CONVEX AND STARLIKE GENERALIZED HYPERGEOMETRIC FUNCTIONS ASSOCIATED WITH THE HARDY SPACE

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Abstract

The object of this paper is to present generalizations of some recent results of S.S. Miller and P.T. Mocanu on the univalence and starlikeness of Kummer's confluent hypergeometric function. Some applications involving generalized hypergeometric functions associated with the Hardy space of analytic functions are also considered.

1. Introduction and Definitions

A function f, analytic in the *open* unit disk $\mathcal{U}$, is said to be convex if it is univalent and $f(\mathcal{U})$ is convex. It is well known that f is convex of order α if and only if

$$f'(0) \neq 0 \quad \text{and} \quad \Re\left(1 + \frac{z f''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1). \tag{1.1}$$

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If $\alpha = 0$, then f is called convex in $\mathcal{U}$.

A function f analytic in $\mathcal{U}$, with $f(0) = 0$, is said to be starlike if it is univalent and $f(\mathcal{U})$ is starlike with respect to the origin. The function f is starlike in $\mathcal{U}$ if and only if

$$f(0) = 0, \quad f'(0) \neq 0, \quad \text{and} \quad \Re\left(\frac{z f'(z)}{f(z)}\right) > 0 \qquad (z \in \mathcal{U}).$$

If, in addition,

$$\Re\left(\frac{z f'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1),$$

then f is called starlike of order α in $\mathcal{U}$.

Let λ_j $(j = 1, \cdots, \rho)$ and μ_j $(j = 1, \cdots, \sigma)$ be complex numbers such that

$$\mu_j \neq 0, -1, -2, \cdots \qquad (j = 1, \cdots, \sigma).$$

Then the generalized hypergeometric function ${}_\rho F_\sigma(z)$ is defined by

$$\begin{aligned} {}_\rho F_\sigma(z) &\equiv {}_\rho F_\sigma(\lambda_1, \cdots, \lambda_\rho;\ \mu_1, \cdots, \mu_\sigma;\ z) \\ &:= \sum_{k=0}^{\infty} \frac{(\lambda_1)_k \cdots (\lambda_\rho)_k}{(\mu_1)_k \cdots (\mu_\sigma)_k} \frac{z^k}{k!} \qquad (\rho \leq \sigma + 1), \end{aligned} \tag{1.2}$$

where $(\lambda)_k$ denotes the Pochhammer symbol defined, in terms of Γ-functions, by

$$(\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k = 0) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N} := \{1, 2, 3, \cdots\}). \end{cases} \tag{1.3}$$

Indeed it is known that the ${}_\rho F_\sigma(z)$ series in (1.2) converges absolutely for $|z| < \infty$ if $\rho < \sigma + 1$, and for $z \in \mathcal{U}$ if $\rho = \sigma + 1$.

Let $\mathcal{H}^p$ $(0 < p \leq \infty)$ denote the Hardy space of analytic functions $f(z)$ in $\mathcal{U}$, and define the integral means

$$M_p(r, f) := \begin{cases} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} & (0 < p < \infty) \\ \max\limits_{|z| \leq r} |f(z)| & (p = \infty). \end{cases} \tag{1.4}$$

Then, by definition, an analytic function $f(z)$ in $\mathcal{U}$ belongs to the Hardy space $\mathcal{H}^p$ $(0 < p \leq \infty)$ if

$$\lim_{r \to 1-} \{M_p(r, f)\} < \infty \qquad (0 < p \leq \infty). \tag{1.5}$$

For $1 \leq p \leq \infty$, $\mathcal{H}^p$ is a Banach space with the norm defined by (*cf.* [1, p. 23])

$$\|f\|_p = \lim_{r \to 1-} \{M_p(r,f)\} \qquad (1 \leq p \leq \infty). \tag{1.6}$$

Furthermore, $\mathcal{H}^\infty$ is the class of bounded analytic functions in $\mathcal{U}$.

In this paper, we generalize certain recent results of Miller and Mocanu [7] on the univalence and starlikeness of Kummer's confluent hypergeometric function. We also give some applications involving generalized hypergeometric functions associated with the Hardy space $\mathcal{H}^p$ defined above.

2. Preliminary Results

The following results will be required in our investigation.

Lemma 1. *Let E be a set in the complex plane $\mathbb{C}$ and let a function $H : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfy the condition:*

$$\begin{gathered} H(is,t,u+iv;z) \notin E \quad \text{for} \quad z \in \mathcal{U} \quad \text{and for real } s,t,u, \text{ and } v \text{ satisfying} \\ t \leq -(1+s^2)/2 \quad \text{and} \quad t+u \leq 0. \end{gathered} \tag{2.1}$$

If p is analytic in $\mathcal{U}$, with $p(0) = 1$ and $H\left(p(z), zp'(z), z^2p''(z); z\right) \in E$ $(z \in \mathcal{U})$, then

$$\Re\{p(z)\} > 0 \qquad (z \in \mathcal{U}).$$

This lemma is a special case of Theorem 1 in [6] with $q(z) = (1+z)/(1-z)$.

If we only have $H : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$, then the condition (2.1) simplifies to

$$H(is,t;z) \notin E \quad \text{for} \quad z \in \mathcal{U} \quad \text{and for real } s \text{ and } t \text{ with} \quad t \leq -(1+s^2)/2. \tag{2.2}$$

Lemma 2 (Eenigenburg and Keogh [2, Theorem 4]). *If a function f, convex of order α $(0 \leq \alpha < 1)$ in $\mathcal{U}$, is not of the form:*

$$\begin{cases} f(z) = a + bz(1-ze^{i\tau})^{2\alpha-1} & (\alpha \neq \frac{1}{2}) \\ f(z) = a + b\log(1-ze^{i\tau}) & (\alpha = \frac{1}{2}), \end{cases}$$

for some complex numbers a and b, and for some real number τ, then the following statements hold true:

(i) *There exists $\delta = \delta(f) > 0$ such that $f'(z) \in \mathcal{H}^{\delta+1/[2(1-a)]}$*

(ii) *If $0 \leq \alpha < \frac{1}{2}$, then there exists $\epsilon = \epsilon(f) > 0$ such that $f(z) \in \mathcal{H}^{\epsilon+1/(1-2\alpha)}$*

(iii) *If $\alpha \geq \frac{1}{2}$, then $f(z) \in \mathcal{H}^{\infty}$.*

Lemma 3 (Kim and Srivastava [5, Theorem 1 and Theorem 3]). *Let the parameters $\alpha_1, \cdots, \alpha_s$ and $\beta_1, \cdots, \beta_s$ be complex numbers such that*

$$\beta_j \neq 0, -1, -2, \cdots \qquad (j = 1, \cdots, s),$$

and let ω be defined by

$$\omega := \sum_{j=1}^{s} \beta_j - \sum_{j=1}^{s} \alpha_j \; . \tag{2.3}$$

Then the generalized hypergeometric function

$${}_{\rho+s}F_{\sigma+s}(z) = {}_{\rho+s}F_{\sigma+s}(\lambda_1, \cdots, \lambda_\rho, \alpha_1, \cdots, \alpha_s; \mu_1, \cdots, \mu_\sigma, \beta_1, \cdots, \beta_s; z) \tag{2.4}$$

defined by (1.2) is in the Hardy space $\mathcal{H}^{\infty}$ if

$$\Re\{({}_\rho F_\sigma(z))'\} > 0 \qquad \textit{and} \qquad \Re(\omega) > 0 \tag{2.5}$$

or if

$$\Re\left\{1 + \frac{z({}_\rho F_\sigma(z))''}{({}_\rho F_\sigma(z))'}\right\} > 0 \qquad \textit{and} \qquad \Re(\omega) > 1. \tag{2.6}$$

Remark 1. For further results involving generalized hypergeometric functions and the Hardy space $\mathcal{H}^{\infty}$, see (for example) a recent work by Srivastava [9].

Lemma 4 (Miller and Mocanu [7, Theorem 2]). *If $a \neq 0$ and c are real numbers and satisfy $c > N(a)$, where*

$$N(a) := \begin{cases} |a| + \frac{1}{2} & (|a| \geq \frac{1}{3}) \\ \frac{3}{2}\,a^2 + \frac{2}{3} & (|a| \leq \frac{1}{3}), \end{cases} \tag{2.7}$$

then $\Phi(a; c; z) := {}_1F_1(a; c; z)$ is convex in $\mathcal{U}$.

3. Two Theorems on Kummer's Confluent Hypergeometric Function

Throughout this paper, we let ${}_1F_1(a;c;z) = \Phi(a;c;z)$, just as in Lemma 4.

By using the method of Miller and Mocanu [7], we obtain the following generalizations of Lemma 4 and of another result of Miller and Mocanu [7, Corollary 2.1]. It should be mentioned here that, in a recent paper, Noor [8] considered the convexity of Kummer's confluent hypergeometric function $\Phi(a;c;z)$, but her assertion [8, Equation (2.3)], corresponding to Equation (3.1) below, seems to be in error.

Theorem 1. *Let* $0 \leq \alpha < 1$. *If* $a \neq 0$ *and* c *are real numbers and satisfy* $c > N(a,\alpha)$, *where*

$$N(a,\alpha) := \begin{cases} \frac{4\alpha^2-7\alpha+3}{6-4\alpha} + \frac{|a+\alpha|}{1-\alpha} & \left(|a+\alpha| \geq \frac{1-\alpha}{3-2\alpha}\right) \\ \frac{1}{6-4\alpha}\left[4\alpha^2 - 7\alpha + 4 + \left\{\frac{(a+\alpha)(3-2\alpha)}{1-\alpha}\right\}^2\right] & \left(|a+\alpha| \leq \frac{1-\alpha}{3-2\alpha}\right), \end{cases} \tag{3.1}$$

then $\Phi(a;c;z)$ *is convex of order* α *in* $\mathcal{U}$.

Proof. The inequality $c > N(a,\alpha)$ can be shown to imply that

(i) $a > -1$ and $c \geq a$

or

(ii) $a \leq -1$ and $c \geq \{1 + (a+1)^2\}^{\frac{1}{2}}$.

Therefore, by Theorem 1 of Miller and Mocanu [7, p. 336], we have

$$\Phi'(z) = \Phi'(a;c;z) \neq 0. \tag{3.2}$$

Setting

$$p(z) = 1 + \frac{z\,\Phi''(z)}{(1-\alpha)\,\Phi'(z)},$$

where $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = 1$, and noting that the Φ-function satisfies the (Kummer's) differential equation:

$$z\,\Phi''(z) + (c-z)\,\Phi'(z) - a\,\Phi(z) = 0 \qquad (\Phi = \Phi(a;c;z)), \tag{3.3}$$

we find that

$$zp'(z) + (1-\alpha)\{p(z)\}^2 + (c-2-z+2\alpha)\,p(z) - \left(\frac{a+\alpha}{1-\alpha}\right) z - c + 1 - \alpha = 0. \tag{3.4}$$

If we let

$$H(w_1,w_2;z) := w_2 + (1-\alpha)\,w_1^2 + (c-2-z+2\alpha)\,w_1 + \left(\frac{a+\alpha}{1-\alpha}\right) z - c + 1 - \alpha$$

and $E := \{0\}$, then

$$\begin{aligned}\Re\left\{H(is,t;z)\right\} &= t-(1-\alpha)\,s^2+ys-\left(\frac{a+\alpha}{1-\alpha}\right)x-c+1-\alpha \\ &\leq -\frac{1}{2}\left[(3-2\alpha)\,s^2-2ys+2\left(\frac{a+\alpha}{1-\alpha}\right)x+2c-1+2\alpha\right] \equiv Q(s) \\ &\qquad \left(t \leq -(1+s^2)/2;\quad z=x+iy\right).\end{aligned} \tag{3.5}$$

Then the discriminant D of $Q(s)$ satisfies

$$\begin{aligned}D &= y^2-\left[\frac{(a+\alpha)(6-4\alpha)}{1-\alpha}\,x+6c-3+7\alpha-4\alpha c-4\alpha^2\right] \\ &< 4-6c-7\alpha+4\alpha c+4\alpha^2-\frac{(a+\alpha)(6-4\alpha)}{1-\alpha}\,x-x^2 \equiv h(x).\end{aligned} \tag{3.6}$$

If

$$|a+\alpha| \leq \frac{1-\alpha}{3-2\alpha},$$

then

$$h'(x_0)=0 \quad \text{for} \quad x_0=-\frac{(a+\alpha)(3-2\alpha)}{1-\alpha}.$$

Using (3.1) we obtain

$$h(x) \leq h(x_0) = 4-6c-7\alpha+4\alpha c+4\alpha^2+\left\{\frac{(a+\alpha)(3-2\alpha)}{1-\alpha}\right\}^2 \leq 0 \qquad (-1<x<1). \tag{3.7}$$

If, on the other hand,

$$|a+\alpha| \geq \frac{1-\alpha}{3-2\alpha},$$

then $h(x)$ is monotone on $(-1,1)$, and from (3.1) we deduce

$$h(x) < 3-6c-7\alpha+4\alpha c+4\alpha^2+\frac{6-4\alpha}{1-\alpha}\,|a+\alpha| \leq 0. \tag{3.8}$$

Hence, in every case, $D<0$ for $x^2+y^2<1$.

Using (3.1) again, we find that $Q(0) < 0$. Hence $\Re\{H(is,t;z)\} < 0$ for $z \in \mathcal{U}$ and for all real numbers s and t with $t \leq -(1+s^2)/2$. By Lemma 1, we conclude that

$$\Re\{p(z)\} = \Re\left[1 + \frac{z\,\Phi''(z)}{(1-\alpha)\,\Phi'(z)}\right] = \Re\left[\frac{1}{1-\alpha}\left(1-\alpha+\frac{z\,\Phi''(z)}{\Phi'(z)}\right)\right] > 0, \tag{3.9}$$

which shows that Φ is convex of order α in $\mathcal{U}$.

Remark 2. If $\alpha = 0$ in Theorem 1, then $N(a,0) = N(a)$, where $N(a)$ is defined by (2.7). Hence Theorem 1 provides a generalization of Lemma 4.

Making use of Theorem 1 and another known result [3, p. 473, Corollary ?], we can prove

Corollary 1. *If* $c > N(a,\alpha)$, *where* $N(a,\alpha)$ *is defined by* (3.1), *then*

$$\Re\{\Phi'(a;c;z)\}^{1/[2(1-\alpha)]} > \frac{1}{2} \qquad (z \in \mathcal{U}). \tag{3.10}$$

Corollary 2. *If* $a \neq 0$ *and* c *are real numbers and satisfy* $c > N(a,\alpha)$, *where* $N(a,\alpha)$ *is defined by* (3.1), *then*

$$z\left[\frac{a}{c}\,\Phi(a+1;c+1;z)\right]^{1/(1-\alpha)} \tag{3.11}$$

is starlike in $\mathcal{U}$.

Proof. From the definition given in Section 1, it is easy to see that f is convex of order α if and only if there exists a convex function g such that

$$f'(z) = [g'(z)]^{1-\alpha}. \tag{3.12}$$

Since $c > N(a,\alpha)$, $\Phi(a;c;z)$ is convex of order α, by Theorem 1.

Hence, by the equation (3.12), there exists a convex function g such that

$$z\,[\Phi'(a;c;z)]^{1/(1-\alpha)} = z\,g'(z). \tag{3.13}$$

Since $g(z)$ is a convex in $\mathcal{U}$, $z\,g'(z)$ is starlike in $\mathcal{U}$. Thus the proof of Corollary 2 is completed.

Corollary 3. *If $a \neq 0$ and c are real and satisfy $c > N(a, \alpha)$, where $N(a, \alpha)$ is defined by* (3.1), *then*

$$\begin{array}{lll} \text{(i)} & \Phi(a;c;z) \in \mathcal{H}^{\infty} & (\alpha \geq \tfrac{1}{2}) \\ \text{(ii)} & \Phi(a;c;z) \in \mathcal{H}^{1/(1-2\alpha)} & (0 \leq \alpha < \tfrac{1}{2}). \end{array}$$

Proof. Since

$$\frac{1}{(1 - ze^{i\tau})^{1-2\alpha}} = {}_2F_1(1, 1-2\alpha; 1; ze^{i\tau}) \qquad \left(\alpha \neq \frac{1}{2}\right) \tag{3.14}$$

for real τ, $\Phi(a;c;z)$ is not of the form:

$$\begin{cases} 1/(1 - ze^{i\tau})^{1-2\alpha} & (\alpha \neq \frac{1}{2}) \\ \log(1 - ze^{i\tau}) & (\alpha = \frac{1}{2}). \end{cases}$$

Hence, by Lemma 2 and Theorem 1, the proof of Corollary 3 is completed.

Theorem 2. *If $a \neq -1$ and c are real numbers and satisfy $c > N(a+1) - 1$, where $N(a)$ is defined by* (2.7), *then $\Phi(a;c;z) \in \mathcal{H}^{\infty}$.*

Proof. From the definition (1.2), we obtain

$$\Phi'(a;c;z) = \frac{a}{c}\, \Phi(a+1; c+1; z). \tag{3.15}$$

Also, by the hypergeometric representation (3.14), we observe that $\Phi(a+1;c+1;z)$ is not of the form $1/(1 - ze^{i\tau})$. Combining (3.15) and Lemma 4, $\Phi'(a;c;z)$ is convex in $\mathcal{U}$ and is not of the form $1/(1 - ze^{i\tau})$.

Hence, by Lemma 2, we have

$$\Phi'(a;c;z) \in \mathcal{H}^1. \tag{3.16}$$

Therefore, by appealing to a known result [1, Theorem 3.11], $\Phi(a;c;z)$ is continuous in

$$\bar{\mathcal{U}} := \mathcal{U} \cup \partial\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| \leq 1\},$$

so that $\Phi(a;c;z)$ is a bounded analytic function in $\mathcal{U}$. This completes the proof of Theorem 2.

4. Results Involving Generalized Hypergeometric Functions

Theorem 3. *Let* $0 \leq \alpha < 1$. *If* a, b, *and* c *are real numbers and satisfy* $-2 \leq a < 0$, $-1 \leq b$, $b \neq 0$, *and* $c > M(a, b, \alpha)$, *where*

$$M(a,b,\alpha) := \max\left\{2 + |a+b|,\ \frac{1-\alpha(2+a+b)-ab}{1-\alpha},\ \frac{\alpha(2\alpha+a+b-1)+ab}{1-\alpha}\right\}, \tag{4.1}$$

then

$$F(z) := {}_2F_1(a, b; c; z)$$

is convex of order α *in* $\mathcal{U}$.

Proof. The proof of Theorem 3 is much akin to that of an earlier result of Miller and Mocanu [7, Theorem 4]. Indeed, instead of assuming

$$p(z) = 1 + \frac{z\,F''(z)}{F'(z)},$$

which is derived from the proof of [7, Theorem 4], if we assume that

$$p(z) = 1 + \frac{z\,F''(z)}{(1-\alpha)\,F'(z)},$$

we would get

$$\begin{aligned} &z(1-z)\,p'(z) + (1-\alpha)(1-z)\,\{p(z)\}^2 + \{c - 2 + 2\alpha - (a+b+2\alpha)z\}p(z) \\ &\qquad - \frac{(\alpha+b)a + (\alpha+b)\alpha}{1-\alpha}\,z - c + 1 - \alpha = 0. \end{aligned} \tag{4.2}$$

By using the condition on $M(a, b, \alpha)$, we can rewrite (4.2) in the form:

$$J(z)\left[z\,p'(z) + (1-\alpha)\{p(z)\}^2\right] + p(z) + \frac{1}{2}\left[J(z) - K(z)\right] = 0, \tag{4.3}$$

where

$$J(z) := \frac{1-z}{c - 2 + 2\alpha - (a+b+2\alpha)\,z}$$

and

$$K(z) := \frac{2c - 1 + 2a - \left\{\frac{1-\alpha(1+2a+2\alpha+2b)-2ab}{1-\alpha}\right\}\,z}{c - 2 + 2\alpha - (a+b+2\alpha)\,z}.$$

Applying the method of proof of the aforementioned result of Miller and Mocanu [7, Theorem 4], we find that $\Re\{p(z)\} > 0$ $(z \in \mathcal{U})$. This implies that

$$\Re\left\{1+\frac{z\,F''(z)}{F'(z)}\right\} > \alpha \qquad (z \in \mathcal{U}; \quad 0 \leq \alpha < 1),$$

which evidently completes the proof of Theorem 3.

Remark 3. If $\alpha = 0$ in Theorem 3, then $M(a,b,0) = M(a,b)$, where $M(a,b)$ is defined by [7, p. 340, Equation (27)]. Hence Theorem 3 provides a slight generalization of the Miller-Mocanu result [7, Theorem 4].

Corollary 4. *If a, b, and c are real numbers and satisfy*

$$-1 \leq a \leq 1, \quad 0 \leq b, \quad \textit{and} \quad c > 1 + M(a-1, b-1, \alpha),$$

where $M(a,b,\alpha)$ is defined by (4.1), *then*

$$z\,{}_2F_1(a,b;c;z)$$

is starlike of order α in $\mathcal{U}$.

Making use of Lemma 3, Lemma 4, and a result of Miller and Mocanu [7, Theorem 1], it is not difficult to prove

Theorem 4. *Let the parameters $\alpha_1, \cdots, \alpha_s$ and $\beta_1, \cdots, \beta_s$ be complex numbers such that*

$$\beta_j \neq 0, -1, -2, \cdots \qquad (j = 1, \cdots, s),$$

and let ω be defined by (2.3). *If $a \neq 0$, b, and c are real numbers and satisfy one of the following conditions*:

(i) $-1 < a \leq c < 0$ *and* $\Re(\omega) > 0$

or

(ii) $c > N(a)$ *and* $\Re(\omega) > 1$,

then

$$_{s+1}F_{s+1}(a, \alpha_1, \cdots, \alpha_s; c, \beta_1, \cdots, \beta_s; z) \in \mathcal{H}^{\infty}, \tag{4.4}$$

where $N(a)$ is defined by (2.7).

Proof. We consider the following two cases:

Case 1. $-1 < a \leq c < 0$ and $\Re(\omega) > 0$. By a known result [7, Theorem 1], we have

$$\Re\left\{\Phi'(a;c;z)\right\} > 0. \tag{4.5}$$

Hence $\Phi(a;c;z)$ satisfies the conditions (2.5) in Lemma 3. Therefore, by Lemma 3, we obtain the assertion (4.4) under the conditions (i).

Case 2. $c > N(a)$ and $\Re(\omega) > 1$. By Lemma 4, $\Phi(a;c;z)$ is convex. Hence $\Phi(a;c;z)$ satisfies the conditions (2.6) in Lemma 3. By Lemma 3, we have the assertion (4.4) under the conditions (ii).

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References

[1] P.L. Duren, *Theory of H^p Spaces*, A Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol. **38**, Academic Press, New York and London, 1970.

[2] P.J. Eenigenburg and F.R. Keogh, The Hardy class of some univalent functions and their derivatives, *Michigan Math. J.* **17**(1970), 335–346.

[3] I.S. Jack, Functions starlike and convex of order α, *J. London Math. Soc.* (2) **3**(1971), 469–474.

[4] Y.C. Kim, K.S. Lee, and H.M. Srivastava, Certain classes of integral operators associated with the Hardy space of analytic functions, *Complex Variables Theory Appl.* **20**(1992), 1–12.

[5] Y.C. Kim and H.M. Srivastava, Some families of generalized hypergeometric functions associated with the Hardy space of analytic functions, *Proc. Japan Acad. Ser. A Math. Sci.* **70**(1994), 41–46.

[6] S.S. Miller and P.T. Mocanu, Differential subordinations and inequalities in the complex plane, *J. Differential Equations* **67**(1987), 199–211.

[7] S.S. Miller and P.T. Mocanu, Univalence of Gaussian and confluent hypergeometric functions, *Proc. Amer. Math. Soc.* **110**(1990), 333–342.

[8] K.I. Noor, The order of starlikeness and convexity of confluent hypergeometric functions, *Nihonkai Math J.* **3**(1992), 109–114.

[9] H.M. Srivastava, A certain class of generalized hypergeometric functions associated with the Hardy space of analytic functions, in *Current Topics in Analytic Function Theory* (H.M. Srivastava and S. Owa, Editors), World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992, 337–348.