x) : x \in \imath(A)^*_n\}$. Since $\psi(M^m(\imath(A)))$ is spanned by $\{r(x) : x \in \imath(A)^*_m\}$, we have $[a, w] \in \{r(x) : x \in \imath(A)^*_m\}$ such that $[[a, w], z]$ is not in the span of $\{r(x) : x \in \imath(A)^*_n\}$ and $a \in \imath(A)$. By properties 3.1.1 and 3.1.2, $[[a, w], z] = k_1[z, [a, w]] = k_1(k_2[a, [w, z]] + k_3[w[z, a]])$ where $k_1, k_2, k_3 \in U(K)$ but $[a, [w, z]]$ & $[w, [z, a]]$ are spanned by $\{r(x) : x \in \imath(A)^*_n\}$ because $a \in \imath(A)$ & $w \in \psi(M^{m-1}(\imath(A))$ which contradicts $[[a, w], z]$ not being in the span of $\{r(x) : x \in \imath(A)^*_n\}$.

(iii)It is clear that $K \cdot \psi(M(Y))$ is a K-module containing $\psi(Y)$ which is closed under $[\ ]$. So it is a subalgebra of $\mathcal{L}$. Let L be a subalgebra of $\mathcal{L}$ containing $\psi(Y)$. Suppose that $\psi(M(Y))$ is not contained in L. Then there is a least positive natural number m such that $x \in M^m(Y)$ and $\psi(x) \notin L$. Since $m > 1$, $x = (yz)$ where y & z are of length less than m. So $\psi(y), \psi(z) \in L$ which implies that $\psi(x) = \psi((xy)) = [\psi(y), \psi(z)] \in L$ which is a contradiction. So $K \cdot \psi(M(Y))$ is the least subalgebra of $\mathcal{L}$ containing $\psi(Y)$.

(iv)We prove the result for the case that $\mathcal{L}(A) = D(A)/I$. Since $\imath$ is injective, this result holds when $m = 0$. Note that $ad(\imath(a))^n(\imath(b)) = h(r(a^n b))$ for all $n \geq 0$. Suppose that m is the least positive integer such that we have $ad(\imath(a))^m(\imath(b)) = 0$. Thus $r(a^m b) \in I$. Since $M(A)$ is a basis of $D(A)$, we have that $I = K \cdot J$ where $J = \bigcup_{n\geq 0} J_n$, $J_0 = \bigcup_{n=1,\dots,4} H_n$ and $J_{\imath+1} = M(A)J_\imath \cup J_\imath M(A)$ for $\imath > 0$. Thus we have that $r(a^m b) = \sum_{\jmath\in J} k_\jmath \jmath = \sum_{\jmath\in F} k_\jmath \jmath$ where $F = \{\jmath \in J : k_\jmath \neq 0\}$. Let $F_0 = \{\jmath \in F : \jmath = (a, x_\jmath) \text{ for some } x_\jmath \in J\}$. Thus we have that

$$r(a^m b) - \sum_{(a,x_\jmath)\in F_0} k_\jmath(a, x_\jmath) = \sum_{\jmath\in F\setminus F_0} k_\jmath \jmath$$

Every element of $F \setminus F_0$ is of the form (x, y) where either $x \in J$ and $y \in M(A)$ or

$x \in M(A) \setminus \{a\}$ and $y \in J$. Consider the direct sum $D(A) = \sum_{i>0} D(A)_i$. For all $x \in J$, $x \in D(A)_i$ where $i > 1$. Thus every $j \in F \setminus F_0$ is represented in the basis $M(A)$ as

$$j = \sum_{z \in M} k_z z$$

where $M = \{z \in M(A) : z = (x, y) \ \&\ x \in M(A) \setminus \{a\}\}$. Thus $F \setminus F_0$ is empty and we have that

$$r(a^m b) - \sum_{(a,x_j) \in F_0} k_j (a, x_j) = 0$$

Since the K-linear map $x \mapsto (a, x)$ is injective,

$$r(a^{m-1} b) - \sum_{(a,x_j) \in F_0} k_j x_j = 0$$

Thus $r(a^{m-1}b) \in I$ which implies that $ad(i(a))^n(i(b)) = 0$. This is a contradiction. □

Proposition 4.1.3. *Let* $\{a\} = \bigcup_{h \in G} A_h$ *be a* G*-graded set such that* $A_g = \{a\}$ *and* $A_h = \emptyset$ *for all* $h \neq g$.

(i)If $\epsilon(g, g) = 1$, *then* $\{i(a)\}$ *is a basis of* $\mathcal{L}(\{a\})$.

(ii) If $\epsilon(g, g) = -1$, *then* $\{i(a), \psi((i(a)i(a))\}$ *is a basis of* $\mathcal{L}(\{a\})$.

Proof. (i)Let $\epsilon(g, g) = 1$. Let L denote the K-module on the basis $\{a\}$ with $[\,]$ as K-bilinear multiplication such that $[a, a] = 0$. Thus $L = \bigoplus_{h \in G} L_h$ where $L_g = K.\{a\}$ and $L_h = \{0\}$ for $h \in G \setminus \{g\}$. It is clear that L is an (ϵ, G)-Lie coloralgebra. Let L' be an (ϵ, G)-Lie coloralgebra and $f : \{a\} \longrightarrow L'$ be a G-map. Consider the map $\overline{f} : L \longrightarrow L'$ defined as $\overline{f}(ka) = kf(a)$ for all $k \in K$. It is clear that K-linear and that $\overline{f}(L_h) \subseteq L'_h$ for all $h \in G$. For any $k_1, k_2 \in K$, $\overline{f}([k_1 a, k_2 a]) = \overline{f}(0) = 0 = [k_1 f(a), k_2 f(a)] = [\overline{f}(k_1 a), \overline{f}(k_2 a)]$. Thus L is a free (ϵ, G)-Lie coloralgebra on $\{a\}$

under the inclusion map. Since L is isomorphic to $\mathcal{L}(\{\imath(a)\})$ under an extension of the map $\imath$, we are done.

(ii)Let $\epsilon(g,g) = -1$. Let L denote the (ϵ, G)-Lie coloralgebra defined as the K-module on the basis $\{a, (a,a)\}$ with $[\ ]$ as K-bilinear multiplication such that $[a,a] = (a,a)$, $[a,(a,a)] = [(a,a),a] = [(a,a),(a,a)] = 0$. Thus $L = \bigoplus_{h\in G} L_h$ where $L_g = K \cdot \{a\}$, $L_{g+g} = K \cdot \{(a,a)\}$ and $L_h = \{0\}$ for $h \in G \setminus \{g, g+g\}$. It is clear that L is an (ϵ, G)-Lie coloralgebra. Let L' be an (ϵ, G)-Lie coloralgebra and $f : \{a\} \longrightarrow L'$ be a G-map. Consider the map $\overline{f} : L \longrightarrow L'$ defined as $\overline{f}(k_1 a + k_2(a,a)) = k_1 f(a) + k_2[f(a), f(a)]$ for all $k_1, k_2 \in K$. It is clear that $\overline{f}$ is K-linear and that $\overline{f}(L_h) \subseteq L'_h$ for all $h \in G$. For all $k_1, k_2, k_3, k_4 \in K$, $\overline{f}([k_1 a + k_2(a,a), k_3 a + k_4(a,a)]) = \overline{f}(k_1 k_3(a,a)) = k_1 k_3[f(a), f(a)] = [k_1 f(a) + k_2[f(a), f(a)], k_3 f(a) + k_4[f(a), f(a)]] = [\overline{f}(k_1 a + k_2(a,a)), \overline{f}(k_3 a + k_4(a,a))]$. Thus L is a free (ϵ, g)-Lie coloralgebra on $\{a\}$ under the inclusion map. Since L is isomorphic to $\mathcal{L}(\{\imath(a)\})$ under an extension of the map $\imath$, we are done. □

4.2 $\mathcal{L}(A)$-derivations

Theorem 4.2.1. *Let A be a nonempty G-graded set and $d : A \longrightarrow \mathcal{L}(A)$ have degree g_d. That is, $\forall g \in G, \forall a \in A_g$, $d(a) \in \mathcal{L}_{g+g_d}(A)$. There exists a unique $\overline{d} \in Der(\mathcal{L}(A))$ such that $\overline{d} \circ \imath = d$. Moreover, $\overline{d} \in Der_{g_d}(\mathcal{L}(A))$.*

Proof. By Proposition 4.1.1, it is enough to prove the result for the case that $\mathcal{L}(A) = D(A)/I$ and $\imath = h \circ \imath d$ where $D(A)/I$ and $h \circ \imath d$ are as in the proof. Since $M(A)$ is a basis of $D(A)$, we have a unique K-linear map $f : D(A) \longrightarrow D(A)/I$ such that,

$$(f \circ \imath d)(a) = d(a)$$

for all $a \in A$ and

$$f((xy)) = [f(x), h(y)] + \epsilon(g_d, g_x)[h(x), f(y)]$$

for all $x \in M_{g_x}(A)$ & $y \in M(A)$. By K-linearity we have that,

$$f((x,y)) = [f(x), h(y)] + \epsilon(g_d, g_x)[h(x), f(y)]$$

for all $x \in D_{g_x}(A)$ & $y \in D(A)$. By induction on length of elements in $M(A)$ it can be shown that for all $x \in M_{g_x}(A)$, $f(x) \in h(M_{g_x+g_d}(A)) \subseteq \mathcal{L}_{g_x+g_d}(A)$ which implies that for all $y \in D_{g_y}(A)$ $f(y) \in D_{g_y+g_d}(A)$. Let $x \in M_{g_x}(A)$, $y \in M_{g_y}(A)$ and $z \in M_{g_z}(A)$. The following calculations show that the kernel of f contains the sets H_1, H_2, H_3 & H_4,

$$\begin{aligned} f((x,y) + \epsilon(g_x, g_y)(y,x)) &= [f(x), h(y)] + \epsilon(g_d, g_x)[h(x), f(y)] \\ &+ \epsilon(g_x, g_y)([f(y), h(x)] + \epsilon(g_d, g_y)[h(y), f(x)]) \\ &= [f(x), h(y)] + \epsilon(g_x + g_d, g_y)[h(y), f(x)] \\ &+ \epsilon(g_d, g_x)[h(x), f(y)] + \epsilon(g_x, g_y)[f(y), h(x)] \\ &= \epsilon(g_d, g_x)[h(x), f(y)] + \epsilon(g_x, g_y)[f(y), h(x)] \\ &= -\epsilon(g_x, g_y)[f(y), h(x)] + \epsilon(g_x, g_y)[f(y), h(x)] \\ &= 0 \end{aligned}$$

Thus $H_1 \subseteq kernel(f)$.

Let σ be the permutation of the $\{x, y, z\}$ defined as $\sigma(x) = y$, $\sigma(y) = z$ & $\sigma(z) = x$

The cyclic group generated by σ is denoted by $\langle\sigma\rangle$

$$f(\sum_{\tau\in\langle\sigma\rangle} \epsilon(g_{\tau z}, g_{\tau x})(\tau x(\tau y \tau z))) = \sum_{\tau\in\langle\sigma\rangle} \epsilon(g_{\tau z}, g_{\tau x}) f(\tau x(\tau y \tau z))$$

$$= \sum_{\tau\in\langle\sigma\rangle} \epsilon(g_{\tau z}, g_{\tau x})([f(\tau x), h(\tau y \tau z)]$$

$$+ \epsilon(g_d, g_{\tau x})[h(\tau x), [f(\tau y), h(\tau z)] + \epsilon(g_d, g_{\tau y})[h(\tau y), f(\tau z)]])$$

$$= \sum_{\tau\in\langle\sigma\rangle} \epsilon(g_{\tau z}, g_{\tau x})([f(\tau x), [h(\tau y), h(\tau z)]]$$

$$+ \epsilon(g_d, g_{\tau x})[h(\tau x), [f(\tau y), h(\tau z)]]$$

$$+ \epsilon(g_d, g_{\tau x})\epsilon(g_d, g_{\tau y})[h(\tau x), [h(\tau y), f(\tau z)]])$$

$$= \sum_{\tau\in\langle\sigma\rangle} (\epsilon(g_{\tau z}, g_{\tau x})[f(\tau x), [h(\tau y), h(\tau z)]]$$

$$+ \epsilon(g_d + g_{\tau z}, g_{\tau x})\epsilon(g_d, g_{\tau y})[h(\tau x), [h(\tau y), f(\tau z)]]$$

$$+ \epsilon(g_d + g_{\tau z}, g_{\tau x})[h(\tau x), [f(\tau y), h(\tau z)]])$$

When we expand the sum and regroup terms we have,

$$= \sum_{\tau\in\langle\sigma\rangle} (\epsilon(g_{\tau z}, g_{\tau x})[f(\tau x)[h(\tau y), h(\tau z)]]$$

$$+ \epsilon(g_d, g_{\tau z})\epsilon(g_d + g_{\tau x}, g_{\tau y})[h(\tau y), [h(\tau z), f(\tau x)]] + \epsilon(g_d + g_{\tau y}, g_{\tau z})[h(\tau z), [f(\tau x), h(\tau y)]])$$

$$= \sum_{\tau\in\langle\sigma\rangle} \epsilon(g_d, g_{\tau z})(\epsilon(g_{\tau z}, g_{\tau x}$$

$$+ g_d)[f(\tau x)[h(\tau y), h(\tau z)]] + \epsilon(g_d + g_{\tau x}, g_{\tau y})[h(\tau y), [h(\tau z), f(\tau x)]]$$

$$+ \epsilon(g_{\tau y}, g_{\tau z})[h(\tau z), [f(\tau x), h(\tau y)]])$$

$$= 0$$

Thus $H_2 \subseteq kernel(f)$

$$
\begin{aligned}
f((xx)) &= [f(x), h(x)] + \epsilon(g_d, g_x)[h(x), f(x)] \\
&= [f(x), h(x)] - \epsilon(g_d, g_x)\epsilon(g_x, g_d + g_x)[f(x), h(x)] \\
&= 0
\end{aligned}
$$

when $\epsilon(g_x, g_x) = 1$ So $H_3 \subseteq kernel(f)$

$$
\begin{aligned}
\epsilon(g_x, g_d + g_x) f((x(xx))) &= \epsilon(g_x, g_d + g_x)([f(x), [h(x), h(x)]] + \epsilon(g_d, g_x)[h(x), [f(x), h(x)] \\
&\quad + \epsilon(g_d, g_x)[h(x), f(x)]]) \\
&= \epsilon(g_x, g_d + g_x)[f(x), [h(x), h(x)]] + \epsilon(g_x, g_x)[h(x), [f(x), h(x)]] \\
&\quad + \epsilon(g_d + g_x, g_x)[h(x), [h(x), f(x)]] \\
&= 0
\end{aligned}
$$

Since $\epsilon(g_x, g_d + g_x)$ is a unit, $f((x(xx))) = 0$ So $H_4 \subseteq kernel(f)$

Let $J_0 = \bigcup_{j=1}^{4} H_j$ and $J_n = J_{n-1}D(A) \cup D(A)J_{n-1}$ for $n > 0$ Thus $I = K \bigcup_{n \geq 0} J_n$ where I is the kernel of h If $x \in I$ and $f(x) = 0$,then $\forall y \in D(A), f(xy) = [f(x), h(y)] + \epsilon(g_d, g_x)[h(x), f(y)] = 0$ Thus $J_n \subseteq kernel(f)$ for all $n \geq 0$ which implies that $I \subseteq kernel(f)$ By Propostion 1 3 1 we have a unique K-linear map $\overline{d}$ $D(A)/I \longrightarrow D(A)/I$ such that $\overline{d} \circ h = f$ Let $x \in D_{g_x}(A)$ and $y \in D_{g_y}(A)$

$$
\begin{aligned}
\overline{d}([h(x), h(y)]) &= (\overline{d} \circ h)(xy) \\
&= f((xy)) \\
&= [f(x), h(y)] + \epsilon(g_d, g_x)[h(x), f(y)] \\
&= [\overline{d}(h(x)), h(y)] + \epsilon(g_d, g_x)[h(x), \overline{d}(h(y))]
\end{aligned}
$$

Note that $\overline{d}(h(x)) = f(x) \in \mathcal{L}_{g_x + g_d}(A)$ So $\overline{d} \in Der_{g_d}(\mathcal{L}(A))$ The following commutative diagram summarizes this argument

$$\begin{array}{ccc}
A & \xrightarrow{d} & D(A)/I \\
\Big\downarrow id_A & & \Big\downarrow id_{D(A)/I} \\
D(A) & \xrightarrow{f} & D(A)/I \\
\Big\downarrow h & & \Big\downarrow id_{D(A)/I} \\
D(A)/I & \xrightarrow{\overline{d}} & D(A)/I
\end{array}$$

To show uniqueness let $\overline{d}'$ be a $\mathcal{L}(A)$-derivation agreeing with $\overline{d}$ on $(h \circ id)(A)$. Let $\overline{d}' = \sum_{g\in G} \overline{d}'_g$ where $\overline{d}'_g \in Der_g(\mathcal{L}(A))$ for all $g \in G$. It is clear that for all $g \in G \setminus \{g_d\}$, $\overline{d}'_g(\imath(a)) = 0$ when $a \in A$. Towards a contradiction suppose that there is a least positive integer m such that there exists $x \in h(M^m(A))$ with $\overline{d}'_g(x) \neq 0$ for some $g \neq g_d$. Thus $m > 1$ and we have that $x = h((y, z)) = [h(y), h(z)]$ which implies that $\overline{d}'_g(x) = \overline{d}'_g([h(y), h(z)]) = [\overline{d}'_g(h(y)), h(z)] + \epsilon(g, g_y)[h(y), \overline{d}'_g(h(z))] = 0$ where $h(y) \in \mathcal{L}_{g_y}(A)$. This is a contradiction. So we have that $\overline{d}'_g(\mathcal{L}(A)) = \overline{d}'_g(K \cdot h(M(A))) = \{0\}$ for all $g \in G \setminus \{g_d\}$. Thus $\overline{d}' = \overline{d}'_{g_d}$. Let $x, y \in h(M(A))$ such that $\overline{d}'(x) = \overline{d}(x)$ & $\overline{d}'(y) = \overline{d}(y)$,

$$\begin{aligned}
\overline{d}'([x, y]) &= [\overline{d}'(x), y] + \epsilon(g_d, g_x)[x, \overline{d}'(y)] \\
&= [\overline{d}(x), y] + \epsilon(g_d, g_x)[x, \overline{d}(y)] \\
&= \overline{d}([x, y])
\end{aligned}$$

So $\overline{d}$ agrees with $\overline{d}'$ on $h(M(A))$. Since $h(M(A))$ spans $\mathcal{L}(A)$, $\overline{d}' = \overline{d}$. □

Corollary 4.2.2. *Let A be a nonempty G-graded set and $d : A \longrightarrow \mathcal{L}(A)$. There is a unique $\overline{d} \in Der(\mathcal{L}(A))$ such that $\overline{d} \circ \imath = d$.*

Proof. For all $g \in G$ and $x \in \mathcal{L}(A)$ we define $proj_g(x)$ to be the g^{th} component of the direct sum decomposition of $x \in \bigoplus_{h\in G} \mathcal{L}_h(A)$. For every $h \in G$ let $d_h : A \longrightarrow \mathcal{L}(A)$

be defined as $\forall a \in A_{g_a}$ $d_h(a) = proj_{h+g_a}(d(a))$. Let $a \in A_{g_a}$. The map $h \mapsto h + g_a$ is a bijection on G because G is a group. Thus,

$$d(a) = \sum_{h \in G} proj_h(d(a)) = \sum_{h \in G} proj_{h+g_a}(d(a)) = \sum_{g \in G} d_g(a)$$

Thus $d = \sum_{g \in G} d_g$. Applying Theorem 4.2.1 we have that for each $g \in G$ there is a unique $\overline{d}_g \in Der_g(\mathcal{L}(A))$ such that $\overline{d}_g \circ \imath = d_g$. Let $\overline{d} = \sum_{g \in G} \overline{d}_g$. Thus $\overline{d}$ is a $\mathcal{L}(A)$-derivation such that $\overline{d} \circ \imath = d$. To show uniqueness let $\overline{d}'$ be a $\mathcal{L}(A)$-derivation such that $\overline{d}' \circ \imath = d$ and let $\overline{d}' = \sum_{g \in G} \overline{d}'_g$ where $\overline{d_g}' \in Der_g(\mathcal{L}(A))$ for all $g \in G$. Let $h \in G$ and $a \in A_{g_a}$. Since $\imath(a) \in \mathcal{L}_{g_a}(A)$, $d_h(a) = proj_{h+g_a}(d(a)) = proj_{h+g_a}(\overline{d}'(\imath(a))) = \overline{d}'_h(\imath(a))$. Thus $d_g = \overline{d}'_g \circ \imath$ for all $g \in G$. By Theorem 4.2.1, $\overline{d}'_g = \overline{d}_g$ for all $g \in G$. Therefore $\overline{d}' = \overline{d}$. □

The analogous result for Lie algebras (see [18] Lemma 0.7(iii)) is derived from Theorem 4.2.1 in the next example.

Example 4.2.1. *A derivation of a Lie algebra L is defined as a K-linear map f on L such that $f([x,y]) = [f(x), y] + [x, f(y)]$ for all $x, y \in L$. Let $\mathcal{L}(A)$ be the Lie algebra from Example 4.1.1. We have that $\mathcal{L}(A)$ is free on A under $\imath$. Since $\mathcal{L}(A)$ is an (ϵ, G)-Lie coloralgebra which is free on A under $\imath$, we have that for every map $d : A \longrightarrow \mathcal{L}(A)$ there is a unique $\mathcal{L}(A)$-derivation $\overline{d}$ such that $\overline{d} \circ \imath = d$. Thus $\overline{d}$ is K-linear and for all $x, y \in \mathcal{L}(A)$, $\overline{d}[x,y] = [\overline{d}(x), y] + \epsilon(0,0)[x, \overline{d}(y)] = [\overline{d}(x), y] + [x, \overline{d}(y)]$ because $\epsilon(0,0) = 1$ and $G = \{0\}$. So $\overline{d}$ is unique as a derivation in the Lie algebra sense.*

Chapter 5

Decomposition of the free (ϵ, G)-Lie Coloralgebra

In this chapter we prove that a free (ϵ, G)-Lie coloralgebra $\mathcal{L}(A)$ is free as a K-module by showing that certain subsets of $M(A)$ derived from Lazard sets represent a basis for $\mathcal{L}(A)$. To show this we prove a generalization of Lazard elimination (Theorem 5.0.4). The results in this chapter are all generalizations of results for Lie algebras which can be found in [18]. In the last two examples we use the results of Viennot in [20] which are proven in Appendix A and Appendix B.

Let $\mathcal{L}$ be an (ϵ, G)-Lie coloralgebra over K and A be a G-graded set. We say that $\mathcal{L}$ is *freely generated by* A if and only if $\mathcal{L}$ is free on A under the identity map. Note that it is implicit in this definition that $A_g \subseteq \mathcal{L}_g$ for all $g \in G$. There are two important observations to make. The first is that the least subalgebra of $\mathcal{L}$ containing A is $\mathcal{L}$ when $\mathcal{L}$ is freely generated by A. The second is that $\mathcal{L}(A)$ is freely generated by $\imath(A)$.

Proposition 5.0.3. *For every nonempty G-graded set A, we can construct an (ϵ, G)-Lie coloralgebra which is freely generated by A.*

Proof Let $D(A)/I$ and $h \circ id$ be as in the construction from the proof of Theorem 4 1 1 Note that $h \circ id$ is injective Let L be the set obtained from $D(A)/I$ by replacing $h(id(a))$ with a for every $a \in A$ This replacement is justified by the fact $h \circ id$ is injective Consider the map $f \; L \longrightarrow D(A)/I$ defined as

$$f(a) = h(id(a)) \; if \; a \in A$$

$$f(x) = x \; if \; x \in L \setminus A$$

Thus f is a bijection We define the operations on L as

$$kx + y = f^{-1}(kf(x) + f(y))$$

$$[x, y] = f^{-1}([f(x), f(y)])$$

for all $k \in K$ & $x, y \in L$ It can be shown that L is an (ϵ, G)-Lie coloralgebra Thus f is an (ϵ, G)-Lie coloralgebra isomorphism Since $D(A)/I$ is freely generated by $h \circ id(A)$, L is freely generated by $f^{-1}(h \circ id(A)) = A$ □

Theorem 5 0 4 *Let A be a G-graded set of cardinality greater than 1 and $\mathcal{L}$ an (ϵ, G)-Lie coloralgebra which is freely generated by A For all $a \in A$, $\mathcal{L} = \mathcal{L}' \oplus \mathcal{L}''$ where $\mathcal{L}'$ & $\mathcal{L}''$ are subalgebras of $\mathcal{L}$, $\mathcal{L}'$ is freely generated by $\{a\}$ and $\mathcal{L}''$ is freely generated by $\{(ad(a))^n b \; b \in A \setminus \{a\}$ & $n \geq 0\}$*

Proof In this proof we say that a map is a homomorphism iff it is an (ϵ, G)-Lie coloralgebra homomorphism Let $a \in A_{g_a}$ and consider $\{a\}$ as a G-graded subset of A Let $B = A \setminus \{a\}$, N denote the natural numbers and $T = N \times B$ be G-graded such that $T_g = \{(n, b) \; \exists g_b \in G, \; b \in B_{g_b} \; \& \; n \; g_a + g_b = h\}$ for all $g \in G$ By Proposition 5 0 3, we let $\mathcal{S}$ & $\mathcal{T}$ be (ϵ, G)-Lie coloralgebras which are freely generated by $\{a\}$ & T respectively By Theorem 4 2 1, we have a unique derivation D of $\mathcal{T}$ such that $D((n,b)) = (n+1,b)$ for all $(n,b) \in T$ and $D \in Der_{g_a}(\mathcal{T})$. Thus we have a unique homomorphism $h : \mathcal{S} \longrightarrow Der(\mathcal{T})$ such that $h(a) = D$. Applying Theorem 3.3.1, we have a unique (ϵ, G)-Lie coloralgebra $\mathcal{R} = \mathcal{S} \oplus \mathcal{T}$ such that the bracket of $\mathcal{R}$ extends that of $\mathcal{S}$ & $\mathcal{T}$ and $\forall x \in \mathcal{S}$, $\forall y \in \mathcal{T}$, $[x,y] = h(x)(y)$. Since $\mathcal{L}$ is free on A, there is a unique homomorphism $\psi : \mathcal{L} \longrightarrow \mathcal{R}$ such that $\psi(a) = a$ and $\psi(b) = (0,b)$ for all $b \in B$. Similarly, there is a unique homomorphism $\phi : \mathcal{T} \longrightarrow \mathcal{L}$ such that $\phi((n,b)) = (ad(a))^n(b)$ for all $(n,b) \in T$. Let $E = \{x \in \mathcal{T} : \phi \circ D(x) = [a, \phi(x)]\}$. It is clear that E is a K-module. The following calculation shows that E contains T.

$$\phi \circ D((n,b)) = \phi((n+1,b)) = (ad(a))^{n+1}(b) = [a, (ad(a))^n(b)] = [a, \phi((n,b))]$$

Now we show that E is closed under [ ]. If $x, y \in E$ and $x \in \mathcal{T}_{g_x}$ & $y \in \mathcal{T}_{g_y}$, then

$$\begin{aligned}
\phi \circ D([x,y]) &= \phi([D(x), y] + \epsilon(g_a, g_x)[x, D(y)]) \\
&= [\phi \circ D(x), \phi(y)] + \epsilon(g_a, g_x)[\phi(x), \phi \circ D(y)] \\
&= [[a, \phi(x)], \phi(y)] + \epsilon(g_a, g_x)[\phi(x), [a, \phi(y)]] \\
&= [a, [\phi(x), \phi(y)]] = [a, \phi[x,y]]
\end{aligned}$$

Since $\mathcal{T}$ is the subalgebra of $\mathcal{T}$ generated by T and E is a subalgebra of $\mathcal{T}$, $E = \mathcal{T}$. Thus $\phi \circ D(x) = [a, \phi(x)]$ for all $x \in \mathcal{T}$. Let $\alpha : \mathcal{S} \longrightarrow \mathcal{L}$ be the unique homomorphism satisfying $\alpha(a) = a$. Define a K-linear map $\theta : \mathcal{R} \longrightarrow \mathcal{L}$ by $\theta(x+y) = \alpha(x) + \phi(y)$ for all $x \in \mathcal{S}$ and $y \in \mathcal{T}$. By Proposition 4.1.2 (ii), the following induction argument shows that $\theta([y,x]) = [\theta(y), \theta(x)]$ for all $y \in \mathcal{S}$ and $x \in \mathcal{T}$. Let $x \in \mathcal{T}$.

$$\theta([a,x]) = \theta(h(a)(x)) = \phi(h(a)(x)) = \phi \circ D(x) = [a, \phi(x)] = [\theta(a), \theta(x)]$$

Let $\theta([r(a^j), x] = [\theta(r(a^j), \theta(x)]$ for all $j \leq n$ be the induction hypothesis.

$$\theta([r(a^{n+1}), x]) = \theta([[a, r(a^n)], x])$$

$$= \theta([a, [r(a^n), x]] - \epsilon(g_a, n \cdot g_a)[r(a^n), [a, x]])$$

$$= \theta([a, h(r(a^n))(x)]) - \epsilon(g_a, n \cdot g_a)\theta([r(a^n), h(a)(x)])$$

By the induction hypothesis we have

$$= [\theta(a), \theta(h(r(a^n))(x))] - \epsilon(g_a, g_n \cdot g_a)[\theta(r(a^n)), \theta(h(a)(x))]$$

$$= [\theta(a), \theta([r(a^n), x])] - \epsilon(g_a, g_n \cdot g_a)[\theta(r(a^n)), \theta([a, x])]$$

By the induction hypothesis we have

$$= [\theta(a), [\theta(r(a^n)), \theta(x)]] - \epsilon(g_a, g_{n\, g_a})[\theta(r(a^n)), [\theta(a), \theta(x)]]$$

$$= [\alpha(a), [\alpha(r(a^n)), \phi(x)]] - \epsilon(g_a, g_{n\, g_a})[\alpha(r(a^n)), [\alpha(a), \phi(x)]]$$

$$= [[\alpha(a), \alpha(r(a^n))], \phi(x)]$$

$$= [\alpha([a, r(a^n)]), \phi(x)] = [\theta(r(a^{n+1})), \theta(x)]$$

Since θ extends homomorphisms of $\mathcal{S}$ & $\mathcal{T}$, we conclude that θ is a homomorphism. Thus $\psi \circ \theta$ is a homomorphism on $\mathcal{R}$. By $\psi \circ \theta(a) = \psi(a) = a$ and the fact that $\mathcal{S}$ is free on $\{a\}$, we have that $\psi \circ \theta$ restricted to $\mathcal{S}$ is the identity homomorphism on $\mathcal{S}$. The following induction argument shows that $\psi \circ \theta$ restricted to $\mathcal{T}$ is the identity on $\mathcal{T}$. Let $b \in B$.

$$\psi \circ \theta((0, b)) = \psi((ad(a))^0(b)) = \psi(b) = (0, b)$$

Let $\psi \circ \theta((j, b)) = (j, b)$ for all $j \leq n$ be the induction hypothesis.

$$\psi \circ \theta((n+1, b)) = \psi((ad(a))^{n+1}(b)) = \psi([a, \theta((n, b))]) = [\psi(a), \psi \circ \theta((n, b))]$$

By hypothesis we have

$$= [\psi(a), (n, b)] = [a, (n, b)] = h(a)((n, b)) = D((n, b)) = (n+1, b)$$

So $\psi \circ \theta$ restricted to $\mathcal{T}$ is the unique homomorphism extending the identity on T which implies that it is the identity homomorphism on $\mathcal{T}$. Thus we conclude that $\psi \circ \theta$ is the identity homomorphism on $\mathcal{R}$ because $\mathcal{R}$ is the direct sum of $\mathcal{S}$ and $\mathcal{T}$. Similarly,

$$\theta \circ \psi(a) = \theta(a) = a$$

and

$$\theta \circ \psi(b) = \theta((0,b)) = \phi((0,b)) = (ad(a))^0(b) = b$$

So $\theta \circ \psi$ is the identity homomorphism on $\mathcal{L}$. Thus θ is an isomorphism. Therefore $\theta(\mathcal{S})$ is freely generated by $\theta(\{a\}) = \{a\}$ and $\theta(\mathcal{T})$ is freely generated by $\theta(T) = \{(ad(a))^n(b) : n \in N \ \&\ b \in B\}$. Since θ is injective, $\theta(\mathcal{R}) = \theta(\mathcal{S}) \oplus \theta(\mathcal{T})$ and we conclude that

$$\mathcal{L} = \theta(\mathcal{R}) = \theta(\mathcal{S}) \oplus \theta(\mathcal{T}) = \mathcal{L}' \oplus \mathcal{L}''$$

□

The analogous result for Lie algebras (see [18] Theorem 0.6) is derived in the following example.

Example 5.0.2. *Consider a free Lie algebra L_A as an (ϵ, G)-Lie coloralgebra which is freely generated by A where $G = \{0\}$ and $\epsilon(0,0) = 1$. By Propostion 4.1.3, Theorem 5.0.4 becomes*

$$L_A = K \cdot \{a\} \oplus L'$$

where $L' = \{(ad(a))^n b : b \in A \setminus \{a\} \ \&\ n \geq 0\}$.

Consider the free magma $M(A)$. Let $\mathcal{L}$ be freely generated by A. We have a unique mapping $\psi : M(A) \longrightarrow \mathcal{L}$ such that $\psi(a) = a$ for all $a \in A$ and $\psi((s,t)) =$

$[\psi(s), \psi(t)]$ for all $s, t \in M(A)$ For example $\psi((a(bc)) = [a[bc]]$ if $a, b, c \in A$ For every $s \in M(A)$ let $\ell(s) : M(A) \longrightarrow M(A)$ such that $\ell(s)(t) = (st)$ for all $t \in M(A)$ For example, $(\ell(s))^3(t) = (s(s(st)))$ for all $s, t \in M(A)$ We denote the power set of $M(A)$ by $\mathcal{P}(M(A))$ and define the function $F : M(A) \times \mathcal{P}(M(A)) \longrightarrow \mathcal{P}(M(A))$ as $F(t, T) = \{(\ell(t))^m(x) : m \geq 0 \;\&\; x \in T \setminus \{t\}\}$

Definition 5.0.1 Let $n \geq 0$ $\{t_0\}, \ldots, \{t_n\}, T_{n+1} \in \mathcal{P}(M(A))$ is a *decomposition sequence* of $M(A)$ if and only if there exists $T_0, \ldots, T_n \in \mathcal{P}(M(A))$ such that

$$t_0 \in T_0 = A$$

$$t_{i+1} \in T_{i+1} = F(t_i, T_i) \; for \; i = 0, \ldots, n-1 \; when \; n > 0$$

$$T_{n+1} = F(t_n, T_n)$$

Corollary 5.0.5 *Let $\mathcal{L}$ be as defined in Theorem 5.0.4 and $\{t_0\}, \ldots, \{t_n\}, T_{n+1}$ be a decomposition sequence of $M(A)$ Then*

$$\mathcal{L} = \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_n \oplus \mathcal{L}'_{n+1}$$

where for $j \in \{0, \ldots, n\}$, $\mathcal{L}_j$ is the (ϵ, G)-Lie coloralgebra freely generated by $\psi(\{t_j\})$ and $\mathcal{L}'_{n+1}$ is the (ϵ, G)-Lie coloralgebra freely generated by $\psi(T_{n+1})$ Moreover, T_{n+1} is infinite and ψ restricted to T_{n+1} is injective

Proof By definition we have $T_0, \ldots, T_n \in \mathcal{P}(M(A))$ such that

$$t_0 \in T_0 = A$$

$$t_{i+1} \in T_{i+1} = F(t_i, T_i) \; for \; i = 0, \ldots, n-1 \; when \; n > 0$$

$$T_{n+1} = F(t_n, T_n)$$

We first prove this for the case when $n = 0$. By Theorem 5.0.4 we have that $\mathcal{L} = \mathcal{L}' \oplus \mathcal{L}''$ where $\mathcal{L}'$ and $\mathcal{L}''$ are freely generated by $\{a\}$ and $\{ad(a)^m(b) : m \geq 0 \,\&\, b \in A \setminus \{a\}\}$ respectively. Note that the set $\{\ell(a)^m(b) : m \geq 0 \,\&\, b \in A \setminus \{a\}\}$ is infinite because $A \setminus \{a\}$ is nonempty. Since $\psi(c) = c$ for all $c \in A$ and $\psi(\ell(a)^m(b)) = ad(a)^m(b)$ for all $m \geq 0$ and $b \in A \setminus \{a\}$, it is enough to show that ψ restricted to the set $\{\ell(a)^m(b) : m \geq 0 \,\&\, b \in A \setminus \{a\}\}$ is injective. Let $\psi(\ell(a)^i(b)) = \psi(\ell(a)^j(c))$, $i, j \geq 0$ and $b, c \in A \setminus \{a\}$. By Proposition 4.1.2 (iv) $\psi(\ell(a)^i(b)) = ad(a)^m(b) \neq 0$. Thus by the decomposition of $\mathcal{L}$ by length we have that $i = j$. Suppose that $b \neq c$. Then there is an (ϵ, G)-Lie coloralgebra homomorphism f on $\mathcal{L}$ such that $f(a) = a$, $f(b) = b$ and $f(c) = 0$. Thus $f(\psi(\ell(a)^i(b))) = \psi(\ell(a)^i(b)) \neq 0$ and $f(\psi(\ell(a)^j(c))) = 0$ which contradicts $\psi(\ell(a)^i(b)) = \psi(\ell(a)^j(c))$. So $b = c$.

Let $n > 0$ and the induction hypothesis be that the result holds for all $m < n$. Thus

$$\mathcal{L} = \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_{n-1} \oplus \mathcal{L}'_n$$

where $\mathcal{L}_i$ is freely generated by $\psi(t_i)$ for $i = 0, \ldots, n-1$ and $\mathcal{L}'_n$ is freely generated by $\psi(T_n)$. Let L be a (ϵ, G)-Lie coloralgebra which is freely generated by T_n where the grading on T_n is such that ψ restricted to T_n is a G-map. Since ψ restricted to T_n is injective, we have an (ϵ, G)-Lie coloralgebra isomorphism $f : L \longrightarrow \mathcal{L}'_n$ such that $f(t) = \psi(t)$ for all $t \in T_n$. By Theorem 5.0.4 we have that $L = L' \oplus L''$ where L' is freely generated by $\{t_n\}$ and L'' is freely generated by $\{ad(t_n)^m(t) : m \geq 0 \,\&\, t \in T_n \setminus \{t_n\}\}$. Thus $\mathcal{L} = f(L') \oplus f(L'')$ where $f(L')$ is freely generated by $f(\{t_n\}) = \psi(\{t_n\})$ and $f(L'')$ is freely generated by $\{\psi(\ell(t_n)^m(t)) : m \geq 0 \,\&\, t \in T_n \setminus \{t_n\}\} = \psi(T_{n+1})$ because $f(ad(t_n)^m(t)) = ad(f(t_n))^m(f(t)) = ad(\psi(t_n))^m(\psi(t)) = \psi(\ell(t_n)^m(t))$ for all $m \geq 0$ and $t \in T_n \setminus \{t_n\}$. Note that T_{n+1} is infinite because $T_n \setminus \{t_n\}$ is not empty. Let $i, j \geq 0$ and $x, y \in T_n \setminus \{t_n\}$. $\psi(\ell(t_n)^i(x)) = \psi(\ell(t_n)^j(y)) \in \mathcal{L}_n$ iff $ad(f(t_n))^i(f(x)) = ad(f(t_n))^j(f(y)) \in \mathcal{L}_n$ iff $f(ad(t_n)^i(x)) = f(ad(t_n)^j(y)) \in \mathcal{L}_n$ iff $ad(t_n)^i(x) = ad(t_n)^j(y) \in L''$ iff $i = j$ and $x = y$ as we saw in the $n = 0$ case. So ψ restricted to T_{n+1} is injective. □

A subset E of $M(A)$ is called *closed* iff $\forall s, t \in M(A), (st) \in E \Rightarrow s, t \in E$. In particular, $M(A)$ is closed.

Definition 5.0.2. A *Lazard set* is a linearly ordered subset Z of $M(A)$ such that for any finite nonempty closed subset E of $M(A)$, Z satisfies the Lazard condition with E. That is,

$$Z \cap E = \{t_0 < t_1 < \cdots < t_n\}$$

and

$$T_{n+1} \cap E = \emptyset$$

for some decomposition sequence $\{t_0\}, \ldots, \{t_n\}, T_{n+1}$ with $n \geq 0$.

For any $E \subseteq M(A)$ we define the *closure* $\overline{E}$ of E to be the least closed subset of $M(A)$ containing E. It is clear that if E is finite, then $\overline{E}$ is finite.

Corollary 5.0.6. *Let Z be a Lazard set and $\mathcal{L}$ be freely generated by a nonempty G-graded set A. $\mathcal{L}$ has the following decomposition as a K-module,*

$$\mathcal{L} = \bigoplus_{z \in Z} \mathcal{L}_z$$

where for every $z \in Z$, $\mathcal{L}_z$ is the subalgebra of $\mathcal{L}$ which is freely generated by $\{\psi(z)\}$.

Proof. First we consider the case that A consists of only one element. It is easy to verify that A is the only Lazard set in $M(A)$ and $\mathcal{L}$ is freely generated by $\psi(A) \equiv A$.

Now we consider the case where A has cardinality greater than 1. For each $m > 0$ we let $M^{\leq m}(A)$ denote the set consisting of all elements of $M(A)$ with length less than or equal to m. First we show that $\sum_{z \in Z} \mathcal{L}_z$ is a direct sum in $\mathcal{L}$. By Propositon 4.1.2 (iii), for each $z \in Z$, $\mathcal{L}_z = K \cdot \psi(M(\{z\}))$. Let E be a finite nonempty subset of Z. Consider $\sum_{z \in Z} x_z = 0$ such that $x_z \in \mathcal{L}_z$ for all $z \in E$ and $x_z = 0$ for all $z \in Z \setminus E$. We have $m \in N^+$ and $B \subseteq A$ such that B is finite and $E \subseteq M^{\leq m}(B)$. Clearly, $M^{\leq m}(B)$ is finite, nonempty and closed. Thus we have $n \in N$ such that $Z \cap M^{\leq m}(B) = \{t_0, \ldots, t_n\}$ and $Z \cap T_{n+1} = \emptyset$ which implies that $E \subseteq \{t_0, \ldots, t_n\}$. By Corollary 5.0.5, $x_z = 0$ for all $z \in E$.

Now we show that the direct sum is in fact $\mathcal{L}$. Let $x \in \mathcal{L}\setminus\{0\}$ and B be a finite subset of A such that $x \in K \cdot \psi(M(B))$. Thus we have $p \in N \setminus \{0\}$ such that $x \in K \cdot \psi(M^{\leq p}(B))$. We note that $M^{\leq p}(B)$ is finite, nonempty and closed. Thus we have $n \in N$ such that $Z \cap M^{\leq p}(B) = \{t_0, \ldots, t_n\}$ and $M^{\leq p}(B) \cap T_{n+1} = \emptyset$. Let $w \in M^{\leq p}(B)$. By Corollary 5.0.5 we have that $\psi(w) = y_1 + \cdots + y_n + v$ where $v \in K \cdot \psi(M(T_{n+1}))$ and $y_j \in K \cdot \psi(M(\{t_j\}))$ for all $j \in \{0, \ldots, n\}$. Since $\psi(w), y_0, \ldots, y_n \in K \cdot \psi(M(B))$, $v \in K \cdot \psi(M(B))$. Thus $v \in K \cdot \psi(M(T_{n+1}) \cap M(B))$. Indeed, when we consider the unique homomorphism ϕ on $\mathcal{L}$ extending the map from A into $\mathcal{L}$ such that it is the identity on B and takes all other elements of A to zero, we see that ϕ fixes v and takes $\psi(M(T_{N+1}) \setminus M(B))$ to zero. Since $M^{\leq p}(B)$ is closed and $M^{\leq p}(B) \cap T_{n+1} = \emptyset$, $M^{\leq p}(B) \cap M(T_{n+1}) = \emptyset$. Therefore $v \in K \cdot \psi(M(B) \setminus M^{\leq p}(B))$. We have that for every $j \in \{0, \ldots, n\}$, $y_j = r_j + s_j$ such that $r_j \in K \cdot \psi(M(\{t_j\}) \cap M^{\leq p}(B))$ and $s_j \in K \cdot \psi(M(\{t_j\}) \setminus M^{\leq p}(B))$. Thus $\psi(w) - r_0 - \cdots - r_n = s_0 + \cdots + s_n + v \in K \cdot (\psi(M(B) \setminus M^{\leq p}(B)))$ and $\psi(w) - r_0 - \cdots - r_n \in K \cdot \psi(M^{\leq p}(B))$ which, by Proposition 4.1.2(i), implies that $\psi(w) - r_0 - \cdots - r_n = 0$.

So $\psi(w) = r_0 + \cdots + r_n$ and $r_j \in \mathcal{L}_{t_j}$ for all $j \in \{0, \ldots, n\}$. We conclude that $\psi(M^{\leq p}(B)) \subseteq \bigoplus_{z\in Z} \mathcal{L}_z$ and hence that $x \in \bigoplus_{z\in Z} \mathcal{L}_z$. □

We note that $\epsilon(g, g) \in \{1, -1\}$ for every $g \in G$. Since $\psi(Z)$ is a set of homogeneous elements in $\mathcal{L}$, we can write Z as a disjoint union of Z_+ and Z_- where $Z_+ = \{z \in Z : \epsilon(g_z, g_z) = 1 \ \&\ \psi(z) \in \mathcal{L}_{g_z}\}$ and $Z_- = \{z \in Z : \epsilon(g_z, g_z) = -1 \ \&\ \psi(z) \in \mathcal{L}_{g_z}\}$.

Corollary 5.0.7 *Let $V = \{(zz) : z \in Z_-\}$. $\psi(Z) \cup \psi(V)$ is a disjoint union and a basis of $\mathcal{L}$.*

Proof. Let $x \in \mathcal{L}$. By Corollary 5.0.6, $x = \sum_{z\in Z} x_z$ where $x_z \in \mathcal{L}_z$ for all $z \in Z$. By Proposition 4.1.3, for every $z \in Z$ there exists $k_{z_1}, k_{z_2} \in K$ such that $x_z = k_{z_1}\psi(z) + k_{z_2}\psi((zz))$ if $z \in Z_-$ and $x_z = k_{z_1}\psi(z)$ if $z \in Z_+$. Thus we have that $\psi(Z) \cup \psi(V)$ spans $\mathcal{L}$. To show independence we let $0 = \sum_{j=1}^{m}(k_j\psi(z_j) + h_j\psi((z_jz_j)))$ where for all $0 < j \leq m$, $k_j, h_j \in K$, $\{z_1, \ldots, z_m\}$ are distinct in Z and $h_j = 0$ when $z_j \in Z_+$. By Corollary 5.0.6, $(k_j\psi(z_j) + h_j\psi((z_jz_j))) = 0$ for all $0 < j \leq m$. By Proposition 4.1.3 and the fact that $h_j = 0$ when $z_j \in Z_+$, we have that $k_j = h_j = 0$ for all $0 < j \leq m$. Thus we have independence. □

The basis for the free Lie algebra which is stated in Corollary 0.9 in [18] is derived from Corollary 5.0.7 in the following example.

Example 5.0.3 *Consider a free Lie algebra L_A as an (ϵ, G)-Lie coloralgebra which is freely generated by A and let $G = \{0\}$ and $\epsilon(0, 0) = 1$. When we apply Corollary 5.0.7 to L_A as an (ϵ, G)-Lie coloralgebra we have that Z_- is empty. Thus $\psi(Z)$ is a basis of L_A and we conclude that the Lazard set represents the basis of the free Lie algebra on A.*

Example 5.0.4 *Let Y denote the Lyndon words and $t : Y \longrightarrow M(A)$ be as it is defined in Example 2.0.2. $t(Y)$ is a Hall set in $M(A)$ (see Definition 2.0.1) under the total ordering $\leq$ as it is defined in Example 2.0.2. By Viennot's Theorem (see Appendix A), $t(Y)$ is a Lazard set. Thus we have that $\psi(t(Y)) \cup \{\psi((t(y), t(y)) : t(y) \in t(Y)_-\}$ is a basis of $\mathcal{L}$.*

Example 5.0.5 *Let Q be the rational numbers, Z_2 be the integers modulo 2 under addition and $\epsilon' : Z_2 \times Z_2 \longrightarrow Q$ be defined as $\epsilon'(m,n) = (-1)^{mn}$. Consider a Lie superalgebra $\mathcal{S}$ which is freely generated by the Z_2-set $A = A_0 \cup A_1$ where $A_0 = \{a\}$ and $A_1 = \{b\}$. Recall that a Lie superalgebra is an (ϵ', Z_2)-Lie coloralgebra. We order A as $a < b$ and let Y denote the Lyndon words as they are defined in Example 2.0.2. For example ab, abb, $aaab$, $abbb$ are all Lyndon words under this ordering. We define the partial degree of a word w with respect to b as the number of occurrences of the letter b in w and we denote this $|w|_b$. For example $|abaab|_b = 2$ and $|abbabb|_b = 4$. Thus $t(Y)_- = \{t(y) : y \in Y \ \&\ |y|_b \text{ is odd }\}$. So $\psi(t(Y)) \cup \{\psi((t(y), t(y))) : y \in Y \ \&\ |y|_b \text{ is odd}\}$ represents a basis of $\mathcal{S}$. For example $[a, b]$, $[[a, b], [a, b]]$, $[[a, b]b]$, $[a, [a, [a, b]]]$, $[[a, [a, [a, b]]], [a, [a, [a, b]]]]$ are basis elements.*

Bibliography

[1] Bergen, J and Grzeszczuk, P , *Invariants of Lie Color Algebras*, Colloquium Mathematicum, 83(2000), no 1, 107-110

[2] Bourbaki, N , *Algebra, Part* 1, Elements of Mathematics, Addison-Wesley, 1974

[3] Bourbaki, N , *Lie Groups and Lie Algebras, Part* 1, Elements of Mathematics, Addison-Wesley, 1974

[4] Cohn, P , *Universal Algebra*, Harper's Series in Modern Mathematics, Harper & Row, New York, Evanston, London, 1965

[5] Enderton, H , *A Mathematical Introduction to Logic*, Academic Press, Boston, San Diego, New York, 1972

[6] Fraleigh, J , *A First Course in Abstract Algebra*, Addison-Wesley, New York, 1993

[7] Grove, L C , *Algebra*, Academic Press, New York, London, Paris, Toronto, 1983

[8] Hall M , Jr , *A Basis for Free Lie Rings and Higher Commutators in Free Groups*, Proceedings of the American Mathematical Society, 1(1950), 575-581

[9] Hartly, B and Hawks, T , *Rings, Modules, and Linear Algebra*, Chapman and Hall Mathematics Series, Chapman and Hall, 1970

[10] Hoffman, K and Kunze, R , *Linear Algebra*, Prentice Hall, Engelwood Cliffs, New Jersey, 1971

[11] Kac, V , *Infinite Dimensional Lie Algebras*, Cambridge University Press, Cambridge, New York, Melbourne, 1985

[12] Lazard, M , *Groupes, anneaux de Lie et probléme de Burnside*, Istituto Matematico dell' Universitá di Roma, 1960

[13] Lothaire, M , *Combinatorics on Words*, Cambridge University Press, Cambridge, New York, Melbourne, 1983

[14] Melancon, G , *Récritures dans le groupe libres, l'algèbre libres et l'algèbre de Lie libres*, Thèse Mathématique de l'Université du Québec à Montréal, 1991

[15] Mikhalev, A , *Free Color Lie Superalgebras*, Soviet Mathematics Doklady 33(1986), no 1, 136-139

[16] Montgomery, S , *Constructing Simple Lie Super Algebras from Associative Graded Algebras*, Journal of Algebra, 195(1997), 558-579

[17] Ree, R , *Generalized Lie Elements*, Canadian Journal of Mathematics, 12(1960), 493-502

[18] Reutenhauer, C., *Free Lie Algebras*, London Mathematical Society Monographs, Clarendon Press, Oxford, 1993.

[19] Scheunert, M., *The Theory of Lie Superalgebras*, Lecture Notes in Mathematics, Vol. 716, Springer-Verlag, Berlin, 1979.

[20] Viennot, X., *Algèbres de Lie libres et monoïdes libres*, Lecture Notes in Mathematics, Vol. 691, Springer, Berlin, 1978.

1 661

Appendix A

Viennot's Theorem

The following theorem is due to Viennot in [20] which was taken from [18]. We prove the result using the definitions of a Hall set (Definition 2.0.1) and a Lazard set (Definition 5.0.2) which are the reverse of those used by Reutenhauer. Note that we only prove half of Viennot's Theorem because it is all that is required for our results but it should be noted that the converse of Theorem A.0.9 is also true.

We define a function $F : M(A) \times \mathcal{P}(M(A)) \longrightarrow \mathcal{P}(M(A))$ as $F(t,T) = \{\ell(t)^m(x) : x \in T \setminus \{t\} \;\&\; m \geq 0\}$. Let $t_0, \ldots, t_n$ be elements in $M(A)$.

Lemma A.0.8. *Let H be a Hall set in $M(A)$, c be a least element of A and $X = \{\ell(c)^n(a) : a \in A \setminus \{c\} \;\&\; n \geq 0\}$. Then*

(i) c is a least element of H

(ii) $H' = H \cap M(X)$ is a Hall set in $M(X)$

(iii) $H = \{c\} \cup H'$

Proof (i) When $h \in A$ this is obvious Suppose that m is the least positive integer such that there exists $h \in H$ with length m and $h < c$ Since h has length ≥ 1, we have that $h = (h_1, h_2)$ and $h_1 < h$ Since the length of h_1 is less than m we have a contradiction $c < h_1 < h$

(ii) It is enough to show that $X \subseteq H'$ It is obvious that $A \setminus \{c\}$ is contained in H' Suppose that m is the least positive integer such that $x = \ell(c)^m(a) \in X \setminus H$ Since $m \geq 1$, $x = (c, \ell(c)^{m-1}(a))$, $c < \ell(c)^{m-1}(a)$ Either $m - 1 > 0$ and $c \geq c$ or $x = (c, a)$ In both cases we have $x \in H$ which is a contadiction

(iii) It is enough to show that $H \subseteq \{c\} \cup H'$ Suppose that m is the least positive integer such that $h \in H \setminus (\{c\} \cup H')$ with length m Note that $m > 1$ because $A \subseteq \{c\} \cup H'$ So we have $(h_1, h_2) = h \in H$ with $h_1, h_2 \in H \cap M(X)$ which implies that $h \in H \cap M(X) = H'$ This is a contradiction

□

Theorem A 0 9 *Every Hall set is a Lazard set*

Proof We want to show that if H is a Hall set in $M(A)$ and E is a nonempty finite closed subset of $M(A)$, then H satisfies the Lazard condition with E (see Definition 5 0 2) We do this by induction on the cardinality of E Let H be a Hall set in $M(A)$ and $E = \{a\}$ $E \cap H = \{a\}$ and $E \cap F(a, A) = \emptyset$ As an induction hypothesis, let every Hall set in $M(C)$ satisfy the Lazard condition with every nonempty closed subset of $M(C)$ with cardinality less than $m > 0$ for every nonempty set C We let H be a Hall set in $M(A)$ and E be a closed subset of $M(A)$ with cardinality $m > 1$ Let B be the finite subset of A equal to $A \cap E$ Note that $E \subseteq M(B)$ First we show that the Hall set $H \cap M(B)$ in $M(B)$ satisfies the Lazard condition with E

Let $X = F(c, B)$ where c is the least element of B with respect to the order of H. Let E' be the closed set $E \cap M(X)$ in $M(X)$ and $H' = H \cap M(X)$ which is Hall set in $M(X)$ by the above Lemma. The cardinality of E' is less than m because $c \notin M(X)$. If E' is empty, then $E \cap X = \emptyset$ and $E \cap B = \{c\}$ which implies that $H \cap E = \{c\}$ and we have that H satisfies the Lazard condition with E. So we let E' be nonempty. By the hypothesis we have that H' satisfies the Lazard condition with E' as $H' \cap E' = \{t_0 <, \ldots, < t_n\}$ and $T_{n+1} \cap E' = \emptyset$. By the previous Lemma $H \cap M(B) = \{c\} \cup H'$ which intersects E at $\{c < t_0 < \ldots t_n\}$ and $T_{n+1} \cap E = \emptyset$ because $T_{n+1} \subseteq M(X)$. Thus $H \cap M(B)$ satisfies the Lazard condition with E in $M(B)$. Thus $H \cap E = \{c = s_0 < \cdots < s_{n+1} = t_n\}$ with $s_i \subseteq S_i$ where $S_0 = A$ and $S_{i+1} = F(s_i, S_i)$ for $i = 0, \ldots, n+1$. We show that $S_j \cap M(B) = T_{j-1}$ for $j = 1, \ldots, n+2$. Note that $S_0 \cap M(B) = B$ and $S_1 \cap M(B) = F(c, B) = T_0$. So $S_k \cap M(B) = F(s_{k-1}, S_{k-1}) \cap M(B) = F(s_{k-1}, S_{k-1} \cap M(B))$ which by hypothesis equals to $F(t_{k-2}, T_{k-2}) = T_{k-1}$. Thus $S_{n+2} \cap E = (S_{n+2} \cap M(B)) \cap E = T_{n+1} \cap E = \emptyset$. So H satisfies the Lazard condition with E. $\square$

Appendix B

Lyndon Words

The definition of a Lyndon word is found in Example 2.0.2.

Proposition B.0.10. *Let A be totally ordered by $<$, Y be the Lyndon words on A^* with respect to this ordering and t as in Example 2.0.2. The set $t(Y)$ is a Hall set with respect to the ordering in Example 2.0.2.*

Proof. Note that it is clear that t is injective. Thus the ordering on $t(Y)$ defined in Example 2.0.2 is a total ordering. Also $A \subseteq Y$ by the definition of Y and t. Let $y \in Y \setminus A$. Thus $t(y) = (t(u), t(v))$ and $u < y$ which implies that $t(u) < t(y)$. So all we must prove is the last property in the list in Definition 2.0.1. In this section we say that a word x is a left factor of another word u iff $u = xv$ for some $v \in A^*$. Moreover, we say x is proper when $x \neq u$.

Let $t(y) = (t(u), t(v))$ where $y \in Y$. Thus we have that $u < y = uv < v$ which implies that $t(u) < t(v)$. Since u is the greatest proper left factor of y, u is a Lyndon word. Let w be a proper left factor of v. Suppose that $w > v$. Thus $v = zw$ for some $z \neq 1$ because $uw < uv = y$. Therefore $uw < uzw = y$ which implies that

$u < uz$ This is a contradiction Thus v is a Lyndon word Now let $t(v) = (t(r), t(s))$ Suppose that $t(r) > t(u)$ Since $ur < u$, $u = xr$ Note that $x \neq 1$ because $u < r$ Thus $xr^2 < xr$ which implies that $xr < x$ This is not possible because $xr = u$ and u is a Lyndon word So we have a contradiction and conclude that $t(r) \leq t(u)$

Now we prove the converse Let u and v be Lyndon words with $u < v$ In the case that $v \in A$, every left factor of u is less than or equal to u which implies that uv is a Lyndon word and that $(t(u), t(v)) = t(uv) \in t(Y)$ Consider the case that $t(v) = (t(r), t(s))$ with $u \geq r$ Thus $ur < r \leq u$ which implies that u is the greatest proper left factor of uv So it is enough to show that $u < uv$ Suppose that $u > uv$ Since $u < v$, $u = xv$ Thus $x > u$ because $xv > xvv$ This is a contradiction because u is a Lyndon word So $u < uv$ □

Vita

Surname: Sadro — Given Names: Paul

Place of Birth: Nelson, British Columbia, Canada — Date of Birth: Aug 12 1974

Educational Institutions Attended:

University of Victoria	1999 to 2001
Simon Fraser University	1992 to 1998

Degree Awarded:

B.S.	Simon Fraser University	1998

Honours and Awards:

Graduate Teaching and Research Fellowship	2000-2001

PARTIAL COPYRIGHT LICENSE