Broadcasts and multipackings in graphs

by

Laura Elizabeth Teshima
B.Sc., Thompson Rivers University, 2010

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ABSTRACT

A *dominating broadcast* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, \operatorname{diam}(G)\}$ with $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$ and such that each vertex is within distance $f(v)$ from a vertex v with $f(v) > 0$. The *cost* of a broadcast f is $\sigma(f) = \sum_{v\in V} f(v)$, and the *broadcast number* $\gamma_b(G)$ is the minimum cost of a dominating broadcast. A set $X \subseteq V(G)$ is said to be *irredundant* if each $x \in X$ dominates a vertex y that is not dominated by any other vertex in X; possibly $y = x$. The *irredundance number* $\operatorname{ir}(G)$ is the cardinality of a smallest maximal irredundant set of G. In the first half of this thesis, we prove the bound $\gamma_b(G) \leq \frac{3}{2}\operatorname{ir}(G)$ for any graph G and provide an infinite class of graphs with $\gamma_b(G) = \frac{3}{2}\operatorname{ir}(G)$.

In the second portion of this thesis, we consider a new dual property to broadcast domination. A set $M \subseteq V$ is called a *multipacking* of a graph $G = (V, E)$ if, for each $v \in V$ and each s such that $1 \leq s \leq \operatorname{diam}(G)$, v is within distance s of at most s vertices in M. The *multipacking number*, denoted $\operatorname{mp}(G)$, is the maximum cardinality of a multipacking of G. We prove that for any tree T, $\gamma_b(T) = \operatorname{mp}(T)$. This generalizes the well-known result of A. Meir and J. W. Moon [Relations between packing and covering numbers of a tree, Pacific J. Math. 61 (1975), no. 1, 225–233; MR0401519 (53 #5346)] that the 2-packing number of any tree equals its domination number. We then present an algorithm for finding a maximum multipacking of a tree, and show how multipackings can be used to certify the minimality of a dominating broadcast. We conclude with thesis by suggesting some open problem in broadcast domination and multipackings.

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Chapter 1

Introduction

Domination in graphs is a popular and extensively researched topic [15, 16]. A popular problem in domination is determining how a telecommunications company might position its radio towers. The company is charged with providing radio coverage to a collection of geographic regions. A single tower transmitting with a strength (or cost) of one unit can provide coverage to the region it is located in and all regions immediately adjacent to it. Obviously, the company would like to minimize its expenses by erecting as few towers as possible. How should the company distribute its towers to minimize its costs? If we consider each region as a vertex of a graph, where two vertices are joined with an edge if their corresponding geographic regions are adjacent, then any minimum dominating set S represents a minimum cost arrangement of radio towers.

Formally, a *dominating set* S of a graph $G = (V, E)$ is a set such that every vertex of G is either in S or adjacent to some vertex in S. That is, S is a dominating set if and only if for each $v \in G$, either $v \in S$ or there exists some $u \in S$ such that $uv \in E$. The smallest cardinality of a dominating set is the *domination number* $\gamma(G)$. For vertices u and v of G, we say that v *dominates* u if $u = v$ or u is adjacent to v.

More recently, broadcast domination has emerged as an interesting generalization of domination. Returning to the radio tower problem in domination, broadcast domination allows the company to build its towers with varying signal strength so that a tower may transmit its signal a greater distance, but at a greater cost. Similarly to the ordinary domination case, the solution of the telecommunications company's quandary is solvable with a minimum cost dominating broadcast f.

A *broadcast* on a connected graph G is a function $f : V \to \{0, 1, \ldots, \text{diam}(G)\}$ such that $f(v) \leq e(v)$ for all $v \in V$, where $e(v) = \max\{d(v, w) : w \in V\}$ is the *eccentricity* of v. We say that f is a *dominating broadcast* of G if every vertex of G is within distance $f(v)$ from a vertex v such that $f(v) > 0$. The *cost* of a broadcast f is $\sigma(f) = \sum_{v \in V} f(v)$, and the *broadcast number* of G is $\gamma_b(G) = \min\{\sigma(f) : f \text{ is a dominating broadcast of } G\}$. A dominating broadcast f of G such that $\sigma(f) = \gamma_b(G)$ is called a γ_b*-broadcast*. If f is a dominating broadcast such that $f(v) \in \{0, 1\}$ for each $v \in V$, then $\{v \in V : f(v) = 1\}$ is an ordinary dominating set of G.

The focus of this thesis is to present two new results in broadcast domination which have well-known analogues in ordinary domination. In Chapter 2, we review some required notation and relevant background results. The first major result, presented in Chapter 3, is a new upper bound for the broadcast number of a graph in terms of its irredundance number. This tightens the previous implicit bound from Bollobás and Cockayne [2], which was derived for the ordinary domination number. We then examine a generalization of packings, called multipackings, and their relationship to broadcasting, in Chapter 4. In particular, we extend an often-cited result of Meir and Moon [23], by proving that the multipacking and broadcast numbers of a tree are equal. Finally, in Chapter 5, we summarize our results and suggest some future research directions.

Chapter 2

Notation and Background

In this chapter we examine some previous results in the topics of broadcasting, irredundance and packings. We follow the notation of [4, 16], but provide additional definitions in the following chapter as required.

2.1 Broadcast Domination

For any positive integer k and any vertex v, we define the (*closed*) *k-neighbourhood* of v by $N_k[v] = \{u \in V : d(u,v) \leq k\}$. The *k-neighbourhood of a set* $X \subseteq V$ is the set $N_k[X] = \bigcup_{v \in X} N_k[v]$.

Now consider a broadcast f on the graph G. The *broadcast vertices* of G form the set $V_f^+ = \{v \in V : f(v) > 0\}$. The *broadcast neighbourhood* of $v \in V_f^+$ is the set $N_f[v] = N_{f(v)}[v] = \{u \in V : d(u,v) \leq f(v)\}$. Thus, f is a dominating broadcast of G, or G is *f-dominated*, if for each $u \in V$ there exists $v \in V_f^+$ such that $u \in N_f[v]$; we say that u *hears* the broadcast from v. If for each $u \in V$, $\left| \left\{ v \in V_f^+ : u \in N_f[v] \right\} \right| = 1$, that is, u hears only one broadcast, we say that f is an *efficient* broadcast.

2.1.1 Basic Results

Broadcast domination was introduced by Erwin in his 2001 doctoral dissertation as "cost domination" [12]. Many of these results are summarized in [13]. In his dissertation, Erwin established several sharp upper and lower bounds for the broadcast number, including the following often-referenced result.

Proposition 2.1. [12] *For every nontrivial connected graph G,*

$$\left\lceil \frac{\operatorname{diam}(G)+1}{3} \right\rceil \leq \gamma_b(G) \leq \min\left\{\operatorname{rad}(G), \gamma(G)\right\}. \tag{2.1}$$

A simple result for paths nicely demonstrates the previous proposition.

Proposition 2.2. [12] *For any integer $n \geq 2$,*

$$\gamma_b(P_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Erwin also characterized graphs with $\gamma_b(G) \leq 3$.

Proposition 2.3. [12] *Let G be a connected graph. If $\min\{\operatorname{rad}(G), \gamma(G)\} = k$, where $1 \leq k \leq 3$, then $\gamma_b(G) = k$.*

These propositions highlight three classes of graphs defined in terms of their broadcast number.

Type I: (*radial*) graphs with $\gamma_b(G) = \operatorname{rad}(G)$,
Type II: (*1-cap*) graphs with $\gamma_b(G) = \gamma(G)$,
Type III: graphs with $\gamma_b(G) < \min\{\operatorname{rad}(G), \gamma(G)\}$.

Thus, Proposition 2.3 shows that all graphs with $\min\{\operatorname{rad}(G), \gamma(G)\} \leq 3$ are radial or 1-cap. Figure 2.1 depicts Erwin's example which shows that a similar statement does not hold for a graph with $\min\{\operatorname{rad}(G), \gamma(G)\} = 4$. Much of the research done on broadcast domination has been in classifying various graph families as radial or 1-cap. In the following sections, we examine the classification of trees.

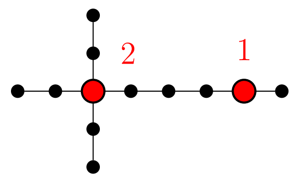


Figure 2.1: A graph G with $\min\{\mathrm{rad}(G), \gamma(G)\} = 4$ and $\gamma_b(G) = 3$.

Another useful result for investigating broadcasts on graphs is from Dunbar et al. concerning efficient broadcasts.

Proposition 2.4. [11] *Every graph G has a γ_b-broadcast which is efficient.*

Proposition 2.4 allows for the examination of broadcasts on graphs without having to worry about the possibility of overlapping broadcast neighbourhoods of broadcast vertices.

2.1.2 Shadow Trees

We now examine a useful construct for investigating the broadcast number of trees, called shadow trees. Let $P : v_0, ..., v_d$ be a diametrical path of a tree T with $\mathrm{diam}(T) = d$. For each $v_i \in V(P)$, let U_i be the set of all vertices of T that are connected to v_i by a (possibly trivial) path internally disjoint from P. Let u_i be a vertex in U_i at maximum distance from v_i, and let B_i be the $v_i - u_i$ path. The *shadow tree $S_{T,P}$ of T with respect to P* is the subtree of T induced by $\bigcup_{i=0}^{d} V(B_i)$. If $T \cong S_{T,P}$ for some diametrical path P of T, then T is also called a *shadow tree*. Note that a shadow tree has maximum degree at most three. In Figures 2.2, 2.3 and 2.4 we illustrate a tree T with diametrical paths P_1 and P_2, and the non-isomorphic shadow trees S_{T,P_1} and S_{T,P_2}.

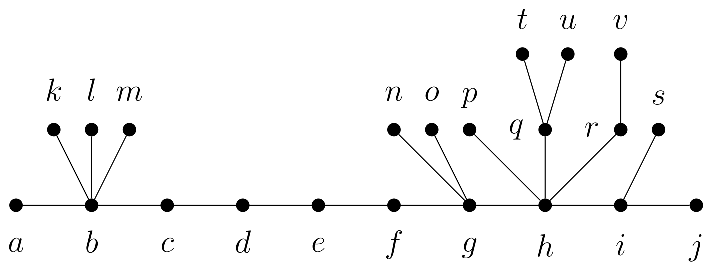


Figure 2.2: A tree T with diametrical paths $P_1 = \{a, b, c, d, e, f, g, h, i, j\}$ and $P_2 = \{k, b, c, d, e, f, g, h, r, v\}$.

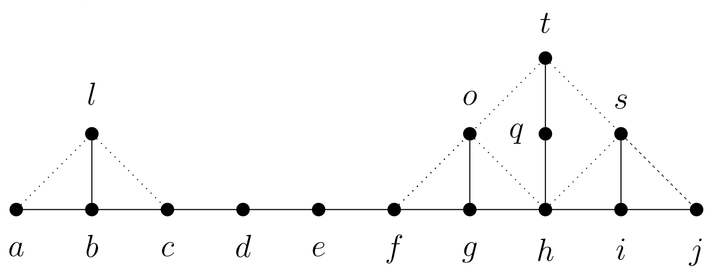


Figure 2.3: The shadow tree S_{T,P_1} of T in Figure 2.2.

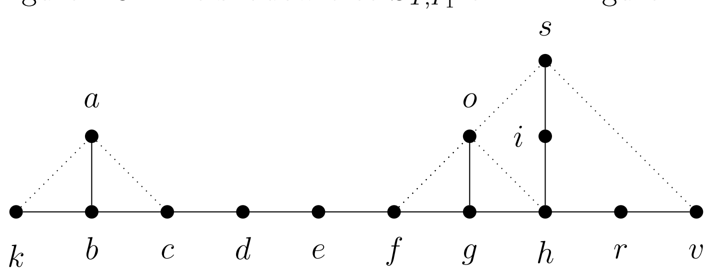


Figure 2.4: The shadow tree S_{T,P_2} of T in Figure 2.2.

Shadow trees are particularly useful in studying broadcasts on trees. Herke and Mynhardt demonstrated the following.

Theorem 2.5. [19] *For any shadow tree S_T of T, $\gamma_b(S_T) = \gamma_b(T)$.*

That is, the broadcast numbers of a tree and any of its shadow trees are equal. This result simplifies the study of broadcasts on trees by allowing the removal of certain branches and leaves.

Before considering more results on shadow trees, we introduce some additional notation concerning their structure. Consider a shadow tree $S_{T,P}$. If $U_i - \{v_i\} \neq \varnothing$, we call v_i a *branch vertex* and the $v_i - u_i$ path B_i a *branch*. Furthermore, for $\alpha_i = d(v_i, u_i) \geq 1$, the tree Δ_i induced by $\{v_{i-\alpha_i}, \ldots, v_{i-1}\} \cup V(B_i) \cup \{v_{i+1}, \ldots, v_{i+\alpha_i}\}$ is called the *triangle at* i. If the vertex subset $\{v_{i-\alpha_i}, \ldots, v_i, \ldots, v_{i+\alpha_i}\}$ of the triangle Δ_i is contained in the vertex subset $\{v_{j-\alpha_j}, \ldots, v_j, \ldots, v_{j+\alpha_j}\}$ of the triangle Δ_j, then Δ_i is called a *nested triangle*. In Figure 2.3, the vertex set $\Delta_g = \{f, g, h, o\}$ forms the triangle at g. Likewise, $\Delta_h = \{f, g, h, i, j, q, t\}$ induces the triangle at h. Since $\{f, g, h\} \subseteq \{f, g, h, i, j\}$, Δ_g is a nested triangle. A *free edge* is an edge of $S_{T,P}$ that is not in any triangle; note that all free edges of $S_{T,P}$ lie on P. The edges cd, de and ef in Figure 2.3 are free edges.

The triangles of $S_{T,P}$ are labeled in order of their occurrence on P and are denoted $\Delta_{i_1}, \Delta_{i_2}, \ldots, \Delta_{i_c}$. For simplicity, we abuse notation and denote Δ_{i_1} as Δ_1, and Δ_{i_c} as Δ_c. A free edge on P that comes before Δ_1 is called a *leading free edge*; likewise, a free edge that comes after Δ_c is called a *trailing free edge*. If e is a free edge of $S_{T,P}$, we also call e a *free edge of* T *with respect to* P. A set M of edges of the diametrical path P of the tree T is a *split-*P *set* if each component T' of $T - M$ has a positive even diameter and $P' = T' \cap P$ is a diametrical path of T'. A *split-set* of T is a split-P set for some diametrical path P of T. An edge in any split-set of T is a *split-edge*. In Figures 2.3-2.4, the sets $M_1 = \{cd\}$ and $M_2 = \{ef\}$ are split-P_1 sets. Thus, cd and ef are split-edges. The requirement that $P' = T' \cap P$ be a diametrical path of T' implies that each split-edge is a free edge. However, not all free edges are split-edges. For example, the free edge de in Figures 2.2-2.4 is not a split-edge because the components of the trees after the deletion of de have odd diameters.

Herke and Mynhardt provide the following result which demonstrates how split-sets determine the broadcast number of a tree.

Theorem 2.6. [19] *If M is a split-set of maximum cardinality m of a tree T, then*

$$\gamma_b(T) = \left\lceil \frac{\operatorname{diam}(T) - m}{2} \right\rceil.$$

Note that $\left\lceil \frac{\operatorname{diam}(T)-m}{2} \right\rceil = \frac{\operatorname{diam}(T)-m}{2}$ unless $m = 0$ and $\operatorname{diam}(T)$ is odd.

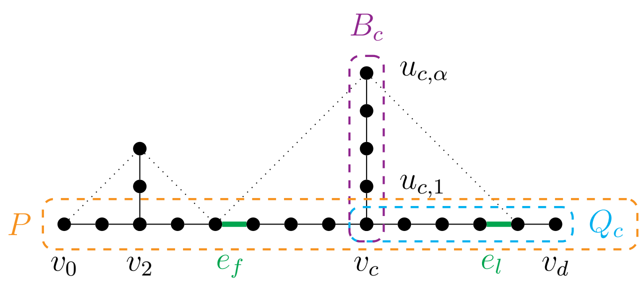


Figure 2.5: A labelled shadow tree $S_{T,P}$.

We illustrate the following notation in Figure 2.5. Let c be the highest index such that v_c is a branch vertex of T. The subpath $Q_c : v_c, ..., v_d$ of P is called the *trailing endpath* of T. The branch of T that starts at v_c is the path $B_c : v_c = u_{c,0}, u_{c,1}, ..., u_{c,\alpha}$ of length α, and is called the *last branch* of T. The triangle Δ_c associated with B_c is called the *last triangle* of T. For brevity we also write B_c and Q_c for $V(B_c)$ and $V(Q_c)$, respectively. We denote the lengths of B_c and Q_c by $\ell(B_c)$ and $\ell(Q_c)$, respectively; note that $\alpha = \ell(B_c) \leq \ell(Q_c) = d - c$. The first and last edges of Δ_c on P are $e_f = v_{c-\alpha}v_{c-\alpha+1}$ and $e_\ell = v_{c+\alpha-1}v_{c+\alpha}$, respectively. We say that the leaf $u_{c,\alpha}$ of B_c *binds* e_f (e_ℓ, respectively) if $T - u_{c,\alpha}$ has a split-set that contains e_f (e_ℓ, respectively), but T has no such split-set.

2.1.3 Broadcasts in Trees

Dunbar, Hedetniemi and Hedetniemi [10] first considered the characterization of trees that are Type I and Type II in an unpublished manuscript in 2003. In 2008, Seager [27] published her characterization of broadcast types of caterpillars. A complete characterization of radial trees was given by Herke in her 2009 thesis.

Theorem 2.7. [18] *A tree T is radial if and only if it has no nonempty split-set.*

Mynhardt and Wodlinger [25] expanded upon these results by providing eight conditions for a tree to be *uniquely radial*; that is, a tree with $\gamma_b(T) = \text{rad}(T)$ and whose only minimum cost broadcast is a broadcast from a central vertex with cost equal to the radius of T. In his 2011 thesis, Lunney [21] continued the characterization of broadcasts in trees by determining a large class of 1-cap trees.

2.1.4 Complexity

Determining the domination number of a general graph is NP-hard. Indeed, determining $\gamma(G)$ for many classes of well-known graphs, including bipartite graphs and chordal graphs, remains NP-hard [16]. Thus, since broadcast domination is a generalization of domination, it was originally speculated that finding $\gamma_b(G)$ would also be NP-hard. However, in 2006 Heggernes and Lokshtanov [17] published an algorithm for finding $\gamma_b(G)$ in $O(n^6)$ time, for a graph on n vertices.

Trees are one of the few classes of graphs with linear algorithms for finding $\gamma(T)$. This algorithm, for computing $\gamma(T)$ for an arbitrary unweighted tree, was discovered by Cockayne, Goodman and Hedetniemi in 1975 [5]. In 2007, Dabney [8] found a linear algorithm for $\gamma_b(T)$ for an arbitrary unweighted tree. This result was published in [9].

2.2 Irredundance

The graph property irredundance is closely related to domination. Formally, a set $X \subseteq V(G)$ is said to be *irredundant* if each $x \in X$ dominates a vertex y that is not dominated by any other vertex in X; it is possible that $y = x$. An irredundant set

X is *maximal irredundant* if, for any $v \in V - X$, $X \cup \{v\}$ is not irredundant. The *irredundance number* $\text{ir}(G)$ is the cardinality of a smallest maximal irredundant set of G. A maximal irredundant set of cardinality $\text{ir}(G)$ is an ir-*set*.

Irredundant sets are often viewed in terms of their private neighbourhoods. For $v \in X$, a vertex u is called a *private neighbour of v with respect to X*, if $N[u] \cap X = \{v\}$. Notice that it is possible that $u = v$; in this case, we call v a *self-private neighbour*. The *private neighbourhood of v with respect to X* is $\text{pn}(v, X) = \{u : N[u] \cap X = \{v\}\}$. The *external private neighbourhood of v with respect to X* is the set $\text{epn}(v, X) = \text{pn}(v, X) - \{v\}$. Note the alternative definition that X is irredundant if and only if $\text{pn}(v, X) \neq \varnothing$ for all $v \in X$.

Irredundance was introduced by Cockayne, Hedetniemi and Miller in 1978 [7]. They observed the following proposition, which allows the use of irredundance to test the minimality of a dominating set.

Proposition 2.8. [7] *A dominating set S is minimal dominating if and only if it is irredundant.*

They also demonstrated the following inequality, a portion of the well-known domination chain.

Proposition 2.9. [7] *For any graph G,* $\text{ir}(G) \leq \gamma(G)$.

The following year, Bollobás and Cockayne [2] showed that for any graph G,

$$\gamma(G) \leq 2\,\text{ir}(G) - 1. \tag{2.2}$$

When investigating irredundant sets, it is helpful to view the vertices of a graph G as partitioned into four subsets (see Figure 2.6):

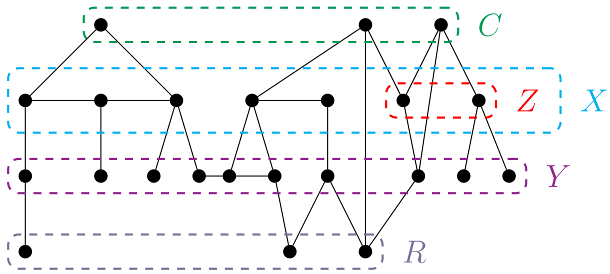


Figure 2.6: A graph H illustrating the subsets of $V(H)$.

X, an irredundant set

$Y = \bigcup_{x \in X} \operatorname{epn}(x, X)$, the set of external private neighbours of vertices in X

$R = V - N[X]$, the set of vertices not dominated by X

$C = V - (X \cup Y \cup R)$, the set of vertices in $V - X$ that are dominated by two or more vertices in X.

Furthermore, let Z be the set of isolated vertices of $G[X]$. Cockayne, Grobler, Hedetniemi and McRae [6] provide a useful necessary and sufficient condition for an irredundant set to be maximal irredundant.

Theorem 2.10. [6] *An irredundant set* X *is maximal irredundant if and only if*

A: *for each* $v \in N[R]$ *there exists* $x \in X$ *such that* $\operatorname{pn}(x, X) \subseteq N[v]$.

If **A** holds, we say that v *annihilates* x, and we call **A** the *annihilation property*. Henceforth we let X denote a maximal irredundant set of G. Then for each $r \in R$ there exists $x \in X$ such that $d(r, x) = 2$, that is, $R \subseteq N_2[X]$.

$$\text{With } x \in X, \text{let } R(x) = \{r \in R : d(r, x) = 2\}.$$

If $x, x' \in X$, then $R(x)$ and $R(x')$ may or may not be disjoint. Moreover, by the annihilation property,

$$\text{for each } r \in R \text{ there exists } x \in X \text{ such that } \operatorname{epn}(x, X) \subseteq N(r).$$

Note that $r \in R(x)$ in this case. Hence

$$R = \bigcup_{x \in X} R(x). \tag{2.3}$$

2.2.1 Complexity

In his 1984 dissertation, Pfaff demonstrated that finding $\operatorname{ir}(G)$ for an arbitrary graph is NP-hard [26]. Determining $\operatorname{ir}(G)$ for other common graph classes, including split-graphs (and hence chordal graphs) [20], line graphs [22], and k-regular graphs [14], has also been shown to be NP-hard. In 1985, Bern, Lawler and Wong [1] constructed an $O(n)$ algorithm for determining $\operatorname{ir}(T)$ for a tree on n vertices.

2.3 Packings

A set of vertices S is a 2-*packing* if for each pair of vertices $u, v \in S$, $N[u] \cap N[v] = \varnothing$. The 2-*packing number* of a graph G, $\rho(G)$, is the cardinality of a maximum 2-packing. The definition provides the following immediate bound.

Theorem 2.11. *For any graph G, $\rho(G) \leq \gamma(G)$.*

For trees, Meir and Moon [23] proved the equality of the 2-packing and domination numbers in 1975.

Theorem 2.12. [23] *For any tree T, $\rho(T) = \gamma(T)$.*

The development of multipackings will be discussed later in Chapter 4. For now, we provide only the basic definitions. For a graph G and an integer k such that $1 \leq k \leq \text{diam}(G)$, a set M of vertices of G is called a *k-multipacking* if, for each $v \in V$ and each integer s such that $1 \leq s \leq k$, the set $N_s[v]$ contains at most s vertices in M. Formally, M is a $k-$multipacking if for all $v \in V$, $|N_s[v] \cap M| \leq s$ for all $1 \leq s \leq k$. The *k-multipacking number* $\text{mp}_k(G)$ is the maximum cardinality of a k-multipacking of G. A 1-multipacking is simply an ordinary 2-packing, and $\text{mp}_1(G)$ is the 2-packing number $\rho(G)$. If M is a k-multipacking, where $k = \text{diam}(G)$, we call M a *multipacking*, and the k-multipacking number the *multipacking number,* denoted $\text{mp}(G)$.

Chapter 3

A New Upper Bound for the Broadcast Number of a Graph

In this chapter we provide a new upper bound for the broadcast number of a graph in terms of its irredundance number. By combining the irredundance and domination number bound of Bollobás and Cockayne (2.2) with Erwin's bound for the broadcast number (2.1), we obtain the following implicit bound:

$$\gamma_b(G) \leq \min\{\gamma(G), \mathrm{rad}(G)\} \leq 2\,\mathrm{ir}(G) - 1$$

for any graph G.

However, this bound is really based upon $\gamma(G)$, not $\gamma_b(G)$. By ignoring the middleman, we obtain a new tight bound for $\gamma_b(G)$. The work in this chapter has been accepted for publication in [3].

Theorem 3.1. *For any graph G, $\gamma_b(G) \leq \frac{3}{2}\,\mathrm{ir}(G)$. Moreover, for each $k \in \mathbb{Z}^+$ there exists a connected graph G_k such that* $\mathrm{ir}(G_k) = 2k$ *and* $\gamma_b(G_k) = 3k$.

Viewed from the perspective of the irredundance number of a graph, Theorem 3.1

provides a tight lower bound for $\mathrm{ir}(G)$ in terms of the more easily computed broadcast number $\gamma_b(G)$.

3.1 Proof of Theorem 3.1

To prove our new upper bound, we first establish two lemmas. We follow the notation defined in Section 2.2.

Lemma 3.2. *For any $z \in Z$ and any $r \in R(z)$, there exists $x \in X - Z$ such that $r \in R(x)$.*

Proof. By the definition of R, $r \notin N[X]$. By the annihilation property of the maximal irredundant set X, r annihilates some $x \in X$. Thus $r \in R(x)$. Since r is not adjacent to x, $x \notin N[r]$. Since r annihilates x, $\mathrm{pn}(x, X) \subseteq N[r]$. It follows that $x \notin \mathrm{pn}(x, X)$, that is, x is not an isolated vertex of $G[X]$. Therefore $x \in X - Z$ as required. ■

By Lemma 3.2 and (2.3),

$$R = \bigcup_{x \in X - Z} R(x),$$

that is,

$$R \subseteq N_2[X - Z]. \tag{3.1}$$

Lemma 3.3. *If a graph G has an* ir*-set X such that $G[X]$ has only isolated vertices, then $\gamma_b(G) \leq \mathrm{ir}(G)$.*

Proof. By Lemma 3.2, $R = \varnothing$. Hence X dominates G so that $\gamma_b(G) \leq \gamma(G) = \mathrm{ir}(G)$. ■

We now prove Theorem 3.1, which we restate here for convenience.

Theorem 3.1 *For any graph G, $\gamma_b(G) \leq \frac{3}{2}\,\mathrm{ir}(G)$. Moreover, for each $k \in Z^+$ there exists a connected graph G_k such that $\mathrm{ir}(G_k) = 2k$ and $\gamma_b(G_k) = 3k$.*

Proof. Let X be an ir-set of G. We proceed by examining three types of components of $G[X]$ and their 2-neighbourhoods. To simplify notation we denote the vertex set of each component X_i of $G[X]$ also by X_i. Below we define a broadcast f in three steps such that $V_f^+ \subseteq X$, and for each $X_i \neq K_2$, $\sum_{v \in X_i} f(v) \leq |X_i|$, while $\sum_{v \in X_i} f(v) = 3$ if $X_i = K_2$. Hence $\sigma(f) \leq \frac{3}{2}|X|$.

Type 1: X_i is a component with order $n \geq 3$.

Let v be a central vertex of X_i. Then, for each $x \in X_i$,

$$d(v,x) \leq \mathrm{rad}(X_i) \leq \mathrm{rad}(P_n) = \left\lceil \frac{n-1}{2} \right\rceil .$$

Define $f(v) = n$ and $f(u) = 0$ otherwise. Since the distance from v to any other vertex in $N_2[X_i]$ is at most $\lceil \frac{n-1}{2} \rceil + 2 \leq n$ (since $n \geq 3$), $N_2[X_i]$ is f-dominated.

Type 2: $X_i = K_2$.

Choose arbitrary $v \in X_i$, define $f(v) = 3$ and $f(u) = 0$ otherwise. Then $N_2[X_i] \subseteq N_f[v]$.

Type 3: $X_i = \{z\}$. Then $z \in Z$.

Define $f(z) = 1$. Then $N[z] = N_f[v]$.

If $X = Z$, then by Lemma 3.3 we are done. If $X \neq Z$, then by (3.1), $R \subseteq N_2[X - Z]$. Hence each vertex in R hears a broadcast from a vertex in a Type 1 or 2 component of $G[X]$. Also, $C \subseteq N[X]$, hence each vertex in C hears a broadcast from a vertex in X. It follows that f is a dominating broadcast of G and so $\gamma_b(G) \leq \sigma(f) \leq \frac{3}{2}|X|$.

Let H be the graph depicted in Figure 3.1. Since H has no universal vertex (a vertex adjacent to every other vertex of H), $\mathrm{ir}(H) \geq 2$. Also, $X = \{v_2, v_3\}$ is irredundant and we only need to show maximality. Since $\mathrm{pn}(v_2, X) = \{v_1\}$ and

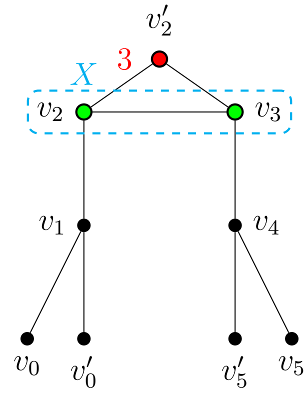


Figure 3.1: A graph H with $\mathrm{ir}(H) = 2$ and $\gamma_b(H) = 3$.

$\mathrm{pn}(v_3, X) = \{v_4\}$, $X \cup \{y\}$ is redundant for each $y \in N[R] = \{v_0, v_0', v_1, v_4, v_5, v_5'\}$. This shows that $\mathrm{ir}(H) = 2$.

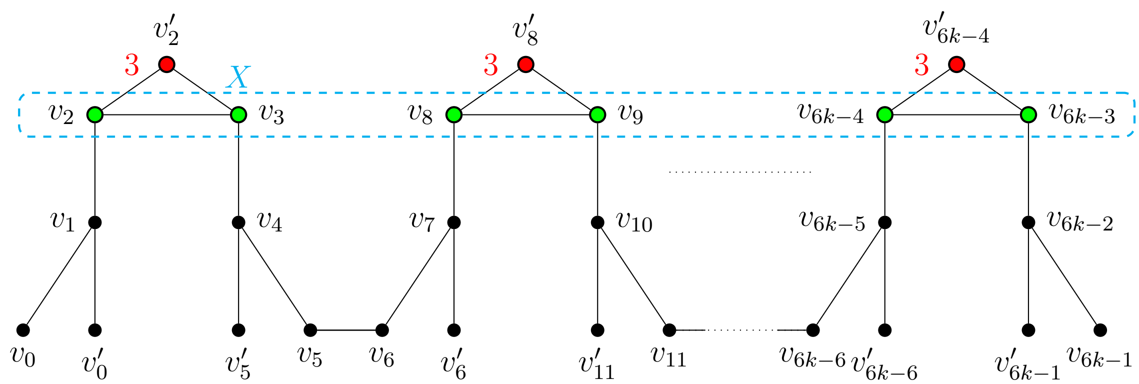


Figure 3.2: A graph G_k with $\mathrm{ir}(G_k) = 2k$ and $\gamma_b(G_k) = 3k$.

The function $f : V(H) \to \{0, 3\}$ defined by $f(v_2') = 3$ and $f(u) = 0$ otherwise is a dominating broadcast of H, hence $\gamma_b(H) \leq 3$. Suppose H has a dominating broadcast g with cost 2. Let w be the vertex that broadcasts to v_0. Since $g(w) \leq 2$, $w \in \{v_0, v_0', v_1, v_2\}$, hence w does not broadcast to v_5. Therefore $g(w) = 1$, $w \in \{v_0, v_1\}$, and there exists a vertex $w' \neq w$ with $g(w') = 1$ that broadcasts to v_5. As for w, $w' \in \{v_4, v_5\}$. Now v_2' hears no broadcast, a contradiction. Therefore $\gamma_b(H) = 3$.

For $i = 1, ..., k$, let $H_i \cong H$, and for $j = 1, ..., 6$, label the vertex of H_i corresponding to the vertex v_{6-j} (v'_{6-j}, where applicable) of H by v_{6i-j} (v'_{6i-j}). Form the graph G_k by joining v_{6i-1} to v_{6i}, $i = 1, ..., k-1$. See Figure 3.2.

As for H, $X = \bigcup_{i=1}^{k} \{v_{6i-4}, v_{6i-3}\}$ is a maximal irredundant set of G_k of cardinality $2k$, hence $\text{ir}(G_k) \leq 2k$. We show that $\gamma_b(G_k) = 3k$; it will follow that $\text{ir}(G_k) \geq \frac{2}{3}\gamma_b(G_k) = 2k$.

The function $f : V(G_k) \to \{0, 3\}$ defined by $f(u) = 3$ if $u \in \{v'_{6i-4} : i = 1, ..., k\}$ and $f(u) = 0$ otherwise is a dominating broadcast of G_k, hence $\gamma_b(G_k) \leq 3k$. For any dominating broadcast f' of G_k, if $u \in V(H_i)$ hears a broadcast from a vertex $w \in V(H_i)$, we say that u *hears a local broadcast*, otherwise we say that u *hears only foreign broadcasts*. Denote the set of vertices that hear only foreign broadcasts of f' by $\text{For}(f')$.

Among all γ_b-broadcasts of G_k, let g be one such that $|\text{For}(g)|$ is minimum. If $\text{For}(g) = \varnothing$, then for each $i = 1, ..., k$ and each vertex $u \in V(H_i)$, u hears a local broadcast, and as with H we show that $\sum_{w \in V(H_i)} g(w) = 3$ for each $i = 1, ..., k$. Thus $\sigma(g) = 3k$ and we are done.

Hence suppose $\text{For}(g) \neq \varnothing$. Let $u' \in \text{For}(g) \cap V(H_{i'})$ for some i' and let $w \in V(H_j)$ broadcast to u'. Assume without loss of generality that $j > i'$. Then there exists an index $i \leq i'$ and a vertex $u \in \text{For}(g) \cap V(H_i)$ at maximum distance from w. If v'_{6i-1} is such a vertex u, then so is v_{6i-3}, if v'_{6i-4} is such a u, then so is v_{6i-4}, and if v'_{6i-6} is such a vertex, then so is v_{6i-6}. Therefore we may assume $u = v_\ell$ for some ℓ. We may similarly assume that $w = v_m$ for some m. Furthermore, we may assume that $g(v_s) = 0$ for all $\ell < s < m$, otherwise there exists a dominating broadcast g' of G_k such that $\sigma(g') < \sigma(g)$, which is a contradiction.

We consider the different values of ℓ. In each case we define a dominating broadcast h such that $\sigma(h) = \sigma(g)$ and $|\text{For}(h)| < |\text{For}(g)|$, thus producing a contradiction on the choice of g. In most cases we use v_{m+t} for some $t \geq 1$ as a broadcast vertex of

h instead of $w = v_m$; note that $m + t \leq 6k$, otherwise we can obtain a lower cost dominating broadcast.

Case 1 $\ell = 6i - 6$. Then $g(w) \geq 6$. Let $h(v_{6i-3}) = 3$, $h(w) = 0$, $h(v_{m+3}) = g(w) - 3$, and $h(v) = g(v)$ otherwise.

Case 2 $\ell \in \{6i - 5, 6i - 4\}$. Then $g(w) \geq 4$. Let $h(v_{6i-3}) = 2$, $h(w) = 0$, $h(v_{m+2}) = g(w) - 2$, and $h(v) = g(v)$ otherwise.

Case 3 $\ell = 6i - 3$. Then $g(w) \geq 3$. Let $h(v_{6i-2}) = 1$, $h(w) = 0$, $h(v_{m+1}) = g(w) - 1$, and $h(v) = g(v)$ otherwise.

Case 4 $\ell \in \{6i - 2, 6i - 1\}$. Since $v_\ell \in \mathrm{For}(g)$, v'_{6i-1} does not hear a broadcast from v_q, $q < 6i - 1$. Since v'_{6i-1} is g-dominated, $g(v'_{6i-1}) = 1$, and thus $\ell \neq 6i - 2$; we need only consider $\ell = 6i - 1$. Let $h(v_{6i-2}) = 1$, $h(v'_{6i-1}) = 0$ and $h(v) = g(v)$ otherwise.

Let p be the largest index such that v_p or v'_p hears the broadcast g from w. Then this vertex also hears the broadcast h from v_{m+t} in each of Cases 1 – 4. All other vertices on $v_\ell - v_p$ paths, and the end-vertices attached to these paths, hear a broadcast from either v_{m+t} or a vertex of H_i, and certainly v_ℓ hears a broadcast from a vertex of H_i in each case. Therefore h is a dominating broadcast such that $\sigma(h) = \sigma(g)$ and $|\mathrm{For}(h)| < |\mathrm{For}(g)|$. This completes the proof of Theorem 3.1. ■

3.1.1 Corollaries

Having proven our new bound in Theorem 3.1, we now present a pair of corollaries concerning graphs with $\gamma_b(G) \leq \mathrm{ir}(G)$ and graphs with $\gamma_b(G) = \frac{3}{2}\,\mathrm{ir}(G)$. In doing so, we often define a broadcast f on a component of an ir-set as for the Type 1, 2 or 3 components in the proof of Theorem 3.1. We then simply say that we use a Type i assignment for f, where $i \in \{1, 2, 3\}$.

Corollary 3.4. *If a graph G has an* ir*-set X such that every nontrivial component of $G[X]$ has order at least three, then $\gamma_b(G) \leq \operatorname{ir}(G)$.*

Proof. By using a Type 1 or Type 3 assignment for each component of $G[X]$, we define a dominating broadcast f with $\sigma(f) = |X|$. ■

Corollary 3.5. *Suppose $\gamma_b(G) = \frac{3}{2}\operatorname{ir}(G)$. Then for any* ir*-set X of G,*

(*i*) *$G[X] = mK_2$ for some integer m,*

(*ii*) *each vertex $c \in C$ is adjacent to exactly two vertices $u, v \in X$, where $uv \in E(G)$, and c is adjacent to no vertices in $N_2[x]$ where $x \in X - \{u, v\}$,*

(*iii*) *for each $u \in X$ there exists $r \in R(u)$ such that $r \notin R(w)$ for any $w \in X - \{u\}$,*

(*iv*) *for each edge uv of $G[X]$ there exists $c \in C$ adjacent to u and v,*

(*v*) *for any two vertices $u, v \in X$, no vertex in $\operatorname{pn}(u, X)$ is adjacent to all vertices in $\operatorname{pn}(v, X)$.*

Proof. Assume $\gamma_b(G) = \frac{3}{2}\operatorname{ir}(G)$.
(*i*) Suppose G has an ir-set X such that $G[X] \ncong mK_2$. As in the proof of Theorem 1, by using a Type 2 assignment on each K_2 component of $G[X]$ and a Type 1 or 3 assignment otherwise, we obtain a dominating broadcast f of G with $\sigma(f) < \frac{3}{2}\operatorname{ir}(G)$, a contradiction.

(*ii*) Suppose $c \in C$ is adjacent to $u, u' \in X$, where $uu' \notin E(G)$. Then there exist distinct vertices $v, v' \in X$ adjacent to u and u', respectively. Define a broadcast f so that $f(c) = 4$, all other components have Type 2 assignments, and $f(w) = 0$ otherwise. Then f is a dominating broadcast with $\sigma_f(G) < \frac{3}{2}\operatorname{ir}(G)$, a contradiction. We now show that c is not adjacent to any vertex in $N_2[X - \{u, v\}]$.

Case 1 c is adjacent to some $c' \in C$. By the above, c' is adjacent to some component $u'v'$ of $G[X]$. Define a broadcast f so that $f(c) = 4$; for each component in $G[X]$ other than uv and $u'v'$, use Type 2 assignments, and $f(w) = 0$ otherwise. Then all vertices in $N_2[\{u, v\}] \cup N_2[\{u', v'\}]$ hear the broadcast from c, and all other vertices in G are broadcast to by a Type 2 assignment.

Case 2 c is adjacent to some $y' \in \text{pn}[u']$, where $u'v'$ is a component of $G[X]$. Define a broadcast f so that $f(y') = 4$; for each component in $G[X]$ other than uv and $u'v'$, use Type 2 assignments, and $f(w) = 0$ otherwise. Since $N_2[\{u, v\}] \cup N_2[\{u', v'\}] \subseteq N_4[y']$, G is dominated by this broadcast.

Case 3 c is adjacent to some $r \in R(u')$, where $u'v'$ is a component of $G[X]$. Define a broadcast f so that $f(c) = 3$ and $f(v') = 2$; for each edge in $G[X]$ other than uv and $u'v'$, use Type 2 assignments, and $f(w) = 0$ otherwise. The vertices in $N_2[\{u, v\}]$, $N_2[v']$ and $N[u']$ hear the broadcast from c or v'. We need only examine $R(u')$. If $R(u') = \{r\}$, we are done, so let $r' \in R(u') - \{r\}$. By the annihilation property, r' must annihilate some $w \in X$. If $w = u'$ then $d(r', r) \leq 2$, and thus r' hears the broadcast from c. If $w \in \{u, v\}$, then $d(r', c) \leq 3$, so it again hears the broadcast from c. If $w = v'$, then $d(r', v') = 2$ and it hears the broadcast from v'. If $w \notin \{u, v, u', v'\}$, then r' hears a Type 2 broadcast from some other component of $G[X]$.

In each case f is a dominating broadcast with $\sigma_f(G) < \frac{3}{2}\,\text{ir}(G)$, a contradiction.

(iii) Suppose no such r exists and assume u is matched to v in $G[X]$. Let $f(v) = 2$, use Type 2 assignments for all other components of $G[X]$, and let $f(w) = 0$ for all other vertices of G. Then u, $\text{pn}(u, X)$, $\text{pn}(v, X)$ and $R(v)$ all hear the broadcast from v. By our assumption, for all $r' \in R(u)$, $r' \in R(x)$ for some other $x \in X$. Thus, the Type 2 assignment on the component containing x dominates r'. Therefore, G is f-dominated and $\sigma(G) < \frac{3}{2}\,\text{ir}(G)$. This again contradicts the assumption of $\gamma_b(G) = \frac{3}{2}\,\text{ir}(G)$.

(iv) Suppose uv is an edge of $G[X]$ such that no vertex in C is adjacent to u and v. Note that $\mathrm{pn}(u,X) = \mathrm{epn}(u,X)$ and $\mathrm{pn}(v,X) = \mathrm{epn}(v,X)$. If $x \in \mathrm{pn}(u,X)$ is not adjacent to a vertex in R, then no vertex in $R(u)$ annihilates u. By the annihilation property each $r \in R(u)$ annihilates a vertex in $X - \{u\}$, that is, $r \in R(w)$ for some $w \in X - \{u\}$, contradicting (iii). Thus each vertex in $\mathrm{pn}(u,X)$ and, similarly, each vertex in $\mathrm{pn}(v,X)$ is adjacent to a vertex in R.

Choose any $a \in \mathrm{pn}(u,X)$ and any $b \in \mathrm{pn}(v,X)$ and let $f(a) = f(b) = 1$. Use Type 2 assignments for all other components of $G[X]$, and let $f(w) = 0$ otherwise. We see immediately that $N_2[X - \{u,v\}]$ is f-dominated and only need to verify that $N_2[\{u,v\}] - N_2[X - \{u,v\}]$ is f-dominated.

Consider $z \in N_2[\{u,v\}] - N_2[X - \{u,v\}]$. Since no vertex in C is adjacent to u and v, $z \in \mathrm{pn}(u,X) \cup \mathrm{pn}(v,X) \cup R$, and thus, $z \in N[R]$. By the annihilation property and the choice of z, $\mathrm{pn}(w,X) \subseteq N[z]$ for $w \in \{u,v\}$ and thus z hears the broadcast from either a or b. Therefore f is a dominating broadcast with $\sigma(f) < \frac{3}{2}\,\mathrm{ir}(G)$, which once again is a contradiction.

(v) Suppose $u, v \in X$ and there is some $a \in \mathrm{pn}(u,X)$ that is adjacent to all vertices in $\mathrm{pn}(v,X)$. If uv is an edge in $G[X]$, define a broadcast f by $f(a) = 2$; for each component of $G[X]$ other than uv, use Type 2 assignments, and $f(w) = 0$ otherwise. Then $\sigma_f(G) < \frac{3}{2}\,\mathrm{ir}(G)$. Immediately, $N[u]$, $N[v]$ and $R(v)$ are all subsets of $N_2[a]$, and hear the broadcast from a. We need only examine $R(u)$. Let $r \in R(u)$. By the annihilation property, r annihilates some $w \in X$. If $w = u$, then $r \in N[a] \subseteq N_2[a]$. If $w = v$, then r is adjacent to all vertices in $\mathrm{pn}(v,X)$ and hence $r \in N_2[a]$. Otherwise, $w \in X - \{u,v\}$, in which case r hears the broadcast from w or its unique neighbour in X. In all cases f is a dominating broadcast.

Now consider the case when u and v are not adjacent. By (i), let $u', v' \in X$ be the vertices adjacent to u and v respectively. Define f by $f(u) = 3$ and $f(v') = 2$;

for any other edge in $G[X]$, use Type 2 assignments, and $f(w) = 0$ otherwise. Then $N_2[X - \{v\}]$ is dominated and we only need to check the vertices in $R(v)$. Consider $r \in R(v)$. By definition, r is adjacent to some $b \in \mathrm{pn}(v, X)$, which in turn is adjacent to a. Thus $d(r, u) \leq 3$ and r hears the broadcast from u. Moreover, $\sigma(f) < \frac{3}{2}\,\mathrm{ir}(G)$. This contradiction completes the proof of the corollary. ■

The converse of Corollary 3.5 is not true. The graph G in Figure 3.3 satisfies conditions (i) – (v) but has $\gamma_b(G) = 5$ and $\mathrm{ir}(G) = 4$.

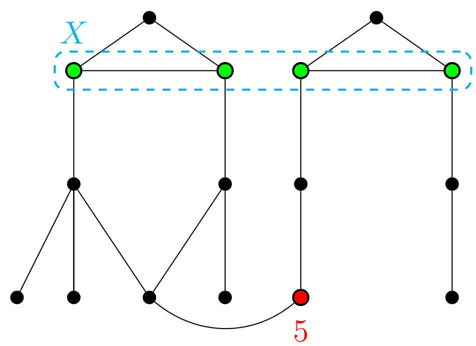


Figure 3.3: A counterexample to the converse of Corollary 3.5.

Theorem 3.1 shows that the ratio $\gamma_b/\,\mathrm{ir}$ is bounded above by $\frac{3}{2}$. However, neither the ratio $\mathrm{ir}\,/\gamma_b$ nor the difference $\mathrm{ir} - \gamma_b$ is bounded above. To see this, consider a tree T obtained from the star $K_{1,m}$ by subdividing each edge exactly once. Then $\mathrm{ir}(T) = \gamma(T) = m$ and $\gamma_b(T) = 2$.

Chapter 4

Broadcast Domination and Multipackings in Graphs

In this chapter we introduce the notion of a multipacking of a graph by formulating broadcast domination as a linear programming problem and then considering its dual. We generalize the famous result of Meir and Moon [23] and show the equality of the multipacking and broadcast numbers of a tree. Finally, we provide an alternative proof to the second half of Theorem 3.1 using multipackings.

4.1 Broadcast Domination as an LP Problem

Like many other graph-theoretic parameters, broadcast domination can be considered as an integer programming (IP) problem. Its fractional relaxation linear program (LP), or primal linear program (PLP), has a dual linear program (DLP) whose IP formulation provides a lower bound for the broadcast number (see (4.1) below). While this bound may be weak in general, we use it to establish $\gamma_b(G_k)$ for the graph G_k in Theorem 3.1. This work has been accepted for publication in [3].

We refer the reader to the chapter by P. J. Slater in [15, Chapter 1] for a discussion of LP-duality in domination-related problems.

A dominating broadcast on a graph G can also been viewed as a covering of G with k-neighbourhoods centred at each of the broadcast vertices. Thus a broadcast can be seen as a collection $\mathcal{B} = \{N_k[v]\}$ such that for each $u \in V$ there exists some $N_k[v] \in \mathcal{B}$ with $u \in N_k[v]$. For this covering form of broadcasting, we denote the cost of the broadcast $\mathcal{B}$ by

$$\sigma_{\mathcal{B}} = \sum_{N_k[v] \in \mathcal{B}} k.$$

Finding the minimum $\sigma_{\mathcal{B}}$ is a natural IP. With each $N_k[v]$ we associate an indicator variable $x_{k,v} \in \{0, 1\}$, where

$$x_{k,v} = \begin{cases} 1 & \text{if } f(v) = k \\ 0 & \text{otherwise.} \end{cases}$$

The IP objective function is given by

$$\min \sum_{v \in V} \sum_{1 \leq k \leq e(v)} k \cdot x_{k,v}.$$

There is one constraint for each vertex u. We define $\mathcal{B}_u = \{(k, v) : u \in N_k[v]\}$, the set of k-neighbourhoods that contain u. Our IP constraints require that each u be in at least one selected k-neighbourhood. That is, for each $u \in V$,

$$\sum_{(k,v) \in \mathcal{B}_u} x_{k,v} \geq 1.$$

The fractional relaxation LP is given by

$$\begin{aligned} &\min \sum_{v\in V} \sum_{1\le k\le e(v)} k\cdot x_{k,v} \\ &\text{s.t.} \sum_{(k,v)\in\mathcal{B}_u} x_{k,v} \ge 1 \text{ for each } u\in V \\ &x_{k,v}\ge 0. \end{aligned}$$

The dual LP has one variable y_u for each vertex u. It is

$$\begin{aligned} &\max \sum_{u\in V} y_u \\ &\text{s.t.} \sum_{u\in N_k[v]} y_u \le k \text{ for each } N_k[v],\ k=1,\dots,e(v) \\ &y_u \ge 0. \end{aligned}$$

That is to say, we assign a weight y_u to each $u \in V$ so that, for each $k \in \{1, ..., e(u)\}$, the total weight in the k-neighbourhood of u does not exceed k. Thus, when we consider the 1-neighbourhood of u, it follows that $0 \le y_u \le 1$. In the case that $y_u \in \{0, 1\}$ this simplifies to choosing a set of vertices Y so that each k-neighbourhood of u has at most k vertices in Y. The following proposition is an immediate consequence of these concepts.

Proposition 4.1. *Let $Y \subseteq V$ such that $|Y \cap N_k[v]| \le k$ for each $v \in V$ and each $k = 1, 2, \dots, e(v)$. Then $\gamma_b \ge |Y|$.*

Proof. Let f be a γ_b-broadcast and let $v \in V^+$. Then $N_f[v]$ contains at most $f(v)$ vertices from Y. Since $V = \bigcup_{v\in V^+} N_f[v]$, we obtain

$$\gamma_b = \sum_{v \in V_f^+} f(v) \geq \sum_{v \in V_f^+} |N_f[v] \cap Y| \geq |Y|.$$

■

The set Y described in Proposition 4.1 is a multipacking of G, as defined in Section 2.3. Hence, by Proposition 4.1,

$$\gamma_b(G) \geq \text{mp}(G). \tag{4.1}$$

For some graphs it turns out that $\gamma_b(G) = \text{mp}(G)$, that is, the PLP and the DLP have optimum integer solutions. However, for other graphs $\gamma_b(G) > \text{mp}(G)$. This is the case when the PLP and DLP have fractional optimal solutions. Consider the example of C_5, where $x_{1,v} = y_v = \frac{1}{3}$ and $x_{2,v} = 0$ for each vertex v. Then $\sum_{v \in V} \sum_{k=1}^{2} k \cdot x_{k,v} = \sum_{u \in V} y_u = \frac{5}{3}$. Thus these labellings, which satisfy the constraints, are primal and dual optimal, and $\min \sum_{v \in V} \sum_{k=1}^{2} k \cdot x_{k,v} = \max \sum y_u = \frac{5}{3}$. However, $\gamma_b(C_5) = 2$ and $\text{mp}(C_5) = 1$.

The following result is immediate from the definition of multipackings.

Proposition 4.2. *If G is 1-cap ($\gamma(G) = \gamma_b(G)$) and does not have an efficient γ-set, then* $\text{mp}(G) < \gamma_b(G)$.

This proposition is again nicely demonstrated with C_5. Although $\gamma_b(C_5) = \gamma(C_5)$, C_5 does not have an efficient γ-set. Hence, $\text{mp}(C_5) < \gamma_b(C_5)$. We can generalize this result for all cycles.

Proposition 4.3. *For any cycle C_n with $n \geq 3$,* $\text{mp}(C_n) = \gamma_b(C_n)$ *if and only if* $n \equiv 0 \pmod 3$.

Proof. ($\Rightarrow$) Let C_n be any cycle with $n \equiv 0 \pmod 3$, $n \geq 4$. Cycles with

$n \equiv 0 \pmod 3$ have efficient γ-sets; let M be such a set. Consider any $v \in V$. Since M is efficient and dominating, $|N[v] \cap M| = 1$. Pick any $2 \leq s \leq e(v)$. Suppose $v \in M$. If $s \equiv 0 \pmod 3$, then $|N_s[v] \cap M| = s$; otherwise, if $s \not\equiv 0 \pmod 3$, then $|N_s[v] \cap M| < s$. Now suppose $v \notin M$. If $s = 2$, $|N_s[v] \cap M| = s$; otherwise, $|N_s[v] \cap M| < s$. It follows that M is a multipacking of C_n.

($\Leftarrow$) The converse follows immediately from Proposition 4.2, since cycles with $n \not\equiv 0 \pmod 3$ do not have efficient γ-sets. ■

Our initial investigations suggested that $\gamma_b(G) - \mathrm{mp}(G) \leq 1$. However, the graph G in Figure 4.1 is an example with $\gamma_b(G) = 4$, $\mathrm{mp}(G) = 2$. This is currently the only graph we know of with $\gamma_b(G) - \mathrm{mp}(G) \geq 2$.

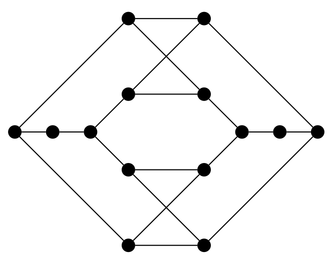


Figure 4.1: A graph G with $\gamma_b(G) = 4$, $\mathrm{mp}(G) = 2$.

4.2 Broadcasts and Multipackings in Trees

In Chapter 2, we introduced a famous result of Meir and Moon which we restate here.

Theorem 2.12 [23] *For any tree T, $\rho(T) = \gamma(T)$.*

Since multipackings generalize 2-packings in the same manner that broadcast domination generalizes ordinary domination, it seems natural that the analogue of Meir and Moon's result should hold for multipackings and broadcasts. We prove next that this is indeed the case. The work in Sections 4.2.1 - 4.2.2 has been submitted for publication in [24].

Theorem 4.4. *For any tree T, $\gamma_b(T) = \mathrm{mp}(T)$.*

Before we prove Theorem 4.4, we require the formulation of some technical lemmas.

4.2.1 Lemmas

Let T be a tree with $\gamma_b(T) = k$. Then $\mathrm{diam}(T) \geq 2k-1$ since $\gamma_b(T) \leq \mathrm{rad}(T)$ (as observed in [12]) and $2\,\mathrm{rad}(T)-1 \leq \mathrm{diam}(T)$. If $\mathrm{diam}(T) \geq 3(k-1)$, then $\mathrm{mp}(T) \geq k$, as shown below in Lemma 4.5.

Lemma 4.5. *For any graph G, if $\mathrm{diam}(G) \geq 3(k-1)$, then $\mathrm{mp}(G) \geq k$.*

Proof. Let G have a diametrical path $P : v_0, ..., v_d$, where $d \geq 3(k-1)$. Define $V_i = \{v : d(v, v_0) = i\}$ for each $1 \leq i \leq d$. We claim that the set $M = \{v_i : i \equiv 0 (\mathrm{mod}\ 3), 0 \leq i \leq 3(k-1)\}$ is a multipacking of G. By our choice of M, any vertex v_i of P satisfies $|N_s[v_i] \cap M| \leq s$ for all $s \geq 1$. Consider any $1 \leq r \leq d$ and any $v \in V_r$. Since $v_r \in V_r$ is on P and $M \subseteq V(P)$, $N_s[v] \cap M \subseteq N_s[v_r] \cap M$. Thus $|N_s[v] \cap M| \leq s$ for all $s \geq 1$. It follows that $\mathrm{mp}(G) \geq |M| = k$. ■

Hence to prove Theorem 4.4 we only need to consider trees T with $\gamma_b(T) = k$ and $2k-1 \leq \mathrm{diam}(T) \leq 3k-4$.

- Let $\mathcal{T}_{k,d}$ be the set of all trees T such that $\mathrm{diam}(T) = d$, $\gamma_b(T) = k$ and $\gamma_b(T-\ell) = k-1$ for each leaf ℓ of T.

Since $\gamma_b(S_{T,P}) = \gamma_b(T)$ (Theorem 2.5), each $T \in \mathcal{T}_{k,d}$ is a shadow tree. Also, if u_{i,α_i} is the leaf on the branch B_i of a nested triangle Δ_i, then an edge e is a free edge of $T - u_{i,\alpha_i}$ if and only if e is a free edge of T. By Theorem 2.6 and the fact that every split-edge is a free edge, $\gamma_b(T - u_{i,\alpha_i}) = \gamma_b(T)$. Thus $T \in \mathcal{T}_{k,d}$ has no nested triangles.

Lemma 4.6. *If the shadow tree T has no trailing free edges, then $\gamma_b(T - u_{c,\alpha}) = \gamma_b(T)$.*

Proof. If T has no trailing free edges, then $e_f = v_{d-2\alpha}v_{d-2\alpha+1}$ and $e_\ell = v_{d-1}v_d$. Suppose to the contrary that $\gamma_b(T - u_{c,\alpha}) < \gamma_b(T)$. Since $\mathrm{diam}(T - u_{c,\alpha}) = \mathrm{diam}(T)$, Theorem 2.6 implies that $T - u_{c,\alpha}$ has a maximum split-set of larger cardinality than a maximum split-set of T. Since e_f and e_ℓ are the only free edges of $T - u_{c,\alpha}$ that are not free edges of T, $u_{c,\alpha}$ binds e_f or e_ℓ. But e_ℓ is not a split-edge of $T - u_{c,\alpha}$ because v_d is an isolated vertex of $T - u_{c,\alpha} - e_\ell$ and thus the subgraph induced by $\{v_d\}$ does not have positive diameter, and e_f is not a split-edge of $T - u_{c,\alpha}$ because the component of $T - u_{c,\alpha} - e_f$ that contains v_d has odd diameter $2\alpha - 1$ and no other split-edges. Thus any split-set of $T - u_{c,\alpha}$ is a split-set of T, a contradiction. ■

Lemma 4.6 shows that all trees in $\mathcal{T}_{k,d}$ have at least one trailing free edge and, similarly, at least one leading free edge. In the next lemma we demonstrate how any multipacking M of a shadow tree T can be transformed into a new multipacking M' of the same size.

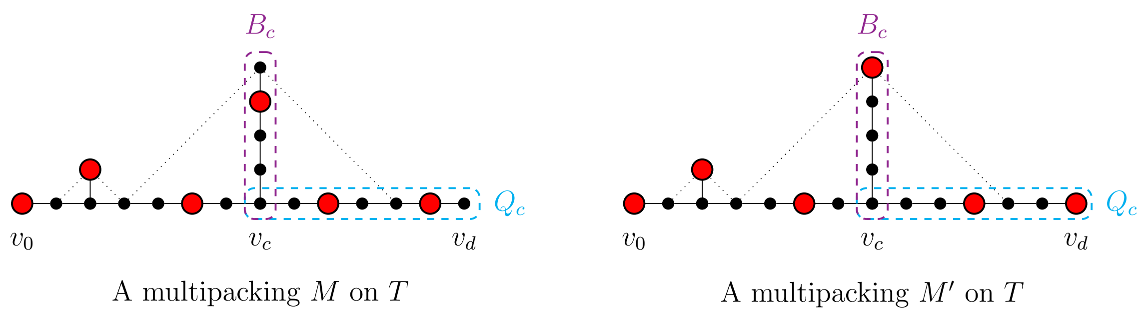


Figure 4.2: Illustration of Lemma 4.7.

Lemma 4.7. *Let M be a multipacking of a shadow tree T and suppose M contains the x vertices $v_{i_1}, ..., v_{i_x}$ of Q_c, where $c \le i_1 \le \cdots \le i_x \le d$, and the y vertices $u_{c,j_1}, ..., u_{c,j_y}$ of $B_c - \{v_c\}$, where $1 \le j_1 \le \cdots \le j_y \le \alpha$. Then there exists a multipacking M' of T such that $|M'| = |M|$ and M' contains the x vertices $v_d, v_{d-3}, ..., v_{d-3x+3}$ of Q_c and the y vertices $u_{c,\alpha}, u_{c,\alpha-3}, ..., u_{c,\alpha-3y+3}$ of B_c.*

Proof. Define M' by

$$M' = (M - B_c - Q_c) \cup \{v_d, v_{d-3}, ..., v_{d-3x+3}\} \cup \{u_{c,\alpha}, u_{c,\alpha-3}, ..., u_{c,\alpha-3y+3}\}.$$

That is, M' is obtained from M by "pushing out" the vertices of $M \cap B_c$ and $M \cap Q_c$ as far as possible towards the leaf, using the leaf and every third vertex from the leaf until the correct number of vertices have been used. This technique is illustrated in Figure 4.2. It is clear that $|M| = |M'|$. If $M = M'$, then we are done. Thus assume $M \neq M'$. We show that M' is a multipacking of T.

Let T' be the component of $T - \{v_c v_{c+1}, v_c u_{c,1}\}$ that contains v_c. For each vertex $u \in V(T')$, each $r = 0, ..., x-1$, and each $\ell = 0, ..., y-1$, $d_T(u, v_{d-3r}) \geq d_T(u, v_{i_{x-r}})$ and $d_T(u, u_{c,\alpha-3\ell}) \geq d_T(u, u_{c,j_{y-\ell}})$. Hence if T has a vertex u such that $|N_s[u] \cap M'| \geq s+1$ for some integer s, then $u \in \{v_{c+1}, ..., v_d\} \cup \{u_{c,1}, ..., u_{c,\alpha}\}$.

First assume that there exists such a vertex $u \in \{v_{c+1}, ..., v_d\}$. Let m be the smallest index, $c+1 \leq m \leq d$, for which there exists an integer s such that $|N_s[v_m] \cap M'| \geq s+1$. By the choice of M', $s \geq 2$. Define

$$X = \begin{cases} \{v_{m+s-1}, v_{m+s}\} & \text{if } m+s \leq d \\ \{v_d\} & \text{if } m+s-1 = d \\ \varnothing & \text{otherwise.} \end{cases}$$

Now $N_{s-1}[v_{m-1}] \cup X = N_s[v_m]$, and again by the choice of M', X contains at most one vertex from M'. Hence $|N_{s-1}[v_{m-1}] \cap M'| \geq s$, contradicting the choice of m. A similar contradiction follows if $u \in \{u_{c,1}, ..., u_{c,\alpha}\}$. Therefore M' is a multipacking of T. ∎

In the next lemma we consider two trees T and T', each with diameter d. We assume that both these trees have a diametrical path $P : v_0, ..., v_d$ labelled with the same labels. Examples of T and T' are shown in Figures 4.3 and 4.4.

Lemma 4.8. *Let T and T' be shadow trees of diameter d such that the last branch $B_c = v_c, u_{c,1}, ..., u_{c,\alpha}$ of T has length $\alpha = d-c-1$, the last branch $B_{c'} = v_{c'}, u_{c',1}, ..., u_{c',\alpha'}$ of T' has length $\alpha' = d - c' - 2$, where $c' = c - 1$ (and thus $\alpha = \alpha'$), and $T - \{u_{c,1}, ..., u_{c,\alpha}\} \cong T' - \{u_{c-1,1}, ..., u_{c-1,\alpha}\}$. Let M' be a multipacking of T' that satisfies the conditions of Lemma 4.7. Define $M \subseteq V(T)$ by*

$$M = \begin{cases} (M' - (B_{c-1} - \{v_{c-1}\})) \cup \{u_{c,i} : u_{c-1,i} \in M',\ i \geq 1\} & \text{if } |\{v_c, u_{c-1,2}\} \cap M'| \leq 1 \\ (M' - B_{c-1} - \{v_c\}) \cup \{u_{c,i} : u_{c-1,i} \in M'\} \cup \{v_{c-1}\} & \text{otherwise.} \end{cases}$$

Then M is a multipacking of T. Moreover, $\gamma_b(T) = \gamma_b(T')$.

Proof. Except for the first edge $e'_f = v_{d-2\alpha-2}v_{d-2\alpha-1}$ of Δ_{c-1} in T', which is a free edge of T but not of T', and the last edge $e_\ell = v_{d-2}v_{d-1}$ of Δ_c in T, which is a free edge of T' but not of T, an edge e of P is a free edge of T if and only if e is a free edge of T'. But e'_f is not a split-edge of T, nor is e_ℓ a split-edge of T', because the component of $T - e'_f$ ($T' - e_\ell$, respectively) that contains v_d has odd diameter. By Theorem 2.6, $\gamma_b(T) = \gamma_b(T')$.

Now consider the sets M and M'. Since $\ell(B_{c-1}) = \ell(Q_{c-1}) - 2$, $\ell(Q_{c-1}) \equiv k \pmod 3$ for some $k \in \{0, 1, 2\}$ and $\alpha = \ell(B_{c-1}) = \ell(B_c) \equiv k + 1 \pmod 3$. Let T_1 be the component of $T - v_{c-1}v_c$ that contains v_{c-1}. We consider two cases, depending on the elements of M'.

Case 1 $|\{v_c, u_{c-1,2}\} \cap M'| \leq 1$. Since M' satisfies the conditions of Lemma 4.7,

$$\begin{aligned} M' \cap V(B_{c-1}) &\subseteq \{u_{c-1,k+1}, u_{c-1,k+4}, ..., u_{c-1,\alpha}\}, \\ M \cap V(B_c) &\subseteq \{u_{c,k+1}, u_{c,k+4}, ..., u_{c,\alpha}\} \text{ and} \\ M \cap V(Q_{c-1}) = M' \cap V(Q_{c-1}) &\subseteq \{v_{c+(k-1)}, v_{c+(k+2)}, ..., v_d\}. \end{aligned} \tag{4.2}$$

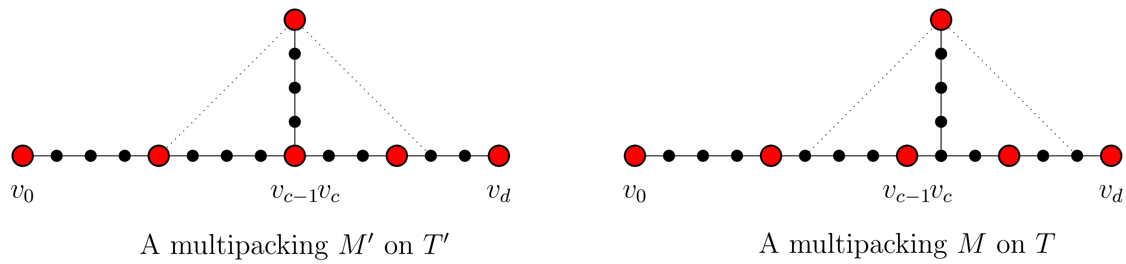


Figure 4.3: Illustration of Case 1 of the proof of Lemma 4.8

This situation is illustrated in Figure 4.3. For each $u \in V(T_1)$ and each $u_{c,i} \in M$, $d_T(u, u_{c,i}) > d_{T'}(u, u_{c-1,i})$. Hence if T has a vertex u such that $|N_s[u] \cap M| \geq s+1$ for some integer s, then $u \in B_c \cup Q_c$.

First assume there exists an integer s such that $|N_s[v_c] \cap M| \geq s+1$ in T. By the choice of M and M', $s \geq 2$. The only vertex of M that is in the s-neighbourhood of v_c in T for which there is no corresponding vertex of M' in the s-neighbourhood of v_{c-1} in T' is v_{c+s}, if $v_{c+s} \in M$. In T', $|N_s[v_{c-1}] \cap M'| \leq s$. Since $|N_s[v_c] \cap M| \geq s+1$ in T, we deduce the following three facts:

(i) $|N_s[v_{c-1}] \cap M'| = s$,

(ii) there is no vertex of M' in T_1 at distance s from v_{c-1}, and

(iii) $v_{c+s} \in M$ and thus $v_{c+s} \in M'$.

Since M' is a multipacking, $v_{c+s-1} \notin M'$. Moreover, by (4.2), $s \equiv k-1 \pmod 3$. Now (4.2) also implies that $u_{c-1,s} \notin M'$. Thus

$$|N_{s-1}[v_{c-1}] \cap M'| = |N_s[v_{c-1}] \cap M'| = s,$$

which is impossible since M' is a multipacking. Hence $|N_s[v_c] \cap M| \leq s$ for all $s \geq 1$.

If there exists a vertex $u \in \{v_{c+1}, ..., v_d\} \cup \{u_{c,1}, ..., u_{c,\alpha}\}$ such that $|N_s[u] \cap M| \geq s+1$ for some s, we now obtain a contradiction as in the proof of Lemma 4.7.

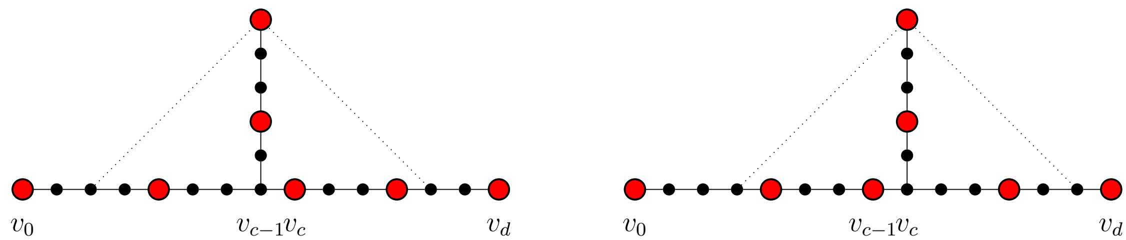


Figure 4.4: Illustration of Case 2 of the proof of Lemma 4.8.

Case 2 $\{v_c, u_{c-1,2}\} \subseteq M'$. By Lemma 4.7, $\ell(Q_{c-1}) \equiv 1 \pmod 3$, $\alpha = \ell(B_{c-1}) = \ell(B_c) \equiv 2 \pmod 3$,

$$\begin{aligned} M' \cap V(B_{c-1}) &= \{u_{c-1,2}, u_{c-1,5}, ..., u_{c-1,\alpha}\}, \\ M \cap V(B_c) &= \{u_{c,2}, u_{c,5}, ..., u_{c,\alpha}\}, \\ M' \cap V(Q_{c-1}) &= \{v_c, v_{c+3}, ..., v_d\} \text{ and} \\ M \cap V(Q_{c-1}) &= \{v_{c-1}, v_{c+3}, ..., v_d\}. \end{aligned} \tag{4.3}$$

This situation is illustrated in Figure 4.4. Thus $v_{c-1} \in M - M'$. Since M' is a multipacking and $\{v_c, u_{c-1,2}\} \subseteq N_2[v_{c-1}] \cap M'$, $\{v_{c-2}, v_{c-3}\} \cap M' = \varnothing$. Hence,

$$N[v_{c-1}] \cap M = N_2[v_{c-1}] \cap M = \{v_{c-1}\},$$

and for each $s \geq 3$,

$$|N_s[v_{c-1}] \cap M| \leq |N_s[v_{c-1}] \cap M'| \leq s.$$

Suppose there exists $u \in V(T_1) - \{v_{c-1}\}$ such that in T, $|N_s[u] \cap M| \geq s + 1$ for some integer s. Since M' is a multipacking of T', $|N_s[u] \cap M'| \leq s$ in T'. Let w be the vertex adjacent to u on the $u - v_d$ path of T. Since $u \neq v_{c-1}$, $w \neq v_c$.

The only vertex $x \in M$ such that u is closer to x in T than to its counterpart $x' \in M'$ in T' is $x = v_{c-1}$, and then $x' = v_c$. It follows that $|N_s[u] \cap M| = s + 1$ and $|N_s[u] \cap M'| = s$ in T'. Therefore

$$N_s[u] \cap \{v_{c-1}, v_c\} = \{v_{c-1}\}. \tag{4.4}$$

Since $v_c \notin N_s[u]$ and $d_{T'}(u, u_{c-1,2}) > d(u, v_c)$, it also follows that $u_{c-1,2} \notin N_s[u]$. Now consider $N_{s+1}[w]$. By (4.4), $v_{c-1} \in N_s[u]$, hence the definition of w implies that $v_c, u_{c-1,2} \in N_{s+1}[w]$. Since $v_c, u_{c-1,2} \in M'$ by assumption and $v_c \notin N_s[u]$ by (4.4), it follows that

$$|N_{s+1}[w] \cap M'| \geq |N_s[u] \cap M'| + 2 = s + 2,$$

contrary to M' being a multipacking. Hence $|N_s[u] \cap M| \leq s$ for all $u \in V(T_1)$ and all $s \geq 1$.

Suppose now there exists an integer s such that $|N_s[v_c] \cap M| \geq s + 1$ in T. By the choice of M and M', $s \geq 2$. We now proceed as in Case 1 and obtain facts (i) – (iii); specifically, by (4.3), $d(v_{c-1}, v_{c+s}) = s + 1 \equiv 1 \pmod 3$, so that $s \equiv 0 \pmod 3$. Then (4.3) implies that $u_{c-1,s} \notin M'$. Thus

$$|N_{s-1}[v_{c-1}] \cap M'| = |N_s[v_{c-1}] \cap M'| = s,$$

which is again impossible since M' is a multipacking. Hence $|N_s[v_c] \cap M| \leq s$ for all $s \geq 1$.

If there exists a vertex $u \in \{v_{c+1}, ..., v_d\} \cup \{u_{c,1}, ..., u_{c,\alpha}\}$ such that $|N_s[u] \cap M| \geq s + 1$ for some s, we once again obtain a contradiction as in the proof of Lemma 4.7. Hence, for all $v \in V(T)$ and all $s \geq 1$, $|N_s[v] \cap M| \leq s$. Therefore M is a multipacking of T and $\mathrm{mp}(T) \geq \mathrm{mp}(T')$. ■

Lemma 4.9 now follows immediately from Lemma 4.8.

Lemma 4.9. *Let T and T' be shadow trees of diameter d such that the last branch $B_c = v_c, u_{c,1}, ..., u_{c,\alpha}$ of T has length $d-c-1$, the last branch $B_{c'} = v_{c'}, u_{c',1}, ..., u_{c',\alpha'}$ of T' has length $d-c'-2$, where $c' = c-1$ (and thus $\alpha = \alpha'$), and $T - \{u_{c,1}, ..., u_{c,\alpha}\} \cong T' - \{u_{c-1,1}, ..., u_{c-1,\alpha}\}$. Then $\gamma_b(T) = \gamma_b(T')$ and $\mathrm{mp}(T) \geq \mathrm{mp}(T')$.*

Lemma 4.10. *Let T be a shadow tree with multipacking M. Then, for each vertex $v \in V(T)$ and each $s \geq d(v, v_d) + 2$, $|N_s[v] \cap M| < s$.*

Proof. We proceed by examining all vertices $v \in V(T)$, using the usual vertex-labelling scheme outlined above. Since M is a multipacking, $|N_s[v] \cap M| \leq s$ for all s.

Suppose $v = v_i$ is on the diametrical path. If $v_i = v_0$, then $N_s[v_0] = N_d[v_0] = V$ for any $s \geq d(v_0, v_d) + 2$. Hence $|N_s[v_0] \cap M| = |N_d[v_0] \cap M| \leq d < s$. Consider $v = v_i$ for $i \in \{1, 2, \ldots, d\}$. Since v_i is on the diametrical path and $s \geq d(v_i, v_d) + 2$, $v_d \in N_{s-1}[v_{i-1}]$ and thus $N_{s-1}[v_{i-1}] = N_s[v_i]$. Hence

$$|N_s[v_i] \cap M| = |N_{s-1}[v_{i-1}] \cap M| \leq s - 1.$$

Now suppose $v = u_{i,j}$ is on a branch. Since $s > d(u_{i,j}, v_d) > d(v_i, v_d)$, $v_d \in N_s[v_i]$ and thus $u_{i,\alpha_i} \in N_s[v_i]$. Therefore $N_s[u_{i,j}] \subset N_s[v_i]$ and so

$$|N_s[u_{i,j}] \cap M| \leq |N_s[v_i] \cap M| < s,$$

as shown above. ■

Note that by symmetry, Lemma 4.10 can be rewritten using v_0 in place of v_d.

Lemma 4.11. *Let $T \in \mathcal{T}_{k,d}$ have at least three trailing free edges. If M' is a multipacking of the tree $T' = T - \{v_{d-2}, v_{d-1}, v_d\}$, then $M = M' \cup \{v_d\}$ is a multipacking of T.*

Proof. For each $v \in T$ we show that $|N_s[v] \cap M| \leq s$ for all $s \in \mathbb{Z}^+$.

Case 1 $v \in T - \{v_{d-2}, v_{d-1}, v_d\}$. If $s \leq d(v, v_{d-3})$, then $N_s[v] \cap M = N_s[v] \cap M'$. But M' is a multipacking, hence $|N_s[v] \cap M| \leq s$. If $s = d(v, v_{d-3}) + 1$, then since v_{d-2} is not in M, $N_s[v] \cap M' = N_s[v] \cap M$ and hence $|N_s[v] \cap M| \leq s$. If $s \geq d(v, v_{d-3}) + 2$, then by Lemma 4.10, $|N_s[v] \cap M'| < s$ and so $|N_s[v] \cap M| \leq s$.

Case 2 $v \in \{v_{d-2}, v_{d-1}, v_d\}$. Since $\{v_{d-2}, v_{d-1}, v_d\} \cap M = \{v_d\}$, the case $s = 1$ is immediate. For $s = 2$ or 3, $N_s[v] \subseteq N[v_{d-4}] \cup \{v_{d-2}, v_{d-1}, v_d\}$. Since M' is a multipacking, $|N[v_{d-4}] \cap M| \leq 1$. Thus

$$|N_s[v] \cap M| \leq |\left(N[v_{d-4}] \cup \{v_{d-2}, v_{d-1}, v_d\}\right) \cap M| = 2.$$

Otherwise, $s \geq 4$ and $N_s[v] \subseteq N_s[v_{d-3}]$. By Case 1,

$$|N_s[v] \cap M| \leq |N_s[v_{d-3}] \cap M| \leq s.$$

■

The figures used in the proof of the next lemma also illustrate some of the ideas in the proof of Theorem 4.4.

Lemma 4.12. *If $T \in \mathcal{T}_{k,d}$, where $k \leq 4$ and $2k - 1 \leq d \leq 3k - 4$, then* $\text{mp}(T) = k$.

Proof. By Lemma 4.6 we need only consider trees with at least one leading and at least one trailing free edge. The result is vacuously true if $k \leq 2$. We exhaustively demonstrate in Figure 4.5 that $k = \text{mp}(T) = \gamma_b(T)$ for $k = 3, 4$. The large (red, if viewed in colour) vertices denote a maximum multipacking of T, and a γ_b-broadcast is shown. ■

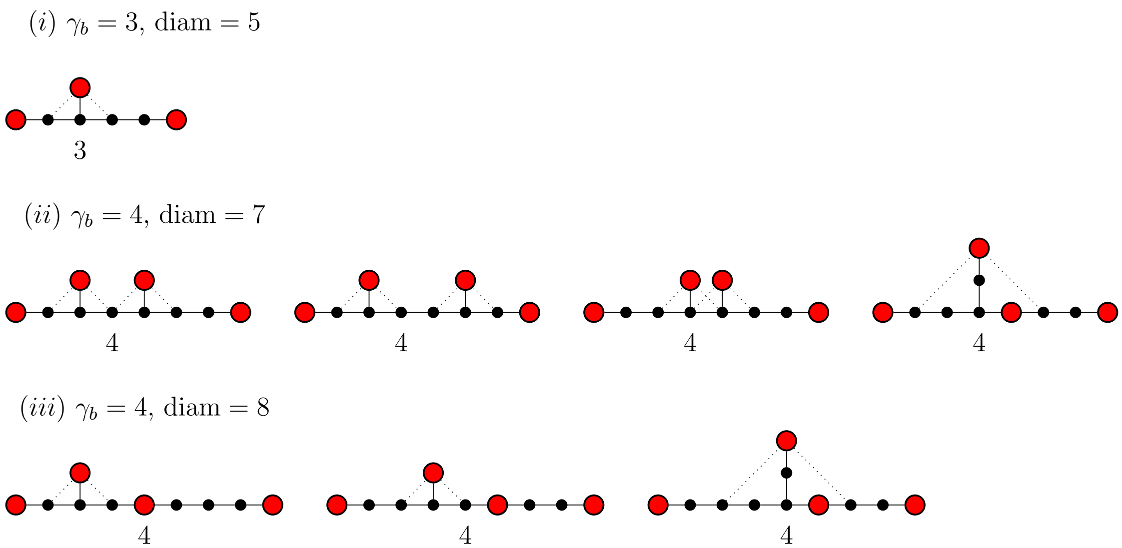


Figure 4.5: Trees with $k = \mathrm{mp}(T) = \gamma_b(T)$ for $k = 3, 4$.

4.2.2 Proof of Theorem 4.4

We restate the theorem for convenience.

Theorem 4.4 *For any tree T, $\gamma_b(T) = \mathrm{mp}(T)$.*

Proof. The proof is by induction on $|V(T)|$. The statement is clearly true for trees of order at most 3. Suppose it holds for all trees of order at most $n-1$ and let T be a tree such that $|V(T)| = n$ and $\gamma_b(T) = k$. If T has a leaf ℓ such that $\gamma_b(T-\ell) = k$, then by the induction hypothesis $T-\ell$ has a multipacking M of cardinality k, which is also a multipacking of T. Hence assume $\gamma_b(T-\ell) = k-1$ for each leaf ℓ. Then $T \in \mathcal{T}_{k,d}$ for some d. As stated above, T is a shadow tree without nested triangles. By Lemma 4.6, T has at least one trailing free edge.

Case 1 T has at least three trailing free edges. Let $T' = T - \{v_{d-2}, v_{d-1}, v_d\}$ and let M' be a maximum multipacking of T'. By the choice of T, $\gamma_b(T') \leq k-1$.

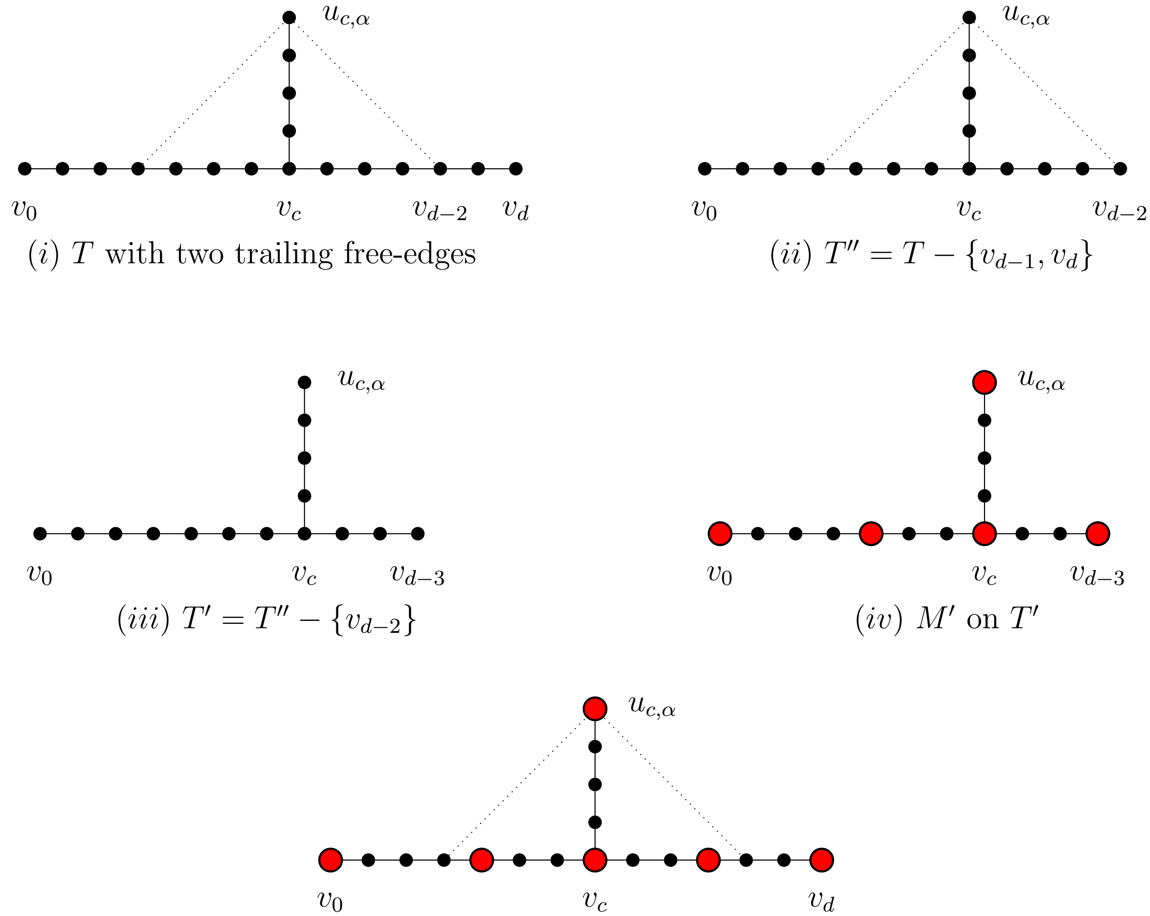
Any γ_b-broadcast f' of T' can be extended to a broadcast f of T such that $\sigma(f) = \sigma(f')+1$ by broadcasting from v_{d-1} with a cost of 1. Hence $\gamma_b(T') = k-1$ (otherwise $\sigma(f) < \gamma_b(T)$). By the induction hypothesis, $|M'| = k-1$, and by Lemma 4.11, $M = M' \cup \{v_d\}$ is a multipacking of T. Hence $\mathrm{mp}(T) = k$.

Case 2 T has two trailing free edges. Let $T'' = T - \{v_{d-1}, v_d\}$. As in Case 1, $\gamma_b(T'') = k-1$. Note that T'' has no trailing free edges. Thus by Lemma 4.6, $\gamma_b(T'' - u_{c,\alpha}) = k-1$. Since $T'' - v_{d-2} \cong T'' - u_{c,\alpha}$, $\gamma_b(T'' - v_{d-2}) = k-1$, and by the induction hypothesis $T'' - v_{d-2}$ has a multipacking M' of cardinality $k-1$. Let $M = M' \cup \{v_d\}$; we claim that M is a multipacking of T. This construction is illustrated in Figure 4.6.

Suppose to the contrary that there exists $v \in V(T)$ such that $|N_s[v] \cap M| \geq s+1$ for some $s \geq 1$. If $v \in T - \{v_{d-2}, v_{d-1}, v_d\}$ then we obtain a contradiction as in Case 1 in the proof of Lemma 4.11. Thus assume that $v \in \{v_{d-2}, v_{d-1}, v_d\}$. By definition of M, $|N_s[v] \cap M| \leq s$ for any $s \leq 2$. For any $s \geq 3$, $N_s[v] \subseteq N_s[v_{d-3}]$. However, by the above, $|N_s[v_{d-3}] \cap M| \leq s$. Therefore $|N_s[v] \cap M| \leq s$. It follows that M is a multipacking of T and $\mathrm{mp}(T) = k$.

Case 3 T has one trailing free edge. Suppose v_{c-1} is a branch vertex of T and let x be the leaf of B_{c-1}. If $\ell(B_{c-1}) < \ell(B_c)$, then Δ_{c-1} is a nested triangle, which is not the case. If $\ell(B_{c-1}) > \ell(B_c)$, then Δ_c is a nested triangle, which is also not the case. If $\ell(B_{c-1}) = \ell(B_c) = \alpha$, then the last edge e'_ℓ of Δ_{c-1} is an edge of Δ_c and thus not a free edge of $T - x$. The first edge e'_f of Δ_{c-1} is $v_{d-2\alpha-2}v_{d-2\alpha-1}$. Since $d(v_{d-2\alpha-1}, v_d) = 2\alpha+1$, which is odd, e'_f is not a split-edge of $T - x$. Therefore the leaf x of B_{c-1} does not bind any edges of P. By Theorem 2.6, $\gamma_b(T-x) = \gamma_b(T)$, which, again, is not the case. Hence v_{c-1} is not a branch vertex of T.

Let T' be the tree obtained by deleting the branch B_c (but not v_c) and adding the branch $B_{c-1} : u_{c-1,1}, ..., u_{c-1,\alpha}$. By Lemma 4.9, $\gamma_b(T') = \gamma_b(T) = k$. Also, T' has



(i) T with two trailing free-edges

(ii) $T'' = T - \{v_{d-1}, v_d\}$

(iii) $T' = T'' - \{v_{d-2}\}$

(iv) M' on T'

(v) M on T

Figure 4.6: The construction of Case 2 of Theorem 4.4.

two trailing free edges and, as shown in Case 2, $\text{mp}(T') = k$. Again by Lemma 4.9, $\text{mp}(T) \geq k$ and we are done. ■

4.2.3 Finding a Maximum Multipacking of a Tree

In Subsection 4.2.2, we proved that $\text{mp}(T) = \gamma_b(T)$ for any tree T. We also noted in Subsection 2.1.4 that $\gamma_b(T)$ (and $\text{mp}(T)$) can be found in linear time [8]. However, we have not yet addressed the problem of finding a maximum multipacking. In this subsection we apply the insight gained in the proof of Theorem 4.4 to develop an algorithm for finding a maximum multipacking of a tree.

By Theorem 2.5, the shadow tree $S_{T,P}$ of T with diametrical path P has $\gamma_b(S_{T,P}) = \gamma_b(T)$. It follows that any maximum multipacking M of $S_{T,P}$ will also be a maximum multipacking of T. Thus our algorithm need only consider shadow trees. Furthermore, as in the proof of Theorem 4.4, we limit our algorithm to trees in $\mathcal{T}_{k,d}$, and hence ignore all nested triangles.

We again rely heavily upon the notation developed in Subsection 2.1.2. Notice that v_c, v_d, Q_c, B_c and Δ_c depend upon the choice of P for a diametrical path. For Δ_{i_1}, the first triangle of T, we define $Q_1 = \{v_0, v_1, \ldots, v_{i_1}\}$ as the *leading endpath of* T. Similarly to Q_c, Q_1 is dependent upon the choice of P.

We first explain the steps of the algorithm informally.

- Begin with a shadow tree T with no nested triangles; we will find M, a maximum multipacking of T.

- Create an empty set of vertex pairs S. As the algorithm progresses, we will store vertices in S that will need to be checked at the end.

- If T is any of the paths P_1, P_2 or P_3, then $M = \{v_0\}$ and we are done.

- Otherwise, repeat the following process of reducing T (and P) until T is P_1, P_2 or P_3, adding vertices to M at each step.
  - Check to ensure that P is still a diametrical path. If at any stage of the process, B_c is the first (and hence only) branch and $l(B_c) > l(Q_1)$, then change P to be $P = B_c \cup Q_c$. Now, the vertices formerly in Q_1 become vertices on a branch.
  - If a nested triangle Δ_{i_j} at v_{i_j} results, delete $B_{i_j} - \{v_{i_j}\}$ from T.
  - Perform one of the following steps depending upon the number of trailing free edges of T.

    Case 1 T has at least three trailing free edges. Add v_d to M and delete $\{v_d, v_{d-1}, v_{d-2}\}$ from P and T. Thus the vertex that was previously labelled v_{d-3} is now labelled v_d.

    Case 2 T has two trailing free edges. Add v_d to M and delete $\{v_d, v_{d-1}\}$ from P and T. Thus T now has no trailing free edges. As mentioned in Case 2 of the proof of Theorem 4.4, $\gamma_b(T) = \gamma_b(T - u_{c,\alpha})$, and $T - u_{c,\alpha} \cong T - v_d$, so delete v_d (formerly v_{d-2}) from T. This makes $l(Q_c) < l(B_c)$, so switch Q_c and B_c in P. Thus the vertices formerly on Q_c now become branch vertices, and the vertices formerly on B_c become vertices on the endpath.

    Case 3 T has one trailing free edge. In Case 3 of the proof of Theorem 4.4 we show that there is no branch B_{c-1} in T. Let T' be the tree obtained from T by deleting B_c (but not v_c) and adding the branch $B_{c-1} = \{u_{c-1,1}, u_{c-1,2,}, \ldots, u_{c-1,\alpha}\}$. That is, $T' = T - (B_c - \{v_c\}) \cup B_{c-1}$. Recall that $\gamma_b(T) = \gamma_b(T')$ and that T' has two trailing free edges. Add (v_c, v_{c-1}) to S. If at the end of the algorithm, v_c is in M, then (as in Case

2 of Lemma 4.8) we will need to swap v_c for $v_{c-1} \in M$. Now set T to T'. In the next iteration of the algorithm, T has at least two trailing free edges and M can be expanded using Case 1 or 2.

Case 4 T has no trailing free edges. By Lemma 4.6, $\gamma_b(T - u_{c,\alpha}) = \gamma_b(T)$; delete $u_{c,\alpha}$ from T. Now T has at least one trailing free edge. In the next iteration, T will have at least two trailing free edges.

Once T has been reduced to P_1, P_2 or P_3, add v_0 to M.

- Lastly, we must check the vertices in S against M. For each $(u, v) \in S$ If $u \in M$, remove u from M and add v to M instead.

  Then we are done.

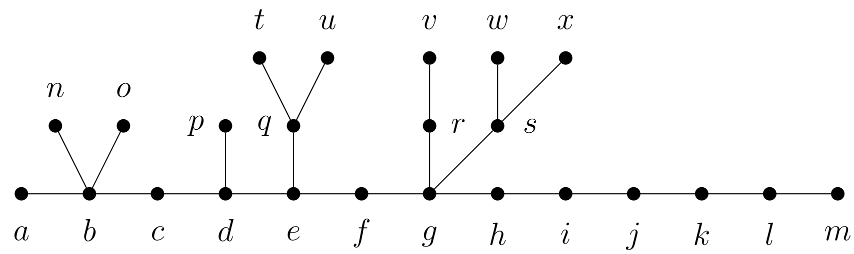
We summarize these steps in Algorithm 4.1 below. An example of this process is given in Figure 4.7.

Algorithm 4.1: FindTreeMP finds a maximum multipacking of a tree

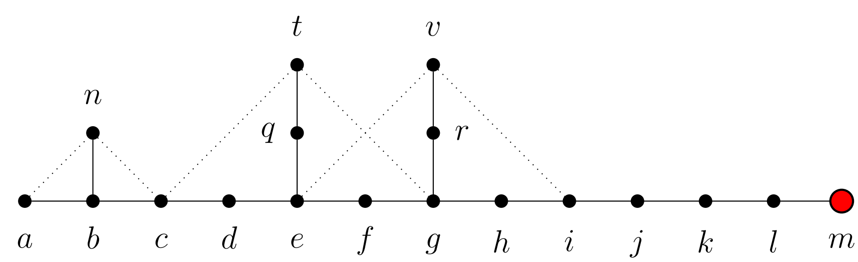
Input: Shadow tree T with no nested triangles, diametrical path $P = \{v_0, v_1, \ldots v_d\}$

Output: A maximum multipacking M of T

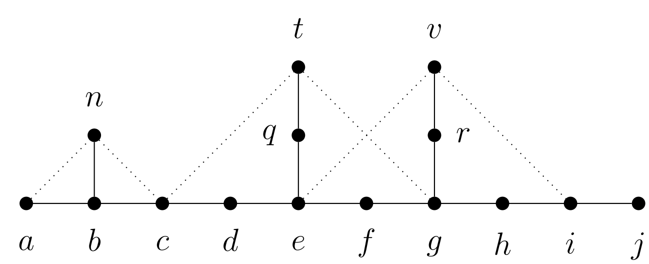
$M \leftarrow \varnothing$
$S \leftarrow \varnothing$
while $T \neq P_1, P_2, P_3$ **do**
    if $c = i_1$ *and* $l(B_c) > l(Q_1)$ **then**
        $P \leftarrow P - Q_1 + B_c$
        comment: Q_1 becomes B_c
    end
    if Δ_{i_j} *is a nested triangle* **then**
        $T \leftarrow T - (B_{i_j} - \{v_{i_j}\})$
    end
    if *number of trailing free edges* ≥ 3 **then**
        $M \leftarrow M \cup \{v_d\}$
        $P \leftarrow P - \{v_d, v_{d-1}, v_{d-2}\}$
        $T \leftarrow T - \{v_d, v_{d-1}, v_{d-2}\}$
    else if *number of trailing free edges* $= 2$ **then**
        $M \leftarrow M \cup \{v_d\}$
        $P \leftarrow P - \{v_d, v_{d-1}, v_{d-2}\}$
        $T \leftarrow T - \{v_d, v_{d-1}, v_{d-2}\}$
        $P \leftarrow P - Q_c + B_c$
        comment: Q_c becomes B_c
    else if *number of trailing free edges* $= 1$ **then**
        $S \leftarrow S \cup \{(v_c, v_{c-1})\}$
        $T \leftarrow T \cup \{u_{c-1,1}, u_{c-1,2}, \ldots, u_{c-1,\alpha}\}$
        $T \leftarrow T - (B_c - \{v_c\})$
    else
        $T \leftarrow T - \{u_{c,\alpha}\}$
    end
end
$M \leftarrow M \cup \{v_0\}$;
forall the $(u, v) \in S$ **do**
    if $u \in M$ **then**
        $M \leftarrow (M - \{u\}) \cup \{v\}$
    end
end
return M



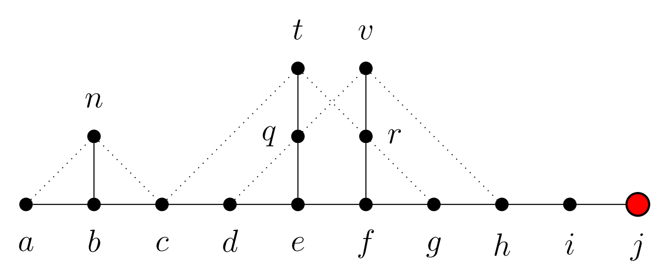
(1) A tree T' with diametrical path $P = \{a, b, c, \ldots, m\}$



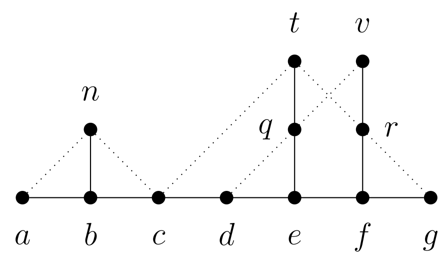
(2) Create shadow tree $T = S_{T',P}$ of T' with no nested triangles. Add m to M.



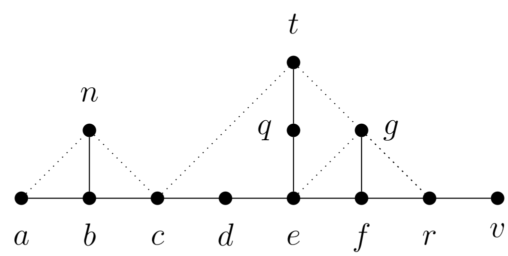
(3) Delete $\{k, l, m\}$ from T.



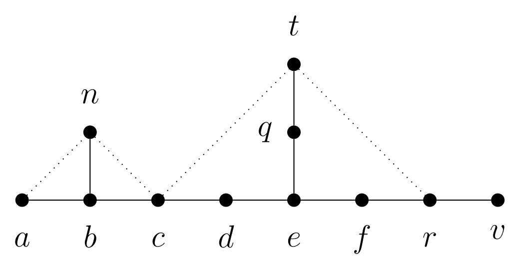
(4) Shift B_c to v_{c-1}. Add (g, f) to S. Add j to M.



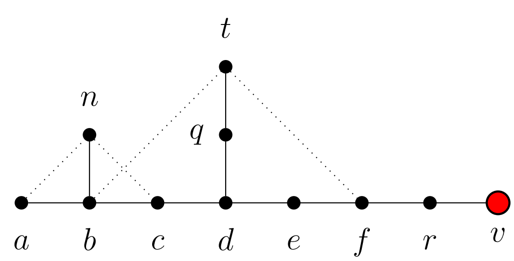
(5) Delete $\{h, i, j\}$ from T.



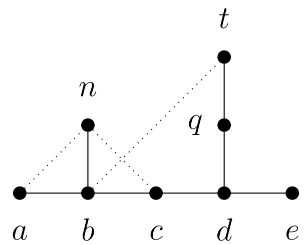
(6) Swap Q_c and B_c. P becomes $P - B_c \cup Q_c$.



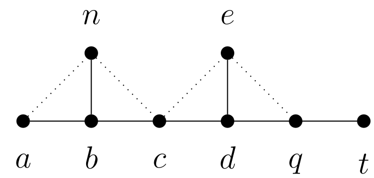
(7) Delete the nested triangle Δ_f.



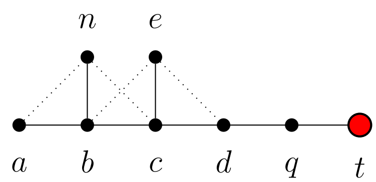
(8) Shift B_c to v_{c-1}. Add (e, d) to S. Add v to M.



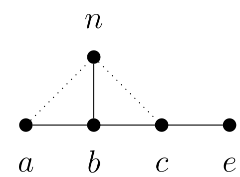
(9) Delete $\{f, r, v\}$ from T.



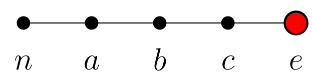
(10) Swap Q_c and B_c.



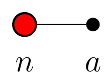
(11) Shift B_c. Add (d, c) to S. Add t to M.



(12) Delete $\{d, q, t\}$ from T.

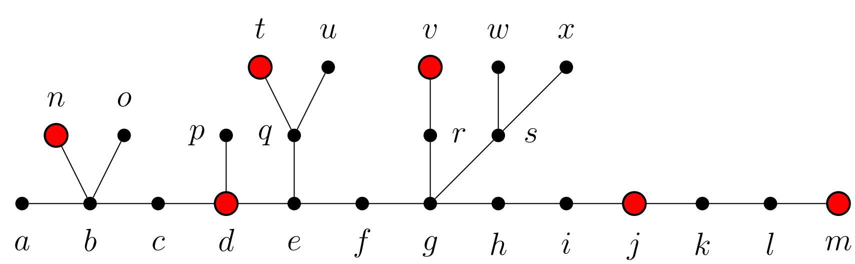


(13) Shift B_c. Add (b, a) to S. Add e to M.



(14) Delete $\{b, c, e\}$ from T. Add n to M.

(15) $M = \{m, j, v, t, e, n\}$, $S = \{(g, f), (e, d), (d, c), (b, a)\}$. Swap e and d in M.



(16) A maximum multipacking M of T'.

Figure 4.7: Example of Algorithm 4.1.

4.3 An Alternative Proof of Theorem 3.1

Having examined the origins of multipackings and their relationship with broadcasting on trees, we now present a useful application of multipackings for calculating $\gamma_b(G)$. In Theorem 3.1, we presented a new bound for $\gamma_b(G)$ in terms of $\text{ir}(G)$, and showed the existence of a graph G_k with $\text{ir}(G_k) = 2k$ and $\gamma_b(G_k) = 3k$, for any positive integer k. However, in this proof we had to demonstrate that our broadcast f was indeed a *minimum* dominating broadcast, a somewhat long and cumbersome endeavor.

Now that we have established in Section 4.1 that for any graph G, $\text{mp}(G) \leq \gamma_b(G)$, if we can find a maximum multipacking with cardinality equal to the cost of our dominating broadcast, we can verify that our broadcast is a minimum dominating broadcast. In this subsection we give a maximum multipacking of G_k, and thereby confirm that $\gamma_b(G_k) = 3k$.

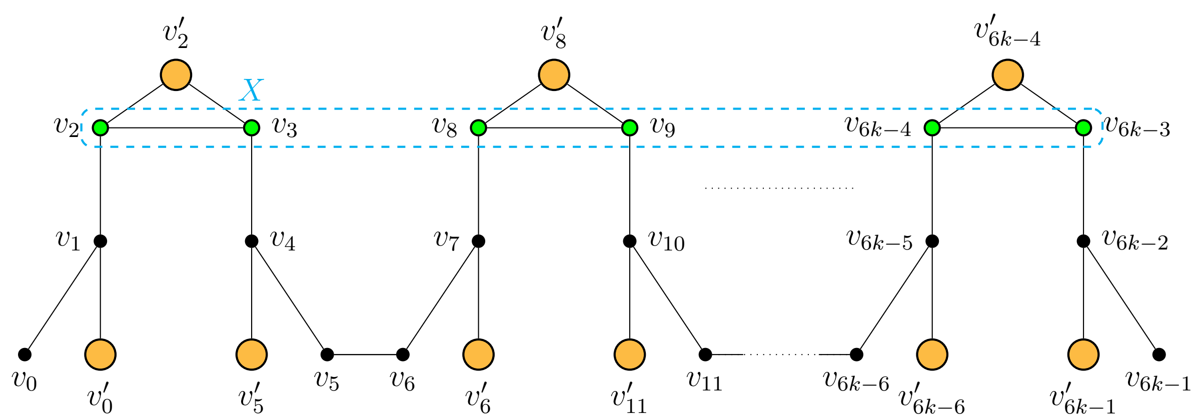


Figure 4.8: A graph G_k with $\text{ir}(G_k) = 2k$ and $\gamma_b(G_k) = 3k$.

Consider the vertex set $M_k = \{v_0', v_2', v_5', v_6', v_8', v_{11}', \ldots, v_{6k-4}', v_{6k-2}', v_{6k-1}'\}$ of G_k. In Figure 4.8, M_k is depicted as the large yellow (if viewed in colour) vertices.

Lemma 4.13. *Let v be a vertex of G_k. For any $s > \min\{d(v, v_0), d(v, v_{6k-1})\}$, if $|N_{s-1}[v] \cap M_k| \leq s - 1$, then $|N_s[v] \cap M_k| \leq s$.*

Proof. Consider arbitrary $v \in V(G_k)$ and assume $\min\{d(v,v_0), d(v,v_{6k-1})\} = d(v,v_{6k-1})$, i.e., $s > d(v,v_{6k-1})$.

$$\text{Let } U = \{u \in V(G_k) : d(u,v) = s\}.$$

If $|U| \leq 1$, the result follows, hence suppose $|U| \geq 2$. Since $s > d(v,v_{6k-1})$, U lies "to the left" of v and is one of the sets $\{v_{6i-3}, v'_{6i-1}\}$, $\{v_{6i-4}, v'_{6i-4}\}$, $\{v_{6i-6}, v'_{6i-6}\}$, for some $1 \leq i \leq k$. In each case $|U \cap M_k| = 1$. Hence $|N_s[v] \cap M_k| = |N_{s-1}[v] \cap M_k| + 1$ and again the result follows. By symmetry the result also holds if $\min\{d(v,v_0), d(v,v_{6k-1})\} = d(v,v_0)$. ■

Claim 4.14. *The set* $M_k = \left\{v'_0, v'_2, v'_5, v'_6, v'_8, v'_{11}, \ldots, v'_{6k-4}, v'_{6k-2}, v'_{6k-1}\right\}$ *is a multipacking of the graph* G_k *in Figure 4.8.*

Proof. Suppose to the contrary that there is some $v \in V(G_k)$ such that $|N_s[v] \cap M_k| \geq s+1$ for some s.

Case 1 v is on the diametrical path $P = \{v_0, v_1, v_2, v_3, \ldots, v_{6k-1}\}$. Let $v = v_i$, where

(a) i is the smallest index and

(b) subject to (a), s the smallest distance

such that $|N_s[v_i] \cap M_k| \geq s+1$. Clearly, $|N_s[v_0] \cap M_k| \leq s$ for all s, hence $v_0 \neq v_i$. Likewise, $v_i \neq v_{6k-1}$. Also, by our choice of M_k, $|N[v_i] \cap M_k| \leq 1$ for each i, thus $s \geq 2$. Let

$$\ell = \begin{cases} i-s & \text{if } i-s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $d(v_\ell, v_i) \leq s$.

Subcase 1.1 $\ell \not\equiv 0 \pmod 6$. We proceed by examining the left side of the s-neighbourhood of v_i. Define

$$V_\ell = \begin{cases} \{v_{\ell-1}, v'_{\ell-1}, v_{\ell-2}\} & \text{if } \ell \equiv 1, 3 \pmod 6 \\ \{v_{\ell-1}, v_{\ell-2}, v'_{\ell-2}\} & \text{if } \ell \equiv 2 \pmod 6 \\ \{v'_{\ell+1}, v_{\ell-1}, v_{\ell-2}, v'_{\ell-2}\} & \text{if } \ell \equiv 4 \pmod 6 \\ \{v'_\ell, v_{\ell-1}, v_{\ell-2}\} & \text{if } \ell \equiv 5 \pmod 6. \end{cases}$$

In each case $|V_\ell \cap M_k| \geq 1$. Furthermore, $N_s[v_i] \cup V_\ell = N_{s+1}[v_{i-1}]$. Since $|N_s[v_i] \cap M_k| \geq s+1$, $|N_{s+1}[v_{i-1}] \cap M_k| \geq s+2$, contradicting the choice of v_i.

Subcase 1.2 $\ell \equiv 0 \pmod{6}$, $i - s \geq 0$. The left side of the s-neighbourhood of v_i extends to v_ℓ and the only vertex in M_k at distance s from v_i and to the left of v_i is v'_ℓ. We now examine the right side of the s-neighbourhood of v_i. By our choice of s, $|N_{s-1}[v_i] \cap M_k| \leq s-1$, thus $|\,(N_s[v_i] - N_{s-1}[v_i]) \cap M_k| \geq 2$. This implies that there is a vertex $x \in M_k$ at distance s from v_i and to the right of v_i. Since $d(v_i, v'_{6(j-1)}) = d(v_i, v_{6j-4})$, $d(v_i, v'_{6j-4}) = d(v_i, v_{6j-3})$ and $d(v_i, v'_{6j-1}) = d(v_i, v_{6j-1})$ for each j,

$$i + s \equiv \begin{cases} 2 \pmod{6} & \text{if } x = v'_{6(j-1)} \\ 3 \pmod{6} & \text{if } x = v'_{6j-4} \\ 5 \pmod{6} & \text{if } x = v'_{6j-1}. \end{cases}$$

However, since $\ell = i - s \equiv 0 \pmod{6}$ is even, $i + s$ is even, hence $i + s \equiv 2 \pmod 6$. Suppose $i + s = 6r - 4$ for some $r \geq 1$. Now

$$N_{s-2}[v_i] \cup \{v_\ell, v'_\ell, v_{\ell+1}, v_{6r-4}, v_{6r-5}, v'_{6r-6}\} = N_s[v_i]$$

and

$$\{v_\ell, v'_\ell, v_{\ell+1}, v_{6r-4}, v_{6r-5}, v'_{6r-6}\} \cap M_k = \{v'_l, v'_{6r-6}\}.$$

Thus $|N_{s-2}[v_i] \cap M_k| \geq s+1-2 = s-1$, contradicting the choice of s.

Subcase 1.3 $\ell \equiv 0 \pmod 6$ and $i - s < 0$. By Lemma 4.13 and Subcase 1.2 when $i - s = 0$, it is immediate that $|N_s[v] \cap M_k| \leq s$ for all $s \geq 1$.

Case 2 v is not on P. Then $v \in V(G_k) - P = M_k$, so let $v = v'_i$. Choose v_x such that

$$v_x = \begin{cases} v_{i+1} & \text{if } i \equiv 0 \pmod 6 \\ v_i & \text{if } i \equiv 2 \pmod 6 \\ v_{i-1} & \text{if } i \equiv 5 \pmod 6. \end{cases}$$

Then v_x is on P and is adjacent to v'_i. Clearly $N_s[v'_i] \subseteq N_s[v_x]$ for all $s \geq 1$. Thus $|N_s[v'_i] \cap M_k| \leq |N_s[v_x] \cap M_k| \leq s$ by Case 1, a contradiction.

Therefore, for all $v \in V(G_k)$ and all $s \geq 1$, $|N_s[v] \cap M_k| \leq s$, and M_k is a multipacking of G_k. ■

Since M_k is a multipacking of G_k with cardinality $3k$, this confirms that $\gamma_b(G_k) = 3k$.

Chapter 5

Conclusions

In Chapter 3 we presented a new upper bound for the broadcast number of a graph in terms of its irredundance number. In Chapter 4 we derived a new dual property of the broadcast number called the multipacking number. We then showed that the multipacking and broadcast numbers of a tree are equal. From this proof, we derived an algorithm for finding a maximum multipacking of a tree. Finally, we demonstrated the usefulness of multipackings in verifying the minimality of a dominating broadcast by providing an alternative proof to the second portion of Theorem 3.1.

5.1 Future Work

We conclude this thesis by presenting some open problems for future research.

Although we demonstrated a few properties of graphs with $\gamma_b(G) = \frac{3}{2}\operatorname{ir}(G)$, we did not completely characterize this class of graphs.

Problem 1. *Characterize graphs G with $\gamma_b(G) = \frac{3}{2}\operatorname{ir}(G)$.*

Problem 2. *Characterize graphs G with $\gamma_b(G) = \operatorname{ir}(G)$.*

Problem 3. *Characterize graphs G with $\gamma_b(G) \leq \operatorname{ir}(G)$.*

The next open problems relate to multipackings. Although we have shown that the broadcast and multipacking numbers of any tree are equal, this is not the case for general graphs. For example, $\gamma_b(C_5) = 2$ but $\mathrm{mp}(C_5) = 1$. Conversely, there are graphs G that are not trees with $\gamma_b(G) = \mathrm{mp}(G)$; for example, $\gamma_b(C_6) = \mathrm{mp}(C_6) = 2$. A characterization of these graphs has not been studied.

Problem 4. *For any integer k, does there exist a connected graph G with $\gamma_b(G) - \mathrm{mp}(G) = k$?*

Problem 5. *Characterize graphs G with $\gamma_b(G) = \mathrm{mp}(G)$, or find other large classes of graphs for which these parameters are equal.*

The Cartesian product $K_n \,\square\, K_n$ satisfies $\rho(K_n \,\square\, K_n) = 1$ and $\gamma(K_n \,\square\, K_n) = n$. The question arises whether a similar result holds for the multipacking and broadcast numbers and some class of graphs. While $\mathrm{mp}(K_n \,\square\, K_n) = 1$, $\gamma_b(K_n \,\square\, K_n) \le 2$ for all $n \ge 2$. Indeed, by Lemma 4.5, if $\mathrm{mp}(G) = 1$, then $\mathrm{diam}(G) \le 2$ and hence $\gamma_b(G) \le \mathrm{rad}(G) \le \mathrm{diam}(G) \le 2$. The following problems remain open, though.

Problem 6. *Does there exist an integer m and an infinite class of graphs $\{G_k\}$ such that $\mathrm{mp}(G_k) = m$ and $\gamma_b(G_k) \ge k$ for all $k \ge m$?*

Problem 7. *Can the ratio γ_b / mp be arbitrary?*

Problem 8. *Determine the complexity of finding $\mathrm{mp}(G)$ for a general graph G, or for special classes of graphs such as a trees, split-graphs, chordal graphs, etc.*

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