

The Toeplitz C^* -algebra of the semigroup of principal ideals in a number field

by

Jason Samuel Peebles
B.Sc., University of Victoria, 2005

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Abstract

We consider the semigroup of principal integral ideals, P , in a number field and study its associated Toeplitz representation. From this specific representation, a certain covariance relation is obtained and subsequently arbitrary isometric representations of P which satisfy this relation are analyzed. This leads to the study of the universal C^* -algebra $C^*(P)$ satisfying these relations and to the following results. We first express $C^*(P)$ as a crossed product of an abelian C^* -algebra by endomorphisms associated to P . We then give an explicit characterization of faithful representations of this crossed product, from which it follows as an immediate corollary that the Toeplitz C^* -algebra is in fact isomorphic to the universal C^* -algebra.

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Jason Peebles

Chapter 1

Introduction

To one studying C^* -algebras, one of the most classical and well-known results in the industry of generating C^* -algebras by various families of isometries is Coburn's theorem [2]. Published in 1967, it states that if S and S' are two non-unitary isometries, then $C^*(S)$ is isomorphic to $C^*(S')$. Of course, this problem can equivalently be considered as a question of representations of the additive semigroup $\mathbb{N} \cup \{0\}$ by isometries. Indeed, such a representation is a homomorphism from $\mathbb{N} \cup \{0\}$ to $\mathcal{B}(\mathcal{H})$ which maps n to S_n , where each S_n is an isometry. It is easily verified that the S_n 's satisfy one extra relation that we get for free. With the standard ordering on $\mathbb{N} \cup \{0\}$, the natural numbers become a lattice-ordered semigroup, meaning that every pair of numbers m, n has a common upper bound in $\mathbb{N} \cup \{0\}$ which is of course $m \vee n := \max \{m, n\}$. Now since $S_n^* S_n = I$ for all n , it follows easily that

$$(S_m S_m^*)(S_n S_n^*) = S_{m \vee n} S_{m \vee n}^* \quad (1.1)$$

for all m, n . This relation is what is known in the literature (see [5, 6]) as a *covariance relation*, and it simply states that the range projections of the S_n 's preserve the lattice structure of $(\mathbb{N} \cup \{0\}, \leq)$.

While it is by no means readily obvious, this type of relation provides a powerful framework to examine C^* -algebras generated by a single isometry. However, even if one changes from \mathbb{N} to another semigroup and a different lattice ordering while still insisting that a relationship analogous to (1.1) holds true, the same arguments, developed in various settings by Cuntz, Douglas, Murphy, and (most generally) Nica, can still be brought to bear on such representations. The full range of semigroups that fall under such arguments is still unknown. In [6], Nica analysed a large family of semigroups, which he called “quasi-lattice” ordered semigroups (P, \leq) and provided evidence of a crossed product structure of the associated C^* -algebras. Later in [5], Laca and Raeburn reinterpreted Nica’s results in the context of semigroup crossed products, giving a concrete realization of the Toeplitz C^* -algebras as crossed products that could be studied using methods developed by Cuntz, Douglas, and others. In this case P is considered to be a subsemigroup of an ambient group G , where the ordering is defined as $x \leq y$ if and only if $x^{-1}y$ lies in P . While we won’t give the exact definition of “quasi-lattice order” here, suffice it to say that it includes all cases where \leq is a lattice ordering (and so includes our above working example) and also some very non-abelian cases, like the free product $\mathbb{N} * \mathbb{N} * \dots * \mathbb{N}$ where \leq is the free product ordering.

This thesis examines a separate family of examples of semigroups, all coming from number theory. We consider the semigroup P of so-called “principal integral ideals” in a number field K . The order on P is that of divisibility. Unless extra assumptions are made on P , this semigroup is, in general, not lattice ordered or even quasi-lattice ordered with respect to divisibility. But P can be thought of as “almost lattice ordered” in the sense that it can be embedded inside the semigroup \mathcal{I} of all integral ideals, the latter being lattice ordered with respect to the same divisibility ordering. What is interesting is that this weaker property is enough to ensure that the arguments of Nica, and Laca and Raeburn can be adapted to this setting, but our familiar covariance relation (1.1) must be abandoned, for in

this setting, with $a, b \in P$, the least common multiple $[a, b]$ no longer necessarily exists in P (which is equivalent to saying that not all integral ideals in a number field are principal) and so an equation like (1.1) no longer makes sense.

Before searching for a new covariance relation which fits the bill, we begin with a primer on algebraic number theory in chapter 2, explicitly defining P and emphasizing those algebraic properties of P which will be most important to the rest of the thesis. In chapter 3 we look at the Toeplitz representation of P and its associated C^* -algebra and deduce several results on its structure along the way, one of which being our new model covariance relation. After this, in chapter 4 we shelve the Toeplitz representation for the time being, and focus instead on the general properties of arbitrary isometric representations of P which satisfy our new covariance relation. This leads us naturally to introduce the universal C^* -algebra for covariant isometric representations of P , denoted $C^*(P)$, and to a reinterpretation of the covariance condition in a C^* -dynamical setting. This reinterpretation quickly leads to expressing $C^*(P)$ as a crossed product C^* -algebra. The fifth and final chapter takes this crossed product and characterizes its faithful representations. This last chapter is by far the most technical and follows closely ideas and arguments given in [5], which have been modified and adapted here to handle this new type of covariance relation. As an immediate corollary of this characterization, it follows that the Toeplitz C^* -algebra is isomorphic to the universal C^* -algebra, this being the main result of the thesis.

Chapter 2

Ideas from Number Theory

Before we dive into the C^* -algebra theory proper, we need to make precise all the number theory terminology involved. No other algebraic number theory results will be used aside from what is in this chapter. To those with no knowledge of this subject, the definitions and results will seem quite natural and reasonable after first looking at the simplest example which is comprised of \mathbb{N} , \mathbb{Z} and \mathbb{Q} .

Here we now consider the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ as a multiplicative semigroup and equip it with the divisibility ordering. That is $m \leq n$ if and only if m divides n , written $m|n$. This new ordering makes \mathbb{N} into a lattice ordered semigroup, where now the least upper bound of two numbers m, n is of course their least common multiple, $[m, n] := \text{lcm} \{m, n\}$. Consider now the rationals \mathbb{Q} . This is the most trivial of examples of what is called a “number field”, and there is a distinguished subring \mathbb{Z} which is called, not surprisingly, “the ring of integers of \mathbb{Q} ”. Now since \mathbb{Z} is a principal ideal domain, every non-zero ideal in \mathbb{Z} can be identified with an integer n , modulo units since $n\mathbb{Z} = (-n)\mathbb{Z}$. With a naturally defined multiplication, the semigroup P of non-zero ideals in \mathbb{Z} , called *integral*

ideals, is then isomorphic to $\mathbb{Z}/\{\pm 1\}$ which is precisely (\mathbb{N}, \times) . Divisibility in \mathbb{N} transfers over quite naturally as well, since m divides n if and only if $n\mathbb{Z}^\times \subset m\mathbb{Z}^\times$. In this picture, taking the least common multiple of two integers corresponds then to taking the intersection of the associated ideals and taking the greatest common divisor of two integers corresponds to taking the sum of the associated ideals. These ideas provide a model for the general case.

Before defining a number field, we first need a couple of a notions from field theory. Suppose we have two fields $E \subset F$. F is said to be an **algebraic extension** if every element of F is the root of a non-zero polynomial with co-efficients in E . If one views F as a vector space over E , then F is said to be of **finite degree over** E if the dimension of F as a vector space over E is finite.

We let K be a **number field**, that is, an algebraic extension of \mathbb{Q} of finite degree, and denote by \mathcal{O} the **ring of integers of** K ,

$$\mathcal{O} := \{\alpha \in K : p(\alpha) = 0 \text{ for some monic } p \in \mathbb{Z}[x]\},$$

where we recall that a polynomial $\sum_0^n c_j x^j$ is *monic* whenever its leading term c_n is 1. A subset \mathfrak{a} of K is said to be a **fractional ideal** of K if $d\mathfrak{a} \subset \mathcal{O}$ is a non-zero ideal of \mathcal{O} for some non-zero algebraic integer d . Throughout the entire paper, we implicitly ignore the zero fractional ideal.

In the simplest case of \mathbb{Q} and its ring of integers \mathbb{Z} , if \mathfrak{f} is a fractional ideal, then $d\mathfrak{f}$ is an ideal of \mathbb{Z} for some non-zero integer d . But since every ideal of \mathbb{Z} is principal, there exists a non-zero integer c such that $d\mathfrak{f} = c\mathbb{Z}$. Thus every fractional ideal of \mathbb{Q} is, not surprisingly, of the form $\frac{c}{d}\mathbb{Z}$ for non-zero integers c, d .

Now let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals. The **product** of these is defined to be $\mathfrak{a}\mathfrak{b} := \{\sum_j a_j b_j : a_j \in \mathfrak{a}, b_j \in \mathfrak{b}\}$ and the **inverse** of \mathfrak{a} is $\mathfrak{a}^{-1} := \{\alpha \in K : \alpha\mathfrak{a} \subset \mathcal{O}\}$. Ideals of the ring \mathcal{O} are fractional ideals and are often referred to as **integral**

ideals, the set of all such objects being denoted \mathcal{I} . Finally, \mathfrak{a} is said to be a **principal fractional ideal** if $\mathfrak{a} = \alpha\mathcal{O}$ for some non-zero α in K . In this case it is more common to simply refer to \mathfrak{a} as a **principal ideal**. Since there are no actual proper ideals in the K (for K is field) and the notion of a principal fractional ideal $\alpha\mathcal{O}$ coincides with the standard notion of a principal ideal of \mathcal{O} when $\alpha \in \mathcal{O}$, no difficulty should arise concerning the context of this abuse of terminology.

As noted in the introduction, \mathcal{O} is not necessarily a principal ideal domain. As an example, consider the ring of integers $\mathbb{Z}[i\sqrt{17}] = \mathbb{Z} + i\sqrt{17}\mathbb{Z}$ in the number field $\mathbb{Q}[i\sqrt{17}]$. In general, for an ideal \mathfrak{a} of $\mathbb{Z}[i\sqrt{17}]$, one defines the *norm* $N(\mathfrak{a})$ of \mathfrak{a} to be the cardinality of the quotient group $\mathbb{Z}[i\sqrt{17}]/\mathfrak{a}$. While this is by no means obvious, it turns out that if \mathfrak{a} is principal, generated by $a = m + ni\sqrt{17}$, then $N(\mathfrak{a}) = m^2 + 17n^2$. Note that this is just the modulus squared of the complex number $m + ni\sqrt{17}$. Now consider the ideal \mathfrak{a} generated by 2 and $1 + i\sqrt{17}$. One can show that \mathfrak{a} consists of elements of the form $m + ni\sqrt{17}$ where m and n are of the same parity. Thus for every x in $\mathbb{Z}[i\sqrt{17}]$, either x is in \mathfrak{a} , or $x = 1 + y$ for some y in \mathfrak{a} . Thus it follows that $N(\mathfrak{a}) = 2$. If \mathfrak{a} is in fact principal then it follows that $2 = N(\mathfrak{a}) = m^2 + 17n^2$ for some integers m, n which is impossible. So \mathfrak{a} is a non-principal integral ideal.

Given integral ideals $\mathfrak{a}, \mathfrak{b}$, we say that \mathfrak{a} **divides** \mathfrak{b} , written $\mathfrak{a}|\mathfrak{b}$ if $\mathfrak{b} \subset \mathfrak{a}$ or, equivalently, $\mathfrak{a}^{-1}\mathfrak{b} \subset \mathcal{O}$. This yields a partial ordering according to which we can easily see that the **least common multiple** and **greatest common divisor** of integral ideals $\mathfrak{a}, \mathfrak{b}$ should respectively be

$$[\mathfrak{a}, \mathfrak{b}] = \mathfrak{a} \cap \mathfrak{b} \text{ and } (\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}.$$

As a slight abuse of notation, given a, b algebraic integers we will write (a, b) to denote $(a\mathcal{O}, b\mathcal{O}) = a\mathcal{O} + b\mathcal{O}$.

Recall that an integer $p > 1$ is prime if and only if whenever p divides mn , p divides m or n . In ring theory, the same definition is taken- in our setting, an integral ideal \mathfrak{p} is a **prime ideal** if for all integral ideals $\mathfrak{a}, \mathfrak{b}$, $\mathfrak{p}|\mathfrak{a}\mathfrak{b}$ implies $\mathfrak{p}|\mathfrak{a}$ or $\mathfrak{p}|\mathfrak{b}$. And just as is the case for \mathbb{Z} , every integral ideal has a unique (up to ordering of the factors) decomposition into prime ideal factors [8] (Theorem 5.5). Now recall that all fractional ideals are of the form $\frac{\mathfrak{a}}{d}$ where \mathfrak{a} is an integral ideal and d is a non-zero algebraic integer. Applying the prime factorization theorem to \mathfrak{a} and $d\mathcal{O}$ (which we identify with d), one sees that there exists a factorization of fractional ideals into primes, where negative powers are allowed.

The operations of taking products and inverses of fractional ideals also yields fractional ideals (this is not altogether obvious; for details see [8]) and this makes the set J_K of fractional ideals into a group, which turns out to be the free abelian group on the set of prime integral ideals, in light of the prime factorization theorem mentioned above. This view of J_K as a free group allows one to define $[f, g]$ and (f, g) for arbitrary fractional ideals in a way that is consistent with the one given above for integral ideals. Indeed, using the prime decomposition of the two fractional ideals

$$f = \prod_{\mathfrak{p} \text{ prime}} \mathfrak{p}^{m_{\mathfrak{p}}} \quad \text{and} \quad g = \prod_{\mathfrak{p} \text{ prime}} \mathfrak{p}^{n_{\mathfrak{p}}}$$

one defines analogously to the case for \mathbb{Z}

$$[f, g] := \prod_{\mathfrak{p} \text{ prime}} \mathfrak{p}^{\max\{m_{\mathfrak{p}}, n_{\mathfrak{p}}\}} \quad \text{and} \quad (f, g) := \prod_{\mathfrak{p} \text{ prime}} \mathfrak{p}^{\min\{m_{\mathfrak{p}}, n_{\mathfrak{p}}\}}.$$

For the most part, we will make use of this extended definition where f is a principal fractional ideal, ie $f = \alpha\mathcal{O}$ for some $\alpha \in K$ and where $g = \mathcal{O} = 1$, the symbol 1 denoting of course that \mathcal{O} is the identity element in the group J_K . In this case $[f, 1] = [\alpha, 1]$ can be thought of as “cutting off” f (or α) to its “integral part” so that the result fits in the integers \mathcal{O} . In fact, one has $[\alpha, 1] = \alpha\mathcal{O} \cap \mathcal{O}$

and, fitting with the nomenclature, it is shown below that every integral ideal \mathfrak{a} is the “integral part” of a ratio of two integers, in the sense that $\mathfrak{a} = [\frac{a}{b}, 1]$. This is Proposition 2.2, and if nothing else is remembered from this section, this is the result which should be taken to heart in what follows. It will be ubiquitous in the most important properties of the Toeplitz representation and the notions of covariance in the following sections.

As a last fact to keep in mind, it easily follows that for any fractional ideals $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$, $\mathfrak{f}[\mathfrak{g}, \mathfrak{h}] = [\mathfrak{f}\mathfrak{g}, \mathfrak{f}\mathfrak{h}]$ and $\mathfrak{f}(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{f}\mathfrak{g}, \mathfrak{f}\mathfrak{h})$. We frequently make use of these two relations also in what follows.

Our concern will be on the subsemigroup P of non-zero *principal integral ideals*. Let \mathcal{O}^\times and \mathcal{O}^* denote, respectively, the non-zero elements and the units of \mathcal{O} . Notice that if $a, b \in \mathcal{O}^\times$ and $a = bu$ for some $u \in \mathcal{O}^*$, then $a\mathcal{O} = b\mathcal{O}$. Hence $P = \mathcal{O}^\times/\mathcal{O}^*$ and from now on we often view an element a in \mathcal{O} also as the principal ideal $a\mathcal{O}$.

One can show that every ideal of \mathcal{O} can be generated by two elements.

Theorem 2.1 ([8]). *If \mathfrak{a} is a non-zero integral ideal of a number field and a is any non-zero element of \mathfrak{a} , then there exists $b \in \mathfrak{a}$ such that $\mathfrak{a} = (a, b)$.*

This allows us to prove the following similar fact which will be relevant in what follows.

Proposition 2.2. *Every integral ideal \mathfrak{a} of a number field is of the form*

$$\mathfrak{a} = [a/b, 1]$$

for some algebraic integers a, b .

Proof. In general, from the prime decomposition of integral ideals $\mathfrak{b}, \mathfrak{c}$ it is clear that $\mathfrak{bc} = (\mathfrak{b}, \mathfrak{c})[\mathfrak{b}, \mathfrak{c}]$.

Now given a non-zero integral ideal \mathfrak{a} , \mathfrak{a}^{-1} is a fractional ideal and so there exists a non-zero a in \mathcal{O} such that $a\mathfrak{a}^{-1} \subset \mathcal{O}$ is an integral ideal. By definition, $1 \in \mathfrak{a}^{-1}$ and hence $a \in a\mathfrak{a}^{-1}$. Thus by Proposition 2.1 there exists $b \in \mathcal{O}$ such that $a\mathfrak{a}^{-1} = (a, b)$. Therefore

$$\mathfrak{a} = \frac{a}{a\mathfrak{a}^{-1}} = \frac{a}{(a, b)} = \frac{[a, b]}{b} = \left[\frac{a}{b}, 1 \right].$$

□

Chapter 3

The Toeplitz Representation of P

We begin our study of isometric representations of P with what is perhaps the first concrete example one would think to consider. We work with the Hilbert space $\ell^2(P)$ with standard orthonormal basis $\{\delta_a\}_{a \in P}$ and represent P on $\ell^2(P)$ as follows. For each $a \in P$, define the operator T_a on the basis vectors of $\ell^2(P)$ by

$$T_a \delta_b := \delta_{ab},$$

and then extend by linearity and continuity to all of $\ell^2(P)$.

The map $a \mapsto T_a$ is the **Toeplitz representation of P** and the **Toeplitz algebra of P** , denoted $\mathcal{T}(P)$ is the C^* -algebra generated by these T_a 's. In this explicit setting we will easily be able to single out four essential properties, listed in Proposition 3.1, which are satisfied by the operators $\{T_a : a \in P\}$ and we will then study all other representations of P which also satisfy these relations in the next section.

One has, for $a, b, c \in P$, $\langle T_a \delta_b, \delta_c \rangle = \langle \delta_{ab}, \delta_c \rangle$. The right-hand side of this

is equal to 1 if and only if $ab = c$, which is precisely when $a|c$. Thus

$$T_a^* \delta_b = \begin{cases} \delta_{b/a} & \text{if } a|b \\ 0 & \text{otherwise} \end{cases} \quad (3.1)$$

and from this one gets the following properties.

Proposition 3.1. *For all a, b in P ,*

$$(T1) \quad T_1 = I;$$

$$(T2) \quad T_a^* T_a = I;$$

$$(T3) \quad T_a T_b = T_{ab};$$

$$(T4) \quad (T_b^* T_a T_a^* T_b)(T_d^* T_c T_c^* T_d) = T_y^* T_x T_x^* T_y \text{ whenever } [\frac{a}{b}, \frac{c}{d}, 1] = [\frac{x}{y}, 1].$$

We will prove this proposition shortly. For the moment, let us assume properties (T1) through (T4) as given and examine the implications of (T4) in particular. First, since $[\frac{a}{b}, \frac{a}{b}, 1] = [\frac{a}{b}, 1]$, it follows that $(T_b^* T_a T_a^* T_b)^2 = T_b^* T_a T_a^* T_b$ and thus $T_b^* T_a T_a^* T_b$ is a projection for each choice of $a, b \in P$. Second, (T4) implies these projections commute with each other, since $[\frac{a}{b}, \frac{c}{d}, 1] = [\frac{c}{d}, \frac{a}{b}, 1]$. Third, if a, b, c, d are such that $[\frac{a}{b}, 1] = [\frac{c}{d}, 1]$ then $[\frac{a}{b}, \frac{1}{1}, 1] = [\frac{c}{d}, 1]$. (T1) and (T4) imply that

$$T_b^* T_a T_a^* T_b = (T_b^* T_a T_a^* T_b)(T_1^* T_1 T_1^* T_1) = T_d^* T_c T_c^* T_d.$$

Now recall from Proposition 2.2 that every integral ideal \mathfrak{a} can be written as $[\frac{a}{b}, 1]$. Thus to each ideal \mathfrak{a} we can associate the projection $T_b^* T_a T_a^* T_b$, and this choice is independent of the a, b such that $[\frac{a}{b}, 1] = \mathfrak{a}$.

To get an idea of why (T4) is the *right* relation to focus on in the general case, let us look back at the special case where \mathcal{O} is assumed to be a principal ideal domain. In this setting, of course, every integral ideal \mathfrak{a} can be written as $a\mathcal{O}$ for

some a , that is $\mathfrak{a} = [a, 1]$ and the resulting semigroup P is lattice-ordered. The covariance relation from [6] and [5], namely

$$(T_a T_a^*)(T_b T_b^*) = T_{[a,b]} T_{[a,b]}^* \text{ for all } a, b \in P,$$

written in a rather convoluted way, states that

$$(T_a T_a^*)(T_b T_b^*) = T_c T_c^* \text{ whenever } [a, b, 1] = [c, 1]$$

for all $a, b, c \in P$. While this is not very illustrative in this restricted case, it is perfectly tailored to be adapted to the general case in light of Proposition 2.2. The result is precisely (T4) of Proposition 3.1, which we now prove.

Proof. Properties (T1), (T2), and (T3) are easily checked.

For each integral ideal \mathfrak{a} , let us define $L_{\mathfrak{a}}$ to be the orthogonal projection of $\ell^2(P)$ onto the subspace

$$\overline{\text{span}}\{\delta_c : \mathfrak{a}|c\}$$

Since $[\mathfrak{a}, \mathfrak{b}]|c$ if and only if both $\mathfrak{a}|c$ and $\mathfrak{b}|c$, it follows immediately that

$$L_{\mathfrak{a}} L_{\mathfrak{b}} = L_{[\mathfrak{a}, \mathfrak{b}]}.$$

We show that if $\mathfrak{a} = [\frac{a}{b}, 1]$, then

$$T_b^* T_a T_a^* T_b = L_{\mathfrak{a}}. \tag{3.2}$$

First let $a \in P$ be arbitrary. By the adjoint computation (3.1), it is easily checked that $T_a T_a^* = L_a$. If $\mathfrak{a} = a\mathcal{O}$ is principal, then $\mathfrak{a} = [\frac{a}{1}, 1]$. Thus $T_1^* T_a T_a^* T_1 = T_a T_a^* = L_a$, proving (3.2) in the special case when \mathfrak{a} is in fact principal. In general, suppose $\mathfrak{a} = [\frac{a}{b}, 1]$. Note that

$$T_b^* T_a T_a^* T_b = T_b^* (T_a T_a^* T_b T_b^*) T_b = T_b^* (L_a L_b) T_b = T_b^* L_{[a,b]} T_b.$$

Let $c \in P$. Suppose that $[\frac{a}{b}, 1] | c$. Then $[a, b] | bc$ and so

$$(T_b^* T_a T_a^* T_b) \delta_c = T_b^* L_{[a,b]} T_b \delta_c = T_b^* L_{[a,b]} \delta_{bc} = T_b^* \delta_{bc} = \delta_c = L_a \delta_c.$$

If $[\frac{a}{b}, 1]$ does not divide c , then $[a, b]$ does not divide bc . Thus

$$(T_b^* T_a T_a^* T_b) \delta_c = T_b^* L_{[a,b]} T_b \delta_c = T_b^* L_{[a,b]} \delta_{bc} = 0 = L_a \delta_c.$$

Hence $T_b^* T_a T_a^* T_b = L_a$.

(T4) follows immediately from (3.2), for if $[\frac{a}{b}, \frac{c}{d}, 1] = [\frac{x}{y}, 1]$, then we set $\mathbf{a} := [\frac{a}{b}, 1]$ and $\mathbf{b} := [\frac{c}{d}, 1]$, and obtain

$$\left[\frac{x}{y}, 1 \right] = \left[\frac{a}{b}, \frac{c}{d}, 1 \right] = \left[\left[\frac{a}{b}, 1 \right], \left[\frac{c}{d}, 1 \right] \right] = [\mathbf{a}, \mathbf{b}].$$

Therefore

$$(T_b^* T_a T_a^* T_b)(T_d^* T_c T_c^* T_d) = L_a L_b = L_{[\mathbf{a}, \mathbf{b}]} = T_y^* T_x T_x^* T_y.$$

□

In the following chapters we will study arbitrary representations of P which satisfy relations (T1) through (T4) in Proposition 3.1. We will see (Corollary 5.5) that the Toeplitz C^* -algebra studied here is actually the universal C^* -algebra with respect to these four relations.

First, though, we take a slight detour and analyze both $\mathcal{T}(P)$, the Toeplitz C^* -algebra of P , and the commutative C^* -algebra, B_P , generated by $\{T_b^* T_a T_a^* T_b : a, b \in P\} = \{L_a : \mathbf{a} \in \mathcal{I}\}$. We obtain dense spanning sets for both of these C^* -algebras and explicitly describe the spectrum of B_P .

In the case of B_P , an obvious choice of a dense spanning set is precisely its generating set $\{L_a : \mathbf{a} \in \mathcal{I}\}$ since the equation $L_a L_b = L_{[\mathbf{a}, \mathbf{b}]}$ means that this set is

multiplicatively closed. Recall that $L_a = T_b^* T_a T_a^* T_b$ where $\mathbf{a} = [\frac{a}{b}, 1]$. In the case of \mathcal{T} we obtain a similar dense spanning set of elements of the form $T_b^* T_a T_a^* T_c$ where b and c are not necessarily equal. Note that $\{T_b^* T_a T_a^* T_c : a, b, c \in P\}$ is self-adjoint and contains $T_a = T_1^* T_1 T_1^* T_a$ for all $a \in P$, and so $\mathcal{T}(P) = C^*(T_b^* T_a T_a^* T_c : a, b, c \in P)$. Thus to prove that the set has dense linear span, we need only verify that it is multiplicatively closed. This is the following result, which handles both B_P and $\mathcal{T}(P)$.

Proposition 3.2. *The sets $\{T_b^* T_a T_a^* T_b | a, b \in P\}$ and $\{T_b^* T_a T_a^* T_c | a, b, c \in P\}$ are both multiplicatively closed.*

Proof. As remarked above, $L_a L_b = L_{[a,b]}$ which proves the result for the first set in question. As for $\{T_b^* T_a T_a^* T_c | a, b, c \in P\}$, let $a, b, c, \alpha, \beta, \gamma$ be elements of P . Then

$$\begin{aligned}
(T_b^* T_a T_a^* T_c)(T_\beta^* T_\alpha T_\alpha^* T_\gamma) &= T_b^* (T_\beta^* T_\beta) T_a T_a^* (T_\beta^* T_\beta) T_c T_\beta^* (T_c^* T_c) T_\alpha T_\alpha^* (T_c^* T_c) T_\gamma \\
&= T_{b\beta}^* T_{a\beta} T_{a\beta}^* T_{c\beta} T_{c\beta}^* T_{c\alpha} T_{c\alpha}^* T_{c\gamma} \\
&= T_{b\beta}^* (T_h^* T_g T_g^* T_h) T_{c\gamma} \text{ for some } g, h \in P \\
&\quad \text{by multiplicative closure of } \{L_a : a \in \mathcal{I}\} \\
&= T_{b\beta h}^* T_g T_g^* T_{hc\gamma}.
\end{aligned}$$

□

To express the commutative C^* -algebra B_P as continuous functions on some compact set, we prove that B_P is isomorphic to an analogously defined C^* -algebra of operators on $\ell^2(\mathcal{I})$ (recall that \mathcal{I} is lattice ordered with respect to divisibility) whose spectrum is explicitly characterized by results in [6]. To this end we define the *Toeplitz operators* on $\ell^2(\mathcal{I})$ with standard orthonormal basis $\{\delta_a\}_{a \in \mathcal{I}}$ in the same way as for P . Namely to each \mathbf{a} we associate the operator \mathcal{T}_a uniquely

determined by

$$T_a \delta_b := \delta_{ab}.$$

Note that each T_a is simply the restriction of \mathcal{T}_a to the subspace $\ell^2(P) \subset \ell^2(\mathcal{I})$. The map $\mathfrak{a} \mapsto T_a$ is an isometric representation of \mathcal{I} where $T_a T_a^* T_b T_b^* = T_{[a,b]} T_{[a,b]}^*$. As we did before, we let \mathcal{L}_a be the orthogonal projection of $\ell^2(\mathcal{I})$ onto the subspace $\overline{\text{span}}\{\delta_b : \mathfrak{a}|b\}$. Note once again that L_a is the restriction of \mathcal{L}_a to the subspace $\ell^2(P)$. Let $B_{\mathcal{I}}$ be the commutative C^* -algebra generated by the \mathcal{L}_a 's. Similar to B_P , $B_{\mathcal{I}}$ is densely spanned by $\{\mathcal{L}_a : \mathfrak{a} \in \mathcal{I}\}$. With all these similarities, it is perhaps not surprising that B_P is the same as $B_{\mathcal{I}}$. To prove this, we first need a lemma.

Lemma 3.3. *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be distinct integral ideals in \mathcal{I} . Suppose $\mathfrak{a} \in \{\mathfrak{a}_1, \dots, \mathfrak{a}_n\}$ is minimal with respect to divisibility, meaning $\mathfrak{a}_j | \mathfrak{a}$ implies $\mathfrak{a}_j = \mathfrak{a}$ for each j . Then*

$$\mathfrak{a} \not\subseteq \bigcup_{\mathfrak{a}_j \neq \mathfrak{a}} \mathfrak{a}_j$$

Proof. Suppose $\mathfrak{a} \in \{\mathfrak{a}_1, \dots, \mathfrak{a}_n\}$ is minimal with respect to divisibility. Without loss of generality, we may assume $\mathfrak{a} = \mathfrak{a}_1$. To find an $a \in \mathfrak{a}_1$ which does not lie in \mathfrak{a}_j for $j \neq 1$, first pick a non-zero c in $[\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n]$. Then $c \in \mathfrak{a}_1$ and by Theorem 2.1 there exists an $a \in \mathfrak{a}_1$ such that $\mathfrak{a}_1 = (c, a)$. Suppose now $\mathfrak{a}_j | a$, that is, $a \in \mathfrak{a}_j$. Since $c \in \mathfrak{a}_k$ for all $k = 1, 2, \dots, n$, it follows that $\mathfrak{a}_j | (c, a) = \mathfrak{a}_1$, which implies that $j = 1$ by minimality of \mathfrak{a}_1 . This completes the proof. \square

Proposition 3.4. *B_P is isomorphic to $B_{\mathcal{I}}$.*

Proof. The obvious correspondence to consider is $F : L_a \mapsto \mathcal{L}_a$. The hard work lies in proving that this extends in a well-defined, bijective and multiplicative way to linear combinations of the L_a 's and hence to all of B_P .

To show that F is well-defined and bijective, it suffices to prove that the sets of spanning elements, $\{L_{\mathfrak{a}} : \mathfrak{a} \in \mathcal{I}\}$ and $\{\mathcal{L}_{\mathfrak{a}} : \mathfrak{a} \in \mathcal{I}\}$, of B_P and $B_{\mathcal{I}}$ respectively are in fact linearly independent.

In both cases, the proof is given by induction on the cardinality n of finite subsets over which the linear combinations are being taken. We handle $\{L_{\mathfrak{a}} : \mathfrak{a} \in \mathcal{I}\}$. The argument for the $\mathcal{L}_{\mathfrak{a}}$'s is identical.

For $n = 1$, the result is trivial since all $L_{\mathfrak{a}}$ are non-zero, whence $\lambda L_{\mathfrak{a}} = 0$ implies $\lambda = 0$.

Now suppose that for some $n \geq 1$ all subsets of cardinality n of $\{L_{\mathfrak{a}} : \mathfrak{a} \in \mathcal{I}\}$ are linearly independent. Note that the correspondence $\mathfrak{a} \mapsto L_{\mathfrak{a}}$ is one-to-one. Let $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_{n+1}$ be distinct integral ideals and suppose

$$\sum_1^{n+1} \lambda_j L_{\mathfrak{a}_j} = 0$$

for some complex numbers λ_j . By the induction hypothesis it suffices to show that at least one λ_j in the above linear combination is zero.

Choose an $\mathfrak{a} \in \{\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_{n+1}\}$ which is minimal with respect to divisibility. In other words \mathfrak{a} is such that $\mathfrak{a}_j | \mathfrak{a}$ implies $\mathfrak{a}_j = \mathfrak{a}$. Without loss of generality, say that $\mathfrak{a} = \mathfrak{a}_1$. By Lemma 3.3 there exists an $a \in \mathfrak{a}_1$ which does not lie in \mathfrak{a}_j for all $j \neq 1$

Thus it follows that

$$|\lambda_1| = \|\lambda_1 \delta_{\mathfrak{a}}\| = \|(\sum_1^{n+1} \lambda_j L_{\mathfrak{a}_j}) \delta_{\mathfrak{a}}\| = 0$$

which yields the result for the $L_{\mathfrak{a}}$'s.

Finally, that F is multiplicative follows from noting that $L_{\mathfrak{a}} L_{\mathfrak{b}} = L_{[\mathfrak{a}, \mathfrak{b}]}$ and similarly $\mathcal{L}_{\mathfrak{a}} \mathcal{L}_{\mathfrak{b}} = \mathcal{L}_{[\mathfrak{a}, \mathfrak{b}]}$. Hence F is multiplicative on all linear combinations of

the $L_{\mathfrak{a}}$'s and hence on B_P , the closure of these linear combinations. This proves the result. \square

From now on, we refer to both B_P and $B_{\mathcal{I}}$ as simply B . To complete our study of B we quote a result from [6] which characterizes the spectrum of B . A subset $S \subset \mathcal{I}$ is called *directed* if given any two $\mathfrak{a}, \mathfrak{b}$ in S , there exists a \mathfrak{c} in S such that both \mathfrak{a} and \mathfrak{b} divide \mathfrak{c} . Secondly, S is called *hereditary* if $\mathfrak{a} \in S$ and $\mathfrak{b}|\mathfrak{a}$ implies that $\mathfrak{b} \in S$. Letting Ω be the set of all non-empty, directed, hereditary subsets of \mathcal{I} endowed with the product topology inherited from $2^{\mathcal{I}}$ (where a set is identified with its characteristic function). The following result is an immediate application of [6, Proposition 1°, p. 47].

Theorem 3.5. *The spectrum of the C^* -algebra $B = \overline{\text{span}}\{L_{\mathfrak{a}} : \mathfrak{a} \in \mathcal{I}\}$ is the space Ω of nonempty, directed, hereditary subsets of \mathcal{I} , viewed as a subspace of $\{0, 1\}^{\mathcal{I}}$. For each integral ideal $\mathfrak{a} \in \mathcal{I}$ let $D(\mathfrak{a}) := \{\mathfrak{b} \in \mathcal{I} : \mathfrak{b}|\mathfrak{a}\}$. Then $D(\mathfrak{a})$ is the nonempty, directed and hereditary subset corresponding to the multiplicative linear functional on B which maps A to $\langle A\delta_{\mathfrak{a}}, \delta_{\mathfrak{a}} \rangle$. Moreover, $\{D(\mathfrak{a}) : \mathfrak{a} \in \mathcal{I}\}$ is dense in Ω .*

Chapter 4

Covariant Representations of P

Recall that \mathcal{I} denotes the semigroup of integral ideals of a number field and P denotes the subsemigroup of integral principal ideals.

Definition 4.1. *A representation $W : P \rightarrow \mathcal{B}(H)$ is called a covariant isometric representation if it satisfies the following.*

$$(1) \quad W_1 = I;$$

$$(2) \quad W_a^* W_a = I \text{ for all } a \in P;$$

$$(3) \quad W_a W_b = W_{ab} \text{ for all } a, b \in P;$$

$$(4) \quad (W_b^* W_a W_a^* W_b)(W_d^* W_c W_c^* W_d) = W_y^* W_x W_x^* W_y \text{ whenever } \left[\frac{a}{b}, \frac{c}{d}, 1\right] = \left[\frac{x}{y}, 1\right].$$

We let $C^*(P)$ denote the universal C^* -algebra with respect to the above four relations. To construct this, we let i be the direct sum of all such covariant

isometric representations. One takes the $*$ -algebra generated by the elements of $i(P)$ and completes this with respect to the supremum norm, that is for a , $\|i(a)\| = \sup\{W_a : W \text{ a covariant isometric representation of } P\}$. Note that condition (2) guarantees that $\|i(a)\| = 1$ for all $a \in P$. Hence the norm extends in well-defined way to all of $i(P)$. We set $C^*(P) = C^*(i(P))$. $C^*(P)$ is non-trivial since, as we have seen, the Toeplitz representation is one such non-trivial covariant isometric representation.

Remark 4.2. *While this is not needed for the rest of the paper, we note at this stage the connection with the work of Exel, Laca and Quigg [4] on partial representations. Suppose W is a covariant isometric representation. For every $a, b, x, y \in P$,*

$$\begin{aligned} W_b^* W_a W_y^* W_x W_x^* W_y &= W_b^* (W_y^* W_y) W_a W_y^* (W_a^* W_a) W_x W_x^* (W_a^* W_a) W_y \\ &= W_{by}^* W_{ay} W_{ay}^* W_{ax} W_{ax}^* W_{ay} \\ &= W_{by}^* W_{ax} W_{ax}^* W_{ay} W_{ay}^* W_{ay} \\ &= W_{by}^* W_{ax} W_x^* W_y. \end{aligned}$$

Thus the map

$$\left(\frac{x}{y} \right) \mapsto W_{\frac{x}{y}} := W_y^* W_x$$

is a partial isometric representation (see [4]) of the whole group K^*/\mathcal{O}^* of principal fractional ideals, meaning that operators $W_{\frac{x}{y}}$ satisfy

$$W_{\frac{a}{b}} W_{\frac{x}{y}} W_{\frac{x}{y}}^* = W_{\frac{ax}{by}} W_{\frac{x}{y}}^*.$$

As further setup, we wish to consider a C^* -dynamical system naturally associated with P (and with \mathcal{I}). We first rewrite our commutative C^* -algebra B from the last section in a way that makes no reference to the Toeplitz representation. Consider the C^* -subalgebra of $\ell^\infty(\mathcal{I})$ generated by the functions 1_a , $a \in \mathcal{I}$, where

$$1_a(b) = \begin{cases} 1 & \text{if } a|b \\ 0 & \text{otherwise} \end{cases}$$

In other words, $1_{\mathfrak{a}}$ is essentially just the characteristic function of \mathfrak{a} (modulo units). It is clear then that this C^* -algebra is precisely $\overline{\text{span}}\{1_{\mathfrak{a}} : \mathfrak{a} \in \mathcal{I}\}$. In fact, it is clearly isomorphic to the C^* -algebra B generated by the $L_{\mathfrak{a}}$'s (defined in chapter 3), because $L_{\mathfrak{a}}$ is the multiplication operator by the function $1_{\mathfrak{a}}$.

On B we have two actions of P by endomorphisms, α and σ , which correspond to left and right translations, respectively. Explicitly, define for $x \in P$ and $f \in B$,

$$(\alpha_x f)(\mathfrak{a}) := \begin{cases} f(\frac{\mathfrak{a}}{x}) & \text{if } x|\mathfrak{a} \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\sigma_x f)(\mathfrak{a}) := f(x\mathfrak{a}).$$

From these definitions one easily checks that $\sigma_x \circ \alpha_x = \text{id}$ for all $x \in P$ and that $\alpha_x(1_{\mathfrak{a}}) = 1_{x\mathfrak{a}}$ and $\sigma_x(1_{\mathfrak{a}}) = 1_{[\frac{\mathfrak{a}}{x}, 1]}$. Therefore, if $\mathfrak{a} = [\frac{x}{y}, 1]$, then $1_{\mathfrak{a}} = (\sigma_y \circ \alpha_x)(1)$.

Definition 4.3. A covariant pair for the dynamical system (B, P, α, σ) is a pair (π, W) where π is a unital representation of B on a Hilbert space H , W is an isometric representation of P on H and where for all $A \in B$ and $x \in P$,

$$\pi(\alpha_x(A)) = W_x \pi(A) W_x^* \quad \text{and} \quad \pi(\sigma_x(A)) = W_x^* \pi(A) W_x. \quad (4.1)$$

The Toeplitz representation gives rise to a non-trivial covariant pair for the system (B, P, α, σ) and by a standard universality argument, using the direct sum of all such (cyclic) covariant pairs as in [5], one gets a triple $(B \rtimes P, i_B, i_P)$ which is universal for covariant pairs for (B, P, α, σ) . $B \rtimes P$ is called the *crossed product* of B by the two actions α, σ of P . This definition entails that given any covariant pair (π, W) , there exists a unique homomorphism, denoted $(\pi \times W)$, from $B \rtimes P$ onto $C^*(\pi, W) := C^*(\pi(A), W_x : A \in B, x \in P)$ such that $(\pi \times W) \circ i_B = \pi$ and $(\pi \times W) \circ i_P = W$.

We now study the connections between covariant isometric representations of P and covariant pairs, which culminates in a characterization of faithful representations of $C^*(P)$. These connections are studied in the remainder of this chapter and are direct analogues of Proposition 1.3 and Proposition 2.3 in [5]. We tackle the question of faithfulness in the next chapter.

Lemma 4.4. (1) *If π is a non-degenerate representation of B , then*

$E_{\mathfrak{a}} := \pi(1_{\mathfrak{a}})$, for $\mathfrak{a} \in \mathcal{I}$ is a family of projections satisfying

$$E_{\mathfrak{a}}E_{\mathfrak{b}} = E_{[\mathfrak{a}, \mathfrak{b}]} \quad (4.2)$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$.

(2) *Conversely, if $\{E_{\mathfrak{a}} : \mathfrak{a} \in \mathcal{I}\}$ is a family of projections satisfying (4.2), then there exists a representation π_E of B satisfying $\pi_E(1_{\mathfrak{a}}) = E_{\mathfrak{a}}$.*

(3) *With E as in (2), π_E is faithful if and only if E satisfies*

$$\prod_1^n (E_{\mathfrak{a}} - E_{\mathfrak{b}_j}) \neq 0 \text{ whenever } \mathfrak{a} | \mathfrak{b}_j, \mathfrak{a} \neq \mathfrak{b}_j \text{ for all } j.$$

Proof. This is [5, Proposition 1.3] for (J_K, \mathcal{I}) , the quasi-lattice ordered group of fractional ideals of K with the semigroup of integral ideals. \square

The fact that the word ‘‘covariant’’ was used in two different forms, once in Definition 4.1 for covariant representations and again in Definition 4.3 for covariant pairs, is no accident. The next result says that covariant representations and covariant pairs capture the same behaviour. What we mean by this is that every covariant isometric representation W induces a natural representation π_W of B such that (π_W, W) is a covariant pair; conversely if (π, W) is a covariant pair, W is then automatically a covariant isometric representation.

Proposition 4.5. (1) *If (π, W) is a covariant pair for (B, P, α, σ) , then W is a covariant isometric representation of P and*
 $\pi(1_{\mathfrak{a}}) = W_y^* W_x W_x^* W_y$ whenever $\mathfrak{a} = [\frac{x}{y}, 1]$, $x, y \in P$.

(2) *If W is a covariant isometric representation of P then there exists a well-defined representation π_W of B such that $\pi_W(1_{\mathfrak{a}}) = W_y^* W_x W_x^* W_y$ whenever $\mathfrak{a} = [\frac{x}{y}, 1]$, for $x, y \in P$ and such that (π_W, W) is a covariant pair for (B, P, α, σ) .*

(3) *If W is as above in (2), then π_W is faithful if and only if*

$$\prod_1^n (I - W_{y_j}^* W_{x_j} W_{x_j}^* W_{y_j}) \neq 0 \text{ whenever } \mathfrak{a}_j := \left[\frac{x_j}{y_j}, 1 \right] \neq (1) \text{ for all } j.$$

Proof. First suppose that (π, W) is a covariant pair for (B, P, α, σ) . Let $\mathfrak{a} \in \mathcal{I}$, with $\mathfrak{a} = [\frac{x}{y}, 1]$. Then

$$\pi(1_{\mathfrak{a}}) = \pi(\sigma_y \circ \alpha_x(1)) = W_y^* \pi(\alpha_x(1)) W_y = W_y^* W_x \pi(1) W_x^* W_y = W_y^* W_x W_x^* W_y.$$

Furthermore, with \mathfrak{a} as above, $\mathfrak{b} = [\frac{a}{b}, 1]$, and $[\mathfrak{a}, \mathfrak{b}] = [\frac{a}{b}, 1]$, we have

$$(W_y^* W_x W_x^* W_y)(W_b^* W_a W_a^* W_b) = \pi(1_{\mathfrak{a}}) \pi(1_{\mathfrak{b}}) = \pi(1_{[\mathfrak{a}, \mathfrak{b}]}) = W_{\beta}^* W_{\alpha} W_{\alpha}^* W_{\beta}$$

and hence W is a covariant isometric representation of P . This proves (1).

Now suppose W is a covariant isometric representation of P . For each integral ideal $\mathfrak{a} = [\frac{a}{b}, 1]$ define $E_{\mathfrak{a}} := W_b^* W_a W_a^* W_b$. By condition (4) of Definition 4.1, it follows immediately that $E_{\mathfrak{a}}$ is a projection defined independently of the choice of a, b for which $[\frac{a}{b}, 1] = \mathfrak{a}$ and that if \mathfrak{b} is another integral ideal, then $E_{\mathfrak{a}} E_{\mathfrak{b}} = E_{[\mathfrak{a}, \mathfrak{b}]}$. Let $\pi_W := \pi_E$, the well-defined representation of B which results from Lemma 4.4 (2).

Now let $x \in P$ and note that $\mathbf{a} = [\frac{a}{b}, 1] = [\frac{xa}{xb}, 1]$. Hence we have

$$\begin{aligned} W_x \pi_W(1_{\mathbf{a}}) W_x^* &= W_x (W_{x_b}^* W_{x_a} W_{x_a}^* W_{x_b}) W_x^* = (W_x W_x^*) (W_b^* W_{x_a} W_{x_a}^* W_b) (W_x W_x^*) \\ &= \pi_W(1_x) \pi_W(1_{[\frac{xa}{b}, 1]}) \pi_W(1_x) \\ &= \pi_W(1_{[x, [\frac{xa}{b}, 1], x]}) \\ &= \pi_W(1_{[\frac{xa}{b}, x, 1]}) = \pi_W(1_{[\frac{xa}{b}, x]}) \\ &= \pi_W(\alpha_x(1_{\mathbf{a}})). \end{aligned}$$

Next, we prove that $W_x^* \pi(1_{\mathbf{a}}) W_x = \pi_W(\sigma_x(1_{\mathbf{a}}))$. First note that

$$\left[\frac{\mathbf{a}}{x}, 1 \right] = \left[\left[\frac{a}{bx}, \frac{1}{x} \right], 1 \right] = \left[\frac{a}{bx}, 1 \right].$$

Therefore $1_{[\frac{a}{xb}, 1]} = \sigma_x(1_{\mathbf{a}})$. Thus

$$W_x^* \pi_W(1_{\mathbf{a}}) W_x = W_{x_b}^* W_a W_a^* W_{x_b} = \pi_W(1_{[\frac{a}{xb}, 1]}) = \pi_W(\sigma_y(1_{\mathbf{a}}))$$

and so (π_W, W) is a covariant pair for (B, P, α, σ) , proving (2).

Suppose now W is a covariant isometric representation and, let $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ be in P . Define

$$b := y_1 y_2 \cdots y_n \quad \text{and} \quad a_j := x_j \prod_{k \neq j} y_k \quad \text{for each } j.$$

Thus

$$\left[\frac{x_j}{y_j}, 1 \right] = \left[\frac{a_j}{b}, 1 \right] \quad \text{for all } j.$$

Hence we may write

$$E_b - E_{[b, a_j]} = W_b (I - W_{y_j}^* W_{x_j} W_{x_j}^* W_{y_j}) W_b^*$$

for each j , using covariance of W . Thus statement (3) follows immediately by Lemma 4.4(3). \square

From now on, given a covariant pair (π, W) , we write $E_{\mathfrak{a}} = \pi(1_{\mathfrak{a}}) = W_y^* W_x W_x^* W_y$ whenever $\mathfrak{a} = [\frac{x}{y}, 1]$. Before seeing that the universal C^* -algebra $C^*(P)$ is the same as our crossed product under study, we first give a corollary of the covariance relations which will be useful later.

Corollary 4.6. *Let (π, W) be a covariant pair. For every x in P and integral ideal $\mathfrak{a} \in \mathcal{I}$ we have*

$$W_x E_{\mathfrak{a}} = E_{x\mathfrak{a}} W_x \quad \text{and} \quad W_x^* E_{\mathfrak{a}} = E_{[\frac{\mathfrak{a}}{x}, 1]} W_x^*.$$

Proof. For the first equation we have

$$W_x E_{\mathfrak{a}} = (W_x E_{\mathfrak{a}} W_x^*) W_x = (W_x \pi(1_{\mathfrak{a}}) W_x^*) W_x = \pi(\alpha_x(1_{\mathfrak{a}})) W_x = E_{x\mathfrak{a}} W_x.$$

For the second equation, first note that

$$W_x^* E_{\mathfrak{a}} = W_x^* (W_x W_x^* E_{\mathfrak{a}}) = W_x^* E_{\mathfrak{a}} (W_x W_x^*).$$

Now write $\mathfrak{a} = [\frac{\mathfrak{a}}{b}, 1]$. Then

$$\begin{aligned} W_x^* E_{\mathfrak{a}} &= W_x^* E_{\mathfrak{a}} (W_x W_x^*) \\ &= W_x^* (W_b^* W_{\mathfrak{a}} W_{\mathfrak{a}}^* W_b) (W_x W_x^*) \\ &= W_{bx}^* W_{\mathfrak{a}} W_{\mathfrak{a}}^* W_{bx} W_x^* = E_{[\frac{\mathfrak{a}}{bx}, 1]} W_x^*. \end{aligned}$$

But let us observe that $[\frac{\mathfrak{a}}{bx}, 1] = [\frac{\mathfrak{a}}{bx}, \frac{1}{x}, 1] = [\frac{[\frac{\mathfrak{a}}{b}, 1]}{x}, 1] = [\frac{\mathfrak{a}}{x}, 1]$. Thus

$$W_x^* E_{\mathfrak{a}} = E_{[\frac{\mathfrak{a}}{bx}, 1]} W_x^* = E_{[\frac{\mathfrak{a}}{x}, 1]} W_x^*.$$

□

Corollary 4.7. (1) *The maps $i_B : B \rightarrow B \rtimes P$ and $i_P : P \rightarrow B \rtimes P$ are both faithful.*

$$(2) \quad (C^*(P), i) \cong (B \rtimes P, i_P).$$

(3) *The set $\text{span}\{i_P(y)^*i_P(x)i_P(x)^*i_P(z) : x, y, z \in P\}$ is dense in $B \rtimes P$.*

Proof. For the first statement, since (i_B, i_P) is canonically a covariant pair for $(B \rtimes P, P, \alpha, \sigma)$, we prove that i_B is faithful using the equivalent condition given in Proposition 4.5(3) above. To show this, it suffices by the direct sum construction of the crossed product, to show the similar statement holds for any one direct summand. We thus use (π_T, T) where T is the Toeplitz representation, for this task.

So suppose that $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ are elements of P such that each $\mathfrak{a}_j := [\frac{x_j}{y_j}, 1] \neq \mathcal{O}$. Then

$$\prod_{j=1}^n (I - T_{y_j}^* T_{x_j} T_{x_j}^* T_{y_j}) = \prod_{j=1}^n (I - L_{\mathfrak{a}_j})$$

is the orthogonal projection of $\ell^2(P)$ onto the subspace

$$\overline{\text{span}}\{\delta_c : c \text{ is not divisible by any } \mathfrak{a}_j\}. \quad (4.3)$$

It suffices to show that this space is not the trivial space $\{0\}$. Since there are infinitely many prime ideals in \mathcal{O} (see [8]) and only finitely many non-trivial ideals \mathfrak{a}_j under consideration, it follows that there exists a prime ideal \mathfrak{p} which does not occur in the prime factorization of any \mathfrak{a}_j . This implies at once that none of the \mathfrak{a}_j 's divide \mathfrak{p} . Therefore \mathfrak{p} is minimal with respect to divisibility for the set $\{\mathfrak{p}, \mathfrak{a}_1, \dots, \mathfrak{a}_n\}$. By Lemma 3.3 there exists an element p in \mathfrak{p} which is not

in any of the α_j 's. It follows at once that δ_p is in the space (4.3) and hence i_B is faithful.

Now suppose that $i_P(x) = i_P(y)$ for some x, y in P . Then by the canonical map from $B \rtimes P$ onto \mathcal{T} , we have $T_x = T_y$. Thus $\delta_x = T_x(\delta_1) = T_y(\delta_1) = \delta_y$, whence $x = y$. Thus i_P is faithful.

Now Proposition 4.5 parts (1) and (2) give a one to one correspondence between covariant isometric representations of P and covariant pairs for the C^* -dynamical system $(B \rtimes P, P, \alpha, \sigma)$. Since (i_B, i_P) is a covariant pair, we have $i_B(1_\alpha) = i_P(y)^* i_P(x) i_P(x)^* i_P(y)$ whenever $\alpha = [\frac{x}{y}, 1]$, it follows that $B \rtimes P$ is generated as a C^* -algebra by the range of i_P . Thus $(B \rtimes P, i_P)$ is universal for isometric covariant representations and hence is isomorphic to $(C^*(P), i)$ by uniqueness.

Statement (3) follows by exactly the same argument used for the Toeplitz representation given in the proof of Proposition 3.2. \square

Chapter 5

Faithful Representations of $C^*(P)$

Now that we have realized $C^*(P)$ as a crossed product, we can more easily answer the question of when, given a covariant isometric representation W of P , the resulting corresponding representation φ of $C^*(P)$ onto $C^*(W_x : x \in P)$ is in fact faithful. This amounts to asking, in the language of crossed products, when the representation $\pi_W \times W : B \rtimes P \rightarrow C^*(W_x : x \in P)$ which arises from the resulting covariant pair (π_W, W) is faithful. Remarkably, the faithfulness of $(\pi_W \times W)$ is equivalent to the faithfulness of π_W , the proof of which we shall see in Theorem 5.4.

The line of attack for this result is that of [5] (p. 424- 425), which we restate here. Given a covariant pair (π, W) , recall that $B \rtimes P$ is the closed linear span of elements of the form $i_P(y)^*i_P(x)i_P(x)^*i_P(z)$ and, similarly $C^*(\pi, W)$ is the closed linear span of elements of the form $W_y^*W_xW_x^*W_z$. Inside these two C^* -algebras live similar C^* -subalgebras, namely $i_B(B) \subset B \rtimes P$ and $\pi(B) \subset C^*(\pi, W)$. Recall also that $i_B(B)$ is the closed linear span of elements of the form $i_P(y)^*i_P(x)i_P(x)^*i_P(y)$ and likewise $\pi(B)$ is the closed linear span of the elements

$$W_y^* W_x W_x^* W_y.$$

We will show that the two maps

$$\Phi(i_P(y)^* i_P(x) i_P(x)^* i_P(z)) := \delta_{y,z} i_P(y)^* i_P(x) i_P(x)^* i_P(z)$$

and

$$\phi(W_y^* W_x W_x^* W_z) := \delta_{y,z} W_y^* W_x W_x^* W_z$$

extend in a well-defined way to homomorphisms $B \rtimes P \rightarrow i_B(B) = \Phi(B \rtimes P)$ and $C^*(\pi, W) \rightarrow \pi(B) = \phi(C^*(\pi, W))$, respectively. Then it will be obvious that the following diagram

$$\begin{array}{ccc} B \rtimes P & \xrightarrow{\pi \times W} & C^*(\pi, W) \\ \downarrow \Phi & & \downarrow \phi \\ i_B(B) & \xrightarrow{\pi \times W} & \pi(B) \end{array}$$

is commutative. Assume further that we know that Φ is faithful on positive elements, meaning that $\Phi(A^*A) = 0$ implies $A = 0$, and that $(\pi \times W)$ is faithful on the range of Φ . Then it turns out that $(\pi \times W)$ is faithful. For if $(\pi \times W)(A) = 0$, then

$$(\pi \times W)(\Phi(A^*A)) = \phi(\pi \times W)(A^*A) = 0,$$

and hence $\Phi(A^*A) = 0$, from which we may infer that $A = 0$.

This breaks the problem into three manageable parts. The first part is the verification that Φ and ϕ extend as we would like. This is tackled in Proposition 5.2 and Lemma 5.1. Once we know this, the second stage is to obtain the faithfulness of Φ on positive elements, which is done in Lemma 5.3. After this, we see that the question of faithfulness of $(\pi \times W)$ on $B \rtimes P$ reduces to the question of its faithfulness on the range of Φ . The main result of this thesis is Theorem 5.4, which tackles the third part, establishing conditions on covariant pairs (π, W) which guarantee (in fact, *characterize*) faithfulness of $(\pi \times W)$ on the range of Φ .

Lemma 5.1. *With the same hypotheses and notation as the above in Proposition 5.2, given a linear combination*

$$\sum_{x,y,z \in F} \lambda_{x,y,z} W_y^* W_x W_x^* W_z$$

there exists a projection $Q \in \pi(B)$ such that

$$Q W_y^* W_x W_x^* W_z Q = 0 \text{ if } y \neq z \quad (5.1)$$

and

$$\|Q(\sum_{x,y \in F} \lambda_{x,y,y} W_y^* W_x W_x^* W_y)Q\| = \|\sum_{x,y \in F} \lambda_{x,y,y} W_y^* W_x W_x^* W_y\|. \quad (5.2)$$

Proof. Let (π, W) be a covariant pair for (B, P, α, σ) with π faithful and suppose we are given a linear combination

$$\sum_{x,y,z \in F} \lambda_{x,y,z} W_y^* W_x W_x^* W_z$$

where $F \subset P$ is non-empty and finite. After relabelling, we may assume without loss of generality that $(x, y, z) \neq (a, b, c)$ implies that $W_y^* W_x W_x^* W_z \neq W_b^* W_a W_a^* W_c$. Finally, let $\mathcal{F} = \{[\frac{x}{y}, 1] : x, y \in F\}$ and set, as above, $E_a = W_y^* W_x W_x^* W_y$ for each $a = [\frac{x}{y}, 1]$ in \mathcal{F} . There exists then, by [5, Lemma 1.4], a non-empty set $A \subset \mathcal{F}$ which satisfies the following two properties. First,

$$\left\| \sum_{a \in \mathcal{F}} \lambda_{x,y,y} W_y^* W_x W_x^* W_y \right\| = \left| \sum_{a \in A} \lambda_{x,y,y} \right|. \quad (5.3)$$

Second, letting $[A]$ denote the lowest common multiple of the ideals in the set A , the projection

$$Q_A^E := E_{[A]} \prod_{b \in \mathcal{F} \setminus A} (I - E_b)$$

is nonzero. Consider the following projection

$$Q_A := 1_{[A]} \prod_{b \in \mathcal{F} \setminus A} (1 - 1_b)$$

in B and note $Q_A^E = \pi(Q_A)$. Thus $Q_A \neq 0$ since $Q_A^E \neq 0$. This gives the following insight into the set A . Suppose that \mathfrak{a} divides $[A]$ for some $\mathfrak{a} \in \mathcal{F} \setminus A$. Then $[A] = [[A], \mathfrak{a}]$. Thus Q_A vanishes, since it contains the factor $1_{[A]}(1 - 1_{\mathfrak{a}}) = 1_{[A]} - 1_{[[A], \mathfrak{a}]} = 0$. Therefore, $Q_A \neq 0$ implies that A is an initial segment of \mathcal{F} , in the sense that

$$A = \{\mathfrak{a} \in \mathcal{F} : \mathfrak{a} \text{ divides } [A]\}. \quad (5.4)$$

From this it immediately follows that

$$\left(\sum_{\mathfrak{a} \in \mathcal{F}} \lambda_{x,y,y} W_y^* W_x W_x^* W_y \right) Q_A^E = \left(\sum_{\mathfrak{a} \in A} \lambda_{x,y,y} \right) Q_A^E. \quad (5.5)$$

We are now set to define the sought-after projection Q . For distinct $\mathfrak{a}, \mathfrak{b}$ in A , set $d_{\mathfrak{a},\mathfrak{b}} := [\frac{\mathfrak{a}}{\mathfrak{b}}, 1]$ if $[\frac{\mathfrak{a}}{\mathfrak{b}}, 1] \neq \mathcal{O}$ and otherwise set $d_{\mathfrak{a},\mathfrak{b}} := [\frac{\mathfrak{b}}{\mathfrak{a}}, 1]$. Since the ideals $\mathfrak{a}, \mathfrak{b}$ are distinct, it follows that $d_{\mathfrak{a},\mathfrak{b}} \neq \mathcal{O}$. Now let

$$Q := \prod_{\mathfrak{h} \in \mathcal{F} \setminus A} (E_{[A]} - E_{[[A], \mathfrak{h}]}) \prod_{\mathfrak{a}, \mathfrak{b} \in A, \mathfrak{a} \neq \mathfrak{b}} (E_{[A]} - E_{[A]d_{\mathfrak{a},\mathfrak{b}}})$$

We first verify that $Q \neq 0$. It suffices by faithfulness of π to show that the corresponding product

$$\prod_{\mathfrak{h} \in \mathcal{F} \setminus A} (1_{[A]} - 1_{[[A], \mathfrak{h}]}) \prod_{\mathfrak{a}, \mathfrak{b} \in A, \mathfrak{a} \neq \mathfrak{b}} (1_{[A]} - 1_{[A]d_{\mathfrak{a},\mathfrak{b}}})$$

is non-zero in B . To see this, we evaluate this product at the point $[A]$. This evaluation is non-zero for the following reasons. $\mathfrak{h} \notin A$ implies that $[[A], \mathfrak{h}]$ does not divide $[A]$ by (5.4). Thus $(1_{[A]} - 1_{[[A], \mathfrak{h}]})([A]) = 1$. Furthermore, if $\mathfrak{a}, \mathfrak{b}$ are distinct elements of A , recall that $d_{\mathfrak{a},\mathfrak{b}} \neq \mathcal{O}$. Thus $[A]d_{\mathfrak{a},\mathfrak{b}}$ divides $[A]$ implies that $d_{\mathfrak{a},\mathfrak{b}} | \mathcal{O}$ which is impossible. Hence $(1_{[A]} - 1_{[A]d_{\mathfrak{a},\mathfrak{b}}})([A]) = 1$. Thus it follows that $Q \neq 0$.

Since Q is a non-zero subprojection of Q_A^E above, (5.2) follows immediately from (5.3) and (5.5).

We now show that (5.1) in the statement holds, that is that $QW_y^*W_xW_x^*W_zQ = 0$ whenever $y \neq z$ in F . So suppose $y \neq z$ and let $\mathfrak{a} = [\frac{x}{y}, 1]$ and $\mathfrak{b} = [\frac{x}{z}, 1]$. Then $\mathfrak{a} \neq \mathfrak{b}$ by our original relabelling assumption. First note that

$$\begin{aligned} W_y^*W_xW_x^*W_z &= W_y^*(W_xW_x^*)^2W_z \\ &= (W_y^*E_x)(E_xW_z) \\ &= (E_{[\frac{x}{y}, 1]}W_y^*)(W_zE_{[\frac{x}{z}, 1]}) \text{ by Corollary 4.6} \\ &= E_{\mathfrak{a}}W_y^*W_zE_{\mathfrak{b}}. \end{aligned}$$

Now suppose that one of $\mathfrak{a}, \mathfrak{b}$ does not lie in A . If $\mathfrak{a} \notin A$, then $QE_{\mathfrak{a}}$ contains the factor

$$(E_{[A]} - E_{[[A], \mathfrak{a}]})E_{\mathfrak{a}} = E_{[[A], \mathfrak{a}]} - E_{[[A], \mathfrak{a}]} = 0,$$

and so $QE_{\mathfrak{a}} = 0$. Similarly, $\mathfrak{b} \notin A$ implies $E_{\mathfrak{b}}Q = 0$. In either case we then have that

$$QW_y^*W_xW_x^*W_zQ = QE_{\mathfrak{a}}W_y^*W_zE_{\mathfrak{b}}Q = 0.$$

Finally, suppose that both \mathfrak{a} and \mathfrak{b} are members of A , and assume $d_{\mathfrak{a}, \mathfrak{b}} = [\frac{\mathfrak{a}}{\mathfrak{b}}, 1]$. The case where $d_{\mathfrak{a}, \mathfrak{b}} = [\frac{\mathfrak{b}}{\mathfrak{a}}, 1]$ is handled similarly. For the rest of the proof, we denote $d_{\mathfrak{a}, \mathfrak{b}}$ by d for brevity. Before we begin our calculations, we note that for integral ideals $\mathfrak{g}, \mathfrak{h}$, $\mathfrak{g}|\mathfrak{h}$ implies that $E_{\mathfrak{g}} \geq E_{\mathfrak{h}}$.

Now $QW_y^*W_xW_x^*W_zQ = QE_{\mathfrak{a}}W_y^*W_zE_{\mathfrak{b}}Q$ contains the factor $(E_{[A]} - E_{[A]d})E_{\mathfrak{a}}W_y^*W_zE_{\mathfrak{b}}E_{[A]}$, which we show equals 0.

$$\begin{aligned} (E_{[A]} - E_{[A]d})E_{\mathfrak{a}}W_y^*W_zE_{\mathfrak{b}}E_{[A]} &= (E_{[[A], \mathfrak{a}]} - E_{[[A]d, \mathfrak{a}]})W_y^*W_zE_{[\mathfrak{b}, [A]]} \\ &= (E_{[A]} - E_{[A]d})W_y^*W_zE_{[A]} \\ &= (E_{[A]} - E_{[A]d})E_{[\frac{x}{y}, 1]}W_y^*W_z \text{ by Corollary 4.6} \\ &= 0 \end{aligned}$$

where the second equality holds because $\mathfrak{a}, \mathfrak{b} \in A$ and so $[[A], \mathfrak{a}] = [[A], \mathfrak{b}] = [A]$. Furthermore, the last equality is true since $(E_{[A]} - E_{[A]d})E_{[\frac{z[A]}{y}, 1]} = 0$. To see this, we show first that $[\frac{\mathfrak{a}[A]}{\mathfrak{b}}, 1]$ divides $[\frac{z[A]}{y}, 1]$. Recall that $\mathfrak{a} = [\frac{x}{y}, 1]$ and that $\mathfrak{b} = [\frac{x}{z}, 1]$. Since $\frac{z}{y} = \frac{x/z}{x/y}$, it follows that $[\frac{\mathfrak{a}}{\mathfrak{b}}, 1]$ divides $[\frac{z}{y}, 1]$ and so $[\frac{\mathfrak{a}[A]}{\mathfrak{b}}, 1]$ divides $[\frac{z[A]}{y}, 1]$.

Therefore we have

$$(E_{[A]} - E_{[A]d})E_{[\frac{z[A]}{y}, 1]} \leq (E_{[A]} - E_{[A]d})E_{[\frac{\mathfrak{a}[A]}{\mathfrak{b}}, 1]} = E_{[[A], \frac{\mathfrak{a}[A]}{\mathfrak{b}}, 1]} - E_{[[A], \frac{\mathfrak{a}[A]}{\mathfrak{b}}, \frac{\mathfrak{a}[A]}{\mathfrak{b}}, 1]} = 0$$

and the result follows. \square

Proposition 5.2. *Suppose (π, W) is a covariant pair for (B, P, α, σ) , where π is faithful. Let F be a finite subset of P . Then*

$$\phi: \sum_{x,y,z \in F} \lambda_{x,y,z} W_y^* W_x W_x^* W_z \mapsto \sum_{x,y \in F} \lambda_{x,y,y} W_y^* W_x W_x^* W_y$$

is contractive and hence extends to a contraction of

$$C^*(W_x : x \in P) = \pi \times W(B \times P) = \overline{\text{span}}(W_y^* W_x W_x^* W_z : x, y, z \in P) \text{ onto } \overline{\text{span}}(W_y^* W_x W_x^* W_y : x, y \in P) = \pi(B).$$

Proof. Consider an arbitrary linear combination

$$\sum_{x,y,z \in F} \lambda_{x,y,z} W_y^* W_x W_x^* W_z.$$

It suffices to prove that

$$\left\| \sum_{x,y \in F} \lambda_{x,y,y} W_y^* W_x W_x^* W_y \right\| \leq \left\| \sum_{x,y,z \in F} \lambda_{x,y,z} W_y^* W_x W_x^* W_z \right\|$$

since the set $\{W_y^* W_x W_x^* W_z : x, y, z \in P\}$ has dense linear span in $C^*(W_x : x \in P)$.

Lemma 5.1 immediately applies to give a projection Q such that $QW_y^*W_xW_x^*W_zQ = 0$ when $y \neq z$ and such that

$$\|Q(\sum_{x,y \in F} \lambda_{x,y,y} W_y^*W_xW_x^*W_y)Q\| = \|\sum_{x,y \in F} \lambda_{x,y,y} W_y^*W_xW_x^*W_y\|.$$

Thus

$$\begin{aligned} \|\sum_{x,y \in F} \lambda_{x,y,y} W_y^*W_xW_x^*W_y\| &= \|Q(\sum_{x,y \in F} \lambda_{x,y,y} W_y^*W_xW_x^*W_y)Q\| \\ &= \|Q(\sum_{x,y,z \in F} \lambda_{x,y,z} W_y^*W_xW_x^*W_z)Q\| \\ &\leq \|\sum_{x,y,z \in F} \lambda_{x,y,z} W_y^*W_xW_x^*W_z\| \end{aligned}$$

and the result follows. \square

Our last major step before Theorem 5.4 is to show that the map $\Phi : B \rtimes P \rightarrow \overline{\text{span}}\{W_y^*W_xW_x^*W_y : x, y \in P\}$ given by Proposition 5.2 is faithful on positive elements.

Lemma 5.3. *Let $\Phi : B \rtimes P \rightarrow \overline{\text{span}}\{W_y^*W_xW_x^*W_y : x, y \in P\}$ be determined by*

$$\Phi(i_P(y)^*i_P(x)i_P(x)^*i_P(z)) = \delta_{y,z}i_P(y)^*i_P(x)i_P(x)^*i_P(z),$$

*as in Proposition 5.2. If $\Phi(A^*A) = 0$ for $A \in B \rtimes P$, then $A = 0$.*

Proof. We let $G = P^{-1}P$, which is precisely the group of principal fractional ideals of K . Equipping G with the discrete topology, we get that the dual group \hat{G} is compact. For each γ in \hat{G} , let $\gamma i_P : P \rightarrow B \rtimes P$ be the map

$$(\gamma i_P)(x) := \gamma(x)i_P(x).$$

One immediately checks that that $(B \rtimes P, P, i_B, \gamma i_P)$ is also universal for covariant pairs, and thus there is an automorphism θ_γ on $B \rtimes P$ such that $\theta_\gamma \circ i_P = \gamma i_P$.

One can easily check that each θ_γ is pointwise continuous on the generating set $i_B(B) \cup i_P(P)$. And since automorphisms are automatically isometric it follows that θ determines a strongly continuous action of \hat{G} on $B \rtimes P$, i.e. that

$$\gamma \mapsto \theta_\gamma(A)$$

is a continuous map from \hat{G} to $B \rtimes P$ for each fixed $A \in B \rtimes P$.

Setting Φ_θ to be the conditional expectation over \hat{G} defined by

$$\Phi_\theta(A) := \int_{\hat{G}} \theta_\gamma(A) d\gamma, \text{ for each } A \in B \rtimes P$$

it follows by Fourier inversion that

$$\begin{aligned} \Phi_\theta(i_P(y)^* i_P(x) i_P(x)^* i_P(z)) &= \int_{\hat{G}} \theta_\gamma(i_P(y)^* i_P(x) i_P(x)^* i_P(z)) d\gamma \\ &= \left(\int_{\hat{G}} \gamma(y^{-1} x x^{-1} z) d\gamma \right) i_P(y)^* i_P(x) i_P(x)^* i_P(z) \\ &= \delta_{y,z} i_P(y)^* i_P(x) i_P(x)^* i_P(z) \end{aligned}$$

on the spanning set and hence $\Phi_\theta = \Phi$ on all of $B \rtimes P$. \square

Theorem 5.4. *Let W be a covariant isometric representation of P . Then the corresponding representation $\pi_W \rtimes W$ is an isomorphism of $B \rtimes P$ onto $C^*(W_x : x \in P)$ if and only if*

$$\prod_1^n (I - W_{y_j}^* W_{x_j} W_{x_j}^* W_{y_j}) \neq 0 \text{ whenever } \mathfrak{a}_j := \left[\frac{x_j}{y_j}, 1 \right] \neq (1) \text{ for all } j. \quad (5.6)$$

Proof. Suppose first that $\pi_W \rtimes W$ is faithful. Recall by Corollary 4.7 that i_B is faithful, whence $(\pi_W \rtimes W) \circ i_B = \pi_W$ is a faithful representation. Thus (5.6) follows by Proposition 4.5(3).

Now suppose that (5.6) holds. Then by Proposition 4.5(3), π_W is faithful. Thus Proposition 5.2 applies to give a contraction ϕ on $C^*(W_x : x \in P)$ defined by

$\phi(W_y^*W_xW_x^*W_z) = \delta_{y,z}W_y^*W_xW_x^*W_z$. With Φ as in Lemma 5.3, we obviously have that $\phi \circ (\pi_W \rtimes W) = (\pi_W \rtimes W) \circ \Phi$. Recall from Corollary 4.7 that $B \rtimes P$ is densely spanned by $\text{span}\{i_P(y)^*i_P(x)i_P(x)^*i_P(z) : x, y, z \in P\}$. Therefore $\Phi(B \rtimes P) = \overline{\text{span}}\{i_P(y)^*i_P(x)i_P(x)^*i_P(y) : x, y \in P\} = i_P(B)$. But i_P is faithful by Corollary 4.7 and so $\Phi(B \rtimes P)$ is isomorphic to B . Thus the faithfulness of π_W on B implies the faithfulness of $(\pi_W \rtimes W)$ on the range of Φ . Thus $(\pi_W \rtimes W)(A) = 0$ implies that

$$(\pi_W \rtimes W)(\Phi(A^*A)) = \phi((\pi_W \rtimes W)(A^*A)) = 0$$

and hence $\Phi(A^*A) = 0$. Now Lemma 5.3 applies to give $A = 0$, and this completes the proof. \square

Corollary 5.5. *$C^*(P)$ is isomorphic to the Toeplitz C^* -algebra.*

Proof. Letting T be the Toeplitz representation of P , it suffices to show that (5.6) holds for the covariant pair (π_T, T) . But this fact was already demonstrated in the proof of Corollary 4.7 above so we are done. \square

Chapter 6

Conclusion

We began with a number field, K , and its associated semigroup of principal integral ideals P and studied its most concrete isometric representation, namely the Toeplitz representation. Just as the principal ideals $[a, 1]$, $a \in P$ were not enough in general to capture all the integral ideals in \mathcal{I} , so too were the associated range projections $T_a T_a^*$ not enough to obtain a covariance relation that could characterize the Toeplitz C^* -algebra. However once division was considered, we saw that every integral ideal in \mathcal{I} was of the form $[\frac{a}{b}, 1]$ for some $a, b \in P$ (Proposition 2.2). This algebraic property was realized in the Toeplitz C^* -algebra by taking the range projection $T_a T_a^*$ and conjugating to get $T_b^* T_a T_a^* T_b$. The result was (T4) in Proposition 3.1 and it turned out to be the correct property on which to focus. Following the Toeplitz representation we studied arbitrary isometric representations of P which satisfied this new covariance relation and showed that the universal C^* -algebra $C^*(P)$ for relations (1) through (4) in Definition 4.1 could be expressed as crossed product $B \rtimes P$ of a C^* -algebra by two actions of P . With this handle on $C^*(P)$ we then proceeded to characterize faithful representations $(\pi \times W)$ of the crossed product. The culmination of this was Theorem 5.4, from

which it followed as an immediate corollary that the Toeplitz C^* -algebra initially studied was canonically isomorphic to $C^*(P)$.

The approach we took only involved representations and actions of P and was direct. There is a more abstract approach to this involving the use of *partial* isometric representations of the group K^*/\mathcal{O}^* of all fractional principal ideals, which includes P as a proper subsemigroup. We touched upon this in Remark 4.2 and it is our belief that the results of this thesis could be obtained alternatively using the results of [4]. This will be investigated at a later date.

Regarding other possible avenues of research, recall that our need to seek out a new covariance stemmed from the fact that in general not all integral ideals in a number field are principal. But there is a construction in number theory which essentially “makes ideals principal” (see [8]). Specifically, given a number field K with ring of integers \mathcal{O}_K there exists a finite field extension $L \supset K$ with ring of integers $\mathcal{O}_L \supset \mathcal{O}_K$ which satisfies the following. Given any subset S of \mathcal{O}_L , let (S) be the ideal in \mathcal{O}_L generated by S . For every integral ideal \mathfrak{a} in \mathcal{O}_K , there exists an $\alpha \in \mathcal{O}_L$ such that $(\mathfrak{a}) = \alpha\mathcal{O}_L$ and where $(\mathfrak{a}) \cap \mathcal{O}_K = \mathfrak{a}$. Note that this does not say that \mathcal{O}_L is a principal ideal domain and in fact it is not always possible, given K , to find such an extension L where \mathcal{O}_L is a principal ideal domain. Nevertheless, if one applies the theory developed in this paper to the principal integral ideals P_K and P_L of K and L , respectively, it remains an interesting question of how $C^*(P_K)$ could fit inside $C^*(P_L)$ in light of the above.

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